MONTY HALL PROBLEM

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Abstract: an exposition on Monty Hall problem.

Contents

1. The Game

Consider the following game. A prize is placed behind door 1, door 2 or door 3 with equal probability. So if the random variable X denotes where the prize is then

$$P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{3}$$

A player then chooses one of the doors with equal probability. If the door he chooses is where the prize is, he wins the prize. So if the random variable Y denotes his choice then

$$P(Y = 1) = P(Y = 2) = P(Y = 3) = \frac{1}{3}$$

After the player has made his choice, the host will go behind the doors. If the prize is behind the chosen door, that is $X = Y$ then he will open one of the remaining two doors with equal probability. If the prize is not behind the chosen door, that is $X \neq Y$ then he will open the remaining door. So if the random variable Z denotes his choice then

$$P(Z = 1 \,|\, X = 1, Y = 1) = 0$$
$$P(Z = 2 \,|\, X = 1, Y = 1) = \frac{1}{2}$$
$$P(Z = 3 \,|\, X = 1, Y = 1) = \frac{1}{2}$$
$$P(Z = 1 \,|\, X = 1, Y = 2) = 0$$
$$P(Z = 2 \,|\, X = 1, Y = 2) = 0$$
$$P(Z = 3 \,|\, X = 1, Y = 2) = 1$$
$$\vdots$$

Gathering all this together, we have the trivariate random variable (X, Y, Z) with probability mass function $p_{X,Y,Z}(x, y, z)$ where

$$p_{X,Y,Z}(1,1,1) = 0, p_{X,Y,Z}(1,1,2) = \frac{1}{18}, p_{X,Y,Z}(1,1,3) = \frac{1}{18}$$

$$p_{X,Y,Z}(1,2,1) = 0, p_{X,Y,Z}(1,2,2) = 0, p_{X,Y,Z}(1,2,3) = \frac{1}{9}$$

$$p_{X,Y,Z}(1,3,1) = 0, p_{X,Y,Z}(1,3,2) = \frac{1}{9}, p_{X,Y,Z}(1,3,3) = 0$$

$$p_{X,Y,Z}(2,1,1) = 0, p_{X,Y,Z}(2,1,2) = 0, p_{X,Y,Z}(2,1,3) = \frac{1}{9}$$

$$p_{X,Y,Z}(2,2,1) = \frac{1}{18}, p_{X,Y,Z}(2,2,2) = 0, p_{X,Y,Z}(2,2,3) = \frac{1}{18}$$

$$p_{X,Y,Z}(2,3,1) = \frac{1}{9}, p_{X,Y,Z}(2,3,2) = 0, p_{X,Y,Z}(2,3,3) = 0$$

$$p_{X,Y,Z}(3,1,1) = 0, p_{X,Y,Z}(3,1,2) = \frac{1}{9}, p_{X,Y,Z}(3,1,3) = 0$$

$$p_{X,Y,Z}(3,2,1) = \frac{1}{9}, p_{X,Y,Z}(3,2,2) = 0, p_{X,Y,Z}(3,2,3) = 0$$

$$p_{X,Y,Z}(3,3,1) = \frac{1}{18}, p_{X,Y,Z}(3,3,2) = \frac{1}{18}, p_{X,Y,Z}(3,3,3) = 0$$

Finally, the player is given a choice to stay with his initial door or switch to the remaining door. So if the random variable W denotes his switch then

$$P(W = 1 \,|\, X = 1, Y = 1, Z = 1) = 0$$
$$P(W = 2 \,|\, X = 1, Y = 1, Z = 1) = 0$$
$$P(W = 3 \,|\, X = 1, Y = 1, Z = 1) = 0$$
$$P(W = 1 \,|\, X = 1, Y = 1, Z = 2) = 0$$
$$P(W = 2 \,|\, X = 1, Y = 1, Z = 2) = 0$$
$$P(W = 3 \,|\, X = 1, Y = 1, Z = 2) = 1$$
$$\vdots$$

Gathering all this together, we have the quadvariate random variable (X, Y, Z, W) with probability mass function $p_{X,Y,Z,W}(x, y, z, w)$ where

$$p_{X,Y,Z,W}(x,y,z,w) = \begin{cases} p_{X,Y,Z}(x,y,z) \text{ if } w \neq y \text{ and } w \neq z \\ 0 \text{ otherwise} \end{cases}$$

2. Staying or Switching

Now we can compute all probabilities in the game.

Example 2.1. We can compare the probability of winning the prize by staying with the initial door against the probability of winning the prize by switching to the remaining door

$$\begin{aligned} P(\text{winning by staying}) &= P(X = Y) \\ &= \sum_{x=y} p_{X,Y,Z}(x,y,z) \\ &= \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned}
P(\text{winning by switching}) &= P(X = W) \\
&= \sum_{x=w} p_{X,Y,Z,W}(x,y,z,w) \\
&= \sum_{x \neq y \text{ and } x \neq z} p_{X,Y,Z}(x,y,z) \\
&= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} \\
&= \frac{2}{3}
\end{aligned}$$

Alternatively, one can reason that

$$\begin{aligned}
P(X = W) &= P(X \neq Y \text{ and } X \neq Z) \\
&= P(X \neq Y) \\
&= 1 - P(X = Y) \\
&= 1 - \frac{1}{3} = \frac{2}{3}
\end{aligned}$$

Example 2.2. We can compute the distribution of Z

$$\begin{aligned}
P(Z = 1) &= \sum_{z=1} p_{X,Y,Z}(x,y,z) \\
&= \frac{1}{18} + \frac{1}{9} + \frac{1}{9} + \frac{1}{18} \\
&= \frac{1}{3}
\end{aligned}$$

Similarly, $P(Z = 2) = P(Z = 3) = \frac{1}{3}$.

Example 2.3. We can compute the distribution of W

$$\begin{aligned}
P(W = 1) &= \sum_{w=1} p_{X,Y,Z,W}(x,y,z,w) \\
&= \sum_{y \neq 1 \text{ and } z \neq 1} p_{X,Y,Z}(x,y,z) \\
&= \frac{1}{9} + \frac{1}{9} + \frac{1}{18} + \frac{1}{18} \\
&= \frac{1}{3}
\end{aligned}$$

Similarly, $P(W = 2) = P(W = 3) = \frac{1}{3}$.

All X, Y, Z, W are identically distributed.

3. Information and Entropy

If we think the host's action makes a difference in this game, can we quantify it? That brings up information and entropy. We limit ourselves to discrete random variables.

Definition 3.1. For each random variable $(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we define its information to be the random variable

$$(\mathbb{R}, \mathcal{B}(\mathbb{R}), X_*(P)) \xrightarrow{I_X} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$
$$x \mapsto \begin{cases} 0 \text{ if } p_X(x) = 0 \\ \log_2\left(\frac{1}{p_X(x)}\right) \text{ otherwise} \end{cases}$$

Note that $p_X(x) = X_*(P)(x)$. Here the base is chosen to be 2, and the unit of I_X is called *bit*. Other popular bases are 10 with unit *nat* and e with unit *hartley*. All results then differ by constant scalars $\log_{10} 2$ and $\log_e 2$ respectively.

Example 3.2. The joint random variable (X, Y) has information

$$(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), (X, Y)_*(P)) \xrightarrow{I_{X,Y}} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$
$$(x, y) \mapsto \begin{cases} 0 \text{ if } p_{X,Y}(x, y) = 0 \\ \log_2\left(\frac{1}{p_{X,Y}(x,y)}\right) \text{ otherwise} \end{cases}$$

Note that $p_{X,Y}(x, y) = (X, Y)_*(P)(x, y)$. We will use this in example 3.10.

Definition 3.3. For each random variable X, we define its entropy $H(X)$ to be the expected value $E(I_X)$ of I_X.

Explicitly, one has

$$\begin{aligned} H(X) &= \sum_{x_i} p_X(x_i) I_X(x_i) \\ &= \sum_{x_i} p_X(x_i) \log_2\left(\frac{1}{p_X(x_i)}\right) \end{aligned}$$

For each pair of random variables X, Y, given event $Y = y$, we can think of

- conditional probability measure $P(- \mid Y = y)$
- its pushforward $X_*(P(- \mid Y = y))$
- random variable $(\mathbb{R}, \mathcal{B}(\mathbb{R}), X_*(P(- \mid Y = y))) \xrightarrow{X \mid Y = y} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$
- its information $I_{X \mid Y = y}$

Note that $p_{X \mid Y=y}(x) = X_*(P(- \mid Y = y))(x)$.

Definition 3.4. For each pair of random variables X, Y, we define the entropy $H(X \mid Y = y)$ of X given an event $Y = y$ to be the expected value $E(I_{X \mid Y=y})$ of $I_{X \mid Y=y}$.

Explicitly, one has

$$\begin{aligned} H(X \mid Y = y) &= \sum_{x_i} p_{X \mid Y=y}(x_i) I_{X \mid Y=y}(x_i) \\ &= \sum_{x_i} p_{X \mid Y=y}(x_i) \log_2\left(\frac{1}{p_{X \mid Y=y}(x_i)}\right) \end{aligned}$$

If we view $(\mathbb{R}, \mathcal{B}(\mathbb{R}), Y_*(P)) \xrightarrow{H(X \mid Y = -)} (\mathbb{R}, \mathcal{B}(\mathbb{R})), y \mapsto H(X \mid Y = y)$ as a random variable then it has expected value

$$\begin{aligned}
E(H(X \mid Y = -)) &= \sum_{y_j} p_Y(y_j) H(X \mid Y = y_j) \\
&= \sum_{y_j} p_Y(y_j) \sum_{x_i} p_{X \mid Y = y_j}(x_i) \log_2 \left(\frac{1}{p_{X \mid Y = y_j}(x_i)} \right) \\
&= \sum_{x_i, y_j} p_Y(y_j) p_{X \mid Y = y_j}(x_i) \log_2 \left(\frac{1}{p_{X \mid Y = y_j}(x_i)} \right) \\
&= \sum_{x_i, y_j} p_Y(y_j) \frac{p_{X,Y}(x_i, y_j)}{p_Y(y_j)} \log_2 \left(\frac{1}{p_{X \mid Y = y_j}(x_i)} \right) \\
&= \sum_{x_i, y_j} p_{X,Y}(x_i, y_j) \log_2 \left(\frac{p_Y(y_j)}{p_{X,Y}(x_i, y_j)} \right)
\end{aligned}$$

Definition 3.5. For each pair of random variables X, Y we define the entropy $H(X \mid Y)$ of X given Y to be the expected value $E(H(X \mid Y = -))$ of $H(X \mid Y = -)$ above.

The difference $H(X) - H(X \mid Y)$ is called mutual information between X and Y and denoted by $I(X, Y)$. It also equals the mutual information $I(Y, X) = H(Y) - H(Y \mid X)$ between Y and X and has explicit formula

$$I(X, Y) = \sum_{i,j} p_{X,Y}(x_i, y_j) \log_2 \left(\frac{p_{X,Y}(x_i, y_j)}{p_X(x_i) p_Y(y_j)} \right)$$

Now we can compute all informations and entropies in the game.

Example 3.6. Without the host's action, the prize location X has entropy

$$\begin{aligned}
H(X) &= \frac{1}{3} \log_2 \left(\frac{1}{\frac{1}{3}} \right) + \frac{1}{3} \log_2 \left(\frac{1}{\frac{1}{3}} \right) + \frac{1}{3} \log_2 \left(\frac{1}{\frac{1}{3}} \right) \\
&= \log_2 3
\end{aligned}$$

Example 3.7. With the host's action of opening door 3, the prize location $X \mid Z = 3$ has entropy

$$\begin{aligned}
H(X \mid Z = 3) &= p_{X \mid Z=3}(1) \log_2 \left(\frac{1}{p_{X \mid Z=3}(1)} \right) + p_{X \mid Z=3}(2) \log_2 \left(\frac{1}{p_{X \mid Z=3}(2)} \right) + p_{X \mid Z=3}(3) \log_2 \left(\frac{1}{p_{X \mid Z=3}(3)} \right) \\
&= \frac{1}{2} \log_2 \left(\frac{1}{\frac{1}{2}} \right) + \frac{1}{2} \log_2 \left(\frac{1}{\frac{1}{2}} \right) + 0 \\
&= 1
\end{aligned}$$

With the host's action of opening door 3, the entropy of X decreases. Similarly $H(X \mid Z = 1) = H(X \mid Z = 2) = 1$.

Example 3.8. With the host's action, the prize location has expected entropy

$$\begin{aligned}
H(X \mid Z) &= p_Z(1) H(X \mid Z = 1) + p_Z(2) H(X \mid Z = 2) + p_Z(3) H(X \mid Z = 3) \\
&= \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 \\
&= 1
\end{aligned}$$

One can verify this directly with the explicit formula in definition 3.5 as well.

Example 3.9. The information the host provides about the prize location is

$$\begin{aligned}
I(X,Z) = 0 + (\frac{1}{18} + 0 + \frac{1}{9}) \log_2 \left(\frac{\frac{1}{18} + 0 + \frac{1}{9}}{\frac{1}{3} \cdot \frac{1}{3}} \right) + (\frac{1}{18} + \frac{1}{9} + 0) \log_2 \left(\frac{\frac{1}{18} + \frac{1}{9} + 0}{\frac{1}{3} \cdot \frac{1}{3}} \right) \\
+ (0 + \frac{1}{18} + \frac{1}{9}) \log_2 \left(\frac{0 + \frac{1}{18} + \frac{1}{9}}{\frac{1}{3} \cdot \frac{1}{3}} \right) + 0 + (\frac{1}{9} + \frac{1}{18} + 0) \log_2 \left(\frac{\frac{1}{9} + \frac{1}{18} + 0}{\frac{1}{3} \cdot \frac{1}{3}} \right) \\
(0 + \frac{1}{9} + \frac{1}{18}) \log_2 \left(\frac{0 + \frac{1}{9} + \frac{1}{18}}{\frac{1}{3} \cdot \frac{1}{3}} \right) + (\frac{1}{9} + 0 + \frac{1}{18}) \log_2 \left(\frac{\frac{1}{9} + 0 + \frac{1}{18}}{\frac{1}{3} \cdot \frac{1}{3}} \right) + 0 \\
= \log_2 \frac{3}{2} \\
= \log_2 3 - 1
\end{aligned}$$

One can see that $I(X,Z) = H(X) - H(X \mid Z)$. This is how much the entropy of X decreases by.

Example 3.10. The joint random variable (X,Z) has entropy

$$\begin{aligned}
H(X,Z) &= E(I_{X,Z}) \\
&= \sum_{x_i, z_k} p_{X,Z}(x_i, z_k) \log_2 \left(\frac{1}{p_{X,Z}(x_i, z_k)} \right) \\
&= \frac{1}{6} \log_2 \left(\frac{1}{\frac{1}{6}} \right) + \frac{1}{6} \log_2 \left(\frac{1}{\frac{1}{6}} \right) + \frac{1}{6} \log_2 \left(\frac{1}{\frac{1}{6}} \right) + \frac{1}{6} \log_2 \left(\frac{1}{\frac{1}{6}} \right) + \frac{1}{6} \log_2 \left(\frac{1}{\frac{1}{6}} \right) + \frac{1}{6} \log_2 \left(\frac{1}{\frac{1}{6}} \right) \\
&= \log_2 6 \\
&= \log_2 3 + 1
\end{aligned}$$

One can see that $H(X,Z) = H(X) + H(Z) - I(X,Z) = H(X \mid Z) + I(X,Z) + H(Z \mid X)$. Draw a Venn diagram.

Exercise 3.11. Compute $H(Y), H(Y \mid Z = 3), H(Y \mid Z), I(Y,Z), H(Y,Z)$. Hint: swap prize and player.

4. Simulation

See Python code and games at github.com/dinhuun/probability_statistics.