CSM 16A **Designing Information Systems and Devices**

Week 11 Worksheet

Term: **Spring 2020** *Name:*

Problem 1: Linear Algebra Review

1. Suppose λ is an eigenvalue for the matrix A. Consider the λ-eigenspace of A: the set of all vectors v satisfying the equation $\mathbf{A}\vec{v} = \lambda\vec{v}$. Show that this eigenspace is a subspace by directly checking the three conditions needed to be a subspace.

2. Solve for the eigenvalue-eigenvector pairs for the following 2 by 2 matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

3. Projection of a vector $\vec{u}$ onto $\vec{v}$ is given by:

$$\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\vec{v}$$

   Prove that projection onto a vector $\vec{v}$ is a linear transformation.

Problem 2: Introduction to Inner Products

1. What is an inner product?

2. What is the dot product between two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$?

   In the next four parts, we prove that the dot product is an inner product. Do note that the dot product is simply a type of inner product, and other inner products are also possible.

3. Prove that the dot product satisfies symmetry, i.e. that $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

4. Prove that the dot product satisfies homogeneity, i.e. that $\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$: $c \in \mathbb{R}$

5. Prove that the dot product satisfies additivity, i.e. that $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

6. Prove that the dot product satisfies positive-definiteness, i.e., that $\langle \vec{x}, \vec{x} \rangle \geq 0$, and is equal to 0 iff $\vec{x} = \vec{0}$

   We will now consider ways to use dot products to do neat things. For each of the following, assume that you're given a $\vec{x}$, and that you get a pick $\vec{y}$ of your choosing. Describe a $\vec{y}$, such that when you compute $\langle \vec{x}, \vec{y} \rangle$, you get:

7. The sum of every element in $\vec{x}$

8. The sum of certain elements in $\vec{x}$

9. The mean of all the items in $\vec{x}$ (for $\vec{x}$ in $\mathbb{R}^n$)

10. The sum of the elements of $\vec{x}$ squared

    We will conclude by making some observations based on that last case.

11. Consider that last case, where we summed the squares of the elements of a vector. Try doing that for a few 2-dimensional vectors (vectors of length 2). What do you notice about the resulting answer? What about for vectors of length 3, or for vectors of any length n?

Problem 3: Eigenspace, Orthogonality, and Symmetric Matrices

Suppose we have a matrix $A \in \mathbb{R}^{n \times n}$.

1. Show that if $\vec{v}$ is an eigenvector of A, then it must also be an eigenvector of A^2.
2. Show that if $\vec{u}$ is an eigenvector of A with associated eigenvalue α, and $\vec{v}$ is an eigenvector of A^T with associated eigenvalue β, if $\alpha \neq \beta$, then $\vec{u}$ and $\vec{v}$ must be orthogonal to each other.

For the following parts, assume A is also symmetric.

3. Show that A has all real eigenvalues.
4. Using the result from part 2, explain why the eigenvectors of A are orthogonal to each other. (If the set of all eigenvectors are orthogonal to each other, we call the set an *orthogonal eigenbasis*)

Problem 4: Robust Linear Systems

Up and till now, we have been extensively studying different examples of linear systems represented by the iconic matrix vector equation $A\vec{x} = \vec{v}$ and how to solve them.

However, we haven't looked much into the sensitivity of a linear system to external changes. In particular, how the solutions to such linear systems react to small changes (we call these changes *perturbations*) in A or b can be of great importance to designing a system *robust* to changes.

In this question, we will work toward deriving a well-know metric used to measure such sensitivity to *perturbations* within the system.

1. To get started, consider the following linear system:

$$\begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 23 \\ 33 \\ 31 \end{bmatrix}$$

   First, find the solution to this system. Then, consider the following linear system with some slight *perturbation* to the right-hand side (i.e. $\vec{b}$).

$$\begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32.1 \\ 22.9 \\ 33.1 \\ 30.9 \end{bmatrix}$$

   Find its solution, and compare how much it has changed from the previous system to how much $\vec{b}$ has changed from the previous system. What did you notice? Is this linear system sensitive to *perturbations*?

2. Before moving forward, let us provide the following definition of **a norm that applies to matrices**.

   We define the spectral norm on a matrix A as the greatest possible value of the vector norm $\|A\vec{v}\|$ for all unit-length vectors $\vec{v}$.

   In other words,

$$\|A\| = \max_{\|\vec{v}\|=1} \|A\vec{v}\|$$

   In addition, assume the following property holds:

$$\|A\vec{v}\| \leq \|A\| \, \|\vec{v}\|$$

   Let's first study the case where we *perturb* $\vec{b}$ slightly. Specifically, given an **invertible** matrix A, we have the following pair of solutions to a linear system and a lightly perturbed one:

$$A\vec{v} = \vec{b}$$

$$A(\vec{v} + \delta\vec{v}) = \vec{b} + \delta\vec{b}$$

   Here, $\delta\vec{v}$ and $\delta\vec{b}$ represents the slight perturbation in the system.

   Show that we can find some constant c such that:

$$\frac{\|\delta\vec{v}\|}{\|\vec{v}\|} \leq c \cdot \frac{\left\|\delta\vec{b}\right\|}{\left\|\vec{b}\right\|}$$

   For those interested, we call this constant the *condition number*.

3. Now, instead of perturbing our measurement of the vector $\vec{b}$, we perturb the matrix A by some amount δA. In particular, we have the following pair of solutions to a linear system and a lightly perturbed one:

$$A\vec{v} = \vec{b}$$
$$(A + \delta A)(\vec{v} + \delta\vec{v}) = \vec{b}$$

Show that we can achieve a similar bound on the relative error of the solution to the perturbed linear system using the **same** condition number from the previous part:

$$\frac{\|\delta\vec{v}\|}{\|\vec{v} + \delta\vec{v}\|} \leq c \cdot \frac{\|\delta A\|}{\|A\|}$$