CSM 16A — Designing Information Systems and Devices I

Week 4 Worksheet Solutions

Term: **Spring 2020** *Name:*

Problem 1: Eigenvalues and Eigenvectors

Consider a square matrix $\mathbf{A}$ that is $n \times n$. Recall that we say λ is an eigenvalue of $\mathbf{A}$ if there exists a **non-zero** vector $\vec{v}$ such that:

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

We call $\vec{v}$ the eigenvector associated with λ.

1. What is the one eigenvalue and eigenvector of the matrix that you can see without solving any equations?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Solution: Since this matrix is clearly not-invertible, it must have an eigenvalue 0.

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$
$$\mathbf{A}\vec{x} = 0\vec{x}$$
$$\mathbf{A}\vec{x} = \vec{0}$$

This equation is precisely the equation for computing the nullspace of $\mathbf{A}$. Therefore, any $\vec{x} \in \text{Nullspace}(\mathbf{A})$ works.

For example, the vector

$$\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is a valid answer.

2. What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: This is a scaling matrix. It scales any vector by a factor of 3. What this means is that any vector $\vec{x} \in \mathbb{R}^3$ when post-multiplied by $\mathbf{A}$ will output $3\vec{x}$. This matrix has only one eigenvalue, $\lambda = 3$ and any $\vec{x} \in \mathbb{R}^3$ is an eigenvector.

3. What are the eigenvalues of

$$\mathbf{C} = \begin{bmatrix} 2 & 0 \\ 3 & 4 \\ 1 & 3 \end{bmatrix}?$$

Solution: A non-square matrix (say $m \times n$) maps a vector of dimension n to a vector of dimension m. So, it is impossible for a non-square matrix to have eigenvalues, because the output cannot be a scaled version of the input. In fact, eigenvalues are defined only for square matrices. For similar reasons, the determinant of a matrix is only well-defined if the matrix is square.

4. Consider a matrix that rotates a vector in $\mathbb{R}^2$ by $45°$ counterclockwise. For instance, it rotates any vector along the x-axis to orient towards the $y = x$ line. Find its eigenvalues and corresponding eigenvectors. This matrix is given as

$$\mathbf{D} = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Solution: Remember that the equation $\mathbf{A}\vec{x} = \lambda\vec{x}$ geometrically means that for the matrix $\mathbf{A}$, there exist some special vectors $\vec{x}$ that are merely scaled by λ when post-multiplied by $\mathbf{A}$. For a matrix that takes a vector and rotates it by $45°$, there are no real-valued vectors that it can simply scale. This means that there are no real eigenvalues for this matrix either.

5. What are the eigenvalues of the following matrix?

$$\mathbf{E} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Solution: Remember that for upper triangular matrices, the eigenvalues can be read from the diagonal. $1, \frac{1}{2}, \frac{1}{3}$ are the three eigenvalues. This is because when multiplied by this matrix, any vector of the form $\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$ will be scaled by 1, $\begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix}$ will be scaled by 1/2 and $\begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$ will be scaled by 1/3.

6. Can you find an eigenvalue of the following matrix without solving any equations?

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Solution: This is a matrix whose rows sum to 1, therefore, it has an eigenvalue 1.

This is proven by letting $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be a potential eigenvector of the matrix $\mathbf{F}$. Looking at the column view of matrix-vector multiplication –

$$\mathbf{F} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$
$$\mathbf{F}\vec{x} = 1 \cdot \vec{x}$$

since the rows sum to one.

Therefore, 1 is an eigenvalue with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

7. Show that a matrix and its transpose have the same eigenvalues

Hint: The determinant of a matrix is the same as the determinant of its transpose

Solution: For any matrix $\mathbf{M}$,

$$det(\mathbf{M}) = det(\mathbf{M}^T)$$

Eigenvalues are found by solving the equation $det(\mathbf{M} - \lambda\mathbf{I}) = 0$.

Note that $(\mathbf{M} - \lambda\mathbf{I})^T = \mathbf{M}^T - \lambda\mathbf{I}^T = \mathbf{M}^T - \lambda\mathbf{I}$.

Let $\mathbf{M} - \lambda\mathbf{I} = \mathbf{G}$.

$$det(\mathbf{G}) = det(\mathbf{G}^T)$$
$$det(\mathbf{M} - \lambda\mathbf{I}) = det(\mathbf{M}^T - \lambda\mathbf{I})$$

If we set the left hand side to 0 to solve for the lambdas, we also extract the lambdas corresponding to the right hand side. Therefore, $\mathbf{M}$ and its transpose have the same eigenvalues.

8. Consider a matrix whose columns sum to one. What is one possible eigenvalue of this matrix?

   Solution: We showed that for any matrix like $\mathbf{F}$ whose rows sum to 1, one eigenvalue is 1. We also showed that a matrix and its transpose have the same eigenvalues. Consider $\mathbf{F}^T$. It has columns summing to 1. Therefore, 1 is an eigenvalue of $\mathbf{F}^T$ too, and by extension of all matrices whose columns sum to one.

Problem 2: Eigenvalue Calculations

1. Solve for the eigenvalue-eigenvector pairs for the following 2 by 2 matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Solution: To solve for eigenvalues and eigenvectors, let's go back and review the definition of eigenvectors and eigenvalues:
If $\vec{x}$ and λ are the eigenvector and eigenvalue of $\mathbf{A}$, respectively, then the following equation holds:

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$

Since the (appropriately sized) identity matrix is analogous to multiplying by 1 in arithmetic, we can say:

$$\mathbf{A}\vec{x} = \lambda\mathbf{I}\vec{x}$$

Rearranging, we get:

$$\mathbf{A}\vec{x} - (\lambda\mathbf{I})\vec{x} = \vec{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}$$

What does this look like? It looks similar to solving for the nullspace of $(\mathbf{A} - \lambda\mathbf{I})$!

Assuming that there is a nontrivial nullspace, that also means that $\mathbf{det}(\mathbf{A} - \lambda\mathbf{I}) = 0$!

Let's solve for λ first:

$$\begin{aligned} (\mathbf{A} - \lambda\mathbf{I}) &= \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix} \\ \mathbf{det}(\mathbf{A} - \lambda\mathbf{I}) &= (3-\lambda)(4-\lambda) - 2 \\ &= 10 - 7\lambda + \lambda^2 \\ &= (\lambda - 5)(\lambda - 2) \end{aligned}$$

By factoring:

$$\lambda = 5, 2$$

Let's check: We've just solved for the eigenvalues. But what about the eigenvectors?

To do that, we plug in λ into $(\mathbf{A} - \lambda\mathbf{I})$ and solve for the nullspace!

For $\lambda = 5$:

$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}$$
$$\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$
$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

By row reduction:

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$
$$x_1 = x_2$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$

So the first pair is

$$\lambda = 5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Repeating for $\lambda = 2$,

$$\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$
$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$
$$x_1 = -2x_2$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

So, the second pair is

$$\lambda = 2, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

2. Find the eigenvectors for matrix $\mathbf{A}$ given that we know that $\lambda_1 = 4, \lambda_2 = \lambda_3 = -2$ and that

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Solution: Once the eigenvalues of a matrix have been found, we can find the eigenvectors by Gaussian Elimination.

Step 1: For each eigenvalue λ, we have

$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = 0$$

where $\vec{x}$ is the eigenvector associated with eigenvalue λ.

Step 2: Find $\vec{x}$ in the nullspace of $(\mathbf{A} - \lambda\mathbf{I})$ by plugging in a value of λ and using Gaussian elimination to solve.

Case 1: $\lambda = 4$. First, form the matrix $\mathbf{A} - 4\mathbf{I}$:

$$\mathbf{A} - 4\mathbf{I} = \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$$

To make our numbers nicer, first let's divide our first row by -3

$$R_1 = R_1 \cdot \frac{-1}{3}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$$

$$R_2 = R_2 - 3 \cdot R_1$$

$$R_3 = R_3 - 6 \cdot R_1$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{bmatrix}$$

$$R_3 = R_3 - R_2$$

$$R_2 = R_2 \cdot \frac{1}{6}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we see that we have reached a row of 0s, which means that our last variable x_3 is the free variable in our system. Now, we can expand this matrix by putting it into a system of linear equations and solving for all the variables in terms of our free variable x_3

$$x_1 + x_2 - x_3 = 0$$

$$-2x_2 + x_3 = 0$$

$$x_2 = \frac{x_3}{2}$$

$$x_1 + \frac{x_3}{2} - x_3 = 0$$

$$x_1 = \frac{x_3}{2}$$

$$\vec{x} = \begin{bmatrix} \frac{x_3}{2} \\ \frac{x_3}{2} \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}, \forall x_3 \in \mathbb{R}$$

So the eigenvector for when $\lambda = 4$ is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Now, let's use this same technique to find the eigenvector for $\lambda = -2$

Solution: Case 2: Now let's plug in $\lambda = -2$ into $\mathbf{A} - \lambda\mathbf{I}$ to get

$$\mathbf{A + 2I} = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}$$

And, just like before, let's use Gaussian elimination to reduce the matrix. We can see that this will only take a few steps.

$$R_2 = R_2 - R_1$$

$$R_3 = R_3 - 2 \cdot R_1$$

$$R_1 = R_1 \cdot \frac{1}{3}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As we can see here, we have two rows of 0s, which means that we have two free variables (x_2 and x_3). Now we can take this matrix and write it as a linear system to get

$$x_1 - x_2 + x_3 = 0 \Rightarrow x_1 = x_2 - x_3$$

Thus,

$$\vec{x} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Which are the two eigenvectors associated with $\lambda = -2$

3. Find the eigenvalues for matrix $\mathbf{A}$ given that we know that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ are the eigenvectors of $\mathbf{A}$, and that

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution: There are 2 ways to go about solving this problem. Either you can plug each eigenvector $\vec{v}_i$ into $\mathbf{A}v = \lambda v$ or the nullspace equation to come up with 3 equations and solve. As you have had a lot of practice with the latter, we will use the former to try to answer this question.

Let's plug in the first eigenvector and solve for the first eigenvalue.

$$\mathbf{A}\vec{v}_1 = \lambda_1 \vec{v}_1$$

$$\begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So, we can see that $\lambda_1 = 1$. Similarly, we can do this for the other two eigenvectors.

$$\begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

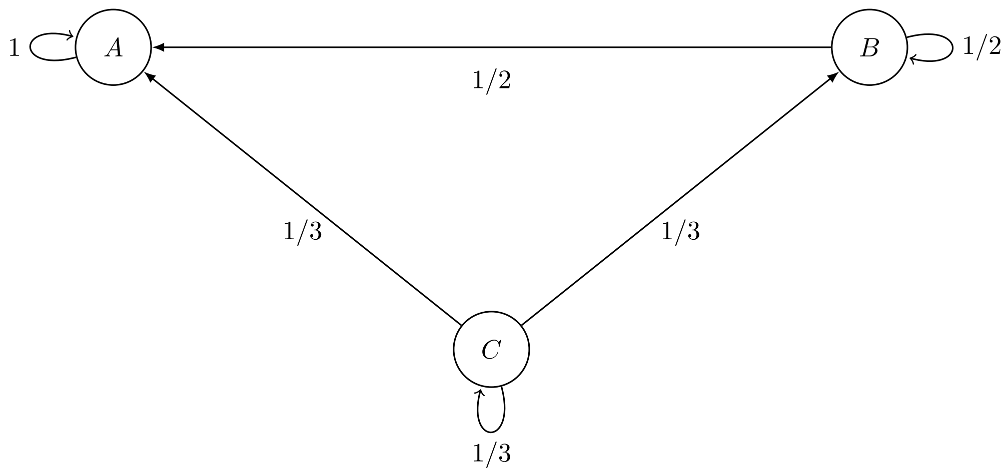
So, we can see that $\lambda_2 = 2$.

$$\begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$$

So, we can see that $\lambda_3 = 3$.

Problem 3: Mechanical PageRank

Now suppose we have a network consisting of 3 websites connected as shown below. Each of the weights on the edges represent the probability of a user taking that edge.



1. Write down the probability transition matrix for this graph, and call it $\mathbf{P}$. Can you say something about the eigenalues/eigenvectors of $\mathbf{P}^T$? (*Hint: Try to recall the properties of transition matrices*).

   Solution: Transition matrix notation: Let $A->B$ represent the fraction of A that goes to B after transition.Then, a general 3x3 transition matrix with states A, B, and C can be written as follows:

   $$\begin{bmatrix} A->A & B->A & C->A \\ A->B & B->B & C->B \\ A->C & B->C & C->C \end{bmatrix}$$

   The transition matrix is:

   $$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

   We know that the columns of a probability transition matrix must sum to 1. This means that the rows of $\mathbf{P}^T$ must sum to 1. So, we have that the matrix-vector product $\mathbf{P}^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. This means that 1 must be an eigenvalue of the matrix $\mathbf{P}^T$, and therefore from part (a), it must also be an eigenvalue of $\mathbf{P}$. This is true for any probability transition matrix.

2. We want to rank these webpages in order of importance. But first, find the eigenvector of $\mathbf{P}$ corresponding to eigenvalue 1.

Solution:

$$P - I = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}$$
$$\xrightarrow[R1 \leftrightarrow R2, R2 \leftrightarrow R3]{R1 \to R1+R2+R3} \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that the pivots lie in the second and third columns. So, we want to solve the equation

$$\begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$
$$-\frac{1}{2}x_2 + \frac{1}{3}x_3 = 0 \text{ and } -\frac{2}{3}x_3 = 0$$
$$\implies x_3 = 0 \text{ and } x_2 = 0$$

This means that the eigenvector is of the form $\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1$. And since x_1 is a free variable, the eigenvectors corresponding to eigenvalue 1 must belong in span$\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$

3. Now looking at the matrix $\mathbf{P}$, can you identify what its other eigenvalues are?

   Solution: $\mathbf{P}$ is an upper-triangular matrix, which means that the diagonal elements are the eigenvalues. So, the eigenvalues are $1, \frac{1}{2}, and \frac{1}{3}$ (we already found the eigenvalue 1 in part (b) through a different method).

4. Suppose that we start with 90 users evenly distributed among the websites. What is the steady-state number of people who will end up at each website?

   Solution: The initial vector of people is $\vec{x} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix}$. We know that since the other eigenvalues are less than 1, those components will die out as we keep applying $\mathbf{P}$ to $\vec{x}$. So we only care about the component of $\vec{x}$ that is in the direction of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This is just the first component of the vector, which is $\begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix}$. However, the total number of people must be conserved, so we multiply by 3 so that the total is 90, the same as before. So, the steady-state distribution is $\begin{bmatrix} 90 \\ 0 \\ 0 \end{bmatrix}$