Coconuts

Dinglong Wang

April 27, 2023

The motivation for this note is to understand comodule better. The author encountered something inexplicable in the appendix 1 of [Rav04]. Some of costuffs are listed below.

Contents

1 Comodule

1.1 Definitions

Here are the definitions which involve inevitable abstract nonsense. Because of the nature of the definition of comodule, it is the categorical dual and the best way to deal with them seems to be that first handle the familiar side (but in a categorical fashion), then reverses all the arrows.

So, fix a commutative ring K, let $\mathcal{C}$ denotes the category $grMod_K^{op}$. In fact, any monoidal category would work for the definition part. Moreover, we will abuse the notations for module and comodule if there is no ambiguity.

Definition 1.1. *(C,Δ,ϵ) is a coalgebra over K if it is a monoid in $\mathcal{C}$. (M,ψ) is a (left) comodule over (C,Δ,ϵ) if in addition, it possesses a coaction map $\psi : M \to C \otimes M$, which needs to be coassociative and counital, i.e. it is a module over the monoid C in $\mathcal{C}$. We will abbreviate the tuples to C and M whenever possible.*

This seems weird and hard to deal with if one only has training in algebra. More concretely, it is not an abelian category in the general case. Let's first begin by discussing examples with mixed characteristics: Hopf algebras.

Definition 1.2. *(Hopf algebra, not yet finished)*

1.2 (Co)module?

The main reference for this subsection is [MM65]

Definition 1.3 ((co)tensor product of (co)modules)**.** *Let A be a K-algebra, if M is a right A-module and N is a left A-module, then the tensor product $M \otimes_A N$ is defined to be the cokernel of the map $\phi_M \otimes N - M \otimes \phi_N$ in $\mathcal{C}$, i.e. the following is an exact sequence:*

$$M \otimes A \otimes N \xrightarrow{\phi_M \otimes N - M \otimes \phi_N} M \otimes N \longrightarrow M \otimes_A N \longrightarrow 0$$

Dually, let A be a K-coalgebra, if M is a right A-comodule and N is a left A-comodule, then the cotensor product $M \Box_A N$ is defined to be the kernel of the map $\psi_M \otimes N - M \otimes \psi_N$ in $\mathcal{C}$, i.e. the following is an exact sequence:

$$0 \longrightarrow M \Box_A N \longrightarrow M \otimes N \xrightarrow{\psi_M \otimes N - M \otimes \psi_N} M \otimes A \otimes N$$

Definition 1.4 (augmentation for (co)algebra)**.** *An augmentation map for an algebra A over K is an algebra map $\epsilon : A \to K$. And, we define the "reduced" algebra $\bar{A}$ to be ker ϵ.*

Dually, an augmentation map for a coalgebra A over K is a coalgebra map $\eta : K \to A$. And, we define the "reduced" coalgebra co$\bar{A}$ to be coker η.

Definition 1.5 (connected (co)algebra)**.** *A K-algebra A is connected if $\eta : K \to A_0$ is an isomorphism.*

Dually, a K-coalgebra A is connected if $\epsilon : A_0 \to K$ is an isomorphism.

Here I abuse notations heavily, the same symbols ϵ and η in respectively two definitions are completely unrelated. Note that a connected (co)algebra has a unique augmentation map. And the corresponding map makes K into a right A-(co)module.

Proposition 1.6. *Let A be a connected K-algebra and N be a left A-module, then $N = 0 \iff K \otimes_A N = 0$.*

Dually, let A be a connected K-coalgebra and N be a left A-comodule, then $N = 0 \iff K \Box_A N = 0$.

Proof. We only prove the first statement. The latter follows by changing every words and symbols, if possible, to their duals. If $N = 0$, certainly $K \otimes_A N = 0$. Since A is a connected K-algebra, $\bar{A}_0 = 0$. Plugging in the definition with $M = K$ and $\phi_M = \epsilon$ the unique augmentation map, we see that $\bar{A} \otimes N \to N$ is an epimorphism. Now we induct on the degree q where N vanishes below. The base case is obvious. Suppose $N_k = 0$, $\forall k \leq q$, Then, $(\bar{A} \otimes N)_k = 0$, $\forall k \leq q + 1$. Therefore, $N_k = 0$, $\forall k \leq q + 1$ because of the epimorphism. □

Until now, all the things work out dually. But this duality will break if we want to consider abelian category, which is the presuming background for homological algebra. The fundamental reason is that the tensor product in $\mathcal{C}$ is right exact instead of exact. This has the effect that the category of left A-modules is an abelian category. But the category of left A-comodules is **not** an abelian category in general. finish this subsection. End with structure theorem.

1.3 Homological Algebra

In this subsection, we assume in the Hopf algebroid (A, Γ), Γ is flat over A as left A-module. So that the category of left Γ-comodules, call it $\mathbf{Comod_\Gamma}$, is abelian.

Proposition 1.7. *There is an adjoint pair of functors*

$$\begin{tikzcd} \mathbf{Mod_A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{Comod_\Gamma} \arrow[l, bend left, "U"] \end{tikzcd}$$

where $F(M) := \Gamma \otimes_A M$ with comultiplication induced by the diagonal map and U is the forgetting functor using the left A-module structure on Γ.

Proposition 1.7 provides a way to produce injectives in our category: take M to be an injective left A-module, F(M) is also an injective left Γ-comodule. Now, we can define some derived functors in $\mathbf{Comod_\Gamma}$.

Remark. *This adjunction (F left adjoint to U) is true for all modules over a monoid and comodules over a comonoid in arbitrary monoidal category.*

Definition 1.8 (Ext and Cotor)**.**

$$Ext^i_\Gamma(M, N) := R^i Hom_\Gamma(M, N)$$

$$Cotor^i_\Gamma(M, N) := R^i M \Box_\Gamma N$$

Develop relative injectives and cobar complex.

1.4 When is a comodule category equivalent to a module category?

In particular, we are asking the category of left A-comodules to be abelian since the abelianness is an invariant property under categorical equivalence. `https://mathoverflow.net/questions/94115/when-a-comodule-category-is-equivalent-to-a-module-category`.

References

[MM65] J. W. Milnor and J. C. Moore. On the structure of hopf algebras. *Annals of Mathematics, 81(2), 211–264*, 1965.

[Rav04] D. C. Ravenel. Complex cobordism and stable homotopy groups of spheres. *AMS Chelsea Pub*, 2004.