Midterm Exam II

1. Consider the ARMA(2,1) process:

$$(1-1.2B+.8B^2)Z_t = (1-.8B)a_t$$

$$\text{or } Z_t = 1.2Z_{t-1} - .8Z_{t-2} + a_t - .8a_{t-1}$$

where $a_t \sim N(0,\sigma_a^2)$, $\phi_1 = 1.2$, $\phi_2 = -.8$, $\theta_1 = .8$

(a) Find ACF ρ_k for $k=0,1,2,3$

$$Z_{t-k}Z_t = 1.2Z_{t-k}Z_{t-1} - .8Z_{t-k}Z_{t-2} + a_tZ_{t-k} - .8a_{t-1}Z_{t-k}$$

$$E(Z_{t-k}Z_t) = 1.2E(Z_{t-k}Z_{t-1}) - .8E(Z_{t-k}Z_{t-2}) + E(Z_{t-k}a_t) - .8E(Z_{t-k}a_{t-1})$$

$$\gamma_k = 1.2\gamma_{k-1} - .8\gamma_{k-2} + E(Z_{t-k}a_t) - .8E(Z_{t-k}a_{t-1})$$

we have $E(Z_ta_t) = E(1.2Z_{t-1}a_t - .8Z_{t-2}a_t + a_t^2 - .8a_{t-1}a_t) = \sigma_a^2$

and $E(Z_ta_{t-1}) = E(1.2Z_{t-1}a_{t-1} - .8Z_{t-2}a_{t-2} + a_ta_{t-1} - .8a_{t-1}^2)$

$$= E(1.2Z_{t-1}a_{t-1}) - 0.8\sigma_a^2$$

$$= E(1.2Z_{t-2}a_{t-1} - 1.2(.8)Z_{t-3}a_{t-1} + 1.2a_{t-1}^2 - 1.2(.8)a_{t-2}a_{t-1}) - 0.8\sigma_a^2$$

$$= 1.2\sigma_a^2 - 0.8\sigma_a^2 = .4\sigma_a^2$$

$$\gamma_0 = 1.2\gamma_1 - .8\gamma_2 + \sigma_a^2 - .8(.4\sigma_a^2) = 1.2\gamma_1 - .8\gamma_2 + .68\sigma_a^2$$

$$\gamma_1 = 1.2\gamma_0 - .8\gamma_1 - .8\sigma_a^2$$

$$\gamma_1 + .8\gamma_1 = 1.2\gamma_0 - .8\sigma_a^2$$

$$\gamma_1 = \frac{1.2\gamma_0 - .8\sigma_a^2}{1.8}$$

$$\gamma_2 = 1.2\gamma_1 - .8\gamma_0$$

$$= 1.2\left(\frac{1.2\gamma_0 - .8\sigma_a^2}{1.8}\right) - .8\gamma_0$$

$$= \frac{2}{3}(1.2\gamma_0 - .8\sigma_a^2) - .8\gamma_0$$

$$= .8\gamma_0 - .533\sigma_a^2 - .8\gamma_0 = -.533\sigma_a^2$$

$$\gamma_0 = 1.2\left(\frac{1.2\gamma_0 - .8\sigma_a^2}{1.8}\right) - .8(-.533\sigma_a^2) + .68\sigma_a^2$$

$$= .8\gamma_0 - .533\sigma_a^2 + .4264\sigma_a^2 + .68\sigma_a^2$$

$$\gamma_0 - .8\gamma_0 = .5734\sigma_a^2$$

$$\gamma_0 = 2.867\sigma_a^2$$

$$\gamma_1 = \frac{1.2(2.867\sigma_a^2) - .8\sigma_a^2}{1.8} = 1.467\sigma_a^2$$

$$\gamma_3 = 1.2\gamma_2 - .8\gamma_1$$

$$= 1.2(-.533\sigma_a^2) - .8(1.467\sigma_a^2)$$

$$= -.1813\sigma_a^2$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{1.467\sigma_a^2}{2.867\sigma_a^2} = .5117$$

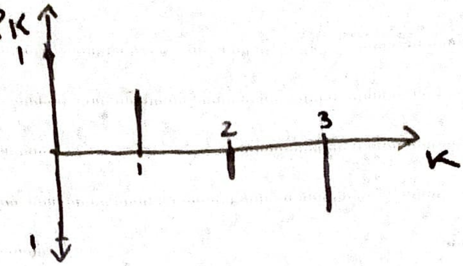
$$\rho_3 = \frac{\gamma_3}{\gamma_0} = \frac{-.1813\sigma_a^2}{2.867\sigma_a^2} = -.6324$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{-.533\sigma_a^2}{2.867\sigma_a^2} = -.1859$$

$\Rightarrow$ ACF for $k = 0, 1, 2, 3$

$$\rho_K = \begin{cases} 1 & K=0 \\ .5117 & K=1 \\ -.1859 & K=2 \\ -.6324 & K=3 \end{cases}$$

(b)



(c) Find PACF for $K = 0, 1, 2, 3$

$\phi_{11} = \rho_1 = .5117$

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} = \frac{-.1859 - (.5117)^2}{1 - .5117^2} = -.6066$$

$$\phi_{33} = \frac{\rho_3 + \rho_1^3 + \rho_1\rho_2^2 - \rho_1\rho_2 - \rho_1\rho_2 - \rho_1^2\rho_3}{1 + \rho_1^2\rho_2 + \rho_1^2\rho_2 - \rho_2^2 - \rho_1^2 - \rho_1^2}$$

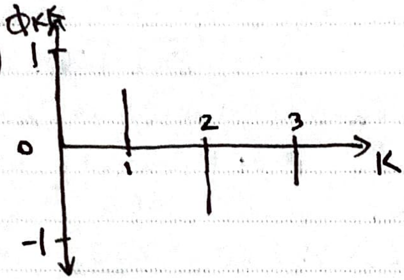
$$= \frac{(-.6324) + .5117^3 + .5117(-.1859)^2 - 2(.5117)(-.1859) - .5117^2(-.6324)}{1 + .5117^2(-.1859) + .5117^2(-.1859) - (-.1859)^2 - .5117^2 - .5117^2}$$

$= -.3626$

$\Rightarrow$ PACF is

$$\phi_{KK} = \begin{cases} 1 & K=0 \\ .5117 & K=1 \\ -.6066 & K=2 \\ -.3626 & K=3 \end{cases}$$

(d)



(f) verify whether ARMA(2,1) is stationary or invertible or both

$$\begin{cases} \phi_2 + \phi_1 = -.8 + 1.2 = .4 < 1 \\ \phi_2 - \phi_1 = -.8 - 1.2 = -2 < 1 \\ -1 < \phi_2 = -.8 < 1 \end{cases} \Rightarrow \text{Stationary}$$

$|\theta_1| = |.8| < 1 \Rightarrow$ invertible

(g) Express an MA representation. Give first 3 ψ_j weight, $j = 1, 2, 3$

$$Z_t = \psi(B)\, a_t = \frac{(1-\theta_1 B)}{(1-\phi_1 B)} a_t$$

$$(1 - \phi_1 B - \phi_2 B^2)(1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = (1 - \theta_1 B)$$

$$1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots$$
$$- \phi_1 B - \phi_1\psi_1 B^2 - \phi_1\psi_2 B^3 - \phi_1\psi_3 B^4 + \dots$$
$$- \phi_2 B^2 - \phi_2\psi_1 B^3 - \phi_2\psi_2 B^4 - \phi_2\psi_3 B^5 + \dots = (1 - \theta_1 B)$$

$B: \psi_1 - \phi_1 = -\theta_1 \Rightarrow \psi_1 = -\theta_1 + \phi_1$

$B^2: \psi_2 - \phi_1\psi_1 - \phi_2 \Rightarrow \psi_2 = \phi_1\psi_1 + \phi_2$

$B^3: \psi_3 - \phi_1\psi_2 - \phi_2\psi_1 \Rightarrow \psi_3 = \phi_1\psi_2 + \phi_2\psi_1$

$$\psi_j = \begin{cases} -.8 + 1.2 = .4 & j=1 \\ 1.2(.4) - .8 = -.32 & j=2 \\ 1.2(-.32) + (-.8)(.4) = -.704 & j=3 \end{cases}$$

$\Rightarrow Z_t = (1 + .4B - .32B^2 - .704B^3 + \dots)a_t$

(h) Express the process in an AR representation. Give the first 3 π_j weights, $j = 1,2,3$

$$\pi(B)Z_t = a_t = \frac{(1-\phi_1 B)}{(1-\theta_1 B)} Z_t$$

$$(1-\phi_1 B - \phi_2 B^2) = (1-\theta_1 B)(1-\pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \dots)$$

$$(1-\phi_1 B - \phi_2 B^2) = 1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \dots$$
$$-\theta_1 B + \theta_1\pi_1 B^2 + \theta_1\pi_2 B^3 - \theta_1\pi_3 B^4 - \dots$$

$B: -\phi_1 = -\pi_1 - \theta_1 \Rightarrow \pi_1 = \phi_1 - \theta_1$

$B^2: -\phi_2 = -\pi_2 + \theta_1\pi_1 \Rightarrow \pi_2 = \phi_2 + \theta_1\pi_1$

$B^3: -\pi_3 + \theta_1\pi_2 \Rightarrow \pi_3 = \theta_1\pi_2$

$$\Rightarrow \pi_j = \begin{cases} 1.2 - .8 = .4 \\ -.8 + (.8)(.4) = -.48 \\ .8(-.48) = -.384 \end{cases}$$

$(1 - .4B + .48B^2 + .384B^3 - \dots)Z_t = a_t$