Washington Experimental Mathematics Lab

Hausdorff Dimension of Random Curves

Department of Mathematics
University of Washington

Alex Forney
Xiyi Yan
Ran Zhao

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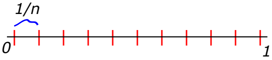
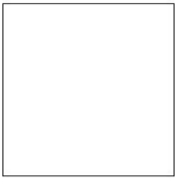
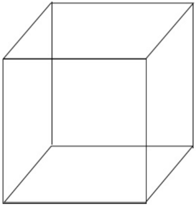
What is Hausdorff Dimension?



We need n boxes of length $1/n$ to cover



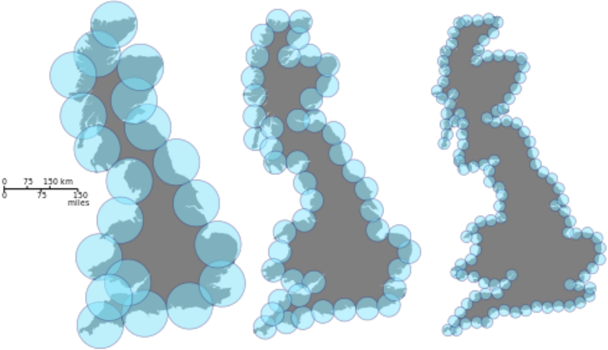
n^2 boxes of length $1/n$ to cover



n^3 boxes of length $1/n$ to cover

Numbers of boxes needed to cover the sets = n^D.
The Hausdorff dimension D for the sets is the power of n as $n \to \infty$ (as the boxes shrink to zero).

Hausdorff Dimension for Irregular Sets



The Hausdorff dimension for very irregular sets is between 1 and 2.

Project Goal

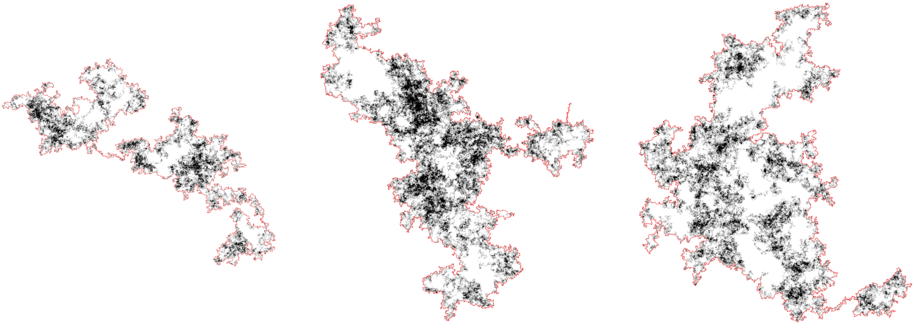
Hausdorff Dimension of Random Curves

Description Our first goal was to approximate the Hausdorff dimension of the *frontier* of lattice random walk in 2D.

Problem Simulate a random walk and find the frontier, i.e. the outer boundary of the whole path.

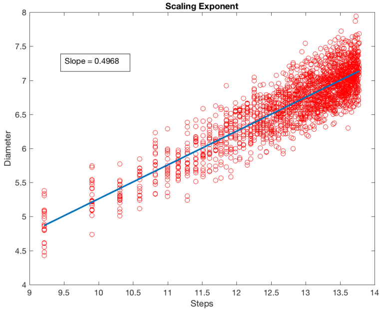
Methods We simulated thousands of random walks and collected data using Python. We then took each point on the walk and determined whether it was a boundary point or not. Finally, we traced the outer contour points to form the frontier.

Random Walks



The *frontier* is outlined in red. Simulation $\approx$ 1 million steps.

Calculations

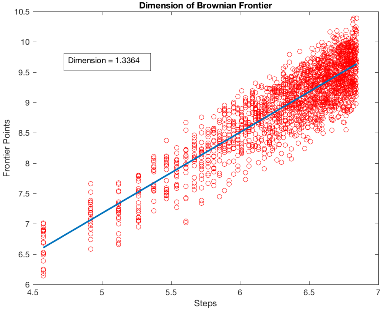


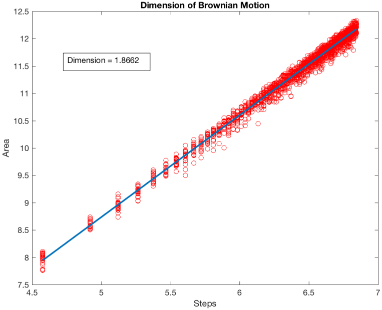
The diameter (maximum height or width) of our walk $\approx n^d$, for some d, where n is the number of steps. We calculated $d = 0.4986$.

The size (the number of points) of the frontier $\approx n^h$, for some h. Then, the dimension is

$$\frac{\log n^h}{\log n^d} = \frac{h}{d}.$$

Pictures





calculated $= 1.3364$, actual $= \frac{4}{3}$

calculated $= 1.8662$, actual $= 2$

Progress

What's worked The computational challenge of finding the frontier was a success. Our calculations for the diameter and the dimension of the frontier were very close to the actual values.

What hasn't The calculated dimension of the area of the walk was not as accurate as we would have liked. This is probably a result of the limitations in the scale of our walks. The sizes of our walks were limited to about 10^5. We gathered hundreds of samples.

Future goals

Trace of Earthworm

Next steps *Estimating the Hausdorff dimension for other models.* We will focus on simulations of the "earthworm" model. The path of the earthworm is just two-dimensional random walk. We put particles on every point of the grid and with the random movement of earthworm, the particles on the grid can be pushed aside and leave a hole. Goal: *Estimate the Hausdorff dimension of the set of holes.*

Challenges Keeping track of the entire grid will be difficult for large walks. We may attempt to dynamically increase our grid, but that could introduce new complications.