CONVERGENCE AND DIVERGENCE IN LINEAR ITERATION SEQUENCES

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Abstract. This report investigates the behavior of linear iteration sequences defined by $f(x) = ax + b$, starting from an initial value x_0. Through algebraic and geometric analyses, we examine the convergence and divergence of these sequences, determined by the parameters a, b, and x_0. Key findings emphasize the role of the fixed point $\frac{b}{1-a}$ and provide insights into the dynamics of iterative processes under various conditions.

1 INTRODUCTION

In this laboratory, we investigate and analyze the behavior of sequences generated by the iterative application of a linear function, defined by $f(x) = ax + b$, starting from a given initial value x_0. Our goal is to develop a deeper understanding of sequences, and the concept of convergence and divergence. We will systematically explore the behavior of these sequences under various parameters of the linear function, learn to formulate definitions, engage in mathematical induction, and utilize essential concepts such as lemmas, propositions, theorems, and corollaries. This exploration will be complemented by the use of Desmos to visualize function behavior.

At the beginning of our analysis are several fundamental definitions like *sequence*, *linear function*, *iteration sequence*, and *linear iteration sequence*. Section 2: Convergence and Divergence defines what it means for sequences to be convergent and divergent. Section 3: Types of Convergence and Divergence categorizes various convergence and divergence behaviors with illustrative examples. Section 4: Analysis explores the mathematical properties of iteration sequences through derivations and proofs, and provides visual representations to complement the algebraic findings. Throughout the report, we will consistently use the notation $\{x_n\}$ to represent our iteration sequences.

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We begin by introducing the general definition of a sequence as follows:

Definition 1.1. A *sequence* is an infinite list of numbers that can be indexed by natural numbers. [4] Unlike an unordered set of numbers, the order of the numbers in a sequence is the essential characteristic of the sequence. Each member of a sequence has an associated position, called an *index*, which represents its place in the order. For this reason, when we use variables to represent members in a sequence they will be denoted as follows:

$$\{a_n\} = \{a_0, a_1, a_2, a_3, ...\} \text{ [3]}.$$

A more formal approach to the definition is as follows: A sequence is a real-valued function whose domain is a set that has the form $\{n \in Z | n \geq m\}$ (m is usually 0 or 1). [2]

Example 1.2. An example of a sequence is given by $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, ..., where here the general term is $a_n = n + 1$ for $n \geq 0$. We might denote this sequence as $\{a_n\} = \{1, 2, 3, ...\}$.

Definition 1.3. *Iteration* is the process of generating a sequence by beginning with some initial value x_0 and then repeatedly applying the same function to the output of the previous step to produce the members of the sequence. The sequence of values obtained in this way is called an *iteration sequence*. [5] We denote this process using the following notation given some initial value x_0:

$$\begin{aligned}
&x_0,\\
&x_1 = f(x_0),\\
&x_2 = f(x_1),\\
&\quad\vdots\\
&x_n = f(x_{n-1}), \ldots \text{ and so on.}
\end{aligned}$$

Here, each subsequent member x_n is obtained by applying the function $f(x)$ to the previous member x_{n-1}. The *members* of the sequence are the values $x_0, x_1, x_2, \ldots$. The members of iteration sequences are often called *iterates* of the function $f(x)$.

Example 1.4. Consider $f(x) = \cos(x)$. Let the initial value be $x_0 = 0$. Using the iterative process, we apply the function repeatedly to get the sequence of values:

$$\text{First iteration}(n = 1) : x_1 = f(x_0) = \cos(0) = 1,$$
$$\text{Second iteration}(n = 2) : x_2 = f(x_1) = \cos(1) \approx 0.5403,$$
$$\text{Third iteration}(n = 3) : x_3 = f(x_2) = \cos(0.5403) \approx 0.8576,$$
$$\vdots$$

In this example, the iteration sequence is approximately $\{x_0, x_1, x_2, x_3, x_4, x_5, \ldots\} = \{0, 1, 0.54, 0.856, \ldots\}$ approximately, and $x_n = \cos(x_{n-1})$ for $n \geq 1$ and $x_0 = 0$.

Now we will define a linear function and use it to define a linear iteration sequence.

Definition 1.5. A *linear function* is a function whose graph is a line. The formula of a linear function can be written in the form $f(x) = ax + b$, where a and b are real numbers. [1] A linear function when graphed represents a straight line.

Example 1.6. For example, $f(x) = 2x + 3$ is a linear function with $a = 2$ and $b = 3$.

Definition 1.7. A *linear iteration sequence* is an iteration sequence generated by a linear function given some initial value x_0.

Example 1.8. If we use the linear function $f(x) = x + 1$ and start with $x_0 = 2$, we get the following linear iteration sequence:

$$x_0 = 2,$$
$$x_1 = f(x_0) = 2 + 1 = 3,$$
$$x_2 = f(x_1) = 3 + 1 = 4,$$
$$x_3 = f(x_2) = 4 + 1 = 5,$$
$$\vdots$$

In this example, $\{x_n\} = \{2, 3, 4, 5, ...\}$.

The purpose of this laboratory is to explore the behavior of these linear iteration sequences and understand the factors that influence their convergence or divergence. We aim to investigate how the initial value (x_0), the slope (a), and the y-intercept (b) of the linear function affect the sequence's behavior by using both algebraic and geometric tools.

2 CONVERGENCE AND DIVERGENCE

In this section, we will define what it means for a sequence to converge or diverge.

Definition 2.1. Informally, a sequence is said to be *convergent* if there exists a number L such that the values of $\{s_n\}$ get closer and closer to L as n increases.

For example, consider the sequence defined by: $f(x) = \frac{1}{2}x$, and initial value $x_0 = 1$. The sequence evolves as follows: $x_0 = 1, x_1 = \frac{1}{2}, x_2 = \frac{1}{4}, x_3 = \frac{1}{8}, \ldots$. As n grows, the sequence members approach 0, suggesting that the limit of the sequence is 0.

Formally, a sequence $\{s_n\}$ converges if there exist a real number L such that for all $\epsilon > 0$ there exists an N_ϵ for which: $|S_n - L| < \epsilon$ whenever $n > N_\epsilon$.

Example 2.2. Consider the same linear iteration sequence: $f(x) = \frac{1}{2}x$, with $x_0 = 1$. The iteration sequence generated by this function has the general term: $x_n = (\frac{1}{2})^n$ for $n = 0, 1, 2, \ldots$.

Given any $\epsilon > 0$, we want to prove that there will always exist an N such that for all $n > N$, $|\frac{1}{2^n} - 0| < \epsilon$, or simply $\frac{1}{2^n} < \epsilon$. Solving for n, we get $n > \log_2\left(\frac{1}{\epsilon}\right)$. Thus, $N = \log_2\left(\frac{1}{\epsilon}\right)$, and $n > N$ will ensure that $|\frac{1}{2^n} - 0| < \epsilon$ for all $\epsilon > 0$.

Now to show that we recently understood the meaning of this definition, suppose that $\epsilon = 0.01$. We want to find N such that for all $n > N$, we have: $|\frac{1}{2^n}| < 0.01$. Solving for n, we get:

$$n > \log_2\left(\frac{1}{0.01}\right) = \log_2(100) \approx 6.64.$$

Therefore, $N \approx 6.64$. Thus, for $n \geq 7$, the inequality $\frac{1}{2^n} < 0.01$ will hold.

Let's verify: $\frac{1}{2^7} = \frac{1}{128} \approx 0.00781$, which is indeed less than 0.01. Hence, for $n \geq 7$, we have $\frac{1}{2^n} < 0.01$, confirming that N exists for $\epsilon = 0.001$.

Definition 2.3. A sequence is said to be *divergent* if it does not converge to any finite limit.

Example 2.4. Consider the linear iteration sequence defined by $f(x) = 2x$, with $x_0 = 1$. The sequence members are $x_0 = 1, x_1 = 2, x_2 = 4, x_3 = 8, \ldots$. These terms seem to be growing without bound. Since the members of this sequence do not seem to approach any finite value, it is a divergent sequence.

3 TYPES OF CONVERGENCE AND DIVERGENCE

In this section, we will define and provide examples of the eight types of convergence and divergence we have observed in linear iterative sequences.

3.1 Types of Convergence

We start with some types of convergence that we have seen so far.

(1) **Increasing Convergence**: An iterative sequence $\{x_n\}$ is increasing if $x_n < x_{n+1}$ for all n, and if the sequence also converges to a limit L we say it exhibits increasing convergence.
  - Example: Let $f(x) = \frac{1}{2}x + \frac{1}{2}$. Consider the sequence defined by $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}$, with $x_0 = 0$. As n increases, the terms of the sequence seem to keep increasing $\{x_n\} = \{0, 0.5, 0.75, 0.875, \ldots\}$, and these terms seem to approach the limit 1.

(2) **Decreasing Convergence**: An iterative sequence $\{x_n\}$ is decreasing if $x_n > x_{n+1}$ for all n, and if the sequence converges to a limit L we say it exhibits decreasing convergence.
  - Example: Let $f(x) = \frac{1}{2}x$, $x_0 = 1$. As n increases, the terms of the sequence decrease steadily: $\{x_n\} = \{1, 0.5, 0.25, 0.125, \ldots\}$ and seem to approach the limit 0.

(3) **Constant Convergence**: An iterative sequence is constantly convergent if $x_n = c$ for some constant c and all $n \geq 1$. This type of sequence converges to c.
  - Example: Let $f(x) = 5$ with $x_0 = 2$. The members of the sequence are: $x_0 = 2, x_1 = 5, x_2 = 5, x_3 = 5, \ldots$. Thus, for all $n \geq 1$, we have $x_n = 5$, and $\lim_{n\to\infty} x_n = 5$. Thus, the sequence $\{x_n\}$ converges to 5.

(4) **Oscillatory Convergence**: An iterative sequence exhibits oscillatory convergence when the terms alternate either side of the limiting values but each member is approaching the limit ever more closely.

- Example: Let $f(x) = -0.5x$, and take $x_0 = 10$, then: $x_1 = -5, x_2 = 2.5, x_3 = -1.25, \ldots$. Here, the terms of the sequence alternate between positive and negative values, but the magnitude of the terms decreases with each iteration. Despite the alternating signs, the sequence members still seem to converge to 0 as n increases.

3.2 Types of Divergence

Now let us move on to explore some types of divergence we have seen.

(1) **Oscillatory Divergence**: An iterative sequence exhibits oscillatory divergence when its terms eventually alternate between positive and negative values, with the absolute value of successive terms increasing without bound.

- Example: Consider $f(x) = -2x$ with $x_0 = 1$. Then the sequence oscillates but terms of the sequence grow without bound in absolute value: $x_0 = 1, x_1 = -2, x_2 = 4, x_3 = -8, x_4 = 16, \ldots$. Here, the absolute value of x_n grows without bound as n increases.

(2) **Increasing Divergence**: An iterative sequence exhibits increasing divergence if $x_n < x_{n+1}$ for all n and the terms grow without bound as n increases.

- Example: Consider $f(x) = 2x + 3$ with $x_0 = 5$. The sequence members are: $x_0 = 5, x_1 = 13, x_2 = 29, x_3 = 61, x_4 = 125, \ldots$. Here, the terms x_n seem to grow without bound as n increases.

(3) **Decreasing Divergence**: An iterative sequence exhibits decreasing divergence if $x_n > x_{n+1}$ for all n and the terms decrease without lower bound as n increases.

- Example: Consider the function $f(x) = 2x - 5$, with $x_0 = 0$. The sequence members are: $x_0 = 0, x_1 = -5, x_2 = -15, x_3 = -35, \ldots$. Here, the terms x_n seem to decrease without a lower bound as n increases.

(4) **Two-cycle Divergence**: An iterative sequence exhibits two-cycle divergence if its terms alternate between two values x_0 and x_1 so that $\{x_n\} = \{x_0, x_1, x_0, x_1, \ldots\}$

- Example: Let $f(x) = -x + 24$ with $x_0 = 10$. The sequence members generated by this function and $x_0 = 10$ are: $x_0 = 10, x_1 = 14, x_2 = 10, x_3 = 14, \ldots$. Here, the sequence members alternate between 10 and 14.

4 ANALYSIS

This section examines the behavior of linear iteration sequences $f(x) = ax + b$ through both algebraic and geometric analyses. Algebraically, the results are presented via lemmas, propositions, theorems, and corollaries, supported by examples to elucidate the resulting sequence behaviors. The analysis begins with a proof of a lemma establishing the closed-form expression for a finite geometric sum. Geometrically, the analysis focuses on the graph of $f(x) = ax + b$, exploring the fixed points—where the graph of $f(x)$ intersects the identity line $y = x$—as well as the long-term behavior of sequences. Visualizations include the identity function $y = x$ as a red line, the function $f(x)$ as a green line, and the initial value x_0 as a blue dot, providing intuitive insights into the dynamics of the sequences.

Lemma 4.1. *If a is any real number then:*

$$1 + a + a^2 + ... + a^{n-1} = \begin{cases} \frac{1-a^n}{1-a} & \text{for } a \neq 1, \\ \\ n & \text{for } a = 1. \end{cases}$$

Proof. We begin by examining the case where $a = 1$. In this situation, the left-hand side of the equation simplifies to a sum of n terms, each equal to 1:

$$\underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ terms}} = n.$$

Thus, when $a = 1$, we have $1 + a + a^2 + \ldots + a^{n-1} = n$. This work directly proves the second case of the lemma for $a = 1$. Next, we consider the case where $a \neq 1$. We will use induction on n to prove this case. Let us start with the base case when $n = 1$. For this case, we need to verify that the left-hand side is equal to the right-hand side. Since $a \neq 1$, the right-hand side in this case is $\frac{1-a^1}{1-a} = \frac{1-a}{1-a} = 1$. Hence the statement is true for $n = 1$.

Assuming that the statement is true for some integer $n \geq 1$, we will show that it also holds for $n + 1$. By the inductive hypothesis, we have $1 + a + a^2 + \ldots + a^{n-1} = \frac{1-a^n}{1-a}$.

Let us consider the expression that we need to prove and we see that we can use our inductive hypothesis to simplify the first n terms as follows:

$$(*) \qquad 1 + a + a^2 + \ldots + a^{n-1} + a^n = \frac{1-a^n}{1-a} + a^n.$$

Now we combine the terms on the right-hand side:

$$(**) \qquad \frac{1-a^n}{1-a} + a^n = \frac{1-a^n + a^n(1-a)}{1-a}.$$

Simplifying the numerator, we find that the number n of the right-hand side in equation (*) becomes $1 - a^n + a^n - a^{n+1} = 1 - a^{n+1}$. This work means that (**) becomes $\frac{1-a^n+a^n(1-a)}{1-a} = \frac{1-a^{n+1}}{1-a}$. Thus, we have shown that $1 + a + a^2 + \ldots + a^{n-1} + a^n = \frac{1-a^{n+1}}{1-a}$. If the statement is true for n, it is also true for $n+1$ when $a \neq 1$. Therefore, by the principle of mathematical induction, we conclude that the lemma is proven for all $n \geq 1$ and all a. □

Following the establishment of this lemma, we now move on to a related proposition. As we computed the iterates of the sequence generated by $f(x) = ax + b$, we observed that the terms associated with b accumulated in a specific manner. Specifically, we noted that the contribution from b after n iterations can be represented as $b(1 + a + \ldots + a^{n-1})$, indicating a geometric series where b is multiplied by the sum of powers of a. This work led us to formulate the general proposition for the n-th iterate as

$$x_n = a^n x_0 + b(1 + a + \ldots + a^{n-1}) \text{ when } n \geq 1.$$

We will now proceed to prove this proposition using the method of induction.

Proposition 4.2. *If $f(x) = ax + b$ and a, b, x_0 are real numbers then the n^{th} iterate of the sequence $\{x_n\}$ has the recursive form $x_n = a^n x_0 + b(1 + a + \ldots + a^{n-1})$ for all $n \geq 1$.*

Proof. We will prove this proposition by induction on n. First, we consider the base case when $n = 1$. By definition of iteration, we know that $x_1 = f(x_0)$ and so we see that

$$x_1 = f(x_0) = a^1 x_0 + b(1) = ax_0 + b.$$

Hence our proposition is true for $n = 1$. Now, let us assume the statement is true for some $n \geq 1$, that is, we assume that $x_n = a^n x_0 + b(1 + a + \ldots + a^{n-1})$. This statement is our induction hypothesis. Now we need to show the formula is true for $n+1$. Let us calculate x_{n+1}. We know that by definition of iteration, we have

$$x_{n+1} = f(x_n) = ax_n + b. \tag{1}$$

Now using our inductive hypothesis and our formula (1) for x_{n+1} we have

$$x_{n+1} = a\left(a^n x_0 + b(1 + a + \ldots + a^{n-1})\right) + b.$$

Expanding this expression, we get that $x_{n+1} = a^{n+1}x_0 + ab(1 + a + \ldots + a^{n-1}) + b$. Next, we simplify the term involving b to get $x_{n+1} = a^{n+1}x_0 + b\left(a(1 + a + \ldots + a^{n-1}) + 1\right)$. Finally, we can express the formula in a more compact form as $x_{n+1} = a^{n+1}x_0 + b(1 + a + \ldots + a^n)$.

Thus, the proposition holds for $n + 1$. Therefore, by the principle of induction, the proposition is true for all $n \geq 1$. □

With the lemma and the proposition established, we are now prepared to present a theorem that gives the closed-form expression for the n^{th} iterate of the sequence:

Theorem 4.3. *If $f(x) = ax + b$ and a, b, x_0 are real numbers then the n^{th} iterate of the sequence $\{x_n\}$ has closed-form expression:*

$$x_n = \begin{cases} a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a} & \textit{for } a \neq 1, \\ \\ x_0 + bn & \textit{for } a = 1, \end{cases}$$

for $n \geq 1$.

Proof. We will prove this theorem by considering two cases: $a \neq 1$ and $a = 1$. In the first case, we have $a \neq 1$. From the proposition, we know that for all a,

$$x_n = a^n x_0 + (1 + a + \ldots + a^{n-1})b. \tag{*}$$

Using our Lemma for $a \neq 1$, we have $1 + a + a^2 + \ldots + a^{n-1} = \frac{1-a^n}{1-a}$. Therefore we can substitute this formula into (*) to get the expression $x_n = a^n x_0 + \frac{1-a^n}{1-a} b$. This expression can be rewritten by separating the rational expression to get $x_n = a^n x_0 + \frac{b}{1-a} - \frac{a^n b}{1-a}$. Next, we rearrange the terms and get the form with a^n in them: $x_n = a^n x_0 - \frac{a^n b}{1-a} + \frac{b}{1-a}$. Finally, we factor out a^n to arrive at the expression:

$$x_n = a^n \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}.$$

Hence we have proved the first case in the theorem when $a \neq 1$. Next, we consider the case when $a = 1$. In this case, $f(x) = x + b$. Using Proposition 4.2 and Lemma 4.1 for $a = 1$, we have $x_n = x_0 + bn$. This statement has proved the second case where $a = 1$. We have proved the theorem in both cases, $a \neq 1$ and $a = 1$ and our Theorem holds. □

Now, we will build upon the foundational principles established in Theorem 4.3 and deepen our analysis by deriving specific corollaries that address the behavior of the iteration sequence in different scenarios. These corollaries will categorize the iteration outcomes based on the value of a and specify the convergence or divergence characteristics of the sequence. By systematically examining each case, we will clarify the nuances of how the initial value x_0 and the parameters a and b influence the trajectory of the iteration sequence.

The following corollaries will be presented, each corresponding to an interval or values of a:

Corollary 4.4. *Given the hypothesis of Theorem 4.3 and suppose $a < -1$, then the iteration sequence $\{x_n\}$ can exhibit two behaviors:*

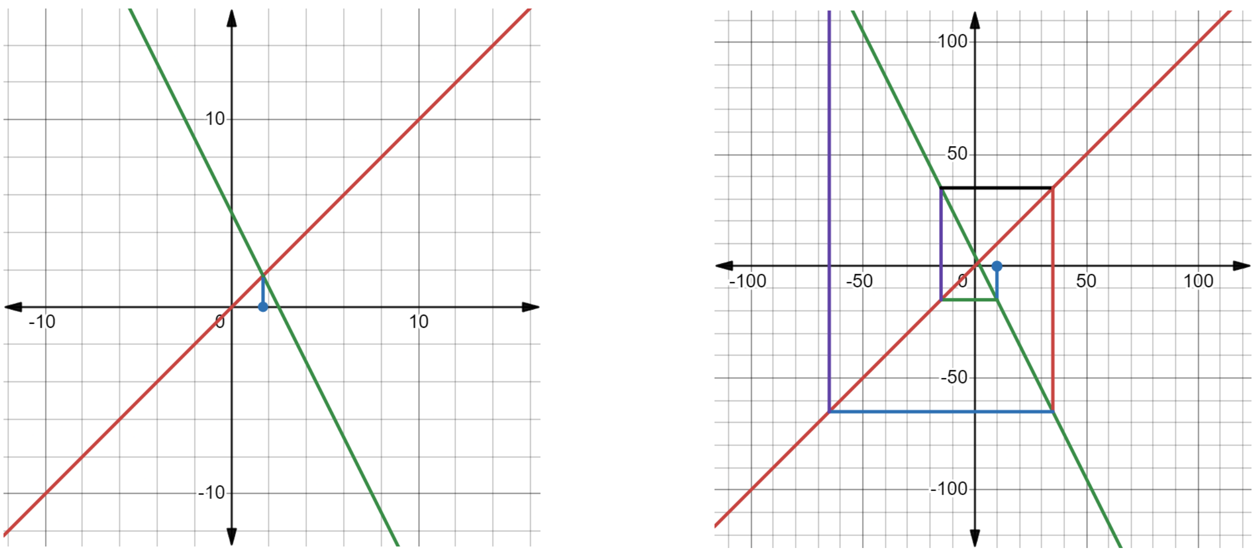
- *Case 1: If $x_0 = \frac{b}{1-a}$ then x_n exhibits constant convergence to $\frac{b}{a-1}$.*
- *Case 2: If $x_0 \neq \frac{b}{1-a}$ then x_n exhibits oscillating divergence where $|x_n|$ grows without bound.*

Proof. First, we recall the closed form for the general term of the iteration sequence as derived from Theorem 4.3 when $a \neq 1$. In this case we have that $x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$.

Now, let's analyze the two cases based on the initial value x_0.

Case 1: If $x_0 = \frac{b}{1-a}$, we substitute this into the expression for x_n. Our first term vanishes so that $x_n = \frac{b}{1-a}$ for $n \geq 1$. Thus, for this choice of x_0, the sequence x_n exhibits constant convergence to $\frac{b}{1-a}$ for all $n \geq 0$.

Case 2: If $x_0 \neq \frac{b}{1-a}$, the sequence takes the form $x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$. Here, we consider the behavior of x_n as n increases. We know that $x_0 - \frac{b}{1-a}$ is constant for this case. Since $a < -1$, we know that $|a|^n$ grows without bound, and a^n oscillates between positive and negative values while increasing in magnitude. Thus, for $x_0 \neq \frac{b}{1-a}$, the sequence x_n will equal a large value $|a^n(x_0 - \frac{b}{1-a}|$ that is alternately positive and negative. Since $|a| > 1$, the absolute value $\left|a^n\left(x_0 - \frac{b}{1-a}\right)\right| = |a|^n \left|x_0 - \frac{b}{1-a}\right|$ grows without bound as n increases. Therefore, the dominant term in the sequence is $a^n\left(x_0 - \frac{b}{1-a}\right)$, which diverges without bound depending on the parity of n. Thus, the sequence exhibits oscillating divergence with increasingly larger magnitudes. □



(A) Initial value $x_0 = \frac{5}{3}$. (B) Initial value $x_0 = 10$.

FIGURE 1. Geometric visualizations of the iteration of $f(x) = -2x + 5$.

Figure 1a shows that the iteration converges to a fixed point where the two lines intersect (case 1). Since the initial value x_0 equals the fixed point ($x_0 = \frac{5}{3}$), the sequence remains constant at this point. Figure 1b shows a cobweb pattern where each iteration alternates between positive and negative values, moving further and further away from the intersection point. This visual representation clearly shows the divergence and oscillation of the sequence as it moves outward in both directions.

Corollary 4.5. *Given the hypothesis of Theorem 4.3, suppose $a = -1$. The sequence $\{x_n\}$ exhibits two behaviors:*

- *Case 1: If $x_0 = \frac{b}{2}$ then x_n exhibits constant convergence to $\frac{b}{2}$.*

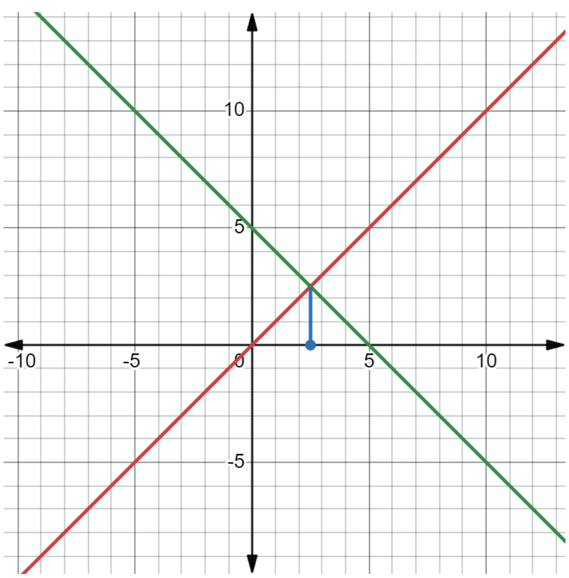
- *Case 2: If $x_0 \neq \frac{b}{2}$ then x_n exhibits two-cycle divergence. Two situations can happen: (1) For n is odd, we have $x_n = -x_0 + b$; and (2) For n is even, we have $x_n = x_0$ and so $x_n = \{x_0, -x_0 + b, x_0, -x_0 + b, \ldots\}$.*

Proof. By Theorem 4.3, when $a \neq 1$, the sequence takes the form $x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$. Substituting $a = -1$ into Theorem 4.3 we get:

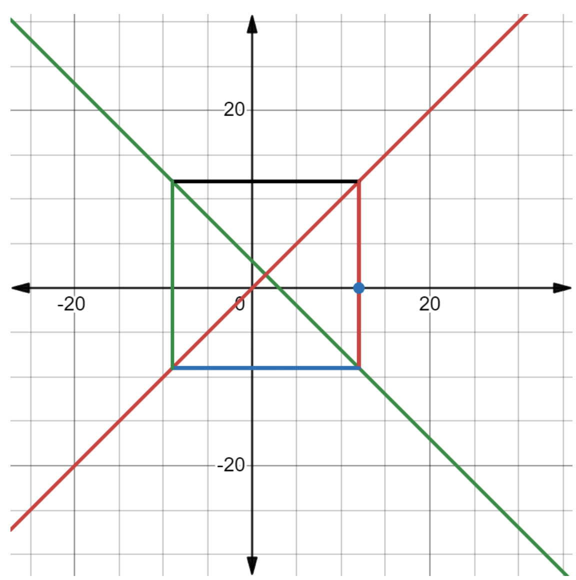
$$x_n = (-1)^n(x_0 - \frac{b}{2}) + \frac{b}{2}. \tag{*}$$

Case 1: Let $x_0 = \frac{b}{2}$. Substituting this value into (*) we get $x_n = \frac{b}{2}$ for all $n \geq 0$. Since $x_1 = x_0$, the iteration sequence remains constant: $x_0 = x_1 = x_2 = \ldots = \frac{b}{2}$ for all n. Therefore, the sequence exhibits constant convergence to $\frac{b}{2}$.

Case 2: Let $x_0 \neq \frac{b}{2}$. For even n, $(-1)^n = 1$, so $x_n = x_0$. For odd n, $(-1)^n = -1$, so $x_n = -x_0 + b$. Thus, the sequence alternates between two values: x_0 (for even n) and $-x_0+b$ (for odd n). Therefore, the sequence exhibits two-cycle divergence. □



(A) Initial value $x_0 = \frac{5}{2}$.



(B) Initial value $x_0 = 12$.

FIGURE 2. Geometric visualizations of the iteration of $f(x) = -1x + 5$.

Figure 2a (case 1) has similar characteristics as Figure 1a.

Let us look at the graph for Case 2. When $x_0 \neq \frac{b}{2}$, the sequence alternates between two points: x_0 and $-x_0 + b$ which are 12 and -7. In Figure 2b, this pattern is visualized as points switch positions on opposite sides of the fixed point at each step of the iteration. The graph shows how these two points are equidistant from the center which is less apparent in algebraic expressions.

Corollary 4.6. *Given the hypothesis of Theorem 4.3 and suppose $-1 < a < 0$. The iteration sequence $\{x_n\}$ can exhibit two behaviors:*

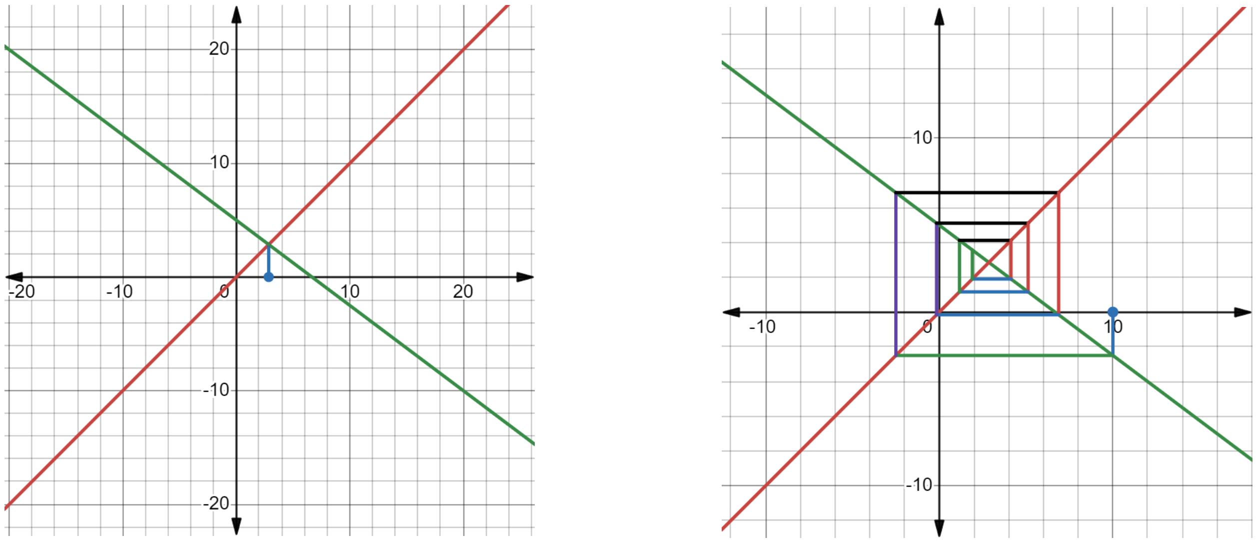
- *Case 1: If $x_0 = \frac{b}{1-a}$, $\{x_n\}$ exhibits constant convergence to $\frac{b}{1-a}$.*
- *Case 2: If $x_0 \neq \frac{b}{1-a}$, $\{x_n\}$ exhibits oscillating convergence to $\frac{b}{1-a}$.*

Proof. By Theorem 4.3, when $a \neq 1$, the sequence takes the form $x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$.

Case 1: If $x_0 = \frac{b}{1-a}$, the first term vanishes, and we get $x_n = \frac{b}{1-a}$ for all $n \geq 0$. Thus, the sequence exhibits constant convergence to $\frac{b}{1-a}$.

Case 2: If $x_0 \neq \frac{b}{1-a}$, the sequence becomes $x_n = a^n\left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}$. Let $y_n = x_n - \frac{b}{1-a}$, so the sequence becomes $y_{n+1} = ay_n$. This is a geometric progression with ratio a and initial term $y_0 = x_0 - \frac{b}{1-a}$. The general term of this sequence is $y_n = a^n y_0$. Since $-1 < a < 0$, $|a^n|$ approaches 0 as n increases. Thus, y_n goes to 0 as n increases. Because $a < 0$, the terms a^n alternate in sign. This alternation causes the sequence x_n to oscillate around the fixed point $\frac{b}{1-a}$, with the oscillations decreasing in magnitude as n increase. Thus, the sequence exhibits oscillating convergence to $\frac{b}{1-a}$ as n increases. Therefore, the sequence exhibits oscillatory convergence toward $\frac{b}{1-a}$. □

In Case 1, when $x_0 = \frac{b}{1-a}$, we observe from Figure 3a the same phenomenon as in previous cases where the graph shows that the iteration process reaches a fixed point. In Case 2, we see a spiral pattern around the intersection point of the two lines in Figure 3b. The sequence alternates between values above and below the fixed point, but as n increases, these oscillations get smaller and smaller, indicating that the sequence is converging to the fixed point in the center.



(A) Initial value $x_0 = \frac{5}{1.75}$.

(B) Initial value $x_0 = 10$.

FIGURE 3. Geometric visualizations of the iteration of $f(x) = -0.75x + 5$.

Corollary 4.7. *Given the hypothesis of Theorem 4.3, suppose $a = 0$. The iteration sequence $\{x_n\}$ exhibits only one behavior which is constant convergence to b.*

Proof. By Theorem 4.3, when $a \neq 1$, the sequence takes the form $x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$. Substituting $a = 0$ into the iterative equation gives $x_n = b$. Thus, for any initial value x_0, we have $x_1 = b$, $x_2 = b$, $x_3 = b$, and so on. This work shows that all terms of the sequence are equal to b after the first iteration. Therefore the sequence exhibits constant convergence to b. □

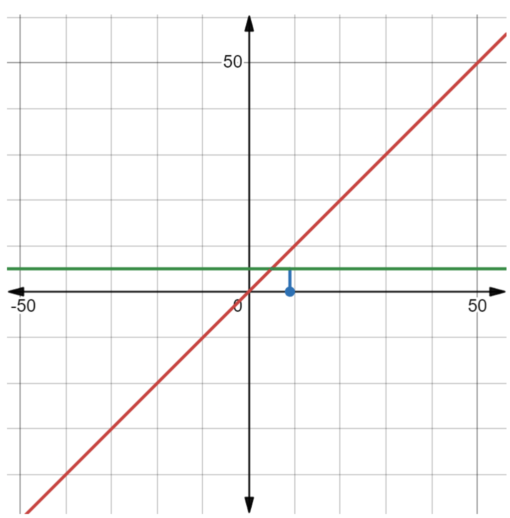


FIGURE 4. Iteration of the function $f(x) = 0x + 5$ with initial value $x_0 = 9$.

Corollary 4.7 can be visualized by Figure 4. Since all iterations after the first one will result in the same value b, the function is represented by a horizontal line. No matter where the starting point is, the sequence will converge to a constant value of b (which is 5 in this example).

Corollary 4.8. *Given the hypothesis of Theorem 4.3, suppose* $0 < a < 1$. *The iteration sequence* $\{x_n\}$ *exhibits three behaviors.*

- *Case 1: If* $x_0 = \frac{b}{1-a}$, $\{x_n\}$ *exhibits constant convergence to* $\frac{b}{1-a}$.
- *Case 2: If* $x_0 < \frac{b}{1-a}$, $\{x_n\}$ *exhibits increasing convergence to* $\frac{b}{1-a}$.
- *Case 3: If* $x_0 > \frac{b}{1-a}$, $\{x_n\}$ *exhibits decreasing convergence to* $\frac{b}{1-a}$.

Proof. By Theorem 4.3, when $a \neq 1$, the sequence takes the form $x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$. Given that $0 < a < 1$, we know that $a^n > 0$ for all n.

Case 1: If $x_0 = \frac{b}{1-a}$, then $x_n = \frac{b}{1-a}$ for all $n \geq 0$. Thus, the sequence exhibits constant convergence to $\frac{b}{1-a}$.

Case 2: If $x_0 < \frac{b}{1-a}$, then $(x_0 - \frac{b}{1-a}) < 0$. From Theorem 4.3: $x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$. Since a^n decreases as n increases, $a^n(x_0 - \frac{b}{1-a})$ becomes less negative, making each term larger than the previous. As n grows, the sequence converges to $\frac{b}{1-a}$. Thus, the sequence exhibits increasing convergence to $\frac{b}{1-a}$.

Case 3: If $x_0 > \frac{b}{1-a}$, then $(x_0 - \frac{b}{1-a}) > 0$. Again by Theorem 4.3: $x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$. Since a^n decreases as n increases, each term of $a^n(x_0 - \frac{b}{1-a})$ becomes smaller than the previous one. As n grows, the sequence converges to $\frac{b}{1-a}$. Thus, the sequence exhibits decreasing convergence to $\frac{b}{1-a}$. $\square$

In Case 1, we see the same phenomenon as in previous cases of constant convergence (Figure 5).

To illustrate Case 2, the graph in Figure 6a shows the iteration sequence starting at $x_0 = 2$, which is less than the fixed point. The graph depicts an upward trajectory, with the magnitude of changes decreasing as x_n approaches 12.5.

To visualize Case 3, the graph in Figure 6b shows the iteration sequence starting at $x_0 = 22$, which is above the fixed point. We can see that the iterations start high and gradually decrease, with the magnitude of changes decreasing as x_n approaches 12.5.

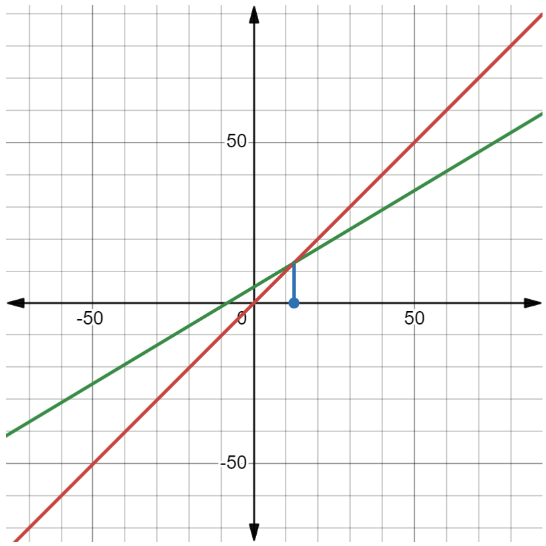
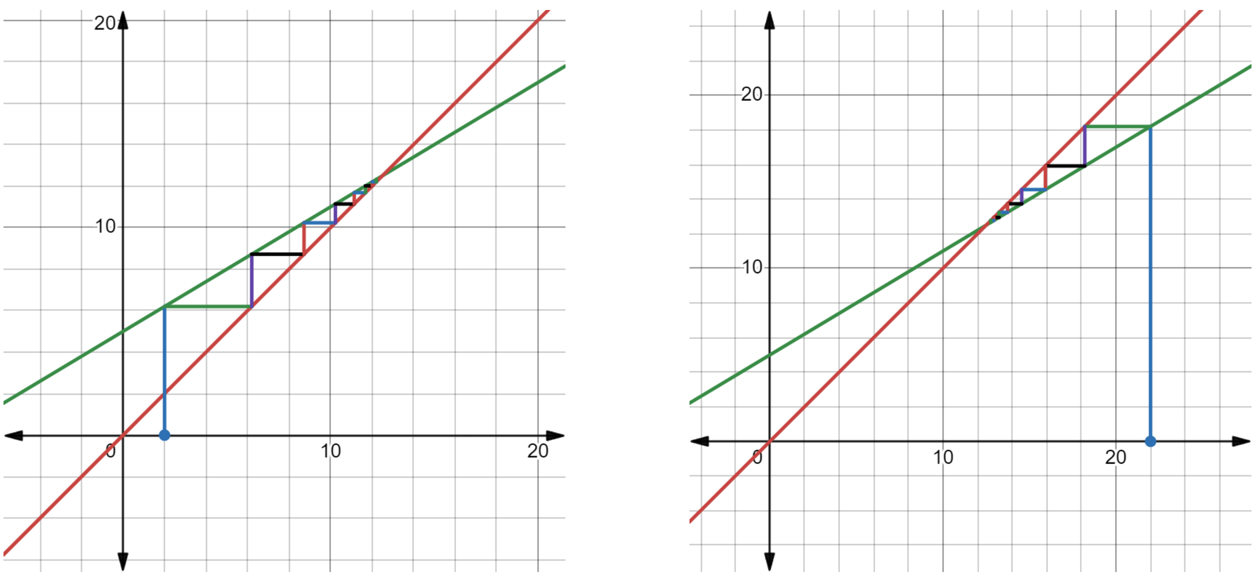


Figure 5. Iteration of the function $f(x) = 0.6x + 5$ with initial value $x_0 = 12.5$.



(a) Initial value $x_0 = 2$.

(b) Initial value $x_0 = 22$.

Figure 6. Visualizations of the iteration of $f(x) = 0.6x + 5$ with different x_0.

Corollary 4.9. *Given the hypothesis of Theorem 4.3, suppose $a = 1$. The iteration sequence $\{x_n\}$ exhibits three behaviors.*

- *Case 1: If $b = 0$, $\{x_n\}$ exhibits constant convergence to x_0.*
- *Case 2: If $b > 0$, $\{x_n\}$ exhibits increasing divergence.*
- *Case 3: If $b < 0$, $\{x_n\}$ exhibits decreasing divergence.*

Proof. By Theorem 4.3, the sequence takes the form $x_n = x_0 + bn$ when $a = 1$.

We will now consider the behavior of this sequence in each case for b.

Case 1: When $b = 0$, the sequence simplifies to $x_n = x_0 + 0n = x_0$ for all $n \geq 1$. Since each term equals x_0, the sequence exhibits constant convergence to x_0.

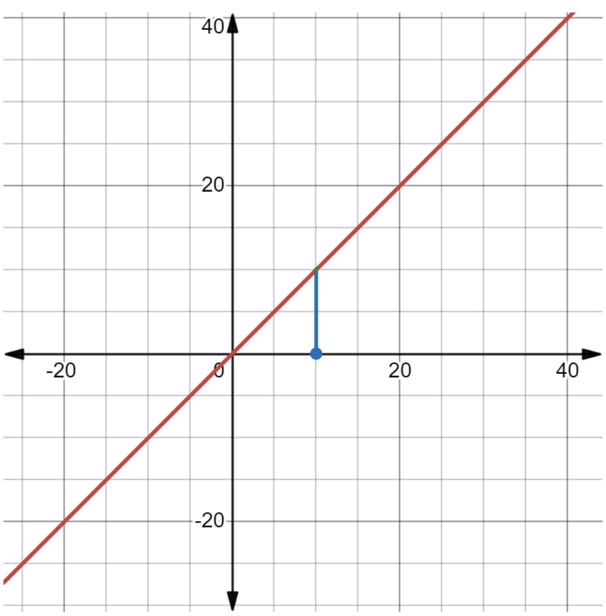
Case 2: If $b > 0$, as n increases, nb increases without bound as b is positive. Thus, the sequence exhibits increasing divergence as x_n increases in steps of size b without bound as n increases.

Case 3: If $b < 0$, since b is negative, as n increases, nb becomes increasingly negative. Consequently, the sequence exhibits increasing divergence as x_n decreases in negative steps of size b without a lower bound as n increases. □

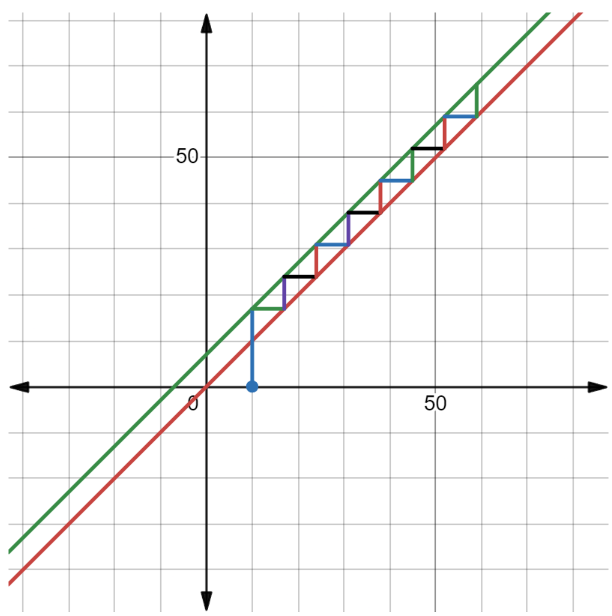
To illustrate Case 1, Figure 7a shows a red line representing $f(x) = x$. No matter how many iterations you take, the output will always be x_0 which is 10 in this example.

To visualize Case 2, Figure 7b shows two parallel lines: the red line represents the identity function), and the green line represents $f(x)$. The iterations are depicted as points moving upwards, bouncing between the red line and the green line. The parallel lines emphasize that our function $f(x)$ and the line $y = x$ are consistently apart by a fixed distance of 7.

For Case 3, Figure 8 shows a similar pattern as Figure 7b, but downward trajectory because of the negative value of b.



(A) Initial value $x_0 = 10$.



(B) Initial value $x_0 = 10$.

FIGURE 7. Visualizations of the iteration of $f(x) = x$.

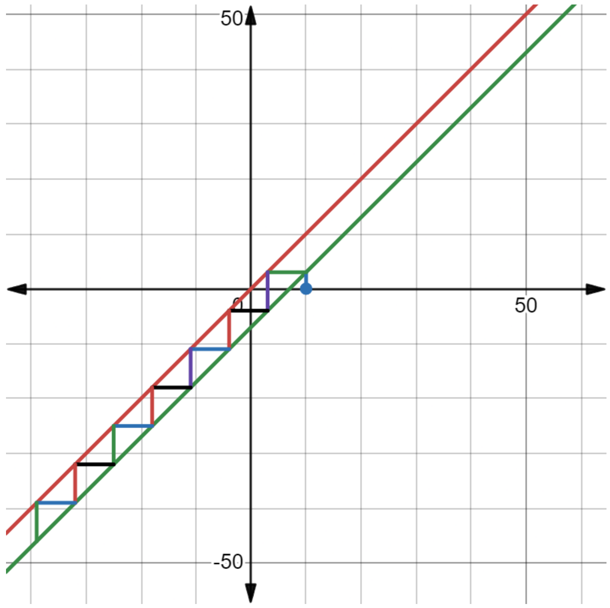


FIGURE 8. Iteration of the function $f(x) = 1x - 7$ with initial value $x_0 = 10$.

Corollary 4.10. *Given the hypothesis of Theorem 4.3 and suppose $a > 1$, then the iteration sequence $\{x_n\}$ can exhibit three behaviors:*

- *Case 1: If $x_0 = \frac{b}{1-a}$, $\{x_n\}$ exhibits constant convergence to $\frac{b}{1-a}$.*
- *Case 2: If $x_0 < \frac{b}{1-a}$, $\{x_n\}$ exhibits decreasing divergence.*
- *Case 3: If $x_0 > \frac{b}{1-a}$, $\{x_n\}$ exhibits increasing divergence.*

Proof. By Theorem 4.3, when $a \neq 1$, the sequence takes the form $x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$.

Case 1: When $x_0 = \frac{b}{1-a}$, we can substitute this value into our expression to get $x_n = \frac{b}{1-a}$ for all $n \geq 0$. Thus, the sequence exhibits constant convergence to $\frac{b}{1-a}$.

Case 2: When $x_0 < \frac{b}{1-a}$, we have $(x_0 - \frac{b}{1-a}) < 0$. Because $a > 1$ we know that a_n will grow without bound. Since a^n grows without bound and is positive, the term $a^n(x_0 - \frac{b}{1-a})$ grows arbitrarily negative as n increases. This result makes x_n decrease from $\frac{b}{1-a}$ without a lower bound, exhibiting decreasing divergence as n increases.

Case 3: When $x_0 > \frac{b}{1-a}$, we have $(x_0 - \frac{b}{1-a}) > 0$. Since a^n grows without bound and is positive, the term $a^n(x_0 - \frac{b}{1-a})$ grows arbitrarily large as n increases. This result makes x_n grow without bound from $\frac{b}{1-a}$, exhibiting increasing divergence as n increases. □

In the first case, we see the same pattern as in previous cases of constant convergence in Figure 9.

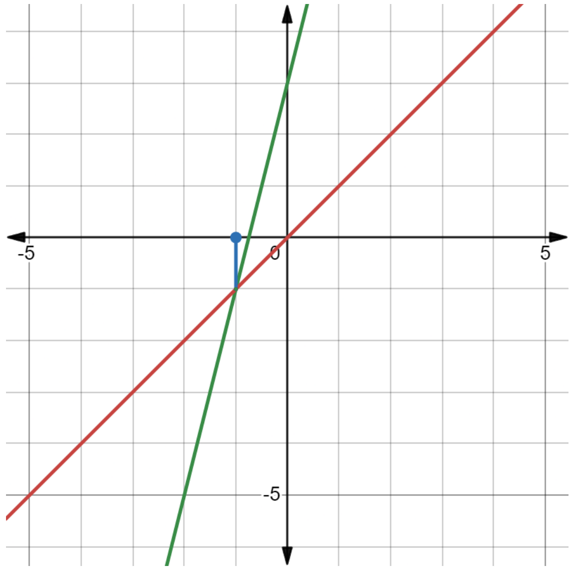
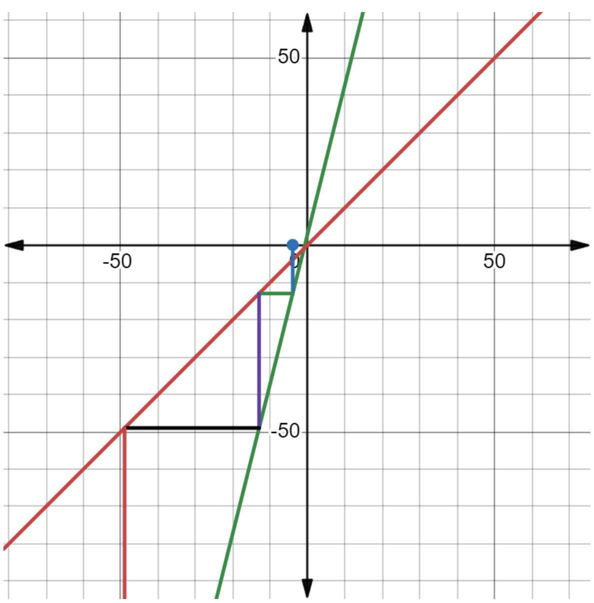
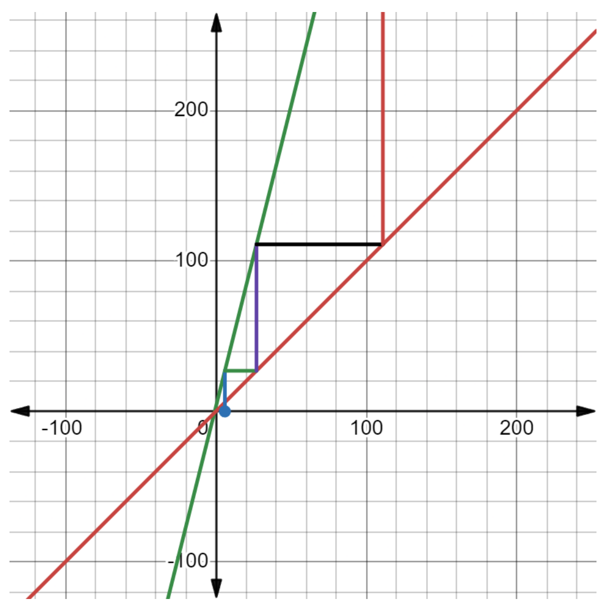


FIGURE 9. Geometrical visualization of the iteration of the function $f(x) = 4x + 3$ with initial value $x_0 = -1$.

In the second case, Figure 10a shows that the initial value lies below the convergence point, and as n increases, the values of x_n fall further and further below $\frac{b}{1-a}$. In the third case, Figure 10b shows that the distance between the green and red lines widens in the positive direction as n increases.



(A) Initial value $x_0 = -4$.



(B) Initial value $x_0 = 6$.

FIGURE 10. Geometric visualizations of the iteration of $f(x) = 4x + 3$ with different initial values.

Through the algebraic analysis, we learned about the specific role that parameters a play in determining whether an iteration sequence converges or diverges. We have provided closed-form expressions that describe the long-term behavior of iterative sequences. Through the geometric analysis, we observed distinct patterns in the behavior of the graphs based on the initial values and the parameters a and b. The geometry emphasizes the relationship between x_n and $f(x_n)$, illustrating that if the sequence converges, the two values must become arbitrarily close.

In the subsequent conclusion, we will summarize our findings on linear iteration and address the key insights gained from both algebraic and geometric perspectives.

5 Conclusion

In this report, we have explored the behavior of linear iteration sequences defined by the function $f(x) = ax + b$ through algebraic and geometric analyses. Our investigations led us to some key insights regarding convergence and divergence based on the parameters a and b, as well as the initial value x_0 in relation to the fixed point.

Algebraically, we learned that the nature of convergence is heavily influenced by the slope a. Some sequences indeed attain their limits while others approach them asymptotically without ever reaching them. The speed of convergence can be highly dependent on the initial conditions and the parameter a. In particular, the closer a is to 1, the slower the convergence tends to be, while sequences diverging from their limits often exhibit exponential divergence.

The geometric analysis clearly showed us that the fixed point of $f(x)$ is where the graph intersects the identity line $y = x$. We observed that when the initial value x_0 is equal to a fixed point, the sequence remains constant. Conversely, when x_0 diverges from the fixed point, the iterations reveal oscillatory or cobweb behaviors depending on the parameter values.

Combining these analyses, we see that the geometric perspective often offers intuitive clarity that complements the algebraic findings. For instance, while algebraic calculations reveal the mechanics of convergence, the geometric representation illustrates the sequence's approach toward the fixed point and highlights the outward oscillatory pattern in divergent cases. Future explorations could address non-linear functions and their iteration dynamics.

References

[1] Dan Kennedy and Randall Charles. Prentice hall mathematics, algebra 2, 2009.

[2] A Ross Kenneth. Elementary analysis: The theory of calculus, 2013.

[3] Oscar Levin. *Discrete mathematics: An open introduction*. Mathematical Sciences University of Northern Colorado, 4 edition, 2024.

[4] J. Rogawski. *Calculus*. W. H. Freeman, 2011.

[5] James Stewart. *Calculus: early transcendentals*. Cengage Learning, 2012.

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