

MA1521 Cheat Sheet

github.com/tysng/ma1521-cheatsheet

Nov 2018

1 MF26 Magic

Trigo

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin P + \sin Q = 2 \sin \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)$$

$$\sin P - \sin Q = 2 \cos \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)$$

$$\cos P + \cos Q = 2 \cos \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)$$

$$\cos P - \cos Q = -2 \sin \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)$$

Derivatives

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

Integrals

Take note of the absolute sign, and always remember to +c

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{x} dx = \ln|x| + c$$

2 Basics

Extreme Values

Points where f can have an extreme value:

- Interior point where $f'(x) = 0$
- Interior points where $f'(x)$ doesn't exist
- End points of the domain of f

L'Hospital's Rule

The $\frac{0}{0}$ form: (1) f and g are differentiable in a neighborhood of x_0 ,
(2) $f(x_0) = g(x_0) = 0$, (3) $g'(x) \neq 0$ except possibly at x_0

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

E.g. $\lim_{x \rightarrow 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1} \Big|_{x=0} = 2$

The $\frac{\infty}{\infty}$ form: when $x \rightarrow a$, $f(x), g(x) \rightarrow \infty$, and both differentiable,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Else, change to these two forms. (e.g

$$\lim_{x \rightarrow 0^+} x \cot x = \lim_{x \rightarrow 0^+} \frac{x}{\tan x} = \lim_{x \rightarrow 0^+} \frac{1}{\sec^2 x} = 1)$$

Fundamental Theorem of Calculus

$$\frac{d}{d\Box} \int_c^\Box f(t) dt = f(\Box)$$

3 Series

Geometric Series

Sum: $S_n = a \frac{1-r^{n+1}}{1-r}$, $r \neq 1$

Ratio test: $\lim_{n \rightarrow \infty} = \left| \frac{a_{n+1}}{a_n} \right| = \rho$;

1. $\rho < 1$: converge;
2. $\rho > 1$: diverge;
3. $\rho = 1$, no conclusion;

For convergent series: $S_n \rightarrow \frac{a}{1-r}$

Power Series

$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots$,

where a is the center of the power series

Convergence: $n \rightarrow \infty, S_n \rightarrow k$

1. $\sum c_n(x-a)^n$ converges at $x = a$ and diverges elsewhere
2. $h \in \mathbb{Z}$ that the series only converges in $(a-h, a+h)$
3. converges for every x

Finding Radius of Convergence

Apply ratio test and find

$$M = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| M < 1$$

and transform it to the form of $|x-a| < b$; a is the center, b is the RoC

Or, if the series converges for all x , the RoC is ∞ ; if it only converges at a , the RoC is 0;

Some magic:

$$\frac{1}{1-\Box} = \sum_{n=0}^{\infty} \Box^n, |\Box| < 1$$

Taylor Series

of f at a :

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \dots$$

$$+ \frac{f^{(n)}(a)}{n!} (x-a)^n + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Finding a specific high order derivative

1. given $\int f dx$
2. evaluate f in polynomial form and integrate the polynomial form
3. Compare the coefficient with the item that contains $f^{(100)}(0)$ in the Taylor expansion

4 Vectors

Angle between two vectors: $\cos \theta = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\|v_1\| \|v_2\|}$

Perpendicular vectors: $\vec{v}_1 \cdot \vec{v}_2 = 0$

5 Partial Diffrentiation

$$f_{xy}(a, b) = f_{yx}(a, b)$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Directional Derivative

$D_{\vec{u}} f(a, b) = f_x(a, b) \cdot u_1 + f_y(a, b) \cdot u_2$, where unit vector

$$\vec{u} = u_1 \vec{i} + u_2 \vec{j}$$

Gradient Vector: $\nabla f = f_x \vec{i} + f_y \vec{j}$

Thus,

$$D_{\vec{u}} f(a, b) = \nabla f(a, b) \cdot \vec{u} = \|\nabla f(a, b)\| \cos \theta$$

f increases most rapidly in $\nabla f(a, b)$, decreases most rapidly in $-\nabla f(a, b)$

Max value of $D_{\vec{u}} f(a, b) = \|\nabla f(a, b)\|$, when \vec{u} and ∇f in the same direction, since $\cos \theta = 0$

Increment in f (approx.): $\Delta f \approx [D_{\vec{u}} f(\vec{p})](\Delta t)$, where p is the origin, u is the unit direction.

Finding Duf

- 1. Find the direction \vec{p}
- 2. Find the unit vector $\vec{u} = \vec{p}/|\vec{p}|$
- 3. Find ∇f , then find $D_u f = \nabla f \cdot \vec{u}$

Critical Points

A point of f that satisfies either is a critical point:

- 1. $f_x(a, b) = 0$ and $f_y(a, b) = 0$
- 2. $f_x(a, b)$ or $f_y(a, b)$ doesn't exist

Perform Second Derivative Test: let $f_x(a, b) = 0$ and $f_y(a, b) = 0$

$D = f_{xx}(a, b)f_{yy}(a, b) - {f_{xy}(a, b)}^2$

- $D > 0, f_{xx} > 0$, f has a local minimum at (a,b)
- $D > 0, f_{xx} < 0$, f has a local maximum at (a,b)
- $D < 0$, f has a saddle point at (a,b)
- $D = 0$, no conclusion

6 Ordinary Differential Equation

Separable Equations

$M(x) - N(y)y' = 0 \implies \int M(x)dx = \int N(y)dy + c$

Reduction to Separable Form

Let $v = y/x \implies y = xv \rightarrow y' = v + xv'$, transform equations of $y' = g(\frac{y}{x})$ to $v + xv' = g(v)$ such that

$\frac{dv}{g(v) - v} = \frac{dx}{x}$

Similarly, $y' = f(ax + by + c)$ can be solved by $u = ax + by + c$

Linear First Order ODE

To solve $y' + Py = Q$: find integration factor

$R = e^{\int Pdx}$

Then, answer

$y = \frac{1}{R} \int RQdx$

Reduction to Linear Form

A Bernoulli equation: $y' + P(x)y = Q(x)y^n$, where $n \in \mathbb{R}$;
(When $n = -1$, try Reduction to Separable Form) To solve it, let $v = y^{1-n}$;
Find and express dv/dx in dy/dx ; find dy/dx and sub that in original equation; transform into

$v' + (1 - n)Pv = Q(1 - n)$

and solve the linear ODE.

Homogeneous Linear Second Order DE

For $y'' + ay' + by = 0$, the characteristic equation is $\lambda^2 + a\lambda + b = 0$
Find $\Delta = a^2 - 4b$:

- 1. $\Delta > 0, y = c_1e^{\lambda_1x} + c_2e^{\lambda_2x}$
- 2. $\Delta = 0, y = (c_1 + c_2x)e^{-\frac{ax}{2}}$
- 3. $\Delta < 0$, it has two complex roots; $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i$;
 $y = c_1e^{\alpha x} \cos \beta x + c_2e^{\alpha x} \sin \beta x$

where,

$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b})$
 $\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$

7 Modeling

Population Growth

Malthus's Model: not an accurate representation

$\frac{dN}{dt} = kN, k = B - D$

$N(t) = N_0e^{kt}$

Logistic Model

Assume $D = sN$, where s is a constant:

$\frac{dN}{dt} = BN - DN = BN - sN^2$

The curve approaches carrying capacity $N = B/S$; point of inflection is at $N = B/2s$

$N = \frac{N_{\infty}}{1 + (\frac{N_{\infty}}{N_0} - 1)e^{-Bt}}, N_{\infty} = \frac{B}{s}$

Harvesting

Basic harvesting model: $\frac{dN}{dt} = BN - sN^2 - E$, where E is fish caught/ year.

Desirable result: $E < \frac{B^2}{4s}$, approaches the second root

$\beta_2 = \frac{B + \sqrt{B^2 - 4Es}}{2s}$, when $dN/dt = 0$

Strategies

- When given dx/dt , find x that $dx/dt = 0$, draw out the axis, determine the sign of dx/dt within each region, and find the flow (+ to the right, - to the left)
- To find E, draw the graph without E and find the line of symmetry; use the product of the roots to find E;

8 PDE

For a PDE in the form of,

$u_x = f(x)g(y)u_y$

Substitute $u(x, y) = X(x)Y(y)$ in the PDE, usually

$u_x = X'Y, u_y = XY', u_{xy} = X'Y'$

and arrange the DE into a from in which X', Y' both has power of 1;
Let both sides be k , or let them be $k, 1/k$; and solve X, Y in k, c ;
