Letter

Higher-order gap ratios of singular values in open quantum systems

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Understanding open quantum systems using information encoded in its complex eigenvalues has been a subject of growing interest. In this Letter, we study higher-order gap ratios of the singular values of generic open quantum systems. We show that the kth-order gap ratio of the singular values of an open quantum system can be connected to the nearest-neighbor spacing ratio of positions of classical particles of a harmonically confined log gas with inverse temperature $\beta'(k)$, where $\beta'(k)$ is an analytical function that depends on k and the Dyson's index $\beta = 1, 2,$ and 4 that characterizes the properties of the associated Hermitized matrix. Our findings are crucial not only for understanding long-range correlations between the eigenvalues but also provide an excellent way of distinguishing different symmetry classes in an open quantum system. To highlight the universality of our findings, we demonstrate the higher-order gap ratios using different platforms such as non-Hermitian random matrices, random dissipative Liouvillians, Hamiltonians coupled to a Markovian bath, and Hamiltonians with built-in non-Hermiticity.

Introduction. Understanding and characterizing various aspects of open quantum systems has been a subject of growing interest both theoretically and experimentally [1–9]. Generic open quantum systems are associated with non-Hermitian matrices whose complex eigenvalues encode information about the underlying setup. Random Matrix Theory (RMT) has proved indispensable in the study of spectral correlations in such systems [2,10–19]. For example, if the correlations between eigenvalues of a system are similar to those of non-Hermitian random matrices, then the system is considered to be chaotic [20–24]. On the other hand, if the eigenvalues of associated matrices have Poisson statistics, the system is considered to be localized [22–25]. Such a complex spectral analysis often helps in predicting several other important aspects such as the nature of quantum transport and the spread of wave packets [26–31]. For Hermitian systems, the spectral statistics of various symmetry classes are well studied and quite distinct [32–44]. In the case of generic open quantum systems, there are various measures extracted from complex eigenvalues of the underlying non-Hermitian matrices [22,23,45–54]. Often certain diagnostics of complex spectra yield very similar behavior, thereby making it difficult to decode the salient features of the underlying setup. A notable example is that of the level spacing [20,21,23] and complex spacing ratios [23,55,56] of widely different symmetry classes of non-Hermitian random matrix theory [45,57–63]. Regardless of whether the symmetry belongs to the Ginibre orthogonal, unitary, or symplectic ensembles (GinOE, GinUE, and GinSE, respectively) [64,65], the distributions for the

level spacing [20–22,25,48,66,67] and complex spacing ratios [19,23,60,68–73] are characterized by the well-known cubic repulsion [56].

Interestingly, a recent study [74] has shown that the singular values (eigenvalues of $\sqrt{M^\dagger M}$ with M being a generic non-Hermitian matrix) of underlying non-Hermitian matrices can markedly distinguish different non-Hermitian symmetry classes for both the level spacing and the gap ratios [75,76]. However, these quantities encode information of only short-range correlations. In contrast, to understand long-range spectral correlations, a natural generalization to the quantities considered in Ref. [74], would be the kth-order spacing ratio of the singular values. Although this has been studied in the context of real eigenvalues associated with Hermitian quantum systems [77–80], Poisson point processes [81], and empirical correlation matrices [82], so far nothing is understood for singular values emerging from generic non-Hermitian systems. In addition to encoding important information about long-range correlations, this quantity is robust to details of nonuniversal features, such as the density of states, which is often cumbersome to factor out in other long-range quantifiers such as the singular value spectral form factor σFF [75,83].

In this Letter, we analyze the singular values λ_i (real and non-negative) emerging from the underlying non-Hermitian matrices through the kth-order nonoverlapping spacing ratio $r_i^{(k)}$ defined as

$$r_i^{(k)} = \frac{\lambda_{i+2k} - \lambda_{i+k}}{\lambda_{i+k} - \lambda_i}, \quad i, k = 1, 2, 3, \dots. \quad (1)$$

We consider scenarios in which the singular values remain in the bulk, i.e., $1 \ll i, k \ll N$ where N is the number of singular values. The non-Hermitian matrices considered here have different origins: (i) non-Hermitian random matrix

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ensembles, (ii) random dissipative Liouvillians, (iii) Hamiltonian systems that are coupled to the Lindblad bath, and finally (iv) Hamiltonians with built-in non-Hermiticity. In all cases, we find remarkable universal connection between the kth-order gap ratios of singular values with that of the nearest-neighbor particle spacing ratio in a one-dimensional classical gas of logarithmically interacting and harmonically confined particles [14,84,85] with the effective inverse temperature $\beta'(k)$ given by

$$\beta'(k) = \frac{k(k+1)}{2}\beta + (k-1), \quad k \geqslant 1. \tag{2}$$

Here, the Dyson index $\beta = 1, 2, 4$ corresponds to the symmetry classes of the singular values of the underlying non-Hermitian matrix [39,86]. In other words, if the position of N confined particles is x_i, with $i = 1, 2, \ldots, N$, then the distribution of the nearest-neighbor spacing ratio of the position of particles $(x_{i+2} - x_{i+1})/(x_{i+1} - x_i)$ drawn from the joint probability distribution,

$$\mathcal{P}_{\beta'(k)}(\{x_i\}) \propto \exp\left[-\frac{1}{2}\sum_{i=1}^{N} x_i^2 - \frac{\beta'(k)}{2}\sum_{i \neq j=1}^{N} \log|x_i - x_j|\right], \tag{3}$$

matches with that of kth-order spacing ratio. In Eq. (3), $\beta'(k)$ is given by Eq. (2). Mathematically, this can be written as

$$P^{(k)}(r, \beta) = P[r, \beta'(k)], \tag{4}$$

where $P[r, \beta'(k)]$ is the standard nearest-level spacing ratio distribution and is given as [77]

$$P[r, \beta'(k)] = C_{\beta'(k)}\frac{(r + r^2)^{\beta'(k)}}{(1 + r + r^2)^{1 + \frac{3}{2}\beta'(k)}}, \tag{5}$$

with $\beta'(k)$ given in Eq. (2) and

$$C_{\beta'(k)} = \frac{3^{3[1+\beta'(k)]/2}\,\Gamma(1 + \beta'(k)/2)^2}{2\pi\,\Gamma(1 + \beta'(k))}, \tag{6}$$

where $\Gamma(x)$ is the gamma function. While a rigorous derivation of the elegant expression in Eq. (2) has been elusive so far, the numerical evidence presented in this Letter is highly compelling. Note that for $\beta = 1, 2,$ and 4, the effective $\beta'(k)$ in Eq. (2) is a positive integer. One can envisage this as a subset of the Gaussian β ensembles [87,88]. To showcase the remarkable universality, we now present and discuss persuasive numerical evidence for various cases that give rise to non-Hermitian matrices.

Non-Hermitian random matrix ensembles. We first discuss the kth-order level spacing ratio given in Eq. (1) for standard non-Hermitian random matrix (NHRM) ensembles, namely GinOE, GinUE, and GinSE [64,65]. In Fig. 1, the kth-order level spacing ratio for the GinOE ensemble is displayed. The singular values of large GinOE matrices are computed. The kth-order level spacing ratios extracted from these singular values match perfectly with the nearest-neighbor spacing ratio of a classical log gas with inverse temperature $\beta'(k)$ given in Eq. (2) with a Dyson index $\beta = 1$. Furthermore, to make a thorough quantitative check, we compute the Kullback-Leibler divergence [89] which is a measure of the
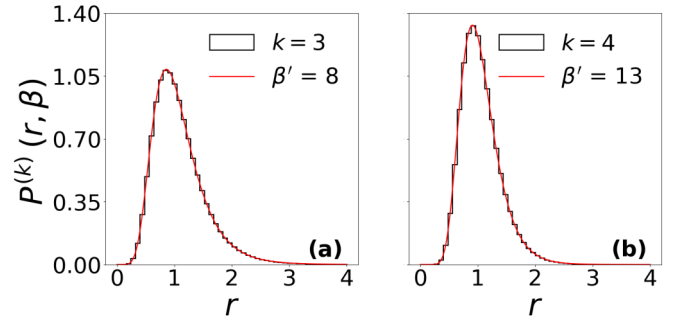

FIG. 1. Plots for kth-order level spacing ratios [Eq. (1)] for the non-Hermitian random matrix (NHRM) that belongs to the GinOE class. (a) and (b) represents $k = 3$ and $k = 4$, respectively (black solid). We notice remarkable agreement between the higher-order level spacing ratio and the nearest-neighbor spacing ratio of Dyson's log gas (red solid) with an effective inverse temperature given in Eq. (2) for $\beta = 1$ and holds for any $k \geqslant 1$. We choose NHRM of size $10^4 \times 10^4$ and obtain the statistics over 500 realizations. We further report the value of KL divergence [Eq. (7)] to be 0.0088 and 0.0005 for (a) and (b), respectively, thereby cementing the remarkable agreement.

distance between two probability distributions $A(x)$ and $B(x)$ and is defined as

$$D_{\mathrm{KL}}(A||B) = \sum_{x} A(x) \log\left[\frac{A(x)}{B(x)}\right]. \tag{7}$$

If the two distributions are identical, then $D_{\mathrm{KL}}(A||B) = 0$. In practice, vanishingly small $D_{\mathrm{KL}}(A||B)$ indicates that A and B are nearly the same. In Fig. 1, KL divergence for $k = 3$ and $k = 4$ is reported. The negligible values demonstrate remarkable agreement between $P^{(k)}(r, \beta)$ and $P[r, \beta'(k)]$ with $\beta'(k)$ given in Eq. (2). The universality of this result for other standard random matrix ensembles such as GinUE and GinSE is provided in the Supplemental Material [56]. Moreover, in the Supplemental Material [56], several other important classes of random matrices are discussed whose symmetries are closely connected with several physical systems, some of which are discussed later.

Random Liouvillian models. We now discuss a construction which describes generic open quantum systems and respects the required mathematical properties such as trace preservation and complete positivity [4]. Such a scenario is well described by a Lindblad quantum master equation $\partial_t \rho_t = \mathcal{L}(\rho_t)$ with dynamics governed by a Liouvillian $\mathcal{L}$ which typically takes the Gorini-Kossakowski-Sudarshan-Lindblad form as given by $\mathcal{L}(\rho) = \mathcal{L}_H(\rho) + \mathcal{L}_D(\rho)$, where $\mathcal{L}_H(\rho) = -i[H, \rho]$ is the unitary part with system Hamiltonian H. We assume that the Hilbert space dimension is 2^M which is the case for, say, a system of M qubits. $\mathcal{L}_D(\rho)$ corresponds to the purely dissipative part given by

$$\mathcal{L}_D(\rho) = \sum_{i,j=1}^{2^{2M}-1} K_{ij}\left[L_i \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_i, \rho\}\right]. \tag{8}$$

Here, L_i, $i = 1, 2, \ldots, 2^{2M} - 1$ is the traceless Lindblad operator that satisfies the orthonormality condition $\mathrm{Tr}[L_i L_j^\dagger] = \delta_{ij}$
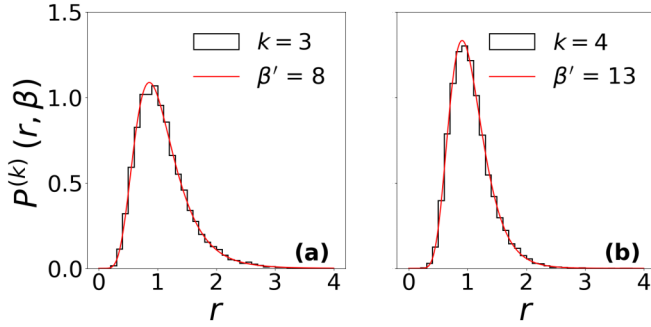
FIG. 2. Plots for kth-order level spacing ratios [Eq. (1)] for the random Liouvillian given in Eq. (8). Here, (a) and (b) represent $k = 3$ and $k = 4$, respectively (black solid). The singular value statistics follows the scaling relation [Eq. (4)] for $\beta = 1$ (red solid). Similar to Fig. 1, we once again observe a remarkable agreement. The results here are obtained for a system size $M = 4$ for 50 realizations. In this case we find the KL divergence [Eq. (7)] to be 0.020 and 0.0019 for (a) and (b), respectively.

[4]. The matrix K is chosen to be random [69,71,90–92] and always satisfies the positive semidefinite property. This guarantees that the time evolution of the density matrix always respects the positivity condition. Furthermore, in our setup we assume that there is no Hamiltonian, i.e., we set $H = 0$ and analyze the kth-order level spacing ratio only in the presence of the dissipative part [Eq. (8)]. We consider a setup of M qubits which allows $2^{2M} - 1$ possible L_i operators consisting of Pauli strings. In other words, each L_i is a direct product of Pauli matrices σ's or identity matrix $I_{2\times2}$ excluding the situation where every element is an identity. More concretely, each L_i is of the form $\frac{1}{2^{M/2}}\sigma_{x_1} \otimes \sigma_{x_2} \cdots \otimes \sigma_{x_n}$, $x_i \in 0, 1, 2, 3$, where 0,1,2,3 represents $I_{2\times2}$, and the Pauli matrices σ_x, σ_y, and σ_z, respectively. The random matrix K is obtained as follows: We first prepare a $(2^{2M} - 1) \times (2^{2M} - 1)$ diagonal matrix D whose entries are chosen from by sampling from a uniform box distribution [0,1]. We then rotate this diagonal matrix with a random unitary matrix U sampled from the Haar measure [93] to give the required $K = U^\dagger D U$ matrix. Given the K matrix and all possible L_i operators, we construct the Liouvillian in a matrix form of dimension $2^{2M} \times 2^{2M}$ and compute its singular values. In Fig. 2, we plot the kth-order level spacing ratios for $k = 3$ and $k = 4$. The Liouvillian constructed in this example falls under the $\text{AI}^\dagger$ symmetry class [94]. Interestingly, we find that the corresponding kth-order spacing ratio for the singular values matches remarkably well with the nearest-neighbor spacing ratio of Dyson's log gas with an effective inverse temperature given by Eq. (2) with a Dyson index $\beta = 1$. It is interesting and important to see whether such observations hold in physical setups where Hamiltonians are coupled to Markovian baths. This is what we address next.

Physical Lindbladian. We now discuss an M-qubit physical setup where a Hamiltonian is coupled to a Markovian bath. This open quantum system is described by the Lindblad equation $\partial_t \rho_t = \mathcal{L}(\rho_t)$ where the Liouvillian $\mathcal{L}$ takes the standard form with unitary dynamics governed by a Hamiltonian H, and dissipative dynamics is governed by Lindbladian L_i's that describe coupling to a Markovian bath. The Lindblad
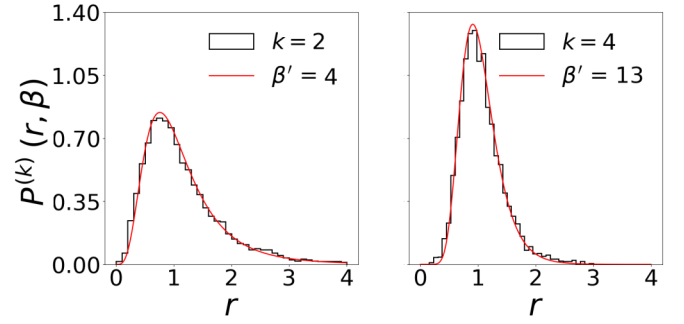

FIG. 3. Plots for kth-order level spacing ratios [Eq. (1)] for the physical Lindbladian given in Eq. (9) with the Hamiltonian given in Eq. (10) and the jump operators given by $L_i = \sqrt{\gamma}\,\sigma_i^-$. Here, (a) and (b) represent $k = 2$ and $k = 4$, respectively (black solid). Similar to Fig. 1, we once again observe a remarkable agreement with $\beta = 1$ (red solid). The results here are obtained for a system size $M = 6$ averaged over 50 different realizations. We choose $J = 1$, $h_x = -1.05$, $h_z = 0.2$, and $\gamma = 0.77$. In this case we find the KL divergence [Eq. (7)] to be 0.008 and 0.032 for (a) and (b), respectively.

equation is given by

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_{i=1}^{M} \left[L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right], \quad (9)$$

where the Hamiltonian H is taken as a variant of the quantum Ising model with open boundary conditions [61,74], and is given by

$$H = -J \sum_{i=1}^{M-1}(1 + \epsilon_i)\sigma_i^z\sigma_{i+1}^z - \sum_{i=1}^{M}\left(h_x\,\sigma_i^x + h_z\,\sigma_i^z\right). \quad (10)$$

Here, $J > 0$ is the nearest-neighbor coupling constant between the Pauli spins with ϵ_i being an additional on-site random disorder parameter which is sampled from a uniform distribution $[-0.1, 0.1]$. h_x and h_z are the constant fields in the x and z directions, respectively. The jump operators L_i, attached at each site, are of the form $L_i = \sqrt{\gamma}\,\sigma_i^-$, with $\sigma_i^- = \sigma_i^x - i\,\sigma_i^y$ representing the damping mechanism of spin excitation at each site with a rate γ. In Fig. 3, we plot the higher-order spacing ratios of the singular values for $k = 2$ and $k = 4$ and observe a good agreement with the nearest-neighbor spacing ratio with the effective $\beta'(k)$ evaluated at $\beta = 1$. So far in all previous cases we discussed the situation when the setup is essentially in an ergodic phase. It will therefore be interesting to consider a situation where the underlying setup can host different phases, in particular, non-ergodic and ergodic. We therefore now analyze the kth-order spacing ratios for singular values of Hamiltonians with built-in non-Hermiticity that contain rich phases.

Non-Hermitian Hamiltonian. We consider an M-site hardcore boson model with nonreciprocal hopping and a complex on-site potential. The Hamiltonian is given by

$$H = \sum_{j=1}^{M}\{-J(e^g c_j^\dagger c_{j+1} + e^{-g} c_{j+1}^\dagger c_j)$$
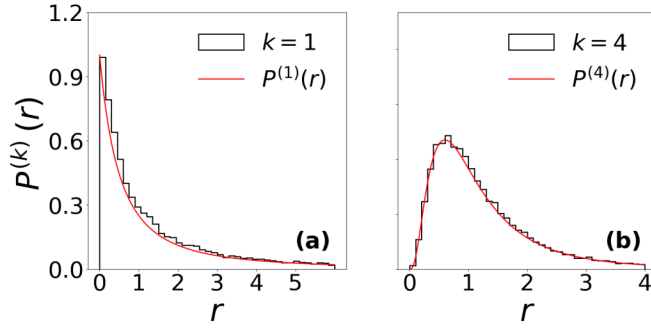$$+ [h_j + i\gamma(-1)^j]n_j + V n_j n_{j+1}\}, \quad (11)$$

FIG. 4. Plots for kth-order level spacing ratios [Eq. (1)] for the non-Hermitian Hamiltonian [Eq. (11)] deep in the localized regime ($h = 14$). Here, (a) and (b) represent $k = 1$ and $k = 4$, respectively (black solid). We observe remarkable agreement with Eq. (12) (red solid). The corresponding KL divergences are 0.1016 and 0.0098, respectively. We consider $M = 16$, $J = 1$, $g = 0.1$, $V = 2$, and $\gamma = 0.1$ and the lattice is half filled. The spacing ratio distributions are obtained by performing an average over 50 different realizations.

FIG. 5. Plots for kth-order level spacing ratios [Eq. (1)] for the non-Hermitian Hamiltonian [Eq. (11)] deep in the ergodic phase ($h = 2$). Here, (a) and (b) represent $k = 2$ and $k = 3$, respectively (black solid). The corresponding KL divergences are 0.0048 and 0.0098, respectively. All other parameters are the same as in Fig. 4. The spacing ratio distributions are obtained by performing an average over 50 different realizations. The numerical results are in excellent agreement with Eq. (5) setting $\beta = 2$ (red solid).

and we use a periodic boundary condition. This many-body system therefore has two different origins of non-Hermiticity, namely, of hopping (g) and of on-site (γ) types. Here, $n_j = c_j^\dagger c_j$ is the number operator. The on-site term in Eq. (11) consists of two parts: a random, real parameter $h_j \in [-h, h]$, and a purely imaginary parameter $\pm i\gamma$ representing alternate gain/loss. In our case, we take $g = 0.1$, $V = 2$, and $\gamma = 0.1$. Different phases of this non-Hermitian model can be realized by tuning the disorder strength h. For the parameters given above, g, V, and γ, we find that when $h = 2$, the system is deep in the ergodic regime, whereas when $h = 14$, it is deep in the nonergodic regime. Therefore, the setup in Eq. (11) is ideally suited for studying the kth-order spacing ratio for singular values in difference phases.

We start from the regime of high disorder ($h = 14$) and study the nature of singular values. We find that the complex spectral statistics of the Hamiltonian follow two-dimensional (2D) Poisson statistics. On the other hand, the kth-order spacing ratios of the singular values follow the kth-order spacing ratios that emerge from 1D Poisson statistics and are given as [95]

$$P^{(k)}(r) = \frac{(2k-1)!}{[(k-1)!]^2} \frac{r^{k-1}}{(1+r)^{2k}}. \tag{12}$$

We show remarkable agreement between the kth level spacing ratios of the singular values with Eq. (12) in Fig. 4.

Now we discuss the regime of low disorder strength ($h = 2$) where the system is deep in the ergodic phase. The singular values of the kth-order level spacing ratios is once again well described by non-Hermitian random matrix theory, in particular, by the Ginibre unitary ensemble (GinUE, $\beta = 2$). Recall that, when one considers a complex spectra analysis, such as complex spacing distributions, it is difficult to differentiate various classes of non-Hermitian random matrix theory. For example, many of the different random matrix ensembles display the same cubic-level repulsion for complex spacing ratios [56]. The power of the singular values lies in the fact that it is uniquely suited to distinguish several
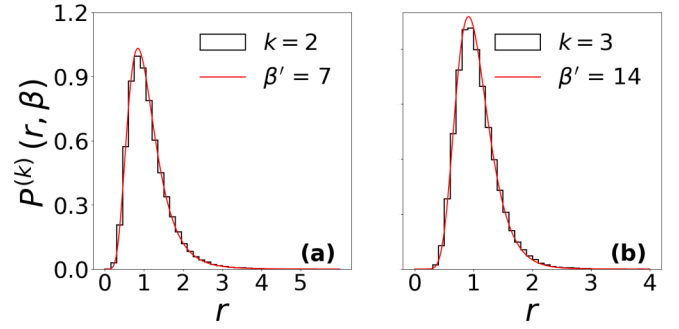
non-Hermitian random matrix ensembles, as we have demonstrated here and as also pointed out for the nearest-neighbor spacing and gap ratios in Ref. [74]. Moreover, the kth-order level spacing ratio seems to be in an excellent agreement with the prediction in Eq. (5) with $\beta = 2$. This is demonstrated in Fig. 5.

Summary. We have highlighted the immense importance of singular values of non-Hermitian random matrices through the lens of higher-order level spacing ratios. By focusing on widely different types of non-Hermitan systems, in all cases, we observe a remarkable agreement between the kth-order level spacing ratio of singular values and the nearest-neighbor spacing ratio of classical log-gas particles [Eq. (3)] confined in a harmonic trap at an effective inverse temperature $\beta'(k)$ through Eq. (2). This intriguing agreement therefore highlights that the use of singular value statistics is a promising route to distinguish different symmetry classes which are otherwise often difficult to demarcate using the conventional procedure of a complex spectral analysis.

In the future, it will be interesting to explore a systematic correspondence between different symmetry classes [61,74] of non-Hermitian random matrices and the corresponding singular value statistics through higher-order gap ratios and other diagnostics such as the spectral form factor [96–99] constructed from singular eigenvalues σFF [75,83]. The deep connection developed in this work can also be adapted to study transitions from delocalized to localized phases in non-Hermitian power-law banded random matrices [100,101].

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