GU4062 Analysis II
Homework 2

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Problem 2. Prove that if $\sum_{n=0}^{\infty} b_n$ converges, then the series $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converges uniformly on $[0, 1]$.

Proof. Let $f_n(x) = \sum_{n=0}^{\infty} b_n x^n$, and $S_n = \sum_{n=0}^{\infty} b_n$, then for $m > n$,

$$
\begin{aligned}
f_m(x) - f_n(x) &= \sum_{k=n+1}^{m} b_k x^k \\
&= \sum_{k=n+1}^{m} (S_k - S_{k-1}) x^k \\
&= \sum_{k=n+1}^{m} S_k x^k - \sum_{k=n+1}^{m} S_{k-1} x^k \\
&= \sum_{k=n+1}^{m} S_k x^k - \sum_{k=n}^{m-1} S_k x^{k+1} \\
&= S_m x^m - S_n x^{n+1} + \sum_{k=n+1}^{m-1} (S_k x^k - S_k x^{k+1}), \text{separating the terms when k=n and m} \\
&= S_m x^m - S_n x^{n+1} + S_n x^{n+1} - S_n x^{n+1} + S_n x^{n+2} - S_n x^{n+2} + S_n x^{n+3} - ... \\
&+ S_n x^m - S_n x^m + \sum_{k=n+1}^{m-1} (S_k x^k - S_k x^{k+1}) \\
&= S_m x^m - S_n x^m + \sum_{k=n+1}^{m-1} S_n(-x^k + x^{k+1}) + \sum_{k=n+1}^{m-1} S_k (x^k - x^{k+1}) \\
&= (S_m - S_n) x^m + (1 - x) \sum_{k=n+1}^{m-1} (S_k - S_n) x^k.
\end{aligned}
$$

Since $\sum_{n=0}^{\infty} b_n$ converges, fix an $\epsilon > 0$, $\exists N = N(\epsilon)$ s.t. for $m, n > N$, $|S_m - S_n| < \frac{\epsilon}{17}$. Then

$$
\begin{aligned}
|f_m - f_n| &\le |(S_m - S_n)x^m| + \left|(1-x)\sum_{k=n+1}^{m-1}(S_k - S_n)x^k\right| \\
&\le \frac{\epsilon}{17}x^m + (1-x)\sum_{k=n+1}^{m-1} x^k\,|S_k - S_n| \\
&\le \frac{\epsilon}{17} + (1-x)\sum_{k=n+1}^{m-1}\frac{\epsilon}{17}x^k \\
&\le \frac{\epsilon}{17} + \frac{x^{n+1}(1-x^{m-n-1})}{1-x}(1-x)\frac{\epsilon}{17}, \text{by geometric series sum formula} \\
&\le \epsilon, \forall x \in [0,1].
\end{aligned}
$$

Thus, $f_n(x)$ is uniformly Cauchy, then $g(x) = \lim_{n\to\infty} f_n(x)$ uniformly converges on $[0,1]$. □

Problem 5. Prove that if

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, |x-x_0| < R$$

converges, and F is an antiderivative of f on $(x_0 - R, x_0 + R)$, then

$$F(x) = C + \sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-x_0)^{n+1}, |x-x_0| < R,$$

where C is a constant.

Proof. Since given F being an antiderivative of $f(x)$ on $(x_0 - R, x_0 + R)$, then F is continuous on this interval, and

$$F(x) = \int_{x_0}^{x}\sum_{n=0}^{\infty} a_n (t-x_0)^n dt + C.$$

This integral exists because $f(x)$ is well-defined.
Define the partial sum $S_n = \sum_{n=0}^{N} a_n (x-x_0)^n$, then S_n converges uniformly to $f(x)$ as $n \to \infty$, since $f(x)$ converges. That is,

$$
\begin{aligned}
F(x) &= \int_{x_0}^{x}\lim_{N\to\infty}\sum_{n=0}^{N} a_n (t-x_0)^n dt + C \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\int_{x_0}^{x} a_n (t-x_0)^n dt + C, \text{since } f(x) \text{ uniformly converges on } [x_0, x] \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{a_n}{n+1}(x-x_0)^{n+1} + C \\
&= \sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-x_0)^{n+1} + C
\end{aligned}
$$

□