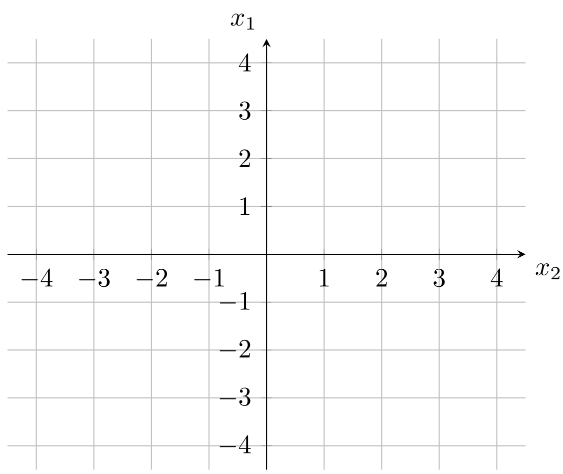
CSM 16A **Designing Information Systems and Devices**

Week 12 Worksheet

Term: **Spring 2020** *Name:*

Problem 1: Projections



1. Consider the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Draw it on the graph provided. Also draw the vectors $\vec{x}$ with the vector $\vec{y_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{y_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Now, find the inner product of $\vec{x}$ with $\vec{y_1}$ and $\vec{y_2}$.

   Interestingly, we notice that the inner products of $\vec{x}$ with each of the unit vectors in the x and y directions gives us the components of the vector in those directions. This is not a coincidence. If we drop perpendiculars from the vector $\vec{x}$ to tne x and y axis, the resulting vectors are just y_1 and y_2. This 'dropping a perpendicular' is what we mean by projection.

2. Now, find the inner product of $\vec{x}$ with the vector $\vec{y_3} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. Is this the same as with $\vec{y_1}$? How can we find the projection of $\vec{x}$ onto $\vec{y_3}$?

3. Now, let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Find the projection of $\vec{x}$ onto $\vec{y}$. Also find the projection of $\vec{y}$ onto $\vec{x}$.

4. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ Suppose we know that $A^T\vec{x} = \vec{0}$. Based on your knowledge of inner products and projections, what does this tell you about the vector $\vec{x}$

Problem 2: Cross Correlation

In this question, we will revisit the definition and some of the properties of cross correlation.

Remember that in class, we mentioned a discrete-time signal over a total of n timestamps can be represented as a vector in $\mathbb{R}^n$. Suppose we receive a signal represented by the vector:

$$\vec{r} = \begin{bmatrix} 0 & 1 & -2 & 0 \end{bmatrix}^{\mathrm{T}}$$

Given another vector $\vec{s} = \begin{bmatrix} 1 & 2 & -1 & 0 \end{bmatrix}^{\mathrm{T}}$, we define the cross correlation of $\vec{r}$ with respect to $\vec{s}$ with an offset of k as:

$$\mathrm{corr}_{\vec{r}}(\vec{s})[k] = \sum_{i=-\infty}^{\infty} r[i]s[i-k]$$

Note: we define the value of the signal for any index that is outside the range of the vector to be 0.

1. Find the value of $\mathrm{corr}_{\vec{r}}(\vec{s})[0]$.

2. Find the value of $\mathrm{corr}_{\vec{r}}(\vec{s})[1]$.

3. Now you have done part (i), you might have realized that the formula for the cross correlation between $\vec{r}$ and $\vec{s}$ when $k = 0$ looks very similar to that of the inner product between $\vec{r}$ and $\vec{s}$. When $k = 0$, is the cross correlation between two vectors the same as their inner product? Why or why not?

4. Note for the three parts above, we have been concerning ourselves with cross correlation **at a particular offset** so far. In reality, all these ”offseted” correlations together define an overall cross correlation vector. To formalize this description, we define the cross correlation (vector) of $\vec{r}$ with respect to $\vec{s}$ as:

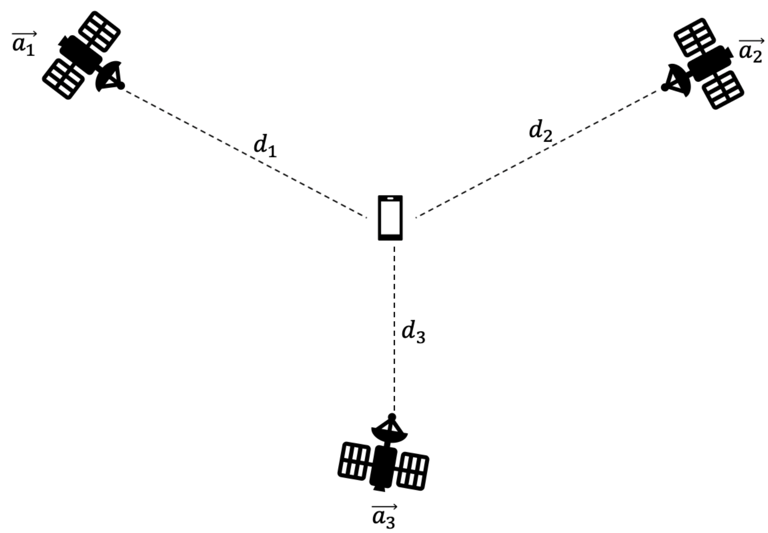
$$\mathrm{corr}_{\vec{r}}(\vec{s}) = \begin{bmatrix} \mathrm{corr}_{\vec{r}}(\vec{s})[i] \\ \mathrm{corr}_{\vec{r}}(\vec{s})[i+1] \\ \vdots \\ \mathrm{corr}_{\vec{r}}(\vec{s})[i+n] \end{bmatrix}.$$

   Given the definition above, if we have a discrete-time signal $\vec{r}$ that is of length x and another discrete-time signal $\vec{s}$ that is of length y, what would be the length of the vector $\mathrm{corr}_{\vec{r}}(\vec{s})$ (In other words, what would be the range $[i, i+n]$ in the definition above) ?

5. Based on all of the parts above, find the cross correlation vector of $\vec{r}$ with respect to $\vec{s}$: $\mathrm{corr}_{\vec{r}}(\vec{s})$.

Problem 3: G(oogle) Positioning System

Suppose that you are the engineer tasked with the job of making Google Maps. For this, you want to be able to determine the position of a user using satellite information. In particular, assume that you know d_1, d_2 and d_3, the distances from the user's cellphone to 3 satellites. You know the positions of these satellites to be $\vec{a_1}, \vec{a_2}$, and $\vec{a_3}$. Here's a simplified figure demonstrating what's been given so far:



Note: What does it mean when we say "position"? You can assume that these positions are taken relative to some common origin. Say, the Google HQ - Mountain View, CA.

1. Suppose the user's location (or the phone's location) is given by the vector $\vec{x}$, write out a system of equations representing the distances from the user to all 3 satellites (Express your answer in terms of $\vec{x}, \vec{a_1}, \vec{a_2}, \vec{a_3}, d_1, d_2$, and d_3).

2. Rewrite these equations in terms of inner products of vectors. Are these equations linear with respect to $\vec{x}$?

3. Are there any non-linear terms in the equations from the previous part? Using **elimination of variables**, rewrite everything as a system of **linear** equations.

4. **Using the system of linear equations we have from the previous part**, if the location of the user (i.e. $\vec{x}$ is a 3-dimensional vector), do we have sufficient information to solve for $\vec{x}$? If not, then how many satellites do you need to locate the user?

5. Suppose now in more generalized terms, we want to not only triangulate the user's position, but also keep track of other information about the user to make more customized analysis. Given that the vector representing the user location now contains a total of n entries, what is the minimum number of satellites we need to find that vector?

> In real life, we won't actually be given the distances from the user to the satellites, either. In other words, we also need to figure out how far away the satellites are from us! Fortunately, as we have already learned in class, **cross correlation** is something that might come in handy for us to figure out the distances. For all the remaining parts of this question, we will use what we have learned about **cross correlation** to figure out what the distances from the user to the satellites are.

6. To figure out how far away the satellites are from us, we can use our phone to receive radio signals from the satellites in the orbit. Once we have received the signals, we can then compare them with a reference signal

on our phone to figure out the time it takes for the signal to reach us. Given our original reference signal:

$$\vec{s} = \begin{bmatrix} -1 & -1 & -1 & 1 & -1 \end{bmatrix}^T,$$

and the three signals we received, each having a period of 4 (we will only show one period of each signal):

$$\vec{r}_1 = \begin{bmatrix} -1 & -1 & -1 & 1 \end{bmatrix}^T$$

$$\vec{r}_2 = \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}^T$$

$$\vec{r}_3 = \begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix}^T$$

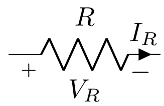
Find the cross correlations $\text{corr}_{\vec{r}_1}(\vec{s})$, $\text{corr}_{\vec{r}_2}(\vec{s})$, and $\text{corr}_{\vec{r}_3}(\vec{s})$ between $\vec{s}$ and all three received signals respectively, and plot them out below.

7. Based on the cross-correlated signals, determine the delays (in seconds) for all 3 received signals $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$.

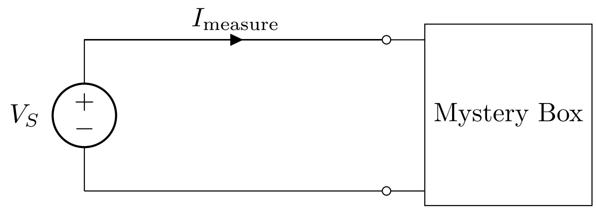
8. Given that the radio signal has a transmission speed of v, and assume all delays are relative to the source signal $\vec{s}$ (this means we assume $\vec{s}$ is received at time $t = 0$), find the distance d_1, d_2, and d_3 between the user location and the 3 satellites in orbit.

Problem 4: Least Sqaures with Ohm's Law

1. Write Ohm's Law for a resistor.



2. You're given the following test setup and told to find R_{Th} between the two terminals of the mystery box. What is R_{Th} of the mystery box between the two terminals in terms of V_S and I_{measure}?



3. You think you've figured out how to find R_{Th}! You've taken the following measurements:

Measurement #	I_{measure}	V_S
1	$1A$	$1.25kV$
2	$2A$	$1kV$
3	$3A$	$4kV$
4	$4A$	$3.5kV$

Using the information above, formulate a least squares problem whose answer provides an estimate of R_{Th}.