CSM 16A **Designing Information Systems and Devices I**

Week 2 Worksheet

Term: **Spring 2020** *Name:*

Problem 1: Matrix * Matrix

Learning Goal: Students should be comfortable working with basic vector operations (such as addition) matrix vector multiplications.

$$\vec{a}_1 = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$\vec{a}_2 = \begin{bmatrix} 7 & 9 \end{bmatrix}$$

$$\vec{b}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\vec{b}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

1. What is $A\vec{b}_1$?

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

$$\vec{b}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

2. What is $A\vec{b}_2$?

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

$$\vec{b}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

3. What is $\vec{a}_1^T$ B?

$$\vec{a}_1^T = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

4. What is $\vec{a}_2^T$ B?

$$\vec{a}_2^T = \begin{bmatrix} 7 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$$

5. What is AB? Do you notice something?

Problem 2: More Proof on Spans

Learning Goal: Proof on spans and linear dependence/independence

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a set of vectors V. Prove that if the set of vectors is linearly dependent, then at least one vector can be deleted from the set without diminishing its span.

Problem 3: Step-by-step Inverse

In this question, we will learn about the underlying transformations that allow us to find the inverse of a given matrix by exploring how matrices can be used to represent different types of row operations.

Learning Goal: Students should understand matrix multiplication, linear transformations, Gaussian elimination

1. What matrix B can we left multiply by a 3×3 matrix M to get a new matrix M' that is the same as M but with row 2 scaled by 1/5?
2. What matrix A can we left multiply by a 3×3 matrix M to get a new matrix M' that is the same as M but with the row 1 and row 3 swapped?
3. What matrix A can we left multiply by a 3×3 matrix M to get a new matrix M' that is the same as M but with the 3 times row 1 added to the row 2?
4. What is the the multiplicative inverse of 2? What is the multiplicative identity? What is the additive inverse of 1? What is the additive identity? What is the identity in matrix/vector multiplication?
5. In what order should we apply the transformations described in parts (a), (b), and (c) to the matrix $M = \begin{bmatrix} 0 & 0 & 1 \\ -15 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ to get the identity matrix?
6. Multiply the matrices for each transformation in the order determined in part (d). What happens when you multiply M by this matrix? What is this matrix called?
7. Are there a set of transformations we can apply to $M = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ to make it the identity? If so, what are they? If not, why is is not possible?
8. Can you find the inverse of a non-square matrix (e.g. a 2×3 matrix)?

Problem 4: Round and Round

In discussion, we talked about rotation matrix as an example of transformation on a given vector.

A rotation matrix is a matrix that takes a vector and rotates it by some number of degrees (counter-clockwise). That matrix looks like:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some angle θ.

In this question, we will explore some of the properties a rotation matrix has in more depth and see how the algebra behind is deeply connected to the geometric transformation we see.

1. Given the following vector:

   $$\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   If we want to rotate $\vec{v}$ counter-clockwise by $\theta = 45^o$, what would the rotation matrix corresponding to this transformation be? What would the resulting vector be?

2. In this part, we will explore the relationship between a series of counter-clockwise rotations applied on a given vector and how the rotation matrix is represented correspondingly.

   Given that we have found the rotation matrix R for $\theta = 45^o$ in the previous part, now find the rotation matrix for $\theta = 90^o$, $\theta = 135^o$. At the same time, evaluate the matrix product R^2, R^3. What pattern did you see?

3. Generalizing from the previous part, if we are given a rotation matrix M that rotates a given vector $\vec{v}$ by k degrees. What would the resulting vector $\vec{v}'$ be if we rotate $\vec{v}$ by a total of $N \times k$ degrees? (N is a positive integer).

4. Backing off from counter-clockwise rotation for a bit, let's now explore **clockwise rotation** instead on a given vector. Given the **counter-clockwise rotation** matrix R_c we provided at the beginning of the question, consider the rotation matrix $R_{c'}$ for a **clockwise rotation** of θ degrees. What is the relationship between $R_{c'}$ and R_c?

5. Now that we have learned about the intrinsic connections between rotation matrix multiplication/inverse and the geometric transformation, in a few sentences, explain why the multiplication of rotation matrices is commutative. i.e.: Explain why given two rotation matrices A and B (A and B are both $N \times N$) , $AB = BA$.

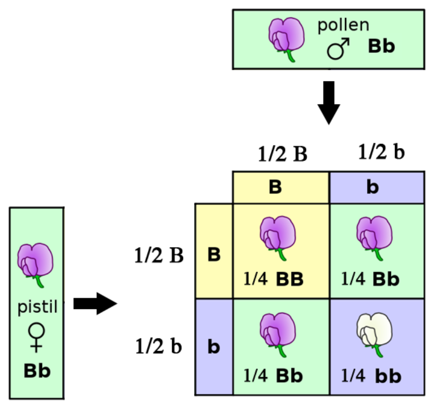
6. Finally, for one additional nice property that results from rotation matrix multiplication, we know that if we rotate a vector each time by 30 degrees for a total of 12 times, eventually it will be a total of 360 degrees rotation, which puts the vector right back to where it was originally! Utilizing this fact, find a 2×2 matrix M such that $M^7 = I$, where I is the identity matrix.

Problem 5: (Challenging Exam-level Question) Gen(e) Z

Living things like you and me inherit from our parents many of their physical characteristics. In the study of population genetics, there are several types of inheritance; one of them is the **autosomal type**, where each heritable trait is assumed to be governed by a single gene.

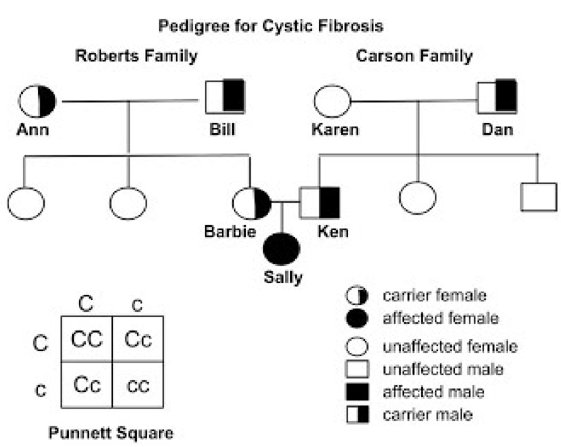
Typically, there are two different forms of genes denoted by A and a. Each individual in a population carries a pair of genes; the pairs are called the individual's genotype. This gives three possible genotypes for each inheritable trait: AA, Aa, and aa (aA is genetically the same as Aa, or in other words, the order of the genes in the genotypes doesn't matter).

As some of you might have recalled from a high school biology class, the Mendel's experiment is one of the earliest genetics studies that explores the possible genotypes and variation in the inheritable traits from crossing different individuals in the population:



Mendel's Experiment using Punnet Square

As you can see in the figure above, each cell in the square represents the chance that you will get a specific genotype for the flower after crossing. **The reason we care to calculate such chances** is because among the majority purple flowers, you can find a white flower (which fully manifests a recessive trait). Unfortunately, recessive traits will sometimes show in the form of disorders or diseases. Here's an example of how studying the likelihood of genotypes on the genotype that can cause *Cystic Fibrosis* (a very serious neural degenerative disease).



Now that you have some background in how popular genetics works, let's dive back to this problem! Suppose we have just discovered a new population of animals on a hypothetical Planet 16A, and our biologist-in-residence Kevin has found that an autosomal model of inheritance controls eye coloration (what colors the eyes have). Here is what Kevin has found:

- Genotypes AA and Aa have brown eyes.
- Genotype aa has blue eyes.

Kevin believes that the A gene dominates the a gene, and he further classifies an animal as **dominant** if it carries AA genes, **hybrid** if it carries Aa genes, and **recessive** if it carries aa genes. We can see that in this case, the dominant and hybrid genes are indistinguishable in appearance.

To further investigate how the distribution of the eye-color genes of this animal change over time, as a leading engineer on the research team, you are tasked with simulating the distribution of genotypes over multiple generations for this animal.

Note: for all of the following parts, we assume that each offspring inherits one gene from each parent in a completely random manner.

1. Given the genotypes of the parents, we can determine the distribution of the genotypes for the offspring. Suppose that in the original sample of 200 animals, 50 of them carry the **dominant** genes, 120 of them carry the **hybrid** genes, and the rest carries the **recessive** genes. We want to represent this distribution as a vector $\vec{v_p}^{(0)}$, where each entry $v_{p,i}$ in $\vec{v_p}^{(0)}$ represents the chance that a randomly selected animal from our sample population carries the genotype i. Find $\vec{v_p}^{(0)}$ (Entries should be in order of **dominant**, **hybrid**, and **recessive**)

2. Now, you woud like to consider a series of simulated experiments where we continuously breed all animals in our sample population **only** with animals that carry a **dominant** genotype. Suppose after 1 round of breeding, the distribution of the genotypes in our population becomes $\vec{v_p}^{(1)}$. Find $\vec{v_p}^{(1)}$.

   Note: For ease of computation, for all later parts of this question, we will assume that the original distribution vector (for the genome)
$$\vec{v_p}^{(0)} = \begin{bmatrix} Pr(AA \text{ at } t=0) \\ Pr(Aa \text{ at } t=0) \\ Pr(aa \text{ at } t=0) \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

3. Now that you have completed one round of breeding with the **dominant** genotype, you are eager to continue more rounds of simulation. However, before going about doing this, you would like to know if you can re-represent this one round of breeding in a more concise and matrix-oriented way. In other words, you would like to see if there exists a matrix A such that it can predict what the $(T+1)$st (next) round's distribution of genotypes will be given the Tth (current) round. Mathematically, we can represent this as the equation below:
$$\vec{v_p}^{(T+1)} = A\vec{v_p}^{(T)}$$
   Does A exist? If so, find A; if not, explain why.

4. **Without explicitly solving for the inverse or row reducing the matrix**, determine if the transformation matrix A is invertible or not. Provide your explanations in a few sentences.