

1. Resistance is not Futile!

Learning Goal: Introduce the concept of resistivity and provide physical intuition for how changes in physical properties/dimensions of a resistor changes its resistivity. Read [Note 12 Section 12.3-12.4](#) to learn more.

Resistivity is a **physical property** of the material that quantifies how much it opposes the flow of electric current.

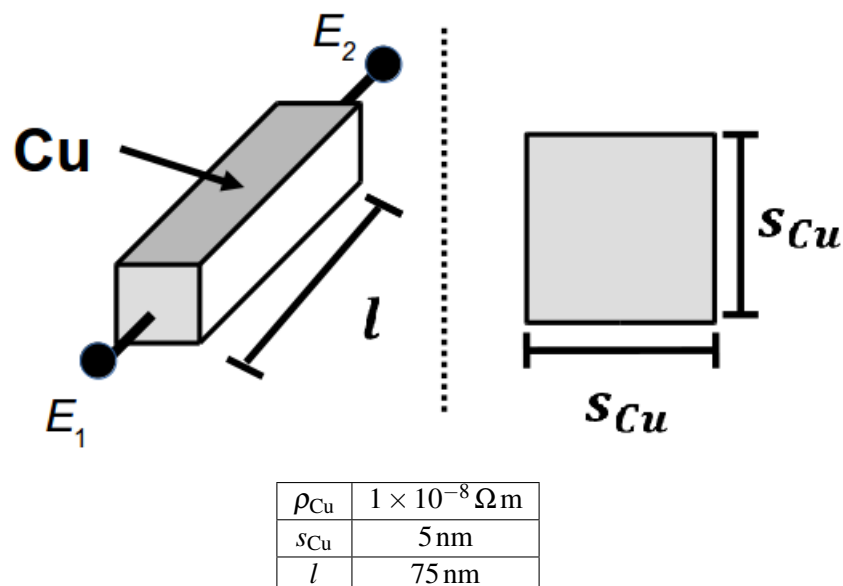
Assume that in an ideal case, the cross-section and physical composition of the wire are uniform, We can find its resistance with the equation below:

$$R = \rho \frac{L}{A}$$

Here, ρ stands for the resistivity of the wire, R stands for its resistance, A stands for the area of the cross section of the wire, and L stands for the length of the wire.

We will be frequently referencing some of the following variables:

- A : the cross section area of a single wire.
 - L : the length of a single wire.
 - ρ_{Cu} : resistivity for the material copper.
 - ρ_{Al} : resistivity for the material aluminum.
- (a) A copper (Cu) structure with a square cross-section is shown below. Given the material parameters, calculate the resistance R_{Cu} of the structure between E_1 and E_2 .

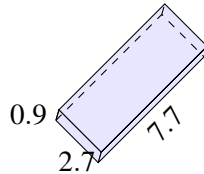


Answer:

Using the formula for resistance R , we can find the resistance of the Cu structure given the resistivity and the dimensions:

$$R = \rho \frac{l}{A} \Rightarrow R_{\text{Cu}} = \rho_{\text{Cu}} \frac{l}{s_{\text{Cu}}^2} = 1 \times 10^{-8} \Omega \text{m} \cdot \frac{75 \times 10^{-9} \text{m}}{(5 \times 10^{-9} \text{m})^2} = 30 \Omega$$

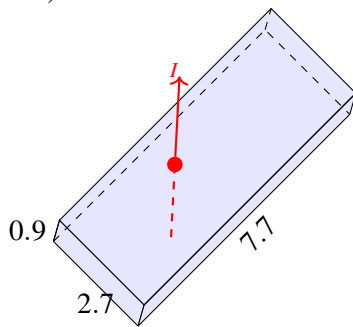
- (b) Now you are given a copper slab with dimensions 0.9cm, 2.7cm, and 7.7cm as denoted on the figure below. The dimensions and the resistivity of the slab remain the same throughout the rest of the problem.



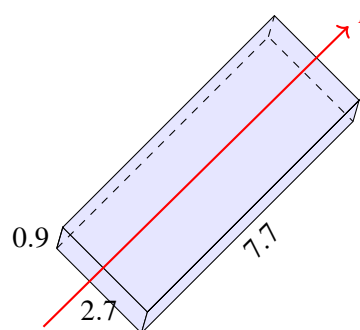
Suppose we connect opposite faces of the slab to a voltage source so that current can flow through it. Notice that there are **3 possible configurations** we can have for this slab, leading to 3 possible directions in which the current can flow (drawn in the answer choices below). Which direction will lead to the **highest** current flow I assuming we use the same voltage source all three times?

Assume the resistivity is the same throughout the slab and does not vary with respect to the direction of current flow.

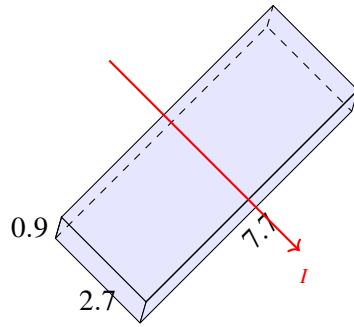
- i. Parallel to 0.9 (current goes through 2.7, 7.7 face):



- ii. Parallel to 7.7 (current goes through 0.9, 2.7 face):



iii. Parallel to 2.7 (current goes through 0.9, 7.7 face):



iv. Choices (i) and (iii) are tied for the maximum current flow.

v. The direction doesn't matter; the resistor is physically the same, so the current will be the same.

Answer: For a given supply voltage from a voltage source, the most current will flow when the resistance of the slab is smallest. We apply the geometrical resistance formula, $R = \frac{\rho_{cu} L}{A}$. To maximize current, we need to minimize resistance, which requires minimizing L and maximizing A . Two of the dimensions will comprise the area, and the third will form the length. Noticing that $0.9\text{cm} < 2.7\text{cm} < 7.7\text{cm}$, it follows that we want to feed current into the $2.7\text{cm} \times 7.7\text{cm}$ face and use 0.9cm as the length. Therefore, we select choice (i).

(c) Suppose we have N wires similar to the one described in part (a). We align them side by side to form a bundle of wires. Find the overall resistance of this bundle. Are the wires connected in series or parallel?

Answer: Since we have all N wires aligned side by side, we are essentially expanding the cross-section area. This means that the new bundle will have a new cross-section area of NA , while its length remains the same (L). Hence, the overall resistance of the mega-wire will be:

$$R_{mega} = \rho_{cu} \frac{L}{NA}$$

This is similar to a parallel combination of N wires. Note that the equivalent resistance is smaller for a parallel combination.

(d) **(PRACTICE)** Again how can we connect these N identical wires so that the equivalent resistance is the highest?

Answer: The key of this question is to start from the resistance equation:

$$R = \rho \frac{l}{A}$$

Algebraically, we want to maximize the value of R for this question. Since ρ is just a physical constant, we can't change its value. Observing the fraction $\frac{l}{A}$, we can see that the overall length of the mega-wire should be as great as possible, while its cross section area should be kept as small as possible. How can we arrange N wires in a way so that the overall new wire is as long as possible?

We can arrange the wires in a single long line! This keeps the cross section area A unchanged, but it has a length of NL . Applying the resistance equation, we have:

$$R_{mega} = \rho_{cu} \frac{NL}{A}$$

This configuration is exactly the same as a series connection of resistors. If you think in terms of the equivalent resistance for a series circuit, it also makes sense since we are summing up all the resistances.

- (e) **(PRACTICE)** Consider part (c) again, but this time, instead of N copper wires, we split the number evenly between aluminum wires and copper wires. We arrange $N/2$ copper and $N/2$ aluminum wires side by side, and push them to form a new bundle of wires. What is the overall resistance of this wire? (In terms of ρ_{Cu} , ρ_{Al} , L , and A)

Answer: As we can see from part (c), when we are aligning the wires side by side, we are essentially arranging the wires to be parallel to each other. We can consider the bundle of wires to be a parallel combination of a bundle of copper wires and a bundle of aluminum wires! For both wires, they will have a length of L and an overall cross section area of $(N/2)A$ (since we have $N/2$ wires for each category). Hence, applying the resistance equation again, we can find that:

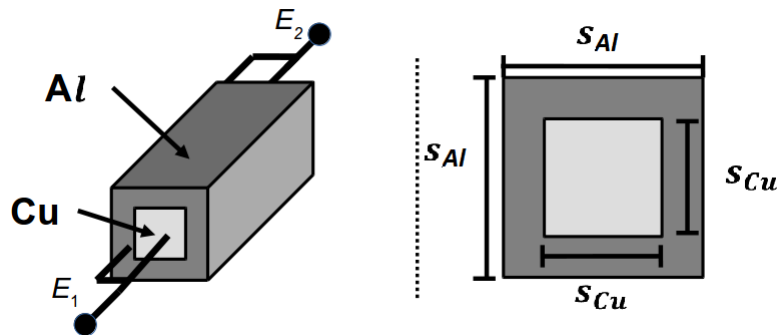
$$R_{Cu-bundle} = \rho_{Cu} \frac{L}{\frac{N}{2}A} = \rho_{Cu} \frac{2L}{NA}$$

$$R_{Al-bundle} = \rho_{Al} \frac{L}{\frac{N}{2}A} = \rho_{Al} \frac{2L}{NA}$$

Now, since these 2 mega wires are parallel to each other, by equivalent resistance, we can find the overall resistance of the wire to be:

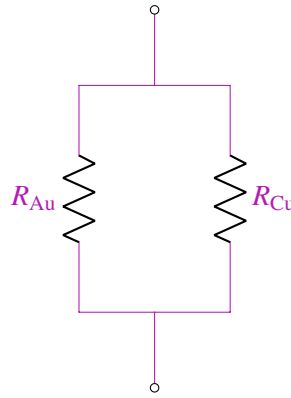
$$R_{overall} = \frac{1}{\frac{1}{R_{Cu-bundle}} + \frac{1}{R_{Al-bundle}}} = \left(\frac{\rho_{Cu}\rho_{Al}}{\rho_{Cu} + \rho_{Al}} \right) \frac{2L}{NA}$$

- (f) Now consider this *core-shell nanowire* structure, where the outside shell made of Al and the inside made of Cu. E_1 and E_2 terminals are both connected to the faces of the Cu and Al structure, as shown in the picture. Draw a circuit diagram of the nanowire as a set of resistors, using R_{Al} for the resistance of the Al layer and R_{Cu} for the resistance of the Cu layer.



Answer:

Given that we are contacting the full area and that current is flowing from end to end, each end can be treated as a node since each end will have the Au and Cu at the same potential. This means that we can model the core-shell nanowire as a set of parallel resistors.



- (g) Based on your model from part (f) and the parameters given below, find the equivalent resistance R_{wire} between E_1 and E_2 .

ρ_{Al}	$2 \times 10^{-8} \Omega \text{m}$
s_{Al}	10 nm
ρ_{Cu}	$1 \times 10^{-8} \Omega \text{m}$
s_{Cu}	5 nm
l	75 nm

Answer: Using the formula for resistance R , we can find the resistance of the Cu structure given the resistivity and the dimensions:

$$R = \rho \frac{l}{A} \Rightarrow R_{\text{Cu}} = \rho_{\text{Cu}} \frac{l}{s_{\text{Cu}}^2} = 1 \times 10^{-8} \Omega \text{m} \cdot \frac{75 \times 10^{-9} \text{m}}{(5 \times 10^{-9} \text{m})^2} = 30 \Omega$$

Using the formula for resistance R , we can find the resistance of the Al structure given the resistivity and the dimensions. In this case, the area of the Al structure is not just a square but rather the area of the Cu square subtracted from the area of the Al square (the area of a shell).

$$R_{\text{Al}} = \rho_{\text{Al}} \frac{l}{s_{\text{Al}}^2 - s_{\text{Cu}}^2} = 2 \times 10^{-8} \Omega \text{m} \cdot \frac{75 \times 10^{-9} \text{m}}{(10 \times 10^{-9} \text{m})^2 - (5 \times 10^{-9} \text{m})^2} = 20 \Omega$$

Because the two structures are in parallel, the total resistance R_{wire} is:

$$R_{\text{wire}} = R_{\text{Al}} \parallel R_{\text{Cu}} = \frac{20 \Omega \cdot 30 \Omega}{20 \Omega + 30 \Omega} = \frac{600 \Omega^2}{50 \Omega} = 12 \Omega$$

2. Equivalence in Resistive Networks

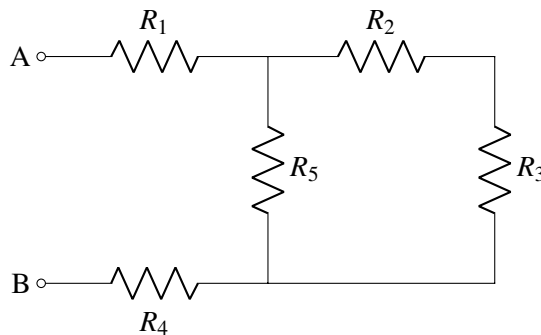
Learning Goal: Students will practice finding equivalences when given circuits with resistors in series and/or parallel.

Please look into [Note 15 Section 15.7.1 - 15.7.2](#) to see examples of finding circuit equivalences with resistors in series and resistors in parallel.

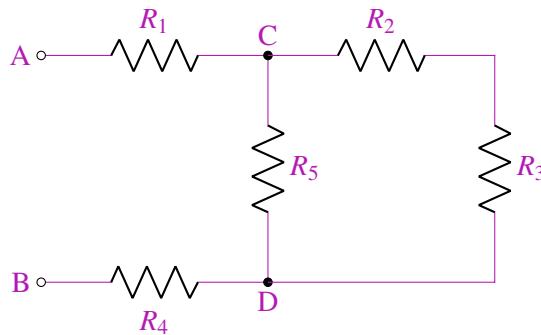
For all of the following networks find an expression or a numerical value for the equivalent resistance between terminals A and B.

Hint: You can use the equivalence formulas for series and parallel combinations of resistors for all of the subparts in this question.

(a)

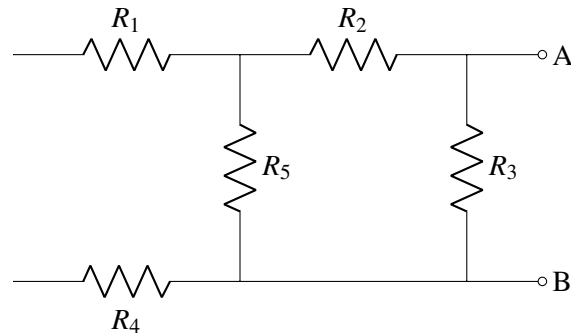


Answer:



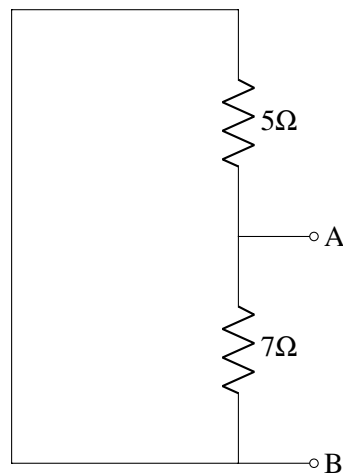
The easiest way to simplify a network if unsure, is to label more nodes and try to make smaller simplifications steps. For example, after introducing terminals C and D, we can see that we can represent the equivalent resistance in between those terminals as R_5 in parallel with the series combination of R_2 and R_3 , so $R_{CD} = R_5 || (R_2 + R_3)$. Finally, we can see that R_{CD} is in series with both R_1 and R_4 , as we look right from terminals A, B. This gives us the final result: $R_{AB} = R_1 + R_5 || (R_2 + R_3) + R_4$.

(b)



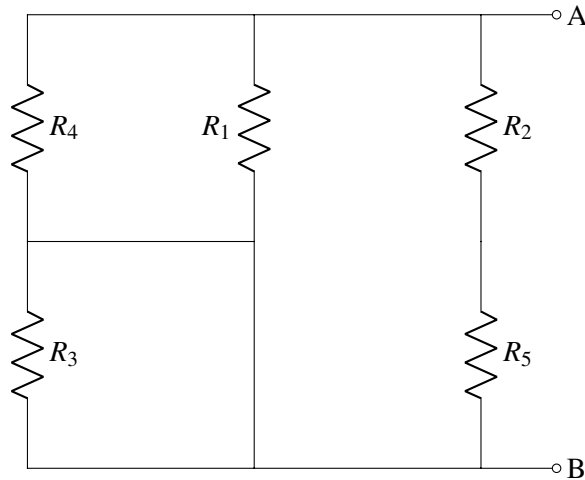
Answer: In this question it is crucial to realize, resistors R_1 and R_4 are connected to an open-circuit so they don't contribute to the overall resistance looking left from terminals A, B. After we delete them from our circuit diagram, we can get the answer as $R_{AB} = R_3 || (R_2 + R_5)$.

(c)



Answer: Here we need to notice that terminal B is connected not only to the bottom of the 7Ω resistor, but also to the top of the 5Ω resistor. Thus, the two resistors have both terminals common, are connected in parallel and the equivalent resistance can be computed as: $R_{AB} = \frac{7\Omega \cdot 5\Omega}{7\Omega + 5\Omega} = 2.917\Omega$.

(d)



Answer: Notice that here resistor R_3 is shorted (in other words connected in parallel with a wire). This means that it effectively can be removed from the circuit diagram, which leaves us with R_4 in parallel with R_1 and in parallel with the series combination of R_2 and R_5 , hence $R_{AB} = R_4 || R_1 || (R_2 + R_5)$.

3. Midterms are a lot of Pressure

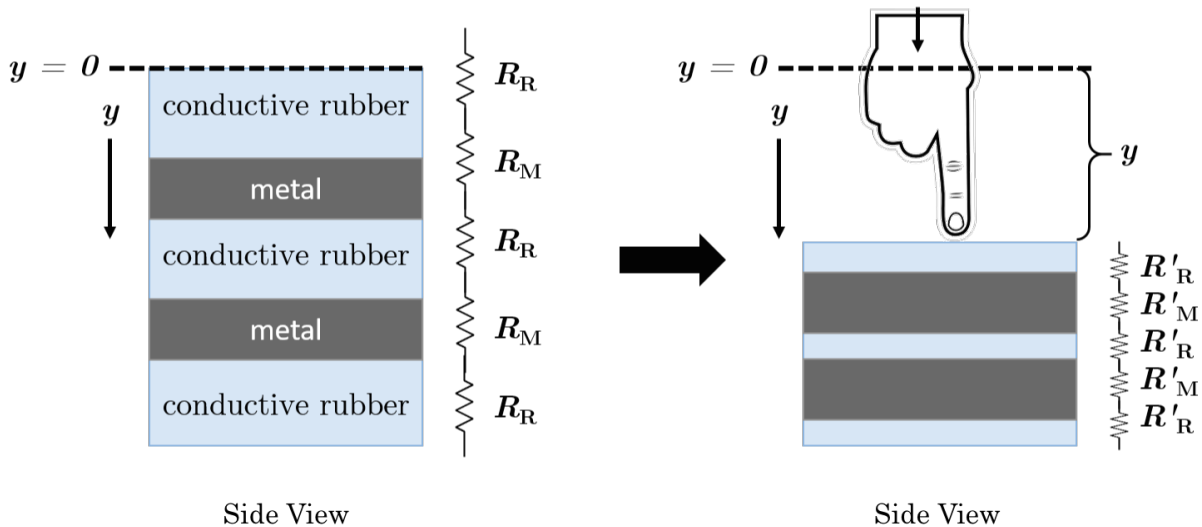
Learning Goal: Students will apply the concept of resistive sensors to a midterm style problem.

Please look into [Note 14 Section 14.4](#) to understand resistive sensors and convert physical models to circuits.

For this problem, we use something called “pressure sensitive rubber,” which incorporates conductive rubber and metal into one system. As the rubber is pressed, the conductive rubber portions are compressed, which changes the resistance. The metal plates do not change dimensions.

The pressure sensitive rubber system is shown below, with a resistive model next to the diagram. The resistivity of rubber and metal are represented by ρ_R and ρ_M respectively. When the system is at rest (no touch), the resistances of the rubber and metal are represented by R_R and R_M . The area of the sensor, as seen from above, is A .

To use the material, a finger presses on top of the system, compressing the rubber regions, creating a change in resistance, also shown below. Please answer the following questions related to the system.



- (a) Is the resistor model implementing resistors in series or parallel?

Answer: The resistors are in series.

- (b) If the values are $R_R = 1 \text{ k}\Omega$ and $R_M = 10 \Omega$, what is the total resistance before pressing the system?

Answer: There are three R_R resistors and two R_M , so $R_{total} = 3R_R + 2R_M = 3 \text{ k}\Omega + 20 \Omega = 3.02 \text{ k}\Omega$.

- (c) During the press, the length of each rubber portion is reduced by a factor of 5. (Its length is now 1/5 of its original value.) The size of the metal plates does not change. What is the new total resistance during a press?

Answer: If the length of conductive rubber is reduced by a factor of 5, our R_R is reduced by a factor of 5:

$$R_R = \rho_R \frac{l}{A} \rightarrow R'_R = \rho_R \frac{l/5}{A} = \frac{R_R}{5}$$

Since the length of metal is not changing, the metal resistance remains the same:

$$R'_M = R_M$$

So the total resistance is:

$$R'_{total} = 3R'_R + 2R'_M = \frac{3 \text{ k}\Omega}{5} + 20 \Omega = 620 \Omega$$

- (d) The force required to compress the rubber is $F = ky$, where k is a constant and y is the distance compressed (from the origin). Derive an expression for the resistance as a function of the pressing force F .

Write your answer in terms of the initial resistances (R_R and R_M), the resistivities (ρ_R and ρ_M), the area of the sensor, A , and the constant, k . Assume all rubber layers compress the same amount and uniformly.

Answer: First, let's consider the total resistance of just the rubber. If we press the rubber such that the rubber compresses by an amount y , this means we've reduced the length of our conductive rubber region to $3l - y$, making our resistance of the rubber region:

$$\begin{aligned}
 R_{R-press} &= \rho_R \frac{3l - y}{A} \\
 &= \rho_R \frac{3l - F/k}{A} \\
 &= \rho_R \frac{3l}{A} - \rho_R \frac{F/k}{A}
 \end{aligned}$$

In the set of equations above, it's important to note that the decrease by a distance of y is applied to all 3 resistors together.

The first term is equivalent to the initial (no press) resistance of one segment of rubber times 3: $3\rho_R \frac{l}{A}$. We can write this as, $3R_R$.

$$R_{R-press} = 3R_R - \rho_R \frac{F}{kA}$$

To get the total resistance of the sensor, we can add the resistance of the metal portions. Since the metal portions do not change, their resistance is still $2R_M$.

$$R_{total}(F) = 3R_R - \rho_R \frac{F}{kA} + 2R_M$$