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VILAB

数字图像处理实验二补充材料

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内容

- 大部分数学变换都可以用矩阵的形式来表示
  - 例如DFT、DCT、DWT等
    - 都属于采用张量积的形式从一维扩展到多维
  - 但有专门设计的二维和多维数学变换，这时无法将其二维变换通过一维变换表示出来，称为不可分离变化

DFT

- DFT

For a vector $\mathbf{x} = (x_0, x_1, x_2, \cdots, x_{N-1})$ with N components, if $\omega = e^{2\pi i/N}$,

Then its DFT is :

$$X = \begin{pmatrix} (\omega^0)^0 & (\omega^0)^1 & & (\omega^0)^{N-2} & (\omega^0)^{N-1} \\ (\omega^1)^0 & (\omega^1)^1 & & (\omega^1)^{N-2} & (\omega^1)^{N-1} \\ (\omega^2)^0 & & \cdot & & \\ & & \cdot & & \\ & & & \cdot & \\ (\omega^{N-1})^0 & (\omega^{N-1})^1 & & (\omega^{N-1})^{N-2} & (\omega^{N-1})^{N-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{N-1} \end{pmatrix}$$

$$F_N$$

DFT-basis vector

- Let $\mathbf{v}_k = (1, \omega^k, \omega^{2k}, \cdots, \omega^{(N-1)k}), k = 0, 1, \cdots, N-1$ then X is a combination of these special basis vectors
- In fact, the vectors $\mathbf{v}_k$ are the columns of the Fourier matrix

  $F = F_N$. It should be noted that these columns are orthogonal.

  Hence, $F^{-1} = F^* / \|\mathbf{v}_k\|^2 = F^* / N$

orthogonal

- For orthogonal, it is straightforward:

$$(\mathbf{v}_k, \mathbf{v}_l) = \sum_{j=0}^{N-1} (\omega^k)^j (\bar{\omega}^l)^j = \frac{(\omega^k \bar{\omega}^l)^N - 1}{\omega^k \bar{\omega}^l - 1}$$

according to the fact $\omega^N = 1$

- Speed of calculation, generally speaking, $\mathbf{y} = F\mathbf{x}, \mathbf{x} = F^{-1}\mathbf{y}$ means N^2 multiplications. However, due to the special form of ω, and $F_{jk} = \omega^{jk}$, this matrix can be factorized into very sparse and simple matrices.

FFT

FFT

- FFT

  when N is a power 2^L, FFT is easiest. The operation count drops from N^2 to $NL/2$. It is noted that the matrix entries are complex.

- DCT

  A special form of DFT which only consider real transforms that involve cosines.

  In fact, there are four types DCT, namely DCT-1 through DCT-4, which differ in the boundary conditions at the ends of the interval. The DCT-2 and DCT-4are commonly used in image processing, and have FFT implementations.

DFT

- First, we give a circulant matrix

$$A_0 = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$$

In fact this matrix is a second difference matrix: the j^{th} entry of $A_0 u$ is $-u_{j-1} + 2u_j - u_{j+1}$, which corresponds to $-u''$. No eigenvalues are negative.

At the first and last rows, this second difference involves u_{-1} and u_N. It reaches beyond the boundary.

Then the periodicity $u_N = u_0$ and $u_{N-1} = u_{-1}$ leads to the -1entries that appear in the corners of A_0

DFT

- Second

Now, let us consider the relationship between A_0 and vector $\mathbf{v}_k$. In fact, $\mathbf{v}_k$ is an eigenvector of A_0. It is periodic due to $\omega^N = 1$. The j^{th} entry of $A_0\mathbf{v}_k = \lambda_k \mathbf{v}_k$ is the second difference

$$-\omega^{(j-1)k} + 2\omega^{jk} - \omega^{(j+1)k} = (-\omega^{-k} + 2 - \omega^{k})\omega^{jk} =$$

$$(-e^{-2\pi ki/N} + 2 - e^{2\pi ki/N})\omega^{jk} = (2 - 2\cos\tfrac{2\pi k}{N})\omega^{jk}$$

DFT

Eigenvalues and eigenvectors

The smallest is $\lambda_0 = 0$, corresponding to the eigenvector $\mathbf{v}_0 = (1, 1, \cdots, 1)$. In applications it is very useful to have this flat DC vector as one of the basis vectors

Since A_0 is a real symmetric matrix, its orthogonal eigenvectors can also be chosen real. In fact, the real and imaginary parts of the $\mathbf{v}_k$ must be eigenvectors:

$$\mathbf{c}_k = \mathrm{Re}\,\mathbf{v}_k = (1, \cos\tfrac{2k\pi}{N}, \cdots, \cos\tfrac{2(N-1)k\pi}{N})$$

$$\mathbf{s}_k = \mathrm{Im}\,\mathbf{v}_k = (0, \sin\tfrac{2k\pi}{N}, \cdots, \sin\tfrac{2(N-1)k\pi}{N})$$

Length of eigenvector

- The equal pair of eigenvalues $\lambda_k = \lambda_{N-k}$ gives the two eigenvectors $\mathbf{c}_k$ and $\mathbf{s}_k$. The exceptions are $\lambda_0 = 0$ with one eigenvector $\mathbf{c}_0 = (1, 1, \cdots, 1)$,and for even N also $\lambda_{N/2} = 4$ with $\mathbf{c}_{N/2} = (1, -1, \cdots, 1, -1)$. These two eigenvectors have length $\sqrt{N}$, while the other $\mathbf{c}_k$ and $\mathbf{s}_k$ have length $\sqrt{N/2}$. It is these exceptions that make the real DFT(sines together with cosines) less attractive than the complex form. That factor $\sqrt{2}$ is familiar from ordinary Fourier series. It will appear in the $k = 0$ term for the DCT-1 and DCT-2, always with the flat basis vector (1,1,…,1)

From DFT to DCT

- DCT only involves cosines
  - How to do?

We expect the cosines alone, without sines, to be complete over a half-period. In Fourier series this changes the interval from $[-\pi,\pi]$ to $[0,\pi]$. Periodicity is gone because $\cos 0 \neq \cos\pi$. The differential equation is still $-u'' = \lambda u$. The boundary condition that produces cosines is $u'(0)=0$. Then there are two possibilities, Neumann and Dirichlet, at the other boundary:

Zero slope: $u'(\pi)=0$ gives eigenfunctions $u_k(x)=\cos kx$

Zero value: $u(\pi)=0$ gives eigenfunctions $u_k(x)=\cos(k+\frac{1}{2})x$

Eigenvalues

- For two cases mentioned above:

The two sets of cosines are orthogonal bases for $L^2[0,\pi]$,

the eigenvalues from $-u_k'' = \lambda u_k$ are $\lambda = k^2$ and $\lambda = (k+\frac{1}{2})^2$

Each continuous problem (differential equation) has many discrete approximations (difference equations). The discrete case has a new level of variety and complexity, often appearing in the boundary conditions.

Boundary conditions

- A matrix

$$A=\begin{pmatrix} \otimes & \otimes & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & \oplus & \oplus \end{pmatrix}$$

- For matrix A , $u'(0)=0$ have two natural choices :

  - Symmetry around the meshpoint $j=0$: $u_{-1}=u_1$ (whole-sample symmetry)

    Extended vector: $(\cdots,u_2,u_1,u_0,u_1,u_2\cdots)$

Boundary conditions

- Second,

➢ Symmetry around the midpoint $j=-\frac{1}{2}$: $u_{-1}=u_0$ (half-sample symmetry)

Extended vector: $(\cdots,u_1,u_0,u_0,u_1,\cdots)$

So, the first row of matrix A ($-u_{-1}+2u_0-u_1$ in difference equation)has two possible choices:

- $\otimes \quad \otimes \rightarrow 2 \quad -2$
- $\otimes \quad \otimes \rightarrow 1 \quad -1$

Another end

- In the similar idea, for $u'(\pi)=0$ the last row of matrix $A\,(-u_{N-2}+2u_{N-1}-u_N$ in difference equation)has two possible choices:
  - $\oplus \quad \oplus \rightarrow -2 \quad 2$
  - $\oplus \quad \oplus \rightarrow -1 \quad 1$

Dirichlet condition at another end

- Of course, for $u(\pi)=0$, the last row of matrix A ($-u_{N-2}+2u_{N-1}-u_N$ in difference equation)has another two possible choices:
  - $\oplus \quad \oplus \rightarrow -1 \quad 2$
  - $\oplus \quad \oplus \rightarrow -1 \quad 3$

Eight combinations of BC

- Hence, there are 8 combinations. Four of them give the standard basis functions of cosines mentioned above, DCT-1-DCT-4.

  In fact, DCT-1 through DCT-4 corresponds to one of four differences matrices, A_1, A_2, A_3, A_4

$$A_1 = \begin{pmatrix} 2 & -2 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 2 & -2 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \quad A_4 = \begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 3 \end{pmatrix}$$

Centers $j = 0$ and $j = N-1$,

components $\cos jk\frac{\pi}{N-1}$,

$D_1 = diag(\sqrt{2}, 1, \cdots, 1, \sqrt{2})$

Centers $j = -\frac{1}{2}$ and $j = N - 1/2$

components $\cos(j+\frac{1}{2})k\frac{\pi}{N}$,

$D_2 = I$

Centers $j = 0$ and $j = N$,

components $\cos j(k+\frac{1}{2})\frac{\pi}{N}$,

$D_3 = diag(\sqrt{2}, 1, \cdots, 1, 1)$

Centers $j = -\frac{1}{2}$ and $j = N - 1/2$

components $\cos(j+\frac{1}{2})(k+\frac{1}{2})\frac{\pi}{N}$

$D_4 = I$

Similar Boundary conditions at two ends!

Four standard types of DCT

- The four standard types of DCT are now studied directly from their basis vectors(recall that j and k go from 0 to $N-1$. The j^{th} component of the k^{th} basis vector is:

DCT-1: $\cos jk\frac{\pi}{N-1}$ (divide by $\sqrt{2}$ when j or k is 0 or $N-1$)

DCT-2: $\cos(j+\frac{1}{2})k\frac{\pi}{N}$ (divide by $\sqrt{2}$ when k is 0)

DCT-3: $\cos j(k+\frac{1}{2})\frac{\pi}{N}$ (divide by $\sqrt{2}$ when j is 0)

DCT-4: $\cos(j+\frac{1}{2})(k+\frac{1}{2})\frac{\pi}{N}$

Entries of DCT matrix

- Those are the orthogonal columns of the 4matrices C_1, C_2, C_3, C_4.The matrix C_3 with top row $\frac{1}{\sqrt{2}}(1,1,\cdots,1)$ is the transpose of C_2. All columns of C_2, C_3, C_4 have length $\sqrt{N/2}$

- Orthogonality

$\lambda = 2 - 2\cos\theta$, $\theta = k\frac{\pi}{N-1}$ for type1(DCT-1), $\theta = k\frac{\pi}{N}$ for type2(DCT-2), $\theta = (k+\frac{1}{2})\frac{\pi}{N}$ for type3 and type 4(DCT-3 and DCT-4)

Orthogonality

- Difference equation:

For interior rows:

$$A_i\mathbf{c}_k = \lambda_k \mathbf{c}_k \rightleftharpoons -c_{(j-1)k} + 2c_{jk} - c_{(j+1)k} = \lambda_k c_{jk}$$

– Only two cases

$$-\cos(j-1)\theta + 2\cos j\theta - \cos(j+1)\theta = (2-2\cos\theta)\cos j\theta$$

$$-\cos(j-\tfrac{1}{2})\theta + 2\cos(j+\tfrac{1}{2})\theta - \cos(j+\tfrac{3}{2})\theta = (2-2\cos\theta)\cos(j+\tfrac{1}{2})\theta$$

For boundary rows:

– All are ok!

– Giving the proof for DCT-4

DCT5-DCT8(ODD DCT)

- Even DCT
  - DCT1-DCT4

DCT-5

Centers $j = 0$ and $N - \frac{1}{2}$
Components $\cos jk \frac{\pi}{N - \frac{1}{2}}$
$D_5 = \mathrm{diag}(\sqrt{2}, 1, \ldots, 1)$

$$A_5 = \begin{bmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

DCT-6

Centers $j = -\frac{1}{2}$ and $N - 1$
Components $\cos\left(j + \frac{1}{2}\right) k \frac{\pi}{N - \frac{1}{2}}$
$D_6 = \mathrm{diag}(1, \ldots, 1, \sqrt{2})$

$$A_6 = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{bmatrix}$$

DCT-7

Centers $j = 0$ and $N - \frac{1}{2}$
Components $\cos j\left(k + \frac{1}{2}\right) \frac{\pi}{N - \frac{1}{2}}$
$D_7 = \mathrm{diag}(\sqrt{2}, 1, \ldots, 1)$

$$A_7 = \begin{bmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 3 \end{bmatrix}$$

DCT-8

Centers $j = -\frac{1}{2}$ and N
Components $\cos\left(j + \frac{1}{2}\right)\left(k + \frac{1}{2}\right) \frac{\pi}{N + \frac{1}{2}}$
$D_8 = I$

$$A_8 = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

End

- Reference：
  - The discrete cosine transform, *SIAM Review* **41** (1999) 135-147