CSM 16A   Designing Information Devices and Systems I
Spring 2021   Week 9

Reference: Op-Amp Example Circuits

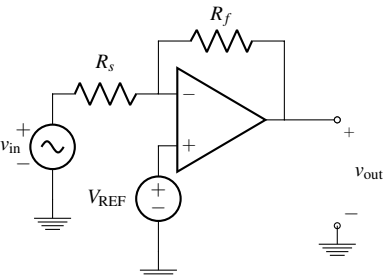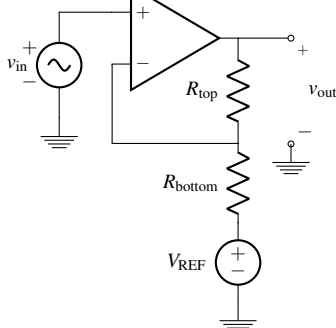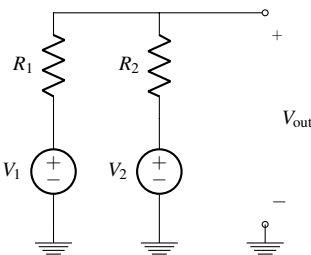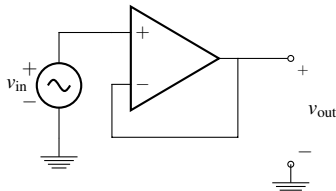| Voltage Divider	Inverting Amplifier	Noninverting Amplifier
$V_{R_2} = V_S \left(\frac{R_2}{R_1 + R_2} \right)$	$v_{\text{out}} = v_{\text{in}} \left(-\frac{R_f}{R_s} \right)$	$v_{\text{out}} = v_{\text{in}} \left(1 + \frac{R_{\text{top}}}{R_{\text{bottom}}} \right)$
Current Divider	Inverting Amplifier with Reference	Noninverting Amplifier with Reference
$I_1 = I_S \left(\frac{R_2}{R_1 + R_2} \right)$	$v_{\text{out}} = v_{\text{in}} \left(-\frac{R_f}{R_s} \right) + V_{\text{REF}} \left(\frac{R_f}{R_s} + 1 \right)$	$v_{\text{out}} = v_{\text{in}} \left(1 + \frac{R_{\text{top}}}{R_{\text{bottom}}} \right) - V_{\text{REF}} \left(\frac{R_{\text{top}}}{R_{\text{bottom}}} \right)$
Voltage Summer	Unity Gain Buffer	
$V_{\text{out}} = V_1 \left(\frac{R_2}{R_1 + R_2} \right) + V_2 \left(\frac{R_1}{R_1 + R_2} \right)$	$v_{\text{out}} = v_{\text{in}}$	

1. Comparators

Learning Goal: This problem will help to understand comparator properties and design circuits with comparators.

Relevant Notes: Note 17C goes over comparator properties.

Comparators are typically drawn like the figure on the left, and their internal workings can be represented by the figure on the right.
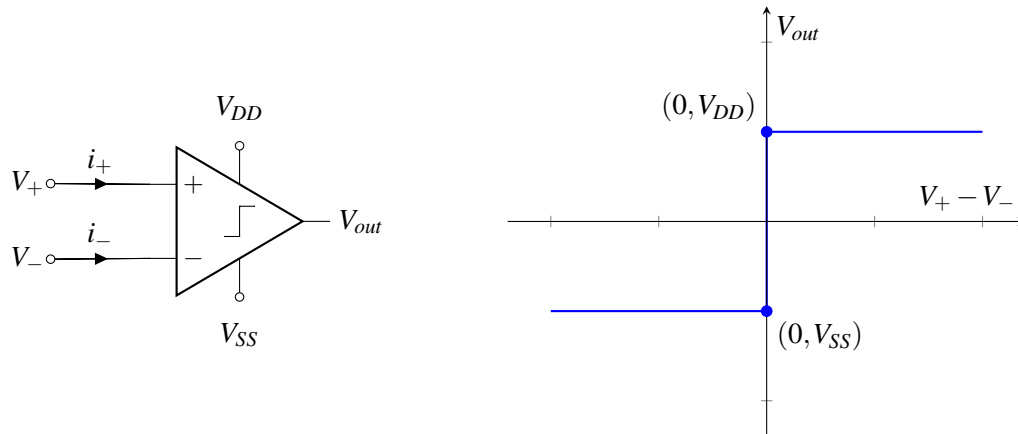
Here, V_+ and V_- are input voltages, V_{DD} and V_{SS} are what we call the "supply rails", and V_{out} is the output voltage. From the diagram and knowing that V_{out} cannot exceed the supply rail voltages, we have a relationship between the outputs and the inputs:

$$V_{out} = \begin{cases} V_{DD} & \text{, if} & V_+ > V_-) \\ V_{SS} & \text{, if} & V_+ < V_-) \\ \text{undefined} & \text{, if} & V_+ = V_-) \end{cases}$$

(a) Identify the output voltage V_o for the following comparator:
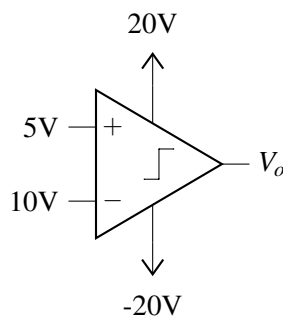
Answer: $V_- = 10\text{V}$ and $V_+ = 5\text{V}$, so $V_+ < V_-$, causing the comparator to rail to V_{SS}. So $V_o = -20\text{V}$.

(b) Identify the output voltage V_o for the following comparator:
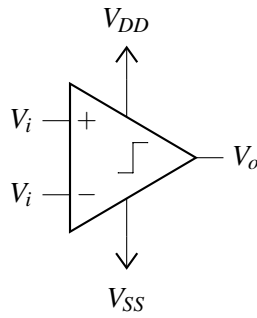
Answer: Since $V_- = V_+ = V_i$, the two input voltages into the comparator are equivalent. This results in undefined behavior, so V_o =undefined.

(c) Design a circuit such that $v_o = 2V$ if $v_c > 0.5V$ and $v_o = -1V$ if $v_c < 0.5V$. Draw your designed circuit.

You can use only up to 3 voltage sources, but other than that you can use whatever circuit components you want (ex. comparator, capacitor, resistors, etc.) and as many as you would like. Label the values of the voltage sources you use. Hint: Can we use the circuit components we talked about in last week's worksheet?

Answer: The problem is asking us to design a circuit that produces the following V_o:

$$V_o = \begin{cases} 2V & \text{, if } v_c > 0.5V \\ -1V & \text{, if } v_c < 0.5V \\ ? & \text{, if } v_c = 0.5V \end{cases}$$

We would like to use a comparator to achieve this. Notice that if we replace $V_{SS} = 2V$, $V_{DD} = -1V$, $V_- = 0.5V$, and $V_+ = v_c$, our comparator will output the desired values.

To construct this, we will connect V_{SS} and V_{DD} to two voltage sources of 2V and $-1V$, respectively. We will also connect V_- to a 0.5V voltage source. Finally, we want $V_+ = v_c$, which we can achieve by connecting the top terminal of v_c to V_+ and the bottom terminal to ground:
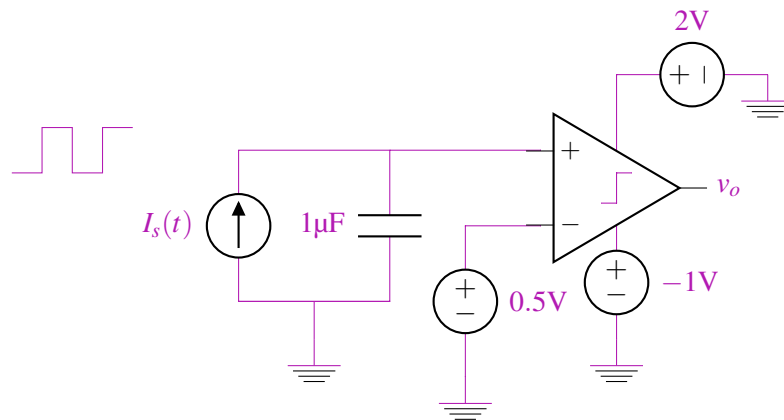
(d) Assume $I_s(t)$ and $v_c(t)$ are given in the following figure. Plot $v_o(t)$ vs. time for the circuit you designed in the last part.
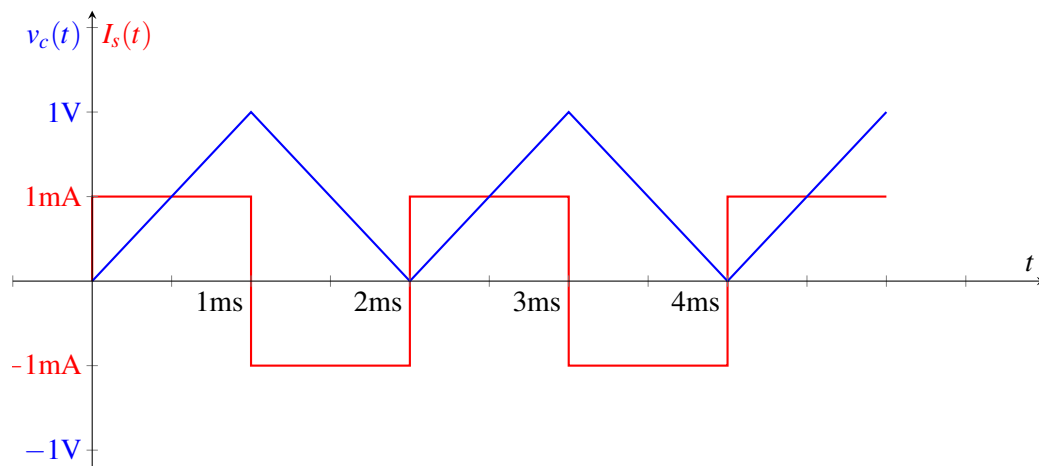
Answer: Since our comparator is a function solely on $v_c(t)$, we look at when $v_c(t)$ is above or below 0.5V:
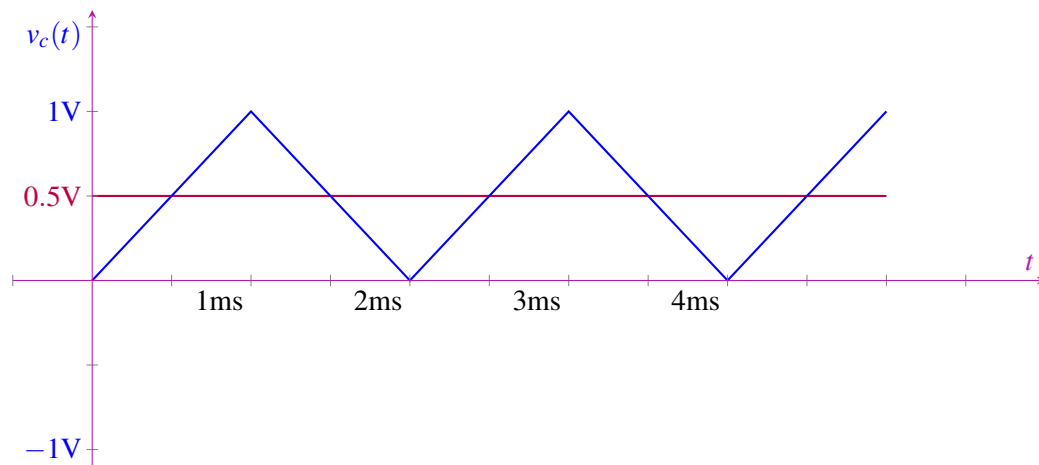
When the blue line is above the dark red one, the comparator rails to 2V. This happens periodically: between $t = 0.5$ms and $t = 1.5$ms, between $t = 2.5$ms and $t = 3.5$ms, etc. When it is below, it rails to -1V. This happens at all other times. Thus we have the following plot for $v_o(t)$:
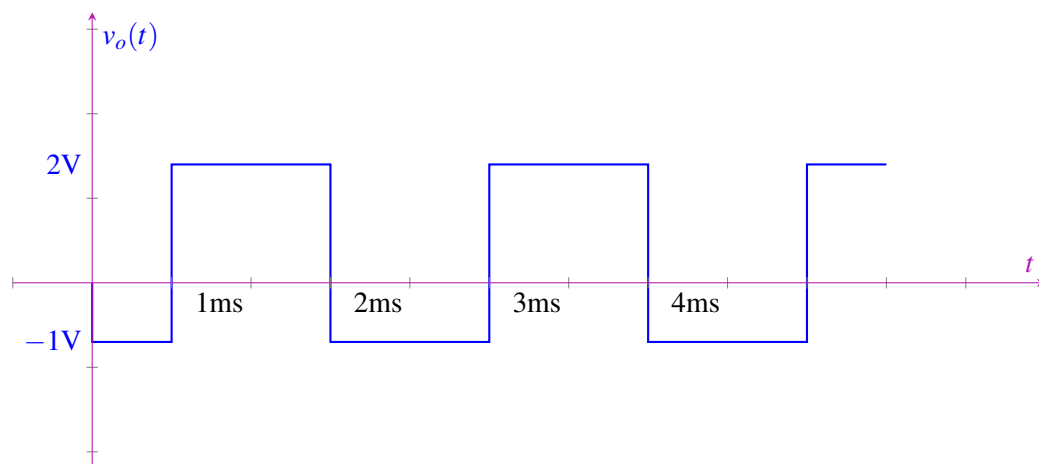
(e) Redesign your circuit such that $v_o = -2\text{V}$ if $v_c > 0.5\text{V}$ and $v_o = 1\text{V}$ if $v_c < 0.5\text{V}$. Draw your redesigned circuit.

You can use as many voltage sources you want. Label the values of the voltage sources you use.

Answer: This circuit exhibits similar comparative behavior to the one in part (c). The major difference is the value of the rails, where the lower voltage is now -2V and the upper one is 1V. Also, the condition on when to rail to the lower or upper voltage is flipped. We can achieve this by flipping the connected elements to v_+ and v_- relative to part (c), so $v_- = v_c$ (connected to the periodic capacitive circuit) and $v_+ = 0.5\text{V}$:
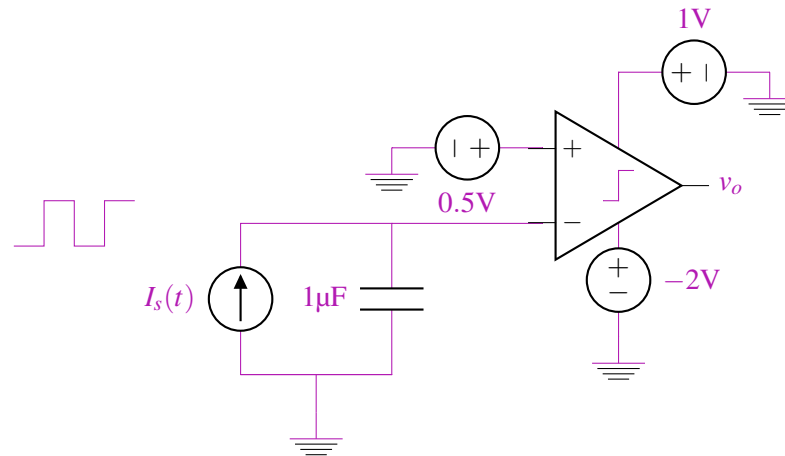

2. Operational Amplifiers

Learning Goal: This problem will help to understand op-amps in negative feedback and the operation of an inverting amplifier.

Relevant Notes: Note 18 goes over op-amp properties and derivation of circuit responses.

Operational amplifiers (op amps for short) are typically drawn like the figure on the left, and their internal workings can be represented by the figure on the right.

$$V_{out} = A(V_+ - V_-) + \frac{V_{DD} + V_{SS}}{2}$$
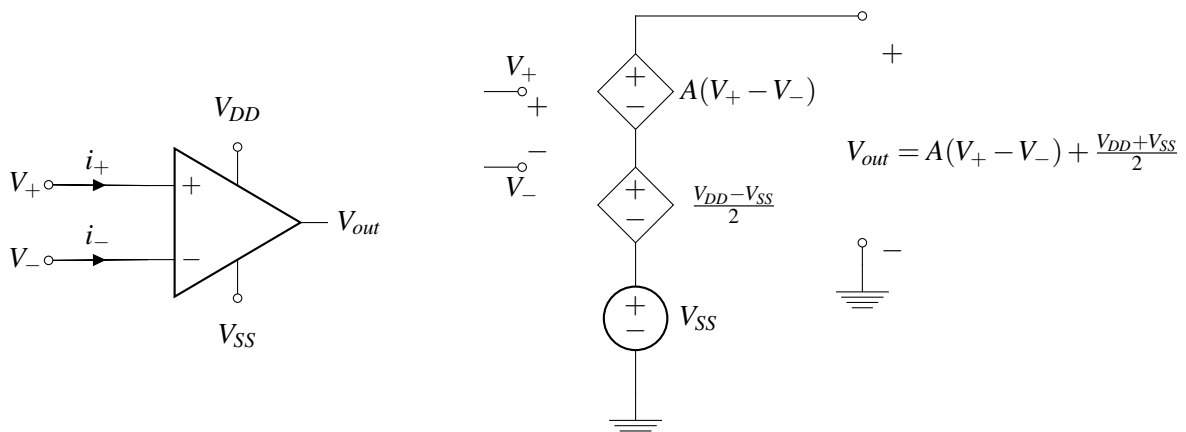
Here, V_+ and V_- are input voltages, V_{DD} and V_{SS} are what we call the "supply rails", and V_{out} is the output voltage. From the diagram and knowing that V_{out} cannot exceed the supply rail voltages, we have a

relationship between the outputs and the inputs:

$$V_{out} = \begin{cases} V_{SS} & \text{, if} & A(V_+ - V_-) + \frac{V_{DD}+V_{SS}}{2} < V_{SS} \\ A(V_+ - V_-) + \frac{V_{DD}+V_{SS}}{2} & \text{, if} & V_{SS} \leq A(V_+ - V_-) + \frac{V_{DD}+V_{SS}}{2} \leq V_{DD} \\ V_{DD} & \text{, if} & V_{DD} < A(V_+ - V_-) + \frac{V_{DD}+V_{SS}}{2} \end{cases}$$
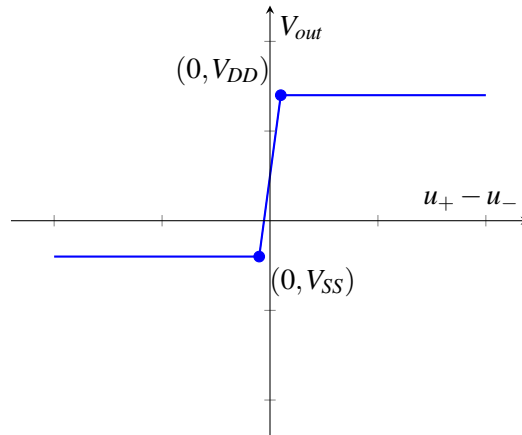
Typically gain A is quite large, meaning the the sloped region in the center is quite narrow.

(a) Much of EE16A will have our analysis be restricted to ideal op-amps. However, it is important to know non-ideal behaviors. What are some main differences between ideal and non-ideal op-amps?

Answer: Ideal op-amps: Infinite gain $A = \frac{V_o}{V_i}$ (where $V_i = V_+ - V_-$). This means that the gain will be infinity if there was no limit on the rails, but it cannot reach infinity because it gets clipped at the positive or negative rails. infinite input resistance, zero input current. Non-ideal op amps: finite gain, finite input resistance, finite input current.

(b) If the gain of an operational amplifier/op amp is given by $A \to \infty$, so we can make some assumptions known as the "Golden Rules". What are the "Golden Rules" and when are they applicable?

Answer:

- Always true: $i_- = i_+ = 0A$
- Negative feedback only: $u_+ = u_-$

(c) Determine if the following system is in negative feedback.
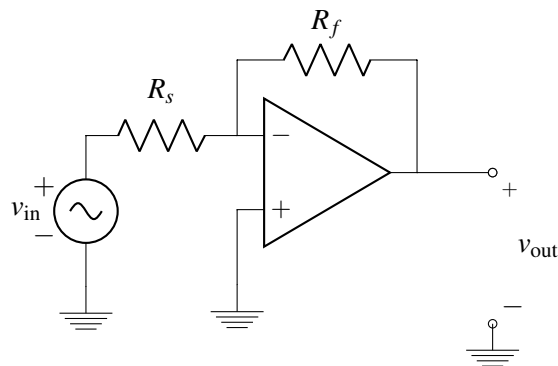
Answer: This is in negative feedback, and in fact is a fairly well-known schematic called the inverting amplifier.

- For our initial stimulus, we'll kick v_{out} up
- How does u_- respond? Well, we can't assume $u_+ = u_-$ (because that requires the system to be in negative feedback), so we'll look at the resistors. Note that this is a voltage divider with u_- in the and, v_{out} at the top, so when v_{out} moves upward, u_- travels upward too
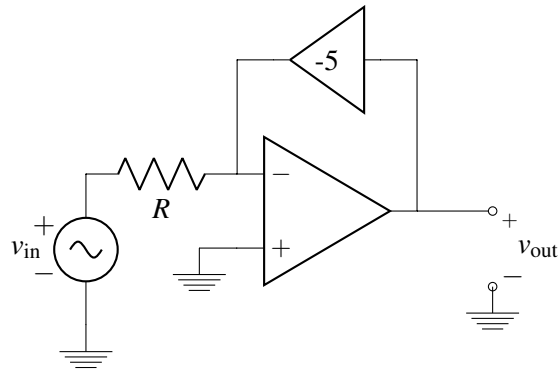- If u_- goes up, that means v_{out} goes down because of the op amp—the opposite direction as the inital stimulus!

(d) Determine if the following system is in negative feedback.

Note: The triangular block with the label "-5" in the figure above represents an amplifier with a factor of -5; In other words, it takes V_{out} as an input from the right side and outputs $-5V_{\text{out}}$ on the left side.
Answer: This is *not* in negative feedback! Going through the process of checking:

- For our initial stimulus, we'll kick v_{out} up
- With the $-5 \cdot v_{out}$, that means u_- goes down
- If u_- goes down, that means v_{out} goes up—the same direction as the initial stimulus!

(e) Find the expression of v_{out} for the following circuit:
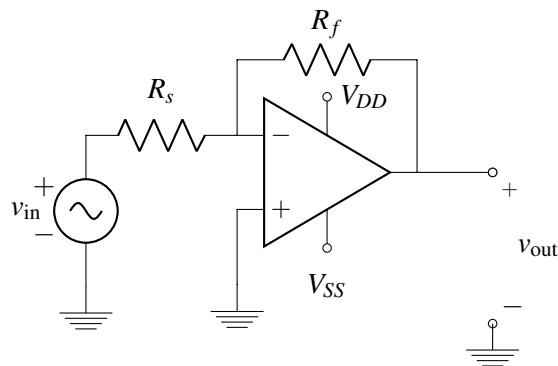
Answer:
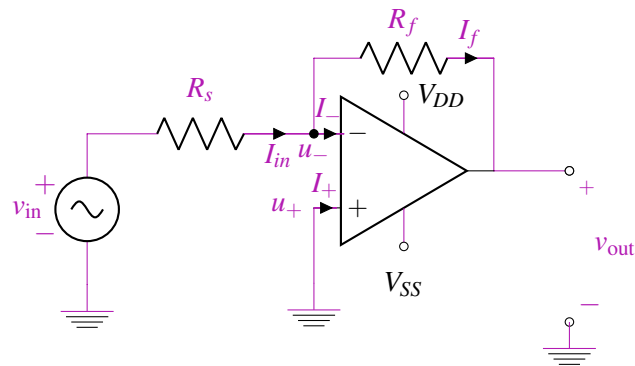
The first golden rule for op amps tell us:

$$I_+ = I_- = 0$$

Since we verified that this circuit is in negative feedback in part (c), the second golden rule also applies:

$$u_+ = u_-$$

Using KCL, we can write the following equations:

$$I_{in} = I_f + I_-$$

Since $I_- = 0$, we can write

$$I_{in} = I_f$$

Since the positive input terminal is connected to ground, $u_+ = 0$, i.e. $u_- = u_+ = 0$.

Using the formula, V = IR:

$$I_{in} = \frac{v_{in} - u_-}{R_s} = \frac{v_{in}}{R_s}$$
$$I_f = \frac{u_- - v_{out}}{R_f} = \frac{-v_{out}}{R_f}$$

From the KCL equations, we know $I_{in} = I_f$. Substituting the equations from above,

$$\frac{v_{in}}{R_s} = \frac{-v_{out}}{R_f}$$
$$v_{out} = -\frac{v_{in} R_f}{R_s}$$

(f) Plot v_{out} vs. time for the following v_{in}. Assume $V_{DD} = 10V$, $V_{SS} = -10V$, $R_s = 100\Omega$, $R_f = 500\Omega$.

Answer:

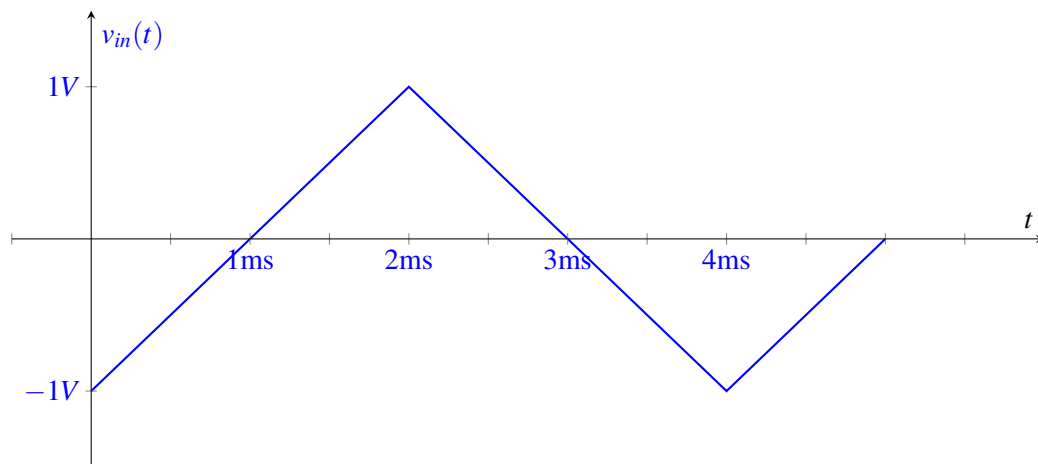Using the formula for v_{out} obtained in part (e) and the values for R_f and R_s, we can find the circuit gain:

$$G = \frac{v_{out}}{v_{in}} = -\frac{R_f}{R_s} = -\frac{500}{100} = -5$$

So the input will be multiplied by a factor of -5. We can substitute to find v_{out} along different times.

v_{in}	Time	v_{out} Calculation	v_{out}
-1	0	$-\frac{(-1)(500)}{100}$	5
0	1	$-\frac{(0)(500)}{100}$	0
1	2	$-\frac{(1)(500)}{100}$	-5
0	3	$-\frac{(0)(500)}{100}$	0
-1	4	$-\frac{(-1)(500)}{100}$	5
0	5	$-\frac{(0)(500)}{100}$	0

Using the v_{out} values we calculated above in the table, we can plot the v_{out} vs. time graph.

(g) What happens if R_f is changed to $R_f = 2000\Omega$. Plot v_{out} vs. time for the same v_{in}, where $V_{DD} = 10$V, $V_{SS} = -10$V, and $R_s = 100\Omega$.
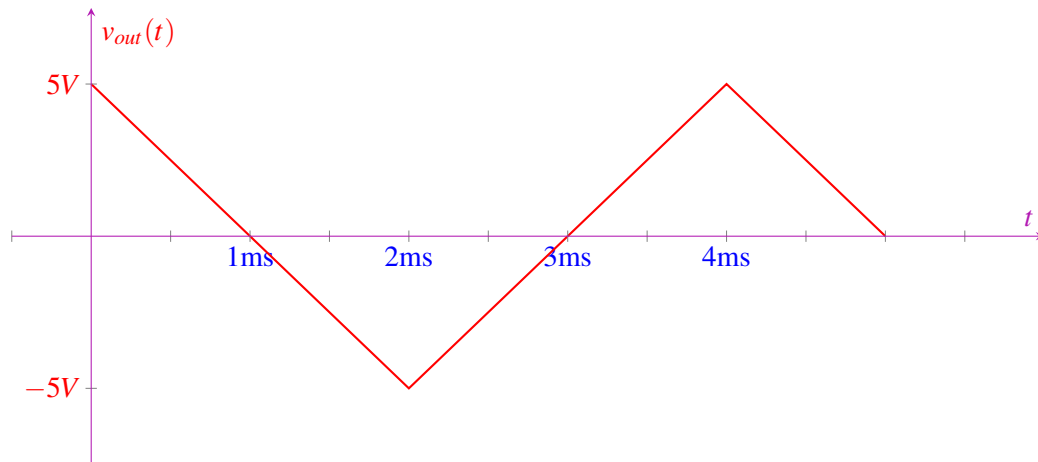
Answer:

Similar to part (f), using the formula for v_{out} obtained in part (e) and the values for R_f and R_s, we can find the circuit gain:

$$G = \frac{v_{out}}{v_{in}} = -\frac{R_f}{R_s} = -\frac{2000}{100} = -20$$

So the input will be multiplied by a factor of -20. We can substitute to find v_{out} along different times. Using the v_{out} values we calculated above in the table, we can plot the v_{out} vs. time graph.

Here, we have to also consider the values given for V_{DD} and V_{SS}. The given bounds are $V_{DD} = 10$V, $V_{SS} = -10$V, but the values we obtained for V_{out} are between -20V and 20V. Therefore, we have to "clip" the graph at these points to not go past -10V and 10V, as shown in the graph below.

v_{in}	Time	v_{out} Calculation	v_{out}
-1	0	$-\frac{(-1)(2000)}{100}$	20
0	1	$-\frac{(0)(2000)}{100}$	0
1	2	$-\frac{(1)(2000)}{100}$	-20
0	3	$-\frac{(0)(2000)}{100}$	0
-1	4	$-\frac{(-1)(2000)}{100}$	20
0	5	$-\frac{(0)(2000)}{100}$	0
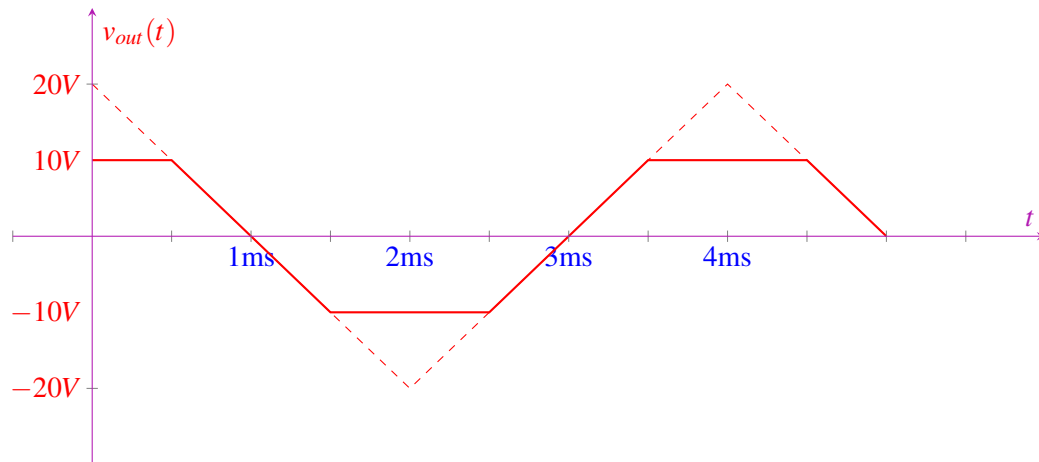
3. Voltage Summers
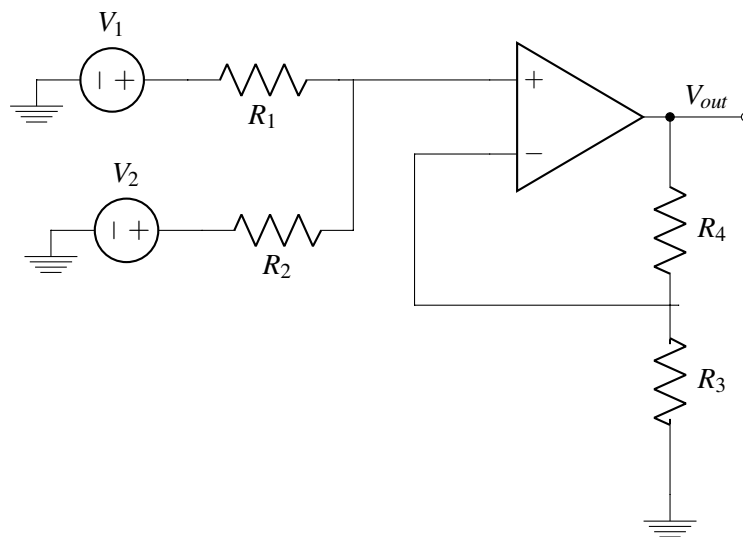
Learning Goal: This problem uses basic circuit analysis techniques to find the response of a summer circuit.

Relevant Notes: Note 19 goes different op-amp circuit topology and corresponding derivations.

(a) Calculate V_{out} in terms of V_1 and V_2. Assume that $R_1 = R_2$. Use superposition.

Answer: Let's first consider the case when only V_1 is active. We deactivate voltage source V_2 by replacing it with a wire:
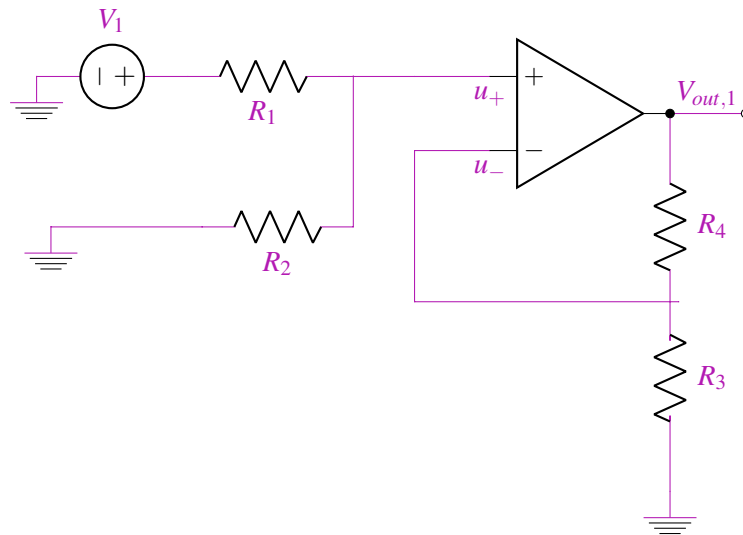
We see that u_+ is the middle node of a voltage divider with resistors R_1 and R_2. We also know that input current $I_+ = 0$. So we can find u_+ using:

$$u_+ = \frac{R_2}{R_1 + R_2} V_1.$$

Similarly u_- is the middle node of a voltage divider with resistors R_3 and R_4. We know that input current $I_- = 0$. We can find u_- as:

$$u_- = \frac{R_3}{R_3 + R_4} V_{out,1}.$$

Now we have to check if the circuit is in negative feedback. If we move the negative input of the op amp u_- upward, $V_{out,1} = A(u_+ - u_-)$ moves downward and as a result $u_- = \frac{R_3}{R_3+R_4} V_{out,1}$ moves downward. So the result of the initial stimulus goes in the opposite direction of the initial stimulus, which is the requirement for negative feedback.

Since the circuit is in negative feedback, we can apply the second golden rule:

$$u_+ = u_-$$
$$\implies \frac{R_2}{R_1 + R_2} V_1 = \frac{R_3}{R_3 + R_4} V_{out,1}$$
$$\implies V_{out,1} = \left(1 + \frac{R_4}{R_3}\right) \frac{R_2}{R_1 + R_2} V_1$$

Next we consider the case when only V_2 is active. We deactivate voltage source V_1 by replacing it with a wire:
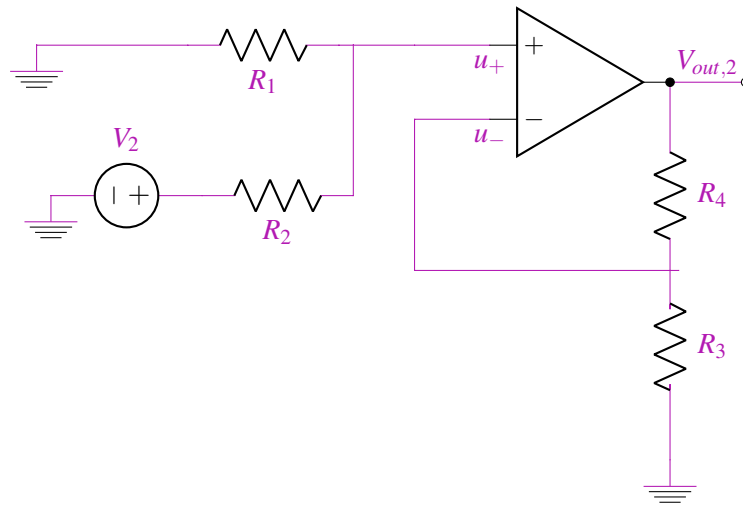
Following a similar process we have:

$$u_+ = \frac{R_1}{R_1 + R_2} V_2;$$

$$u_- = \frac{R_3}{R_3 + R_4} V_{out,2}.$$

(Note that u_+ is now the voltage across R_1, while for the previous scenario u_+ was the voltage across R_2.)

Using the second golden rule we have:

$$u_+ = u_-$$

$$\implies \frac{R_1}{R_1 + R_2} V_2 = \frac{R_3}{R_3 + R_4} V_{out,2}$$

$$\implies V_{out,2} = \left(1 + \frac{R_4}{R_3}\right) \frac{R_1}{R_1 + R_2} V_2$$

Now using the superposition theorem, we can find V_{out}:

$$V_{out} = V_{out,1} + V_{out,2}$$

$$\implies V_{out} = \left(1 + \frac{R_4}{R_3}\right) \frac{R_2}{R_1 + R_2} V_1 + \left(1 + \frac{R_4}{R_3}\right) \frac{R_1}{R_1 + R_2} V_2$$

Note that a similar expression can be reached by using the equations for the voltage summer and the non-inverting amplifier. Pattern matching this circuit with the voltage summer and the non-inverting amplifier and then employing the given equations is a valid technique to solve this problem.

(b) What values should we select for R_1, R_2, R_3, and R_4 such that $V_{out} = V_1 + 2V_2$?

Answer: From the last part, we have

$$V_{out} = \left(1 + \frac{R_4}{R_3}\right) \frac{R_2}{R_1 + R_2} V_1 + \left(1 + \frac{R_4}{R_3}\right) \frac{R_1}{R_1 + R_2} V_2$$

Comparing with $V_{out} = V_1 + 2V_2$, we have:

$$\left(1 + \frac{R_4}{R_3}\right)\frac{R_2}{R_1 + R_2} = 1 \tag{1}$$

$$\left(1 + \frac{R_4}{R_3}\right)\frac{R_1}{R_1 + R_2} = 2 \tag{2}$$

Dividing equation (**??**) with equation (**??**) we get:

$$\frac{R_2}{R_1} = \frac{1}{2}$$

$$R_1 = 2R_2$$

Now if we choose $R_2 = 100\Omega$ and $R_1 = 200\Omega$, we have

$$\frac{R_2}{R_1 + R_2} = \frac{100}{200 + 100} = \frac{1}{3} \tag{3}$$

$$\frac{R_1}{R_1 + R_2} = \frac{200}{200 + 100} = \frac{2}{3} \tag{4}$$

Plugging in the values of equation (**??**) in equation (**??**) (or equation (**??**) in equation (**??**)), we have:

$$\left(1 + \frac{R_4}{R_3}\right)\frac{1}{3} = 1$$

$$1 + \frac{R_4}{R_3} = 3$$

$$\frac{R_4}{R_3} = 2$$

$$R_4 = 2R_3$$

We can choose $R_3 = 100\Omega$ and $R_4 = 200\Omega$, so that the output voltage is:

$$V_{out} = \left(1 + \frac{200}{100}\right)\frac{100}{200 + 100}V_1 + \left(1 + \frac{200}{100}\right)\frac{200}{200 + 100}V_2 = (1+2)\frac{1}{3}V_1 + (1+2)\frac{2}{3}V_2 = V_1 + 2V_2.$$

4. **Multi-stage Amplifier**

Learning Goal: The objective of this problem is to understand how multiple stages of op-amp circuits can be used to achieve a specific circuit gain.

Relevant Notes: Note 19 Section 19.5 goes over inverting and non-inverting amplifiers.

(a) What is the range of values that we can scale V_{in} by when using a non-inverting op amp? (What are possible values for the gain?)

Answer: Recall that when using a non-inverting op amp, the equation for V_{out} is given by

$$V_{out} = V_{in}\left(1 + \frac{R_2}{R_1}\right)$$

where R_2 is the upper resistor and R_1 is the lower one (connected to ground). The circuit gain G is represented by

$$G = \frac{V_{out}}{V_{in}} = 1 + \frac{R_2}{R_1}$$

We can choose any values for resistance from $[0, \infty)$; then, the minimum gain would be 1 if we chose $R_2 = 0$; the maximum gain approaches infinity as we choose some very large R_2 and/or a very small R_1. Hence, the range of gains is from 1 to infinity, and our range of values for V_{out} is $[V_{in}, \infty)$.

(b) What is the range of values that we can scale V_{in} by when using an inverting op amp? (What are the possible values for the gain?)

Answer: Recall that when using an inverting op amp, the equation for V_{out} is given by

$$V_{out} = -V_{in}\frac{R_2}{R_1}$$

. The circuit gain is represented by

$$G = -\frac{R_2}{R_1}$$

.

Again, we can choose any values for resistance from $[0, \infty)$; then, the minimum absolute value scaling would be 0 if we chose $R_2 = 0$; the maximum absolute value scaling approaches infinity as we choose some very large R_2 and/or a very small R_1. Hence, the range of gains is from 0 to -∞, and our range of values for V_{out} is also $(-\infty, 0]$.

(c) Can you design a single inverting/ non-inverting amplifier with circuit gain $G = 0.5$? If not, what range of gain values is not reachable using a single inverting op amp or a single non-inverting op amp?

Answer: From part (a), we found that a non-inverting op amp can only reach gain values from $[1, \infty)$. From part (b), we found that an inverting op amp can only reach gain values from $(-\infty, 0]$. So using only a single op-amp, we cannot reach values between $(0, 1)$, specifically $G = 0.5$.

(d) How would you construct a circuit using inverting/ non-inverting amplifiers so that the overall circuit gain is $G = 0.5$?

Answer: We can use two inverting op-amps to achieve overall gain $G = 0.5$. Say the first op-amp has gain G_1 and the second op-amp G_2, so $G = G_1 G_2$. From part (b), these gains can take on values from $(-\infty, 0]$, so multiplying them together helps reach $G = 0.5$.
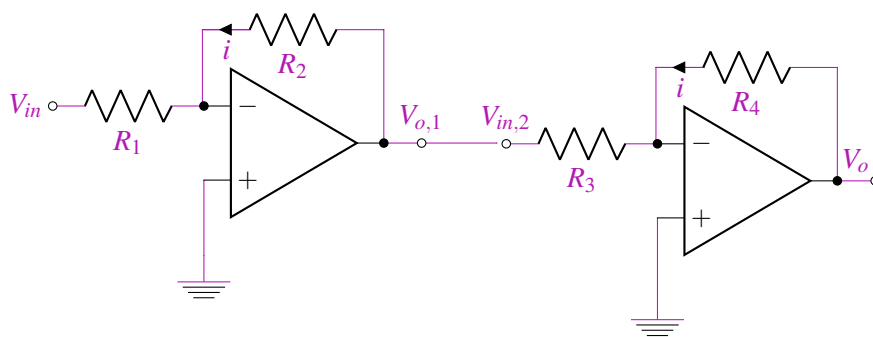
For the first inverting op-amp, we have $G_1 = -\frac{R_2}{R_1}$, and for the second, $G_2 = -\frac{R_4}{R_3}$. Plugging these into the equation for G:

$$G = G_1 G_2 = \frac{R_2 R_4}{R_1 R_3} = 0.5$$

$$R_1 R_3 = 2R_2 R_4$$

To reach out desired gain, we can pick any combination of R_1, R_2, R_3, and R_4 that satisfies this equation. One such solution is $R_1 = 200\Omega$ and $R_2 = R_3 = R_4 = 100\Omega$.

5. Op Amps as Buffers

Learning Goal: This problem helps understand the operating principle of an op-amp buffer and how it helps with loading.

Relevant Notes: Note 19 Section 19.7 goes different op-amp circuit topology and corresponding derivations.

Now we will revisit a problem that you might have seen before, with our new knowledge of op-amps. We have access to a circuit inside a 'black box' as shown below, with two terminal coming out of it.

(a) We need a voltage of 6V power a light bulb with resistance R_L. Design R_1 and R_2 inside the black box so that the voltage across R_2 is exactly equal to this required voltage **when the bulb is not connected**; i.e. $V_{R_2} = V_{out} = 6V$.
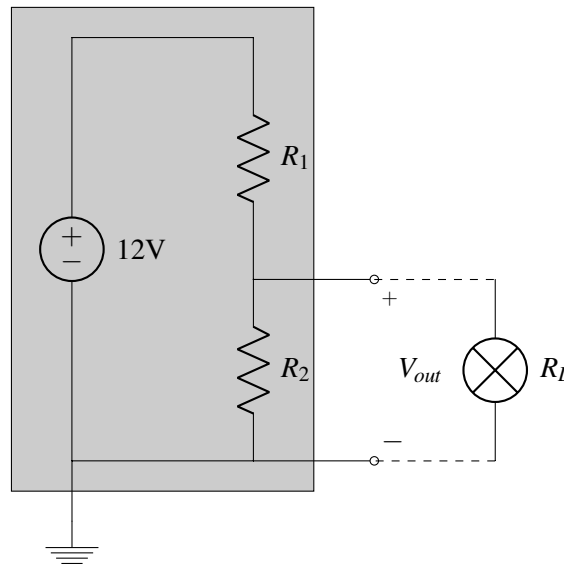
Answer: The voltage across R_2 is given by

$$V_{R_2} = \frac{R_2}{R_1 + R_2} \times 12V$$
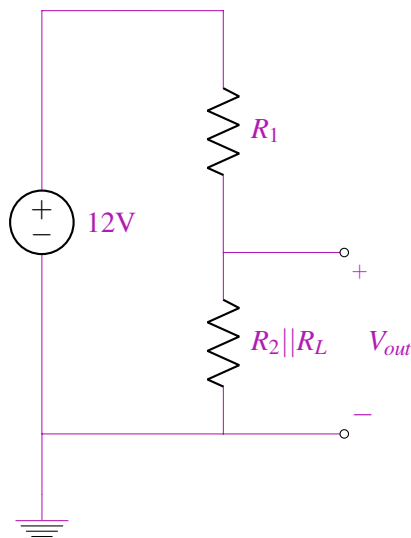
If we set $V_{R_2} = 6V$, we get

$$6V = \frac{R_2}{R_1 + R_2} \times 12V$$
$$\implies R_1 = R_2$$

For example, we can choose $R_1 = R_2 = 1\text{k}\Omega$.

(b) Now let us connect the bulb R_L across R_2. What is the voltage across R_1, R_2 and R_L when the bulb is connected when $R_L = R_2$? Use the values of R_1 and R_2 from the last part. Will the light bulb turn on? What happens if $R_L = 2R_2$?

Answer: When we connect the bulb across R_2, the equivalent circuit is the following:

The voltage V_{out} given by

$$V_{out} = \frac{R_2 || R_L}{R_1 + R_2 || R_L} \times 12\text{V}$$

When $R_L = R_2$:

$$R_2 || R_L = R_2 || R_2 = \frac{R_2}{2}.$$

Hence V_{out} is:

$$V_{out} = \frac{R_2 || R_L}{R_1 + R_2 || R_L} \times 12\text{V} = \frac{\frac{R_2}{2}}{R_1 + \frac{R_2}{2}} \times 12\text{V} = \frac{\frac{R_2}{2}}{R_2 + \frac{R_2}{2}} \times 12\text{V} = 4\text{V} < 6\text{V}$$

So the bulb will not turn on.

When $R_L = 2R_2$:

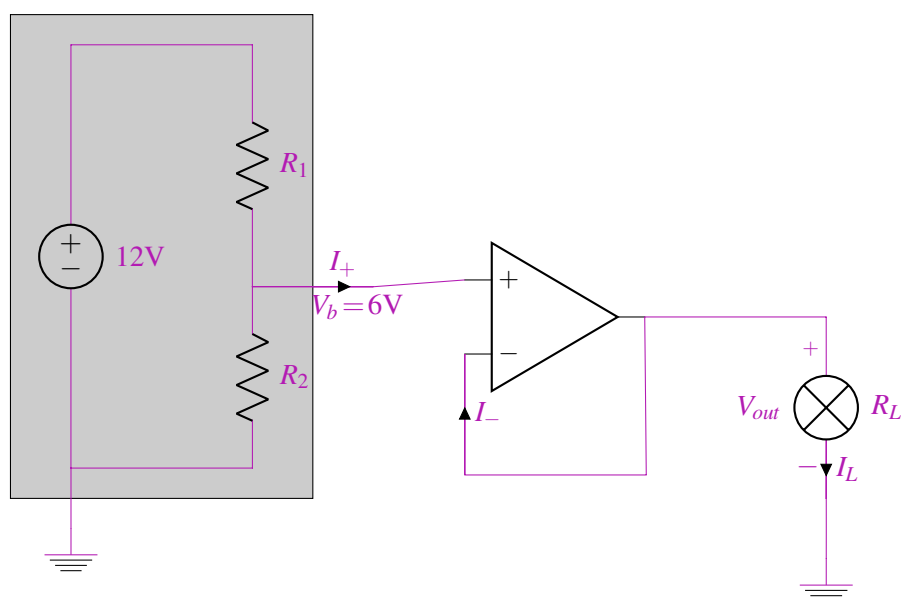$$R_2 || R_L = R_2 || 2R_2 = \frac{2R_2}{3}.$$

Hence V_{out} is:

$$V_{out} = \frac{R_2 || R_L}{R_1 + R_2 || R_L} \times 12\text{V} = \frac{\frac{2R_2}{3}}{R_1 + \frac{2R_2}{3}} \times 12\text{V} = \frac{\frac{2R_2}{3}}{R_2 + \frac{2R_2}{3}} \times 12\text{V} = 4.8\text{V} < 6\text{V}$$

Although, V_{out} is now closer to 6V, the bulb will still not turn on.

(c) Using your knowledge of op-amps, how could you resolve this issue of V_{out} changing based on the value of R_L? Think about how you might use an op-amp buffer.

Answer: We can introduce a 'buffer' op-amp as shown below:

A buffer op amp effectively 'decouples' its input and output, It does this by preventing the bulb from drawing current from the black box circuit. Remember that the input currents of an ideal op-amp:

$$I_+ = I_- = 0,$$

while the output current I_L can adjust itself to any extent, depending on the demand placed by V_{out}.
Let us use nodal analysis and the golden rules to formally solve this circuit. Firstly, we observe that the op-amp is in negative feedback configuration. Using the golden rules, we have that $V_+ = V_- = V_b = 6V$. Also, because the feedback connection is a short, $V_{out} = V_-$. Therefore, $V_{out} = V_b = 6V$. This is exactly what we want! The voltage across R_L equals $V_{out} = 6V$, which is above the required voltage.