CSM 16A Designing Information Devices and Systems I
Spring 2021 Week 4

1. Eigen Introduction

Learning Goal: The goal of this problem is to practice both intuitively and mechanically finding eigenvalues and their corresponding eigenvectors/eigenspaces.

Relevant Notes: **Note 9 Sections 9.2, 9.4, and 9.6** cover the process of finding eigenvalue-eigenvector pairs.

(a) What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{B} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Answer: $\mathbf{B}$ is a scaling matrix which scales any vector by a factor of 5. Formally, this means that given any vector $\vec{x}$,

$$\mathbf{B}\vec{x} = 5\vec{x}$$

No matter what value of $\vec{x} \in \mathbb{R}^3$ we put in to this equation, the same scaling will be performed. Thus, this matrix has only one eigenvalue, $\lambda = 5$ and any $\vec{x} \in \mathbb{R}^3$ is an eigenvector.

(b) What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

Answer: $\mathbf{C}$ is a scaling matrix which scales each entry of a vector by a different amount: the first entry is kept constant, the second entry is scaled by 6, and the third entry is scaled by 10:

$$\mathbf{C}\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 6x_2 \\ 10x_3 \end{bmatrix}$$

Consider only the third row (we choose the third row here randomly - this idea works for any of the rows). The value of x_3 is inputted into the transformation, which outputs $10x_3$. So if we can somehow ignore the scaling operations of the other two rows, which scale by different amounts, we have found a λ, or eigenvalue. It turns out that there is a way to "select" for this, by setting $x_1 = x_2 = 0$. Then

$$\mathbf{C}\vec{x} = \mathbf{C} \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10x_3 \end{bmatrix} = 10\vec{x}$$

So one eigenvalue is 10 and one eigenvector is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) Consider a matrix that rotates a vector in $\mathbb{R}^2$ by $45°$ counterclockwise about the origin in a coordinate plane. For instance, it rotates any vector along the x-axis to orient towards the $y = x$ line. This matrix is given as

$$\mathbf{E} = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

What are the eigenvalues and eigenvectors of this matrix?

Answer: We recall the equation $\mathbf{A}\vec{x} = \lambda\vec{x}$ which describes eigenvalues and eigenvectors. Geometrically, the right hand side means that there exist some vectors $\vec{x}$ that are scaled by λ. The left hand side represents the transformation $\mathbf{A}$ which when applied to $\vec{x}$ causes this scaling. The act of "scaling" in a coordinate plane preserves the angle or direction of the vector, only changing its magnitude.

For our matrix $\mathbf{E}$ that takes a vector and rotates it by $45°$, it would be changing the direction of any original input vector. So there are no possible vectors that this matrix could scale, which also means that there are no real eigenvalues for this matrix either.

Side note: However, this doesn't mean that $\mathbf{E}$ has no eigenvalues at all. There are actually *complex* eigenvalues! If you are curious about this topic, we recommend looking into **Note 9: Section 9.6** to learn more.

(d) Solve for the eigenvalue-eigenvector pairs for the following 2 by 2 matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Also find the eigenspaces.

Answer: To solve for eigenvalues and eigenvectors, let's go back and review the definition of eigenvectors and eigenvalues:
If $\vec{x}$ and λ are the eigenvector and eigenvalue of $\mathbf{A}$, respectively, then the following equation holds:

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$

Since the (appropriately sized) identity matrix is analogous to multiplying by 1 in arithmetic, we can say:

$$\mathbf{A}\vec{x} = \lambda\mathbf{I}\vec{x}$$

Rearranging, we get:

$$\mathbf{A}\vec{x} - (\lambda\mathbf{I})\vec{x} = \vec{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}$$

What does this look like? It looks similar to solving for the nullspace of $(\mathbf{A} - \lambda\mathbf{I})$!
Assuming that there is a nontrivial nullspace, that also means that $\mathbf{det}(\mathbf{A} - \lambda\mathbf{I}) = 0$!
Let's solve for λ first:

$$(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$
$$= \begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix}$$
$$\mathbf{det}(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda)(4-\lambda) - 2$$

$$= 10 - 7\lambda + \lambda^2$$
$$= (\lambda - 5)(\lambda - 2)$$

By factoring:

$$\lambda = 5, 2$$

Let's check: We've just solved for the eigenvalues. But what about the eigenvectors?
To do that, we plug in λ into $(\mathbf{A} - \lambda\mathbf{I})$ and solve for the nullspace!
For $\lambda = 5$:

$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}$$
$$\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$
$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

By row reduction:

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$
$$x_1 = x_2$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$

So the first pair is

$$\lambda, \vec{x} = 5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Repeating for $\lambda = 2$,

$$\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$
$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$
$$x_1 = -2x_2$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

So, the second pair is

$$\lambda, \vec{x} = 2, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The eigenspace is the space of all vectors $\vec{v}$ that satisfy the equation $\mathbf{A}\vec{v} = \lambda\vec{v}$. In this problem, we have two eigenspaces (one for each eigenvalue), and their basis vectors are the eigenvectors we found:

$$E_{\lambda=5} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$
$$E_{\lambda=2} = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

(e) Find the eigenvectors for matrix $\mathbf{A}$ given that we know that $\lambda_1 = 4, \lambda_2 = \lambda_3 = -2$ and that

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Also find the eigenspaces.

Answer: Once the eigenvalues of a matrix have been found, we can find the eigenvectors by Gaussian Elimination.

Step 1: For each eigenvalue λ, we have

$$(\mathbf{A} - \lambda \mathbf{I})\vec{x} = \vec{0}$$

where $\vec{x}$ is the eigenvector associated with eigenvalue λ.

Step 2: Find $\vec{x}$ in the nullspace of $(\mathbf{A} - \lambda \mathbf{I})$ by plugging in a value of λ and using Gaussian elimination to solve.

Case 1: $\lambda = 4$. First, form the matrix $\mathbf{A} - 4\mathbf{I}$:

$$\mathbf{A} - 4\mathbf{I} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$$

Then we use augmented matrix to solve for $\mathbf{A} - 4\mathbf{I} = \vec{0}$:

$$\left[\begin{array}{ccc|c} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right] \text{using } R_1 \leftarrow R_1(\tfrac{-1}{3})$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{array}\right] \text{using } R_2 \leftarrow R_2 - 3R_1 \text{ and } R_3 \leftarrow R_3 - 6R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{using } R_3 \leftarrow R_3 - R_2 \text{ and } R_2 \leftarrow R_2(\tfrac{1}{6})$$

We see that we have reached a row of 0s, which means that our last variable x_3 is the free variable in our system. Now, we can expand this matrix by putting it into a system of linear equations and solving for all the variables in terms of our free variable x_3:

$$x_1 + x_2 - x_3 = 0$$

$$-2x_2 + x_3 = 0$$

$$x_2 = \frac{x_3}{2}$$

$$x_1 + \frac{x_3}{2} - x_3 = 0$$

$$x_1 = \frac{x_3}{2}$$

$$\vec{x} = \begin{bmatrix} \frac{x_3}{2} \\ \frac{x_3}{2} \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}, \quad \text{where} \quad x_3 \in \mathbb{R}$$

So an eigenvector for $\lambda = 4$ is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. The corresponding eigenspace is

$$E_{\lambda=4} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Now, let's use this same technique to find the eigenvector for $\lambda = -2$.

Case 2: Now let's plug in $\lambda = -2$ into $\mathbf{A} - \lambda \mathbf{I}$ to get

$$\mathbf{A} + 2\mathbf{I} = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}$$

Just like before, let's use Gaussian elimination to reduce the matrix. We can see that this will only take a few steps:

$$\left[\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{using } R_2 \leftarrow R_2 - R_1;\, R_3 \leftarrow R_3 - 2 \cdot R_1;\, R_1 \leftarrow R_1 \cdot \tfrac{1}{3}$$

As we can see here, we have two rows of 0s, which means that we have two free variables (x_2 and x_3). Now we can take this matrix and write it as a linear system to get

$$x_1 - x_2 + x_3 = 0 \Rightarrow x_1 = x_2 - x_3$$

Thus,

$$\vec{x} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

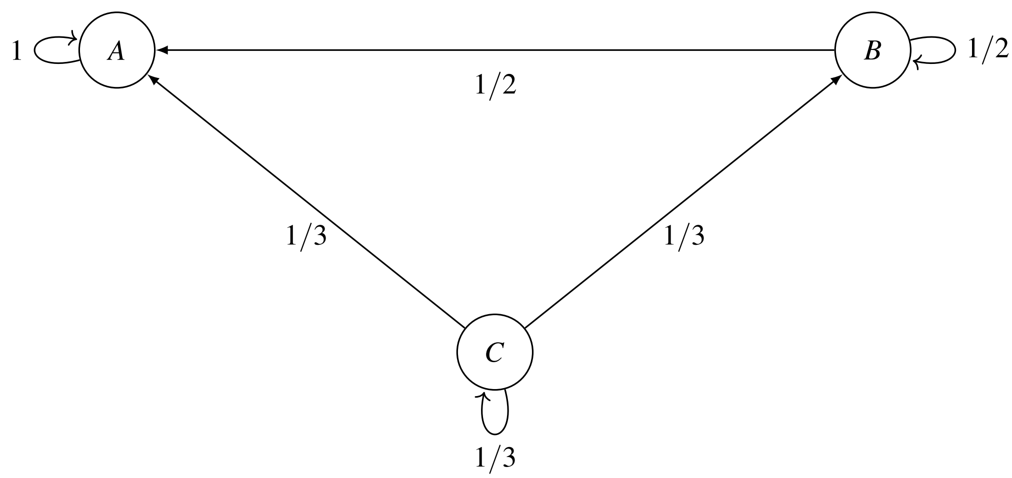
which are the two eigenvectors associated with $\lambda = -2$. These eigenvectors form a basis for a two-dimensional eigenspace,

$$E_{\lambda=-2} = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

2. Page Rank

Learning Goal: This problem is designed to provide insight into state transition. We will observe how the steady state depends of the eigenvalue and eigenvectors of a state-transition matrix.

Now suppose we have a network consisting of 3 websites connected as shown below. Each of the weights on the edges represent the probability of a user taking that edge.



(a) Call the transition matrix for this system **P** Write down **P** from this graph. (*Hint: Try to recall the properties of transition matrices and observe the sum of each column*).

Answer: Transition matrix notation: Let $A \to B$ represent the fraction of A that goes to B after transition.Then, a general 3×3 transition matrix with states A, B, and C can be written as follows:

$$\begin{bmatrix} A \to A & B \to A & C \to A \\ A \to B & B \to B & C \to B \\ A \to C & B \to C & C \to C \end{bmatrix}$$

The transition matrix is:

$$\mathbf{P} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

(b) We want to rank these webpages in order of importance. Can you predict at least one of the eigenvalues of **P**? Verify your predicted eigenvalue by calculation and then find the corresponding eigenvectors of **P**.

Answer:

We see that the columns of the transition matrix sum to 1. This means that population is conserved in this system, and the system might reach a steady-state. This hints that $\lambda_1 = 1$ might be an eigenvalue of the matrix **P**.

$$\mathbf{P}\vec{v_1} = \lambda_1 \vec{v_1}$$
$$\mathbf{P}\vec{v_1} = \lambda_1 \mathbf{I}\vec{v_1}$$
$$(\mathbf{P} - \lambda_1 \mathbf{I})\vec{v_1} = \vec{0}$$
$$(\mathbf{P} - \mathbf{I})\vec{v_1} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\vec{v_1} = \vec{0}$$
$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}\vec{v_1} = \vec{0}$$
$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}\vec{v_1} = \vec{0} \quad \text{using } R_2 \leftarrow R_2 + R_1$$
$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}\vec{v_1} = \vec{0} \quad \text{using } R_3 \leftarrow R_3 + R_2$$

We can see that the pivots lie in the second and third columns. So, we want to solve the equation:

$$\begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$
$$\implies -\frac{1}{2}x_2 + \frac{1}{3}x_3 = 0 \text{ and } -\frac{2}{3}x_3 = 0$$
$$\implies x_3 = 0 \text{ and } x_2 = 0$$

This means that the eigenvector is of the form $\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1$. And since x_1 is a free variable, the eigenvectors corresponding to eigenvalue 1 must belong in span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(c) Now you are told that the other two eigenvalues of $\mathbf{P}$ are $\lambda_2 = \frac{1}{2}$ and $\lambda_3 = \frac{1}{3}$, and the corresponding eigenvectors are $\vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v_3} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, respectively.

Suppose we start with just 30 users in A and no users in B and C. Can you express the initial state, $\vec{x}[0]$, as a linear combination of $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$?

Answer: The initial vector of users is $\vec{x}[0] = \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix}$. We assume

$$\vec{x}[0] = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \alpha_3 \vec{v_3},$$

where α_1, α_2 and α_3 are scalars. We can see by inspection that $\alpha_1 = 30$, $\alpha_2 = 0$ and $\alpha_3 = 0$ for this problem.

(d) Now use the results from the previous part to express the state at time step n as a function of the eigenvectors and eigenvalues. What is the steady-state? Is the steady-state different from the initial state? Why?

Relevant Notes: **Note 9: Subsection 9.8.2** are helpful for this problem.

Answer:

$$\begin{aligned}
&\vec{x}[0] = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 \\
&\vec{x}[1] = \mathbf{P}\vec{x}[0] = \alpha_1\mathbf{P}\vec{v}_1 + \alpha_2\mathbf{P}\vec{v}_2 + \alpha_3\mathbf{P}\vec{v}_3 = \alpha_1\lambda_1\vec{v}_1 + \alpha_2\lambda_2\vec{v}_2 + \alpha_3\lambda_3\vec{v}_3 \\
&\vec{x}[2] = \mathbf{P}\vec{x}[1] = \alpha_1\lambda_1\mathbf{P}\vec{v}_1 + \alpha_2\lambda_2\mathbf{P}\vec{v}_2 + \alpha_3\lambda_3\mathbf{P}\vec{v}_3 = \alpha_1\lambda_1^2\vec{v}_1 + \alpha_2\lambda_2^2\vec{v}_2 + \alpha_3\lambda_3^3\vec{v}_3 \\
&\quad\ldots \\
&\vec{x}[n] = \alpha_1\lambda_1^n\vec{v}_1 + \alpha_2\lambda_2^n\vec{v}_2 + \alpha_3\lambda_3^n\vec{v}_3 \\
&\vec{x}[n] = 30\lambda_1^n\vec{v}_1 + 0\lambda_2^n\vec{v}_2 + 0\lambda_3^n\vec{v}_3 = 30(1)^n\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}30\\0\\0\end{bmatrix}
\end{aligned}$$

The steady state is given by:

$$\vec{x}_{steady} = \lim_{n\to+\infty} 30(1)^n\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}30\\0\\0\end{bmatrix}$$

The steady state is the same as the initial state, because the initial state vector is in the eigenspace corresponding to $\lambda_1 = 1$, i.e. it is in the direction of $\vec{v}_1$.

(e) Now suppose we start with 30 users in A, 30 users in B and no users in C. Express the initial state, $\vec{x}[0]$, as a linear combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ and find the steady state. Is the steady-state different from the initial state? Why?

Answer: The initial vector of users is $\vec{x}[0] = \begin{bmatrix}30\\30\\0\end{bmatrix}$. Let us assume:

$$\vec{x}[0] = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3,$$

where α_1, α_2 and α_3 are scalars. We can see by inspection that $\alpha_1 = 60$, $\alpha_2 = -30$ and $\alpha_3 = 0$ for this problem. Otherwise, we can solve the following system to find α_i's:

$$\begin{bmatrix}1 & 1 & 1\\0 & -1 & -2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}30\\30\\0\end{bmatrix}.$$

We look at the previously derived expression of $\vec{x}[n]$:

$$\begin{aligned}
&\vec{x}[n] = \alpha_1\lambda_1^n\vec{v}_1 + \alpha_2\lambda_2^n\vec{v}_2 + \alpha_3\lambda_3^n\vec{v}_3 \\
&\vec{x}[n] = 60\lambda_1^n\vec{v}_1 - 30\lambda_2^n\vec{v}_2 + 0\lambda_3^n\vec{v}_3 = 60(1)^n\begin{bmatrix}1\\0\\0\end{bmatrix} - 30(\frac{1}{2})^n\begin{bmatrix}1\\-1\\0\end{bmatrix}
\end{aligned}$$

The steady state is given by:

$$\vec{x}_{steady} = \lim_{n\to+\infty} 60(1)^n \begin{bmatrix}1\\0\\0\end{bmatrix} - 30(\frac{1}{2})^n \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}60\\0\\0\end{bmatrix}$$

The steady state is different than the initial state, because the initial state vector had components in the direction of both $\vec{v}_1$ and $\vec{v}_2$. The population component in the direction of $\vec{v}_2$ diminishes, while the population is the direction of $\vec{v}_1$ stays steady (and is equal to the steady state). Note that the total population in the system stays the same!

(f) Suppose that we start with 90 users evenly distributed among the websites. Without doing any calculations, can you estimate the steady-state number of people who will end up at each website?

Answer: The initial vector of users is $\vec{x} = \begin{bmatrix}30\\30\\30\end{bmatrix}$. Since the other eigenvalues are less than 1, as we keep applying $\mathbf{P}$, this transition is scaling those corresponding components a value < 1, which will eventually zero out those components. So the only remaining component of $\vec{x}[0]$ will be in the direction of $\begin{bmatrix}1\\0\\0\end{bmatrix}$. The total number of users must be conserved, so the total steady state population will be 90.

Thus, the steady-state distribution is $\begin{bmatrix}90\\0\\0\end{bmatrix}$.

3. Diagonalization and Change of Basis

Learning Goal: The goal of this problem is to understand how to perform change of basis and diagonalization computations. Please look into **Note 10** for more on Diagonalization and Change of Basis.

(a) Let $A = \mathbb{R}^2$, $\mathbf{B} = \left\{\begin{bmatrix}-3\\4\end{bmatrix}, \begin{bmatrix}3\\-2\end{bmatrix}\right\}$, $\mathbf{C} = \left\{\begin{bmatrix}3\\-1\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}\right\}$, and $\vec{x} = \begin{bmatrix}3\\0\end{bmatrix}$.

i. Find $[x]_B$.

Answer: To find $[x]_B$, we need to find out what linear combination of the vectors in $\mathbf{B}$ yield $\vec{x}$.

$$\begin{bmatrix}3\\0\end{bmatrix} = \alpha_1\left(\begin{bmatrix}-3\\4\end{bmatrix}\right) + \alpha_2\left(\begin{bmatrix}3\\-2\end{bmatrix}\right)$$

Solving for α_1 and α_2, we get $\alpha_1 = 1$ and $\alpha_2 = 2$. Thus $[x]_B = \begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$

ii. Find $P_{C\leftarrow B}$.

Answer: To solve this problem, we have to represent each of the vectors in $\mathbf{B}$ as a linear combination of the vectors in $\mathbf{C}$.

$$\begin{bmatrix}-3\\4\end{bmatrix} = \beta_1\left(\begin{bmatrix}3\\-1\end{bmatrix}\right) + \beta_2\left(\begin{bmatrix}3\\3\end{bmatrix}\right)$$

$$\begin{bmatrix}3\\-2\end{bmatrix} = \gamma_1\left(\begin{bmatrix}3\\-1\end{bmatrix}\right) + \gamma_2\left(\begin{bmatrix}3\\3\end{bmatrix}\right)$$

Using system of equations to solve for the weights, we get, $\beta_1 = -1, \beta_2 = 1, \gamma_1 = 1, \gamma_2 = \frac{-1}{3}$. This means, $\begin{bmatrix}-3\\4\end{bmatrix}_C = \begin{bmatrix}\beta_1\\\beta_2\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$ and $\begin{bmatrix}3\\-2\end{bmatrix}_C = \begin{bmatrix}\gamma_1\\\gamma_2\end{bmatrix} = \begin{bmatrix}1\\\frac{-1}{3}\end{bmatrix}$.

Hence, $P_{C\leftarrow B} = \begin{bmatrix}\beta_1 & \gamma_1\\\beta_2 & \gamma_2\end{bmatrix} = \begin{bmatrix}-1 & 1\\1 & \frac{-1}{3}\end{bmatrix}$

iii. Compute $[x]_C$, given that you only know $[x]_B$ and $P_{C\leftarrow B}$.

Answer: Using the relationship $[x]_C = P_{C\leftarrow B}[x]_B$, we can solve for $[x]_C$.

$$[x]_C = \begin{bmatrix} -1 & 1 \\ 1 & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$[x]_C = \begin{bmatrix} -1+2 \\ 1+\frac{-2}{3} \end{bmatrix}$$

$$[x]_C = \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$$

iv. What is the relationship between matrices $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$. Prove it.

Answer: Matrices $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$ are inverses of each other.

We know that this relationship, $[x]_C = P_{C\leftarrow B}[x]_B$, exists. Multiplying both sides by $P_{B\leftarrow C}$ yields:

$$P_{B\leftarrow C}[x]_C = P_{B\leftarrow C}P_{C\leftarrow B}[x]_B$$

Reusing the relationship, we know that $P_{B\leftarrow C}[x]_C$ is equal to $[x]_B$. So, we can write:

$$[x]_B = P_{B\leftarrow C}P_{C\leftarrow B}[x]_B$$

Now, this must be true for every vector $[x]_B$, so:

$$P_{B\leftarrow C}P_{C\leftarrow B} = I$$

Since the only matrix with the property $A\vec{x} = \vec{x}$ is the identity matrix, $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$ are inverses of each other.

(b) Let matrix $\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

i. Find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $A = PDP^{-1}$

Answer: Start by computing the characteristics polynomial.

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda+2 & -1 & -1 \\ 1 & \lambda & -1 \\ 1 & -1 & \lambda \end{bmatrix}$$

Find the eigenvalues by first taking the determinant of the matrix above: $\det\left(\begin{bmatrix} \lambda+2 & -1 & -1 \\ 1 & \lambda & -1 \\ 1 & -1 & \lambda \end{bmatrix}\right)$

$$\begin{aligned} &= (\lambda+2)\,(\lambda^2-1) - (-\lambda-1)(1+\lambda) \\ &= \lambda^3 + 2\lambda^2 - \lambda - 2 + 2\lambda + 2 \\ &= \lambda^3 + 2\lambda^2 + \lambda \\ &= \lambda(\lambda+1)^2 \end{aligned}$$

So, $\lambda_1 = 0$ and $\lambda_2 = -1$.

For eigenvalue $\lambda_1 = 0$, we can use Gaussian elimination to find the eigenvector.

$$
\begin{aligned}
\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \text{using } R_1 \leftarrow R_2/2 \\
&\rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1/2 & -1/2 \\ 1 & -1 & 0 \end{bmatrix} \text{using } R_2 \leftarrow R_2 - R_1 \\
&\rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1/2 & -1/2 \\ 1 & -1 & 0 \end{bmatrix} \text{using } R_3 \leftarrow R_3 - R_1 \\
&\rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1/2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \text{using } R_1 \leftarrow R_1 + R_2 \\
&\rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \text{using } R_3 \leftarrow R_3 + R_2 \\
&\rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \text{using } R_2 \leftarrow 2(R_2) \\
&\rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

So, $x_1 = x_3, x_3 = x_3, x_2 = x_3$. Using this relationship, we can say $x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a eigenvector for eigenvalue $\lambda = 0$.

Similarly, for eigenvalue $\lambda_1 = -1$, we can use Gaussian elimination again to find the eigenbasis.

$$
\begin{aligned}
\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} &\rightarrow \begin{bmatrix} & -1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix} \text{using } R_2 \leftarrow R_1 - R_2 \\
&\rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{using } R_3 \leftarrow R_1 - R_3
\end{aligned}
$$

So, $x_2 = x_2, x_3 = x_3$, and $x_1 = x_2 + x_3$. Thus, we can write $x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

The basis for the eigenspace when $\lambda = -1$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Using the vectors we obtained by doing Gaussian Elimination, we can now find our invertible matrix **P** and diagonal matrix **D**.

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that the columns of invertible matrix **P** can be in any order, as long as the eigenvalue placement in **D** correspond.

For example, if you write invertible matrix as $\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, then $\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

ii. What is $\mathbf{D}^{2021}$?

Answer: A property of a diagonal matrix is that $\mathbf{D}^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$.

Therefore, $\mathbf{D}^{2021} = \begin{bmatrix} 0^{2021} & 0 & 0 \\ 0 & -1^{2021} & 0 \\ 0 & 0 & -1^{2021} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

