# COLUMN RANK AND ROW RANK ARE AQUAL

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Abstract: a proof that column rank and row rank are equal.

**Example 0.1.** The following matrix has column rank 1 because it has row rank 1.

$$X_{1\times *} = \begin{pmatrix} 0 & 1 & 2 & 3 & \dots \end{pmatrix}$$

**Example 0.2.** The following dataset has 2 linearly independent samples because it has 2 linearly independent features.

$$X_{*\times 2} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \\ \vdots & \vdots \end{pmatrix}$$

Consider a matrix

$$A_{n\times m} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix}$$

**Definition 0.3.** We define its column rank $r$ to be its maximal number of linearly independent columns.

This is the dimension of the span of its columns dim(span(columns of $A$)), also the dimension of its image dim(im($A$)).

**Definition 0.4.** We define its row rank $s$ to be its maximal number of linearly independent rows.

This is the dimension of the span of its rows dim(span(rows of $A$)), also the dimension of its transpose's image dim(im($A^t$)).

**Proposition 0.5.** *Its column rank and its row rank are equal $r = s$.*

*Proof.* Take a basis

$$\begin{pmatrix} c_{11} \\ \vdots \\ c_{n1} \end{pmatrix}, \dots, \begin{pmatrix} c_{1r} \\ \vdots \\ c_{nr} \end{pmatrix}$$

for span(columns of $A$) and write the columns of $A$ as linear combinations of these basis vectors

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} = b_{11} \begin{pmatrix} c_{11} \\ \vdots \\ c_{n1} \end{pmatrix} + \dots + b_{r1} \begin{pmatrix} c_{1r} \\ \vdots \\ c_{nr} \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix} = b_{1m} \begin{pmatrix} c_{11} \\ \vdots \\ c_{n1} \end{pmatrix} + \cdots + b_{rm} \begin{pmatrix} c_{1r} \\ \vdots \\ c_{nr} \end{pmatrix}$$

Written together

$$\begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1r} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nr} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{r1} & \dots & b_{rm} \end{pmatrix}$$

$$A = CB$$

$$\begin{tikzcd} \mathbb{R}^m \arrow[rr, "A"] \arrow[dr, "B"'] & & \mathbb{R}^n \\ & \mathbb{R}^r \arrow[ur, "C"'] & \end{tikzcd}$$

After transposition

$$A^t = B^t C^t$$

$$\begin{tikzcd} \mathbb{R}^m & & \mathbb{R}^n \arrow[ll, "A^t"'] \arrow[dl, "C^t"] \\ & \mathbb{R}^r \arrow[ul, "B^t"] & \end{tikzcd}$$

Hence $s = \dim(\operatorname{im}(A^t)) = \dim(\operatorname{im}(B^t C^t)) \leq \dim(\operatorname{im}(C^t)) \leq \dim(\mathbb{R}^r) = r$.
Repeat the above discussion for $A^t$ to get the reverse inequality $r \leq s$.
So $r = s$. $\square$