MATHEMATICAL METHODS - III (2017)
TUTORIAL 1

Problem 1. *Let $F(x) = \cos(x)$. Consider the sequence generated by the recipe,*

$$x_0 = a$$
$$x_n = F(x_{n-1})$$

(1) *Write down the first few terms of this sequence for different initial values a.*
(2) *Check whether the sequence satisfies any of the following properties.*

(a) *Bounded above*
(b) *Bounded below*
(c) *Monotonically increasing*
(d) *Monotonically decreasing*

(3) *Use a program or an excel sheet to check whether this sequence converge to some value.*
(4) *One the same axes plot $y = F(x)$ and $y = x$ and use this plot to graphically represent the sequence you generated.*
(5) *When the sequence converge, to which value does it converge?*

Problem 2. *Let $F(x) = x^2 - 12$. Consider the sequence generated by the recipe,*

$$x_0 = a$$
$$x_n = x_{n-1} - \frac{F(x_{n-1})}{F'(x_{n-1})}$$

(1) *Write down the first few terms of this sequence for different initial values a.*
(2) *Check whether the sequence satisfies any of the following properties.*

(a) *Bounded above*
(b) *Bounded below*
(c) *Monotonically increasing*
(d) *Monotonically decreasing*

(3) *Use a program or an excel sheet to check whether this sequence converge to some value.*
(4) *When the sequence converge, to which value does it converge?*

Problem 3. *Consider the sequence given by*

$$x_1 = 1, \qquad x_{n+1} = 3 - \frac{1}{x_n}$$

(1) *Show that the sequence is both bounded above and bounded below.*
(2) *Show, using induction, that the sequence is monotonically increasing.*
(3) *Conclude that the sequence converge to some limit and find it.*
(4) *Explain how you can use this sequence to approximate the value of $\sqrt{5}$.*

Problem 4. *Consider the sequence given by*

$$x_1 = 1, \qquad x_{n+1} = 1 + \frac{1}{x_n}$$

(1) *Is the sequence monotonic (i.e. monotonically decreasing or increasing)?*
(2) *Is the sequence bounded bellow and/or bounded above?*

(3) *Show that the sequence converge to some limit and find it. You may use the fact that every Monotonic bounded sequence converge to some limit.*

Problem 5. *Consider the sequence given by* $x_1 = 1$ *and* $x_{n+1} = \frac{1}{2}\left(\frac{3}{x_n^2} + x_n\right)$.

(1) *Is the sequence monotonic (i.e. monotonically decreasing or increasing)?*
(2) *Is the sequence bounded bellow and/or bounded above?*
(3) *Show that the sequence converge to some limit and find it. You may use the fact that every Monotonic bounded sequence converge to some limit.*

Problem 6. (1) *Show that the sequence* $\left(\frac{1}{n}\right)$ *is a Cauchy sequence.*
(2) *Show that the sequence* $((-1)^n)$ *is* ***not*** *a Cauchy sequence.*

Problem 7. *Consider the following recepie to construct a sequence based in a function* f *and two points* a *and* b. *This is called the* ***Bisection Method***.

- *Begin with two points,* a *and* b, *such that* $f(a)$ *and* $f(b)$ *have opposite signs.*
- *Consider the midpoint of* a *and* b, $c = \frac{a+b}{2}$.
- *Not all three have the same sign.*
- *Pick the two with the opposite signes.*
- *Repeat the process.*
- *In each step, you can take the mid-point of the chosen interval to be the next term in the sequence.*

Let $f(x) = sin(x) - x^3 + 2x + 2 = 0$, $a = 0$ *and* $b = 3$.

(1) *Find the first few terms of the sequence* (x_n) *generated by this recipe.*
(2) *Show that* $|x_n - x_{n+1}| = \alpha^n$ *for some* α *and explicitly mention the value of* α.
(3) *Show that this sequence is Cauchy. There is no need to use the actual definition of* f *and the values of* a *and* b *to show this. Use the previous result.*
(4) *To which value does the sequence converge?*

Problem 8. *Find the limit of the sequence* $(x_n)_{n\in\mathbb{N}}$, *where* $x_1 = 1$ *and* $x_{n+1} = \frac{1}{2}\left(\frac{1}{ax_n} + x_n\right)$.

(1) $\frac{1}{\sqrt{a}}$
(2) a
(3) $\sqrt{a}$
(4) *limit does not exist*
(5) *none of the above*

Problem 9. *Consider the following sequences.*

(i) $x_1 = 0$ *and* $x_{n+1} = x_n + 1$
(ii) $y_n = (-1)^n$
(iii) $z_n = \frac{1}{n^2}$

Which of the above sequence(s) is/are

(1) *Monotonicaly increasing?*
(2) *Monotonicaly decreasing?*
(3) *Bounded above?*
(4) *Bounded below?*
(5) *Converge?*
(6) *Cauchy?*