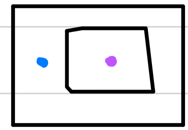
# Proofs!

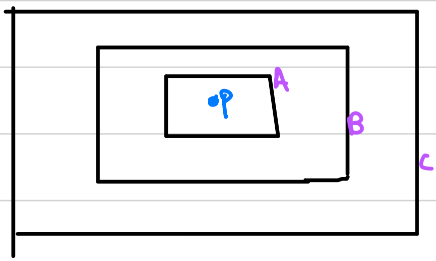
## Types of Proofs



### Direct

$A \to B \to C \to \dots \to X \to Y \to Z$ proves $A \to Z$

Spatial intuition:




if point P is inside rectangle A, then point P must be in rectangle C.

### Constructive

"proof by example"

can be useful if
- proving a "there exists" stmt
- disproving a "for all" stmt → known as a "counterexample"

$\exists$ = there exists
$\forall$ = for all

examples:

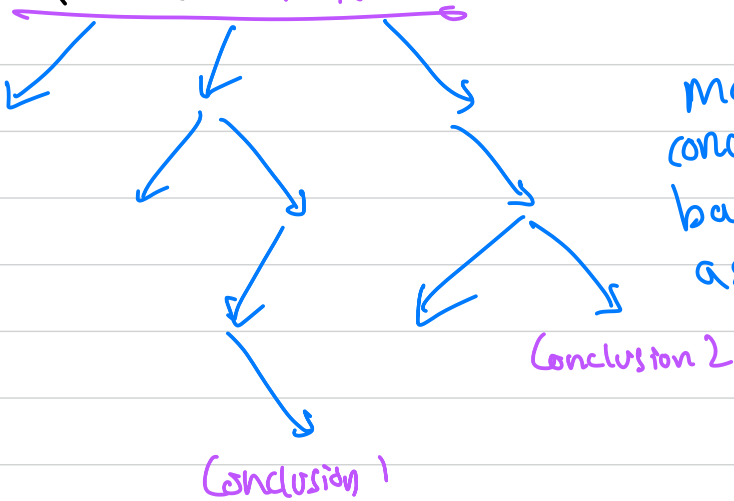
↳ seeing one alien is enough to prove existence of aliens

↳ disprove all integers are even by giving one odd integer

## Contradiction

want to prove X

Assume not X




make many conclusions based on the assumption not X

Main idea:
- if assumption not X, then Conc. 1
- also if assumption not X, then Conc. 2
- if Conc. 1 and Conc 2 cannot both be true, we cannot possibly have not X
- therefore, X

**Extra** Contraposition

$A \Rightarrow B$ is equivalent to $!B \Rightarrow !A$

1. Comparison (direct proofs)

START | END

$f(x, y, z)$ express $x$ in terms of $a$ $g(a, b, c)$

I don't see $x$

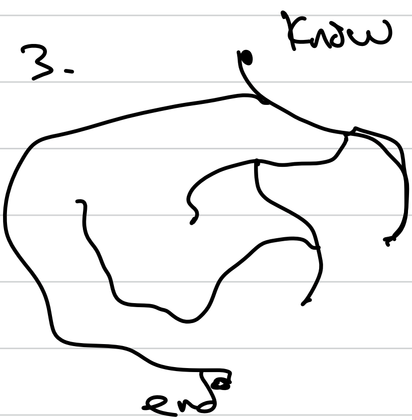
2. Show that all integers are positive.

$\Downarrow$

↳ $x$ abstract container

$x > 0$

3.




organize

① path f → ② path

↳ $x$

1. Proofs on Linear In Dependence

(a)

WHAT WE KNOW $\quad A \in \mathbb{R}^{m \times n}$

$\vec{v}$ a solution of $A\vec{x} = \vec{0}$

$\underline{A\vec{v} = \vec{0}}$ $\quad \vec{v} \neq \vec{0}$ $\quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

GO FROM ↑ TO ↓

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \vec{a_3} & \dots & \vec{a_n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\vec{a_1} + v_2\vec{a_2} + \dots + v_n\vec{a_n} = \vec{0}$$

$\exists \alpha$ because $\alpha_i = v_i$

WHAT WE NEED TO SHOW

$\alpha_1\vec{a_1} + \alpha_2\vec{a_2} + \dots + \alpha_n\vec{a_n} = \vec{0}$ $\quad A = \begin{bmatrix} | & | & & | \\ \vec{a_1} & \vec{a_2} & \dots & \vec{a_n} \\ | & | & & | \end{bmatrix}$

not all $\alpha$'s are equal to zero

→ there exists a set of $\alpha$'s

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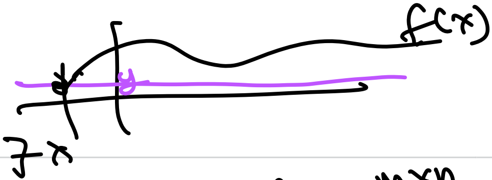
Linear dependence $\quad \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$

$(\beta_1\vec{v_1} + \beta_2\vec{v_2} + \dots) = \quad \vec{v_n}$

$\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$

at least one $\alpha$ must be $\neq 0$




## WHAT DO WE KNOW

$A \in \mathbb{R}^{m \times n}$

$\vec{u}$ solution $\quad A\vec{x}=\vec{b} \iff f(x)=y$

$\exists \vec{u} \quad A\vec{u}=\vec{b}$

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \dots & \vec{a_n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \vec{b}$$

$$u_1\vec{a_1} + \dots + u_n\vec{a_n} = \vec{b}$$

## WHAT WE NEED TO SHOW

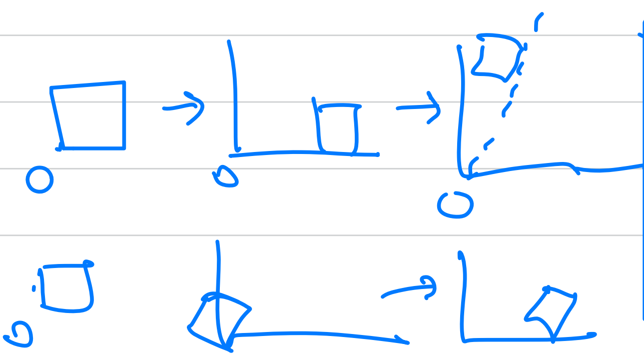
$$A = \begin{bmatrix} | & | & & | \\ \vec{a_1} & \vec{a_2} & \dots & \vec{a_n} \\ | & | & & | \end{bmatrix}$$

$\vec{b} \in \text{span}\{\text{cols}(A)\}$

$\vec{b} = \alpha_1\vec{a_1} + \alpha_2\vec{a_2} + \dots + \alpha_n\vec{a_n}$

$\exists \alpha_i$'s $\longrightarrow$

$$\alpha_1\vec{a_1} + \dots + \alpha_n\vec{a_n} + (-1)\vec{b} = 0$$



$AB \neq BA$

$AB\vec{x} = BA\vec{x}$

github.com/dinganthony/teaching

## Problem #4: Invertibility + Row Operations

$$M = \begin{bmatrix} \vec{m_1}^T \\ \vec{m_2}^T \\ \vec{m_3}^T \end{bmatrix}$$

(a) AM

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{dividing second row by 5}$$

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} a_1 \vec{m_1}^T \\ a_2 \vec{m_2}^T \\ a_3 \vec{m_3}^T \end{bmatrix}$$

(b) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{m_1}^T \\ \vec{m_2}^T \\ \vec{m_3}^T \end{bmatrix} = \begin{bmatrix} \vec{m_3}^T \\ \vec{m_2}^T \\ \vec{m_1}^T \end{bmatrix}$

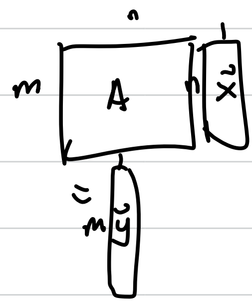
swapping 1 and 3

(c) $C = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → adding 3x row 1 to row 2

(d) $Q = \begin{bmatrix} 0 & 0 & 1 \\ -15 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[R_2\ 1/5]{\text{Scaling}} \begin{bmatrix} 0 & 0 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\xrightarrow[R_1\ R_3]{\text{flip}} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(c)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $A\vec{x} = \vec{y}$ $\qquad \vec{x} \to \boxed{A} \to \vec{y}$




$D = CBA$

$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{5} & 3 \\ 1 & 0 & 0 \end{bmatrix}$

$$DQ = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{5} & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -15 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$QD = \begin{bmatrix} 0 & 0 & 1 \\ -15 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{5} & 3 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$DQ = QD = I$$

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$$A\vec{x} = \vec{b}$$

$$E_1 A\vec{x} = E_1\vec{b} \qquad E_1 \left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$$

$$E_2E_1A\vec{x} = E_2E_1\vec{b}$$

$$\vdots \qquad \left[\begin{array}{c|c} E_1A & E_1b \end{array}\right]$$

$$\underbrace{E_{10}E_9 \dots E_2E_1}_{E} A\vec{x} = E_{10} \dots E_1\vec{b}$$

$$E = A^{-1}$$

$$EA\vec{x} = E\vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b} \quad \Rightarrow \quad \vec{x} = A^{-1}\vec{b}$$

(e) $Q = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

$\begin{bmatrix} \vec{q_1} & \vec{q_2} & \vec{q_3} \end{bmatrix}$

$\vec{q_2} = 1 \cdot \vec{q_1} + 0 \cdot \vec{q_3}$

By IMT, if columns matrix are linearly dep., that matrix is not invertible.

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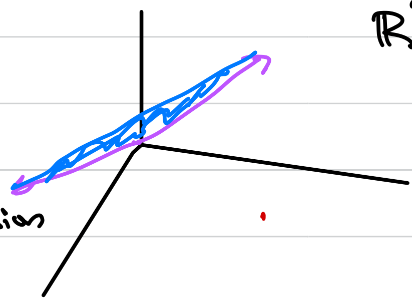
$\text{span}(Q) = \left\{ \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 15 \\ 4 \\ 4 \end{bmatrix} \right\}$

$\hookrightarrow$ 2 dimension

$\mathbb{R}^3$



$A\vec{x} = \vec{y}$

$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \vec{a_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= x_1\vec{a_1} + x_2\vec{a_2} + x_3\vec{a_3}$

A takes any $\vec{x} \in \mathbb{R}^3$

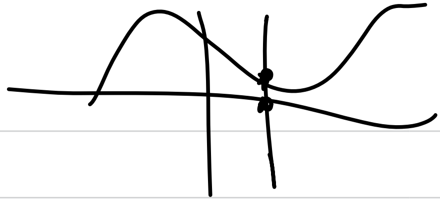
Spits out $\vec{y} \in$ plane

$\vec{x}_1$

$\vec{x}_2$

$\vec{y}$

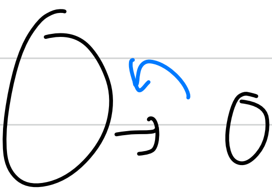
$\vec{y} \begin{cases} \vec{x}_1 \\ \vec{x}_2 \end{cases}$



$$A^{-1}A\vec{x} = \vec{x}$$

if span(A) = # columns, A is invertible
span(A) < # columns, A is invertible

PNG
lossless compression



JPEG
lossy compression