CSM 16A Designing Information Devices and Systems I
Fall 2020 Week 8

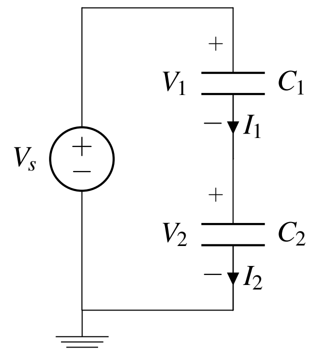
1. Series and Parallel Capacitors

Learning Goal: This problem will help to understand how capacitors in series or parallel combination respond to a voltage source or a current source.

Relevant Notes: **Note 16: Section 16.3** goes over the capacitance equivalence.

Find the voltage across and current through each capacitor for each of the following scenarios. Consider all the capacitors to be initially uncharged.

(a) C_1 and C_2 are in series:



Answer: The capacitors are in series combination, so the equivalent capacitance is given by

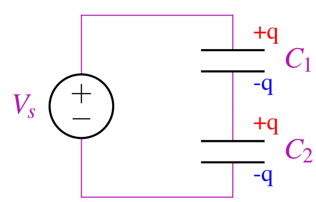
$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}.$$

The total charge for the equivalent capacitance is given by:

$$q = C_{eq} V_S$$

Now in series connection, both capacitors have the same charge on them, which is equal to $q = C_{eq} V_S$:

$$q_1 = q_2 = C_{eq} V_S$$



Since we know the charges on both capacitors, we can calculate both voltages by

$$\begin{aligned} q_1 = C_1 V_1 &\implies V_1 = \frac{q_1}{C_1} \\ &\implies V_1 = \frac{q}{C_1} \\ &\implies V_1 = \frac{C_{eq} V_S}{C_1} \\ &\implies V_1 = \frac{C_{eq}}{C_1} V_S \\ &\implies V_1 = \frac{C_2}{C_1 + C_2} V_S \end{aligned}$$

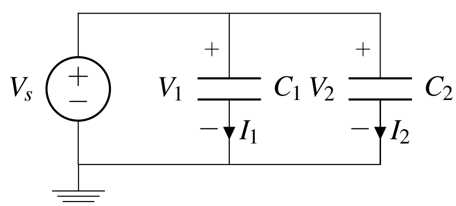
Similarly we have,

$$q_2 = C_2 V_2 \implies V_2 = \frac{q_2}{C_2} \implies V_2 = \frac{C_1}{C_1 + C_2} V_S$$

(Note that this is very similar to the current divider equation.)
Since the rate of change of voltage is zero for both of the capacitors, the current through them is zero, i.e.

$$\begin{aligned} I_1 &= C_1 \frac{dV_1}{dt} = 0 \\ I_2 &= C_2 \frac{dV_2}{dt} = 0 \end{aligned}$$

(b) C_1 and C_2 are in parallel:



Answer: The voltages across the capacitors are the same and they are the same as V_s, i.e.

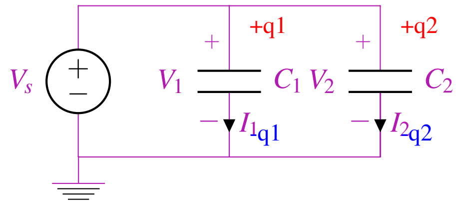
$$V_1 = V_2 = V_s$$

Since the rate of change of voltage is zero for both of the capacitors, the current through them is zero, i.e.

$$\begin{aligned} I_1 &= C_1 \frac{dV_1}{dt} = C_1 \frac{dV_s}{dt} = 0 \\ I_2 &= C_2 \frac{dV_2}{dt} = C_2 \frac{dV_s}{dt} = 0 \end{aligned}$$

Since we know the voltages across both capacitors, we can calculate the charge on both capacitors:

$$\begin{aligned} q_1 &= C_1 V_1 = C_1 V_s \\ q_2 &= C_2 V_2 = C_1 V_s \end{aligned}$$



The capacitors are in parallel combination, so the equivalent capacitance is given by

$$C_{eq} = C_1 + C_2.$$

The total charge for the equivalent capacitance is given by:

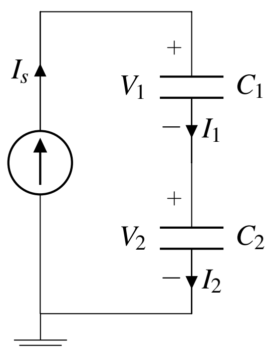
$$q = C_{eq} V_S$$

Now in parallel connection, the sum of charges on both capacitors is equal to $q = C_{eq} V_S$:

$$q = q_1 + q_2 = C_{eq} V_S = (C_1 + C_2) V_s,$$

which is the same

(c) C_1 and C_2 are in series:



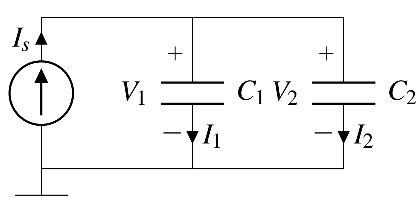
Answer: The capacitors are in series combination, so the current through them is the same as the current source I_s:

$$I_1 = I_2 = I_s$$

Since we know the current through the capacitors, we can calculate the voltages by:

$$V_1 = \int_0^t \frac{I_1}{C_1} dt = \int_0^t \frac{I_s}{C_1} dt = \frac{I_s}{C_1} t - V_1(0) = \frac{I_s}{C_1} t$$
$$V_2 = \int_0^t \frac{I_2}{C_2} dt = \int_0^t \frac{I_s}{C_2} dt = \frac{I_s}{C_2} t - V_2(0) = \frac{I_s}{C_2} t$$

(d) C_1 and C_2 are in parallel:



Answer: The capacitors are in parallel, so the voltage across them is the same as the voltage source V_s:

$$V_1 = V_2 = V_{eq}.$$

The equivalent capacitance is given by

$$C_{eq} = C_1 + C_2.$$

Since we know the total current through equivalent C_{eq} is equal to I_s, we can calculate the voltage V_{eq} by:

$$V_{eq} = \int_0^t \frac{I_s}{C_{eq}} dt = \frac{I_s}{C_{eq}} t - V_{eq}(0) = \frac{I_s}{C_{eq}} t = \frac{I_s}{C_1 + C_2} t$$

So the voltages on the capacitors are:

$$V_1 = V_{eq} = \frac{I_s}{C_1 + C_2} t;$$

$$V_2 = V_{eq} = \frac{I_s}{C_1 + C_2} t.$$

Since we found the voltages for both capacitors, the current can be calculated as:

$$I_1 = C_1 \frac{dV_1}{dt} = C_1 \frac{d}{dt}\left(\frac{I_s}{C_1 + C_2} t\right) = C_1 \frac{I_s}{C_1 + C_2} = \frac{C_1}{C_1 + C_2} I_s$$
$$I_2 = C_2 \frac{dV_2}{dt} = C_2 \frac{d}{dt}\left(\frac{I_s}{C_1 + C_2} t\right) = C_2 \frac{I_s}{C_1 + C_2} = \frac{C_2}{C_1 + C_2} I_s$$

(Note that this is very similar to the voltage divider equation.)

2. Capacitive Touchscreen

Learning Goal: The goal of this problem is to model the capacitive touchscreen covered in lecture.

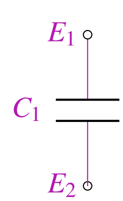
Relevant Notes: **Note 17 Section 17.1** introduces the capacitive touchscreen and its circuit model. **Note 16 Section 16.3** is helpful for creating the model, as it goes over capacitor equivalence.

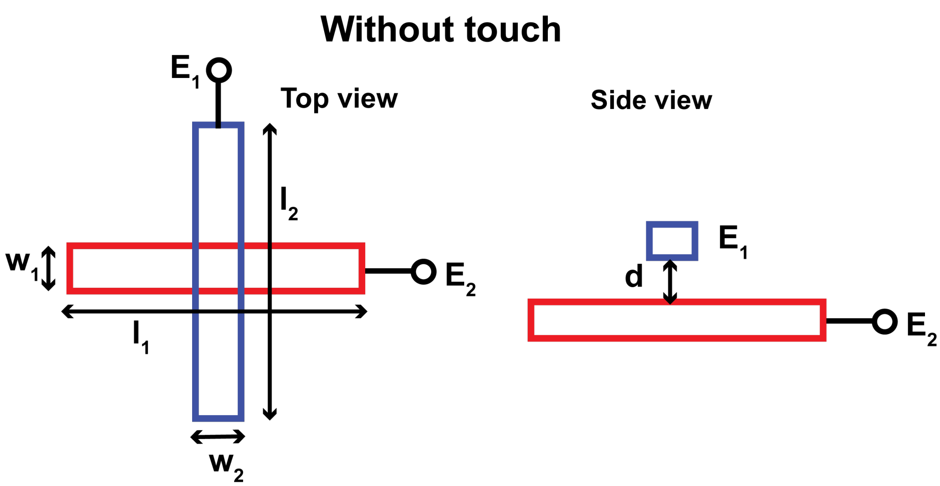
Consider the following capacitive touchscreen configuration from lecture.

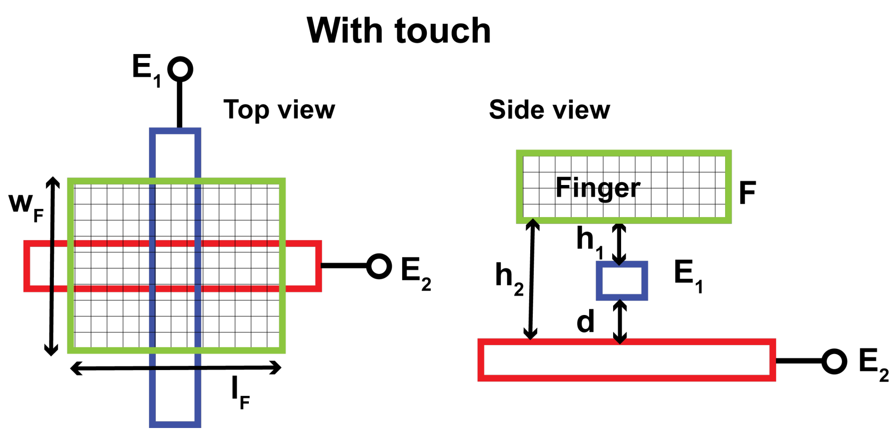
For the following parts, let $\varepsilon = 10^{-11}$F/m, $w_1 = 1$cm, $w_2 = 1$cm,$w_F = 3$cm, $l_1 = 5$cm, $l_2 = 5$cm, $l_F = 4$cm, $d = 5$mm, $h_1 = 5$mm, and $h_2 = 15$mm.

(a) Draw a diagram representing the capacitance between E_1 and E_2 when there is no touch on the screen.

Answer: When there is no touch, we can construct a model where the distance d between E_1 and E_2 has a capacitance. We call this capacitor C_1.







(b) Calculate the value of the capacitance between the two electrodes E_1 and E_2 when the screen is not being touched. Remember that $\varepsilon = 10^{-11}\text{F/m}$, $w_1 = 1\text{cm}$,$w_2 = 1\text{cm}$, and $d = 5\text{mm}$.

Answer: We use the formula

$$C_1 = \varepsilon A/d$$

where we are given $\varepsilon = 10^{-11}\text{F/m}$, area of overlap between $E_1$and E_2 is $A = w_1 \times w_2 = 1 \times 1\text{cm}^2$, and distance between $E_1$and E_2 is $d = 5\text{mm}$:

$$C_1 = \frac{(10^{-11}F/m)(1\text{cm}\times 1\text{cm})}{5\text{mm}} = \frac{(10^{-11}F/m)(10^{-2}\text{m}\times 10^{-2}\text{m})}{5\times 10^{-3}\text{m}} = 2\times 10^{-13}\text{F}$$

(c) Calculate (i) the capacitance between the finger and the top electrodes and (ii) the capacitance between the finger and the bottom electrodes, when the screen is being touched.

Answer: The overlapping area between electrodes F and E_1 is given by

$$A_{F-E1} = w_F \times w_2$$

while the distance between them is given by $h_1 = 5\text{mm}$. So we can calculate:

$$C_{F-E_1} = \frac{\varepsilon A_{F-E1}}{h_1} = \frac{\varepsilon w_F \times w_2}{h_1}$$

$$\implies C_{F-E_1} = \frac{(10^{-11}F/m)(3\text{cm}\times 1\text{cm})}{5\text{mm}} = \frac{(10^{-11}F/m)(3\times 10^{-2}\text{m}\times 10^{-2}\text{m})}{5\times 10^{-3}\text{m}} = 6\times 10^{-13}\text{F}.$$

The overlapping area between electrodes F and E_2 is given by

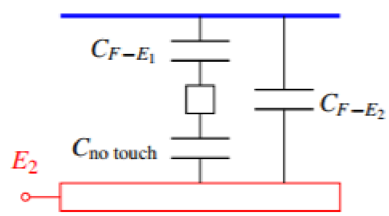
$$A_{F-E2} = l_F \times w_1 - w_2 \times w_1$$

while the distance between them is given by $h_2 = 15\text{mm}$. Note that the area of overlap is only the area that has a dielectric in between, i.e. any area between F and E_2 that has E_1 in it is excluded. So we can calculate:

$$C_{F-E_2} = \frac{\varepsilon A_{F-E2}}{h_2} = \frac{\varepsilon(l_F \times w_1 - w_2 \times w_1}{h_2})$$

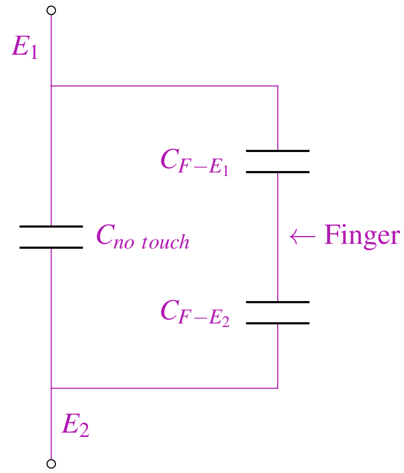
$$\implies C_{F-E_2} = \frac{(10^{-11}F/m)(4\text{cm}\times 1\text{cm} - 1\text{cm}\times 1\text{cm})}{15\text{mm}} = \frac{(10^{-11}F/m)(3\times 10^{-4}\text{m})}{15\times 10^{-3}\text{m}} = 2\times 10^{-13}\text{F}.$$

(d) Now consider what happens when we touch the screen. Let the blue line represent our finger, and assume there is a capacitance between your finger and each of the electrodes. The diagram looks like this:



Redraw the circuit diagram representing the capacitive touchscreen after being touched, so that the nodes representing E_1 and E_2 are on opposite ends of the diagram.

Answer: In order to model this situation as a circuit where E_1 and E_2 are on opposite sides, we attempt to find all paths from E_1 to E_2 (where E_1 is the middle bar between the blue line (our finger) and the red bar). The first path goes down from the middle bar and passes through $C_{no\ touch}$. The second path goes up to the finger, passing through C_{F-E_1}, and then down to the red bar, passing through C_{F-E_2}. So, we can construct a circuit where each path is a branch:



Notice that the left branch represents the circuit segment we got in part (a). Touching the capacitive touchscreen adds on the two new capacitances due to the finger.

(e) Calculate the new capacitance between E_1 and E_2. Remember that $C_{F-E_1} = 6 \times 10^{-13}$F and $C_{F-E_2} = 2 \times 10^{-13}$F. Has the effective capacitance changed from when there was no touch?

Answer: We begin by writing equations based on series and parallel capacitor equivalence. We know that

$$C_{E_1-E_2} = C_{no\ touch} + C_{finger}$$

where C_{finger} is the equivalent capacitance of the right branch. This is composed of two capacitors in series, so we can use the parallel operator ∥ to combine them.

$$C_{E_1-E_2} = C_{no\ touch} + C_{F-E_1} || C_{F-E_2}$$

Now we expand the parallel operator:

$$C_{E_1-E_2} = C_{no\ touch} + \frac{C_{F-E_1} C_{F-E_2}}{C_{F-E_1} + C_{F-E_2}}$$

Plugging the values of capacitors in we get:

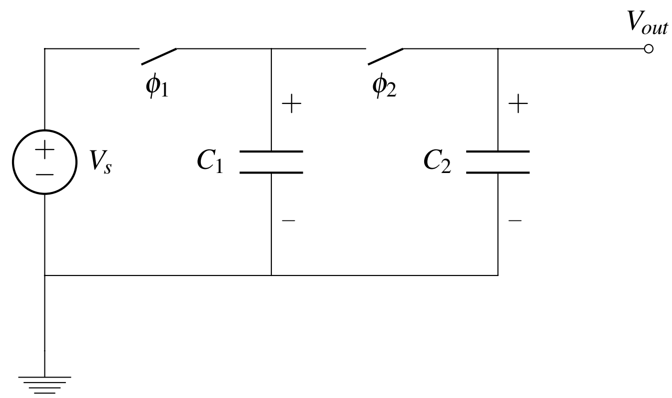
$$C_{E_1-E_2} = 2 \times 10^{-13}\text{F} + \frac{(2 \times 10^{-13}\text{F})(6 \times 10^{-13}\text{F})}{(2 \times 10^{-13}\text{F}) + (6 \times 10^{-13}\text{F})} = 3.5 \times 10^{-13}\text{F}$$

The new capacitance (after touch) is larger than the previous capacitance (before touch). Since the capacitance value has changed by a measurable amount, we can use the information gained in this part to work backwards and find the distance at which the screen was touched.

3. Charge Sharing and Conservation

In this question, we will explore how charges are conserved and shared when multiple charged (or uncharged) capacitors are connected together. Charge sharing and conservation is useful not only for dividing up the charges for different components (with different power demands) within a system, but also for storing and transferring the power in the case of limited access to the original voltage source.

Given the following circuit containing 2 switches ϕ_1, ϕ_2, the circuit repeatedly goes through a cycle of 2 phases (described below), continuously supplying voltage to the node V_{out}.



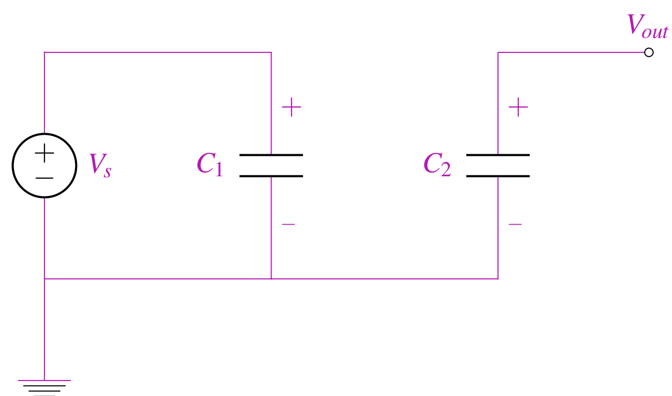
The two phases the circuit goes through are as follows:

(a) Close switch ϕ_1 until C_1 (initial uncharged) is fully charged. Switch ϕ_2 remains open.

(b) Open switch ϕ_1 and close switch ϕ_2. Maintain this configuration until the charges on both capacitors stabilize.

i. Draw out what the circuit would look like in Phase (1).

Answer: Since ϕ_2 is open, there will be no current flowing to the right half of the circuit (consider it as an open circuit). The circuit in Phase 1 would look like the following:

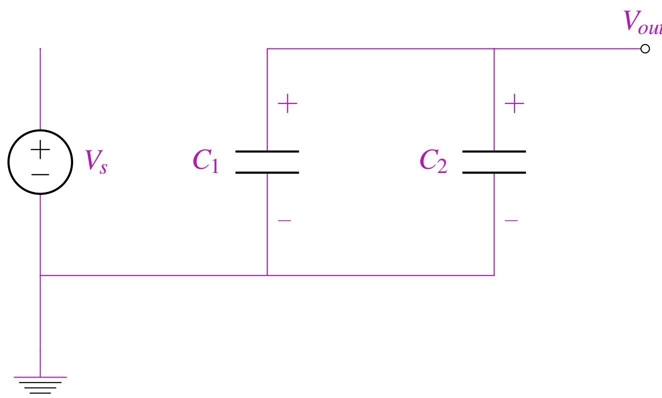


ii. Given that C_1 and C_2 are both initially uncharged, what would be the charges Q_1 and Q_2 on capacitors C_1 and C_2 respectively by the end of Phase 1? What would V_{out} be?

Answer: As we can see, by the end of Phase (1), capacitor C_1 would be fully charged: $Q_1 = C_1 V_s$. On the other hand, since C_2 is in an open circuit, it receives no voltage, the charge Q_2 on C_2 will remain 0. V_{out} is currently measuring the voltage across C_2, so $V_{out} = V_{C_2} = 0$.

(c) Draw out what the circuit would look like in Phase (2).

Answer: Since ϕ_1 is open, the voltage source is considered "disconnected" from the rest of the circuit since current cannot flow through between the voltage source and the capacitor C_1. The circuit in Phase ii would look like the following:



(d) Continuing from what the charges were on both capacitors at the end of Phase 1, what would the charges on both capacitors C_1 and C_2 be by the end of Phase 2? What would V_{out} be?

Answer:

Let's analyze the circuit and look at the charge on each capacitor and the output voltage step by step:

Coming from Phase 1, initially at the beginning of Phase 2, capacitor C_1 has a charge of $Q_1 = C_1V_s$, while capacitor C_2 is uncharged.

i. To redistribute the charges among both capacitors (since the circuit is no longer connected to the voltages), we make use of the following two key observations:
   - The total amount of charges ($Q_{total} = Q_1 = C_1V_s$) is **conserved**. We are not adding in additional charges (no external voltage source connected), and we are not losing any charges (no external closed circuit to route the charges to).
   - Capacitors C_1 and C_2 are in **parallel** with each other, meaning that they will have the same voltage.

ii. Utilizing the these two observations, we can create two equations in terms of Q_1 and Q_2 (the final charges on both capacitors by the end of Phase 2):

$$\begin{aligned} Q_1 + Q_2 = C_1V_s \quad & \text{Conservation of Charges from Phase 1} \\ \tfrac{Q_1}{C_1} = \tfrac{Q_2}{C_2} \quad & \text{Equi-voltage in a Parallel Circuit} \end{aligned}$$

Solving this system of equations, we find Q_1 and Q_2 to be:

$$Q_1 = \tfrac{C_1^2}{C_1+C_2}V_s$$
$$Q_2 = \tfrac{C_1C_2}{C_1+C_2}V_s$$

iii. Using the second equation, we can also find V_{out} to be:

$$V_{out} = V_1 = V_2 = \frac{C_1}{C_1+C_2}V_s.$$