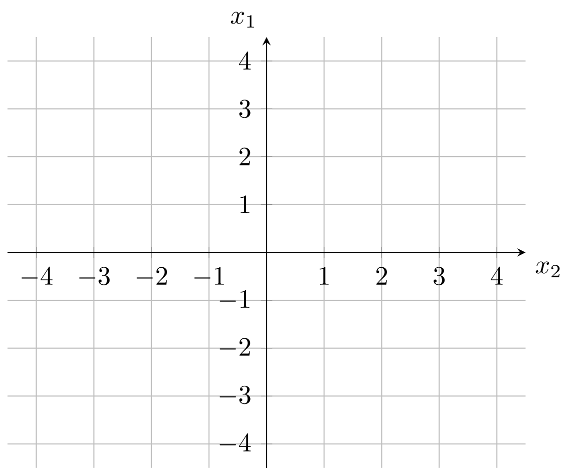
CSM 16A — Designing Information Systems and Devices

Week 12 Worksheet Solutions

Term: **Spring 2020** — *Name:*

Problem 1: Projections



1. Consider the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Draw it on the graph provided. Also draw the vectors $\vec{x}$ with the vector $\vec{y_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{y_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Now, find the inner product of $\vec{x}$ with $\vec{y_1}$ and $\vec{y_2}$.

   Solution:

   $$\langle x, y_1 \rangle = 2 \cdot 1 + 4 \cdot 0 = 2$$
   $$\langle x, y_2 \rangle = 2 \cdot 0 + 4 \cdot 1 = 4$$

   Interestingly, we notice that the inner products of $\vec{x}$ with each of the unit vectors in the x and y directions gives us the components of the vector in those directions. This is not a coincidence. If we drop perpendiculars from the vector $\vec{x}$ to tne x and y axis, the resulting vectors are just y_1 and y_2. This 'dropping a perpendicular' is what we mean by projection.

2. Now, find the inner product of $\vec{x}$ with the vector $\vec{y_3} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. Is this the same as with $\vec{y_1}$? How can we find the projection of $\vec{x}$ onto $\vec{y_3}$?

   Solution:

   $$\langle x, y_3 \rangle = 2 \cdot 5 + 4 \cdot 0 = 10$$

   However, by dropping the perpendicular from $\vec{x}$ to $\vec{y_1}$, we can see that the projection should be the same as it was for $\vec{x_1}$. So, we have to divide this inner product by 5 to get the correct length of the projection (since $10/5 = 2$). It is not a coincidence that 5 is also the length of $\vec{x_3}$! We have to divide the inner product by the magnitude of the vector we are projecting onto, because while the inner product scales with both of its inputs, the projection should only scale with $\vec{x}$, and it should only depend on the direction of $\vec{y_3}$. That is, the projection of $\vec{x}$ onto any vector of the form $\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for nonzero α should be the same.

So, we have that the length of the projection of $\vec{x}$ onto $\vec{y}$ equals
$\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|}$.
Another way of interpreting this is that by dividing by $\|\vec{y}\|$, we are converting $\vec{y}$ into a unit vector. That is, $\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|} = \left\langle \vec{x}, \frac{\vec{y}}{\|\vec{y}\|} \right\rangle$

3. Now, let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Find the projection of $\vec{x}$ onto $\vec{y}$. Also find the projection of $\vec{y}$ onto $\vec{x}$.

   Solution: The length of the projection of $\vec{x}$ onto $\vec{y}$ will be $\frac{\langle \vec{x}, \vec{y} \rangle}{||\vec{y}||} = \frac{1 \cdot 1 + 0 \cdot 2 + 3 \cdot -1}{\sqrt{1^2 + 0^2 + 1^2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$.

   But we don't just want the length of the projection, we want the actual projection vector. We know the projection's length, and we know that it must lie along the direction characterized by $\vec{y}$. So to get the projection, we can simply scale the unit vector in the direction of $\vec{y}$ by the length we found above. So, the projection is $-\sqrt{2} \cdot \frac{\vec{y}}{||\vec{y}||} = -\sqrt{2} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

4. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ Suppose we know that $A^T \vec{x} = \vec{0}$. Based on your knowledge of inner products and projections, what does this tell you about the vector $\vec{x}$

   Solution: It tells you that the vector $\vec{x}$ is orthogonal to the columns of $\mathbf{A}$. Or, in other words, the projection of $\vec{x}$ onto each of the columns of $\mathbf{A}$ is of length 0. This also means that the projection of $\vec{x}$ onto the subspace spanned by the columns of $\mathbf{A}$, also known as the range of $\mathbf{A}$, is the zero vector.

Problem 2: Cross Correlation

In this question, we will revisit the definition and some of the properties of cross correlation.

Remember that in class, we mentioned a discrete-time signal over a total of n timestamps can be represented as a vector in $\mathbb{R}^n$. Suppose we receive a signal represented by the vector:

$$\vec{r} = \begin{bmatrix} 0 & 1 & -2 & 0 \end{bmatrix}^{\mathrm{T}}$$

Given another vector $\vec{s} = \begin{bmatrix} 1 & 2 & -1 & 0 \end{bmatrix}^{\mathrm{T}}$, we define the cross correlation of $\vec{r}$ with respect to $\vec{s}$ with an offset of k as:

$$\text{corr}_{\vec{r}}(\vec{s})[k] = \sum_{i=-\infty}^{\infty} r[i]s[i-k]$$

Note: we define the value of the signal for any index that is outside the range of the vector to be 0.

1. Find the value of $\text{corr}_{\vec{r}}(\vec{s})[0]$.

   Solution: With an offset of $k = 0$, we can see the expression for cross correlation simplified to:

   $$\text{corr}_{\vec{r}}(\vec{s})[k] = \sum_{i=-\infty}^{\infty} r[i]s[i].$$

   We have:

   $$\begin{aligned} \text{corr}_{\vec{r}}(\vec{s})[0] &= \vec{r}[1]\vec{s}[1] + \vec{r}[2]\vec{s}[2] + \vec{r}[3]\vec{s}[3] + \vec{r}[4]\vec{s}[4] \\ &= (0)(1) + (1)(2) + (-2)(-1) + (0)(0) \\ &= 4. \end{aligned}$$

2. Find the value of $\text{corr}_{\vec{r}}(\vec{s})[1]$.

   Solution: With an offset of $k = 1$, we can see the expression for cross correlation simplified to:

   $$\text{corr}_{\vec{r}}(\vec{s})[1] = \sum_{i=-\infty}^{\infty} r[i]s[i-1].$$

   We have:

   $$\begin{aligned} \text{corr}_{\vec{r}}(\vec{s})[1] &= \vec{r}[1]\vec{s}[0] + \vec{r}[2]\vec{s}[1] + \vec{r}[3]\vec{s}[2] + \vec{r}[4]\vec{s}[3] \\ &= (0)(0) + (1)(1) + (-2)(2) + (0)(-1) \\ &= -3. \end{aligned}$$

3. Now you have done part (i), you might have realized that the formula for the cross correlation between $\vec{r}$ and $\vec{s}$ when $k = 0$ looks very similar to that of the inner product between $\vec{r}$ and $\vec{s}$. When $k = 0$, is the cross correlation between two vectors the same as their inner product? Why or why not?

   Solution: As similar as their formulas look alike, these two concepts are actually **not the same**. The key difference lies in that inner product only applies to two vectors **of the same dimensions**, while cross correlation can be actually applied on two vectors (or signals) of different sizes! An example would be given two discrete-time signals represented by the vectors

   $$\begin{aligned} \vec{u} &= \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}^{\mathrm{T}} \\ \vec{v} &= \begin{bmatrix} 2 & -1 \end{bmatrix}^{\mathrm{T}}, \end{aligned}$$

   we cannot compute the inner product $\langle \vec{u}, \vec{v} \rangle$ because the two vectors have different sizes, but we can compute their cross correlation at $k = 0$:

   $$\begin{aligned} \text{corr}_{\vec{r}}(\vec{s})[0] &= \vec{r}[1]\vec{s}[1] + \vec{r}[2]\vec{s}[2] \\ &= (1)(2) + (2)(-1) \\ &= 0. \end{aligned}$$

   Note we leave out the third term in $\vec{r}$ since it will be out of range for $\vec{s}$, which will just be the value 0 (does not contribute to the overall sum so we leave it out).

4. Note for the three parts above, we have been concerning ourselves with cross correlation **at a particular offset** so far. In reality, all these "offseted" correlations together define an overall cross correlation vector. To formalize this description, we define the cross correlation (vector) of $\vec{r}$ with respect to $\vec{s}$ as:

$$\text{corr}_{\vec{r}}(\vec{s}) = \begin{bmatrix} \text{corr}_{\vec{r}}(\vec{s})[i] \\ \text{corr}_{\vec{r}}(\vec{s})[i+1] \\ \vdots \\ \text{corr}_{\vec{r}}(\vec{s})[i+n] \end{bmatrix}.$$

Given the definition above, if we have a discrete-time signal $\vec{r}$ that is of length x and another discrete-time signal $\vec{s}$ that is of length y, what would be the length of the vector $\text{corr}_{\vec{r}}(\vec{s})$ (In other words, what would be the range $[i, i+n]$ in the definition above) ?

Solution: Given $\vec{r}$ has a length of x, and $\vec{s}$ has a length of y, we can see that the smallest value of offset k can be $-(y-1) = -y+1$ since $i-k = i+y-1$ at that point, and for any even smaller offset, we will be always out of range for $\vec{s}$. At the same time, we can see that for any positive offset k, we should always start sampling the signal $\vec{r}$ at the index $1+k$ because any earlier index would mean negative indices for $\vec{s}$, and that will always be undefined (which is 0). This means $1+k$ can be at most $x-1$, because for any offset greater than that, the index will be out of range for $\vec{u}$. Hence, we have found our range of indices to be $[-y, x-1]$, and the length is: $x-1-(-y+1)+1 = x+y-1$.

Therefore, if we have a vector $\vec{r}$ of length x and another vector $\vec{s}$ of length y, the cross correlation (vector) of $\vec{r}$ with respect to $\vec{s}$ will have a length of $x+y-1$, and we can also improve the indexing of the offset k in our original formula to be:

$$\text{corr}_{\vec{r}}(\vec{s}) = \begin{bmatrix} \text{corr}_{\vec{r}}(\vec{s})[-y+1] \\ \text{corr}_{\vec{r}}(\vec{s})[-y+2] \\ \vdots \\ \text{corr}_{\vec{r}}(\vec{s})[x-1] \end{bmatrix}.$$

5. Based on all of the parts above, find the cross correlation vector of $\vec{r}$ with respect to $\vec{s}$: $\text{corr}_{\vec{r}}(\vec{s})$. **Solution:** Since both $\vec{r}$ and $\vec{s}$ have lengths 4, we know that our cross correlation vector will have the following form:

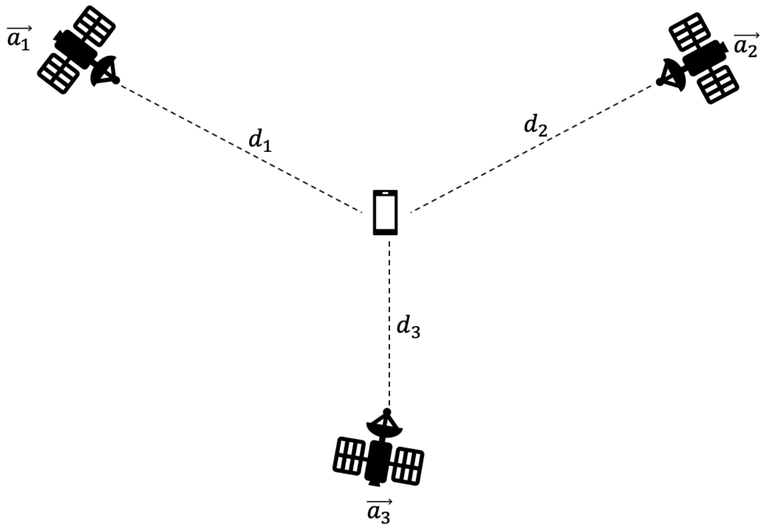
$$\text{corr}_{\vec{r}}(\vec{s}) = \begin{bmatrix} \text{corr}_{\vec{r}}(\vec{s})[-4] \\ \text{corr}_{\vec{r}}(\vec{s})[-3] \\ \vdots \\ \text{corr}_{\vec{r}}(\vec{s})[3] \end{bmatrix}.$$

Computing the correlation with the given offset entry by entry, we can find our cross correlation vector to be:

$$\text{corr}_{\vec{r}}(\vec{s}) = \begin{bmatrix} 0 & 0 & -1 & 4 & -3 & -2 & 0 \end{bmatrix}^{\mathrm{T}}.$$

Problem 3: G(oogle) Positioning System

Suppose that you are the engineer tasked with the job of making Google Maps. For this, you want to be able to determine the position of a user using satellite information. In particular, assume that you know d_1, d_2 and d_3, the distances from the user's cellphone to 3 satellites. You know the positions of these satellites to be $\vec{a_1}, \vec{a_2}$, and $\vec{a_3}$. Here's a simplified figure demonstrating what's been given so far:



Note: What does it mean when we say "position"? You can assume that these positions are taken relative to some common origin. Say, the Google HQ - Mountain View, CA.

1. Suppose the user's location (or the phone's location) is given by the vector $\vec{x}$, write out a system of equations representing the distances from the user to all 3 satellites (Express your answer in terms of $\vec{x}, \vec{a_1}, \vec{a_2}, \vec{a_3}, d_1, d_2$, and d_3).

   Solution:

$$\|\vec{x} - \vec{a_1}\| = d_1$$
$$\|\vec{x} - \vec{a_2}\| = d_2$$
$$\|\vec{x} - \vec{a_3}\| = d_3$$

   These equations relate the position of the user to the distance between the user and each satellite.

2. Rewrite these equations in terms of inner products of vectors. Are these equations linear with respect to $\vec{x}$?

   Solution: By squaring each side, we get the following:

$$(\vec{x} - \vec{a_1})^T(\vec{x} - \vec{a_1}) = d_1^2$$
$$(\vec{x} - \vec{a_2})^T(\vec{x} - \vec{a_2}) = d_2^2$$
$$(\vec{x} - \vec{a_3})^T(\vec{x} - \vec{a_3}) = d_3^2$$

   Then, expanding the LHS, we get:

$$\vec{x}^T\vec{x} - 2\vec{a_1}^T\vec{x} + \vec{a_1}^T\vec{a_1} = d_1^2$$
$$\vec{x}^T\vec{x} - 2\vec{a_2}^T\vec{x} + \vec{a_2}^T\vec{a_2} = d_2^2$$
$$\vec{x}^T\vec{x} - 2\vec{a_3}^T\vec{x} + \vec{a_3}^T\vec{a_3} = d_3^2$$

   These equations are not linear in $\vec{x}$, because they contain an $\vec{x}^T\vec{x}$ term.

3. Are there any non-linear terms in the equations from the previous part? Using **elimination of variables**, rewrite everything as a system of **linear** equations.

   Solution: The non-linear terms in the equations are $\vec{x}^T\vec{x}$, but we can use variable elimination to get rid of them.

   Subtract the first equation from the other 2 in order to eliminate the $\vec{x}^T\vec{x}$ term. We get the following:

$$2(\vec{a_1}^T\vec{x} - \vec{a_2}^T\vec{x}) + (\vec{a_2}^T\vec{a_2} - \vec{a_1}^T\vec{a_1}) = d_2^2 - d_1^2$$
$$2(\vec{a_1}^T\vec{x} - \vec{a_3}^T\vec{x}) + (\vec{a_3}^T\vec{a_3} - \vec{a_1}^T\vec{a_1}) = d_3^2 - d_1^2$$

   We can rewrite these as a linear equations:

$$2(\vec{a_1} - \vec{a_2})^T\vec{x} = \vec{a_1}^T\vec{a_1} - \vec{a_2}^T\vec{a_2} + d_2^2 - d_1^2$$
$$2(\vec{a_1} - \vec{a_3})^T\vec{x} = \vec{a_1}^T\vec{a_1} - \vec{a_3}^T\vec{a_3} + d_3^2 - d_1^2$$

   Or, in matrix-vector form,

$$2\begin{bmatrix}(\vec{a_1} - \vec{a_2})^T \\ (\vec{a_1} - \vec{a_3})^T\end{bmatrix}\vec{x} = \begin{bmatrix}\|a_1\|^2 - \|a_2\|^2 + d_2^2 - d_1^2 \\ \|a_1\|^2 - \|a_2\|^2 + d_3^2 - d_1^2\end{bmatrix}$$

4. **Using the system of linear equations we have from the previous part**, if the location of the user (i.e. $\vec{x}$ is a 3-dimensional vector), do we have sufficient information to solve for $\vec{x}$? If not, then how many satellites do you need to locate the user?

   Solution: No, this is not sufficient. We have only 2 equations, but 3 variables (Equivalently, in a matrix-vector equation representation $A\vec{x} = \vec{b}$, this would correspond to a 2×3 matrix A, implying that not all the column vectors in A are linearly independent, which means A is not invertible). So, this system is underdetermined, meaning that we don't have sufficient information to know the exact location of the user. In other words, **there could be more than one possible location for the user!** If instead we had 4 satellites, then by subtracting one equation from all the rest, we would have 3 equations, and we could then solve the system.

5. Suppose now in more generalized terms, we want to not only triangulate the user's position, but also keep track of other information about the user to make more customized analysis. Given that the vector representing the user location now contains a total of n entries, what is the minimum number of satellites we need to find that vector?

   Solution: First of all, we know that in order to solve for all n unique entries in the user location vector, we need a total of n linear equations (linearly independent).

   Now, based on the previous part, we can see that during the process of variable elimination, we lose a total of 1 equation to reduce the equations down to a linear system. This means we need to have 1 more equation than n equations we originally planned.

   Therefore, we would need at least a total of $n + 1$ satellites.

> In real life, we won't actually be given the distances from the user to the satellites, either. In other words, we also need to figure out how far away the satellites are from us! Fortunately, as we have already learned in class, **cross correlation** is something that might come in handy for us to figure out the distances. For all the remaining parts of this question, we will use what we have learned about **cross correlation** to figure out what the distances from the user to the satellites are.

6. To figure out how far away the satellites are from us, we can use our phone to receive radio signals from the satellites in the orbit. Once we have received the signals, we can then compare them with a reference signal on our phone to figure out the time it takes for the signal to reach us. Given our original reference signal:

$$\vec{s} = \begin{bmatrix}-1 & -1 & -1 & 1 & -1\end{bmatrix}^T,$$

and the three signals we received, each having a period of 4 (we will only show one period of each signal):

$$\vec{r}_1 = \begin{bmatrix}-1 & -1 & -1 & 1\end{bmatrix}^T$$

$$\vec{r}_2 = \begin{bmatrix}1 & -1 & 1 & 1\end{bmatrix}^T$$

$$\vec{r}_3 = \begin{bmatrix}1 & 1 & -1 & 1\end{bmatrix}^T$$

Find the cross correlations $\text{corr}_{\vec{r}_1}(\vec{s})$, $\text{corr}_{\vec{r}_2}(\vec{s})$, and $\text{corr}_{\vec{r}_3}(\vec{s})$ between $\vec{s}$ and all three received signals respectively, and plot them out below.

Solution: Using the formula for cross correlation:

$$\text{corr}_{\vec{r}}(\vec{s})[k] = \sum_{i=-\infty}^{\infty} \vec{r}[i]\vec{s}[i-k],$$

we can find the cross correlations between $\vec{s}$ and all three received signals to be:

$$\text{corr}_{\vec{r}_1}(\vec{s}) = \begin{bmatrix}1 & 0 & 1 & 0 & 4 & 1 & 0 & -1\end{bmatrix}^T,$$
$$\text{corr}_{\vec{r}_2}(\vec{s}) = \begin{bmatrix}-1 & 2 & -3 & 0 & 0 & -1 & -2 & -1\end{bmatrix}^T,.$$
$$\text{corr}_{\vec{r}_3}(\vec{s}) = \begin{bmatrix}-1 & 0 & 1 & -4 & 0 & -1 & 0 & -1\end{bmatrix}^T$$

The plots for the cross-correlated signals are as follows:

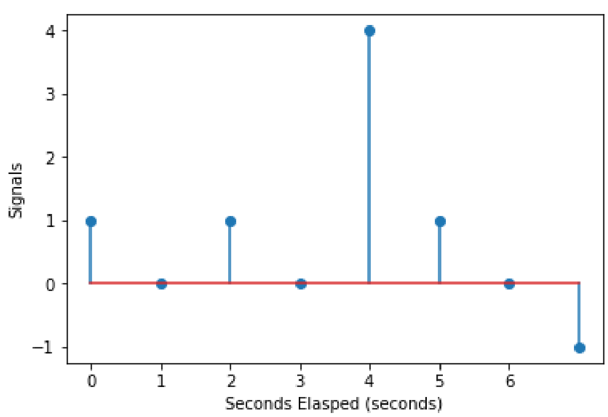


Figure 1: $\text{corr}_{\vec{r}_1}(\vec{s})$

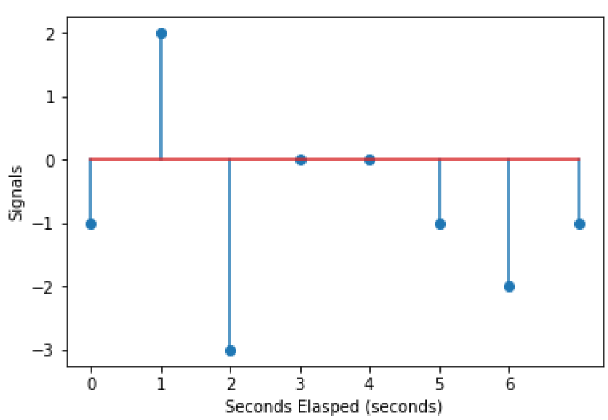


Figure 2: $\text{corr}_{\vec{r}_2}(\vec{s})$

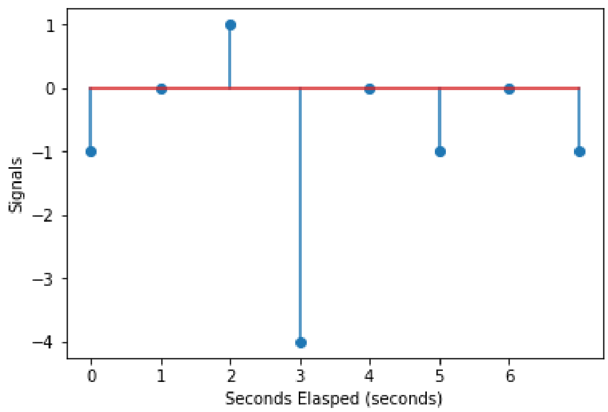


Figure 3: $\text{corr}_{\vec{r}_3}(\vec{s})$

7. Based on the cross-correlated signals, determine the delays (in seconds) for all 3 received signals $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$.

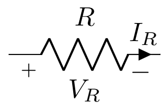
   Solution: Observing the plots for all 3 signals, we can see that the first received signal has a delay of 4 seconds, the second received signal has a delay of 1 second, and the last received signal has a delay of 2 seconds.

8. Given that the radio signal has a transmission speed of v, and assume all delays are relative to the source signal $\vec{s}$ (this means we assume $\vec{s}$ is received at time $t = 0$), find the distance d_1, d_2, and d_3 between the user location and the 3 satellites in orbit.

   Solution: Using the distance formula:
   $$d = v \cdot \tau,$$
   where v is the transmission speed, τ is the delay (in seconds). we can find the distances between the user location and the satellites to be:
   $$d_1 = 4v, \quad d_2 = v, \quad d_3 = 2v.$$

Problem 4: Least Sqaures with Ohm's Law

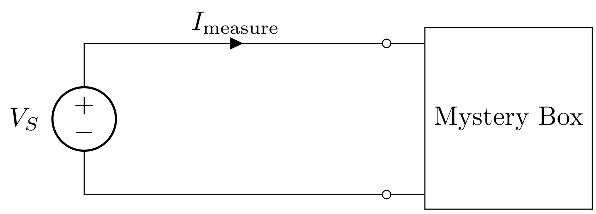
1. Write Ohm's Law for a resistor.



Solution: For a resistor, we have:

$$V_R = I_R R$$

2. You're given the following test setup and told to find R_{Th} between the two terminals of the mystery box. What is R_{Th} of the mystery box between the two terminals in terms of V_S and I_{measure}?



Solution:

$$R_{Th} = \frac{V_S}{I_{\text{measure}}}$$

3. You think you've figured out how to find R_{Th}! You've taken the following measurements:

Measurement #	I_{measure}	V_S
1	$1A$	$1.25kV$
2	$2A$	$1kV$
3	$3A$	$4kV$
4	$4A$	$3.5kV$

Using the information above, formulate a least squares problem whose answer provides an estimate of R_{Th}.

Solution: According to Ohm's Law, $V = IR$. We are estimating the resistance so R_{Th} corresponds to $\vec{x}$ in the $A\vec{x} = \vec{b}$ Least Squares equation. Additionally, I corresponds to A and V corresponds to $\vec{b}$.

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1.25 \\ 1 \\ 4 \\ 3.5 \end{bmatrix}$$

By running Least Squares, $R_{Th} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b} = 0.975\,\mathrm{k\Omega}$