Washington Experimental Mathematics Lab

Hausdorff Dimension of the Brownian Earthworm

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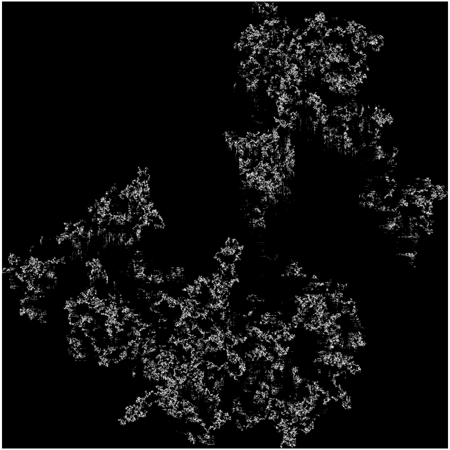
What is the Brownian Earthworm

Motivation Our goal is to find the Hausdorff Dimension of Brownian earthworm model. The earthworm does a random walk on a 2-D grid with particles on each point, moving particles away and creating/filling holes. The number of holes is non-decreasing with the number of steps.

Problem At what rate does the number of holes grow with the number of steps? Estimate the Hausdorff Dimension of the set of holes.

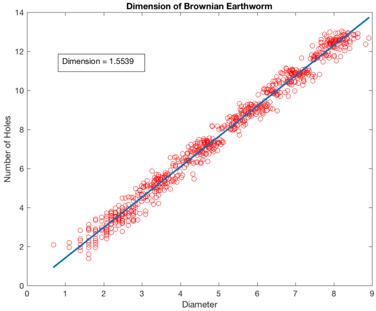
Methods We use two methods of estimating the Hausdorff dimension and compare their results. Both require the simulation of a large number of random walks.

Method 1



We want to find the rate at which the number of holes grows as the number of steps increases. Since Brownian motion is the *limit* of this random walk, we need to scale our calculations.

Results of Method 1



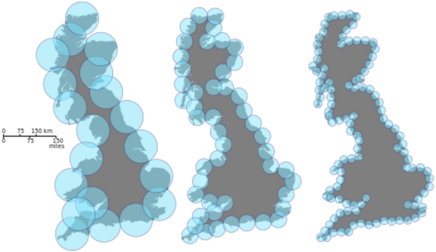
750 simulations with 10 to 10,000,000 steps

Specifically, we want to calculate

$$\frac{\log n^h}{\log n^d} = \frac{h}{d},$$

where the number of holes $\approx n^h$ and the diameter $\approx n^d$ for a walk of n steps.

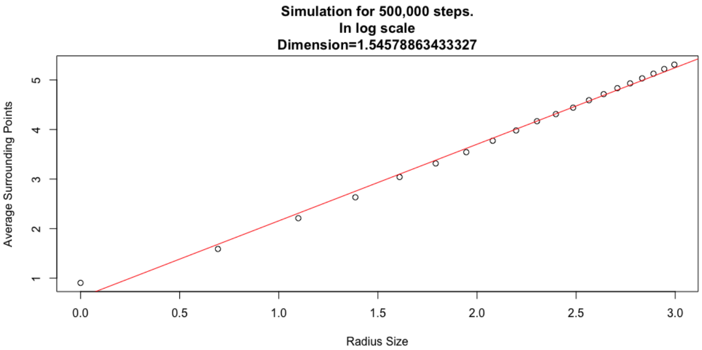
Method 2



Introduction

We want to support our estimates from the first method by using a second one. For the second method we find the rate at which the average number of frontier holes surrounding each frontier hole increases as we increase our search radius.

Method 2



Results

The second method estimated the dimension to be similar to our estimations using the first method.

Conclusion

Results Method 1: 1.5539, Method 2: 1.5457. Both methods of estimation strongly support the conjecture that the Hausdorff dimension of the "holes" for the Brownian earthworm is close to 1.5. *We were not told the conjecture in advance!*

Growth rate For a walk of length n, this suggests the number of holes grows no less than $n^{3/4}$.

Other directions There are many other questions which one could ask: How many connected components are there? Does the number of connected components converge to some distribution upon some proper scaling? Will a law of large numbers hold?