PHENOMENA IN HIGH-DIMENSIONAL SPACES $\mathbb{R}^m$

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Abstract: we look at some phenomena in high-dimensional spaces.

Contents

1. Introduction

Let $\mathbb{R}^m$ be a m-dimensional Euclidean space at let $x = (x_1, \dots, x_m)$ be a point in $\mathbb{R}^m$.

Definition 1.1. We define the ball centered at 0 of radius r as

$$B_m(0, r) = \{x \in \mathbb{R}^m, \|x\| \le r\}$$

Definition 1.2. We define the inner ball centered at 0 of radius $r - \epsilon$ as

$$B_m(0, r - \epsilon) = \{x \in \mathbb{R}^m, \|x\| \le r - \epsilon\}$$

Definition 1.3. We define the spherical shell between the ball $B_m(0, r)$ and its inner ball $B_m(0, r - \epsilon)$ as

$$S_m(0, r - \epsilon, r) = \{x \in \mathbb{R}^m, r - \epsilon \le \|x\| \le r\}$$

Definition 1.4. We define the sphere centered at 0 of radius r as

$$S_m(0, r) = \{x \in \mathbb{R}^m, \|x\| = r\}$$

One can see the spherical shell $S_m(0, r - \epsilon, r)$ consists of spheres $S_m(0, q)$ for all $r - \epsilon \le q \le r$.

Definition 1.5. We define the equator of the sphere $S_m(0, r)$ as

$$E_m(0, r) = \{x \in \mathbb{R}^m, \|x\| = r, x_m = 0\}$$

Definition 1.6. We define the belt of width ϵ around the equator $E_m(0, r)$ of the sphere $S_m(0, r)$ as

$$E_m(0, r, \epsilon) = \{x \in \mathbb{R}^m, \|x\| = r, -\epsilon \le x_m \le \epsilon\}$$

Theorem 1.7. *The ball $B_m(0, r)$ has volume*

$$vol_m(B_m(0, r)) = \frac{\pi^{m/2} r^m}{\Gamma(m/2 + 1)}$$

where Γ is the gamma function.

Proof. literature. □

Example 1.8. We know $vol_2(B_2(0, r)) = \frac{\pi^{2/2} r^2}{\Gamma(2/2+1)} = \pi r^2$.

Example 1.9. We know $vol_3(B_3(0, r)) = \frac{\pi^{3/2} r^3}{\Gamma(3/2+1)} = \frac{\pi^{3/2} r^3}{\frac{3}{4}\pi^{1/2}} = \frac{4\pi r^3}{3}$.

2. When dimension m is large

Example 2.1. Regardless of how small ϵ is, much of the volume of the ball $B_m(0,r)$ is in the spherical shell $S_m(0,r-\epsilon,r)$, since

$$\begin{aligned}\frac{vol_m(S_m(0,r-\epsilon,r))}{vol_m(B_m(0,r))} &= \frac{vol_m(B_m(0,r)) - vol_m(B_m(0,r-\epsilon))}{vol_m(B_m(0,r))}\\ &= 1 - \frac{vol_m(B_m(0,r-\epsilon))}{vol_m(B_m(0,r))}\\ &= 1 - \frac{\pi^{m/2}(r-\epsilon)^m}{\Gamma(m/2+1)}\frac{\Gamma(m/2+1)}{\pi^{m/2}r^m}\\ &= 1 - \frac{(r-\epsilon)^m}{r^m}\\ &= 1 - \left(\frac{r-\epsilon}{r}\right)^m\end{aligned}$$

goes to 1 as m goes to ∞.

Example 2.2. Regardless of how small ϵ is, much of the area of the sphere $S_m(0,r)$ is in the belt $E_m(0,r,\epsilon)$, since

$$\begin{aligned}\frac{area(E_m(0,r,\epsilon))}{area(S_m(0,r))} &= \frac{area(\{x \in \mathbb{R}^m, \|x\| = r, -\epsilon \le x_m \le \epsilon\})}{area(\{x \in \mathbb{R}^m, \|x\| = r\})}\\ &= \frac{area(\{(x_1,\dots,x_m) \in \mathbb{R}^m, x_1^2+\cdots+x_m^2 = r^2, -\epsilon \le x_m \le \epsilon\})}{area(\{(x_1,\dots,x_m) \in \mathbb{R}^m, x_1^2+\cdots+x_m^2 = r^2, -r \le x_m \le r\})}\\ &= \frac{area(\{(x_1,\dots,x_m) \in \mathbb{R}^m, x_1^2+\cdots+x_m^2 = r^2, r^2-\epsilon^2 \le x_1^2+\cdots+x_{m-1}^2 \le r^2\})}{area(\{(x_1,\dots,x_m) \in \mathbb{R}^m, x_1^2+\cdots+x_m^2 = r^2, 0 \le x_1^2+\cdots+x_{m-1}^2 \le r^2\})}\\ &= \frac{\int\limits_{r^2-\epsilon^2 \le x_1^2+\cdots+x_{m-1}^2 \le r^2} f(x_1,\dots,x_{m-1})\,dx_1\dots dx_{m-1}}{\int\limits_{0 \le x_1^2+\cdots+x_{m-1}^2 \le r^2} g(x_1,\dots,x_{m-1})\,dx_1\dots dx_{m-1}}\\ &= \frac{\int\limits_{S_{m-1}(0,\sqrt{r^2-\epsilon^2},r)} f(x_1,\dots,x_{m-1})\,dx_1\dots dx_{m-1}}{\int\limits_{B_{m-1}(0,r)} g(x_1,\dots,x_{m-1})\,dx_1\dots dx_{m-1}}\\ &= \dots\\ &= \frac{vol_{m-1}(S_{m-1}(0,\sqrt{r^2-\epsilon^2},r))}{vol_{m-1}(B_{m-1}(0,r))}\end{aligned}$$

goes to 1 as m goes to ∞ as in example 2.1.

Example 2.3. Two randomly sampled vectors x, x' of norm r in $\mathbb{R}^m$ are almost perpendicular. Suppose $x = (0, \dots, 0, r)$ and view it as the north pole. Then

$$\begin{aligned} angle(x, x') &= \arccos\left(\frac{x \cdot x'}{||x||||x'||}\right) \\ &= \arccos\left(\frac{r x'_m}{rr}\right) \\ &= \arccos\left(\frac{x'_m}{r}\right) \end{aligned}$$

Fix $\delta > 0$ and choose ϵ such that $\arccos\left(\frac{\epsilon}{r}\right) = \delta$. Then

$$\begin{aligned} P(-\delta \leq angle(x, x') \leq \delta) &= P\left(\arccos\left(\frac{-\epsilon}{r}\right) \leq angle(x, x') \leq \arccos\left(\frac{\epsilon}{r}\right)\right) \\ &= P(-\epsilon \leq x'_m \leq \epsilon) \\ &= P(x' \in E_m(0, r, \epsilon) \,|\, x' \in S_m(0, r)) \\ &= \frac{area(E_m(0, r, \epsilon))}{area(S_m(0, r))} \end{aligned}$$

goes to 1 as m goes to ∞ by example 2.2.

Example 2.4. Things are almost linear in $\mathbb{R}^m$.