ROSC: Robust Spectral Clustering on Multi-scale Data

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This technical report proves the grouping effect of $|Z^*|$,$|(Z^*)^T|$ and $\tilde{Z}$. In the following discussion, we use $\boldsymbol{z}_q^*$ to denote the q-th column vector of Z^*.

Lemma 1. *Given a set of objects $\mathcal{X}$, the matrix $X \in \mathcal{R}^{p\times n}$ that is composed of the pseudo-eigenvectors as rows, the reachability matrix $\mathcal{W}$, and the optimal soution Z^*,*

$$Z_{ip}^* = \frac{\boldsymbol{x}_i^T(\boldsymbol{x}_p - X\boldsymbol{z}_p^*) + \alpha_2 \mathcal{W}_{ip}}{\alpha_1 + \alpha_2}, \quad \forall 1 \le i, p \le n. \tag{1}$$

Proof. For $1 \le p \le n$, let $J(z_p) = ||\boldsymbol{x}_p - Xz_p||_2^2 + \alpha_1||z_p||_2^2 + \alpha_2||z_p - \boldsymbol{w}_p||_2^2$. Since Z^* is the optimal solution, we have $\frac{\partial J}{\partial Z_{ip}}|_{z_p = z_p^*} = 0 \ \forall 1 \le i \le n$. Hence, $-2\boldsymbol{x}_i^T(\boldsymbol{x}_p - X\boldsymbol{z}_p^*) + 2\alpha_1 Z_{ip}^* + 2\alpha_2(Z_{ip}^* - \mathcal{W}_{ip}) = 0$, which induces Equation 1. □

Lemma 2. *$\forall 1 \le i, j, p \le n$,*

$$|Z_{ip}^* - Z_{jp}^*| \le \frac{c\sqrt{2(1-r)} + \alpha_2|\mathcal{W}_{ip} - \mathcal{W}_{jp}|}{\alpha_1 + \alpha_2}, \tag{2}$$

where $c = \sqrt{1 + \alpha_2||\boldsymbol{w}_p||_2^2}$ and $r = \boldsymbol{x}_i^T\boldsymbol{x}_j$.

Proof. From Equation 1, we have

$$Z_{ip}^* - Z_{jp}^* = \frac{(\boldsymbol{x}_i^T - \boldsymbol{x}_j^T)(\boldsymbol{x}_p - X\boldsymbol{z}_p^*) + \alpha_2(\mathcal{W}_{ip} - \mathcal{W}_{jp})}{\alpha_1 + \alpha_2}.$$

That implies

$$\begin{aligned} |Z_{ip}^* - Z_{jp}^*| &\le \frac{|(\boldsymbol{x}_i^T - \boldsymbol{x}_j^T)(\boldsymbol{x}_p - X\boldsymbol{z}_p^*)| + \alpha_2|(\mathcal{W}_{ip} - \mathcal{W}_{jp})|}{\alpha_1 + \alpha_2} \\ &\le \frac{||(\boldsymbol{x}_i^T - \boldsymbol{x}_j^T)||_2||(\boldsymbol{x}_p - X\boldsymbol{z}_p^*)||_2 + \alpha_2|(\mathcal{W}_{ip} - \mathcal{W}_{jp})|}{\alpha_1 + \alpha_2} \end{aligned} \tag{3}$$

Since the column vectors of X are normalized (i.e., $\boldsymbol{x}_q^T\boldsymbol{x}_q = 1 \ \forall 1 \le q \le n$), we have $||(\boldsymbol{x}_i^T - \boldsymbol{x}_j^T)||_2 = \sqrt{2(1-r)}$, where $r = \boldsymbol{x}_i^T\boldsymbol{x}_j$. As Z^* is the optimal solution, we have

$$\begin{aligned} J(\boldsymbol{z}_p^*) = ||\boldsymbol{x}_p - X\boldsymbol{z}_p^*||_2^2 + \alpha_1||\boldsymbol{z}_p^*||_2^2 + \alpha_2||\boldsymbol{z}_p^* - \boldsymbol{w}_p||_2^2 \le \\ J(\boldsymbol{0}) = ||\boldsymbol{x}_p||_2^2 + \alpha_2||\boldsymbol{w}_p||_2^2 = 1 + \alpha_2||\boldsymbol{w}_p||_2^2. \end{aligned} \tag{4}$$

Hence, $||\boldsymbol{x}_p - X\boldsymbol{z}_p^*||_2 \le \sqrt{1 + \alpha_2||\boldsymbol{w}_p||_2^2} = c$. Equation 3 can be further simplified as

$$|Z_{ip}^* - Z_{jp}^*| \le \frac{c\sqrt{2(1-r)} + \alpha_2|(\mathcal{W}_{ip} - \mathcal{W}_{jp})|}{\alpha_1 + \alpha_2}.$$

□

Lemma 3. *Z^* and $|Z^*|$ have grouping effect.*

Proof. Given two objects x_i and x_j such that $x_i \to x_j$, we have, (1) $\boldsymbol{x}_i^T\boldsymbol{x}_j \to 1$ and (2) $||\boldsymbol{w}_i - \boldsymbol{w}_j||_2 \to 0$. These imply $r = \boldsymbol{x}_i^T\boldsymbol{x}_j \to 1$ and $|\mathcal{W}_{ip} - \mathcal{W}_{jp}| \to 0$. Hence, the two terms of the numerator of the R.H.S of Equation 2 are close to 0. Therefore, $|Z_{ip}^* - Z_{jp}^*| \to 0$ and thus Z^* has grouping effect. Further, we have $\left||Z_{ip}^*| - |Z_{jp}^*|\right| \le |Z_{ip}^* - Z_{jp}^*|$, so $\left||Z_{ip}^*| - |Z_{jp}^*|\right| \to 0$ and $|Z^*|$ has grouping effect. □

Lemma 4. *$(Z^*)^T$ and $|(Z^*)^T|$ have grouping effect.*

Proof. The problem is equivalent to if $x_i \to x_j$, $Z_{pi}^* \to Z_{pj}^*$. From Lemma 1,

$$z_{pi}^* - z_{pj}^* = \frac{\boldsymbol{x}_p^T(\boldsymbol{x}_i - \boldsymbol{x}_j - X(\boldsymbol{z}_i^* - \boldsymbol{z}_j^*)) + \alpha_2(W_{pi} - W_{pj})}{\alpha_1 + \alpha_2}.$$

Since $\boldsymbol{z}_i^* = (X^TX + \alpha_1 I + \alpha_2 I)^{-1}(X^T\boldsymbol{x}_i + \alpha_2\boldsymbol{w}_i)$, $\boldsymbol{z}_j^* = (X^TX + \alpha_1 I + \alpha_2 I)^{-1}(X^T\boldsymbol{x}_j + \alpha_2\boldsymbol{w}_j)$, let $Y = X(X^TX + \alpha_1 I + \alpha_2 I)^{-1}$. Then

$$\begin{aligned} |Z_{pi}^* - Z_{pj}^*| &\le \frac{||\boldsymbol{x}_p^T(\boldsymbol{x}_i - \boldsymbol{x}_j)||_2 + ||\boldsymbol{x}_p^T X(\boldsymbol{z}_i^* - \boldsymbol{z}_j^*)||_2 + \alpha_2|(W_{pi} - W_{pj})|}{\alpha_1 + \alpha_2} \\ &\le \frac{||\boldsymbol{x}_p^T(\boldsymbol{x}_i - \boldsymbol{x}_j)||_2 + ||\boldsymbol{x}_p^T Y X^T(\boldsymbol{x}_i - \boldsymbol{x}_j)||_2}{\alpha_1 + \alpha_2} \\ &+ \frac{\alpha_2||\boldsymbol{x}_p^T Y(\boldsymbol{w}_i - \boldsymbol{w}_j)||_2 + \alpha_2|W_{pi} - W_{pj}|}{\alpha_1 + \alpha_2} \end{aligned} \tag{5}$$

If $x_i \to x_j$, i.e., $\boldsymbol{x}_i^T\boldsymbol{x}_j \to 1$ and $||\boldsymbol{w}_i - \boldsymbol{w}_j||_2 \to 0$, we have $||\boldsymbol{x}_i - \boldsymbol{x}_j||_2 \to 0$ and $|W_{pi} - W_{pj}| \to 0$. Then $|Z_{pi}^* - Z_{pj}^*| \to 0$ and $(Z^*)^T$ has grouping effect. Since $\left||Z_{pi}^*| - |Z_{pj}^*|\right| \le |Z_{pi}^* - Z_{pj}^*|$, $|(Z^*)^T|$ also has grouping effect. □

Lemma 5. *Matrix $\tilde{Z}$ has grouping effect.*

Proof. From Lemma 3 and 4, both $|Z^*|$ and $|(Z^*)^T|$ have the grouping effect. Since $\tilde{Z} = (|Z^*| + |(Z^*)^T|)/2$,

$$\begin{aligned} |\tilde{Z}_{ip} - \tilde{Z}_{jp}| &= \frac{\left|(|Z_{ip}^*| + |Z_{pi}^*|) - (|Z_{jp}^*| + |Z_{pj}^*|)\right|}{2} \\ &\le \frac{|Z_{ip}^* - Z_{jp}^*| + |Z_{pi}^* - Z_{pj}^*|}{2} \end{aligned} \tag{6}$$

If $x_i \to x_j$, both $|Z_{ip}^* - Z_{jp}^*| \to 0$ and $|Z_{pi}^* - Z_{pj}^*| \to 0$, so $|\tilde{Z}_{ip} - \tilde{Z}_{jp}| \to 0$ and $\tilde{Z}$ has grouping effect. □

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