CSM 16A | **Designing Information Systems and Devices I**

Week 1 Worksheet

Term: **Spring 2020** | *Name:*

Problem 1: Vector operations and Matrix-vector multiplication

Consider the following:

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 7 \\ -5 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

1. What is the transpose of $\vec{v}_1$?
2. What is $\vec{v}_1 + \vec{v}_2$?
3. What is $2\vec{v}_1 - 3\vec{v}_2$?
4. What is $\vec{v}_1^T \vec{v}_2$?
5. What is $A\vec{v}_3$?
6. What is AB?

Problem 2: Gaussian Eliminations, Span, Pivots and Free Variables

1. Consider the following set of linear equations:

$$1x - 3y + 1z = 4$$
$$2x - 8y + 8z = -2$$
$$-6x + 3y - 15z = 9$$

   Place these equations into a matrix A, and row reduce A to solve the equations.

2. Consider another set of linear equations:

$$2x + 3y + 5z = 0$$
$$-1x - 4y - 10z = 0$$
$$x - 2y - 8z = 0$$

   Place these equations into a matrix A, and row reduce A.

3. Convert the row reduced matrix back into equation form.

4. Intuitively, what does the last equation from the previous part tell us?

5. How many pivots are there in the row reduced matrix? What are the free variables?

6. What is the dimension of the span of all the column vectors in A?

7. Now that we've established that this system has infinite solutions, does every possible combination of $x, y, z \in \mathbb{R}$ solve these equations

8. What is the general form (in the form of a constant vector multiplied by a variable t) of the infinite solutions to the system?

Problem 3: Proof on Consistency of $A\vec{x} = \vec{b}$

Let A be an $m \times n$ matrix. Show that the following 4 statements about A are all **logically equivalent**. That is, for a particular A, either these statements are all true or they are all false.

1. For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
2. Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$,
4. A has a pivot position in every row.

Hint: Specifically, **show that the following statements are equivalent**:

- Statement 1 is equivalent to statement 2
- Statement 2 is equivalent to statement 3
- Statement 1 is equivalent to statement 3
- Statement 1 is equivalent to statement 4

Another Hint: In general, to show two statements are equivalent, we can take either of the approaches below:

- Show that the statements carry the same meaning by transforming the definitions/properties in one of the statements into those expressed in the other.
- Show that:
  1. If one statement is true, the other one must also be true.
  2. If one statement is false, the other one must also be false.

  It is important that both cases be justified.

1. Show that statement 1 is equivalent to statement 2.

2. Show that statement 2 is equivalent to statement 3.

3. Show that statement 1 is equivalent to statement 3.

4. Finally, show statement 1 is equivalent to statement 4.

Problem 4: Proof on Linear Dependence/Independence

Prove that a subset of a linear independent set of vectors is linearly independent.

Hint 1: A subset intuitively means part of something bigger. If you have a set of vectors $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, then $\{\vec{v}_1, \vec{v}_2\}$ is a subset of S.

Hint 2: Recall the definition of linear independence. If a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent, and there exist a set of constants $c_1, c_2, \ldots, c_n$ such that:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0},$$

then it must be true that $c_1 = c_2 = \ldots = c_n = 0$.