Determining the in-plane elastic modulus of two-dimensional 3-Archimedean, 3-uniform (3^6; $3^3 4^2$; $3^2 4.3.4$)- and ($3^4 6$; $3^3 4^2$; $3^2 4.3.4$)-tilings

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This project reviews the method of determining the relative density and elastic modulus of a lattice material, and then further examine its implementation to 2 two-dimensional 3-Archimedean, 3-uniform tilings: (3^6; $3^3 4^2$; $3^2 4.3.4$) and ($3^4 6$; $3^3 4^2$; $3^2 4.3.4$). The relative densities are calculated analytically in this project, and the constants of the scaling law are determined using finite elements method with the help of Abaqus software and MATLAB.

1. Introduction:

Lattice material, as described by Fleck & Deshpande & Ashby (2010), are cellular, reticulated, truss or lattice structure made up of many uniform lattice elements (e.g., slender beams or rods) and generated by tessellating a unit cell, comprised of just a few lattice elements, throughout space. That is, a lattice material can be constructed from a single unit cell, and with appropriate arrangement we can duplicate the cell to recreate the material. Some common examples of lattice materials are triangular lattice, hexagonal lattice (regular lattice) and Kagome lattice (semi-regular lattice), as shown in Figure 1:

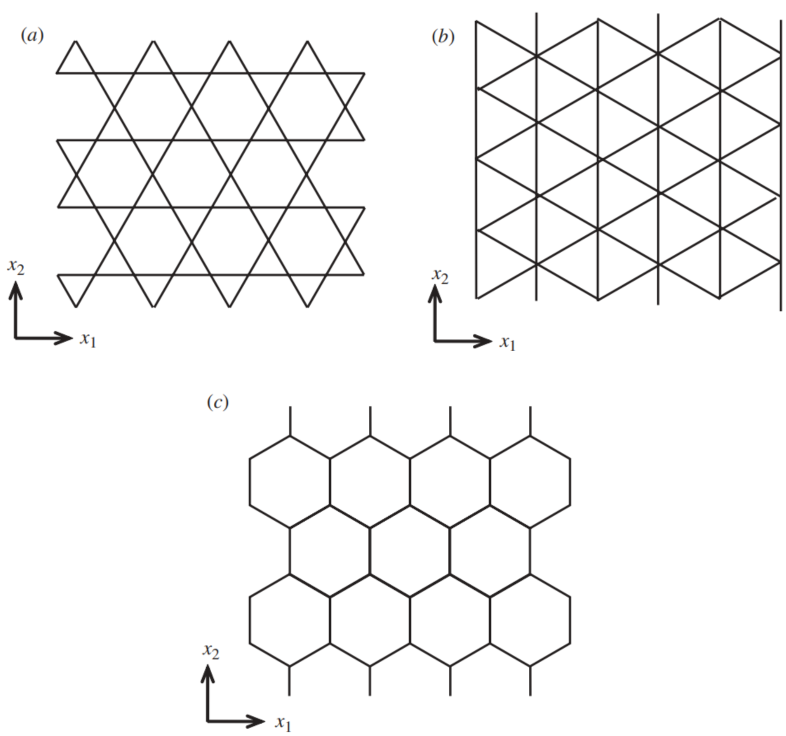


Figure 1. (a) Kagome lattice, (b) triangular lattice and (c) hexagonal lattice.

Lattice materials' mechanical properties can be divided into two distinct species: stretching-dominated and bending-dominated. Stretch-dominated structures offer greater stiffness and strength per unit weight than those in which the dominant mode of deformation is by bending, according to Deshpande & Ashby & Fleck (2001), and this distinction is governed by the nodal connectivity of the structures: typically, the more beams connected to a joint, the more likely it is for the structure to be stretching-dominated.

The relative density $\bar{\rho}$ of a lattice material is defined as the ratio of the density of the lattice material to the density of the solid. Given t is the thickness of the beam and l is the length of a beam, then $\bar{\rho}$ can be defined as:

$$\bar{\rho} = A\left(\frac{t}{l}\right) \tag{1.1}$$

for a two-dimensional lattice. The constant of proportionality A depends upon the geometry, and the determination of value A for the two structures in question will be further discussed in later parts. Furthermore, applying simple beam theory can determine the macroscopic Young modulus E of the structure in term of the relative density $\bar{\rho}$ and the Young modulus E_S of the material as following:

$$\frac{E}{E_S} = B\bar{\rho}^b \tag{1.2}$$

where coefficient B is the scaling factor and the exponent b is determined by the mechanical property of the material. In particular, $b = 1$ indicate that the material is stretching-dominated and $b = 3$ indicate that material is bending-dominated. For example, the following table gives the coefficient values for the hexagonal, triangular and Kagome geometry:

Table 1. Value of the constants A, B and b of the materials in Figure 1

topology	A	B	b
hexagonal	$2/\sqrt{3}$	$3/2$	3
triangular	$2\sqrt{3}$	$1/3$	1
Kagome	$\sqrt{3}$	$1/3$	1

In this report, we will mainly discuss the determination of the constants A, B, and b for two tilings: (3^6; $3^3 4^2$; $3^2 4.3.4$)-tiling and ($3^4 6$; $3^3 4^2$; $3^2 4.3.4$)-tiling (figure 2). Section 2 will cover the methodology of evaluating constant A analytically from the definition and constant B, b numerically by finite elements method. Section 3 will go through the figures used in the simulation, and Section 4 will apply the discussed method to calculate the constants of the two tilings. The final part will give a brief comparison between the two structures as well as the posibility of further research on other properties of the materials.

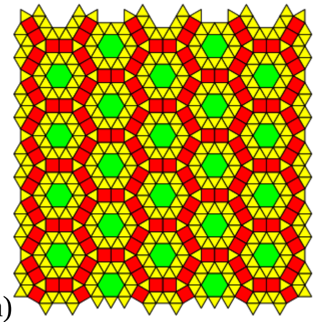
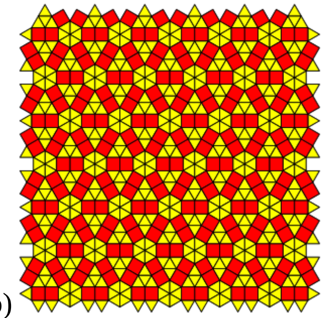

(a)

(b)

Figure 2. (a) (3^6; $3^3 4^2$; $3^2 4.3.4$)-tiling and (b) ($3^4 6$; $3^3 4^2$; $3^2 4.3.4$)-tiling

2. Methodology:

In this part, we will go though the methods of determining constant A from the definition of relative density $\bar{\rho}$, constant b using the results of the simulation on Abaqus software and the approximation of constant B by finite elements method.

2.1. Relative density $\bar{\rho}$ and constant A:

Using the definition of relative density $\bar{\rho}$ we have that:

$$\bar{\rho} = \frac{V_M}{V_S} \tag{2.1.1}$$

where V_M is the volume of the material (or the total volume of the beams combined) and V_S is the volume of the whole structure. Since a lattice material is a tessellation from one cell, therefore we can determine the constant by scaling it down to one single cell. Consider a unit cell with width w and height h (assuming the tessellating pattern is rectangular), and the component beams have length l_b, width w_b and thickness t. If the unit cell comprises of n beams, then the relative density can be defined as:

$$\bar{\rho} = \frac{V_M}{V_S} = \frac{n(l_b w_b t)}{w_b w h} = \frac{n l_b t}{w h} \tag{2.1.2}$$

If we rearrange the width $w = w' l_b$ and height $h = h' l_b$, then we have:

$$\bar{\rho} = \frac{nl_b t}{wh} = \frac{nl_b t}{w'l_b \cdot h'l_b} = \frac{n}{w'h'} \cdot \frac{t}{l_b} \tag{2.1.3}$$

Therefore from (1.1) and (2.1.3) we can determine the constant A as:

$$A = \frac{n}{w'h'} \tag{2.1.4}$$

Note that n, w' and h' are calculated from the shape of one unit cell, therefore the value A is dependent to the geometry of the lattice material and is constant when both the thickness t and length l_b of the beam change.

2.2. Scaling law and constants B, b:

From the equation 1.2, we can see that given a set of relative density value $\bar{\rho}$, if we can calculate the corresponding value $\frac{E}{E_S}$ to the given $\bar{\rho}$ using computational simulation and map all points $\left(\bar{\rho}^b, \frac{E}{E_S}\right)$ to the plane, then we can determine the value of exponent b. Since b can only be 1 (stretching-dominated) or 3 (bending-dominated), the suitable value b would make $\frac{E}{E_S}$ linearly proportional to $\bar{\rho}^b$. Finally, using least square method we can find the value B as the scaling factor, which fulfills the equation 1.2.

In order to find the value $\frac{E}{E_S}$ corresponding to the value $\bar{\rho}$, we can set up a simulation in which the material is put in tension with a total force of F evenly distributed across the area A_M. Under the effect of stress, the material with have a displacement of ΔH. Applying Hooke's law we have the macroscopic Young modulus of the material is:

$$E = \frac{\sigma}{\varepsilon} = \frac{\left(\frac{F}{A_M}\right)}{\left(\frac{\Delta H}{H}\right)} = \frac{F}{Ww_b} \cdot \frac{H}{\Delta H} = \frac{FH}{Ww_b \Delta H} \tag{2.2.1}$$

where W and H is the width and height of the whole structure used in the simulation, and w_b is the width of the component beams. Ideally the material in simulation must be large enough compared to the size of the unit cell ($w \ll W$ and $h \ll H$). Let W, H, w_b and ΔH be constants during the simulation, and assume that the Young modulus E_S of the material is constant:

$$\frac{E}{E_S} = \frac{FH}{E_S W w_b \Delta H} \tag{2.2.2}$$

We can see from 2.2.2 that $\frac{E}{E_S}$ only changes dependently to the variable F, and from 1.1 if we keep the length of the bar l_b constant then $\bar{\rho}$ changes linearly dependent to the variable t. Therefore, if we apply a fixed displacement ΔH to all simulations and change the value of t of the model in each of them, then using finite elements analysis will give us the force F correspondingly. Finally, applying the equation 2.2.2 as formulated above will give us the value $\frac{E}{E_S}$ for each t, and with simple scaling we will have the relationship between $\bar{\rho}$ and $\frac{E}{E_S}$.

3. Simulation and figures used in the simulation:

From the method discussed we can see that in the simulation, the thickness t of the beam is the only variable changing, therefore we can keep other variables constant to facilitate the simulation as well as the calculation of the forces. The following table summarise the figures used in the simulation of the project:

variable	***symbol***	***value***	***unit***
Young modulus of the material	E	200000.0	N/m^2
Poisson ratio	v	0.3	
Width of each beam	w_b	1.0	m
Length of each beam	l_b	10.0	m
Displacement of the structure	ΔH	1.0	m

For the simulation of both (3^6; $3^3 4^2$; $3^2 4.3.4$)-tiling and ($3^4 6$; $3^3 4^2$; $3^2 4.3.4$)-tiling, we use a lattice model consist of 64 unit cells with size 8x8 and 16x4, respectively. The bottom of the structure is fixed, and the top of the structure is applied a uniform displacement of $\Delta H = 1.0$. The size of the model is also fixed in order to avoid rotational effect. The material used in this simulation is deformable beam, with the relative density $\bar{\rho}$ kept below 0.3 in order to guarantee the linearity between the stress and strain, and finally the finite element method is applied with a mesh size of 0.1m.

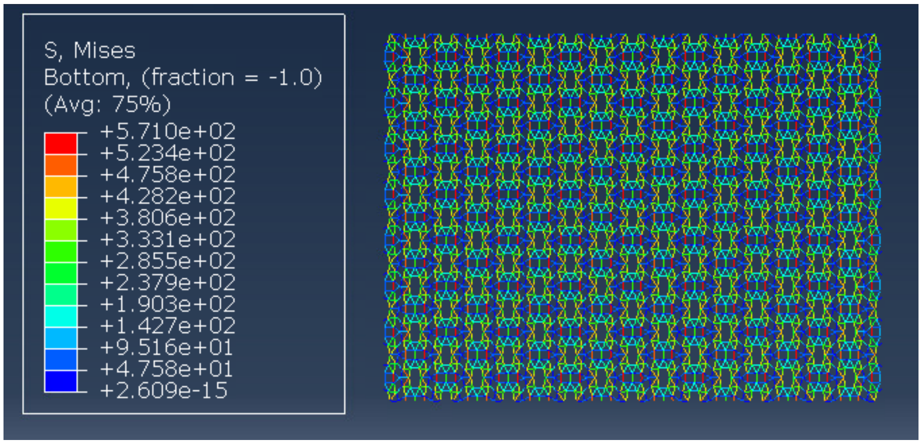


Figure 3. Simulation of the $(3^6; 3^3 4^2; 3^2 4.3.4)$-tiling

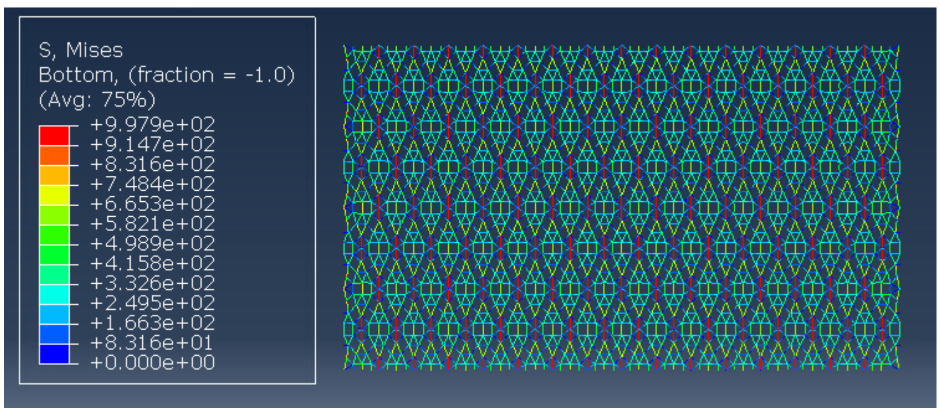


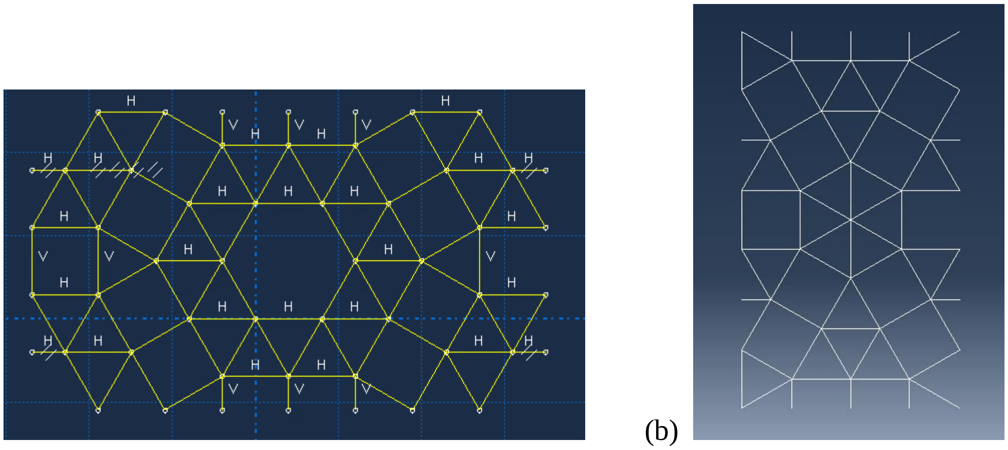
Figure 4. Simulation of the $(3^4 6; 3^3 4^2; 3^2 4.3.4)$-tiling

Throughtout the simulations, the value $\bar{\rho}$ of both models is kept below 0.3 as a benchmark, with the thickness t of the beams taking the value of 0.1, 0.3, 0.5, 0.7 and 0.9m respectively. Therefore, for each tiling, a set of 5 models with different beam thickness will be put into simulation to find the corresponding total force F and $\frac{E}{E_S}$ for the mapping.

4. Relative density constant and the in-plane stiffness of the tilings:

4.1. Determining the constant A from the relative density $\bar{\rho}$:

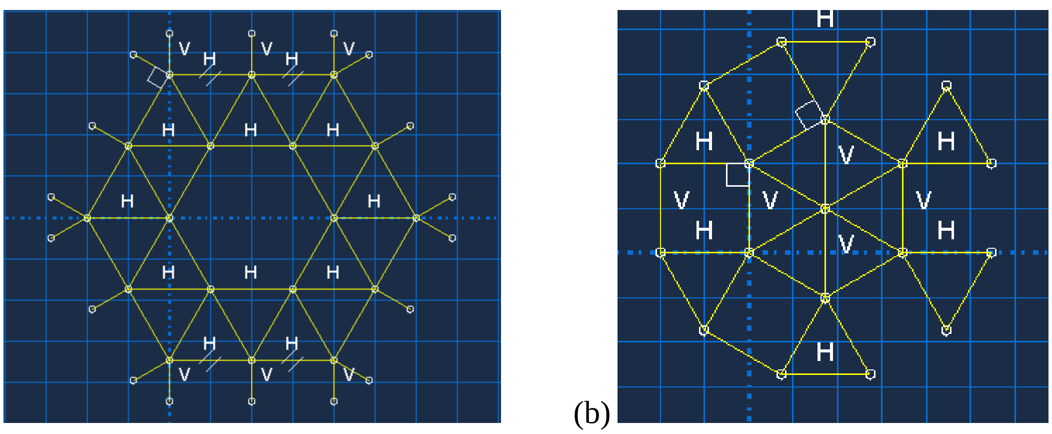
In order to calculate the constant A, first we have to determine the unit cell of each tiling:



(a) (b)

Figure 5. The unit cell of the (a) (3^6; 3^34^2; $3^24.3.4$)-tiling and (b) (3^46; 3^34^2; $3^24.3.4$)-tiling

Note that for each tiling, the unit cell is not unique, but rather dependent on the tessellating method. For example, figure 5 provide the unit cell for the rectangular arrangement method (that is a new cell can be added in 4 directions). Figure 6 provides the unit cell for the hexagonal arrangement method:



(a) (b)

Figure 6. Another unit cell of the (a) (3^6; 3^34^2; $3^24.3.4$)-tiling and (b) (3^46; 3^34^2; $3^24.3.4$)-tiling

Although the hexagonal unit cells consist of fewer beams, this part will analyze the solution using the unit cells in figure 5 since rectangular unit cells give a much simpler formula for computing the area, as discussed in part 2.1. Let w and h be the width and height of the unit cells, then we have the following table:

tiling	w	h	n
(3^6; 3^34^2; $3^24.3.4$)	$(6+\sqrt{3})l_b$	$(1+2\sqrt{3})l_b$	90
(3^46; 3^34^2; $3^24.3.4$)	$(2+\sqrt{3})l_b$	$(3+2\sqrt{3})l_b$	66

Applying the method discussed in 2.1, if we take th coefficient $w' = \frac{w}{l_b}$ and $h' = \frac{h}{l_b}$, then using the equation 2.1.4 we have the value of constants A are:

tiling	w'	h'	$A = \frac{n}{w'h'}$
(3^6; $3^3 4^2$; $3^2 4.3.4$)	$(6+\sqrt{3})$	$(1+2\sqrt{3})$	$\frac{30}{121}(13\sqrt{3}-12)$
($3^4 6$; $3^3 4^2$; $3^2 4.3.4$)	$(2+\sqrt{3})$	$(3+2\sqrt{3})$	$154\sqrt{3}-264$

That is, the coefficient of (3^6; $3^3 4^2$; $3^2 4.3.4$)-tiling and ($3^4 6$; $3^3 4^2$; $3^2 4.3.4$)-tiling are approximately 2.6074 and 2.7358, respectively.

4.2. Determining the constants B, b from the scaling law:

For both simulations of the 2 tilings, we examine the change of total forces on the same models as in figure 3 and 4, with the thickness t of the beams taking the value of 0.1, 0.3, 0.5, 0.7 and 0.9. Then the next step is to use the calculated value F to calculate $\frac{E}{E_S}$ using the equation 2.2.2 and the values stated in Section 3. The following two tables show the result of the simulations:

Table 3. Relative density $\bar{\rho}$, total force F and the ratio $\frac{E}{E_S}$ of the (3^6; $3^3 4^2$; $3^2 4.3.4$)-tiling

t	$\bar{\rho}$	F	$\frac{E}{E_S} = \frac{FH}{E_S W w_b \Delta H}$
0.1	0.02607	2155.76	$6.2231431 \cdot 10^{-3}$
0.3	0.07822	6479.66	$1.8705167 \cdot 10^{-2}$
0.5	0.13037	10840.2	$3.1292962 \cdot 10^{-2}$
0.7	0.18252	15260.2	$4.4052403 \cdot 10^{-2}$
0.9	0.23467	19760.0	$5.7042207 \cdot 10^{-2}$

Table 4. Relative density $\bar{\rho}$, total force F and the ratio $\frac{E}{E_S}$ of the (3^6; $3^3 4^2$; $3^2 4.3.4$)-tiling

t	$\bar{\rho}$	F	$\frac{E}{E_S} = \frac{FH}{E_S W w_b \Delta H}$
0.1	0.02736	3703.49	$8.0182911 \cdot 10^{-3}$
0.3	0.08207	11122.0	$2.4079836 \cdot 10^{-2}$
0.5	0.13679	18574.9	$4.0215838 \cdot 10^{-2}$
0.7	0.19151	26083.6	$5.6472651 \cdot 10^{-2}$
0.9	0.24622	33667.9	$7.2893142 \cdot 10^{-2}$

The final step is to map the pair $\left(\bar{\rho}, \frac{E}{E_S}\right)$ onto the plane and determine the value of the constants B and b. The mapping and the approximation of B is carried out using MATLAB as following:

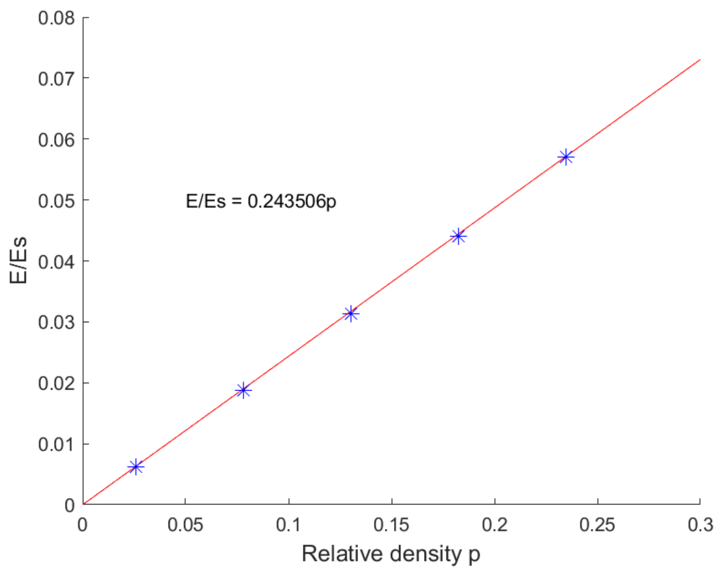


Figure 7. The relationship between the relative density $\bar{\rho}$ and the ratio $\frac{E}{E_S}$ of the $(3^6; 3^34^2; 3^24.3.4)$-tiling

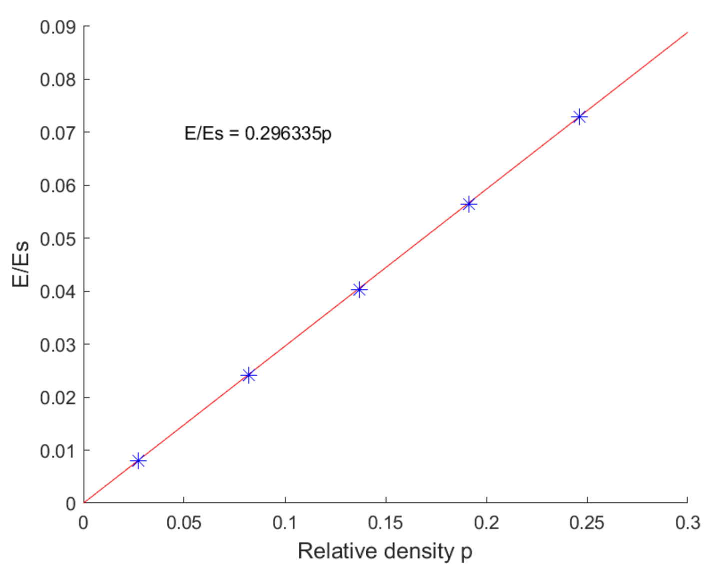


Figure 8. The relationship between the relative density $\bar{\rho}$ and the ratio $\frac{E}{E_S}$ of the $(3^46; 3^34^2; 3^24.3.4)$-tiling

Since the relationship between $\bar{\rho}$ and $\frac{E}{E_S}$ in both models are linear, therefore for both geomertries we have $b = 1$. Using the bulit-in function $\mathrm{polyfit}$ in MATLAB we can determine the value of B by the least-square method as following:

tiling	B
(3^6; $3^3 4^2$; $3^2 4.3.4$)	0.243505820979724
($3^4 6$; $3^3 4^2$; $3^2 4.3.4$)	0.296334734487406

Therefore, the constant B for the (3^6; $3^3 4^2$; $3^2 4.3.4$)-tiling is approximately 0.24351, and for the ($3^4 6$; $3^3 4^2$; $3^2 4.3.4$)-tiling is approximately 0.29633, while the constant b for both geometries is $b = 1$.

5. Conclusion:

In conclusion, both (3^6; $3^3 4^2$; $3^2 4.3.4$)-tiling and ($3^4 6$; $3^3 4^2$; $3^2 4.3.4$)-tiling are stretching-dominated as the value of b are both equal to 1. The tiling (3^6; $3^3 4^2$; $3^2 4.3.4$) has the constant $B = 0.24351$ which is less than $B = 0.29633$ of the tiling ($3^4 6$; $3^3 4^2$; $3^2 4.3.4$), implying that the latter structure has higher stiffness than the former one, and both of which have lower stiffness than the regular lattices discussed in Section 1 (triangular and hexagonal). More calculation and research can be further carried out on other properties of the structures (such as strength, fracture toughness and damage tolerance), giving a better insight to the material for better implementation of them into engineering.

References:

1. Fleck, N. & Deshpande, V.S. & Ashby, M. (2010). Micro-architectured materials: Past, present and future. Proceedings of The Royal Society A: Mathematical, Physical and Engineering Sciences. 466: 2495-2516. 10.1098/rspa.2010.0215.
2. Deshpande, V.S. & Ashby, M. & Fleck, N. (2001). Foam topology: Bending versus stretching dominated architectures. Acta Materialia. 49. 1035-1040. 10.1016/S1359-6454(00)00379-7.