CSM 16A **Designing Information Systems and Devices**

Week 1 Worksheet Solution

Term: **Spring 2020** *Name:*

Problem 1: Vector operations and Matrix-vector multiplication

Consider the following:

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 7 \\ -5 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

1. What is the transpose of $\vec{v}_1$?

   Solution:

$$\vec{v}_1' = \begin{bmatrix} 4 & 7 & -5 \end{bmatrix}$$

2. What is $\vec{v}_1 + \vec{v}_2$?

   Solution:

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 4+1 \\ 7+3 \\ -5-1 \end{bmatrix}$$

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 5 \\ 10 \\ -6 \end{bmatrix}$$

3. What is $2\vec{v}_1 - 3\vec{v}_2$?

   Solution:

$$2\vec{v}_1 - 3\vec{v}_2 = \begin{bmatrix} 2*4-3*1 \\ 2*7-3*3 \\ 2*(-5)-3*(-1) \end{bmatrix}$$

$$2\vec{v}_1 - 3\vec{v}_2 = \begin{bmatrix} 5 \\ 5 \\ -7 \end{bmatrix}$$

4. What is $\vec{v}_1^T \vec{v}_2$?

   Solution:

$$\vec{v}_1^T \vec{v}_2 = \begin{bmatrix} 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ -5 \end{bmatrix}$$

$$\vec{v}_1 \cdot \vec{v}_2 = 4 * 1 + 7 * 3 + (-5) * (-1)$$
$$\vec{v}_1 \cdot \vec{v}_2 = 30$$

   As some of you might have recognized, the expression you just evaluate is in fact the same as the dot product between two vectors. For two vectors with the **same dimensions**, we can calculate the sum of products of corresponding terms in the vectors.

5. What is $A\vec{v}_3$?

   Solution:

$$A\vec{v}_3 = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$A\vec{v}_1 = \begin{bmatrix} 1 * 2 + 3 * 4 \\ 7 * 2 + 9 * 4 \end{bmatrix}$$

   Note: Matrix vector multiplication is just stacked vector vector dot products. The first row of the product is the same as the answer to the last problem.

$$A\vec{v}_1 = \begin{bmatrix} 14 \\ 50 \end{bmatrix}$$

6. What is AB?

   Solution:

$$AB = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1 * 5 + 3 * 3) & (1 * 1 + 3 * 6) \\ (7 * 5 + 9 * 3) & (7 * 1 + 9 * 6) \end{bmatrix}$$
$$AB = \begin{bmatrix} 14 & 19 \\ 62 & 61 \end{bmatrix}$$

Problem 2: Gaussian Eliminations, Span, Pivots and Free Variables

1. Consider the following set of linear equations:

$$1x - 3y + 1z = 4$$
$$2x - 8y + 8z = -2$$
$$-6x + 3y - 15z = 9$$

Place these equations into a matrix A, and row reduce A to solve the equations.

Solution:

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix}$$

$$R_2 = R_2 - 2 * R_1$$
$$R_3 = R_3 + 6 * R_1$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ 0 & -15 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 33 \end{bmatrix}$$

$$R_2 = R_2/2$$
$$R_3 = R_3/3$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 11 \end{bmatrix}$$

$$R_3 = R_3 - 5 * R_2$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 36 \end{bmatrix}$$

$$R_2 = R_2 * -1$$
$$R_3 = R_3 / -18$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix}$$

This form of the matrix is called the row echelon form or the REF.

$$R_2 = R_2 + 3 * R_3$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$$

$$R_1 = R_1 + 3 * R_2 - R_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Now, we have reduced the matrix to the reduced row echelon form or the RREF.

$$z = -2$$
$$y = -1$$
$$x = 3$$

2. Consider another set of linear equations:

$$2x + 3y + 5z = 0$$
$$-1x - 4y - 10z = 0$$
$$x - 2y - 8z = 0$$

Place these equations into a matrix A, and row reduce A.

Solution:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

$$R_2 = R_2 + \frac{1}{2} R_1$$
$$R_3 = R_3 - \frac{1}{2} R_1$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & -2.5 & -7.5 \\ 0 & -3.5 & -10.5 \end{bmatrix}$$

Remember that we can only do row operations without caring about the RHS because the RHS is all zeroes. Hence, any linear row operations won't affect the RHS i.e. it will remain the zero vector.

Make the numbers nicer by dividing row 2 by -2.5, and multiplying row 3 by -2. This is always a good thing to do if you realize your numbers are getting messy! (Also, feel free to keep all the numbers as non-fractional values by finding the least common multiple of the two numbers you are trying to cancel out.)

$$R_2 = \frac{1}{-2.5} R_2$$
$$R_3 = -2R_3$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 7 & 21 \end{bmatrix}$$

$$R_3 = R_3 - 7R_2$$

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 = R_1/2$$

$$A = \begin{bmatrix} 1 & 1.5 & 2.5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the REF of the equation matrix.

$$R_1 = R_1 - 1.5 * R_2$$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the RREF of the equation matrix.

3. Convert the row reduced matrix back into equation form.

Solution:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1x + 0y - 2z = 0$$

$$0x + 1y + 3z = 0$$

$$0x + 0y + 0z = 0$$

4. Intuitively, what does the last equation from the previous part tell us?

Solution: If students are confused at this point about why we can infer this, their confusion is well justified. Suppose that there were **4** equations in 3 variables – 3 of them were linearly independent, and the fourth one was $0x + 0y + 0z = 0$, then the system still has just 1 solution. The last equation is never *used* in some sense. Feel free to talk about this with students. Present it as: what if you had 4 equations, you wrote them in matrix form, got pivots in all rows except for one where you got a row of all 0s – are there still infinite solutions? The answer is no.

5. How many pivots are there in the row reduced matrix? What are the free variables?

   Solution: There are 2 pivots in this row reduced matrix, and the corresponding pivot columns (following Gaussian Elimination's convention) are column 1 and 2. There are no more pivots since the third row are all zeros, and we require a non-zero element at the position of the third column (following column 2) and the third row for there to be one more pivot.

   The free variables can be y or z in this case, but we choose z as our free variable by Gaussian Elimination's convention.

6. What is the dimension of the span of all the column vectors in A?

   Solution: As we can see, in the row reduced form of A, since the third row are all zeros, and there are only 2 pivots with z as the free variable, the dimension of the span of all the column vectors in A is equal to 2 (number of pivots).

   Alternatively, we can follow the definition of a span and algebraically write out the linear combinations of all the column vectors in the row reduced form of A, and we can see that the third entry in the resulting linear combination will always be 0 (since all 3 column vectors have 0's in their third entries), hence there are only 2 dimensions (entries) in the resulting vectors whose values we have control over.

7. Now that we've established that this system has infinite solutions, does every possible combination of $x, y, z \in \mathbb{R}$ solve these equations

   Solution: No. $x = 1, y = 1, z = 1$ doesn't work, for instance.

8. What is the general form (in the form of a constant vector multiplied by a variable t) of the infinite solutions to the system?

   Solution: z is a free variable. If $z = t$, then

   $$y = -3z = -3t$$

   $$x - 2z = 0 \implies x = 2z = 2t$$

   The general solution is then $t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. What this means is that any multiple of the vector $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ will satisfy the equations. Try it!

Problem 3: Proof on Consistency of $A\vec{x} = \vec{b}$

Let A be an $m \times n$ matrix. Show that the following 4 statements about A are all **logically equivalent**. That is, for a particular A, either these statements are all true or they are all false.

1. For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
2. Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$,
4. A has a pivot position in every row.

Hint: Specifically, **show that the following statements are equivalent**:

- Statement 1 is equivalent to statement 2
- Statement 2 is equivalent to statement 3
- Statement 1 is equivalent to statement 3
- Statement 1 is equivalent to statement 4

Another Hint: In general, to show two statements are equivalent, we can take either of the approaches below:

- Show that the statements carry the same meaning by transforming the definitions/properties in one of the statements into those expressed in the other.
- Show that:
  1. If one statement is true, the other one must also be true.
  2. If one statement is false, the other one must also be false.

  It is important that both cases be justified.

1. Show that statement 1 is equivalent to statement 2.

   Solution:
   Let's first show **statement 1 is equivalent to statement 2**.

   By definitions of the matrix vector product $A\vec{x}$ and span of a set of vectors in $\mathbb{R}^m$, define:

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

   If the equation $A\vec{x} = \vec{b}$ has a solution, that means **the equality holds**. Substituting the more specific expressions we defined above (replace A with its column vectors $\vec{a}_1$ through $\vec{a}_n$, and replace $\vec{x}$ with its entries x_1 through x_n), we have:

$$\begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \vec{b}$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = \vec{b}$$

Since $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n$ is a linear combination of the column vectors of A by definition, from this equation, we have shown that $\vec{b}$ is expressed as a linear combination of the columns of A if $A\vec{x} = \vec{b}$ has a solution. □

2. Show that statement 2 is equivalent to statement 3.

   Solution:
   Next, let's show **statement 2 is equivalent to statement 3**.

   Since we are looking at all $\vec{b}$ in $\mathbb{R}^m$, and a linear combination of the columns of A has the form $c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n$, if every such vector $\vec{b}$ can be expressed as $c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n$, that means the linear combination of the columns vectors of A can reach any vector in $\mathbb{R}^m$.

   Equivalently, this implies the columns of A span $\mathbb{R}^m$. □

3. Show that statement 1 is equivalent to statement 3.

   Solution:
   Now, to show **statement 1 is equivalent to statement 3**, notice we have shown that **statement 1 is equivalent to statement 2** and **statement 2 is equivalent to statement 3**, and we know that logical equivalences (just like equalities) are transitive (i.e. If $a = b$, $b = c$, then $a = c$), we conclude that **statement 1 must also be equivalent to statement 3** as well. □

4. Finally, show statement 1 is equivalent to statement 4.

   Solution:
   Finally, let's show **statement 1 is equivalent to statement 4**.

   Let U be an echelon form of A. Given the vector $\vec{b}$ in $\mathbb{R}^m$, we can row reduce the augmented matrix $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ to an augmented matrix $\begin{bmatrix} U & \vec{d} \end{bmatrix}$ for some different vector $\vec{d}$ in $\mathbb{R}^m$:

   $$\begin{bmatrix} A & \vec{b} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} U & \vec{d} \end{bmatrix}.$$

   If statement 4 is indeed true, then each row of U must contain a pivot position and **there can be no pivot in the augmented column**. So $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$, and statement 1 must also be true.

   On the other hand, if statement 4 is false, the last row of U will be all zeros. Let $\vec{d}$ be any vector with a 1 in its last entry, then $\begin{bmatrix} U & \vec{d} \end{bmatrix}$ represents an inconsistent system. Since **the row operations we use when reducing a matrix are reversible**, this means $\begin{bmatrix} U & \vec{d} \end{bmatrix}$ can be transformed back into the form $\begin{bmatrix} A & \vec{b} \end{bmatrix}$. Hence, the new system $A\vec{x} = \vec{b}$ is also inconsistent, and statement 1 will be false as well. □

   Hence, we have shown that all of the 4 statements are logically equivalent. □

Problem 4: Proof on Linear Dependence/Independence

Prove that a subset of a linear independent set of vectors is linearly independent.

Hint 1: A subset intuitively means part of something bigger. If you have a set of vectors $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, then $\{\vec{v}_1, \vec{v}_2\}$ is a subset of S.

Hint 2: Recall the definition of linear independence. If a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent, and there exist a set of constants $c_1, c_2, \ldots, c_n$ such that:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0},$$

then it must be true that $c_1 = c_2 = \ldots = c_n = 0$.

Solution:
This problem can be tackled from two different approaches: **Direct Proof** and **Proof By Contradiction**. If you are not sure what the second approach means, we have also provided an explanation at the beginning of the second approach.

First Approach:
The problem can be translated mathematically to be the following:
Given a set of n vectors $N = \{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$ that are linearly independent, we want to show that any subset of vectors in N is also linearly independent.
Since N is a linearly independent set, we know that if there exists a set of constants $c_1, c_2, \ldots, c_n$ such that:

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = 0$$

then it must be true that $c_1 = c_2 = \cdots = c_n = 0$.
To also symbolically represent a subset of N, since we can shuffle the array or arrange the vectors as much as we want, the following form generalizes to any susbet of vectors in N:
We want to show that for some $k \leq n$, $\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_k}\}$ is also a set of linearly independent vectors.
Following the same setup based on the definition of linear independence, suppose there exists a set of constants $b_1, b_2, \ldots, b_k$ such that:

$$b_1\vec{v_1} + b_2\vec{v_2} + \cdots + b_k\vec{v_k} = 0$$

Now, we can add more "zeros" to both sides of the equations such that we are extending the linear combinations all the way from v_k to v_n:

$$b_1\vec{v_1} + b_2\vec{v_2} + \cdots + b_k\vec{v_k} + (0)\vec{v_{k+1}} + \cdots + (0)\vec{v_n} = 0$$

Now, since we know that the set $N = \{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$ is linearly independent, we know that all coefficients of the given vectors must be equal to 0 (per definition of linear independence). This means that $b_1 = b_2 = \cdots = b_k = 0$ as well.
Hence, it follows that $\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_k}\}$ is also a set of linearly independent vectors.

Second Approach:
This problem can also be approached from another direction using a technique called **Proof by Contradiction**.

More Explanation on Proof by Contradiction: In essence, the technique assumes the opposite of what we are trying to prove (so if the property we are proving is called P, we want to prove *not* P is true), and then reaches two **mutually contradictory** assertions (statements) (i.e., Property A is true and also false at the same time). Since both statements can't be simultaneously true, this leads us to conclude the property not P is in fact wrong, so P must be true.

Given: $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent. This, by definition of linear independence, means that if there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that:

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots + \alpha_n\vec{v}_n = 0$$

then

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = \vec{0}$$

In other words, the only solution to the above αs is that the αs are all 0.
To Prove: $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k = 0 \implies \beta_1 = \beta_2 = \ldots = \beta_k = \vec{0}$.
Note that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are a subset of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

Assume that $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k = 0$ is true but not $\beta_1 = \beta_2 = \ldots = \beta_k = 0$.

Consider $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k + 0\vec{v}_{k+1} + 0\vec{v}_{k+2} + \ldots + 0\vec{v}_n$. If $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k = 0$ then

$$\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k + 0\vec{v}_{k+1} + 0\vec{v}_{k+2} + \ldots + 0\vec{v}_n = 0$$

However, since we assumed that not all $\beta_1, \beta_2, \ldots, \beta_k$ are 0, this means that the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is not linearly independent, which is a contradiction because it is given that the set is linearly independent. Therefore, $\beta_1 = \beta_2 = \ldots = \beta_k = 0$ must have been true.