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BALANCING FREE SQUARE–ROOT ALGORITHM FOR COMPUTING SINGULAR PERTURBATION APPROXIMATIONS

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Abstract

The paper proposes a new algorithm with enhanced numerical robustness for computing singular perturbation approximations of linear, continuous or discrete systems. The new algorithm circumvents the computation of possibly ill–conditioned balancing transformations. Instead, well–conditioned projection matrices are determined for computing state–space representations suitable for applying the singular perturbation formulas. The projection matrices are computed using the Cholesky (square–root) factors of the gramians. The proposed algorithm is intended for efficient computer implementation. It can handle both minimal and non–minimal systems.

I. INTRODUCTION

The method of balanced singular perturbation approximation (BSPA) plays an increased role in the recently developed model and controller reduction procedures [1]–[3]. The main advantage over the popular balance and truncate (B&T) method [4] is its better approximation property at low frequencies. In particular, BSPAs retain the DC gain of the original system. Moreover, the models resulted using BSPAs are further balanced (for both continuous and discrete systems), a feature useful in some applications.

The computation of BSPAs requires, as a first step, the reduction of the given system to a minimal order balanced state–space representation. Numerically reliable computational algorithms are available for this purpose [5], [6]. The algorithm of [6] is also applicable to non–minimal systems. It computes projection matrices by which reduced order models are determined which are identical with those computed by the B&T method. However, the computed projection matrices may be very ill–conditioned when the minimal order system has modes which are either nearly uncontrollable or nearly unobservable. The application of such projections may induce a substantial loss of accuracy in the resulted reduced order model.

A model reduction method equivalent with the algorithm of [6], which however avoids balancing, was proposed in [7]. Well–conditioned projection matrices can be determined by this method for computing minimal order models. The computation of projection matrices involves the determination of orthogonal bases for two invariant subspaces of the product of the gramians. An improved accuracy version of this algorithm, which avoids the computation of gramians as well as of their product, was proposed in [8]. The projection matrices are computed by this algorithm using the Cholesky (square–root) factors of gramians. The resulted minimal order models are however not suitable for computing BSPAs.

In this paper we propose an algorithm for computing singular perturbation approximations (SPAs) which extends the techniques from [7] and [8]. It circumvents the need for computing possibly ill–conditioned projection matrices. Instead, well-conditioned projections are constructed such that the resulted minimal state–space representation is suitable for applying the SPA formulas. These projection matrices are computed using only the square–root factors of the gramians. The resulted SPAs are generally not balanced. However, the proposed algorithm can be easily adapted to compute optionally also BSPAs.

The proposed algorithm has enhanced numerical robustness as compared with the existing methods. It can handle both minimal and non–minimal, continuous and discrete, systems. It is devised for efficient computer implementation by using available, high quality numerical software. Implementations of the proposed algorithm are available as MATLAB *.m* functions [13].

II. PRELIMINARIES

Let (A,B,C,D) an n–th order stable, continuous or discrete state–space system with reachability and observability gramians P and Q, respectively. It is well known that P and Q satisfy the following Lyapunov equations:

$$AP + PA^T + BB^T = 0 \qquad (1)$$

$$A^TQ + QA + C^TC = 0 \qquad (2)$$

for a continuous system, or

$$APA^T + BB^T = P \qquad (3)$$

$$A^TQA + C^TC = Q \qquad (4)$$

for a discrete system. P and Q are symmetric and positive semi–definite matrices. Thus, there exist the Cholesky (square–root) decompositions

$$P = S^TS, \quad Q = R^TR. \qquad (5)$$

Numerically reliable algorithms can be used to compute S and R, without forming the products BB^T and C^TC in (1)–(4) [9], [10].

It is known [11] that there exists an invertible transformation matrix such that the transformed system

$$(\tilde{A},\tilde{B},\tilde{C},\tilde{D}) =: (T^{-1}AT, T^{-1}B, CT, D) \qquad (6)$$

has the corresponding gramians $\tilde{P}$ and $\tilde{Q}$

$$\tilde{P} = T^{-1}PT^{-T} = \mathrm{diag}(\Sigma,\Sigma') \qquad (7)$$

$$\tilde{Q} = T^TQT = \mathrm{diag}(\Sigma,\Sigma'') \qquad (8)$$

where $\Sigma > 0$, $\Sigma' \geq 0$, $\Sigma'' \geq 0$ are diagonal matrices and $\Sigma\Sigma''=0$. The system (6) is called *balanced* and T is the corresponding *balancing transformation matrix*. The *(Hankel) singular values* of the system are defined as

$$\sigma_i = [\lambda_i(PQ)]^{\frac{1}{2}}$$

where $\lambda_i(PQ)$ is the i–th eigenvalue of PQ. We shall assume that

$$\sigma_1 \geq \cdots \geq \sigma_k \geq \sigma_{k+1} \geq \cdots \geq \sigma_m \geq \sigma_{m+1} = \cdots = \sigma_n = 0 \qquad (10)$$

and thus Σ in (7) and (8) is

$$\Sigma = \mathrm{diag}(\sigma_1,\cdots,\sigma_m). \qquad (11)$$

Now partition Σ as

$$\Sigma = \mathrm{diag}(\Sigma_1,\Sigma_2), \qquad (12)$$

where

$$\Sigma_1 = \mathrm{diag}(\sigma_1,\cdots,\sigma_k), \;\; \Sigma_2 = \mathrm{diag}(\sigma_{k+1},\cdots,\sigma_m). \qquad (13)$$

Partition also the matrices of the balanced system $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ in accordance with the structure of the gramians

$\tilde{P} = \mathrm{diag}(\Sigma_1,\Sigma_2,\Sigma'),\ \ \tilde{Q} = \mathrm{diag}(\Sigma_1,\Sigma_2,\Sigma'')$

as

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix},\ \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{bmatrix},\ \tilde{C} = [\,\tilde{C}_1\ \tilde{C}_2\ \tilde{C}_3\,] \tag{14}$$

$$\tilde{P}\tilde{Q} = \mathrm{diag}(\Sigma_1^2,\Sigma_2^2,0), \tag{15}$$

it follows [11] that the system

$$\left[\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, [\,\tilde{C}_1\ \tilde{C}_2\,], D\right] \tag{16}$$

is a balanced minimal realization of (A,B,C,D) with both gramians equal to Σ. The BSPA of the system (16) is $(\bar{A},\bar{B},\bar{C},\bar{D})$ [1], given by

$$\begin{aligned} \bar{A} &= \tilde{A}_{11}+\tilde{A}_{12}(\gamma I-\tilde{A}_{22})^{-1}\tilde{A}_{21} \\ \bar{B} &= \tilde{B}_1+\tilde{A}_{12}(\gamma I-\tilde{A}_{22})^{-1}\tilde{B}_2 \\ \bar{C} &= \tilde{C}_1+\tilde{C}_2(\gamma I-\tilde{A}_{22})^{-1}\tilde{A}_{21} \\ \bar{D} &= D+\tilde{C}_2(\gamma I-\tilde{A}_{22})^{-1}\tilde{B}_2 \end{aligned} \tag{17}$$

where $\gamma = 0$ for a continuous system and $\gamma = 1$ for a discrete system. The properties of the BSPA (17) are discussed in [1] (see also [3]).

The computation of the balanced minimal state–space representation (16) by using for example the algorithm of [6] may lead to numerical difficulties due to the use of possibly ill–conditioned projection matrices. In the next section, we propose a new method which circumvents the computation of balancing projection matrices. The development of the new method is based on the following two properties.

Proposition 1. Let P and Q the reachability and observability gramians, respectively, of the system (A,B,C,D) and assume that PQ has the block–diagonal form (BDF)

$$PQ = \mathrm{diag}(X_1,X_2,0) \tag{18}$$

where X_1 and X_2 are non singular matrices and $\lambda(X_1)\cap\lambda(X_2) = \phi$ (the empty set). Let T be the balancing transformation matrix such that the balanced system $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ given by (6) has the reachability and observability gramians $\tilde{P}$ and $\tilde{Q}$ in the forms (7) and (8), respectively. If $\lambda(X_1) = \lambda(\Sigma_1^2)$ and $\lambda(X_2) = \lambda(\Sigma_2^2)$, then T has a block–diagonal structure

$$T = \mathrm{diag}(T_1,T_2,T_3)$$

where T_1, T_2 and T_3 are non–singular matrices.

Proof. It follows from (15) that

$$\tilde{P}\tilde{Q} = T^{-1}PQT = \mathrm{diag}(\Sigma_1^2,\Gamma^2) \tag{19}$$

where $\Gamma^2 = \mathrm{diag}(\Sigma_2^2,0)$. Partition T conformally with the structure of $\tilde{P}\tilde{Q}$ as

$$T = \begin{bmatrix} T_1 & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

where T_1 is square, having the same order as Σ_1. By denoting $X=\mathrm{diag}(X_2,0)$, (19) can be written equivalently as

$$\begin{bmatrix} X_1 & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} T_1 & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} T_1 & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} \Sigma_1^2 & 0 \\ 0 & \Gamma^2 \end{bmatrix}$$

from which we obtain

$$\begin{aligned} X_1T_1 &= T_1\Sigma_1^2 \\ XT_{22} &= T_{22}\Gamma^2 \\ X_1T_{12} &= T_{12}\Gamma^2 \\ XT_{21} &= T_{21}\Sigma_1^2. \end{aligned}$$

Taking into account that $\lambda(X_1)\cap\lambda(\Gamma^2) = \phi$ and $\lambda(X)\cap\lambda(\Sigma_1^2) = \phi$, from the last two equations it follows that $T_{12}=0$ and $T_{21}=0$. T being non–singular, T_1 and T_{22} are also non–singular. In the same way it can be shown that T_{22} has a BDF, $T_{22}=\mathrm{diag}(T_2,T_3)$. ■

Proposition 2. For a given system (A,B,C,D) with gramians P and Q, let Z a non–singular matrix such that

$$Z^{-1}PQZ = \mathrm{diag}(X_1,X_2,0) \tag{20}$$

with X_1 and X_2 non–singular and let

$$(\hat{A},\hat{B},\hat{C},\hat{D}) =: (Z^{-1}AZ,Z^{-1}B,CZ,D) \tag{21}$$

If we partition the matrices of the system $(\hat{A},\hat{B},\hat{C},\hat{D})$ in the form (14) (with ˆ replacing ˜ overall) conformally with the structure of $Z^{-1}PQZ$ in (20), then the system $(\check{A},\check{B},\check{C},\check{D})$ given by

$$\begin{aligned} \check{A} &= \hat{A}_{11}+\hat{A}_{12}(\gamma I-\hat{A}_{22})^{-1}\hat{A}_{21} \\ \check{B} &= \hat{B}_1+\hat{A}_{12}(\gamma I-\hat{A}_{22})^{-1}\hat{B}_2 \\ \check{C} &= \hat{C}_1+\hat{C}_2(\gamma I-\hat{A}_{22})^{-1}\hat{A}_{21} \\ \check{D} &= D+\hat{C}_2(\gamma I-\hat{A}_{22})^{-1}\hat{B}_2 \end{aligned} \tag{22}$$

is a SPA of the given system, that is, there exists a non–singular matrix T_1 such that

$$(\bar{A},\bar{B},\bar{C},\bar{D}) = (T_1^{-1}\check{A}T_1,T_1^{-1}\check{B},\check{C}T_1,\check{D}) \tag{23}$$

where $(\bar{A},\bar{B},\bar{C},\bar{D})$ is the BSPA in (17).

Proof. From Proposition 1 it follows that the systems $(\hat{A},\hat{B},\hat{C},\hat{D})$ and $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ are related by

$$(\tilde{A},\tilde{B},\tilde{C},\tilde{D}) = (T^{-1}\hat{A}T,T^{-1}\hat{B},\hat{C}T,\hat{D})$$

where

$$T = \mathrm{diag}(T_1,T_2,T_3).$$

Therefore, the matrices of the minimal system (16) are given by

$$\left[\begin{bmatrix} T_1^{-1}\hat{A}_{11}T_1 & T_1^{-1}\hat{A}_{12}T_2 \\ T_2^{-1}\hat{A}_{21}T_1 & T_2^{-1}\hat{A}_{22}T_2 \end{bmatrix}, \begin{bmatrix} T_1^{-1}\hat{B}_1 \\ T_2^{-1}\hat{B}_2 \end{bmatrix}, [\,\hat{C}_1T_1\ \hat{C}_2T_2\,], \hat{D}\right]$$

Evaluating for example $\bar{A}$ from (17), we obtain

$$\bar{A} = T_1^{-1}\hat{A}_{11}T_1+T_1^{-1}\hat{A}_{12}T_2(\gamma I-T_2^{-1}\hat{A}_{22}T_2)^{-1}T_2^{-1}\hat{A}_{21}T_1 = T_1^{-1}\check{A}T_1$$

where $\check{A}$ is given by (22). The other relations from (23) can be verified similarly. ■

Remark. The resulting SPA $(\check{A},\check{B},\check{C},\check{D})$ is generally not balanced. However, a BSPA can be computed if Z determined such that $X_1=\Sigma_1^2$ in (18). The corresponding $T_1=I$, and the resulting SPA is the BSPA $(\bar{A},\bar{B},\bar{C},\bar{D})$ from (19). ■

The main computational problem consists of determining in a numerically sound way a well–conditioned transformation matrix Z for reducing PQ to the BDF (20). This problem is addressed in the next section.

III. THE ALGORITHM

There are several possible numerical approaches for computing the block–diagonalizing transformation matrix Z from (20). The most straightforward method seems to be first to reduce PQ to an ordered real Schur form (RSF) by an orthogonal similarity transformation and then to enter zero blocks above the diagonal blocks by suitably chosen non–orthogonal matrices [12]. The main disadvantage of this approach is the unnecessary loss of accuracy due to the need of forming the product PQ. The consequences of this accuracy loss could be that some small eigenvalues become negative or, even worse, two small eigenvalues, although distinct, result coupled into a complex conjugated pair, preventing thus the possibility of their separation.

A second method can be devised by extending the projection approach of [7]. We can determine the matrices Z and Z^+ such that $Z^+Z = I$ and

$$Z^+PQZ = \mathrm{diag}(X_1, X_2)$$

Thus we obtain directly the minimal part of the system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ from (21) as

$$Z^+AZ =: \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \; Z^+B =: \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \; CZ =: [\, \hat{C}_1 \; \hat{C}_2 \,]. \qquad (25)$$

Let V_{L_1} and V_{R_1} matrices whose columns form bases for the invariant subspaces of QP and PQ, respectively, corresponding to the eigenvalues $\lambda(\Sigma_1^2)$. Let V_{L_2} and V_{R_2} the analogous matrices corresponding to the eigenvalues $\lambda(\Sigma_2^2)$. The matrices Z^+ and Z can be determined in the form

$$Z^+ = \begin{bmatrix} Z_1^+ \\ Z_2^+ \end{bmatrix}, \; Z = [\, Z_1 \; Z_2 \,] \qquad (26)$$

where the submatrices can be computed by formulas analogous to those from [7]:

$$Z_i^+ = \Sigma_{E_i}^{-\frac{1}{2}} U_{E_i}^T V_{L_i}^T, \quad Z_i = V_{R_i} V_{E_i} \Sigma_{E_i}^{-\frac{1}{2}} \text{ for } i = 1, 2,$$

where

$$V_{L_i}^T V_{R_i} = U_{E_i} \Sigma_{E_i} V_{E_i}^T$$

is the singular value decomposition (SVD) of the matrix $V_{L_i}^T V_{R_i}$. Although V_{L_1}, V_{R_1}, V_{L_2} and V_{R_2} can be determined as matrices with orthonormal columns, this method has the same inconveniences as the previous one, due to the accuracy loss which occurs when PQ is constructed explicitly.

An alternative, numerically more robust approach is suggested by the square–root method of [8]. Let S and R the square–root factors of P and Q from (5) and let

$$SR^T = [\, U_1 \; U_2 \; U_3 \,] \,\mathrm{diag}(\Sigma_1, \Sigma_2, 0)\, [\, V_1 \; V_2 \; V_3 \,]^T \qquad (27)$$

the SVD of SR^T where Σ_1 and Σ_2 are given by (13). Note that by the ordering of singular values assumed in (10), Σ_1 and Σ_2 have no common diagonal entries. It can verified easily that for $i=1,2$ and $j=1,2$

$$PQS^TU_i = S^TU_i\Sigma_i^2,$$

$$QPR^TV_i = R^TV_i\Sigma_i^2,$$

$$V_i^TRS^TU_j = 0, \; i \neq j$$

which show that V_{L_1}, V_{R_1}, V_{L_2} and V_{R_2} can be defined as

$$V_{L_i} = R^TV_i, \; V_{R_i} = S^TU_i, \; i = 1,2$$

But, as suggested in [8], R^TV_i and S^TU_i can be replaced by orthogonal matrices resulted from their QR–decompositions. These ideas form the basis for the balancing–free square–root (BFSR) algorithm for computing SPAs of a given stable system (A,B,C,D):

BFSR Algorithm.

1) Compute the square–root factors S and R of gramians in (5) by solving the Lyapunov equations (1) and (2) for a continuous system, or (3) and (4) for a discrete system.
2) Compute the SVD of SR^T in the form (27), where $\lambda(\Sigma_1) \cap \lambda(\Sigma_2) = \phi$.
3) Compute for $i=1,2$ the QR–decompositions
   $S^TU_i = M_iX_i, \; R^TV_i = N_iY_i,$
   where M_i, N_i, are matrices with orthonormal columns and X_i, Y_i, are non–singular matrices.
4) Compute for $i=1,2$ the SVDs: $N_i^TM_i = U_{E_i}\Sigma_{E_i}V_{E_i}^T$.
5) Compute Z^+ and Z in the form (26), where
   $Z_i^+ = \Sigma_{E_i}^{-\frac{1}{2}} U_{E_i}^T N_i^T, \quad Z_i = M_i V_{E_i} \Sigma_{E_i}^{-\frac{1}{2}}, \; i = 1, 2.$
6) Compute the minimal system (Z^+AZ, Z^+B, CZ, D) partitioned as in (25).
7) Compute the SPA $(\check{A}, \check{B}, \check{C}, \check{D})$ given by (22).

All computations in this algorithm can be performed using numerically reliable or numerically stable algorithms. Its enhanced numerical robustness is the consequence of avoiding the application of possibly ill–conditioned transformations as well as the exclusive usage of square–root factors instead of the gramians themself. The algorithm is well suited for efficient computer implementation.

Remark. The resulting SPA can be obtained directly in a balanced state–space representation by choosing the matrices Z_1^+ and Z_1 at step 5 as

$$Z_1^+ = \Sigma_1^{-\frac{1}{2}} V_1^T R, \quad Z_1 = S^TU_1\Sigma_1^{-\frac{1}{2}}.$$

No computations involving S^TU_1 or R^TV_1 should be performed at previous steps. By using this option, it is possible to determine alternatively at step 7 of the proposed algorithm, BSPAs of order less than k, where k is the order of Σ_1. This is possible because in this case

$$Z^+PQZ = \mathrm{diag}(\Sigma_1^2, X_2)$$

with Σ_1^2 diagonal, and thus any partition in (25) with $\hat{A}_{11}$ of order q, $1 \le q \le k$, will give a BSPA. ■

IV. NUMERICAL EXAMPLES

Two examples illustrate the basic feature of the proposed algorithm to determine well–conditioned projection matrices. The increased accuracy of the BFSR projection method has been illustrated elsewhere [8].

Example 1. This is the 10–th order continuous model used in [7]. The minimal system has order six as revealed by the computed singular values of the system:

$\sigma_1 = 2.5001 \times 10^2$, $\sigma_2 = 2.5168 \times 10^{-2}$, $\sigma_3 = 5.5789 \times 10^{-3}$,

$\sigma_4 = 2.4142 \times 10^{-3}$, $\sigma_5 = 9.2182 \times 10^{-4}$, $\sigma_6 = 1.3086 \times 10^{-5}$,

$\sigma_{7\text{-}10} = 10^{-23} \div 10^{-19}$.

We computed fourth order SPAs by two methods: 1) the B&T method; and 2) the proposed BFSR method. The Table I contains the condition numbers of the resulting projection matrices. As is expected, the BFSR algorithm determines much better conditioned projection matrices than those resulted using the existing method.

TABLE I. Condition numbers of projection matrices for Example 1.

Method	B&T	BFSR
cond(Z)	3.5×10^4	39.3
cond(Z^+)	5.5×10^4	28.0

The resulted SPA with the BFSR method is

$$\check{A} = \begin{bmatrix} -12.0094 & 0.0053 & 10.7734 & -1.8618 \\ 1.2558 & -6.3432 & -7.2160 & 1.3985 \\ 2.9977 & 1.4058 & -13.3384 & 1.3195 \\ 1.4838 & 0.9386 & 1.2700 & -2.1579 \end{bmatrix}$$

$$\check{B} = \begin{bmatrix} -0.3922 & -0.8025 \\ 0.1125 & 0.4613 \\ -0.2730 & 0.2827 \\ 0.1110 & 0.0231 \end{bmatrix}$$

$$\check{C} = 10^4 \times \begin{bmatrix} 1.5239 & 1.7785 & 1.0592 & 6.6343 \\ 1.5237 & 1.7783 & 1.0594 & 6.6343 \end{bmatrix}$$

$$\check{D} = \begin{bmatrix} -0.0005 & 0.0008 \\ 0.0008 & -0.0013 \end{bmatrix}.$$

Example 2. This is a discrete–time example obtained by discretizing the model of Example 1 with a sampling period of 0.1 seconds. The singular values of the discrete system are:

$\sigma_1 = 2.7493\times10^2$, $\sigma_2 = 4.0898\times10^{-2}$, $\sigma_3 = 5.5469\times10^{-3}$,

$\sigma_4 = 3.241\times10^{-3}$, $\sigma_5 = 8.4077\times10^{-4}$, $\sigma_6 = 1.1591\times10^{-5}$,

$\sigma_{7\text{-}10} = 10^{-21} \div 10^{-19}$.

The Table II. contains the condition numbers of the resulted projection matrices. As in continuous case, the projection matrices determined by the BFSR method are much better conditioned than those determined by the B&T method.

TABLE II. Condition numbers of projection matrices for Example 2.

Method	B&T	BFSR
cond(Z)	3.36×10^4	48.7
cond(Z^+)	5.15×10^4	28.9

The matrices of the resulted SPA are:

$$\check{A} = \begin{bmatrix} 0.3936 & 0.1134 & 0.1631 & 0.1405 \\ -0.0073 & 0.4614 & -0.1908 & -0.0109 \\ 0.1139 & 0.0636 & 0.3361 & 0.1532 \\ 0.0970 & 0.0603 & 0.0139 & 0.7881 \end{bmatrix}$$

$$\check{B} = \begin{bmatrix} -0.0227 & -0.0439 \\ 0.0116 & 0.0285 \\ -0.0197 & 0.0086 \\ 0.0032 & 0.0036 \end{bmatrix}$$

$$\check{C} = 10^4 \times \begin{bmatrix} 1.5172 & 1.6225 & 0.0647 & 6.6953 \\ 1.5170 & 1.6224 & 0.0648 & 6.6953 \end{bmatrix}$$

$$\check{D} = 10^{-4} \times \begin{bmatrix} 0.5475 & -0.4340 \\ -0.2658 & 0.2164 \end{bmatrix}.$$

V. CONCLUSIONS

A new algorithm for computing SPA of both continuous and discrete systems has been proposed. Its enhanced numerical robustness is due to two main features: 1) it avoids the computation of possibly ill–conditioned balancing transformations; and 2) it circumvents the computation of gramians as well as of their product by working instead with their square–root factors. This new algorithm can be used independently as a model reduction tool or in conjunction with other techniques, as for example, as a subprocedure in the normalized coprime factors model reduction method [2]. In this context, the new method is particularly useful because, as it was shown in [3], the resulting factors computed by the SPA method are also normalized.

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REFERENCES

[1] Y.Liu, and B.D.O.Anderson, "Singular perturbation approximation of balanced systems," *Int. J. Contr.*, vol. 50, pp.1379–1405, 1989.

[2] D.McFarlane, K.Glover, and M.Vidyasagar, "Reduced–order controller design using coprime factor model reduction," *IEEE Trans. Autom. Contr.*, vol. AC–35, pp.369–373, 1990.

[3] A.Varga, "Balanced singular perturbation approximation of coinner transfer function matrices," *Research Report 1.90*, Dept. Mech. Eng, Ruhr–Univ. Bochum, Nov. 1990.

[4] B.C.Moore, "Principal component analysis in linear systems: controllability, observability and model reduction," *IEEE Trans. Autom. Contr.*, vol. AC–26, pp.17–32, 1981.

[5] A.Laub, M.T.Heath, C.C.Paige, and R.C.Ward, "Computation of system balancing transformations and other applications of simulataneous diagonalization algorithms," *IEEE Trans. Autom. Contr.*, vol. AC–32, pp.115–122, 1987.

[6] M.S.Tombs, and I.Postlethwaite, "Truncated balanced realization of a stable non–minimal state–space system," *Int. J. Control*, vol. 46, pp.1319–1330, 1987.

[7] M.G.Safonov, and R.Y.Chiang, "A Schur method for balanced–truncation model reduction," *IEEE Trans. Autom. Contr.*, vol. AC–34, pp.729–733, 1989.

[8] A.Varga, "Efficient minimal realization procedure based on balancing," *Preprints of IMACS Symp. on Modelling and Control of Technological Systems*, Lille, France, May 1991.

[9] S.J.Hammarling, "Numerical solution of the stable, non–negative definite Lyapunov equation," *IMA J. Numer. Anal.*, vol. 2, pp.303–323, 1982.

[10] A.Varga, "A note on Hammarling's algorithm for the discrete Lyapunov equation," *Syst. Contr. Lett.*, vol.15, pp.273–275, 1990.

[11] K.Glover, "All optimal Hankel–norm approximations of linear multivariable systems and their L_∞–error bounds," *Int. J. Contr.*, vol. 39, pp.115–1193, 1984.

[12] G.H.Golub, and C.F.Van Loan, "*Matrix Computations*," The John Hopkins Univ. Press, Baltimore, Maryland, 1983.

[13] A.Varga, and V.Ionescu, "HTOOLS – a Toolbox for solving H^∞ and H^2 synthesis problems," *Prepr. of IFAC/IMACS Symp. CADCS'91*, Swansea, July 1991.