Week 4 Cribsheet

Definitions

- Given $A \in \mathbb{R}^{n \times n}$, A has an eigenvector $\vec{x} \in \mathbb{R}^n$ and an eigenvalue $\lambda \in \mathbb{R}$ if $A\vec{x} = \lambda\vec{x}$
- The determinant of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is written either $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. For 2x2 matrices, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- Given an eigenvalue, the eigenspace is the space spanned by the corresponding eigenvectors.
- The characteristic polynomial of a matrix $A \in \mathbb{R}^{2 \times 2}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A - \lambda I) = \lambda^2 - (a+d)\lambda + (ad - bc)$, and is used to find λ's.
- Given a system or state transition matrix P, the steady state vector $\vec{x}^* \in \mathbb{R}^n$ satisfies $\vec{x}^* = P\vec{x}^*$
- Notation: $A : U \to V$ means the transformation A takes in a vector from input vector space U and outputs a vector from vector space V
- The standard basis of $\mathbb{R}^n$ is the columns of the $n \times n$ identity matrix.
- A matrix $T \in \mathbb{R}^{n \times n}$ is diagonalizable if it has n linearly independent eigenvectors

Given a basis B as a "coordinate system" of a vector space V,

$\forall \vec{x} \in V$ we can write $\vec{x} = a_0 B[0] + a_1 B[1] + \dots \Rightarrow$ coordinates of $\vec{x}$ in the new space are $(a_0, a_1, \dots, a_{n-1})$

Change of Basis

$$\text{Basis } \{\vec{a_1}, \dots, \vec{a_n}\} \underset{A^{-1}}{\overset{A}{\rightleftarrows}} \text{Standard basis} \underset{B^{-1}}{\overset{B}{\rightleftarrows}} \text{Basis } \{\vec{b_1}, \dots, \vec{b_n}\}$$

where $A = \begin{bmatrix} | & & | \\ \vec{a_1} & \cdots & \vec{a_n} \\ | & & | \end{bmatrix}$ and $B = \begin{bmatrix} | & & | \\ \vec{b_1} & \cdots & \vec{b_n} \\ | & & | \end{bmatrix}$

For transformations: given $T: \mathbb{R}^n \to \mathbb{R}^n$:

$$\begin{tikzcd}
\vec{u} \arrow[r, "T"] \arrow[d, shift right, "A^{-1}"'] & \vec{v} \arrow[d, shift right, "A^{-1}"'] \\
\vec{u_a} \arrow[r, "T_a"'] \arrow[u, shift right, "A"'] & \vec{v_a} \arrow[u, shift right, "A"']
\end{tikzcd}$$

$$T_a = A T A^{-1}$$
$$T = A^{-1} T_a A$$

Diagonalization

$A = V D V^{-1}$ $\quad V = \begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_n} \\ | & & | \end{bmatrix}$ $\quad D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$

where $(\lambda_i, \vec{v_i})$ $\forall i \in [1, n]$ are eigen_'s.