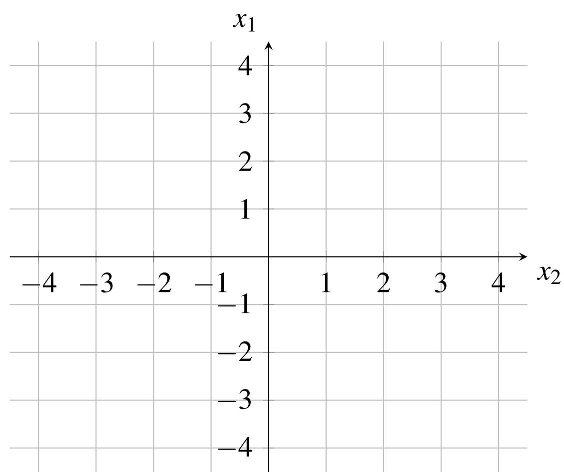
CSM 16A Spring 2021 Designing Information Devices and Systems I Week 11

1. **Projections**

Learning Goal: The goal of this problem is to understand the properties of projection.

Relevant Notes: **Note 23** walks through mathematical derivations for projection.



(a) Consider the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Draw it on the graph provided. Also draw the vector $\vec{y_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{y_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Now, find the projections of $\vec{x}$ on $\vec{y_1}$ and $\vec{y_2}$ geometrically. Compare with mathematical calculations.

(b) Calculate the projection of $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ on $\vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Is it the same as the projection of $\vec{y}$ on $\vec{x}$?

(c) Now consider the vectors $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Now, find the projections of $\vec{x}$ on $\vec{y}_1$ and $\vec{y}_2$. Also find the projection of $\vec{x}$ on $\text{span}\{\vec{y}_1, \vec{y}_2\}$. Is $\text{proj}_{\vec{y}_1}\vec{x} + \text{proj}_{\vec{y}_2}\vec{x}$ equal to $\text{proj}_{\text{span}\{\vec{y}_1,\vec{y}_2\}}\vec{x}$? Explain your answer.

(d) Find the expression for projection of $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ on the columnspace of matrix $\mathbf{A} = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Is $\text{proj}_{\vec{a}_1}\vec{b} + \text{proj}_{\vec{a}_2}\vec{b}$ equal to $\text{proj}_{\text{Col}\{\mathbf{A}\}}\vec{b}$? (No need to do the calculations.)

If we set up a system of linear equations $\mathbf{A}\vec{x} = \vec{b}$, will there be a unique solution? (No need to solve the system.)

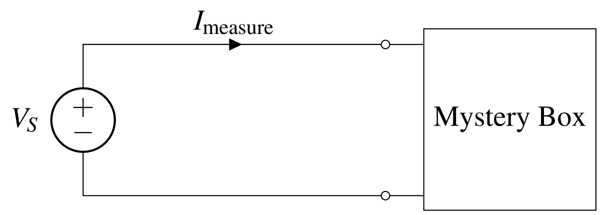
2. And You Thought You Could Ignore Circuits Until Dead Week

Learning Goal: The objective of this problem is to practice solving a noisy system using least squares method.

Relevant Notes: Note 23 covers the details of least squares method.

(a) Write Ohm's Law for a resistor.

(b) You're given the following test setup and told to find R_{eq} between the two terminals of the mystery box. What is R_{eq} of the mystery box between the two terminals in terms of V_S and I_{measure}?



(c) You think you've figured out how to find R_{eq}! You've taken the following measurements:

Measurement #	V_S	I_{measure}
1	2V	1A
2	4V	2A
3	6V	2A
3	8V	4A

Using the information above, formulate a least squares problem whose answer provides an estimate of R_{eq}.

(d) Find the least squares error vector $\|\vec{e}\|$.

3. Least Squares Fitting

Learning Goal: The objective of this problem is to set up a least squares problem for coefficients of non-linear equations.

Relevant Notes: Note 23 covers the details of least squares method.

In an upward career move, you join the starship USS Enterprise as a data scientist. One morning the Chief Science Officer, Mr. Spock, hands you some data for the position (y) of a newly discovered particle at different times (t). The data has three points and **contains some noise**:

$$(t=0, y=0.5), \quad (t=1, y=3), \quad (t=2, y=18.5)$$

Your research shows that the path of the particle is represented by the function:

$$y = e^{w_1 + w_2 t} \tag{1}$$

You decide to fit the collected data to the function in Equation (1) using the Least Squares method.

(a) You need to find the coefficients w_1 and w_2 that *minimize the squared error* between the fitted curve and the collected data points. So you set up a system of linear equations, $\mathbf{A}\hat{\vec{\alpha}} \approx \vec{b}$ in order to find the approximate value of $\hat{\vec{\alpha}} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. What are the values of $\mathbf{A}$ and $\vec{b}$?

(b) Mr. Spock thinks one of the data points is wrong and asks you to redo the fit with only two data points. What will happen to the norm of the error, $\|\vec{e}\| = \left\|\vec{b} - \mathbf{A}\hat{\vec{\alpha}}\right\|$?

(c) Your colleague tries to repeat your fitting process with the same four data points in part (a), but they misread the equation relating t and y, i.e. they use the following function (which is **different than part (a)**):

$$y = e^{w_1 t + w_2 t} \tag{2}$$

Your colleague tries to find w_1 and w_2 by setting up a system of equations $\mathbf{A}\hat{\vec{\alpha}} \approx \vec{b}$ and utilizing the equation:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \hat{\vec{\alpha}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}. \tag{3}$$

What will happen when your colleague tries to solve the above equation?

4. Cactus Care [FINAL QUESTION SP18]

On Midterm 2 you designed a light sensor to check that there is sufficient light in your room for your cactus to be happy and healthy. But you want to monitor the light levels over the course of the day, when you aren't around. You design a transmitter that sends the following periodic code of length $N = 5$:

$$\vec{c} = \begin{bmatrix} 1 & -3 & 2 & 1 & 2 \end{bmatrix}^T$$

You encode information about the light by multiplying the code with the light intensity (y). With your cell phone, you receive a shifted version of the code (since it had to travel an unknown distance), multiplied by the light intensity.

(a) (4 points) Write a matrix A such that

$$A\vec{y} = \vec{r}$$

where $\vec{r}$ is the received signal (length 5) and $\vec{y}$ is a vector of all zeros except one entry which contains the light intensity y. (*Hint: The position of y in the vector $\vec{y}$ will depend on the unknown shift in the signal.*)

(b) (6 points) This semester your learned several techniques for solving linear systems of equations. For each of the following techniques, could you use it to solve the matrix equation from Part A? Justify your answer in 1-2 sentences. Assume there is no noise.

Gaussian Elimination ◯ yes ◯ no

Explain:

Least Squares ◯ yes ◯ no

Explain:

Orthogonal Matching Pursuit ◯ yes ◯ no

Explain:

You're not sure that your room is really the right place for your cactus, so you set up another light detector in the lab to see if it's better. Each of the two light detectors has a transmitter with a different periodic code c_1, c_2.

$$\vec{c}_1 = \begin{bmatrix} 1 & -3 & 2 & 1 & 2 \end{bmatrix}^T$$

$$\vec{c}_2 = \begin{bmatrix} 3 & 1 & 2 & -2 & -1 \end{bmatrix}^T$$

As before, the codes are multiplied by the light intensities at each location, y_1 and y_2, and your cell phone receives the sum of shifted codes, each weighted by the light at that location.

(c) (5 points) Write a new matrix A such that

$$A\vec{y} = \vec{r}$$

where $\vec{r}$ is the received signal (length 5) and $\vec{y}$ is a vector of all zeros except two entries which contain y_1 and y_2.
Hint: The positions of y_1 and y_2 in the vector $\vec{y}$ will depend on the unknown shifts in c_1 and c_2, respectively.

(d) (6 points) For each of the following techniques, could you use it to solve the matrix equation from Part D, with two different light sensors? Justify your answer in 1-2 sentences. Assume there is no noise.

Gaussian Elimination ◯ yes ◯ no
Explain:

Least Squares ◯ yes ◯ no
Explain:

Orthogonal Matching Pursuit ◯ yes ◯ no
Explain:

(e) (3 points) In order to judge if your codes are "good", you want to calculate the autocorrelations and cross-correlation of your codes. Professor Waller helps you calculate the following:

$$\text{autocorrelation of } \vec{c}_1 = \begin{bmatrix} 19 & -3 & \mathbf{??} & -2 & -3 \end{bmatrix}^T$$

$$\text{autocorrelation of } \vec{c}_2 = \begin{bmatrix} 19 & 0 & -5 & -5 & 0 \end{bmatrix}^T$$

$$\text{cross-correlaton of } \vec{c}_1 \text{ with } \vec{c}_2 = \begin{bmatrix} 0 & -10 & 12 & 11 & -4 \end{bmatrix}^T$$

Finish the set by calculating the unknown term in the autocorrelation of c_1.

(f) (6 points) Consider the following set of codes (c_3 and c_4).

$$\vec{c}_3 = \begin{bmatrix} 1 & -2 & -3 & 2 & 1 \end{bmatrix}^T \qquad \vec{c}_4 = \begin{bmatrix} 1 & 1 & 2 & -2 & -3 \end{bmatrix}^T$$

$$\text{autocorr. of } \vec{c}_3 = \begin{bmatrix} 19 & 1 & -10 & -10 & 1 \end{bmatrix}^T$$

$$\text{autocorr. of } \vec{c}_4 = \begin{bmatrix} 19 & 2 & -11 & -11 & 2 \end{bmatrix}^T$$

$$\text{cross-correlation of } \vec{c}_3 \text{ with } \vec{c}_4 = \begin{bmatrix} -14 & -16 & 5 & 18 & -2 \end{bmatrix}^T$$

If you use OMP to solve for the light intensities, which set of codes (c_1,c_2 OR c_3,c_4) is more robust to noise in the received signal? Justify your answer. For the set of codes that is worse, what mistake will is most likely to happen during the OMP algorithm in the presence of noise?

◯ c_1, c_2 are more robust ◯ c_3, c_4 are more robust