## COLUMN RANK AND ROW RANK ARE AQUAL

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Abstract: a proof that column rank and row rank are equal.

**Example 0.1.** The following matrix has column rank 1 because it has row rank 1.

$$X_{1\times *} = (0 \ 1 \ 2 \ 3 \dots)$$

**Example 0.2.** The following dataset has 2 linearly independent samples because it has 2 linearly independent features.

$$X_{*\times 2} = \begin{pmatrix} 0 & 1\\ 2 & 3\\ 4 & 5\\ \vdots & \vdots \end{pmatrix}$$

Consider a matrix

$$A_{n \times m} = \left(\begin{array}{ccc} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{array}\right)$$

**Definition 0.3.** We define its column rank r to be its maximal number of linearly independent columns.

This is the dimension of the span of its columns  $\dim(\text{span}(\text{columns of }A))$ , also the dimension of its image  $\dim(\text{im}(A))$ .

**Definition 0.4.** We define its row rank s to be its maximal number of linearly independent rows.

This is the dimension of the span of its rows  $\dim(\text{span}(\text{rows of }A))$ , also the dimension of its transpose's image  $\dim(\text{im}(A^t))$ .

**Proposition 0.5.** Its column rank and its row rank are equal r = s.

Proof. Take a basis

$$\left(\begin{array}{c}c_{11}\\\vdots\\c_{n1}\end{array}\right),\ldots,\left(\begin{array}{c}c_{1r}\\\vdots\\c_{nr}\end{array}\right)$$

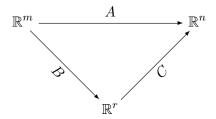
for span(columns of A) and write the columns of A as linear combinations of these basis vectors

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} = b_{11} \begin{pmatrix} c_{11} \\ \vdots \\ c_{n1} \end{pmatrix} + \dots + b_{r1} \begin{pmatrix} c_{1r} \\ \vdots \\ c_{nr} \end{pmatrix}$$

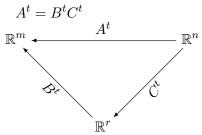
$$\begin{pmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix} = b_{1m} \begin{pmatrix} c_{11} \\ \vdots \\ c_{n1} \end{pmatrix} + \dots + b_{rm} \begin{pmatrix} c_{1r} \\ \vdots \\ c_{nr} \end{pmatrix}$$

Written together

$$\begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1r} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nr} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{r1} & \dots & b_{rm} \end{pmatrix}$$



After transposition



Hence  $s = \dim(\operatorname{im}(A^t)) = \dim(\operatorname{im}(B^tC^t)) \leq \dim(\operatorname{im}(C^t)) \leq \dim(\mathbb{R}^r) = r$ . Repeat the above discussion for  $A^t$  to get the reverse inequality  $r \leq s$ . So r = s.