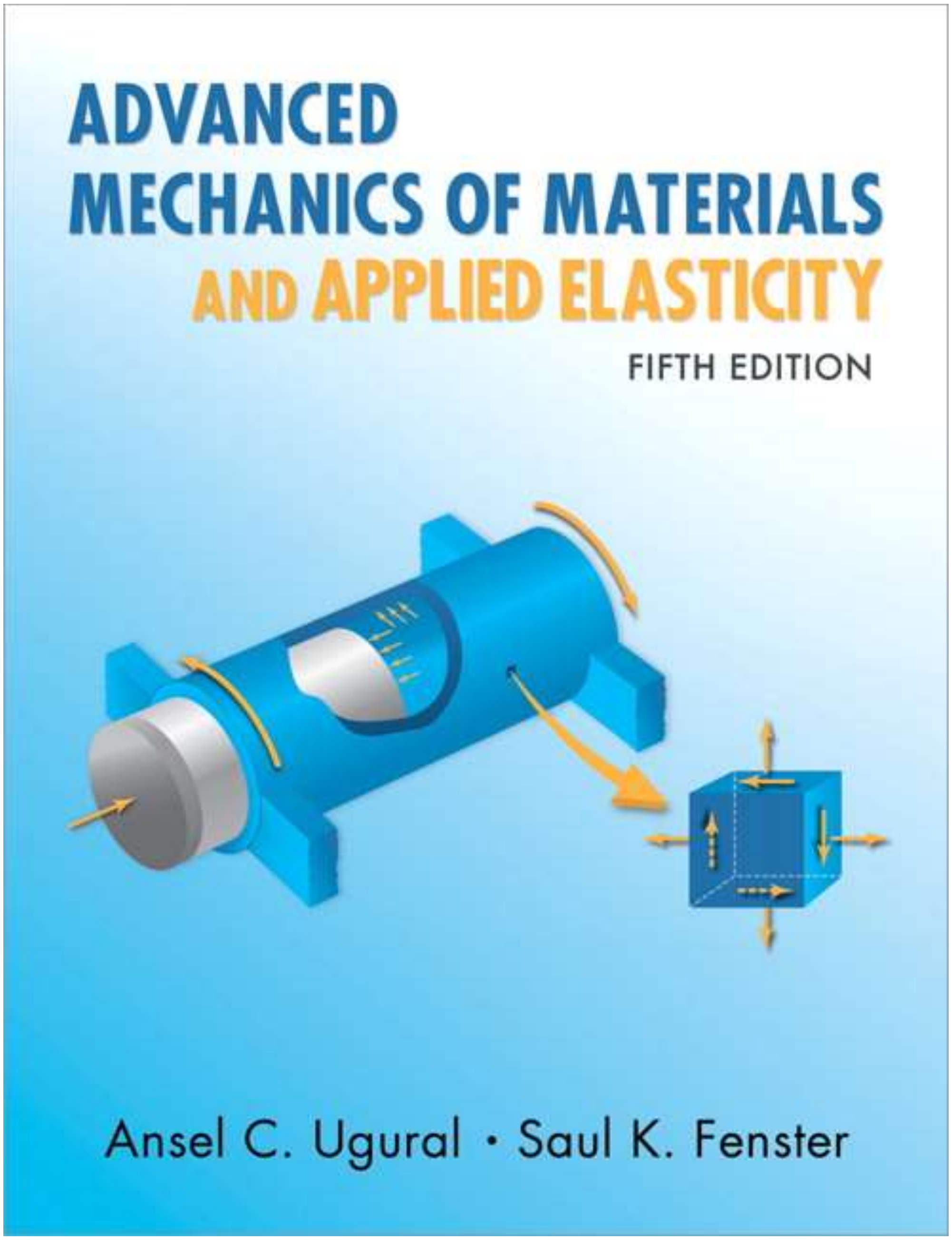
For a brief introduction to Euler-Bernoulli beam theory and torsion of bars, we have included the relevant pages from the book “Advanced Mechanics of Materials and Applied Elasticity” by Ugural and Fenster.



5.6 Elementary Theory of Bending

We may conclude, on the basis of the previous sections, that exact solutions are difficult to obtain. It was also observed that for a slender beam the results of the exact theory do not differ markedly from that of the mechanics of materials or elementary approach provided that solutions close to the ends are not required. The bending deflection was found to be very much larger than the shear deflection. Thus, the stress associated with the former predominates. We deduce therefore that the normal strain ε_y resulting from transverse loading may be neglected. Because it is more easily applied, the elementary approach is usually preferred in engineering practice. The exact and elementary theories should be regarded as complementary rather than competitive approaches, enabling the analyst to obtain the degree of accuracy required in the context of the specific problem at hand.

The basic assumptions of the elementary theory [Ref. 5.2], for a slender beam whose cross section is symmetrical about the vertical plane of loading, are

$$\varepsilon_y = \frac{\partial v}{\partial y} = 0, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \tag{5.26}$$

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \text{(independent of } z\text{)}$$
$$\varepsilon_z = 0, \qquad \gamma_{yz} = \gamma_{xz} = 0 \tag{5.27}$$

The first equation of (5.26) is equivalent to the assertion $v = v(x)$. Thus, all points in a beam at a given longitudinal location x experience identical deformation. The second equation of (5.26), together with $v = v(x)$, yields, after integration,

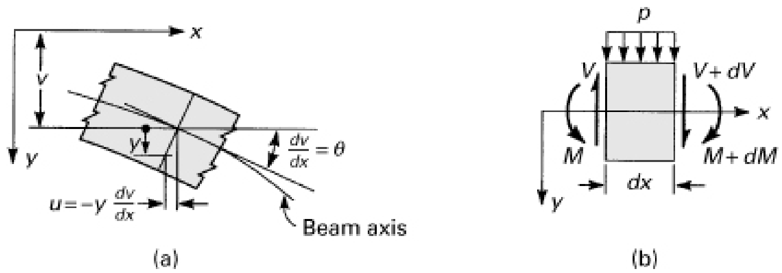
$$u = -y\frac{dv}{dx} + u_0(x) \tag{a}$$

The third equation of (5.26) and Eqs. (5.27) imply that the beam is considered *narrow*, and we have a case of *plane stress*.

At $y = 0$, the bending deformation should vanish. Referring to Eq. (a), it is clear, therefore, that $u_0(x)$ must represent axial deformation. The term dv/dx is the *slope* θ of the beam axis, as shown in Fig. 5.7a, and is very much smaller than unity. Therefore,

$$u = -y\frac{dv}{dx} = -y\theta$$

Figure 5.7. (a) Longitudinal displacements in a beam due to rotation of a plane section; (b) element between adjoining sections of a beam.



The slope is *positive* when *clockwise*, provided that the x and y axes have the directions shown. Since u is a linear function of y, this equation restates the kinematic hypothesis of the elementary theory of bending: *Plane sections perpendicular to the longitudinal axis of the beam remain plane subsequent to bending*. This assumption is confirmed by the exact theory *only* in the case of *pure bending*.

Method of Integration

In the next section, we obtain the stress distribution in a beam according to the elementary theory. We now derive some useful relations involving the shear force V, the bending moment M, the load per unit length p, the slope θ, and the *deflection* v. Consider a beam element of length dx subjected to a distributed loading (Fig. 5.7b). Note that as dx is small, the variation in the load per unit length p is omitted. In the free-body diagram, all the forces and the moments are positive. The shear force obeys the sign convention discussed in Section 1.4; the bending moment is in agreement with the convention adopted in Section 5.2. In general, the shear force and bending moment vary with the distance x, and it thus follows that these quantities will have different values on each face of the element. The increments in shear force and bending moment are denoted by dV and dM, respectively. Equilibrium of forces in the vertical direction is governed by $V - (V + dV) - p\,dx = 0$, or

$$\frac{dV}{dx} = -p$$

(5.28)

That is, the rate of change of shear force with respect to x is equal to the algebraic value of the distributed loading. Equilibrium of the moments about a z axis through the left end of the element, neglecting the higher-order infinitesimals, leads to

$$\frac{dM}{dx} = -V$$

(5.29)

This relation states that the rate of change of bending moment is equal to the algebraic value of the shear force, valid only if a distributed load or no load acts on the beam segment. Combining Eqs. (5.28) and (5.29), we have

$$\frac{d^2M}{dx^2} = p$$

(5.30)

The basic equation of bending of a beam, Eq. (5.10), combined with Eq. (5.30), may now be written as

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = p$$

(5.31)

For a beam of *constant* flexural rigidity *EI*, the beam equations derived here may be expressed as

$$\begin{aligned}
EI\frac{d^4v}{dx^4} &= EIv^{\mathrm{IV}} = p \\
EI\frac{d^3v}{dx^3} &= EIv''' = -V \\
EI\frac{d^2v}{dx^2} &= EIv'' = M \\
EI\frac{dv}{dx} &= EIv' = \int M\,dx
\end{aligned}$$

(5.32)

These relationships also apply to *wide beams* provided that $E/(1-\nu^2)$ is substituted for E (Table 3.1).

In many problems of practical importance, the deflection due to transverse loading of a beam may be obtained through successive integration of the beam equation:

$$\begin{aligned}
EIv^{\mathrm{IV}} &= p \\
EIv''' &= \int_0^x p\,dx + c_1 \\
EIv'' &= \int_0^x dx \int_0^x p\,dx + c_1x + c_2 \\
EIv' &= \int_0^x dx \int_0^x dx \int_0^x p\,dx + \tfrac{1}{2}c_1x^2 + c_2x + c_3 \\
EIv &= \int_0^x dx \int_0^x dx \int_0^x dx \int_0^x p\,dx + \tfrac{1}{6}c_1x^3 + \tfrac{1}{2}c_2x^2 + c_3x + c_4
\end{aligned}$$

(5.33)

Alternatively, we could begin with $EIv'' = M(x)$ and integrate twice to obtain

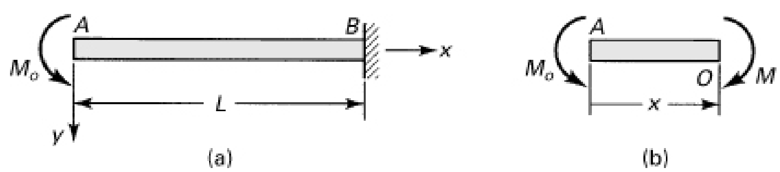
$$EIv = \int_0^x dx \int_0^x M\,dx + c_3 x + c_4 \tag{5.34}$$

In either case, the constants, c_1, c_2, c_3, and c_4, which correspond to the homogeneous solution of the differential equations, may be evaluated from the boundary conditions. The constants c_1, c_2, c_3/EI, and c_4/EI represent the values at the origin of V, M, θ, and v, respectively. In the method of successive integration, there is no need to distinguish between statically determinate and statically indeterminate systems (Sec. 5.11), because the equilibrium equations represent only two of the boundary conditions (on the first two integrals), and because the *total* number of boundary conditions is always equal to the total number of unknowns.

Example 5.2. Displacements of a Cantilever Beam

A cantilever beam AB of length L and constant flexural rigidity EI carries a moment M_o at its free end A (Fig. 5.8a). Derive the equation of the deflection curve and determine the slope and deflection at A.

Figure 5.8. Example 5.2. (a) A cantilever beam is subjected to moment at its free end; (b) free-body diagram of part AO.



Solution

From the free-body diagram of Fig. 5.8b, observe that the bending moment is $+M_o$ throughout the beam. Thus, the third of Eqs. (5.32) becomes

$$EIv'' = M_o$$

Integrating, we obtain

$$EIv' = M_o x + c_1$$

The constant of integration c_1 can be found from the condition that the slope is zero at the support; therefore, we have $v'(L) = 0$, from which $c_1 = -M_o L$. The slope is then

$$v' = \frac{M_o}{EI}(x - L)$$

(5.35)

Integrating, we obtain

$$v = \frac{M_o}{2EI}(x^2 - 2Lx) + c_2$$

The boundary condition on the deflection at the support is $v(L) = 0$, which yields $c_2 = M_o L^2/2EI$. The equation of the deflection curve is thus a parabola:

$$v = \frac{M_o}{2EI}(L^2 + x^2 - 2Lx)$$

(5.36)

However, every element of the beam experiences equal moments and deforms alike. The deflection curve should therefore be part of a circle. This inconsistancy results from the use of an approximation for the curvature, Eq. (5.7). The error is very small, however, when the deformation v is small [Ref. 5.1].

The slope and deflection at A are readily found by letting $x = 0$ into Eqs. (5.35) and (5.36):

$$\theta_A = -\frac{M_o L}{EI}, \qquad v_A = \frac{M_o L^2}{2EI}$$

(5.37)

The minus sign indicates that the angle of rotation is counterclockwise (Fig. 5.8a).

5.11 Statically Indeterminate Systems

A large class of problems of considerable practical interest relates to structural systems for which the equations of statics are not sufficient (though are necessary) for determination of the reactions or other unknown forces. Such systems are *statically indeterminate*, requiring supplementary information for solution. Additional equations usually describe certain geometrical conditions associated with displacement or strain. These *equations of compatibility* state that the strain owing to deflection or rotation must be such as to preserve continuity. With this additional information, the solution proceeds in essentially the same manner as for statically determinate systems. The number of reactions in excess of the number of equilibrium equations is called the *degree of statical indeterminacy*. Any reaction in excess of that which can be obtained by statics alone is said to be *redundant*. Thus, the number of redundants is the same as the degree of indeterminacy.

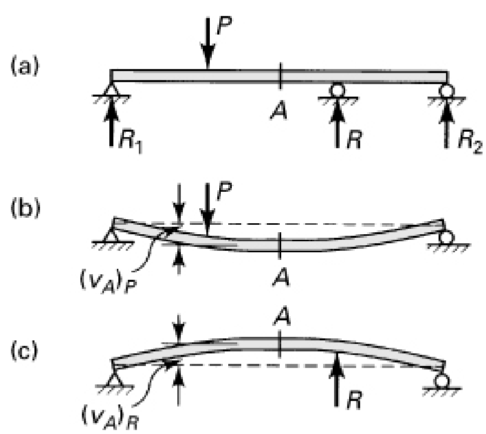
Several methods are available to analyze statically indeterminate structures. The principle of superposition, briefly discussed next, offers for many cases an effective approach. In Section 5.6 and in Chapters 7 and 10, a number of commonly employed methods are discussed for the solution of the indeterminate beam, frame, and truss problems.

The Method of Superposition

In the event of complicated load configurations, the method of *superposition* may be used to good advantage to simplify the analysis. Consider, for example, the continuous beam of Fig. 5.19a, replaced by the beams shown in Fig. 5.19b and c. At point A, the beam now experiences the deflections $(v_A)_P$ and $(v_A)_R$ due respectively to P and R. Subject to the restrictions imposed by small deformation theory and a material obeying Hooke's law, the deflections and stresses are linear functions of transverse loadings, and superposition is valid:

$$v_A = (v_A)_P + (v_A)_R$$
$$\sigma_A = (\sigma_A)_P + (\sigma_A)_R$$

Figure 5.19. Superposition of displacements in a continuous beam.

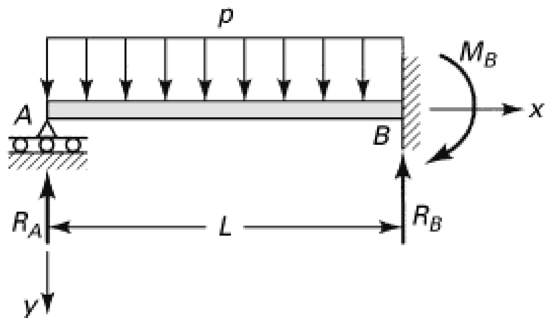


The procedure may in principle be extended to situations involving any degree of indeterminacy.

Example 5.9. Displacements of a Propped Cantilever Beam

A propped cantilever beam AB subject to a uniform load of intensity p is shown in Fig. 5.20. Determine (a) the reactions, (b) the equation of the deflection curve, and (c) the slope at A.

Figure 5.20. Example 5.9. A propped beam under uniform load.



Solution

Reactions R_A, R_B, and M_B are statically indeterminate because there are only two equilibrium conditions ($\Sigma F_y = 0$, $\Sigma M_z = 0$); the beam is statically indeterminate to the first degree. With the origin of coordinates taken at the left support, the equation for the beam moment is

$$M = -R_A x + \tfrac{1}{2}px^2$$

The third of Eqs. (5.32) then becomes

$$EIv'' = -R_A x + \tfrac{1}{2}px^2$$

and successive integrations yield

$$EIv' = -\tfrac{1}{2}R_A x^2 + \tfrac{1}{6}px^3 + c_1$$
$$EIv = -\tfrac{1}{6}R_A x^3 + \tfrac{1}{24}px^4 + c_1 x + c_2 \tag{a}$$

There are three unknown quantities in these equations (c_1, c_2, and R_A) and three boundary conditions:

$$v(0) = 0, \qquad v'(L) = 0, \qquad v(L) = 0 \tag{b}$$

a. Introducing Eqs. (b) into the preceding expressions, we obtain $c_2 = 0$, $c_1 = pL^3/48$, and

$$R_A = \tfrac{3}{8}pL \tag{5.58a}$$

We can now determine the remaining reactions from the equations of equilibrium:

$$R_B = \tfrac{5}{8}pL, \qquad M_B = \tfrac{1}{8}pL^2$$

(5.58b, c)

b. Substituting for R_A, c_1, and c_2 in Eq. (a), the equation of the deflection curve is obtained:

$$v = \frac{p}{48EI}(2x^4 - 3Lx^3 + L^3x)$$

(5.59)

c. Differentiating the foregoing with respect to x, the equation of the angle of rotation is

$$\theta = \frac{p}{48EI}(8x^3 - 9Lx^2 + L^3)$$

(5.60)

Setting $x = 0$, we have the slope at A:

$$\theta_A = \frac{pL^3}{48EI}$$

(5.61)

Example 5.10. Reactions of a Propped Cantilever Beam

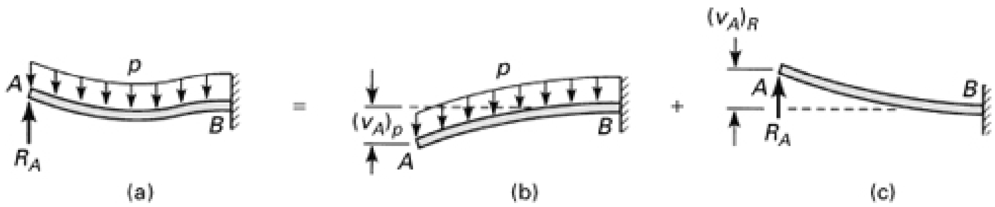
Consider again the statically indeterminate beam of Fig. 5.20. Determine the reactions using the method of superposition.

Solution

Reaction R_A is selected as redundant and is considered an unknown load by eliminating the support at A (Fig. 5.21a). The loading is resolved into those shown in Fig. 5.21b. The solution for each case is (see Table D.4)

$$(v_A)_P = \frac{pL^4}{8EI}, \qquad (v_A)_R = -\frac{R_A L^3}{3EI}$$

Figure 5.21. Example 5.10. Method of superposition: (a) reaction R_A is selected as redundant; (b) deflection at end A due to load P; (c) deflection at end A due to reaction R_A.



The compatibility condition for the original beam requires that

$$v_A = \frac{pL^4}{8EI} - \frac{R_A L^3}{3EI} = 0$$

from which $R_A = 3pL/8$. Reaction R_B and moment M_B can now be found from the equilibrium requirements. The results correspond to those of Example 5.9.

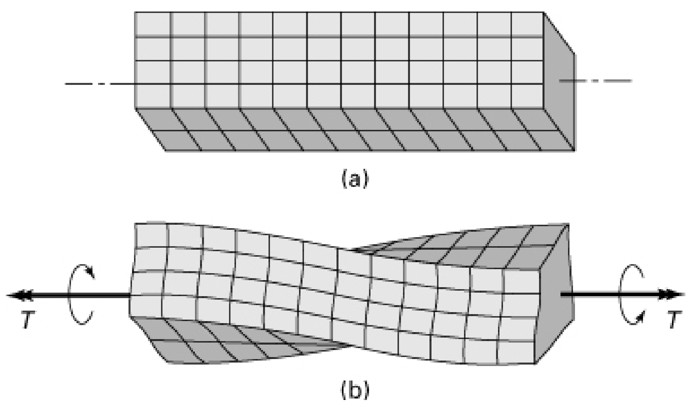
Chapter 6. Torsion of Prismatic Bars

6.1 Introduction

In this chapter, consideration is given to stresses and deformations in prismatic members subject to equal and opposite end torques. In general, these bars are assumed free of end constraint. Usually, members that transmit torque, such as propeller shafts and torque tubes of power equipment, are circular or tubular in cross section. For circular cylindrical bars, the torsion formulas are readily derived employing the method of mechanics of materials, as illustrated in the next section. We shall observe that a shaft having a circular cross section is *most efficient* compared to a shaft having an arbitrary cross section.

Slender members with other than circular cross sections are also often used. In treating noncircular prismatic bars, cross sections initially plane (Fig. 6.1a) experience out-of-plane deformation or *warping* (Fig. 6.1b), and the basic kinematic assumptions of the elementary theory are no longer appropriate. Consequently, the theory of elasticity, a general analytic approach, is employed, as discussed in Section 6.4. The governing differential equations derived using this method are applicable to both the linear elastic and the fully plastic torsion problems. The latter is treated in Section 12.10.

Figure 6.1. Rectangular bar: (a) before loading; (b) after a torque is applied.



For cases that cannot be conveniently solved analytically, the governing expressions are used in conjunction with the membrane and fluid flow analogies, as will be treated in Sections 6.6 through 6.9. Computer-oriented numerical approaches (Chap. 7) are also very efficient for such situations. The chapter concludes with discussions of warping of thin-walled open cross sections and combined torsion and bending of curved bars.

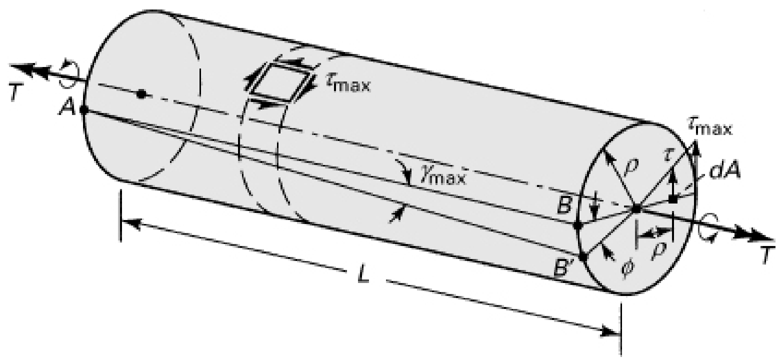
6.2 Elementary Theory of Torsion of Circular Bars

Consider a torsion bar or shaft of circular cross section (Fig. 6.2). Assume that the right end twists relative to the left end so that longitudinal line AB deforms to AB'. This results in a shearing stress τ and an angle of twist or angular deformation φ. The reader will recall from an earlier study of mechanics of materials [Ref. 6.1] the *basic assumptions* underlying the formulations for the torsional loading of *circular bars*:

1. All plane sections perpendicular to the longitudinal axis of the bar remain plane following the application of torque; that is, points in a given cross-sectional plane remain in that plane after twisting.
2. Subsequent to twisting, cross sections are undistorted in their individual planes; that is, the shearing strain γ varies linearly from zero at the center to a maximum on the outer surface.

   The preceding assumptions hold for both elastic and inelastic material behavior. In the elastic case, the following also applies:
3. The material is homogeneous and obeys Hooke's law; hence, the magnitude of the maximum shear angle γ_{max} must be less than the yield angle.

Figure 6.2. Variation of stress and angular rotation of a circular member in torsion.



We now derive the elastic stress and deformation relationships for circular bars in torsion in a manner similar to the most fundamental equations of the mechanics of materials: by employing the previously considered procedures and the foregoing assumptions. In the case of these elementary formulas, there is complete agreement between the experimentally obtained and the computed quantities. Moreover, their validity can be demonstrated through application of the theory of elasticity (see Example 6.3).

Shearing Stress

On any bar cross section, the resultant of the stress distribution must be equal to the applied torque T (Fig. 6.2). That is,

$$T = \int \rho(\tau dA) = \int \rho\left(\frac{\rho}{r}\tau_{max}\right)dA$$

where the integration proceeds over the entire area of the cross section. At any given section, the maximum shearing stress τ_{max} and the distance r from the center are constant. Hence, the foregoing can be written

$$T = \frac{\tau_{max}}{r}\int \rho^2 dA \tag{a}$$

in which $\int \rho^2 dA = J$ is the polar moment of inertia of the circular cross section (Sec. C.2). For circle of radius r, $J = \pi r^4/2$. Thus,

$$\tau_{\max} = \frac{Tr}{J} \tag{6.1}$$

This is the well-known *torsion formula* for circular bars. The shearing stress at a distance ρ from the center is

$$\tau = \frac{T\rho}{J} \tag{6.2}$$

The *transverse* shearing stress obtained by Eq. (6.1) or (6.2) is accompanied by a *longitudinal* shearing stress of equal value, as shown on a surface element in the figure.

We note that, when a shaft is subjected to torques at several points along its length, the *internal torques* will vary from section to section. A graph showing the variation of torque along the axis of the shaft is called the *torque diagram*. That is, this diagram represents a plot of the internal torque T versus its position x along the shaft length. As a *sign convention*, T will be positive if its vector is in the direction of a positive coordinate axis. However, the diagram is not used commonly, because in practice, only a few variations in torque occur along the length of a given shaft.

Angle of Twist

According to Hooke's law, $\gamma_{\max} = \tau_{\max}/G$; introducing the torsion formula, $\gamma_{\max} = Tr/JG$, where G is the modulus of elasticity in shear. For small deformations, $\tan \gamma_{\max} = \gamma_{\max}$, and we may write $\gamma_{\max} = r\varphi/L$ (Fig. 6.2). The foregoing expressions lead to the angle of twist, the angle through which one cross section of a circular bar rotates with respect to another:

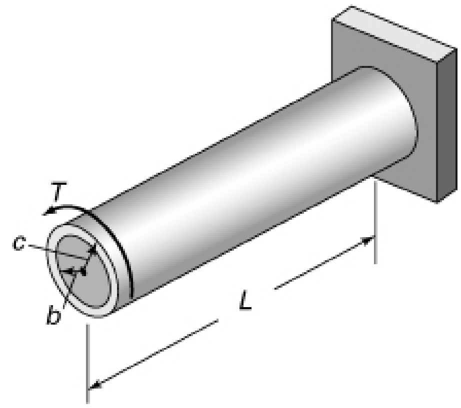
$$\phi = \frac{TL}{JG} \tag{6.3}$$

Angle φ is measured in radians. The product JG is termed the *torsional rigidity* of the member. Note that Eqs. (6.1) through (6.3) are valid for both solid and hollow circular bars; this follows from the assumptions used in the derivations. For a circular tube of inner radius r_i and outer radius r_o, we have $J = \pi(r_o^4 - r_i^4)/2$.

Example 6.1. Stress and Deformation in an Aluminum Shaft

A hollow aluminum alloy 6061-T6 shaft of outer radius $c = 40$ mm, inner radius $b = 30$ mm, and length $L = 1.2$ m is fixed at one end and subjected to a torque T at the other end, as shown in Fig. 6.3. If the shearing stress is limited to $\tau_{max} = 140$ MPa, determine (a) the largest value of the torque; (b) the corresponding minimum value of shear stress; (c) the angle of twist that will create a shear stress $\tau_{min} = 100$ MPa on the inner surface.

Figure 6.3. Example 6.1. A tubular circular bar in torsion.



Solution

From Table D.1, we have the shear modulus of elasticity $G = 72$ GPa and $\tau_{yp} = 220$ MPa.

a. Inasmuch as $\tau_{max} < \tau_{yp}$, we can apply Eq. (6.1) with $r = c$ to obtain

$$T = \frac{J\tau_{max}}{c} \tag{b}$$

By Table C.1, the polar moment of inertia of the hollow circular tube is

$$J = \frac{\pi}{2}(c^4 - b^4) = \frac{\pi}{2}(40^4 - 30^4) = 2.749(10^6)\text{mm}^4$$

Inserting J and τ_{max} into Eq. (b) results in

$$T = \frac{2.749(10^{-6})(140 \times 10^6)}{0.04} = 9.62 \text{ MPa}$$

b. The smallest value of the shear stress takes place on the inner surface of the shaft, and τ_{min} and τ_{max} are respectively proportional to b and c. Therefore,

$$\tau_{\min} = \frac{b}{c}\tau_{\max} = \frac{30}{40}(140) = 105 \text{ MPa}$$

c. Through the use of Eq. (2.27), the shear strain on the inner surface of the shaft is equal to

$$\gamma_{\min} = \frac{\tau_{\min}}{G} = \frac{100 \times 10^6}{72 \times 10^9} = 1389\ \mu$$

Referring to Fig. 6.2,

$$\phi = \frac{L\gamma_{\min}}{b} = \frac{1200}{30}(1389 \times 10^{-6}) = 55.6 \times 10^{-3} \text{ rad}$$

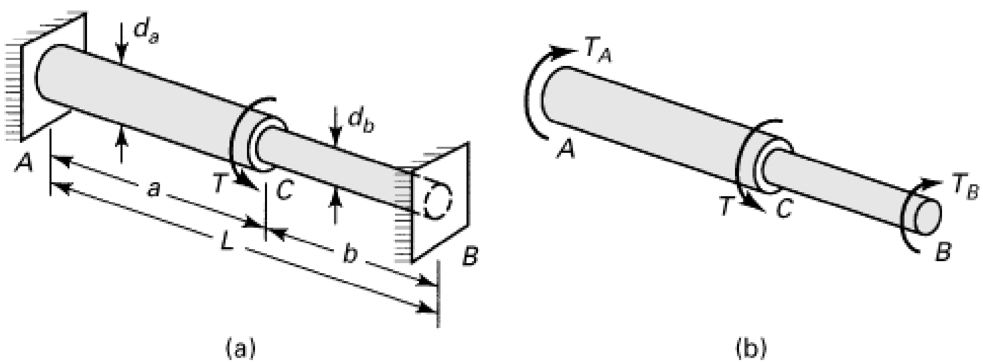
or

$$\varphi = 3.19°$$

Example 6.2. Redundantly Supported Shaft

A solid circular shaft AB is fixed to rigid walls at both ends and subjected to a torque T at section C, as shown in Fig. 6.4a. The shaft diameters are d_a and d_b for segments AC and CB, respectively. Determine the lengths a and b if the maximum shearing stress in both shaft segments is to be the same for d_a = 20 mm, d_b = 12 mm, and L = 600 mm.

Figure 6.4. Example 6.2. (a) A fixed-ended circular shaft in torsion; (b) free-body diagram of the entire shaft.



Solution

From the free-body diagram of Fig. 6.4a, we observe that the problem is *statically indeterminate* to the first degree; the one equation of equilibrium available is not sufficient to obtain the two unknown reactions T_A and T_B.

Condition of Equilibrium. Using the free-body diagram of Fig. 6.4b,

$$\Sigma T = 0, \qquad T_A + T_B = T$$

Torque-Displacement Relations. The angle of twist at section C is expressed in terms of the left and right segments of the solid shaft, respectively, as

$$\phi_a = \frac{T_A a}{J_a G}, \qquad \phi_b = \frac{T_B b}{J_b G} \tag{d}$$

Here, the polar moments of inertia are $J_a = \pi d_a^4/32$ and $J_b = \pi d_b^4/32$.

Condition of Compatibility. The two segments must have the same angle of twist where they join. Thus,

$$\phi_a = \phi_b \quad \text{or} \quad \frac{T_A a}{J_a G} = \frac{T_B b}{J_b G} \tag{e}$$

Equations (c) and (e) can be solved simultaneously to yield the reactions

$$T_A = \frac{T}{1 + (aJ_b/bJ_a)}, \qquad T_B = \frac{T}{1 + (bJ_a/aJ_b)} \tag{f}$$

The maximum shearing stresses in each segment of the shaft are obtained from the torsion formula:

$$\tau_a = \frac{T_A d_a}{2J_a}, \qquad \tau_b = \frac{T_B d_b}{2J_b} \tag{g}$$

For the case under consideration, $\tau_a = \tau_b$, or

$$\frac{T_A d_a}{J_a} = \frac{T_B d_b}{J_b}$$

Introducing Eqs. (f) into the foregoing and simplifying, we obtain

$$\frac{a}{b} = \frac{d_a}{d_b}$$

from which

$$a = \frac{d_a L}{d_a + d_b}, \qquad b = \frac{d_b L}{d_a + d_b} \tag{h}$$

where $L = a + b$. Insertion of the given data results in

$$a = \frac{20(600)}{20 + 12} = 375 \text{ mm}, \qquad b = \frac{12(600)}{20 + 12} = 225 \text{ mm}$$

For a prescribed value of torque T, we can now compute the reactions, angle of twist at section C, and maximum shearing stress from Eqs. (f), (d), and (g), respectively.
