# Completing the square

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## 1 Introduction

Completing the square is a method of solving quadratic equations, it is not a formula. Yet, we still get presented with the funky equation $C = (\frac{B}{2})^2$. What does the equation mean? Why does it work? Why am I tensing this up? All of these questions shall be answered today, with the latter being the exception.

## 2 The Algebraic Proof

Let's get straight to the proof, first we have to consider what the point of completing the square is. What completing the square does is, it makes the equation into a squared form, allowing us to factor the equation. For example, if we have something like $x^2 + 6x = 16$, we can complete the square by adding 9 to both sides.

$$x^2 + 6x = 16 \tag{1}$$
$$x^2 + 6x + 9 = 25 \tag{2}$$
$$(x + 3)^2 = 25 \tag{3}$$
$$x + 3 = 5 \tag{4}$$
$$\implies x = 2 \tag{5}$$

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As you can see, we were able to solve $x$ by modifying the equation so that it becomes factor-able. In other words, we want our equation to go from $x + Bx + C$ to $(x + a)^2$. **We just need to find** $a$. Luckily, we can do that by correlating our current equation $(x + Bx + C)$ to our preferred equation $((x + a)^2)$. Since the simplified version of $(x + a)^2$ equals $x^2 + 2ax + a^2$, we can start matching the equations together. Just look at the two your self.

$$x^2 + Bx + C$$
$$x^2 + 2ax + a^2$$

In fact they are so similar, that we can just say $Bx = 2ax$. This might sound crazy, but if you think about it, it's completely logical. If we can agree that $x^2 + Bx + C = x^2 + 2ax + a^2$, then we can agree that $Bx = 2ax$. Why can't we say that $B$ instead equals $x^2$ or $a^2$? It's because of the mighty power of $x$. Let's go by order, $Bx \neq x^2$ because $x^2$ has the power of 2, while $Bx$ is only to the power of 1. $Bx = 2ax$ because both are to the power of 1. $Bx \neq a^2$ because $a^2$ doesn't even have an x. Shame on $a^2$. Anyways, now that we have established why $Bx = 2ax$, we can now solve for $a$!

$$Bx = 2ax \tag{6}$$
$$B = 2a \tag{7}$$
$$\implies \frac{B}{2} = a \tag{8}$$

And there you have it, that is completing the square. It's that simple. Let's solve another completing the square to problem to really show you that all you needed to know to factor each equation is $a$. Let's solve the following equation: $x^2 + x - 3.75 = 0$. Keep in mind that $C = a^2$

$$x^2 + x - 3.75 = 0 \qquad a = \frac{B}{2} \tag{9}$$
$$x^2 + x = 3.75 \qquad a = \frac{1}{2} \tag{10}$$
$$x^2 + x + (\frac{1}{2})^2 = 3.75 + (\frac{1}{2})^2 \tag{11}$$
$$(x + \frac{1}{2})^2 = 4 \tag{12}$$
$$x + \frac{1}{2} = \pm 2 \tag{13}$$
$$\implies x = \pm 2 - 0.5 \tag{14}$$

Just from knowing $a$, we were able to solve for $x$ by factoring. But what if we wanted to go further? Imagine if we can make an equation where we can just plug everything into and get the answer for $x$, wouldn't that be wonderful?

## 3 Quadratic Equation

That is right, you can get the quadratic equation using completing the square! We have covered everything you need to know to derive it, so I recommend you try that on your own. Otherwise, here is the full proof for the quadratic equation.

$$Ax^2 + Bx + C = 0 \tag{15}$$
$$Ax^2 + Bx = -C \tag{16}$$
$$x^2 + \frac{B}{A}x = -\frac{C}{A} \tag{17}$$
$$x^2 + \frac{B}{A}x + (\frac{B}{2A})^2 = (\frac{B^2}{4A^2}) - \frac{C}{A} \tag{18}$$
$$(x + \frac{B}{2A})^2 = \frac{B^2 - 4AC}{4A^2} \tag{19}$$
$$x + \frac{B}{2A} = \frac{\pm\sqrt{B^2 - 4AC}}{2A} \tag{20}$$
$$\implies x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{21}$$

Using the quadratic equation to solve for $k$ in $50k^2 - 20k - 6 = 0$:

$$50k^2 - 20k - 6 = 0 \tag{22}$$
$$k = \frac{20 \pm \sqrt{20^2 - 4 * 50 * -6}}{2 * 50} \tag{23}$$
$$k = \frac{20 \pm \sqrt{1600}}{100} \tag{24}$$
$$\implies k = \frac{1 \pm 2}{5} \tag{25}$$

Well isn't that much nicer? Compared to what we solved above using completing the square, I would say this is much faster. Some people call it laziness, I personally call it efficiency.

# 4 The Visual Proof

In this section, you are gonna finally see where the name comes from, completing the square. We are going to geometrically model the equation $x^2 + 2x = -1$

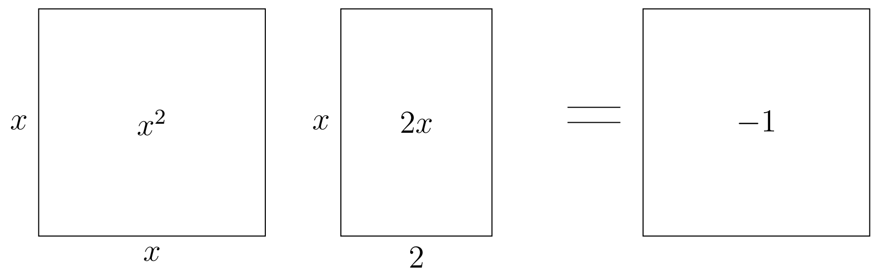


Figure 1: Setting up the equation – geometrically

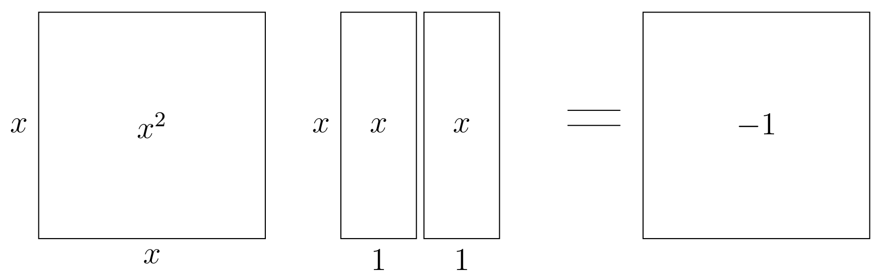


Figure 2: Split the Bx rectangle into two

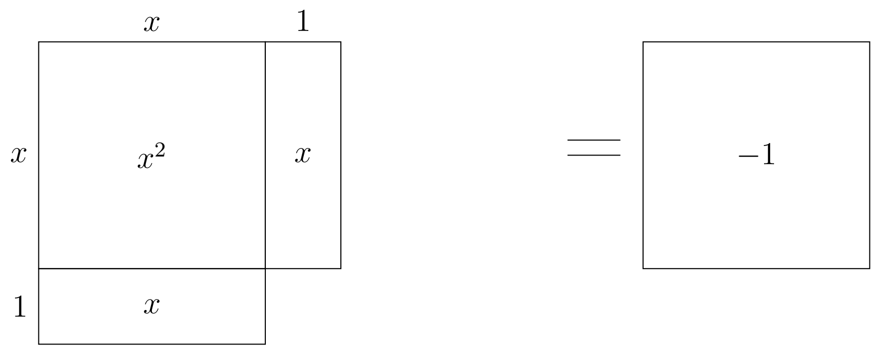


Figure 3: Transformed and rotated the two new rectangles

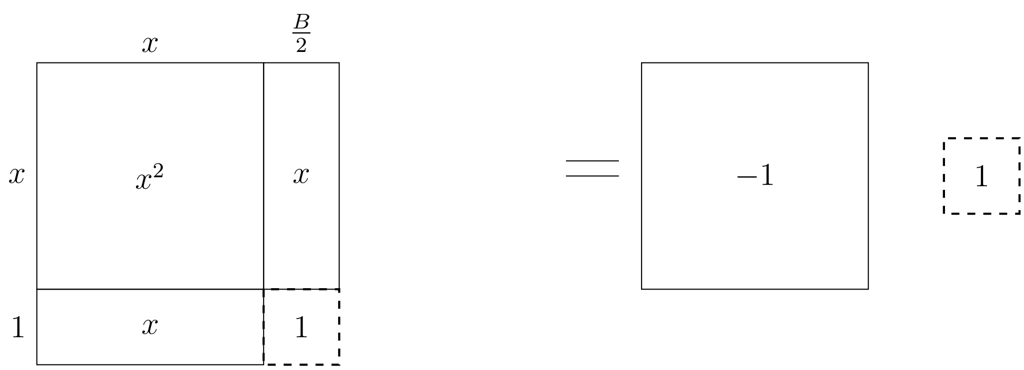


Figure 4: Filling in the piece, to finally complete the square!

Now that we have completed the square, we can figure out the length of a side to figure out $x$. So it would look something like $(x+1)^2 = -1+1$. Or if you would like a more abstract square:

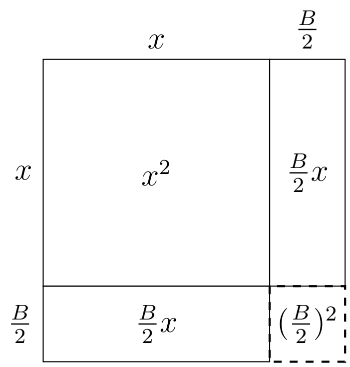


Figure 5: A very abstract square!

I think this is the best way to show completing the square, as it is true to its own name and it's most simple. It's also visual, and most people are better at comprehending things through visuals than by just seeing numbers on paper.