This paper is intended to be educational, it is written to teach the reader about wave theory and vibrational theory, while focusing on a real world situation as an example.

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Investigation into Vibration and Wave theory with practical examples using the Chladni effect

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Abstract

This paper has been designed to explore the vibrations and wave theory associated with the Chladni effect, and to focus upon mathematically describing the patterns caused by the effects in two dimensions. A secondary aim is to explore if these patterns can be accurately modeled using various programming techniques.

Involved in the paper are the theoretical explanations and discussions which describe the effect, in both experimental conditions (as noted by Chladni), while also implementing physical theory using mathematics. This concept is then discussed, and related to waves and vibrations whereupon the derivations and formulas are developed. These are used to show the basic concepts behind the effect, and to provide a background for taking this theory further.

1.0 Introduction

A Chladni Pattern is created when a source - the driving force for the system - is applied upon a flat plate, which through vibrations, causes a certain organisation of nodes upon the plate. One of the most simple methods to visualise this process is to place a substance with a fine grain upon a metallic plate, and to apply a frequency to this system. An example of this is shown in figure 1. The pattern is caused when different points upon the plate vibrate with different amplitudes, while other areas do not vibrate at all, these are called *nodes*. This causes the material to settle and collect upon the nodes, creating the patterns. The patterns can be altered by manipulating the source, this driving force is changed by the frequency applied to give different nodes in different locations, causing unique formations on the grain upon the plate.

This concept was first recorded by Ernst Chladni (1756 - 1827) as he was the first to demonstrate the effect while considering the relationship of pattern to frequency. At the point of discovery the mechanics were not well understood. While on Chladni's travels, he encountered Napoleon who was so intrigued, offered one kilo of gold as a reward to anybody who could describe the effect with equations (Stöckmann, 2007). After a third extension of the competition, Sophie Germain of the University of St Andrews, Scotland, 1996 entered a paper and won the prize.

The vertical component of the plate's movement has such a small distance, it typically proves challenging to view and understand with the eye alone. This is why fine powder is often used, as the particles have low mass, they are greatly affected by the force of the plate. Because these particles are relatively larger than sections of the plate, and are unbound, it is far easier to observe their movement. This is an example of using the vibrational potential energy to create displacement through the transfer to kinetic energy of the grains, which can be seen via a direct relationship with the vertical aspect of the displacement.

Generally powders could not be used because of the static caused between the particles, causing clustering. This means that the patterns created by the plate are not well defined and obvious when using powder.

2.0 The Mathematics

The behaviour of the plate is defined by the driving force, in order to fully understand and comprehend the relationship between these elements, theory about vibrations and waves must be taken into account. There are several variables which influence this relationship, these are shown below:

A = amplitude, which is the maximum displacement from the equilibrium position.

T = time period, this is the time required for a cycle of one period to occur.

f = frequency. This is the number of cycles within a certain time period, T, f= 1/T.

ω= angular frequency

λ= wavelength, this is the distance between two successive peaks of a wave

v = wave speed, which can also be expressed as $v = \frac{\gamma}{T}$

Simple Harmonic Motion

To understand the mechanisms which govern these effects, simple harmonic motion should be investigated as it allows the study of the most basic form of vibrations.

First consider Newton's second law:

$$F = ma \qquad (1)$$

Then compare this to the equation gained by considering a mass upon a light spring (French, 1971) :

$$-kx = m\frac{\partial^2 x}{\partial t^2}$$

Where k is the spring constant, x is the displacement, and m represents mass. When -kx are considered together, this can be thought of as the net force acting in the opposite direction to the applied displacement, hence the negative symbol. The right side of the equation shows mass multiplied with acceleration (seen as $a$in the previous equation, displayed with delta here to indicate the change in distance divided by the change in time).

In order to find a general solution, it is helpful to perform the second derivative of formula (3):

$$x = sin\,(\omega t)$$

(3)

$$\frac{\partial\ x}{\partial t} = -\omega cos(\omega t)$$

$$\frac{\partial^2 x}{\partial t^2} = -\omega^2 sin(\omega t)$$

The exact same relationship can be seen for the equivalent $x = cos\,(\omega t)$, which after taking the second derivative gives $\frac{\partial^2 x}{\partial t^2} = -\omega^2 sin(\omega t)$.

Combining equations (2) and (3) gives:

$$-k\ sin(\omega t) = m[-\omega^2 sin\,(\omega t)]$$

$$\omega^2 = \frac{k}{m}$$

This equation described in (4) shows the relationship between angular frequency, the spring constant, and mass. This was achieved by using trigonometric identities, arranging, and simplifying terms.

This general solution can also be written in a different form:

$$x(t) = x + A\,sin(\omega t) + B\,cos(\omega t)$$

This can be carried out because of the similar nature of sine, and cosine.

To find the unknown constants A, and B, the equation can be written in another form where $t = 0, x = x_0$, and the velocity $\frac{\partial^2 x}{\partial t^2} = v_0$.

$$A = \frac{v_0}{\omega} \text{ and } B = x_0 - x$$

Where in this instance, x is the equilibrium position. These can then be substituted back into the equation.

$$x(t) = x + \frac{v_0}{\omega}\,sin(\omega t) + (x_0 - x)\,cos(\omega t)$$

These equations describe the behaviour characteristic of simple harmonic motion, which is a key concept that governs each variation connected with Chladni's plate patterns.

Now consider boundary conditions. It should be noted that these conditions can either act as a source, naturally, or as nodes. This then leads to another set of mathematics which separately are concerned with the functions required for the Chladni patterns.

Where u (the *vertical displacement*) is a function of any given point (x, y) on the plate at a given time$t$. The initial conditions, and boundary conditions are defined below:

$$u\ (\ x, y, 0)\ =\ 0,$$

Which states that at $t = 0$, the vertical component of displacement is zero at any point (x, y).

$$u\ (\ \mathbf{1.0}, \mathbf{1.0}, t)\ =\ f(t),$$

Showing that at the center of the plate $(1.0, 1.0)$ is vibrating vertically with the behaviour defined by $f(t)$,which is a function described by simple harmonic motion as either a sine, or cosine function. (Rossing, 1982).

Next the wave equation for a string should be considered, which is the area the previous section has been leading to (Elmore & Heald). However, this is a more complex model which varies so greatly with each situation, it becomes unnecessary to explore each of these as they are not relevant theory for the Chladni effect.

So firstly, the assumptions are made that the mass is consistently and evenly distributed along and around the string. The second is that the string can only move vertically. The last assumption to be made is that gravity has no influence upon the string and the tension of the string . So $T_x = \tau$, where T_x is the tension in the x direction, each of these are in terms of force. The force in the y direction should also be considered, $F_y = T_y(x + z) - T_y(x)$. Where z is the gradient.

Then this can be substituted into Newtons second law:

$$T_y(x+z) - T_y(z) = \mu(m)a_y$$

Where $\mu = \frac{M}{L}$, and M is the mass, and L is the length of the string. The term a_y is used to denote the y-component of the acceleration.
Making the further assumption that m is a small value, the equation can be rewritten which leaves the tangent of x = $\frac{\partial x}{\partial t}$, and the x-component of tension as τ. After more derivation and substitution as noted by Waller, a solution is reached:

$$y = A * N(kx)\ N(\omega_n t)$$

Where A is a constant, and N is either sine, or cosine. This equation of course depends upon the boundary conditions set up originally.

This leads to the next equation which implements theory gained in the past two sections. The wave equation acting upon a plate can be considered to be two string waves (as discussed in the previous section) who are both independent to one another.

$$\frac{1}{C}\frac{\partial^2 U(x,y,t)}{\partial t^2} = \frac{\partial^2 U(x,y,t)}{\partial x^2} + \frac{\partial^2 U(x,y,t)}{\partial y^2}$$

(Pain H. J, 2005 P:246).

This equation states that $U(x,y,t)$ is a function which describes the z-component of displacement at every point (x,y) at a specific time on the plate.

Now if each variable is considered,

$$X = X(x)$$
$$Y = Y(y)$$
$$T = T(t)$$

which can then be used in conjunction to the previous equation with each component.

$$\frac{\partial^2 U(x,y,t)}{\partial t^2} = X(x)Y(y)\frac{\partial^2 T(t)}{\partial t^2}$$
$$\frac{\partial^2 U(x,y,t)}{\partial x^2} = Y(y)T(t)\frac{\partial^2 X(x)}{\partial x^2}$$
$$\frac{\partial^2 U(x,y,t)}{\partial y^2} = X(x)T(t)\frac{\partial^2 Y(y)}{\partial y^2}$$

Using each of these and substituting them back into the original equation.

$$\frac{1}{C} X(x)Y(y)\frac{\partial^2 T(t)}{\partial t^2} = Y(y)T(t)\frac{\partial^2 X(x)}{\partial x^2} + X(x)T(t)\frac{\partial^2 Y(y)}{\partial y^2}$$

which can be divided by $X(x)$, $Y(y)$,$T(t)$. Which allows the left side of the equation to be concerned only with the value t, where x, y,and t are independent variables, indicating that both the left and the right must be equivalent to an identical constant. This can be rewritten once again:

$$-(k^2{}_1 + k^2{}_2) = -k^2$$

Where k is the constant selected. This can occur because x and y only vary by that constant. Then the equations produced by this process are solved. The final solution is as follows, where u is the vertical displacement:

$$u = A\ N(k_1 x)\ N(k_2 y)\ N(c\,kt)$$

Where A is a constant , N is again sine or cosine, c is the wave speed, and t is the time.

3.0 Considerations

The paper has investigated the experimental theory and noted upon the observations and the general systems by which Chladni patterns are created. It has then built up from the basic theory which unspecifically describes simple mechanical systems. The mathematics of these have also been explored in a simple manner to show the development of the equations. These have also been carefully designed to take start with the simplest idea, and then build up gently to the theory involved in creating Chladni patterns.

The next steps would be to investigate the Laplace eigenfunctions of the system to create an even more accurate idea of the plates. However this is beyond the scope of the paper and is not mathematically explored further. After this point it is beneficial to implement fourier eigenfunctions also. FInally it would be useful to account for Poisson's ratio, which would describe how a material (the plate in this case) would expand or compress in all dimensions at various points.

However, because of the complex nature of Chladni patterns, this is still an exciting and unexplored area of both mathematics and physics, which is still actively researched, and can be seen from the volume of recent papers on the topic.

4.0 The programming model

The final aspect of this paper is to model the patterns using a computer program. Due to the mathematical nature of this challenge, the program MatLab® has been selected to plot several examples of the phenomenon in order to optimise the visibility of the physical events as the program allows complex mathematical functions to be plotted in a short time frame, while producing high-quality, high-accuracy figures. It was also selected as it has a large collection of mathematical functions and a wide library of functions, which improves the efficiency of the program and reduces the likelihood of bugs. It also allows quick review and compilation of code, as well as useful debugging guidance on the command line. Although these features aren't necessary, they are good practices to become accustomed to, especially when working with larger projects. In figure 4.1 the pseudocode is shown for the model. Below this, figure 4.3 to 4.5 show the output from the computer model with no boundary conditions.

```
function [u, x, y, t]  = Name();
        Output: user define var ←u (conditions)
n ← 1
m ← 1
define u ← equation
plot (x,y, u)
```

Figure 4.1: the general pseudocode for the model.

This model was created and different factors defined. These were compared to the figures shown in figure 4.2.

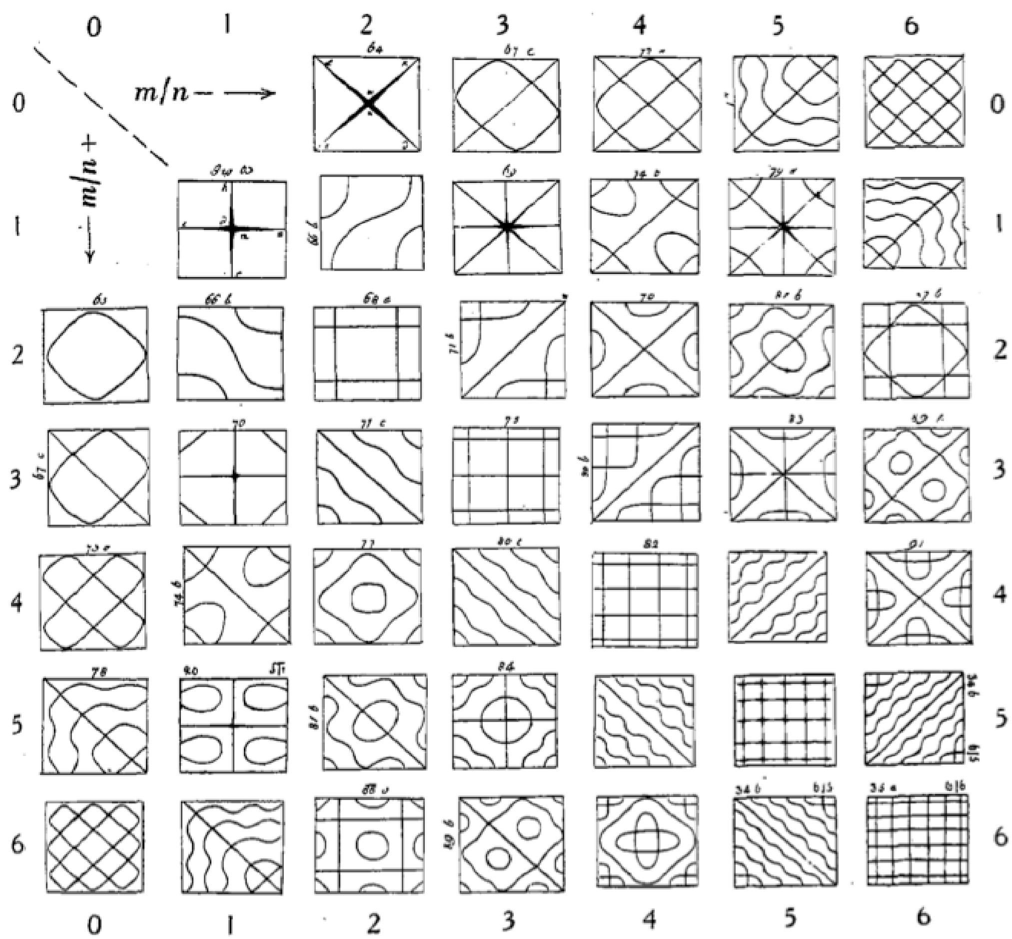
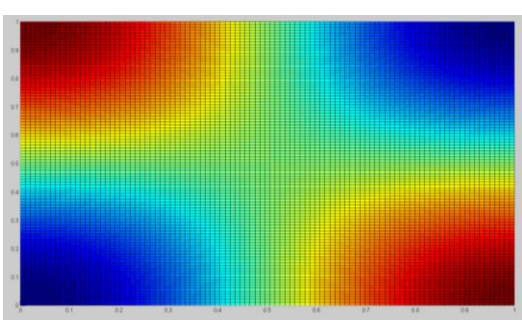


Figure 4.2: From Chladni's the plates are shown with the patterns (Waller). The numbers surrounding the figure indicate the numerical values necessary to gain this pattern.



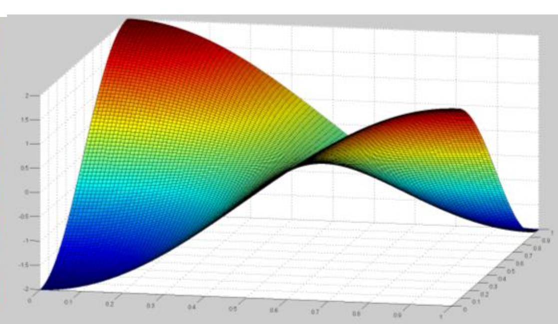


figure 4.3: This is an example of the output of the model at n = 1; and m = 1; Both images are the same output from different angles of view. The blue areas depict a low set of values, and red indicates a high set of values.

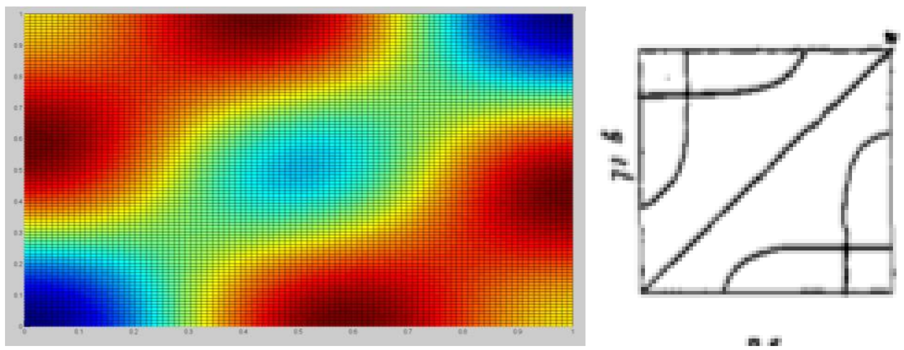


figure 4.4: This is an example of the output of the model at n = 2; and m = 3; The image on the left is from the model, and the image from the right, from the experimental images, as shown by Chladni.

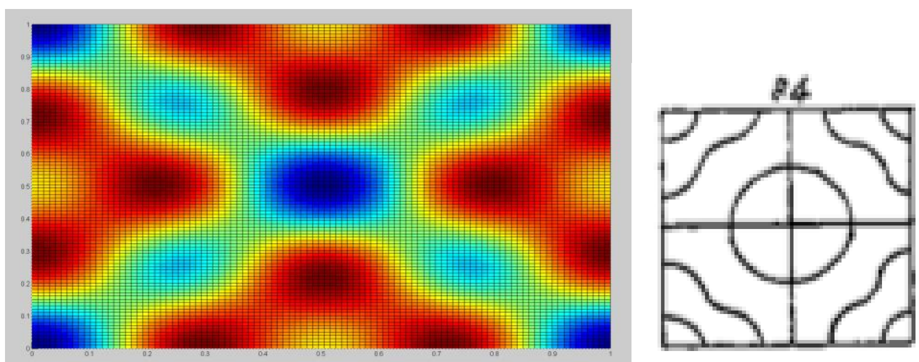


figure 4.5: This is an example of the output of the model at n = 5; and m = 3; The image on the left is from the model, and the image from the right, from the experimental images, as shown by Chladni.

These images show the output and directly compare them to the Chladni patterns which have been achieved experimentally. It can be seen that the computer model is imperfect, and does not directly mimic the images seen on the right. This is due to the many factors that influence the real world scenario, such as boundary conditions, the behaviour of the plate, and the dimensions of the plate. The images on the right were created using a violin bow, which could have produced imperfect sound, compared to the images on the right, whose mathematical input is accurate, causing another source of discrepancies. Overall the models could be improved further in order to gain the most accurate output by understanding and accounting for each variable.

5.0 References

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