ON COMPONENTS OF THE TENSOR SQUARE OF A WEYL MODULE

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ABSTRACT. For a simple Lie algebra $\mathfrak{g}$ of type A_n, B_n, C_n or D_n, we give a characterization of the set of dominant integral weights λ such that for any rational point μ in the fundamental Weyl chamber, $2\lambda - \mu$ is a non-negative rational combination of the simple roots if and only if $V_{m\mu} \subseteq V_{m\lambda} \otimes V_{m\lambda}$ for some positive integer m.

1. INTRODUCTION

Let $\mathfrak{g}$ be a simple Lie algebra of rank n over $\mathbb{C}$ with Borel subalgebra $\mathfrak{b}$ and Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let $\Phi \subset \mathfrak{h}^*$ be the root system, W be the Weyl group, $\Pi_{\mathfrak{g}} = \{\alpha_1, \ldots, \alpha_n\}$ be its simple roots, $\{\omega_1, \ldots, \omega_n\}$ be the corresponding fundamental weights and ρ be the half sum of positive roots. Let Λ^+ be the set of dominant integral weights and set $\Lambda^+_{\mathbb{Q}} = \Lambda^+ \otimes \mathbb{Q}_{\geq 0}$.

For any $\lambda \in \Lambda^+$, let V_λ be the irreducible representation of $\mathfrak{g}$ with highest weight λ. V_λ admits a weight space decomposition given by

$$V_\lambda = \bigoplus_{\mu \in \Lambda^+} V_\lambda(\mu).$$

For $\lambda, \mu \in \Lambda^+$, we write $\lambda \geq \mu$ if $V_\lambda(\mu) \neq 0$. Equivalently, this is when $\lambda - \mu = \sum_{i=1}^n c_i \alpha_i$ where $c_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$. We extend this partial order to $\Lambda^+_{\mathbb{Q}}$ where we only require $c_i \in \mathbb{Q}_{\geq 0}$.

A long-standing conjecture by Kostant states that:

Conjecture 1.1 (Kostant). *For $\mu \in \Lambda^+, V_\mu \subseteq V_\rho \otimes V_\rho$ if and only if $2\rho \geq \mu$.*

In the case of $\mathfrak{g} = \mathfrak{sl}_{n+1}$, Conjecture 1.1 was proved by Berenstein-Zelevinsky [3]. Chirivì-Kumar-Maffei [7] proved a weakening of Kostant's conjecture. That is, if $\mu \in \Lambda^+$ then $2\rho \geq \mu$ if and only if $V_{m\mu} \subseteq V_{m\rho} \otimes V_{m\rho}$ for some $m \in \mathbb{Z}_{>0}$. In fact, the latter statement is also equivalent to $V_{d\mu} \subseteq V_{d\rho} \otimes V_{d\rho}$ where d is a saturation factor for $\mathfrak{g}$. For $\mathfrak{g}$ of type A_n, $d = 1$ by the work of Knutson-Tao [12]. For $\mathfrak{g}$ of type B_n, C_n, d can be taken to be 2 by result of Belkale-Kumar [2], Sam [18], and Hong-Shen [10]. For $\mathfrak{g}$ of type D_n, d can be taken to be 2 by work of Sam [18]. We refer the readers to [[14], Section 10] for definition and further discussion of the saturation factor. In this paper, we study a variation of the result in [7]:

Problem 1.2. *Characterize the set of $\lambda \in \Lambda^+$ such that for all $\mu \in \Lambda^+_{\mathbb{Q}}$,*

$$2\lambda \geq \mu \iff V_{m\mu} \subseteq V_{m\lambda} \otimes V_{m\lambda} \text{ for some } m \in \mathbb{Z}_{>0}. \tag{1}$$

Our main theorem is a solution to Problem 1.2 when $\mathfrak{g}$ is of type A_n, B_n, C_n or D_n.

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Theorem 1.3. *λ satisfy* (1) *if and only if*

- *$\lambda = N\rho$ for some $N \in \mathbb{Z}_{\geq 0}$ when $\mathfrak{g}$ is of type A_n or D_n;*
- *$\lambda = N\rho + k\omega_n$ for $N \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq -N}$ when $\mathfrak{g}$ is of type B_n or C_n.*

Let $\mathcal{K}(\mathfrak{g})_{\mathbb{Q}}$ be the set of rational points in the *Kostka cone* associated to $\mathfrak{g}$;

$$\mathcal{K}(\mathfrak{g})_{\mathbb{Q}} := \{(\lambda, \nu) \in (\Lambda_{\mathbb{Q}}^+)^2 : \lambda \geq \nu\}.$$

Define the *saturated tensor semigroup* to be

$$\mathfrak{g}\text{-sat} := \{(\lambda, \mu, \nu) \in (\Lambda_{\mathbb{Q}}^+)^3 : dV_\nu \subseteq dV_\lambda \otimes dV_\mu \text{ for some } d \in \mathbb{Z}_{>0}\}$$

Fix $\lambda \in \Lambda^+$, define the *intersection polytope* to be the affine slice of $\mathcal{K}(\mathfrak{g})_{\mathbb{Q}}$ given by

$$\mathrm{IP}_{2\lambda} := \{\mu \in \Lambda_{\mathbb{Q}}^+ : (2\lambda, \mu) \in \mathcal{K}(\mathfrak{g})_{\mathbb{Q}}\}.$$

Define similarly the *tensor polytope* to be the affine slice of $\mathfrak{g}\text{-sat}$:

$$\mathrm{TP}_\lambda = \{\mu \in \Lambda_{\mathbb{Q}}^+ : (\lambda, \lambda, \mu) \in \mathfrak{g}\text{-sat}\}.$$

Problem 1.2 is then equivalent to the following:

Problem 1.4. *Characterize the set of $\lambda \in \Lambda^+$ such that $\mathrm{IP}_{2\lambda} = \mathrm{TP}_\lambda$.*

We note that $\mathrm{TP}_\lambda \subseteq \mathrm{IP}_{2\lambda}$ for all $\lambda \in \Lambda^+$. Our strategy is to study the vertices of $\mathrm{IP}_{2\lambda}$ and check if they lie in TP_λ.

2. NOTATION AND BACKGROUND

2.1. **Root system, weights and Weyl groups.** Following Section 1, we continue with more background on root systems in classical Lie types.

We adopt the following convention on root systems for classical types:

- Type A_{n-1}: $\alpha_i = e_i - e_{i+1}, \omega_i = \sum_{j=1}^i e_j$ for $i \in [n-1]$;
- Type B_n: $\alpha_i = e_i - e_{i+1}, \omega_i = \sum_{j=1}^i e_j$ for $i \in [n-1]$, $\alpha_n = e_n$ and $\omega_n = \frac{1}{2}\sum_{j=1}^n e_j$;
- Type C_n: $\alpha_i = e_i - e_{i+1}$ for $i \in [n-1]$, $\alpha_n = e_n$ and $\omega_i = \sum_{j=1}^i e_j$ for $i \in [n]$;
- Type D_n: $\alpha_i = e_i - e_{i+1}$ for $i \in [n-1]$, $\alpha_n = e_{n-1} + e_n$, $\omega_i = \sum_{j=1}^i e_j$ for $i \in [n-2]$, $\omega_{n-1} = \sum_{j=1}^{n-1} \frac{1}{2}e_j - \frac{1}{2}e_n$ and $\omega_n = \frac{1}{2}\sum_{j=1}^n e_j$.

Let $\mathfrak{g}$ be of type A_n through D_n, using the above convention, we will identify $\lambda = \sum_{i=1}^n \lambda_i e_i \in \Lambda^+$ with the sequence $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Let $\langle -, - \rangle$ be the inner product on $\mathfrak{h}$ so that the fundamental weight ω_i is dual to the simple coroots, i.e. $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$.

2.2. **Intersection polytopes and their vertices.** In [6], the authors studied the intersection polytope IP_λ to understand the extremal rays of $\mathcal{K}(\mathfrak{g})$. Here we recall their result on the vertices of IP_λ:

Theorem 2.1 ([6], Proposition 3.9). *For $I \subseteq [n]$, let W_I be the parabolic subgroup of W generated by $\{s_{\alpha_i} : i \in I\}$. Then for any $\lambda \in \Lambda^+$, the vertices of IP_λ is given by*

$$v_I(\lambda) = \frac{1}{|W_I|}\sum_{w \in W_I} w \cdot \lambda.$$

An immediate corollary is that the set of vertices depends linearly on λ:

Corollary 2.2 ([6], Corollary 3.11). *Any vertex of* $\mathrm{IP}_{\lambda+\mu}$ *is a sum of a vertex of* IP_λ *and a vertex of* IP_μ.

2.3. **Embedding of Dynkin diagrams and Schubert structure constants.** For any $\Pi' \subset \Pi$, let $\Phi' \subset \Phi_{\mathfrak{g}}$ be the root subsystem generated by Π'. Let $\mathfrak{g}' \subset \mathfrak{g}$ be the subalgebra defined by

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{\alpha\in\Phi'} \mathfrak{g}_\alpha, \tag{2}$$

where $\mathfrak{h}'$ is spanned by $\{h_\alpha \in \mathfrak{h} : \alpha \in \Phi'\}$. Let $p : \mathfrak{h}^* \to (\mathfrak{h}')^*$ be the natural orthogonal projection induced by $\langle -, - \rangle$.

Lemma 2.3 ([4], Proposition 1.3). *Let* λ, ν, μ *be dominant integral weights of* $\mathfrak{g}$ *such that* $\lambda + \nu - \mu \in \Phi'$. *Denote* $c^{p(\nu)}_{p(\lambda),p(\mu)}(\mathfrak{g}')$ *the multiplicity of* $V_{p(\nu)} \subset V_{p(\lambda)} \otimes V_{p(\mu)}$ *as* $\mathfrak{g}'$*-module. Then*

$$c^\nu_{\lambda,\mu}(\mathfrak{g}) = c^{p(\nu)}_{p(\lambda),p(\mu)}(\mathfrak{g}').$$

In particular, if $|\lambda| + |\mu| = |\nu|$ and $\lambda, \mu, \nu \in \Lambda^+$ are partitions; that is, $\lambda_i, \mu_i, \nu_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$, then

$$c^\nu_{\lambda,\mu}(\mathfrak{g}) = c^\nu_{\lambda,\mu}(\mathfrak{sl}_n). \tag{3}$$

2.4. **Inclusion of Dynkin diagrams and Schubert structure constants.** Let G be the connected complex semisimple algebraic group with Lie algebra $\mathfrak{g}$. Let B be the Borel subgroup of G with Lie algebra $\mathfrak{b}$. The generalized flag variety G/B has finitely many orbits under the left action of Borel subgroup B_-. They are indexed by $w \in W$ where W is the Weyl group of G. The *opposite Schubert varieties* Y_w are the closure of these orbits. Denote $[Y_w]$ the Poincaré dual of the fundamental class of Y_w. We have

$$[Y_w] \in H^{\ell(w)}(G/B),$$

where $\ell(w)$ is the Coxeter length of w. The set $\{[Y_w] : w \in W\}$ form a $\mathbb{Z}$-basis of the cohomology ring $H^*(G/B)$.

Define the *Schubert structure constant* $c^w_{u,v}(G/B)$ to be the structure constant of $H^*(G/B)$ with respect to the opposite Schubert basis $\{[Y_w] : w \in W\}$:

$$[Y_u] \cdot [Y_v] = \sum_{w\in W} c^w_{u,v}(G/B)[Y_w].$$

Let H_1 and H_2 be (connected) finite Dynkin diagrams and $\iota : H_1 \hookrightarrow H_2$ be an inclusion of diagrams; that is, a graph theoretic injection that respects arrows. Label the nodes of H_1 and H_2 by their corresponding simple roots

$$\Pi_{H_1} = \{\alpha_1, \ldots, \alpha_{r(H_1)}\} \text{ and } \Pi_{H_2} = \{\beta_1, \ldots, \beta_{r(H_2)}\}$$

such that $\iota(\alpha_i) = \beta_i$ for all $i \in [r(H_1)]$.

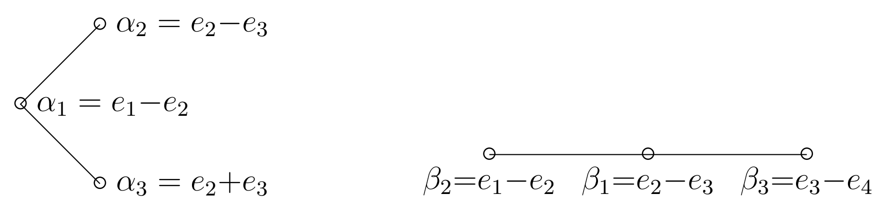
Denote W_1 and W_2 the Weyl groups corresponding to H_1 and H_2 respectively. The inclusion ι then induces an injection $\iota : W_1 \hookrightarrow W_2$ by sending s_{α_i} to s_{β_i}. Let X_1, X_2 be the generalized flag varieties corresponding to H_1, H_2 respectively.

Proposition 2.4 ([17], Theorem 2.1). *For any* $u, v, w \in W_1$,

$$c^w_{u,v}(X_1) = c^{\iota(w)}_{\iota(u),\iota(v)}(X_2).$$

Remark 2.5. We note that the convention in [17] is different from the one used in [5, 7]. The former studies the structure constant $c_{u,v}^{w}$ in the opposite Schubert basis (one where $\ell(w)$ is the codimension) whereas the later two (see also equation (4)) work in the Schubert basis (where $\ell(w)$ is the dimension). The two basis are related by $[X_w] = [Y_{w_0 w}]$ where w_0 is the longest element in the Weyl group W. As a result, Proposition 2.4 does not hold if we simply replace $c_{u,v}^{w}$ in the statement with the structure constant under Schubert basis. However, as we will see later, the inclusion of Dynkin diagram we consider here is an isomorphism and the choice of u, v are self dual. In this case we can apply Proposition 2.4 in the setting of (4).

For the purpose of this paper, we focus on the map of the Dynkin diagrams $\iota : D_3 \leftrightarrow A_3$ where we identify $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_2 + e_3$ with $\beta_1 = e_2 - e_3, \beta_2 = e_1 - e_2, \beta_3 = e_3 - e_4$ respectively.



(A) Dynkin diagram D_3 (B) Dynkin diagram A_3

FIGURE 1. Identification of Dynkin diagrams of type A_3 and D_3

Let $X_1 = SL(4)/B$ and $X_2 = SO(6)/B$ be the generalized flag varieties of type A_3 and D_3.

2.5. **Inequalities defining the saturated tensor cones.** Let $\{x_i : i \in [n]\} \subset \mathfrak{h}$ be the dual to the simple roots $\{\alpha_i : i \in [n]\}$, namely $\alpha_i(x_j) = \delta_{i,j}$. For $\mu \in \Lambda^+$, the dual of μ is $\mu^* = -w_0\mu$. The following characterization of $\mathfrak{g}$-sat is due to Berenstein-Sjamaar.

Proposition 2.6 ([5]). $(\lambda, \mu, \nu) \in \mathfrak{g}\text{-sat}$ *if and only if for all maximal parabolic subgroup* $P = P_\alpha \subset G$ *and all triple* $(u, v, w) \in (W^P)^3$ *such that product of the corresponding Schubert classes in* G/P:

$$\left[X_u^P\right] \cdot \left[X_v^P\right] \cdot \left[X_w^P\right] = k\left[X_e^P\right] \in H^*(G/P, \mathbb{Z}) \text{ for some } k > 0, \tag{4}$$

the following inequality holds:

$$\lambda(ux_P) + \mu(vx_P) + \nu^*(wx_P) \leq 0. \tag{5}$$

Here we set $x_P := x_{i_P}$ *where* $\alpha_{i_P} = \alpha$.

In the case where $\mathfrak{g} = \mathfrak{sl}_n$, a description of $\mathfrak{g}$-sat was conjectured by Horn and proved by combining the result of Klyachko [11] and of Knutson-Tao [12]. We note that the set of *Horn inequalities* and the set of inequalities (5) are the same and we encourage the readers to look at Section 4 of [14] for a detailed exposition. We include a description of the Horn inequalities below for both completeness and convenience to use in the proof of Theorem 1.3.

Let $[n] := \{1, 2, \dots n\}$. For any $I = \{i_1 < i_2 < \cdots < i_d\} \subseteq [n]$ define the partition

$$\tau(I) := (i_d - d \geq \cdots \geq i_2 - 2 \geq i_1 - 1).$$

Theorem 2.7 (Horn inequalities). *Let $\mathfrak{g} = \mathfrak{sl}_n$ and let $\lambda, \mu, \nu \in \Lambda^+_{\mathbb{Q}}$ be such that $\lambda + \mu - \nu$ lies in the root lattice. Then $(\lambda, \mu, \nu) \in \mathfrak{g}\text{-sat}$ if and only if for every $d < n$, and every triple of subsets $I, J, K \subseteq [n]$ of cardinality d such that $c^{\tau(K)}_{\tau(I),\tau(J)} > 0$,*

$$\sum_{k \in K} \nu_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j. \tag{6}$$

Belkale-Kumar [1] described a subset of inequalities (5) that characterize $\mathfrak{g}\text{-sat}$ using a deformed cohomology product $\odot$ on $H^*(G/P)$. It is proved by Ressayre [16] proved that the subset is irredundant, namely any proper subset defines a different cone. We refer the readers to Section 6 of [1] for the precise definition of $\odot$.

Theorem 2.8. [1, 16] *Proposition 2.6 still holds if we replace* (4) *with*

$$\left[X_u^P\right] \odot \left[X_v^P\right] \odot \left[X_w^P\right] = \left[X_e^P\right] \in (H^*(G/P, \mathbb{Z}), \odot). \tag{7}$$

Moreover, replacing (4) *with* (7) *yields a minimal set of inequalities describing* $\mathfrak{g}\text{-sat}$*.*

In [9], Orelowitz, Ressayre, Yong, and the first author introduced *extended Horn inequalities* in their study of tensor product multiplicities in the stable range for $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$. They contain the Horn inequalities (6) and the Belkale-Kumar inequalities when $\mathfrak{g} = \mathfrak{sp}_{2n}$ (as in Theorem 2.8) as special cases (see Section 3 of [9]). Specifically, they proved the following:

Theorem 2.9 (Extended Horn inequalities). *Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $\lambda, \mu, \nu \in \Lambda^+_{\mathbb{Q}}$. Then $(\lambda, \mu, \nu) \in \mathfrak{g}\text{-sat}$ if and only if*

$$0 \leq \sum_{a \in A} \lambda_a - \sum_{a' \in A'} \lambda_{a'} + \sum_{b \in B} \mu_b - \sum_{b' \in B'} \mu_{b'} + \sum_{c \in C} \nu_c - \sum_{c' \in C'} \nu_{c'} \tag{8}$$

for any subsets $A, A', B, B', C, C' \subset [n]$ such that

(1) $A \cap A' = B \cap B' = C \cap C' = \emptyset$;
(2) $|A| + |A'| = |B| + |B'| = |C| + |C'| = |A'| + |B'| + |C'| =: r$;
(3) the Littlewood-Richardson coefficients $c^{\tau(A)}_{\alpha_1,\alpha_2}, c^{\tau(C')}_{\alpha_2,\alpha_3}, c^{\tau(B)}_{\alpha_3,\alpha_4}, c^{\tau(A')}_{\alpha_4,\alpha_5}, c^{\tau(C)}_{\alpha_5,\alpha_6}, c^{\tau(B')}_{\alpha_6,\alpha_1} > 0$ for some partitions $\alpha_1, \ldots, \alpha_6$.

3. Proof of Theorem 1.3

Fix $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+$. We begin with a useful lemma that holds in type A through D.

Lemma 3.1. *If $\lambda_m + \lambda_{m+2} \neq 2\lambda_{m+1}$ for some $m \in [n-2]$ then $\mathrm{TP}_\lambda \neq \mathrm{IP}_{2\lambda}$.*

Proof. Let $\mu = v_{\{m,m+1\}}(\lambda)$ as in Theorem 2.1. More explicitly,

$$\mu_j = \begin{cases} \frac{2(\lambda_m + \lambda_{m+1} + \lambda_{m+2})}{3} & \text{if } m \leq j \leq m+2 \\ 2\lambda_j & \text{otherwise} \end{cases}. \tag{9}$$

Notice that for any $\mathfrak{g}$ of classical Lie type, μ is a vertex of $\mathrm{IP}_{2\lambda}$ and $|\mu| = 2|\lambda|$.

We first study the case where $\mathfrak{g} = \mathfrak{sl}_n$. Consider the inequalities

$$\sum_{k \in [m+2] \setminus \{m\}} \mu_k \leq \sum_{i \in [m+2] \setminus \{m+1\}} \lambda_i + \sum_{j \in [m+2] \setminus \{m+1\}} \lambda_j, \tag{10}$$

and

$$\sum_{k\in[m+2]\setminus\{m,m+1\}} \mu_k \leq \sum_{i\in[m+2]\setminus\{m,m+2\}} \lambda_i + \sum_{j\in[m+2]\setminus\{m,m+2\}} \lambda_j. \tag{11}$$

Observe that both (10) and (11) are cases of Horn inequalities (6) for the triple (λ, λ, μ). Indeed, in the case of (10), we have $\tau(K) = (1,1), \tau(I) = \tau(J) = (1)$, and in (11), we have $\tau(K) = (2), \tau(I) = \tau(J) = (1)$. In both cases, $c_{\tau(I),\tau(J)}^{\tau(K)} = 1 > 0$. Since $\mu_j = 2\lambda_j$ for all $j \notin [m, m+2]$, we can rewrite (10) and (11) as

$$\mu_{m+1} + \mu_{m+2} \leq 2(\lambda_m + \lambda_{m+2}) \tag{7'}$$

and

$$\mu_{m+2} \leq 2\lambda_{m+1}. \tag{8'}$$

By (9), Equation (7′) holds if and only if $\lambda_m + \lambda_{m+2} \geq 2\lambda_{m+1}$, and Equation (8′) holds if and only if $\lambda_m + \lambda_{m+2} \leq 2\lambda_{m+1}$. Since $\lambda_m + \lambda_{m+2} \neq 2\lambda_{m+1}$, (7′) and (8′) cannot hold simultaneously, and thus $\mu \notin \mathrm{TP}_\lambda$. Since μ is a vertex of $\mathrm{IP}_{2\lambda}$, we have $\mathrm{TP}_\lambda \neq \mathrm{IP}_{2\lambda}$ when $\mathfrak{g} = \mathfrak{sl}_n$.

For $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$, we use Lemma 2.3. Let $\lambda \in \Lambda^+$ and let μ be constructed as in (9). Notice that μ is always a vertex of the intersection polytope $\mathrm{IP}_{2\lambda}$, and

$$\sum_{i\in[n]} 2\lambda_i = \sum_{i\in[n]} \mu_i.$$

Since there is some $d \in \mathbb{Z}_{>0}$ such that $d\lambda_i$ and $d\mu_i$ are integers for all $i \in [n]$, $d\lambda + d\lambda - d\mu$ lies in the root lattice of $\mathfrak{sl}_n$. By (3),

$$c_{dk\lambda,dk\lambda}^{dk\mu}(\mathfrak{g}) = c_{dk\lambda,dk\lambda}^{dk\mu}(\mathfrak{sl}_n) = 0,$$

for all $k \in \mathbb{Z}_{>0}$. As a result, $dk\mu \notin \mathrm{TP}_{dk\lambda}$ for any $\mathfrak{g}$ of classical lie type. Since the inequalities defining TP_λ (See equation (5)) are linear, we can conclude that $\mu \notin \mathrm{TP}_\lambda$. □

Proof of Theorem 1.3: We first prove the ($\Longrightarrow$) direction. By Lemma 3.1, we are left to show that for $\mathfrak{g} = \mathfrak{so}_{2n}$, $\mathrm{IP}_{2\lambda} \neq \mathrm{TP}_\lambda$ for all $\lambda = N\rho + k\omega_n$ with $k \neq 0$.

Claim 3.2. *0 is a vertex of $\mathrm{IP}_{2\lambda}$.*

Proof of Claim 3.2: This follows directly from Theorem 2.1 by setting $I = [n]$. □

We wish to show that $0 \notin \mathrm{TP}_\lambda$. By Proposition 2.6, it is enough to show the existence of $u, w \in W^P$ such that

$$\lambda(ux_P) > 0 \text{ and } [X_u^P] \cdot [X_u^P] \cdot [X_w^P] = k[X_e^P] \in H^*(SO(2n)/B) \text{ for some } k > 0.$$

Consider first the case $n = 3$. Let P be the maximal parabolic subgroup associated to the simple root $\alpha_1 = e_1 - e_2$ as in Figure 1(A). Define $u_1 = s_3s_1, u_2 = s_2s_1 \in W^P$ and set w_0^P be the minimal W_P-coset representative of the longest permutation w_0. In one line notation, $u_1 = -3\ 1\ {-2}$ and $u_2 = 312$. Let $X_1 = SO(6)/B$ and $X_2 = SL(4)/B$.

Claim 3.3. *$c_{u_i,u_i}^{w_0^P}(X_1) = 1$ and thus $[X_{u_i}^P] \cdot [X_{u_i}^P] \cdot [X_{w_0^P}^P] = [X_e^P] \in H^*(SO(6)/P)$ for $i \in \{1, 2\}$.*

Proof of Claim 3.3: Consider the identification of Dynkin diagrams of type A_3 and D_3 as in Figure 1. Then $\iota(u_1) = s_3s_2 = 1423 \in S_4, \iota(u_2) = s_1s_2 = 2314 \in S_4$ and $\iota(w_0^P) = 3412 \in S_4$. Since the Weyl group of type A_3 and D_3 are isomorphic under the map ι, we have $\iota(w_0) =$

w_0 and thus the difference in convention as mentioned in Remark 2.5 is not an issue. We can then apply Proposition 2.4 to get

$$c^{w_0^P}_{u_1,u_1}(X_1) = c^{3412}_{1423,1423}(X_2) = 1$$

and

$$c^{w_0^P}_{u_2,u_2}(X_1) = c^{3412}_{2314,2314}(X_2) = 1$$

Since $u_1, u_2, w_0^P \in W^P$, we have

$$[Y^P_{u_i}] \cdot [Y^P_{u_i}] \cdot [Y^P_e] = [Y^P_{w_0^P}] \in H^*(SO(6)/P) \text{ for } i \in \{1,2\}.$$

where $[Y^P_u]$ is the opposite Schubert class dual to $[X^P_u]$ for $u \in W^P$. Notice that both $[Y^P_{u_i}]$ are both self dual. We can therefore conclude that

$$[X^P_{u_i}] \cdot [X^P_{u_i}] \cdot [X^P_{w_0^P}] = [X^P_e] \text{ for } i \in \{1,2\}.$$ □

Suppose $\lambda = N\rho + k\omega_n$ for some nonzero $k \in \mathbb{Z}_{\geq -N}$. If $k > 0$, then

$$\lambda(u_2 x_P) = \langle (2N + \frac{k}{2}, N + \frac{k}{2}, \frac{k}{2}), (0,0,1) \rangle = \frac{k}{2} > 0.$$

If $k < 0$, then

$$\lambda(u_1 x_P) = \langle (2N + \frac{k}{2}, N + \frac{k}{2}, \frac{k}{2}), (0,0,-1) \rangle = -\frac{k}{2} > 0.$$

Therefore $\mathrm{IP}_{2\lambda} \neq \mathrm{TP}_\lambda$ if $k \neq 0$ and we conclude the ($\Longrightarrow$) direction in type D_3. For general n, consider the projection as in Section 2.3 where $\mathfrak{g}'$ is of type D_3. For $\lambda = N\rho + k\omega_n$ with $k \in \mathbb{Z}_{\geq -N}$, if $c^0_{m\lambda,m\lambda} > 0$, then by Lemma 2.3,

$$c^0_{mp(\lambda),mp(\lambda)}(\mathfrak{g}') > 0.$$

By Lemma 3.1 of [8], as a dominant integral weight of $\mathfrak{g}'$, $p(\lambda) = N\omega_1 + N\omega_2 + (N+k)\omega_3$ is of the form $(2N + \frac{k}{2}, N + \frac{k}{2}, \frac{k}{2})$ as seen above. Therefore

$$c^0_{m\lambda,m\lambda}(\mathfrak{g}) = c^0_{mp(\lambda),mp(\lambda)}(\mathfrak{g}') = 0 \text{ unless } k = 0.$$

By Claim 3.2, we conclude that $\mathrm{IP}_{2\lambda} \neq \mathrm{TP}_\lambda$ for all $\lambda \neq N\rho$ when $\mathfrak{g} = \mathfrak{so}_{2n}$.

($\Longleftarrow$) : We divide into two cases:

Case I ($\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{so}_{2n}$): By Corollary 2.2, if μ is a vertex of $\mathrm{IP}_{2N\rho}$, then

$$\mu = \sum_{i=1}^{N} \mu^{(i)},$$

where each $\mu^{(i)}$ is a vertex of $\mathrm{IP}_{2\rho}$. In particular, $2\rho \geq \mu^{(i)}$ for all $i \in [N]$. By Theorem 3 of [7], $\mu^{(i)} \in \mathrm{TP}_\rho$ and thus inequalities in (5) are satisfied. Since these inequalities are linear, $(N\rho, N\rho, \mu)$ also satisfy all inequalities in (5). Therefore $\mu \in \mathrm{TP}_{N\rho}$ and we conclude that $\mathrm{IP}_{2N\rho} = \mathrm{TP}_{N\rho}$ for all $N \in \mathbb{Z}_{>0}$.

Case II ($\mathfrak{g} = \mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$): By Theorem 4.1 and 4.2 of [15] (See also Corollary 7.5 of [13]), the saturated tensor cone and thus the tensor polytope are the same for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$ where we embed the root lattice of $\mathfrak{sp}_{2n}$ into the root lattice of $\mathfrak{so}_{2n+1}$ by sending e_i to e_i using the coordinate as defined in Section 2.1.

We first prove that if $\lambda = \rho - \omega_n = (n-1, n-2, \ldots, 1, 0)$, then $\mathrm{IP}_{2\lambda} = \mathrm{TP}_\lambda$. The proof here mainly follows the proof strategy of Theorem 3 in [7]. Let W be the signed permutation group on $[n]$. This is the Weyl group for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$. For $v \in W$, let $v_1 \dots v_n$ where $v_i = v(i)$ be the one-line notation of u. Let $P = P_r, v \in W_P$ and $\mu \in \Lambda_{\mathbb{Q}}^+$, then

$$\mu(vx_P) = \sum_{i=1}^{r} \mu_{v_i}, \tag{12}$$

where we set $\mu_i := -\mu_{-i}$ if $i < 0$. Let (u, v, w) be any triple satisfying (7).

Claim 3.4. $(\omega_n + u^{-1}\omega_n + v^{-1}\omega_n + w^{-1}\omega_n)(x_P) = 0$

Proof of Claim 3.4: We prove this claim in the case $\mathfrak{g} = \mathfrak{sp}_{2n}$. The case $\mathfrak{so}_{2n+1}$ follows immediately as the n-th fundamental weight is exactly half the n-th fundamental weight of $\mathfrak{sp}_{2n}$ under the natural embedding of the root lattice. Since u, v, w satisfy (7), by Theorem 2.9, the inequality (5) is an extended Horn inequality and can be written in the form of (8). Since the longest element $w_0 \in W$ is -1, $\nu^* = -w_0\nu = \nu$ for all $\nu \in \Lambda^+$. Using (12), we can write

$$A = \{u_i : i \in [r], u_i < 0\}, A' = \{u_i : i \in [r], u_i > 0\},$$

and similarly for the sets B, B', C, C'. Therefore

$$(u^{-1}\omega_n)(x_P) = |A'| - |A|, (v^{-1}\omega_n)(x_P) = |B'| - |B| \text{ and } (w^{-1}\omega_n)(x_P) = |C'| - |C|.$$

By condition (2) in Theorem 2.9,

$$(\omega_n + u^{-1}\omega_n + v^{-1}\omega_n + w^{-1}\omega_n)(x_P) = r + |A'| + |B'| + |C'| - |A| - |B| - |C| = 0. \qquad \square$$

By Equation (5) of [7]

$$(\rho + u^{-1}\rho + v^{-1}\rho + w^{-1}\rho)(x_P) \leq 0$$

for all triples (u, v, w) satisfying (7). Combining with Claim 3.4, we have

$$(\lambda + u^{-1}\lambda + v^{-1}\lambda + w^{-1}\lambda)(x_P) \leq 0, \tag{13}$$

with $\lambda = \rho - \omega_n$ as defined above. Let $\mu = v_I(2\lambda)$ be any vertex of $\mathrm{IP}_{2\lambda}$. Here we will not distinguish $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$ since their corresponding Weyl groups are the same and thus the vertices of the intersection polytope $\mathrm{IP}_{2\lambda}$ are the same by Theorem 2.1.

Claim 3.5. $\lambda(ux_P) + \lambda(vx_P) + \mu^*(wx_P) \leq (\lambda + u^{-1}\lambda + v^{-1}\lambda + w^{-1}\lambda)(x_P)$.

Proof of Claim 3.5: Since $\mu^* = \mu$, it is enough to show that $w^{-1}(\mu - \lambda) \leq \lambda$. Let $W_I \subseteq W$ be the subgroup generated by $\{s_{\alpha_i} : i \in I\}$ and let w_0^I be the longest element in W_I. Since $\lambda = (n-1, n-2, \dots, 1, 0)$, by Theorem 2.1,

$$\mu - \lambda = w_0^I\lambda.$$

Since $\lambda \in \Lambda^+$,

$$w^{-1}(\mu - \lambda) = w^{-1}w_0^I(\lambda) \leq \lambda. \qquad \square$$

Combining Claim 3.5 and (13), (λ, λ, μ) satisfy inequalities in (5). Therefore $(\lambda, \lambda, \mu) \in \mathfrak{g}\text{-sat}$ and $\mathrm{IP}_{2\lambda} = \mathrm{TP}_\lambda$ when $\lambda = \rho - \omega_n$.

We now prove $\mathrm{IP}_{2\omega_n} = \mathrm{TP}_{\omega_n}$ in the case $\mathfrak{g} = \mathfrak{sp}_{2n}$. The $\mathfrak{so}_{2n+1}$ case follows from the same reasoning. Let μ be any vertex of $\mathrm{IP}_{2\omega_n}$, by Theorem 2.1, $\mu = (2, \dots, 2, 0, \dots, 0) = 2\omega_k$ for some $k \in [n]$. To prove $\mu \in \mathrm{TP}_{\omega_n}$, it is enough to show that $(\omega_n, \omega_n, 2\omega_k)$ satisfy the Extended Horn inequalities as in (8). Indeed,

$$\begin{aligned}
0 &\leq |C'| + |C'| + 2|C \cap [k]| - 2|C' \cap [k]| \\
&= |A| - |B'| + |B| - |A'| + 2|C \cap [k]| - 2|C' \cap [k]| \\
&= |A| - |A'| + |B| - |B'| + 2|C \cap [k]| - 2|C' \cap [k]| \\
&= \sum_{a \in A} (\omega_n)_a - \sum_{a' \in A'} (\omega_n)_{a'} + \sum_{b \in B} (\omega_n)_b - \sum_{b' \in B'} (\omega_n)_{b'} + \sum_{c \in C} (2\omega_k)_c - \sum_{c' \in C'} (2\omega_k)_{c'}
\end{aligned}$$

as we desired.

We now finish the ($\Longleftarrow$) direction when $\mathfrak{g} = \mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$. For any λ of the form $N\rho + k\omega_n$ where $k \in \mathbb{Z}_{\geq -N}$, we can write $\lambda = N(\rho - \omega_n) + (k+N)\omega_n$. Since $\mathrm{IP}_{2(\rho-\omega_n)} = \mathrm{TP}_{\rho-\omega_n}, \mathrm{IP}_{2\omega_n} = \mathrm{TP}_{\omega_n}$, by Corollary 2.2 and linearity of the defining inequalities of $\mathfrak{g}$-sat, we conclude that $\mathrm{IP}_{2\lambda} = \mathrm{TP}_\lambda$. $\square$

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REFERENCES

[1] Prakash Belkale and Shrawan Kumar. Eigenvalue problem and a new product in cohomology of flag varieties. *Invent. Math.*, 166(1):185–228, 2006.

[2] Prakash Belkale and Shrawan Kumar. Eigencone, saturation and Horn problems for symplectic and odd orthogonal groups. *J. Algebraic Geom.*, 19(2):199–242, 2010.

[3] A. D. Berenstein and A. V. Zelevinsky. Triple multiplicities for $\mathrm{sl}(r+1)$ and the spectrum of the exterior algebra of the adjoint representation. *J. Algebraic Combin.*, 1(1):7–22, 1992.

[4] A.D. Berenstein and A.V. Zelevinsky. Tensor product multiplicities and convex polytopes in partition space. *Journal of Geometry and Physics*, 5(3):453–472, 1988.

[5] Arkady Berenstein and Reyer Sjamaar. Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion. *J. Amer. Math. Soc.*, 13(2):433–466, 2000.

[6] Marc Besson, Sam Jeralds, and Joshua Kiers. Vertices of intersection polytopes and rays of generalized Kostka cones. *J. Lie Theory*, 31(4):1055–1070, 2021.

[7] Rocco Chirivì, Shrawan Kumar, and Andrea Maffei. Components of $V(\rho) \otimes V(\rho)$. *Transform. Groups*, 22(3):645–650, 2017.

[8] Shiliang Gao and Yibo Gao. Degrees of the stretched kostka quasi-polynomials, 2022.

[9] Shiliang Gao, Gidon Orelowitz, Nicolas Ressayre, and Alexander Yong. Newell-littlewood numbers iii: eigencones and git-semigroups, 2022.

[10] Jiuzu Hong and Linhui Shen. Tensor invariants, saturation problems, and Dynkin automorphisms. *Adv. Math.*, 285:629–657, 2015.

[11] Alexander A. Klyachko. Stable bundles, representation theory and Hermitian operators. *Selecta Math. (N.S.)*, 4(3):419–445, 1998.

[12] Allen Knutson and Terence Tao. The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.*, 12(4):1055–1090, 1999.

[13] Shrawan Kumar. Tensor product decomposition. In *Proceedings of the International Congress of Mathematicians. Volume III*, pages 1226–1261. Hindustan Book Agency, New Delhi, 2010.

[14] Shrawan Kumar. A survey of the additive eigenvalue problem. *Transform. Groups*, 19(4):1051–1148, 2014. With an appendix by M. Kapovich.

[15] Shrawan Kumar and John R. Stembridge. Special isogenies and tensor product multiplicities. *Int. Math. Res. Not. IMRN*, (20):Art. ID rnm081, 13, 2007.
[16] N. Ressayre. Geometric invariant theory and the generalized eigenvalue problem. *Invent. Math.*, 180(2):389–441, 2010.
[17] Colleen Robichaux, Harshit Yadav, and Alexander Yong. The A·B·C·Ds of Schubert calculus. *Sém. Lothar. Combin.*, 85:Art. B85a, 12, [2020–2021].
[18] Steven V. Sam. Symmetric quivers, invariant theory, and saturation theorems for the classical groups. *Adv. Math.*, 229(2):1104–1135, 2012.

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