STRUCTURES-COMPUTER INTERACTION LABORATORY

Module 10

Conservative Force and Potential Energy – General Formulation

- Notations
- Discrete Equations of Motion

Goal

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- Become familiar with notations and conventions
- Formulate and solve discrete equations of motion

Resources

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- Read *“Chapter 4: Conservative force and potential energy”* of Course Notes

When only conservative forces are involved

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$$m_i \ddot{q}_i + \frac{\partial E_{\text{potential}}}{\partial q_i} = 0$$

$$\mathbf{M}\ddot{\mathbf{q}} + \frac{\partial E_{\text{potential}}}{\partial \mathbf{q}} = \mathbf{0}$$

Mass matrix $(N = 3)$:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = m_i \delta_{ij}$$

DOF Vector $(N = 3)$:

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Conservative forces and Viscous Forces

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$$m_i \ddot{q}_i + \frac{\partial E_{\text{potential}}}{\partial q_i} + \boxed{c_i \dot{q}_i} = 0$$

Damping force: $-c_i \dot{q}_i$, where c_i (unit: N-s/m) is related to viscous damping.

$$\mathbf{M}\ddot{\mathbf{q}} + \frac{\partial E_{\text{potential}}}{\partial \mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} = \mathbf{0}$$

Damping matrix ($N = 3$):

$$\mathbf{C} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} = c_i \delta_{ij}$$

Discrete (in time) equations of motion

$$m_i \ddot{q}_i + \frac{\partial E_{\text{potential}}}{\partial q_i} + c_i \dot{q}_i = 0$$

$$f_i \equiv \frac{m_i}{\Delta t}\left[\frac{q_i(t_{k+1}) - q_i(t_k)}{\Delta t} - \dot{q}_i(t_k)\right] + \frac{\partial E_{\text{potential}}}{\partial q_i} + c_i \frac{q_i(t_{k+1}) - q_i(t_k)}{\Delta t} = 0$$

$$\mathbb{J}_{ij} = \frac{\partial f_i}{\partial q_j} = \mathbb{J}_{ij}^{\text{inertia}} + \mathbb{J}_{ij}^{\text{potential}} + \mathbb{J}_{ij}^{\text{viscous}},$$

where

$$\mathbb{J}_{ij}^{\text{inertia}} = \frac{m_i}{\Delta t^2}\delta_{ij},$$

$$\mathbb{J}_{ij}^{\text{potential}} = \frac{\partial^2 E_{\text{potential}}}{\partial q_i \partial q_j},$$

$$\mathbb{J}_{ij}^{\text{viscous}} = \frac{c_i}{\Delta t}\delta_{ij}.$$

Module 11

Example Problem: Rigid Spheres Connected by Springs in Viscous Flow

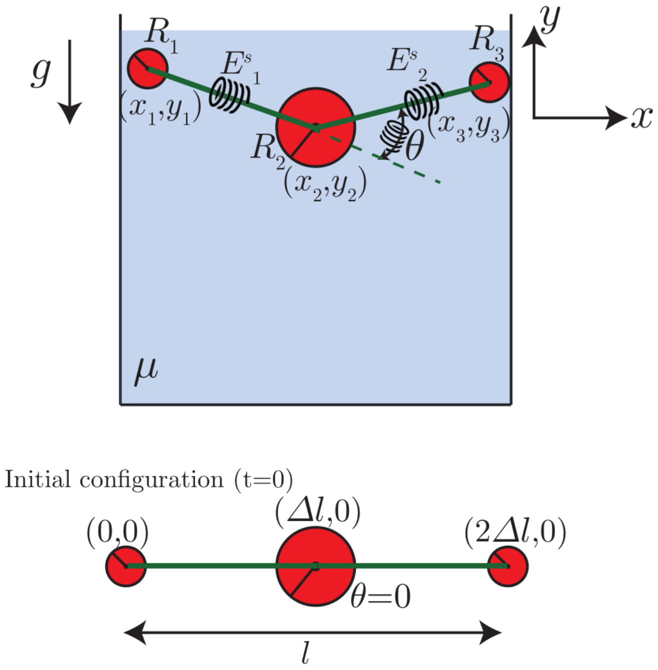
- Equations of Motion
- Advice on Programming Implementation

Goal

- Setup a simple problem with three rigid spheres connected by springs in a viscous flow
- Formulate equations of motion

Resources

- Read *"Chapter 4: Conservative force and potential energy"* of Course Notes

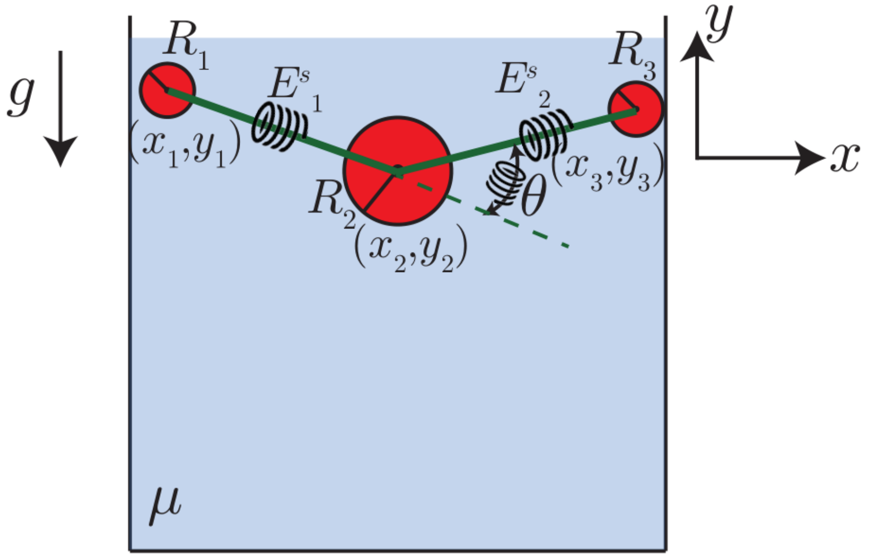


Problem Setup

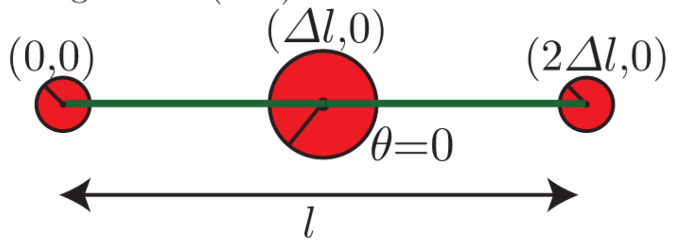
Three spheres are connected with two linear springs and one torsional spring.

- Radii of the sphere (R_1, R_2, R_3) are known
- Viscosity of the fluid (μ) is known
- Initial configuration (Δl) is known
- Densities of the spheres (ρ_{metal}) and fluid (ρ_{fluid}) are known

Problem: simulate the shape of the system as a function of time



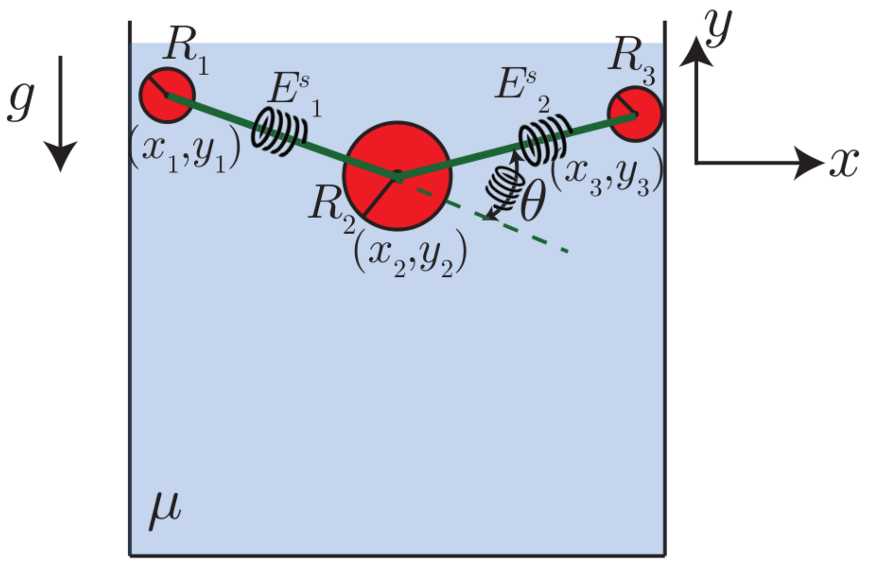
Initial configuration (t=0)



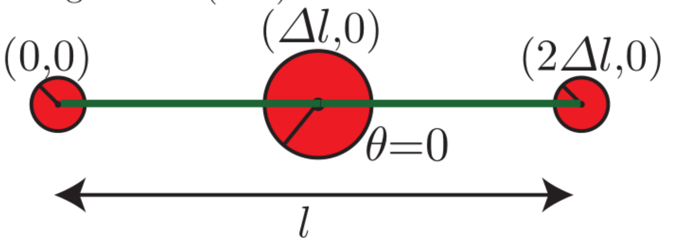
Degrees of Freedom

DOF Vector ($N = 6$):

$$\mathbf{q} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$



Keep in mind: θ can be determined from $\mathbf{q}$ and we do not need θ as an additional DOF



Continuous Equations of Motion

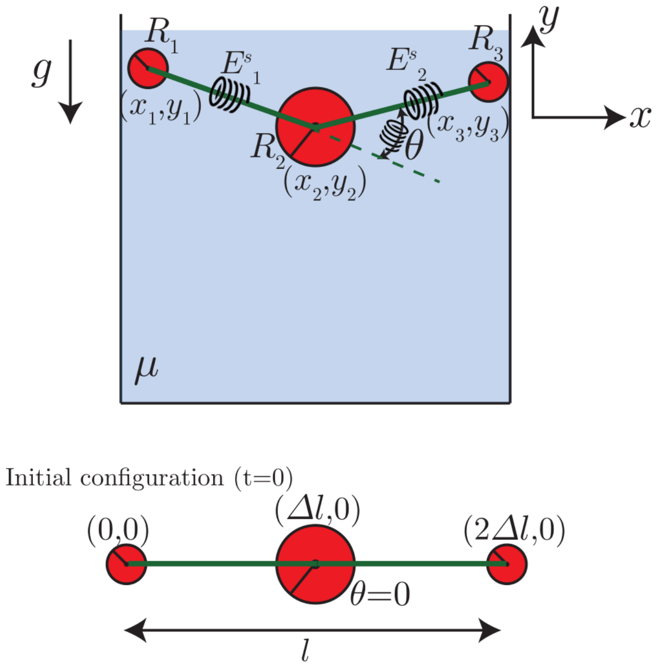
$$m_i\ddot{q}_i + \frac{\partial E^{\text{elastic}}}{\partial q_i} + c_i\dot{q}_i - \boxed{w_i} = 0$$

Weight (unit: N) associated with the i-th DOF.

Alternative notation: $\boxed{\mathbf{M}}\ddot{\mathbf{q}} + \frac{\partial E^{\text{elastic}}}{\partial \mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} - \mathbf{W} = \mathbf{0}$

Mass matrix:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_3 \end{bmatrix},$$



where

$$m_1 = \frac{4}{3}\pi R_1^3 \rho_{\text{metal}}, m_2 = \frac{4}{3}\pi R_2^3 \rho_{\text{metal}}, m_3 = \frac{4}{3}\pi R_3^3 \rho_{\text{metal}},$$

Continuous Equations of Motion

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$$\mathbf{M}\,\ddot{\mathbf{q}} + \boxed{\frac{\partial E^{\text{elastic}}}{\partial \mathbf{q}}} + \mathbf{C}\dot{\mathbf{q}} - \mathbf{W} = \mathbf{0}$$

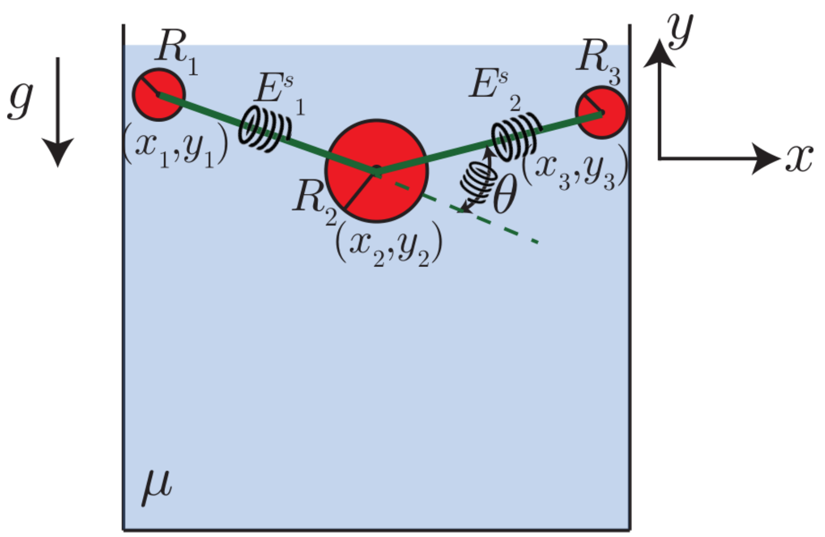
$$E^{\text{elastic}} = E_1^s + E_2^s + E^b,$$

where superscript s refers to *stretching*, b corresponds to bending, and

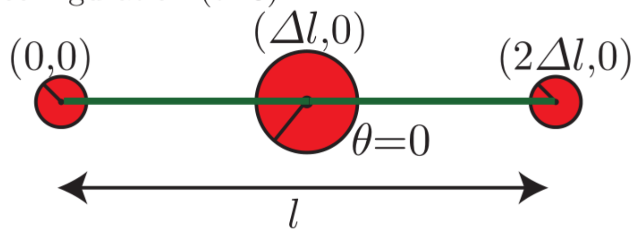
$$E_1^s = \frac{1}{2}EA\left(1 - \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{\Delta l}\right)^2 \Delta l,$$

$$E_2^s = \frac{1}{2}EA\left(1 - \frac{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}}{\Delta l}\right)^2 \Delta l,$$

$$E^b = \frac{1}{2}\frac{EI}{\Delta l}\left(2\tan\left(\frac{\theta}{2}\right)\right)^2.$$

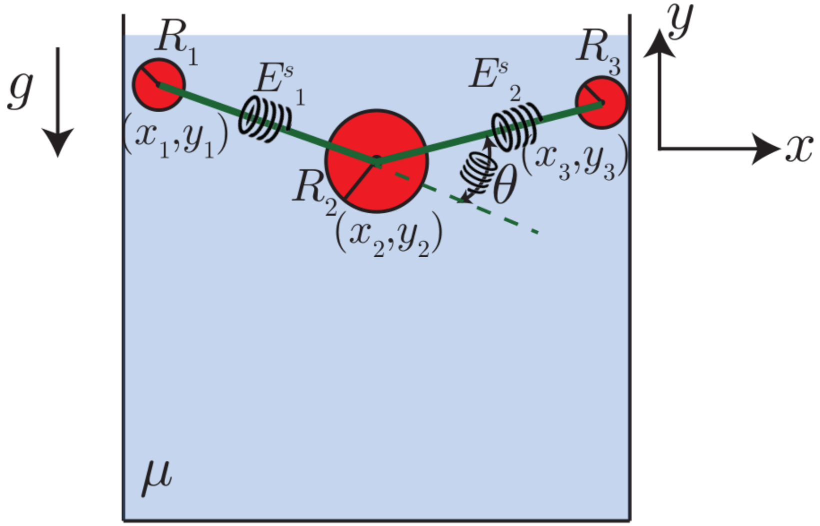


Initial configuration (t=0)



Continuous Equations of Motion

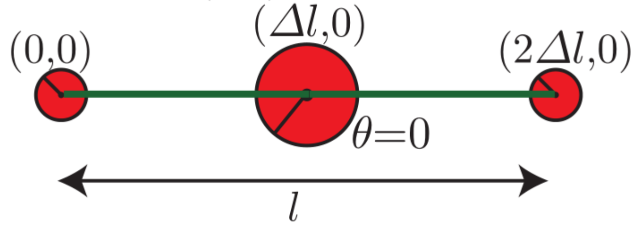
$$\mathbf{M}\,\ddot{\mathbf{q}} + \boxed{\frac{\partial E^{\text{elastic}}}{\partial \mathbf{q}}} + \mathbf{C}\dot{\mathbf{q}} - \mathbf{W} = \mathbf{0}$$



$$\frac{\partial E^{\text{elastic}}}{\partial \mathbf{q}} = \frac{\partial E^s_1}{\partial \mathbf{q}} + \frac{\partial E^s_2}{\partial \mathbf{q}} + \frac{\partial E^b}{\partial \mathbf{q}} \rightarrow \text{Vector of size 6.}$$

Codes that compute this gradient are provided to you.

Initial configuration (t=0)

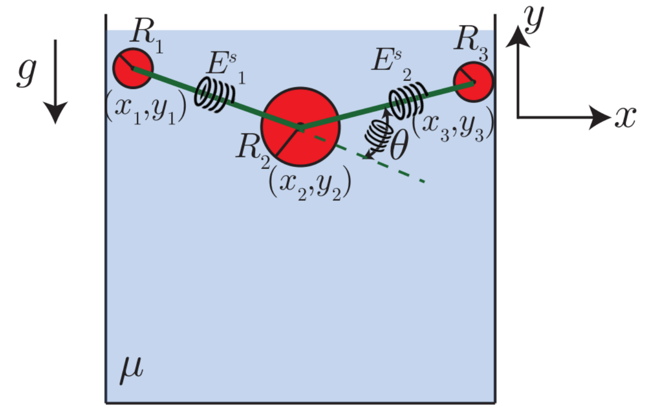


Continuous Equations of Motion

$$\mathbf{M}\,\ddot{\mathbf{q}} + \frac{\partial E^{\mathrm{elastic}}}{\partial \mathbf{q}} + \boxed{\mathbf{C}}\,\dot{\mathbf{q}} - \mathbf{W} = \mathbf{0}$$

Damping matrix:

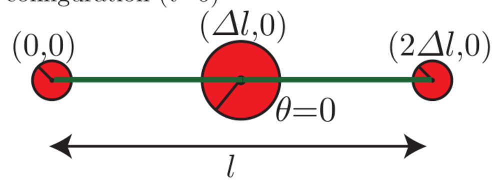
$$\mathbf{C} = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_3 \end{bmatrix},$$



where

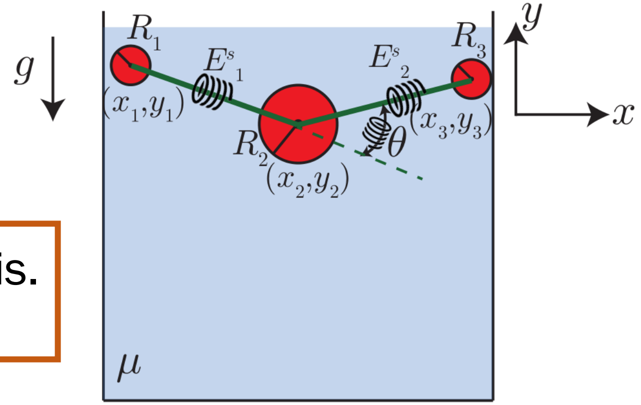
$$C_1 = 6\pi\mu R_1, C_2 = 6\pi\mu R_2, C_3 = 6\pi\mu R_3,$$

Initial configuration (t=0)



Continuous Equations of Motion

$$\mathbf{M}\,\ddot{\mathbf{q}} + \frac{\partial E^{\text{elastic}}}{\partial \mathbf{q}} + \mathbf{C}\,\dot{\mathbf{q}} - \boxed{\mathbf{W}} = \mathbf{0}$$

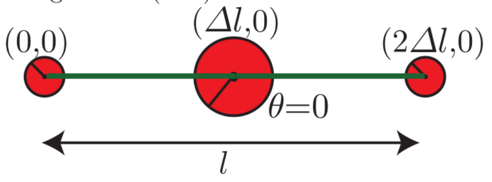


Weight vector:

$$\mathbf{W} = \begin{bmatrix} \boxed{\begin{matrix} 0 \\ W_1 \end{matrix}} \\ 0 \\ W_2 \\ 0 \\ W_3 \end{bmatrix},$$

Weight only acts along the y-axis.
Weight along the x-axis is 0.

Initial configuration (t=0)



where

$$W_1 = -\frac{4}{3}\pi R_1^3(\rho_{\text{metal}} - \rho_{\text{fluid}})g,\ W_2 = -\frac{4}{3}\pi R_2^3(\rho_{\text{metal}} - \rho_{\text{fluid}})g,\ W_3 = -\frac{4}{3}\pi R_3^3(\rho_{\text{metal}} - \rho_{\text{fluid}})g$$

Warning: be consistent with your sign convention.

Advice on Programming Implementation

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- Elastic (conservative) forces

$$\frac{\partial E^{\text{elastic}}}{\partial \mathbf{q}} = \frac{\partial E_1^s}{\partial \mathbf{q}} + \frac{\partial E_2^s}{\partial \mathbf{q}} + \frac{\partial E^b}{\partial \mathbf{q}} \rightarrow \text{Vector of size 6.}$$

E_1^s only depends on x_1, y_1, x_2, y_2. Therefore,

$$\frac{\partial E_s^1}{\partial \mathbf{q}} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ 0 \\ 0 \end{bmatrix}$$

Non-zero vector of size 4.
Code that computers this non-zero portion is provided with the Course Notes.

Advice on Programming Implementation

STRUCTURES-COMPUTER INTERACTION LABORATORY

- Elastic (conservative) forces

$$\frac{\partial E^{\text{elastic}}}{\partial \mathbf{q}} = \frac{\partial E_1^s}{\partial \mathbf{q}} + \frac{\partial E_2^s}{\partial \mathbf{q}} + \frac{\partial E^b}{\partial \mathbf{q}} \rightarrow \text{Vector of size 6.}$$

$$\frac{\partial E_s^1}{\partial \mathbf{q}} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ 0 \\ 0 \end{bmatrix} \qquad \frac{\partial E_s^2}{\partial \mathbf{q}} = \begin{bmatrix} 0 \\ 0 \\ \times \\ \times \\ \times \\ \times \end{bmatrix} \qquad \frac{\partial E^b}{\partial \mathbf{q}} = \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}$$

Advice on Programming Implementation

- Jacobian associated with elastic (conservative) forces
  - Also called the <u>**Hessian**</u> of elastic energy

$$\boxed{\frac{\partial^2 E^{\text{elastic}}}{\partial \mathbf{q} \partial \mathbf{q}}} = \frac{\partial E_1^s}{\partial \mathbf{q} \partial \mathbf{q}} + \frac{\partial E_2^s}{\partial \mathbf{q} \partial \mathbf{q}} + \frac{\partial E^b}{\partial \mathbf{q} \partial \mathbf{q}} \rightarrow \text{Matrix of size } 6 \times 6.$$

$$\text{Alternative notation: } \frac{\partial^2 E^{\text{elastic}}}{\partial \mathbf{q}_i \partial \mathbf{q}_j} \text{ with } i = 1, 2, 3 \text{ and } j = 1, 2, 3.$$

$$\frac{\partial^2 E_s^1}{\partial \mathbf{q} \partial \mathbf{q}} = \begin{bmatrix} \times & \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \frac{\partial^2 E_s^2}{\partial \mathbf{q} \partial \mathbf{q}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \end{bmatrix} \quad \frac{\partial^2 E^b}{\partial \mathbf{q} \partial \mathbf{q}} = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix}$$

Advice on Programming Implementation

- Take advantage of the “block” nature of the forces and Jacobian

```
J = zeros(6,6);

a = [ 1 2 3 4;
      5 6 7 8;
      9 1 2 3;
      4 5 6 7 ];

J(3:6, 3:6) = J(3:6, 3:6) + a;


J =

     0     0     0     0     0     0
     0     0     0     0     0     0
     0     0     1     2     3     4
     0     0     5     6     7     8
     0     0     9     1     2     3
     0     0     4     5     6     7
```