2D Numerical Study of Two-fluid Effects on Magnetic Reconnection

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1 Introduction

Magnetic reconnection is the topological rearrangement of magnetic field lines, which commonly takes place in magnetized astrophysical and laboratory plasmas. Accompanied by the rearrangement of macroscopic plasma quantities such as mass and energy, it is a mechanism of great interest.

Magnetic reconnection plays an important role in various physical systems. Here are two examples. In ideal MHD, the plasma is frozen to the magnetic field lines, which makes it impossible for the plasma embedded on the incoming solar wind lines to penetrate the earth's magnetosphere. However, the finite plasma conductivity leads to magnetic reconnection. Some of the solar-wind lines can break and reattach to lines in the magnetosphere, allowing the plasma to penetrate and eventually detected from the earth. The other example is the self-reorganization process in current carrying fusion plasmas. The topology change of a set of field lines leads to a new equilibrium with lower magnetic energy. In tokamaks, particularly, it is characterized by cycles of ramping up and sudden drop of core electron density and temperature, which may lead to disruptions.

The reconnection observed in solar flares happens much faster than the simple estimate of the resistive decay time $t_R = L^2/v_R \approx SL/v_A$. To explain this, Sweet and Parker established a model with a thin layer of high current density. The Ohmic dissipation acts differentially in this region to change the shape of the field lines so that they develop a strong curvature. The curved lines folding and unfolding the plasma out of the ends of the current layer, transfering the magnetic energy to kinetic energy. The estimated $t_{SP} = L/V_{rec} \approx \sqrt{S}(L/v_A)$. Yet this is way too slow. Then Petschek modified this model by including slow shocks, which shortened the length of the Sweet-Parker layer. This leads to results compatible with observations, but the shock origin remains unclear and the validity is challenged.

In high temperature plasmas, the two-fluid effect may explain the fast reconnection. Unlike in previous resistive MHD models where electrons an ions move together at the same velocity, now ions become demagnetized hence the drift between the two species are not negligible. The electron pressure introduces a new physical scale length ρ_s, the ion gyroradius based on the electron temperature to the resistive MHD equations, splitting the single dissipation layer into two distinct layers: a very small inner current layer and a larger flow layer, which can avoid the secondary magnetic islands at low values of resistivity and open up magnetic nozzle controlling the outflow. [1]

With one more term added to the equation, the difficulty of getting an analytical solution largely increases. It will be helpful to obtain an insight of the two-fluid effect with preliminary numerical studies.

2 Mathematical Model

With electron pressure acting on electrons, the simple Ohm's law in the isothermal limit is given by

$$E_\parallel = \eta J_\parallel - \frac{T_e}{n_0 e} \nabla_\parallel n \tag{1}$$

The scale length ρ_s enters the equations due to the requirement that the ions charge neutralize the electron flow. Now we have a smaller inner current layer where the parallel current $J_\parallel \neq 0$ embedded in a larger flow layer where the parallel electric field $E_\parallel \neq 0$.

Consider the reconnection of two cylindrical flux bundles in Cartesian coordinates (x, y, z). Suppose a large uniform B_y, the problem is reduced to 2D in the $x - z$ plane. The total magnetic field in terms of the parallel flux function ψ is given by

$$\mathbf{B} = B_y\hat{y} + \hat{y} \times \nabla\psi \tag{2}$$

The plasma motion is dominated by the $E \times B$ drift

$$\mathbf{V} = (c/B_y)\hat{y} \times \nabla\phi \tag{3}$$

where ϕ is the electric potential.

The expression for ion density n_i comes from the ion continuity equation

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = -n_0\nabla \cdot \mathbf{V}_i \tag{4}$$

The variation in the ion density is generated by the polarization drift

$$\mathbf{V}_i = -\frac{c}{B_y\Omega_i}\frac{\mathrm{d}}{\mathrm{d}t}\nabla_\perp\phi \tag{5}$$

where the ion gyrofrequency $\Omega_i = eB_y/m_ic$, and $\nabla_\perp$ is the gradient perpendicular to the magnetic field. Combining the previous equations of ion density and ion velocity, we have

$$\frac{dn_i}{dt} = \frac{n_0c}{B\Omega_i}\frac{\mathrm{d}}{\mathrm{d}t}\nabla^2_\perp\phi \tag{6}$$

Ohm's law can be written in the form

$$E_\parallel = \eta J_\parallel - \rho_s^2\nabla_\parallel\nabla^2_\perp\phi \tag{7}$$

where $\rho_s = v_s/\Omega_i$, $v_s = (T_e/m_i)^{1/2}$, and $\nabla_\parallel = \hat{y} \times \nabla\psi \cdot \nabla$ is the gradient parallel to the magnetic field. When ρ_s exceeds the resistive scale length of the dissipation region, this term becomes important.

Plug $\mathbf{B} = B_y\hat{y} + \hat{y} \times \nabla\psi$ into the induction equation, we have $E_\parallel = d\psi/dt$. Together with the parallel current $J_\parallel = \nabla^2_\perp\psi$, the MHD equations can be expressed in terms of two scalar potentials

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \eta\nabla^2_\perp\psi - \rho_s^2\nabla_\parallel\nabla^2_\perp\phi \tag{8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla^2_\perp\phi = \nabla_\parallel\nabla^2_\perp\psi + \mu\nabla^4_\perp\phi \tag{8b}$$

3 Proposed Work

The original work by Kleva and Drake gave a 2D numerical solution to this problem. To resolve the fine scale structure in the dissipation region, a variable mesh is applied in the x direction on the Cartesian grid. Spatial derivatives are evaluated to fourth order in the finite difference scheme, and the time derivatives are evaluated to second order in a leapfrog trapezoidal scheme. Only the $x > 0$ region is simulated, and the $x < 0$ region is obtained by symmetry. The spacing in x is finer near the reversal surface $x = 0$. The total number of grid points is 801×2048.

To reproduce this work, first we are going to apply the same algorithm. If this proves to work, then we will consider other appropriate schemes in time and space to improve the efficiency while keeping a comparable accuracy.

4 Problem Statement

4.1 Equations

With only variations in the $x - z$ plane, $\partial/\partial y = 0$. The gradients

$$\nabla_\parallel = \left(\frac{\partial\psi}{\partial z}, -\frac{\partial\psi}{\partial x}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z} \tag{9a}$$

$$\nabla_\perp = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial z}\right)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = \frac{\partial\psi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial z}\frac{\partial}{\partial z} \tag{9b}$$

Let $\xi = \nabla_\perp^2\phi$, then Eq.8 in the (x,z) plane becomes

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}t} &= \eta\nabla_\perp^2\psi - \rho_s^2\nabla_\parallel\xi = \alpha(\psi,\xi)\\ &= 2\eta\left[\left(\frac{\partial\psi}{\partial x}\right)^2\frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial x\,\partial z} + \left(\frac{\partial\psi}{\partial z}\right)^2\frac{\partial^2\psi}{\partial z^2}\right] - \rho_s^2\left(\frac{\partial\psi}{\partial z}\frac{\partial\xi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\xi}{\partial z}\right)\end{aligned} \tag{10a}$$

$$\begin{aligned}\frac{d\xi}{dt} &= \nabla_\parallel\nabla_\perp^2\psi + \mu\nabla_\perp^2\xi = \beta(\psi,\xi)\\ &= 2\Bigg\{\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial z}\left[2\left(\frac{\partial^2\psi}{\partial x^2}\right)^2 - 2\left(\frac{\partial^2\psi}{\partial z^2}\right)^2 + \frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x^3} - \frac{\partial\psi}{\partial z}\frac{\partial^3\psi}{\partial z^3} + \left(2\frac{\partial\psi}{\partial z} - \frac{\partial\psi}{\partial x}\right)\frac{\partial^3\psi}{\partial x^2\,\partial z} + \left(\frac{\partial\psi}{\partial z} - 2\frac{\partial\psi}{\partial x}\right)\frac{\partial^3\psi}{\partial x\,\partial z^2}\right]\\ &\quad + 2\frac{\partial^2\psi}{\partial x\,\partial z}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right)\left[\left(\frac{\partial\psi}{\partial z}\right)^2 - \left(\frac{\partial\psi}{\partial x}\right)^2\right]\Bigg\}\\ &\quad + \mu\left[\left(\frac{\partial\psi}{\partial x}\right)^2\frac{\partial^2\xi}{\partial x^2} + 2\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial z}\frac{\partial^2\xi}{\partial x\,\partial z} + \left(\frac{\partial\psi}{\partial z}\right)^2\frac{\partial^2\xi}{\partial z^2}\right]\\ &\quad + \mu\left[\left(\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial x\,\partial z}\right)\frac{\partial\xi}{\partial x} + \left(\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial z^2} + \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\,\partial z}\right)\frac{\partial\xi}{\partial z}\right]\end{aligned} \tag{10b}$$

To realize 4th-order accuracy in space, we apply the 5-point central scheme for the time derivatives

$$g'(x_i) = \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12h} \tag{11a}$$

$$g^{(2)}(x_i) = \frac{-f_{i+2} + 16f_{i+1} - 30f_i + 16f_{i-1} - f_{i-2}}{12h^2} \tag{11b}$$

Take the derivatives of ψ for example

$$\frac{\partial\psi_{i,j}}{\partial x} = \frac{-\psi_{i+2,j} + 8\psi_{i+1,j} - 8\psi_{i-1,j} + \psi_{i-2,j}}{12\Delta x}$$

$$\frac{\partial\psi_{i,j}}{\partial z} = \frac{-\psi_{i,j+2} + 8\psi_{i,j+1} - 8\psi_{i,j-1} + \psi_{i,j-2}}{12\Delta z}$$

$$\frac{\partial^2\psi_{i,j}}{\partial x^2} = \frac{-\psi_{i+2,j} + 16\psi_{i+1,j} - 30\psi_{i,j} + 16\psi_{i-1,j} - \psi_{i-2,j}}{12\Delta x^2}$$

$$\frac{\partial^2\psi_{i,j}}{\partial z^2} = \frac{-\psi_{i,j+2} + 16\psi_{i,j+1} - 30\psi_{i,j} + 16\psi_{i,j-1} - \psi_{i,j-2}}{12\Delta z^2}$$

$$\begin{aligned}\frac{\partial^2\psi_{i,j}}{\partial x\,\partial z} = &-\frac{-\psi_{i+2,j+2} + 8\psi_{i+2,j+1} - 8\psi_{i+2,j-1} + \psi_{i+2,j-2}}{144\Delta x\Delta z} + 8\frac{-\psi_{i+1,j+2} + 8\psi_{i+1,j+1} - 8\psi_{i+1,j-1} + \psi_{i+1,j-2}}{144\Delta x\Delta z}\\ &- 8\frac{-\psi_{i-1,j+2} + 8\psi_{i-1,j+1} - 8\psi_{i-1,j-1} + \psi_{i-1,j-2}}{144\Delta x\Delta z} + \frac{-\psi_{i-2,j+2} + 8\psi_{i-2,j+1} - 8\psi_{i-2,j-1} + \psi_{i-2,j-2}}{144\Delta x\Delta z}\end{aligned}$$

$$\frac{\partial^3\psi_{i,j}}{\partial x^3} = \frac{\psi_{i+2,j} - 2\psi_{i+1,j} + 2\psi_{i-1,j} - \psi_{i-2,j}}{2\Delta x^3}$$

$$\frac{\partial^3\psi_{i,j}}{\partial z^3} = \frac{\psi_{i,j+2} - 2\psi_{i,j+1} + 2\psi_{i,j-1} - \psi_{i,j-2}}{2\Delta z^3}$$

$$\frac{\partial^3\psi_{i,j}}{\partial x^2\,\partial z} = \frac{\partial}{\partial z}\frac{\partial^2\psi_{i,j}}{\partial x^2}, \quad \frac{\partial^3\psi_{i,j}}{\partial x\,\partial z^2} = \frac{\partial}{\partial x}\frac{\partial^2\psi_{i,j}}{\partial z^2}$$

The 2nd-order leap-frog trapezoidal scheme in time is given by

$$
\begin{aligned}
kick: \xi^{n+1/2} &= \xi^n + \beta(\psi^n, \xi^n)\frac{\Delta t}{2} \\
drift: \psi^{n+1} &= \psi^n + \alpha(\psi^n, \xi^{n+1/2})\Delta t \\
kick: \xi^{n+1} &= \xi^{n+1/2} + \beta(\psi^{n+1}, \xi^{n+1/2})\frac{\Delta t}{2}
\end{aligned} \tag{12}
$$

4.2 Initial Conditions

The magnetic field is generated by two parallel line currents perpendicular to the $x-z$ plane at $(x_0, z_0) = (\pm 0.25, 0.5)$. According to Biot-Savart law

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{r}}{r^3} dV' \tag{13}$$

$\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is the magnetic vector potential and $\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')}{r} dV'$. Then we have

$$\psi(x, z, t=0) = -\frac{\mu_0}{4\pi}\frac{I}{\sqrt{(x-x_0)^2 + (z-z_0)^2}} \tag{14}$$

Let the odd scalar potential

$$\phi(x, z, t=0) = kx \tag{15}$$

Since $\xi = \nabla_\perp^2 \phi$,

$$\xi(x, z, t=0) = -6k\left(\frac{\mu_0 I}{4\pi}\right)^2 \frac{(x-x_0)^2}{[(x-x_0)^2 + (z-z_0)^2]^4} \tag{16}$$

4.3 Boundary Conditions

In this geometry, the impact of the boundaries on the reconnection process is minimized, so we use the double-periodic boundary conditions for the sake of simplicity. But since we only simulate the $x > 0$ region, we need to be careful about the boundaries in z direction. This is determined by ψ being even and ϕ being odd about $x = 0$.

4.4 Parameters

For the reduced MHD equations, $\rho_s = 0$; for the effect of electron pressure driven currents indicated by Eqs. 8, $\rho_s = 8 \times 10^{-3}$. In both cases, $\eta = \mu = 7.5 \times 10^{-5}$.

5 Numerical Results

The initial vector potential ψ leads to a singularity at the location of the line current in the $x-z$ plane, and the numerical results blow up rapidly at the first few time steps.

Another difficulty is caused by the boundary condition of ξ. ϕ is the scalar potential, but its high order derivatives makes the LHS of Eq. 8b complicated. So we introduced the transformation $\xi = \nabla_\perp^2 \phi$. Although ϕ was taken to be odd about $x = 0$, ξ was neither even or odd about $x = 0$. We have to reconsider the boundary condition for it.

References

[1] Robert G. Kleva, J. F. Drake, and F. L. Waelbroeck. "Fast reconnection in high temperature plasmas". In: *Physics of Plasmas* 2.1 (1995), pp. 23–34. DOI: `10.1063/1.871095`. eprint: `https://doi.org/10.1063/1.871095`. URL: `https://doi.org/10.1063/1.871095`.