CHROMATIC POLYNOMIALS: DERIVATION, APPLICATION, AND THE BIRKHOFF-LEWIS REDUCTION ALGORITHM IN GRAPH COLORING

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ABSTRACT. This report explores the derivation and application of chromatic polynomials in graph theory, focusing on the Birkhoff-Lewis Reduction Algorithm. We begin by introducing key concepts and the algorithm, followed by applying combinatorial reasoning and mathematical induction to derive chromatic polynomials for paths, complete graphs, and cycles. These results provide insight into the relationship between a graph's structure and its colorability.

1. INTRODUCTION

Graph coloring has applications in scheduling, frequency assignment, and various optimization problems. In this laboratory, we first introduce the definitions of graphs and graph coloring, then investigate the *graph coloring existence problem* and the *graph coloring counting problem*. Next, we define the chromatic polynomial and demonstrate how to compute it using the *Birkhoff-Lewis Reduction Algorithm*. Then we prove five theorems about the chromatic polynomial, including the relationship between the number of vertices, edges, and the leading terms of the polynomial. Finally, we will explore the chromatic polynomials of specific types of graphs, such as paths, cycles, and complete graphs, and apply the Birkhoff-Lewis Reduction Algorithm to derive these polynomials. We begin by defining a graph:

Definition 1.1 (Graph)**.** A *simple graph* G is a pair consisting of a vertex set $V(G)$ and an edge set $E(G)$, where the elements of $E(G)$ are subsets of $V(G)$ of size 2, called edges. In a simple graph, there are no loops (edges of the form $\{v_i, v_i\}$) and no multiple (or double) edges, meaning that each pair of distinct vertices can be connected by at most one edge [2].

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Example 1.2. For example, consider a graph G with $V(G) = \{v_1, v_2, v_3\}$ and $E(G) = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$. This graph forms a simple triangle as shown below:

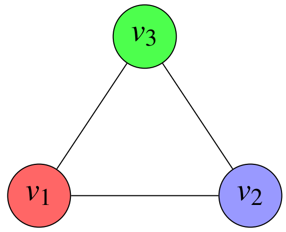


FIGURE 1. Triangle graph with vertices v_1, v_2, v_3

Definition 1.3 (Proper Coloring)**.** A *proper coloring* of a graph is an assignment of colors to its vertices such that no two adjacent vertices share the same color [1].

Example 1.4. In the triangle graph shown in Figure 1, we can properly color it using three colors: assign v_1 red, v_2 blue, and v_3 green.

Definition 1.5 (Graph Coloring Existence Problem and Chromatic Number)**.** The *graph coloring existence problem* involves determining whether it is possible to properly color a graph using a given number of colors. The minimum positive number of colors required to properly color a graph G is known as its *chromatic number* of G [1].

Example 1.6. Consider the following examples:

- For the complete graph (see Definition 3.5) with 3 vertices in Figure 1, the chromatic number is 3. To properly color the graph without any adjacent vertices sharing the same color, it requires three distinct colors. Thus, the Existence Problem can be solved with 3 colors.
- For a cycle graph (see Definition 3.7) with 4 vertices in Figure 2, the chromatic number is 2 because we can alternate between two colors around the square.

Definition 1.7 (Graph Coloring Counting Problem and Chromatic Polynomial)**.** Another significant question is the *graph coloring counting problem*, which asks: How many distinct ways can a graph be properly colored using exactly x colors? This problem is solved with the concept of

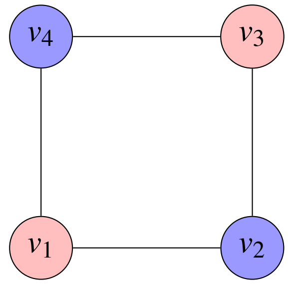


FIGURE 2. Cycle graph with 4 vertices

the *chromatic polynomial* $P(G,x)$. The **chromatic polynomial** is a polynomial in x (where x represents the number of colors) that evaluates to the total number of proper colorings of G with x colors [1].

Example 1.8. For example, consider a path graph (see Definition 3.3) with three vertices v_1, v_2, v_3:

- The chromatic polynomial is $P(G,x) = x(x-1)^2$ since the first vertex can be colored with any of x colors, the second vertex with any of the remaining $x-1$ colors, and the third vertex with $x-1$ colors.

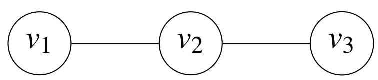


FIGURE 3. A path graph with 3 vertices

It is important to note that the chromatic polynomial of an **empty graph**, which contains no edges, is given by $P(G,x) = x^k$ where k is the number of vertices in the graph G. Since an empty graph has no edges, there are no restrictions on the coloring of its vertices; each vertex can be independently colored in any of the x colors. Thus, the chromatic polynomial is simply the number of ways to assign x colors to each of the k vertices, which is x^k.

Here we have used a combinatorial argument to get the formula for $P(G,x)$. Later we will also show how to use the Birkhoff-Lewis Reduction Algorithm to prove these formulas.

2. ALGORITHM

The Birkhoff-Lewis Reduction Algorithm breaks down the original graph into progressively simpler graphs until only empty graphs (with no edges) remain. The key idea behind the algorithm is that the number of proper colorings of the original graph can be determined by first calculating the total number of colorings, which includes both proper and improper colorings, and then subtracting the number of improper colorings. [1] Below is an explanation of how the algorithm works, followed by an example:

(1) **Focus on an edge:** The algorithm begins by selecting an edge in our original graph G, let us call it "e," in the graph.
(2) **Create two new graphs:** Two new graphs are generated based on the chosen edge "e":
   - **Graph 1 (Edge deletion):** The first graph, denoted as H_1, is created by removing the edge "e" from G. The chromatic polynomial of H_1 then would allow the vertices connected by "e" to be colored either the same or differently.
   - **Graph 2 (Vertex identification):** The second graph, denoted H_2, is formed by merging the two vertices that were originally connected by the edge "e." In H_2, these two vertices must be assigned the same color. The chromatic polynomial of H_2 corresponds to the number of improper colorings of G that were allowed by the deleted edge in H_1 where the two vertices connected by "e" are assigned the same color.
(3) **Apply the formula:** The number of proper colorings of the original graph is determined by the formula:

$$P(G,x) = P(H_1,x) - P(H_2,x)$$

where $P(G,x)$ represents the chromatic polynomial of graph G using x colors. The formula counts all possible proper colorings of H_1 and then subtracts the proper colorings of H_2 (which corresponds to improper colorings of G). In H_1, the two vertices connected by the edge "e" can be colored either the same or differently, which means we count both proper and some improper colorings of G. In H_2, the two vertices connected by "e" are forced to be the same color by the identification. This identification allows the proper colorings of H_2 to correspond $1-1$ with the improper colorings of G counted in coloring H_1. Therefore, this formula will result in the proper colorings of the original graph G.
(4) **Recursion:** The algorithm recursively applies the same process of edge deletion and vertex identification to the newly created graphs H_1 and H_2 until all the resulting graphs are empty.

(5) **Final Calculation:** Since at each application of the Birkhoff-Lewis Reduction Algorithm one edge is deleted, the process terminates when after $|E|$ steps, where $|E|$ is the number of edges in G. Since an empty graph with k vertices can be colored in x^k ways, the chromatic polynomials of these simple graphs are easily calculated. By combining these chromatic polynomials and the formula in (3), the algorithm calculates the chromatic polynomials of the original graph G.

Example 2.1. Let us consider the following graph G:

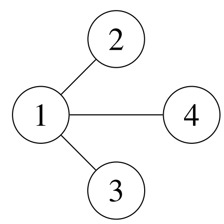


FIGURE 4. Graph G with vertices 1, 2, 3, 4

To compute the chromatic polynomial $P(G,x)$ we will use the Birkhoff-Lewis Reduction Algorithm as shown below.

After applying the algorithm to the original graph G, we generate its edge-deleted graph and vertex-identified graph. We then continue to apply the algorithm to these graphs to get $H_{1.1}$ (edge-deleted) and $H_{1.2}$ (vertex-identified) for the first decomposition, and $H_{1.3}$ (edge-deleted) and $H_{1.4}$ (vertex-identified) for the second. Using the formula, the number of proper colorings of G is now given by:

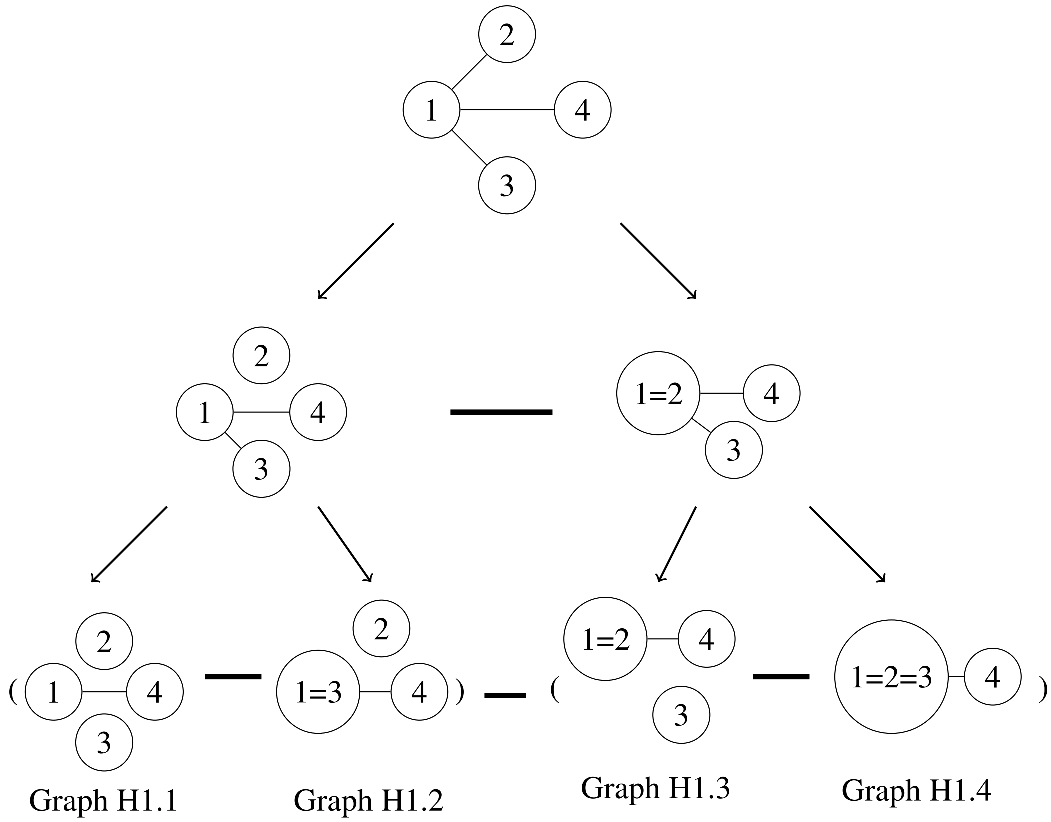
$$P_G(x) = (P_{H1.1}(x) - P_{H1.2}(x)) - (P_{H1.3}(x) - P_{H1.4}(x)).$$

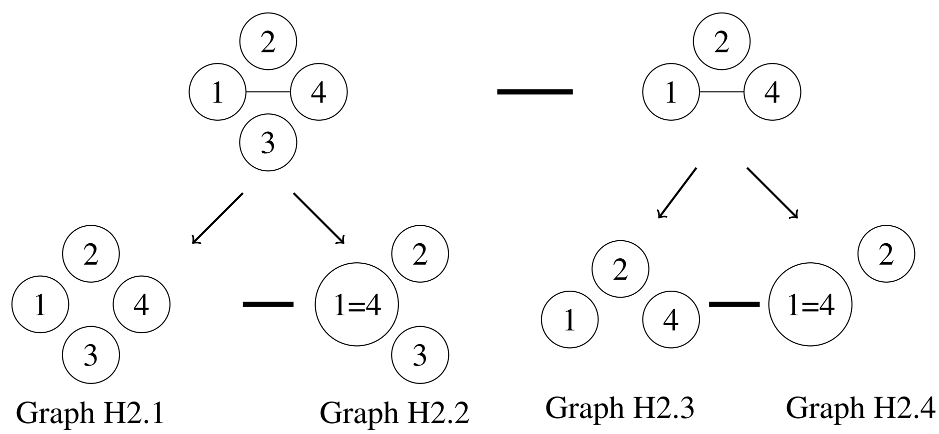
Next, let us apply the algorithm recursively to $H_{1.1}$ and $H_{1.2}$ to get four new graphs:

- $H_{2.1}$ and $H_{2.3}$ are the edge-deleted graphs of $H_{1.1}$ and $H_{1.2}$, respectively.
- $H_{2.2}$ and $H_{2.4}$ are the vertex-identified graphs of $H_{1.1}$ and $H_{1.2}$, respectively.

Now, we move on to applying the algorithm on $H_{1.3}$ and $H_{1.4}$. We have:

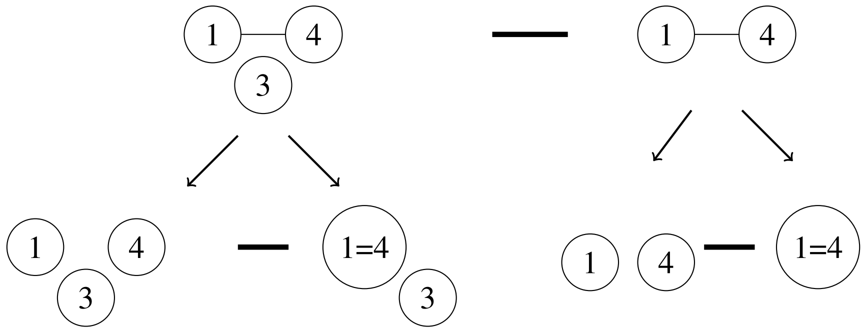
- $H_{2.5}$ and $H_{2.7}$ are the edge-deleted graphs of $H_{1.3}$ and $H_{1.4}$, respectively.
- $H_{2.6}$ and $H_{2.8}$ are the vertex-identified graphs of $H_{1.3}$ and $H_{1.4}$, respectively.





After recursively applying the Birkhoff-Lewis Reduction Algorithm and simplifying all graphs to empty graphs, we arrive at the following expression for the chromatic polynomial of the original graph G:

$$P_G(x) = (P_{H2.1}(x) - P_{H2.2}(x)) - (P_{H2.3}(x) - P_{H2.4}(x)) - (P_{H2.5}(x) - P_{H2.6}(x) - (P_{H2.7}(x) - P_{H2.8}(x))).$$



Graph H2.5 Graph H2.6 Graph H2.7 Graph H2.8

We can simplify this expression to:

$$P_G(x) = P_{H2.1}(x) - P_{H2.2}(x) - P_{H2.3}(x) + P_{H2.4}(x) - P_{H2.5}(x) + P_{H2.6}(x) + P_{H2.7}(x) - P_{H2.8}(x)$$

Since each H-subgraph is an empty graph, the chromatic polynomial for an empty graph with k vertices is given by x^k. Substituting these values, we calculate:

$$P_G(x) = x^4 - x^3 - x^3 + x^2 - x^3 + x^2 + x^2 - x = x^4 - 3x^3 + 3x^2 - x$$

Thus, the chromatic polynomial of the original graph G is $P_G(x) = x^4 - 3x^3 + 3x^2 - x$, which means the graph G can be properly colored using x colors in $x^4 - 3x^3 + 3x^2 - x$ ways.

3. Proofs

In the previous section, we introduced the Birkhoff-Lewis Reduction Algorithm as a tool for deriving chromatic polynomials. In this section, we apply this algorithm, mathematical induction, and combinatorial arguments when applicable to prove five results on the chromatic polynomials of general graphs. We begin by establishing two important theorems:

Theorem 3.1. *The chromatic polynomial of a graph G with n vertices and e edges always begins with the leading terms* $x^n - ex^{n-1} + \ldots$.

Theorem 3.2. *The coefficient of* x^{n-1} *in the chromatic polynomial of a graph G with n vertices is* $-e$*, where e is the number of edges in the graph G.*

Proof. We will prove both theorems by induction on the number of edges e.

Base case: When $e = 0$, the graph consists of n isolated vertices. The chromatic polynomial for such a graph is $P(G,x) = x^n$. Clearly, in this case, the leading term is x^n, and the coefficient of x^{n-1} is 0, which satisfies both the first and second theorems.

Inductive step: Assume that for any graph with n vertices and e edges, the chromatic polynomial has the form $P(G,x) = x^n - ex^{n-1} + \cdots$, where the first two leading terms are $x^n - ex^{n-1}$, and the coefficient of x^{n-1} is $-e$.

Consider a graph G with n vertices and $e+1$ edges. We will use the Birkhoff-Lewis Reduction Algorithm to relate the chromatic polynomial of G to the chromatic polynomials of two simpler graphs: one with n vertices and e edges and the other with $n-1$ vertices and fewer than e edges.

Let G' be a graph obtained by removing an edge from G, and let G'' be the graph obtained by identifying the two vertices connected by the deleted edge. Note that G' has n vertices and e edges, while G'' has $n-1$ vertices and some number of edges less than e.

From the inductive hypothesis, we know: $P(G',x) = x^n - ex^{n-1} + \cdots$. And the chromatic polynomial of G'' is: $P(G'',x) = x^{n-1} + \cdots$.

By the Birkhoff-Lewis Reduction Algorithm, we can express the chromatic polynomial of G as: $P(G,x) = P(G',x) - P(G'',x)$.

Substituting the known forms of $P(G',x)$ and $P(G'',x)$, we get:

$$P(G,x) = \left(x^n - ex^{n-1} + \cdots\right) - \left(x^{n-1} + \cdots\right).$$

This equation can be simplified to:

$$P(G,x) = x^n - (e+1)x^{n-1} + \cdots.$$

Thus, the first two leading terms of $P(G,x)$ are $x^n - (e+1)x^{n-1}$, which completes the inductive step. By induction, we conclude that for any graph with n vertices and e edges, the chromatic polynomial begins with $x^n - ex^{n-1}$. This completes the proof for both theorems. □

Definition 3.3 (Path)**.** An n-path graph P_n is a graph on n vertices and edges in which all vertices are distinct $V = \{v_1, v_2, \ldots v_n\}$ where the edges are $E = \{(v_1,v_2),(v_2,v_3),(v_3,v_4),\ldots,(v_{n-1},v_n)\}$[2]. Figure 3 shows an example for the 3-path graph.

Theorem 3.4. *The formula for the chromatic polynomial for an n-path P_n is $P_{P_n}(x) = x(x-1)^{n-1}$ for $n \geq 1$.*

Proof. We will prove this theorem using two approaches: a combinatorial argument and a proof by induction on the number of vertices, using the Birkhoff-Lewis Reduction Algorithm.

Combinatorial Argument: To find the chromatic polynomial $P_{P_n}(x)$, we count the number of ways to color the graph using x colors such that adjacent vertices receive different colors.

- The first vertex (at the start of the path) can be colored in x ways.
- Each subsequent interior vertex can be colored in $x-1$ ways because it must be different from its adjacent vertex.
- The last vertex (at the end of the path) is adjacent to only the second-to-last vertex, so it also has $x-1$ available color choices.

Therefore, the total number of ways to color the path is the product of the choices for each vertex, which is $P_{P_n}(x) = x \cdot (x-1)^{n-1}$. Thus, the chromatic polynomial for an n-path is $P_{P_n}(x) = x(x-1)^{n-1}$.

Inductive Proof: We now prove the same result using induction on the number of vertices.

Base case: When $n = 1$, the graph P_1 consists of a single vertex. The chromatic polynomial for a single vertex is $P_{P_1}(x) = x$. Thus, the base case holds with the chromatic polynomial $P_{P_1}(x) = x(x-1)^0 = x$.

Inductive step: Assume that the chromatic polynomial for a path graph with k vertices P_k is given by $P_{P_k}(x) = x(x-1)^{k-1}$.

Consider a path graph P_{k+1} with $k+1$ vertices. We will use the Birkhoff-Lewis Reduction Algorithm to express the chromatic polynomial of P_{k+1} as that of simpler graphs.

First, remove one of the edges from P_{k+1}, resulting in a graph P' consisting of k-path P_k and a single vertex. By the inductive hypothesis, the chromatic polynomial for P_k is $P_{P_k}(x) = x(x-1)^{k-1}$.

Because the separate vertex can be of any color, the chromatic polynomial for P' is:

$$P_{P'_k}(x) = x^2(x-1)^{k-1}.$$

Next, remove one of the vertices (along with the incident edge) from P_{k+1}, resulting in a path graph with k vertices. This graph has the same chromatic polynomial as P_k, so $P_{P_k''}(x) = x(x-1)^{k-1}$.

Using the Birkhoff-Lewis Reduction Algorithm, the chromatic polynomial of the graph P_{k+1} is given by the difference:

$$P_{P_{k+1}}(x) = P_{P_k'}(x) - P_{P_k''}(x) = x^2(x-1)^{k-1} - x(x-1)^{k-1}.$$

Factor out $(x-1)^{k-1}$ from the right-hand side, we get:

$$P_{P_{k+1}}(x) = (x-1)^{k-1}\left(x^2 - x\right) = x(x-1)^{k-1}(x-1).$$

Simplifying this result further gives us:

$$P_{P_{k+1}}(x) = x(x-1)^k = x(x-1)^{k+1-1}.$$

Thus we proved the theorem by induction. □

Definition 3.5 (Complete Graph)**.** A complete graph is a graph where every vertex is directly connected to every other vertex by an edge. Thus for a complete graph on n vertices, there will be $\frac{n(n-1)}{2}$ edges (since $1+2+...+(n-1) = \frac{n(n-1)}{2}$. Figure 1 shows an example of a complete graph with 3 vertices.

Theorem 3.6. *The formula for the chromatic polynomial for a complete graph with n vertices is:*

$$P_{G_n}(x) = x(x-1)(x-2)\dots(x-(n-1))$$

for $n \geq 1$.

Proof. We will prove this theorem using two approaches: a combinatorial argument and a proof by induction on the number of vertices.

Combinatorial Argument: The goal is to count the number of valid colorings for G_n using x colors, where adjacent vertices must receive different colors.

- The first vertex can be colored in x ways.

- The second vertex is adjacent to the first, so it must be colored differently. Thus, it can be colored in $x-1$ ways.
- The third vertex is adjacent to both the first and second vertices, so it must be colored differently from both of them. It can be colored in $x-2$ ways.
- Continuing this reasoning, the n-th vertex is adjacent to all the previous $n-1$ vertices and must be colored differently from each of them. It can be colored in $x-(n-1)$ ways.

Thus, the total number of valid colorings is the product of the available color choices for each vertex is $P_{G_n}(x) = x(x-1)(x-2)\ldots(x-(n-1))$. This formula is the chromatic polynomial of the complete graph with n vertices.

Inductive Proof: We now prove this result by induction on the number of vertices.

Base Case $n = 1$: For a complete graph with a single vertex, there is only one way to color the graph with x colors, provided $x \geq 1$. Therefore, the chromatic polynomial for G_1 is $P_{G_1}(x) = x$, so the base case holds.

Inductive Step: Assume the formula holds for a complete graph G_k with k vertices. We will show that the formula holds for G_{k+1}.

Consider a complete graph G_{k+1} with $k+1$ vertices. We obtain G_{k+1} by adding a new vertex v_{k+1} to G_k and adding an edge between v_{k+1} and all k vertices in G_k. To find the chromatic polynomial $P_{G_{k+1}}(x)$, we will apply the Birkhoff-Lewis Reduction Algorithm.

First, we delete an edge e connecting v_{k+1} to a vertex v_i in G_k. This creates G'_{k+1} with $k+1$ vertices and one fewer edge. By performing vertex identification on v_{k+1} and v_i, the graph is reduced to G_k, whose chromatic polynomial is $P_{G_k}(x)$ by the inductive hypothesis.

By the Birkhoff-Lewis reduction, the chromatic polynomial of G_{k+1} satisfies:

$$P_{G_{k+1}}(x) = P_{G'_{k+1}}(x) - P_{G_k}(x).$$

We apply the reduction recursively to all k edges connecting v_{k+1} to the vertices of G_k. In each step, the chromatic polynomial of a graph obtained by identifying v_{k+1} with a vertex of G_k (which has chromatic polynomial $P_{G_k}(x)$) is subtracted from the chromatic polynomial of a subgraph with one fewer edge. The chromatic polynomial of the graph with v_{k+1} isolated but containing no edges to G_k is $x \cdot P_{G_k}(x)$, since v_{k+1} can be independently colored in x ways. Subtracting $P_{G_k}(x)$ for each of the k edges connecting v_{k+1} to G_k, we get:

$$P_{G_{k+1}}(x) = x \cdot P_{G_k}(x) - k \cdot P_{G_k}(x).$$

Simplifying, we obtain $P_{G_{k+1}}(x) = (x-k) \cdot P_{G_k}(x)$.

Substituting the inductive hypothesis $P_{G_k}(x) = x(x-1)(x-2)\dots(x-(k-1))$, we get:

$$P_{G_{k+1}}(x) = x(x-1)(x-2)\dots(x-(k-1))(x-k).$$

Thus, by induction, the chromatic polynomial for a complete graph G_n with n vertices is $P_{G_n}(x) = x(x-1)(x-2)\dots(x-(n-1))$. This work completes the proof.

□

Definition 3.7 (Cycle)**.** A *cycle graph* is a graph in which all vertices are connected in a single closed loop, such that each vertex has exactly two neighbors. A cycle graph with n vertices, denoted as C_n, consists of the vertex set $V = \{v_1, v_2, \dots, v_n\}$, and the edge set $E = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$. Figures 2 and 1 show cycle graphs with 4 and 3 vertices, respectively.

Theorem 3.8. *Prove that the formula for the chromatic polynomial for an n-cycle C_n is $P_{C_n}(x) = (x-1)^n + (-1)^n(x-1)$ for $n \geq 2$.*

Proof. We will prove this theorem by induction on n, the number of vertices in the cycle C_n.

Base Case: When $n = 2$, the graph C_2 is just an edge between two vertices, and the chromatic polynomial is $P_{C_2}(x) = x(x-1)$. The formula in the theorem gives us: $(x-1)^2 + (-1)^2(x-1) = (x-1)^2 + (x-1) = x(x-1)$. Thus, the theorem holds for $n = 2$.

Inductive Step: Assume that for an n-cycle C_n, the chromatic polynomial is given by $P_{C_n}(x) = (x-1)^n + (-1)^n(x-1)$. Consider the $(n+1)$-cycle C_{n+1}. We will apply the Birkhoff-Lewis Reduction Algorithm to express the chromatic polynomial of C_{n+1} as that of an $(n+1)$-path graph (the deleted-edge graph) and an n-cycle graph (the vertex-identified graph).

We can reduce the $(n+1)$-cycle C_{n+1} by removing one of the edges, resulting in a graph G' that is an $(n+1)$-path. By Theorem 3.4, the chromatic polynomial of the $n+1$-path graph $P_{P_{n+1}}(x)$ is $P_{P_{n+1}}(x) = x(x-1)^n$. Next, we identify the two adjacent vertices connecting the deleted edge, resulting in an n-cycle graph C_n which by our induction assumption for the chromatic polynomial G: $P_{C_n}(x) = (x-1)^n + (-1)^n(x-1)$.

By the Birkhoff-Lewis Reduction Algorithm, we have $P_{C_{n+1}}(x) = P_{P_{n+1}}(x) - P_{C_n}(x)$. Substituting the known expressions for $P_{P_{n+1}}(x)$ and $P_{C_n}(x)$, we get $P_{C_{n+1}}(x) = x(x-1)^n - ((x-1)^n + (-1)^n(x-1))$, which can be simplified to $P_{C_{n+1}}(x) = x(x-1)^n - (x-1)^n - (-1)^n(x-1)$. Then we factor out $(x-1)^n$ from the first two terms to get that $P_{C_{n+1}}(x) = (x-1)^n\,(x-1) - (-1)^n(x-1)$, and finally that $P_{C_{n+1}}(x) = (x-1)^{n+1} - (-1)^n(x-1)$. Thus, by induction, the chromatic polynomial for an n-cycle C_n is $P_{C_n}(x) = (x-1)^n + (-1)^n(x-1)$. This work completes our proof. □

4. Conclusion

In this report, we explored the derivation and application of chromatic polynomials in graph theory using the Birkhoff-Lewis Reduction Algorithm. By combining this algorithm with mathematical induction and combinatorial reasoning, we derived chromatic polynomials for various classes of graphs, including paths, cycles, and complete graphs. If I had more time, I would probably work on exploring the chromatic polynomials of more complex graph structures, such as planar graphs or graphs with specific symmetry properties, and investigate the applicability of the Birkhoff-Lewis Reduction Algorithm to these cases. Additionally, I would like to learn about some applications of this topic in real life, such as in scheduling problems and network design.

References

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