STRUCTURES-COMPUTER INTERACTION LABORATORY

Module 19

Space Parallel Reference Frame

- Definition of twist using space parallel reference frame

Goal

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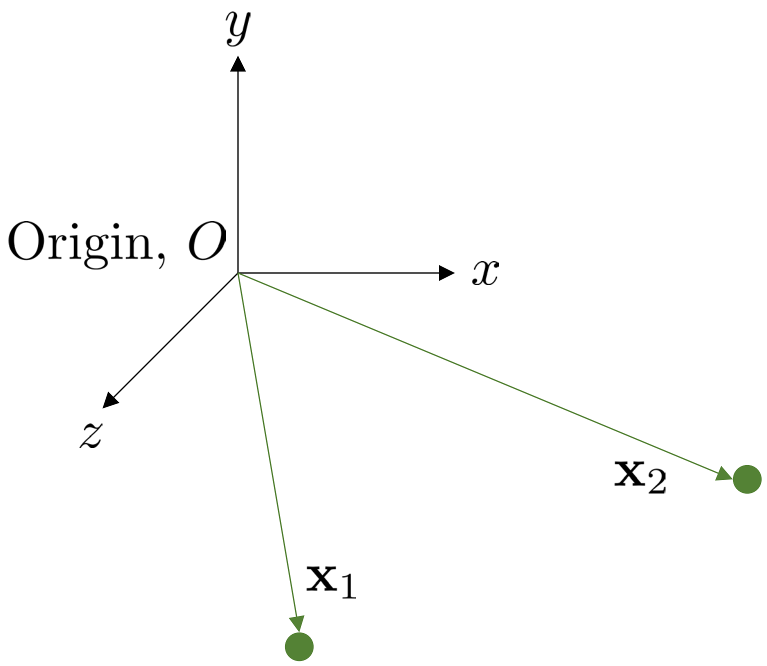
- Formulate a reference frame on the rod centerline using parallel transport along the arc-length (<u>space parallel transport</u>)
- Compute the twist along the rod using space parallel reference frame
- At the end of this module, you should be able to compute the twisting energy of a rod given its centerline and material frame

Resources

- Read *"Chapter 6: Discrete Twist"*
- Review Bergou, Miklós, et al. "Discrete elastic rods" *ACM SIGGRAPH (2008)* 1-12.

Relative Position

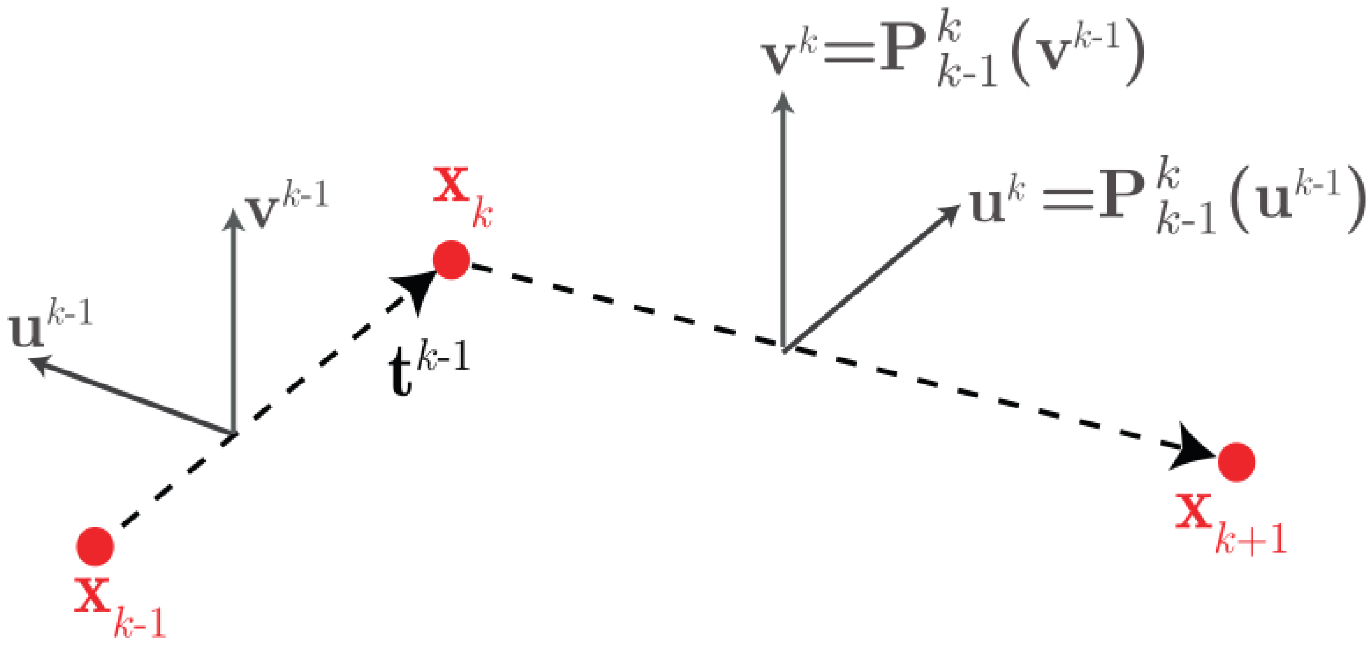


When describing the nodal coordinates, e.g. $\mathbf{x_1}$, we implicitly assume a reference frame with origin O.

Relative Rotation - Twist

When describing the rotation of the material frame, we need a reference frame

- How to compute the reference frame, $(\mathbf{u}^k, \mathbf{v}^k, \mathbf{t}^k)$?
- Known quantities: the nodal coordinates, $\mathbf{x}_k$



Reference Frame

- Since the positions of the nodes are known, the tangent on each edge is known:

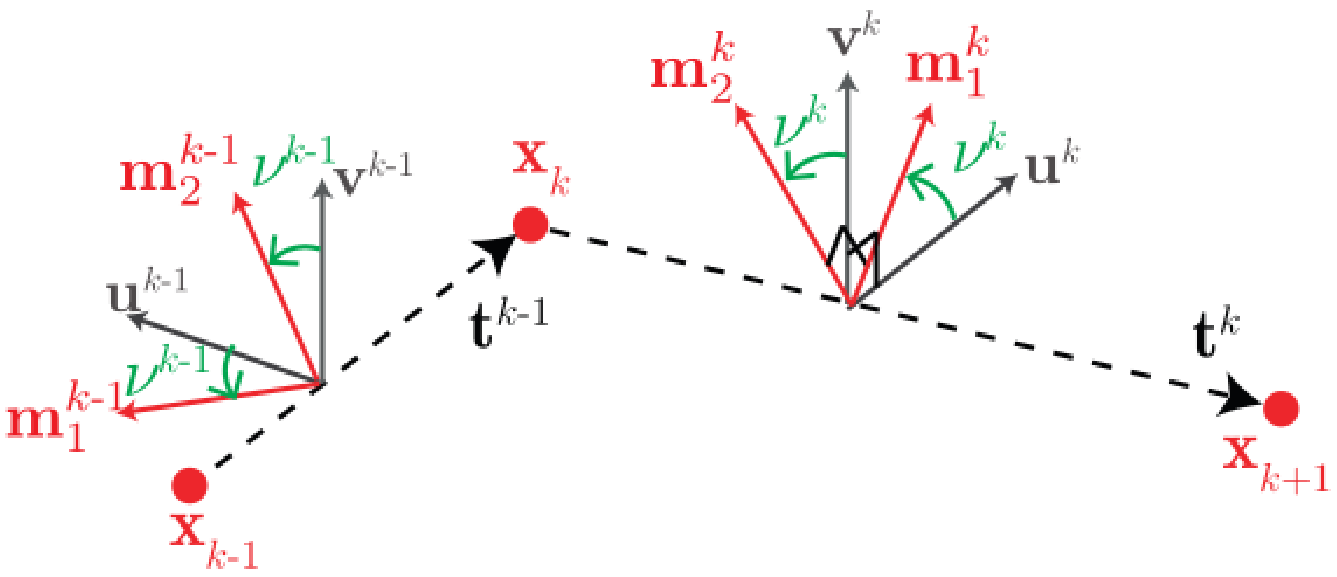
$$\mathbf{t}^k = \frac{\mathbf{e}^k}{\|\bar{\mathbf{e}}^k\|} = \frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{\|\bar{\mathbf{e}}^k\|}$$

- Start with an arbitrary (but adapted) orthonormal frame at the first edge, $\mathbf{e}^1$. For that purpose, choose any unit vector $\mathbf{u}^1$ that is orthogonal to $\mathbf{t}^1$. The second director is $\mathbf{v}^1 = \mathbf{t}^1 \times \mathbf{u}^1$ and the frame is $(\mathbf{u}^1, \mathbf{v}^1, \mathbf{t}^1)$.

- Sequentially compute the reference frames for the subsequent edges using $\mathbf{u}^k = P_{k-1}^k(\mathbf{u}^{k-1}) \equiv \texttt{parallel\_transport}(\mathbf{u}^{k-1}, \mathbf{t}^{k-1}, \mathbf{t}^k)$ and $\mathbf{v}^k = \mathbf{t}^k \times \mathbf{u}^k$.

How to compute the discrete integrated twist?

- Known quantities: configuration of the rod
  - Nodal coordinates, $\boldsymbol{x}_k$
  - Material frame on each edge, $(\boldsymbol{m}_1{}^k, \boldsymbol{m}_2{}^k, \boldsymbol{t}^k)$
  - We learned how to compute the reference frame, $(\boldsymbol{u}^k, \boldsymbol{v}^k, \boldsymbol{t}^k)$, using space parallel transport



How to compute the discrete integrated twist?

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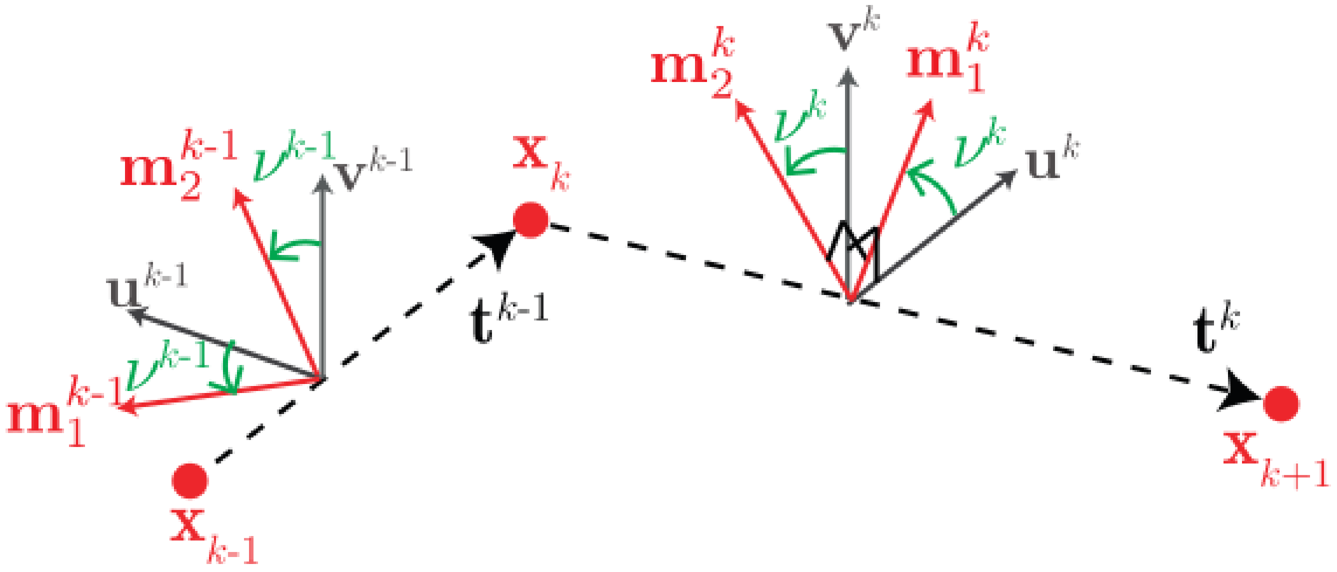
- Steps to compute the twist at the node, x_k
  1. Compute the signed angles from the reference frame to the material frame, ν^k
  2. Discrete integrated twist is the difference between the signed rotation angles of two consecutive edges: $\tau_k = \nu^k - \nu^{k-1}$



Degrees of Freedom of a Rod

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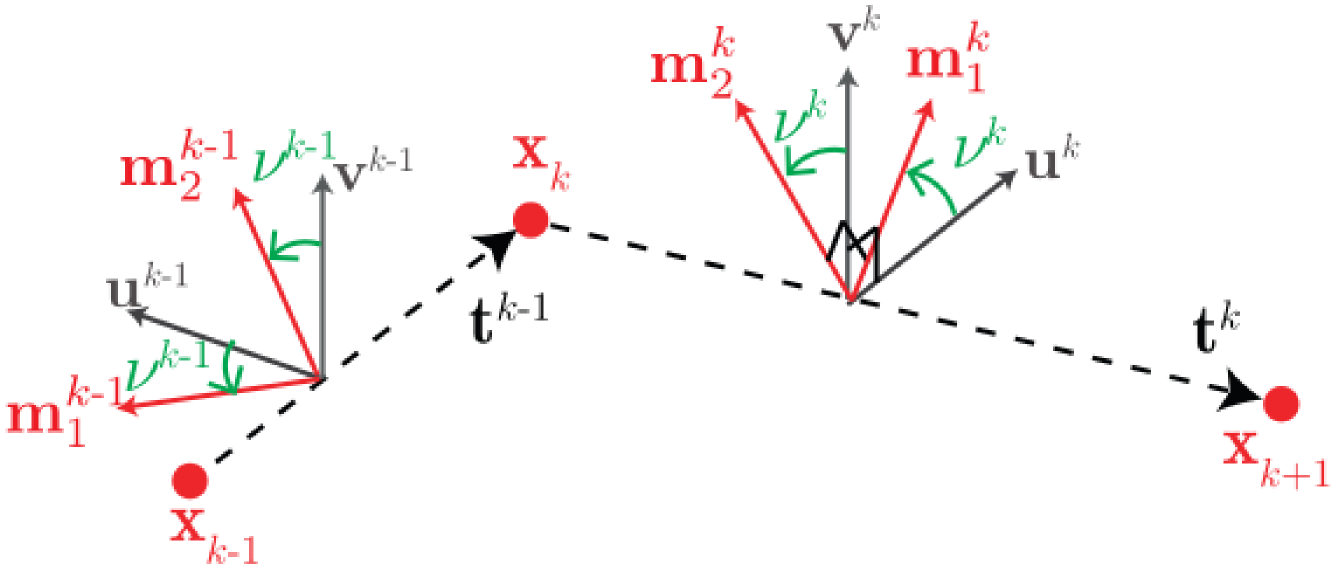
- If only the nodal coordinates, $\boldsymbol{x}_k$, and twist angles, ν_k, are known, the complete configuration of the rod can be determined
  - First, determine the space parallel reference frame
  - Rotate the reference frame director, $\boldsymbol{u}^{\boldsymbol{k}}$, by an angle ν_k to get the material frame director, $\boldsymbol{m}_{\mathbf{1}}^{\boldsymbol{k}}$. Recall that $\boldsymbol{m}_{\mathbf{2}}^{\boldsymbol{k}} = \boldsymbol{t}^{\boldsymbol{k}} \times \boldsymbol{m}_{\mathbf{1}}^{\boldsymbol{k}}$.



Degrees of Freedom (DOF) of a Rod

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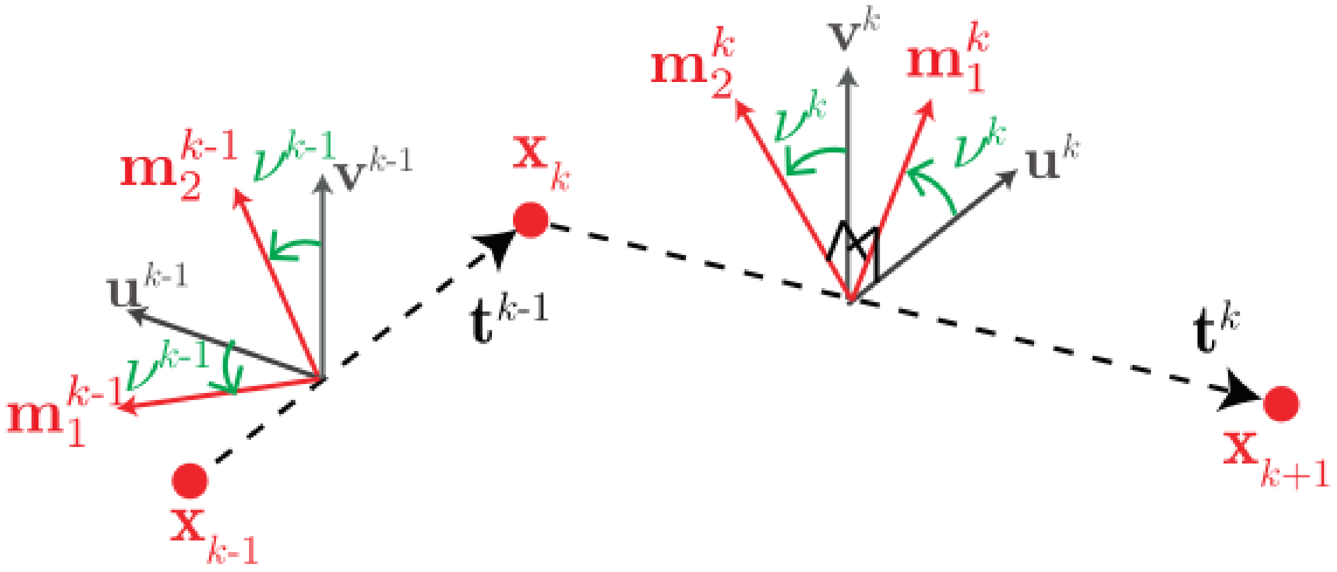
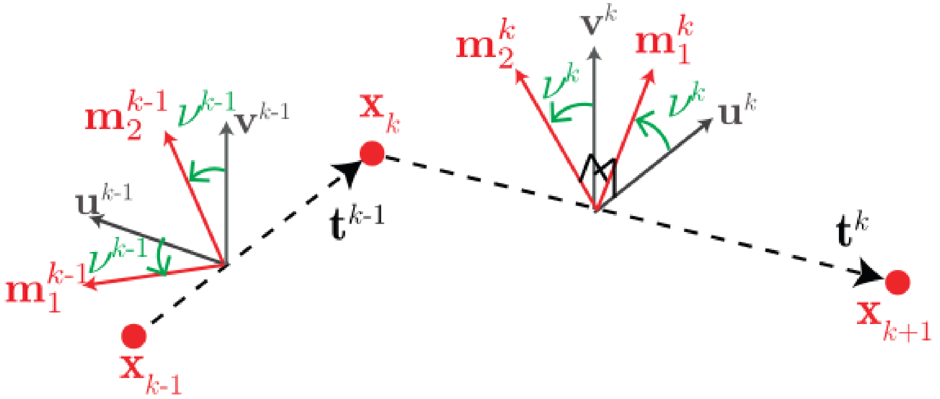
- A rod with N nodes has *(4N-1)* DOFs
  1. *3N* DOFs associated with the positions of N nodes, $\boldsymbol{x_k}$ $(1 \leq k \leq N)$
  2. *(N-1)* DOFs for the twist angles of the edges, $\boldsymbol{\nu^k}$ $(1 \leq k \leq N-1)$



Discrete Elastic Twisting Energy



$$E_t = \sum_{k=2}^{N-1} E_{t,k}$$

where $E_{t,k} = \frac{1}{2}\frac{GJ}{\bar{l}_k}\left(\tau_k\right)^2 = \frac{1}{2}\frac{GJ}{\bar{l}_k}\left(\nu^k - \nu^{k-1}\right)^2$

- G = shear modulus = $\frac{\text{Young's modulus}}{2(1+\text{Poisson ratio})}$
- J = polar moment of area ($= \frac{\pi r_0^4}{2}$ if circular)
- τ_k = signed angle from $P_{k-1}^k(\mathbf{m}_1^{k-1})$ to $\mathbf{m}_1^k$ about the axis $\mathbf{t}^k$

Future Plan

- Bergou et al. *SIGGRAPH* (2008) used space-parallel reference frame
- In 2010 (Bergou et al. *SIGGRAPH* (2010)), the authors formulated another formulation of reference frame: time-parallel reference frame
    - This leads to significant speed-up in computation
- In this class, we will follow the formulation in the 2010 version and discuss time-parallel reference frame in next module

Module 20

Time Parallel Reference Frame

- Definition of twist using time parallel reference frame

Goal

STRUCTURES-COMPUTER INTERACTION LABORATORY

- Formulate a reference frame on the rod centerline using parallel transport along time (time parallel transport)
- Compute the twist along the rod using time parallel reference frame
- At the end of this module, you should be able to compute the twisting energy of a rod using time parallel reference frame

Resources

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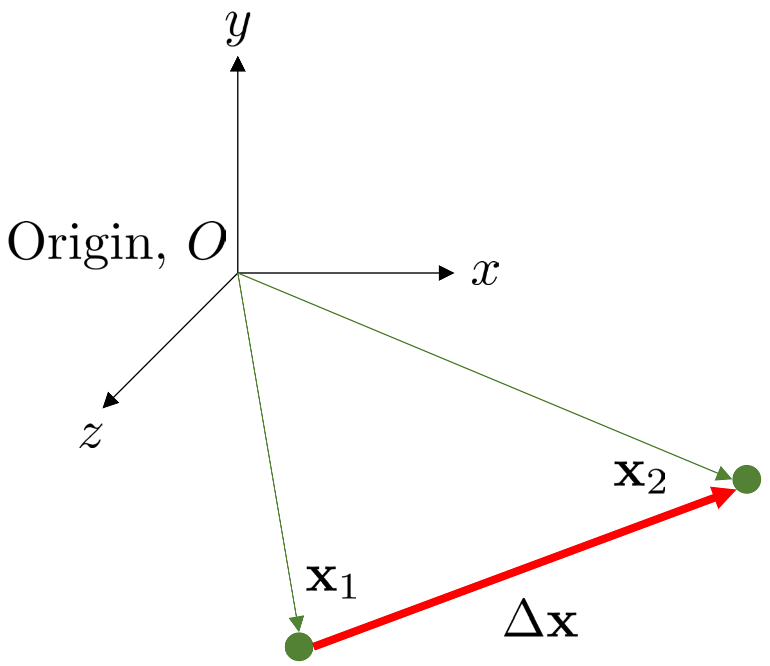
- Read *"Chapter 6: Discrete Twist"*
- Review Bergou, Miklós, et al. "Discrete viscous threads" *SIGGRAPH (2010)* 29(4) 1-10

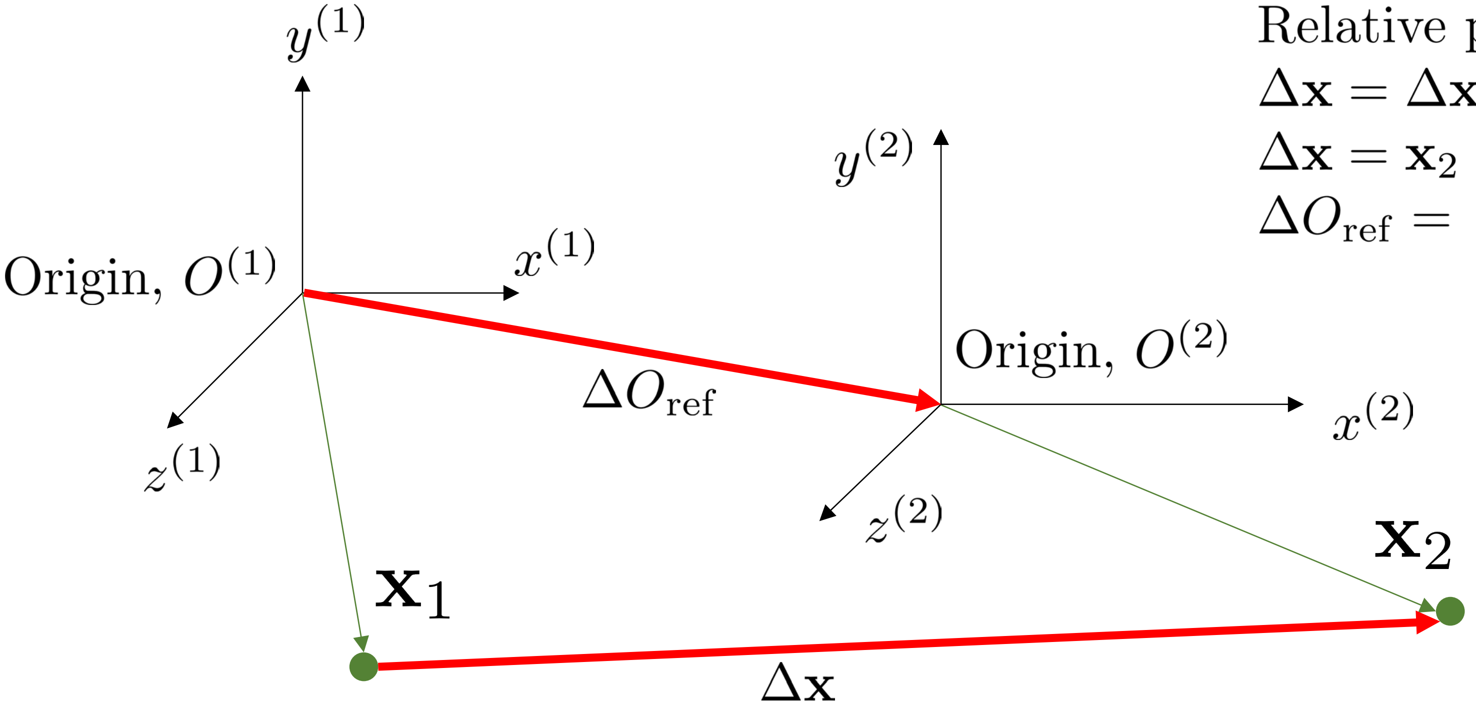
Warm up: Relative Position



Relative position of node # 2 with respect to # 1:
$\Delta\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$

Warm up: Relative Position

What if $\mathbf{x}_1$ and $\mathbf{x}_2$ are measured in two different Cartesian frames?



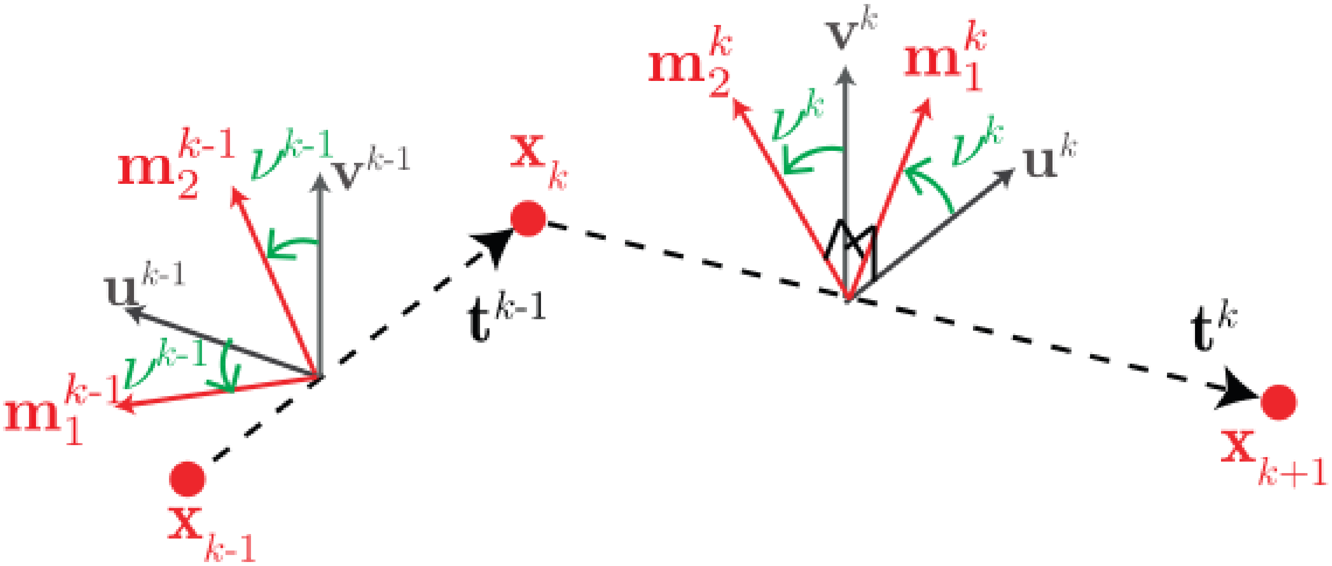
Relative position of node # 2 with respect to # 1:
$\Delta\mathbf{x} = \Delta\mathbf{x} + \Delta O_{\text{ref}}$ where
$\Delta\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$ and
$\Delta O_{\text{ref}} =$ "reference" displacement

Recall: Twist using Space-Parallel Reference Frame

- The frame from the $(k-1)$-th edge is transported *without twist* to the k-th edge

- Twist between $(\mathbf{u}^{k-1}, \mathbf{v}^{k-1}, \mathbf{t}^{k-1})$ and $(\mathbf{u}^{k}, \mathbf{v}^{k}, \mathbf{t}^{k})$ is zero (analogous to $\Delta O_{\text{ref}} = 0$)

- Discrete twist at the k-th node $(\mathbf{x}_k)$ is $\tau_k = \nu^k - \nu^{k-1}$ (analogous to $\Delta x = \mathbf{x}_2 - \mathbf{x}_1$)



Time-Parallel Reference Frame

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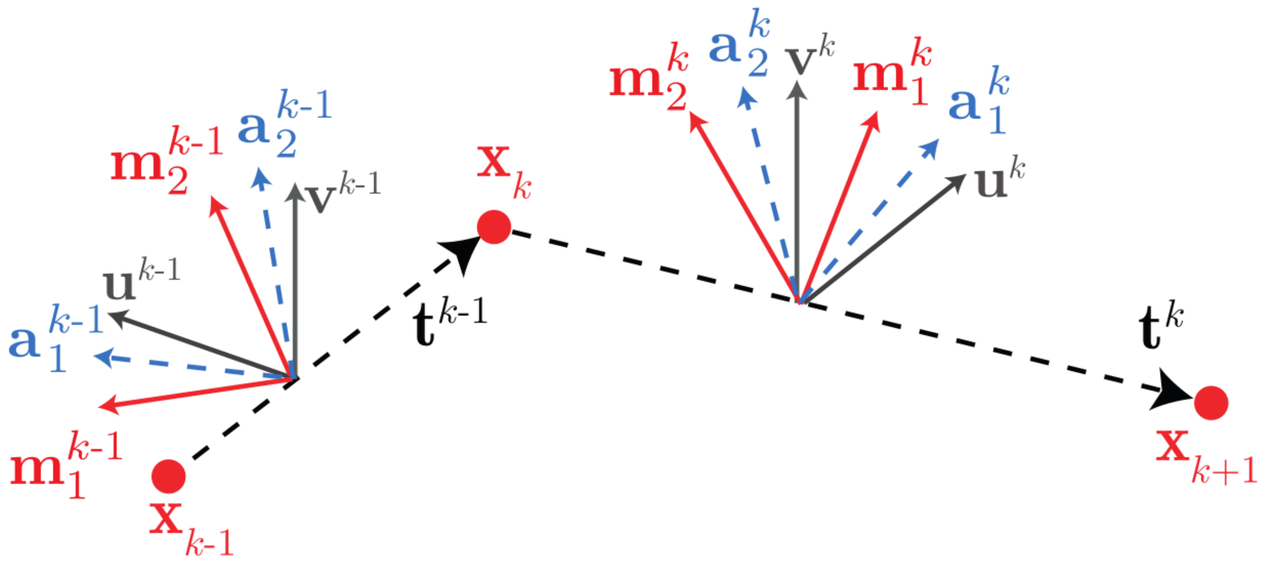
- At time $t = t_{j+1}$, space-parallel reference frame is $\mathbf{u}^k(t_{j+1}) = \texttt{parallel\_transport}\left(\mathbf{u}^{k-1}(t_{j+1}), \mathbf{t}^{k-1}(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$ and $\mathbf{v}^k(t_{j+1}) = \mathbf{t}^k(t_{j+1}) \times \mathbf{u}^k(t_{j+1})$

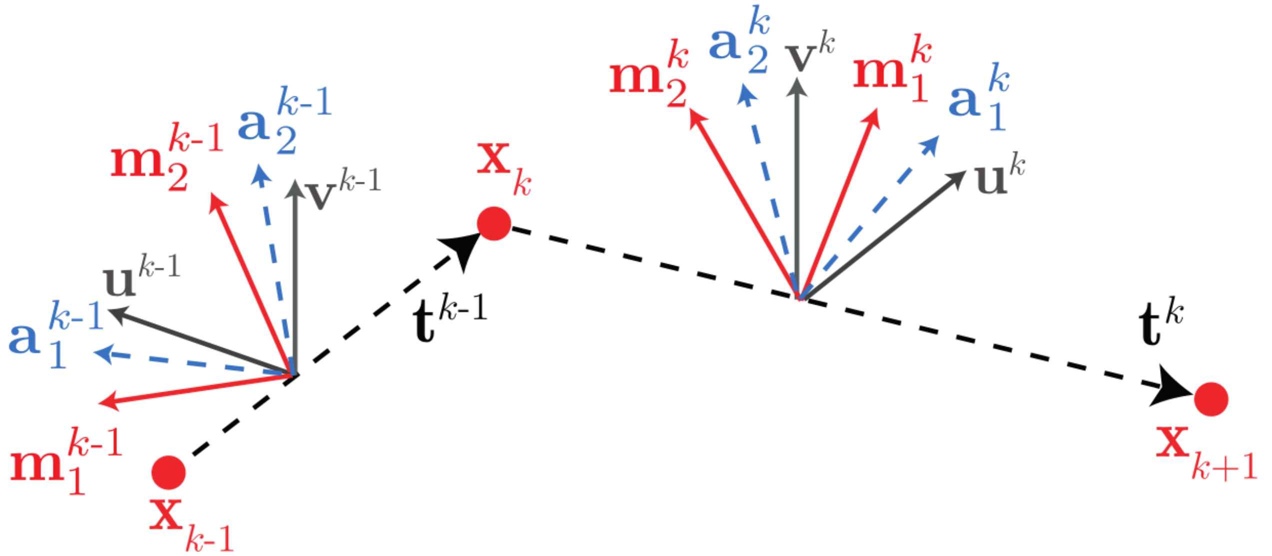
- At time $t = t_{j+1}$, time-parallel reference frame is $\mathbf{a}_1^k(t_{j+1}) = \texttt{parallel\_transport}\left(\mathbf{a}_1^k(t_j), \mathbf{t}^k(t_j), \mathbf{t}^k(t_{j+1})\right)$ and $\mathbf{a}_2^k(t_{j+1}) = \mathbf{t}^k(t_{j+1}) \times \mathbf{a}_1^k(t_{j+1})$



Time-Parallel Reference Frame

- Space-parallel frame $\mathbf{u}^k(t_{j+1}), \mathbf{v}^k(t_{j+1}), \mathbf{t}^k(t_{j+1})$ moves forward in *space*
- Time-parallel frame $\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})$ moves forward in *time*

- Space-parallel frame has no *reference twist*
- Time-parallel frame may have *reference twist* (analogous to $\Delta O_{\text{ref}} \neq 0$)



New Definition of Twist Angle

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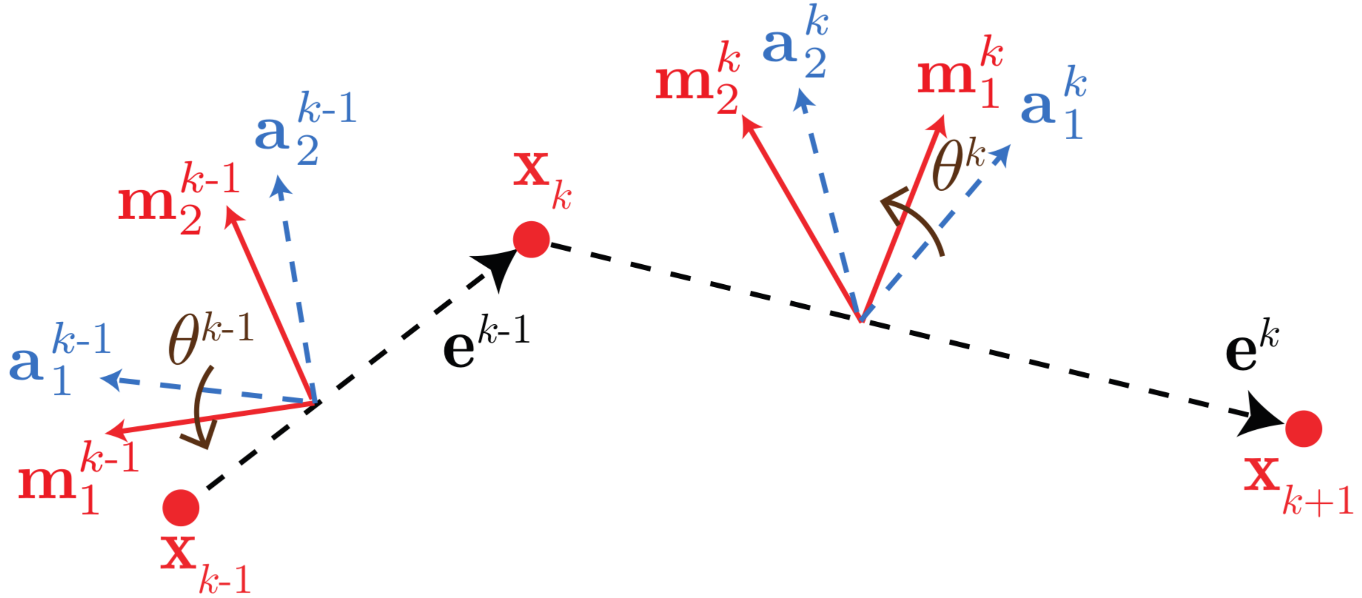
- Recall: signed angle from space-parallel frame director $\mathbf{u}^k(t_{j+1})$ to the material frame director $\mathbf{m}_1^k(t_{j+1})$ is the *twist angle* ν^k
- Signed angle from time-parallel frame director $\mathbf{a}_1^k(t_{j+1})$ to the material frame director $\mathbf{m}_1^k(t_{j+1})$ is the new *twist angle* θ^k
- We will use θ^k $(1 \leq k < N)$ as the DOFs of the rod

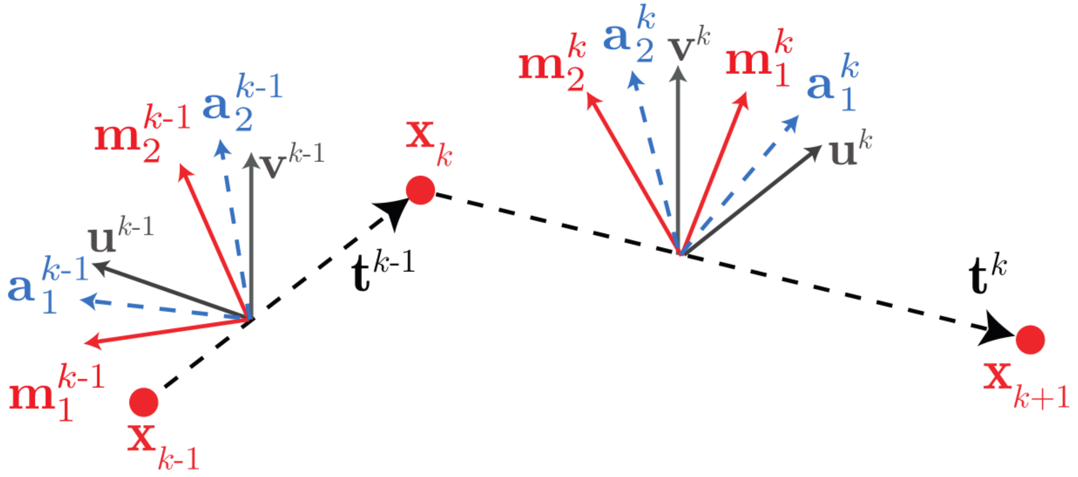


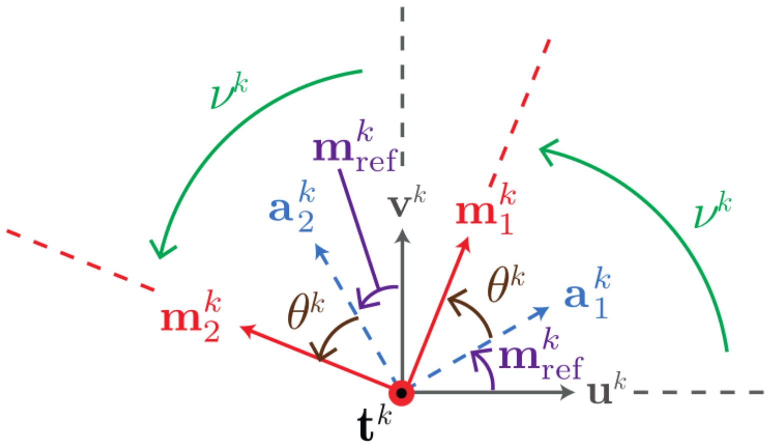
Reference Twist

- m_{ref}^k is the signed angle from space-parallel frame director, $\mathbf{u}^k$, to time-parallel frame director, $\mathbf{a}_1^k$

$$\nu^k = \theta^k + m_{\text{ref}}$$

- Reference twist at the k-th node is $\Delta m_{\text{ref}} = m_{\text{ref}}^k - m_{\text{ref}}^{k-1}$

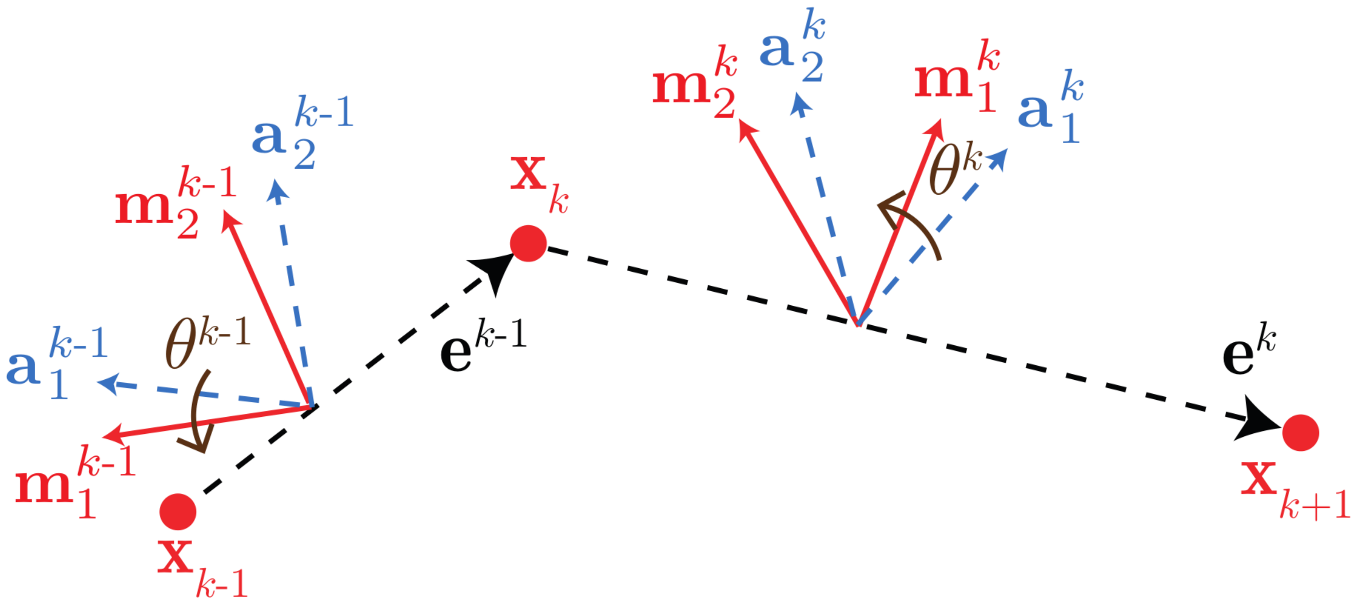




Measurement of Discrete Twist using Time-Parallel Reference Frame

Twist at the k-th node is
$\tau_k = \nu^k - \nu^{k-1} = \left(\theta^k + m^k_{\text{ref}}\right) - \left(\theta^{k-1} + m^{k-1}_{\text{ref}}\right) = \Delta\theta_k + \Delta m_{k,\text{ref}}$

The reference twist, $\Delta m_{k,\text{ref}}$, at node $\mathbf{x}_k$ is the signed angle from $P^k_{k-1}\left(\mathbf{a}^{k-1}_1(t_{j+1})\right)$ to $\mathbf{a}^k_1(t_{j+1})$ about the axis $\mathbf{t}^k(t_{j+1})$, i.e.
$\Delta m_{k,\text{ref}} = \texttt{signedAngle}\left(P^k_{k-1}\left(\mathbf{a}^{k-1}_1(t_{j+1})\right), \mathbf{a}^k_1(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$

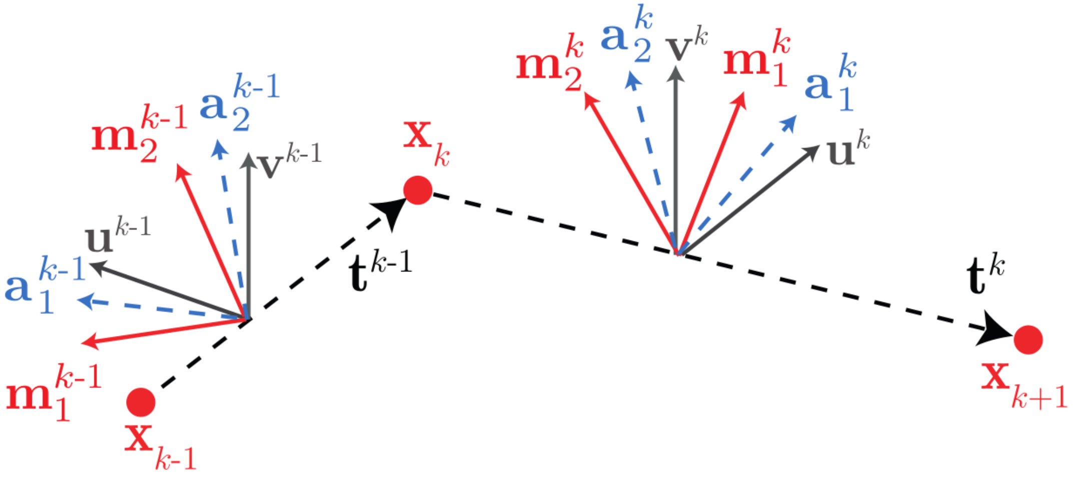


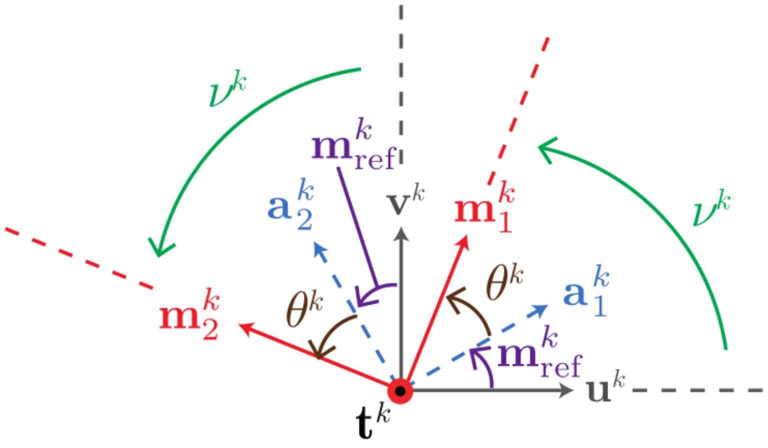
Summary: Time-Parallel Reference Frame and Twist

- At time $t = 0$, start with a space-parallel reference frame on each edge of the rod. This is also our time-parallel frame at $t = 0$

- Parallel transport the frame from previous time step $(t = t_j)$ to the next time step $(t = t_{j+1})$

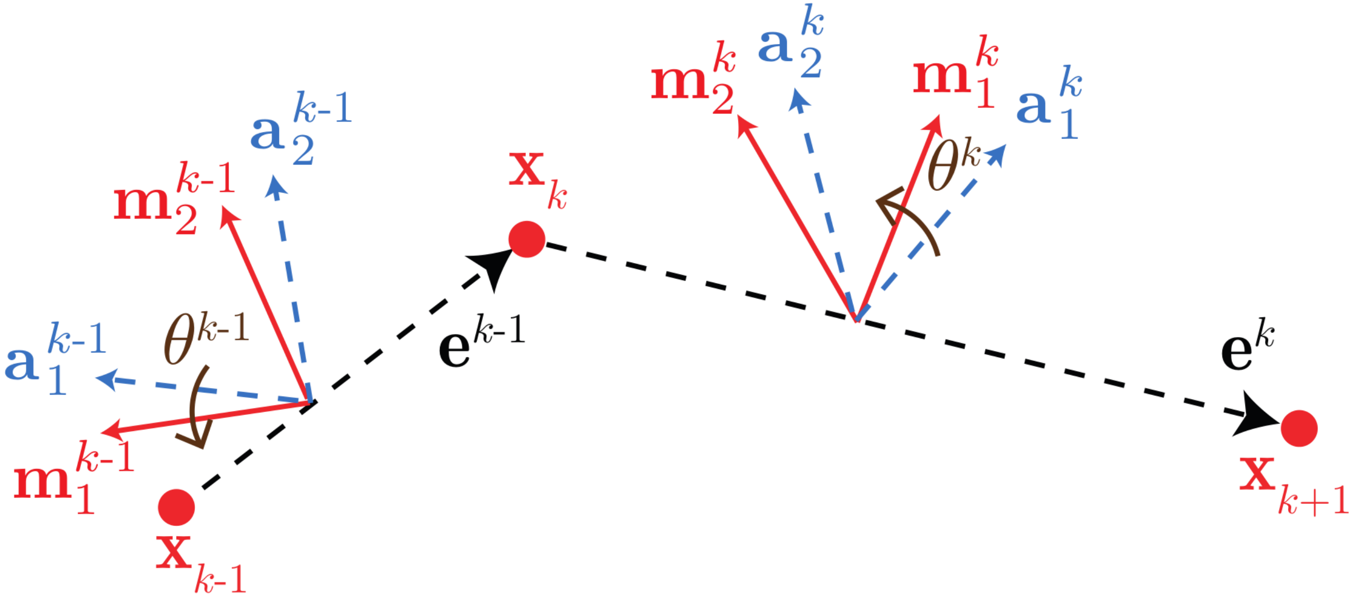
- Twist at the k-th node is
$\tau_k = \nu^k - \nu^{k-1} = \left(\theta^k + m^k_{\text{ref}}\right) - \left(\theta^{k-1} + m^{k-1}_{\text{ref}}\right) = \Delta\theta_k + \Delta m_{k,\text{ref}}$
where $\Delta m_{k,\text{ref}} = \texttt{signedAngle}\left(P^k_{k-1}\left(\mathbf{a}^{k-1}_1(t_{j+1})\right), \mathbf{a}^k_1(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$



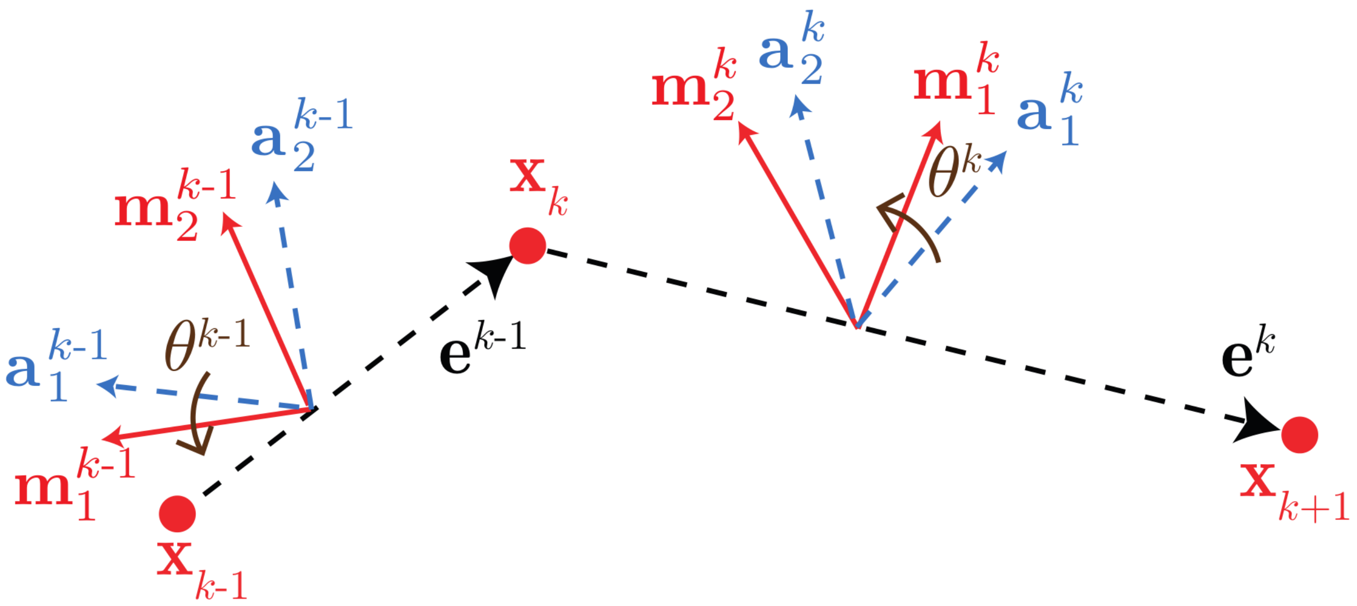


Degrees of Freedom (DOF) of a Rod

- A rod with N nodes has $(4N\text{-}1)$ DOFs
  1. $3N$ DOFs associated with the positions of N nodes, $\boldsymbol{x_k}$ $(1 \le k \le N)$
  2. $(N\text{-}1)$ DOFs for the twist angles of the edges, $\boldsymbol{\theta^k}$ $(1 \le k \le N-1)$



Discrete Elastic Twisting Energy



$$E_t = \sum_{k=2}^{N-1} E_{t,k}$$

where $E_{t,k} = \frac{1}{2}\frac{GJ}{\bar{l}_k}\left(\tau_k\right)^2 = \frac{1}{2}\frac{GJ}{\bar{l}_k}\left(\Delta\theta_k + \Delta m_{k,\text{ref}}\right)^2$

- G = shear modulus = $\frac{\text{Young's modulus}}{2(1+\text{Poisson ratio})}$
- J = polar moment of area ($= \frac{\pi r_0^4}{2}$ if circular)

Future Plan

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- Follow Bergou et al. *SIGGRAPH* (2010) to formulate a discrete simulation algorithm for elastic rods

Module 21

Discrete Elastic Rods (DER) Algorithm

Goal

- Equations of motion of an elastic rod
- Programming implementation of the Discrete Elastic Rods (DER) algorithm

Resources

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- Read *“Chapter 7: Discrete Elastic Rods Algorithm”*
- Review Bergou, Miklós et al. “Discrete viscous threads” *SIGGRAPH (2010)* 29(4) 1-10
- Review Jawed, M. K. et al. “A primer on the kinematics of discrete elastic rods” (2018)

Degrees of Freedom (DOF)

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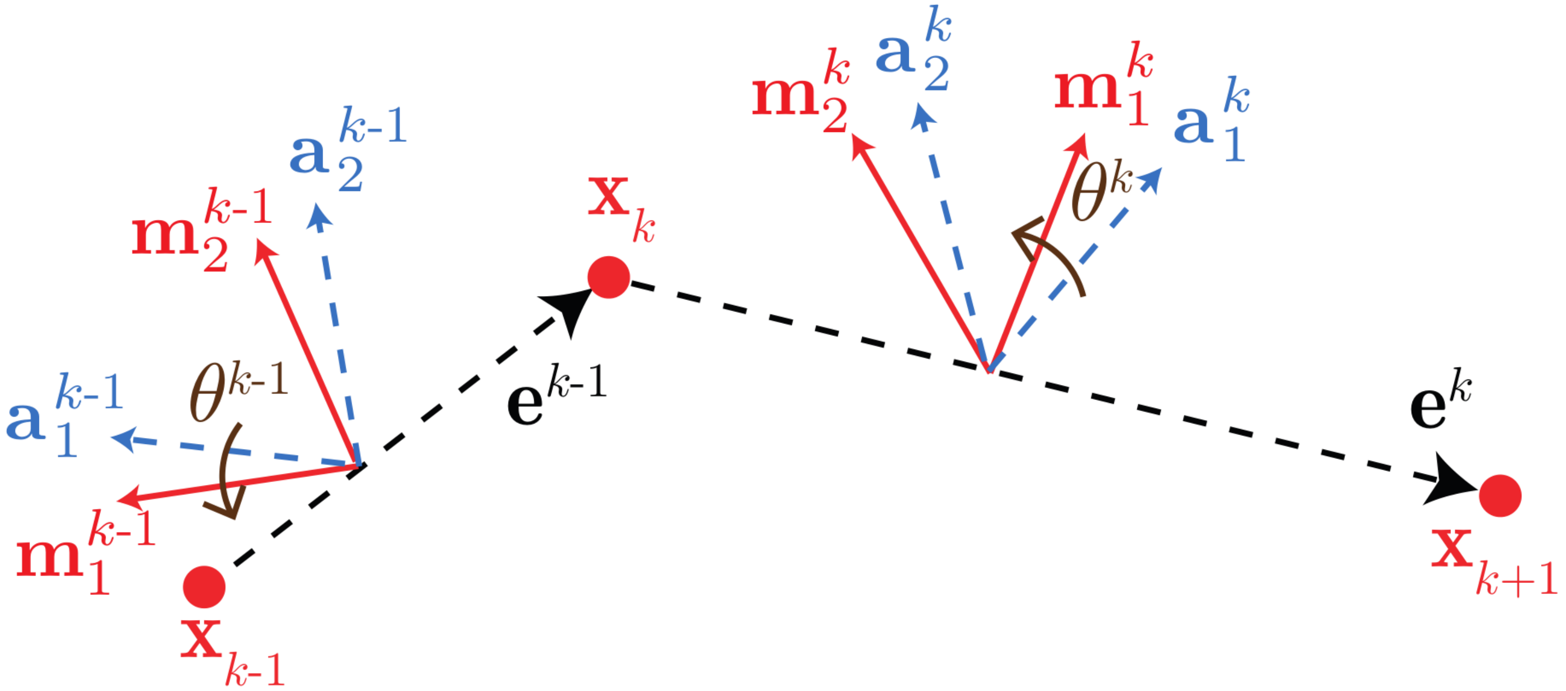
$(4N-1)$ DOF for a rod with N nodes:
$3N$ nodal coordinates and $N-1$ twist angles

Degrees of Freedom (DOF) vector,

$$\mathbf{q} = \begin{bmatrix} \boxed{\begin{matrix} x_1 \\ y_1 \\ z_1 \end{matrix}} \\ \theta^1 \\ \boxed{\begin{matrix} x_2 \\ y_2 \\ z_2 \end{matrix}} \\ \theta^2 \\ \ldots \\ \ldots \\ \ldots \\ \theta^{N-1} \\ x_N \\ y_N \\ z_N \end{bmatrix} = \begin{bmatrix} \boxed{\mathbf{x}_1} \\ \theta^1 \\ \boxed{\mathbf{x}_2} \\ \theta^2 \\ \ldots \\ \ldots \\ \ldots \\ \theta^{N-1} \\ \mathbf{x}_N \end{bmatrix}$$



Discrete Equations of Motion

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$$\boxed{\mathbf{M}\,\ddot{\mathbf{q}} + \frac{\partial E_{\text{elastic}}}{\partial \mathbf{q}} - \mathbf{f}_{\text{external}} = \mathbf{0}}$$

$$f_i \equiv \frac{m_i}{\Delta t}\left[\frac{q_i(t_{k+1}) - q_i(t_k)}{\Delta t} - \dot{q}_i(t_k)\right] + \boxed{\frac{\partial E_{\text{elastic}}}{\partial q_i}} - F_{i,\text{external}} = 0$$

$$\mathbb{J}_{ij} = \frac{\partial f_i}{\partial q_j} = \mathbb{J}_{ij}^{\text{inertia}} + \boxed{\mathbb{J}_{ij}^{\text{elastic}}} + \mathbb{J}_{ij}^{\text{external}},$$

where

$$\mathbb{J}_{ij}^{\text{inertia}} = \frac{m_i}{\Delta t^2}\delta_{ij},$$

$$\mathbb{J}_{ij}^{\text{elastic}} = \frac{\partial^2 E_{\text{elastic}}}{\partial q_i \partial q_j},$$

$$\mathbb{J}_{ij}^{\text{external}} = -\frac{\partial f_i^{\text{ext}}}{\partial q_j}.$$

Discrete Elastic Energies

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$$f_i \equiv \frac{m_i}{\Delta t}\left[\frac{q_i(t_{k+1}) - q_i(t_k)}{\Delta t} - \dot{q}_i(t_k)\right] + \boxed{\frac{\partial E_{\text{elastic}}}{\partial q_i}} - F_{i,\text{external}} = 0$$

$$\mathbb{J}_{ij} = \frac{\partial f_i}{\partial q_j} = \mathbb{J}_{ij}^{\text{inertia}} + \boxed{\mathbb{J}_{ij}^{\text{elastic}}} + \mathbb{J}_{ij}^{\text{external}},$$

$$E_{\text{elastic}} = \underbrace{\sum_{k=1}^{N-1} E_k^s}_{\text{stretching energy}} + \underbrace{\sum_{k=2}^{N-1} E_k^b}_{\text{bending energy}} + \underbrace{\sum_{k=2}^{N-1} E_k^t}_{\text{twisting energy}}$$

$$\mathbb{J}_{ij}^{\text{inertia}} = \frac{m_i}{\Delta t^2}\delta_{ij},$$

$$\mathbb{J}_{ij}^{\text{elastic}} = \frac{\partial^2 E_{\text{elastic}}}{\partial q_i \partial q_j},$$

$$\mathbb{J}_{ij}^{\text{external}} = -\frac{\partial f_i^{\text{ext}}}{\partial q_j}.$$

$$\boxed{\frac{\partial}{\partial q_i} E_{\text{elastic}} = \sum_{k=1}^{N-1} \frac{\partial}{\partial q_i} E_k^s + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^b + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^t}$$

$$\boxed{\frac{\partial^2}{\partial q_i \partial q_j} E_{\text{elastic}} = \sum_{k=1}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^s + \sum_{k=2}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^b + \sum_{k=2}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^t}$$

Reminder 1: Time Parallel Reference Frame

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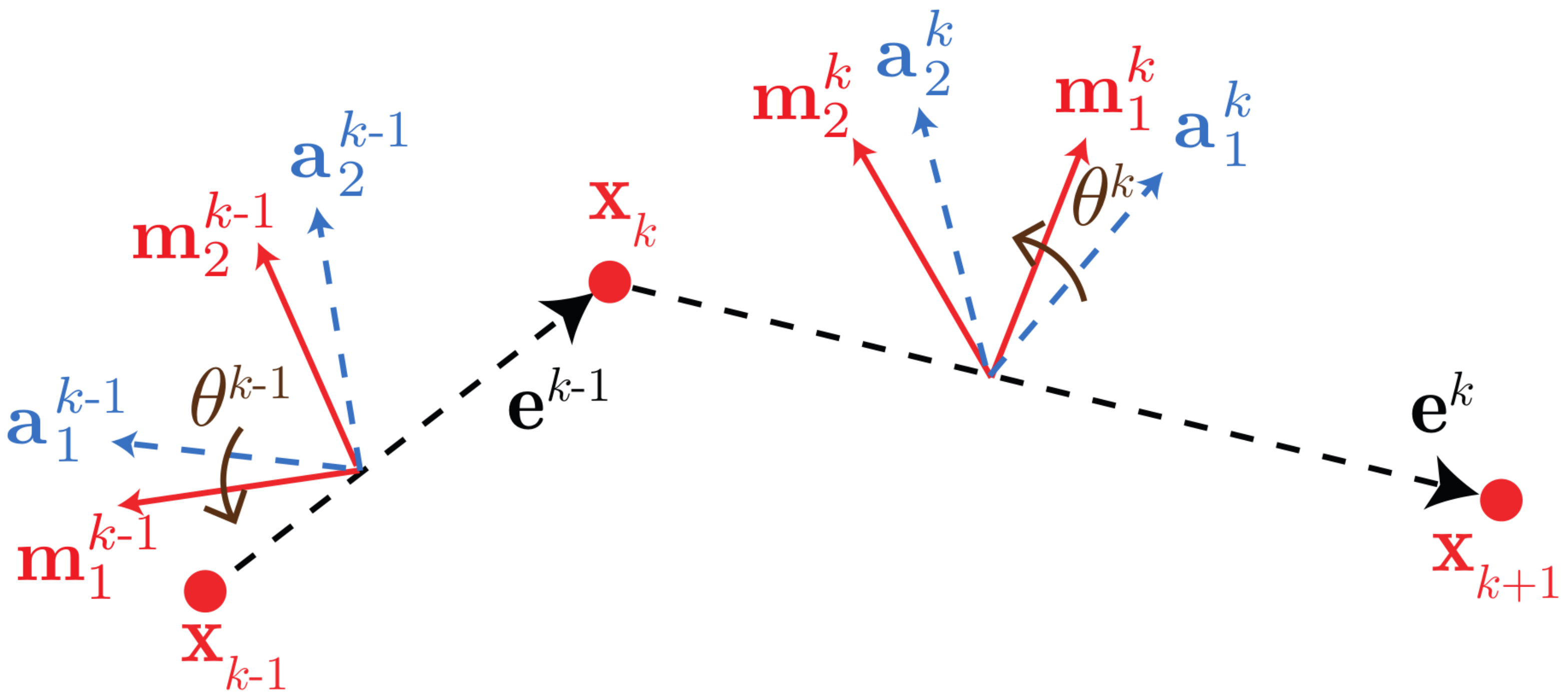
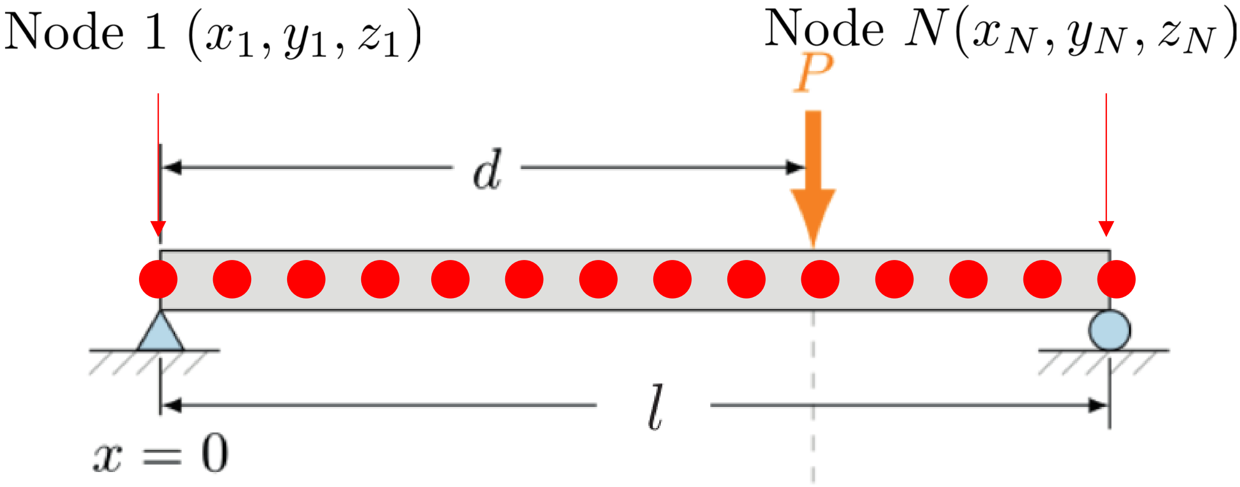
- At every time step, we need to compute the reference frame along the rod to measure twist



Reminder 2: Fixed and Free DOFs

- Some DOFs are fixed



`fixed_index`: $\boxed{[1, 2, 3}, \boxed{4N-2, 4N-1]}$

`free_index`: $[4, 5, \ldots, 4N-3]$

We will need this later.

Fixed DOF:

$$\boxed{\begin{aligned} x_1 &= 0 \\ y_1 &= 0 \\ z_1 &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} y_N &= 0 \\ z_N &= 0 \end{aligned}}$$

Discrete Elastic Rods (DER)

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Algorithm 1 Discrete Elastic Rods

Require: $\mathbf{q}(t_j), \dot{\mathbf{q}}(t_j)$ ▷ DOFs and velocities at $t = t_j$
Require: $\left(\mathbf{a}_1^k(t_j), \mathbf{a}_2^k(t_j), \mathbf{t}^k(t_j)\right)$ ▷ Reference frame at $t = t_j$
Require: `free_index` ▷ index of the free DOFs
Ensure: $\mathbf{q}(t_{j+1}), \dot{\mathbf{q}}(t_{j+1})$ ▷ DOFs and velocities at $t = t_{j+1}$
Ensure: $\left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$ ▷ Reference frame at $t = t_{j+1}$

Algorithm 1 Discrete Elastic Rods

Require: $\mathbf{q}(t_j), \dot{\mathbf{q}}(t_j)$ ▷ DOFs and velocities at $t = t_j$
Require: $\left(\mathbf{a}_1^k(t_j), \mathbf{a}_2^k(t_j), \mathbf{t}^k(t_j)\right)$ ▷ Reference frame at $t = t_j$
Require: `free_index` ▷ index of the free DOFs
Ensure: $\mathbf{q}(t_{j+1}), \dot{\mathbf{q}}(t_{j+1})$ ▷ DOFs and velocities at $t = t_{j+1}$
Ensure: $\left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$ ▷ Reference frame at $t = t_{j+1}$

function DISCRETE_ELASTIC_RODS($\mathbf{q}, \dot{\mathbf{q}}(t_j), \left(\mathbf{a}_1^k(t_j), \mathbf{a}_2^k(t_j), \mathbf{t}^k(t_j)\right)$)
  Guess: $\mathbf{q}^{(1)}(t_{j+1}) \leftarrow \mathbf{q}(t_j)$
  $n \leftarrow 1$
  while `error` $>$ `tolerance` **do**

Only update free DOFs

    Compute $\mathbf{f}$ and $\mathbb{J}$
    $\mathbf{f}_{\text{free}} \leftarrow \mathbf{f}$ `(free_index)`
    $\mathbb{J}_{\text{free}} \leftarrow \mathbb{J}$ `(free_index, free_index)`
    $\Delta\mathbf{q}_{\text{free}} \leftarrow \mathbb{J}_{\text{free}} \backslash \mathbf{f}_{\text{free}}$
    $\mathbf{q}^{(n+1)}$ `(free_index)` $\leftarrow \mathbf{q}^{(n)}$ `(free_index)` $- \Delta\mathbf{q}_{\text{free}}$
    `error` $\leftarrow$ `sum ( abs(` f_{free} `) )`
    $n \leftarrow n + 1$
  end while

Newton-Raphson

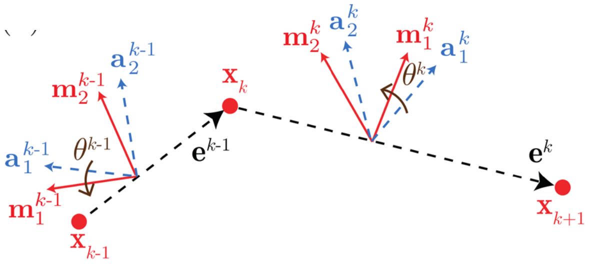
  return $\mathbf{q}(t_{j+1}), \dot{\mathbf{q}}(t_{j+1}), \left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$
end function

Algorithm 1 Discrete Elastic Rods

Require: $\mathbf{q}(t_j), \dot{\mathbf{q}}(t_j)$ ▷ DOFs and velocities at $t = t_j$
Require: $\left(\mathbf{a}_1^k(t_j), \mathbf{a}_2^k(t_j), \mathbf{t}^k(t_j)\right)$ ▷ Reference frame at $t = t_j$
Require: `free_index` ▷ index of the free DOFs
Ensure: $\mathbf{q}(t_{j+1}), \dot{\mathbf{q}}(t_{j+1})$ ▷ DOFs and velocities at $t = t_{j+1}$
Ensure: $\left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$ ▷ Reference frame at $t = t_{j+1}$

function DISCRETE_ELASTIC_RODS($\mathbf{q}, \dot{\mathbf{q}}(t_j), \left(\mathbf{a}_1^k(t_j), \mathbf{a}_2^k(t_j), \mathbf{t}^k(t_j)\right)$)
  Guess: $\mathbf{q}^{(1)}(t_{j+1}) \leftarrow \mathbf{q}(t_j)$
  $n \leftarrow 1$
  while `error` $>$ `tolerance` **do**
    Compute $\mathbf{f}$ and $\mathbb{J}$
    $\mathbf{f}_{\text{free}} \leftarrow \mathbf{f}$ `(free_index)`
    $\mathbb{J}_{\text{free}} \leftarrow \mathbb{J}$ `(free_index, free_index)`
    $\Delta \mathbf{q}_{\text{free}} \leftarrow \mathbb{J}_{\text{free}} \backslash \mathbf{f}_{\text{free}}$
    $\mathbf{q}^{(n+1)}$ `(free_index)` $\leftarrow \mathbf{q}^{(n)}$ `(free_index)` $-\Delta \mathbf{q}_{\text{free}}$
    `error` $\leftarrow$ `sum ( abs(` f_{free} `) )`
    $n \leftarrow n + 1$
  end while
  $\mathbf{q}(t_{j+1}) \leftarrow \mathbf{q}^{(n)}(t_{j+1})$
  $\dot{\mathbf{q}}(t_{j+1}) \leftarrow \frac{\mathbf{q}(t_{j+1}) - \mathbf{q}(t_j)}{\Delta t}$
  $\left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right) \leftarrow \left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)^{(n)}$
  return $\mathbf{q}(t_{j+1}), \dot{\mathbf{q}}(t_{j+1}), \left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$
end function

Return once we are below tolerance



Algorithm 1 Discrete Elastic Rods

Require: $\mathbf{q}(t_j), \dot{\mathbf{q}}(t_j)$ ▷ DOFs and velocities at $t = t_j$
Require: $\left(\mathbf{a}_1^k(t_j), \mathbf{a}_2^k(t_j), \mathbf{t}^k(t_j)\right)$ ▷ Reference frame at $t = t_j$
Require: `free_index` ▷ index of the free DOFs
Ensure: $\mathbf{q}(t_{j+1}), \dot{\mathbf{q}}(t_{j+1})$ ▷ DOFs and velocities at $t = t_{j+1}$
Ensure: $\left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$ ▷ Reference frame at $t = t_{j+1}$

function DISCRETE_ELASTIC_RODS($\mathbf{q}, \dot{\mathbf{q}}(t_j), \left(\mathbf{a}_1^k(t_j), \mathbf{a}_2^k(t_j), \mathbf{t}^k(t_j)\right)$)
  Guess: $\mathbf{q}^{(1)}(t_{j+1}) \leftarrow \mathbf{q}(t_j)$
  $n \leftarrow 1$
  while `error > tolerance` **do**
    Compute reference frame $\left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)^{(n)}$
    Compute reference twist $\Delta m_{k,\text{ref}}^{(n)}$ $(k = 2, \ldots, N-1)$
    Compute material frame $\left(\mathbf{m}_1^k(t_{j+1}), \mathbf{m}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)^{(n)}$
    Compute $\mathbf{f}$ and $\mathbb{J}$
    $\mathbf{f}_{\text{free}} \leftarrow \mathbf{f}$ `(free_index)`
    $\mathbb{J}_{\text{free}} \leftarrow \mathbb{J}$ `(free_index, free_index)`
    $\Delta\mathbf{q}_{\text{free}} \leftarrow \mathbb{J}_{\text{free}} \backslash \mathbf{f}_{\text{free}}$
    $\mathbf{q}^{(n+1)}$ `(free_index)` $\leftarrow \mathbf{q}^{(n)}$ `(free_index)` $-\Delta\mathbf{q}_{\text{free}}$
    `error` $\leftarrow$ `sum ( abs(` f_{free} `) )`
    $n \leftarrow n + 1$
  end while

  $\mathbf{q}(t_{j+1}) \leftarrow \mathbf{q}^{(n)}(t_{j+1})$
  $\dot{\mathbf{q}}(t_{j+1}) \leftarrow \frac{\mathbf{q}(t_{j+1}) - \mathbf{q}(t_j)}{\Delta t}$
  $\left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right) \leftarrow \left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)^{(n)}$
  return $\mathbf{q}(t_{j+1}), \dot{\mathbf{q}}(t_{j+1}), \left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$
end function

We need the frames to compute bending and twisting energies

Next Step

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- Programming implementation of gradient and Hessian of elastic energies

$$\boxed{\frac{\partial}{\partial q_i} E_{\text{elastic}} = \sum_{k=1}^{N-1} \frac{\partial}{\partial q_i} E_k^s + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^b + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^t}$$

$$\boxed{\frac{\partial^2}{\partial q_i \partial q_j} E_{\text{elastic}} = \sum_{k=1}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^s + \sum_{k=2}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^b + \sum_{k=2}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^t}$$

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Module 22

Gradient and Hessian of Elastic Energies in Discrete Rod

- Programming Implementation

Goal

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- Understand the computation of gradient and Hessian of elastic energies in Discrete Elastic Rods (DER)

$$\frac{\partial}{\partial q_i} E_{\text{elastic}} = \sum_{k=1}^{N-1} \frac{\partial}{\partial q_i} E_k^s + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^b + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^t$$

$$\frac{\partial^2}{\partial q_i \partial q_j} E_{\text{elastic}} = \sum_{k=1}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^s + \sum_{k=2}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^b + \sum_{k=2}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^t$$

Resources

- Read *"Chapter 7: Discrete Elastic Rods Algorithm"*
- Review Bergou, Miklós et al. "Discrete viscous threads" *SIGGRAPH (2010)* 29(4) 1-10
- Review Jawed, M. K. et al. "A primer on the kinematics of discrete elastic rods" (2018)

Degrees of Freedom (DOF)

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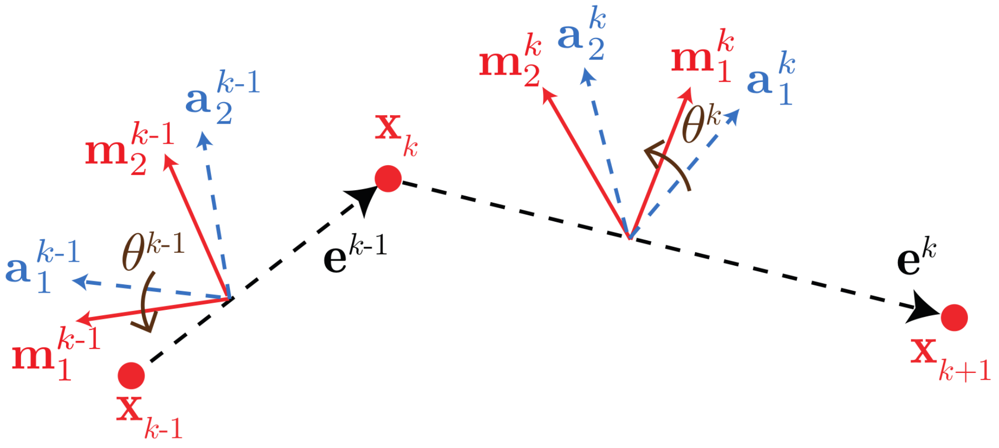
$(4N-1)$ DOF for a rod with N nodes:
$3N$ nodal coordinates and $N-1$ twist angles

Degrees of Freedom (DOF) vector,

$$\mathbf{q} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ \theta^1 \\ x_2 \\ y_2 \\ z_2 \\ \theta^2 \\ \cdots \\ \cdots \\ \cdots \\ \theta^{N-1} \\ x_N \\ y_N \\ z_N \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \theta^1 \\ \mathbf{x}_2 \\ \theta^2 \\ \cdots \\ \cdots \\ \cdots \\ \theta^{N-1} \\ \mathbf{x}_N \end{bmatrix}$$



Index of the k-th node, $\mathbf{x}_k$, in DOF vector:
$4k-3, 4k-2, 4k-1$, where $1 \le k \le N$

Index of the k-th twist angle, θ^k, in DOF vector:
$4k$, where $1 \le k \le N-1$

We will need this during programming implementation.

Discrete Elastic Energies

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$$f_i \equiv \frac{m_i}{\Delta t}\left[\frac{q_i(t_{k+1}) - q_i(t_k)}{\Delta t} - \dot{q}_i(t_k)\right] + \boxed{\frac{\partial E_{\text{elastic}}}{\partial q_i}} - F_{i,\text{external}} = 0$$

$$\mathbb{J}_{ij} = \frac{\partial f_i}{\partial q_j} = \mathbb{J}_{ij}^{\text{inertia}} + \boxed{\mathbb{J}_{ij}^{\text{elastic}}} + \mathbb{J}_{ij}^{\text{viscous}},$$

$$E_{\text{elastic}} = \underbrace{\sum_{k=1}^{N-1} E_k^s}_{\text{stretching energy}} + \underbrace{\sum_{k=2}^{N-1} E_k^b}_{\text{bending energy}} + \underbrace{\sum_{k=2}^{N-1} E_k^t}_{\text{twisting energy}}$$

$$\mathbb{J}_{ij}^{\text{inertia}} = \frac{m_i}{\Delta t^2}\delta_{ij},$$

$$\mathbb{J}_{ij}^{\text{elastic}} = \frac{\partial^2 E_{\text{elastic}}}{\partial q_i \partial q_j},$$

$$\mathbb{J}_{ij}^{\text{external}} = -\frac{\partial f_i^{\text{ext}}}{\partial q_j}.$$

$$\boxed{\frac{\partial}{\partial q_i} E_{\text{elastic}} = \sum_{k=1}^{N-1} \frac{\partial}{\partial q_i} E_k^s + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^b + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^t}$$

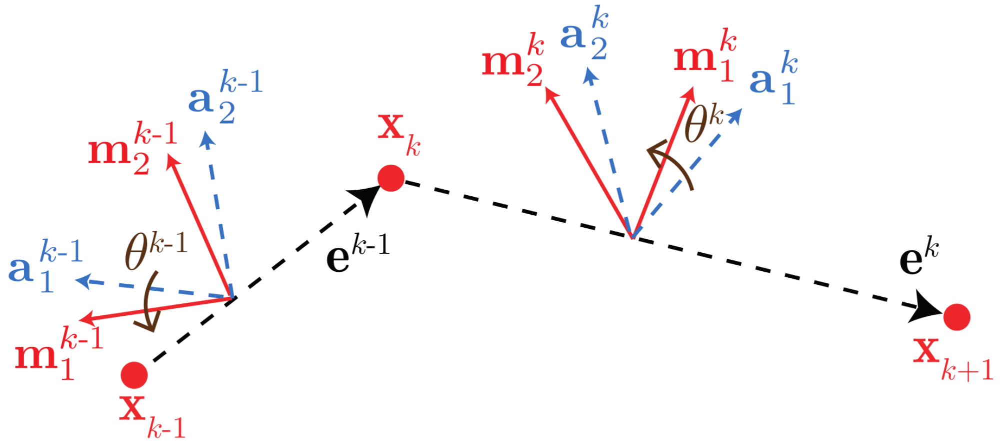
$$\boxed{\frac{\partial^2}{\partial q_i \partial q_j} E_{\text{elastic}} = \sum_{k=1}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^s + \sum_{k=2}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^b + \sum_{k=2}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^t}$$

Gradient of Elastic Energy Computation

$$\frac{\partial}{\partial q_i} E_{\text{elastic}} = \sum_{k=1}^{N-1} \boxed{\frac{\partial}{\partial q_i} E_k^s} + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^b + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^t$$

$$\mathbf{F} \equiv \frac{\partial}{\partial q_i} \rightarrow (4N-1) \text{ sized vector}$$

"Big gradient vector" F



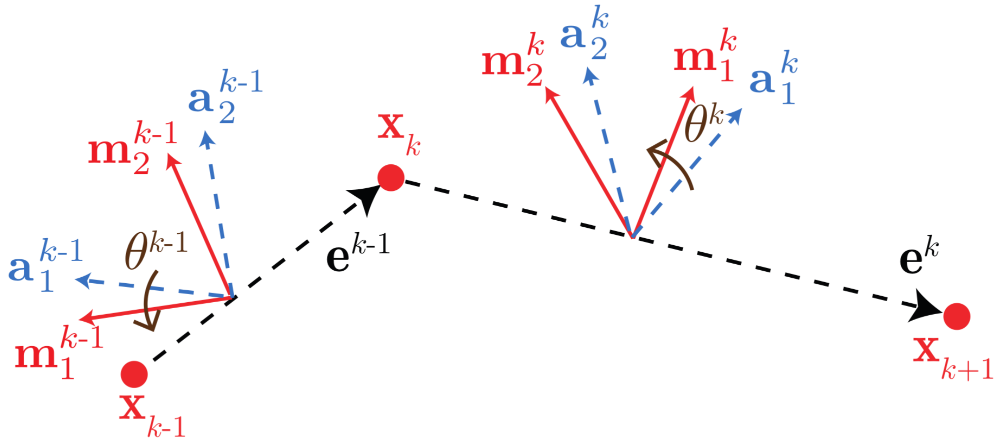
Gradient of Elastic Energy Computation

STRUCTURES-COMPUTER INTERACTION LABORATORY

$$\frac{\partial}{\partial q_i} E_{\text{elastic}} = \sum_{k=1}^{N-1} \boxed{\frac{\partial}{\partial q_i} E_k^s} + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^b + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^t$$

$$\mathtt{F} \equiv \frac{\partial}{\partial q_i} \to (4N-1) \text{ sized vector}$$

“Big gradient vector” F



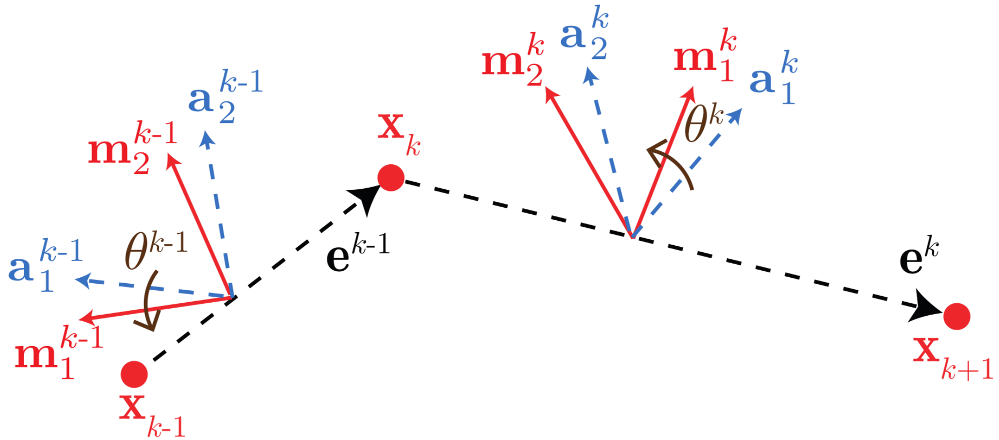
Stretching energy, E_k^s, only depends on $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$.

The corresponding indices in the DOF vector are:
$\mathtt{ind} = [4k-3, 4k-2, 4k-1, 4k+1, 4k+2, 4k+3]$

Gradient of Elastic Energy Computation

$$\frac{\partial}{\partial q_i} E_{\text{elastic}} = \sum_{k=1}^{N-1} \boxed{\frac{\partial}{\partial q_i} E_k^s} + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^b + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^t$$

$\mathtt{F} \equiv \frac{\partial}{\partial q_i} \rightarrow (4N-1)$ sized vector

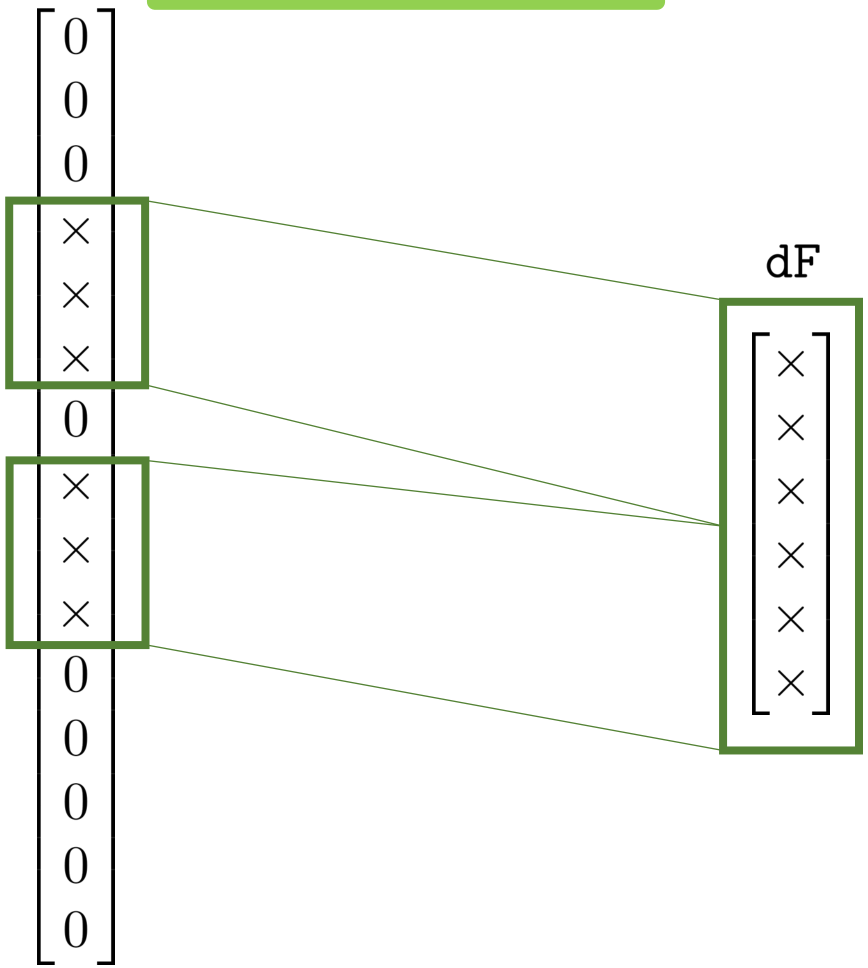


Stretching energy, E_k^s, only depends on $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$.

The corresponding indices in the DOF vector are:
$\mathtt{ind} = [4k-3, 4k-2, 4k-1, 4k+1, 4k+2, 4k+3]$

$\frac{\partial}{\partial q_i} E_k^s$ has 6 non-zero elements.
See Appendix for codes to compute non-zero "small" gradient, $\mathtt{dF}$.

“Big gradient vector” F



Gradient of Elastic Energy Computation

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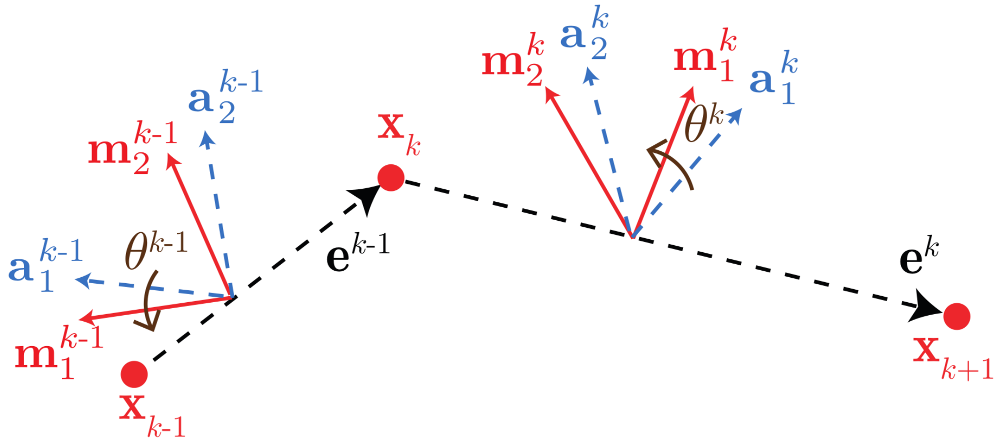
$$\frac{\partial}{\partial q_i} E_{\text{elastic}} = \sum_{k=1}^{N-1} \boxed{\frac{\partial}{\partial q_i} E_k^s} + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^b + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^t$$

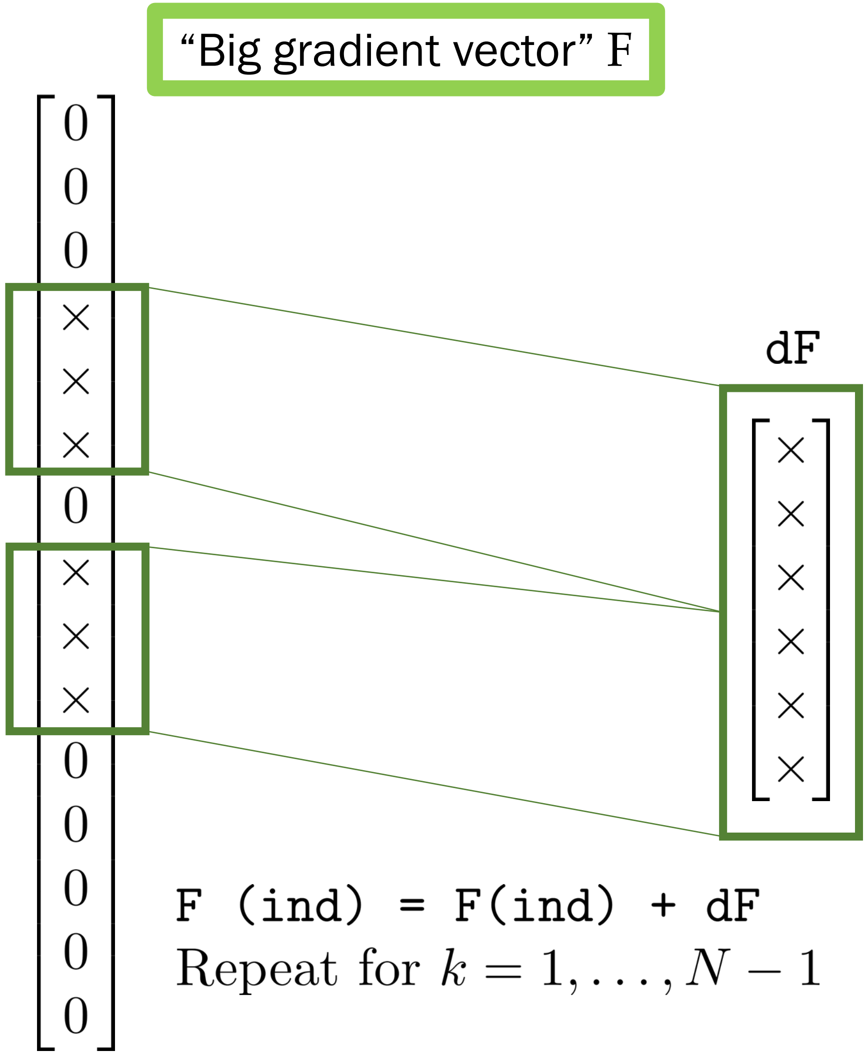
$\mathtt{F} \equiv \frac{\partial}{\partial q_i} \rightarrow (4N-1)$ sized vector



Stretching energy, E_k^s, only depends on $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$.

The corresponding indices in the DOF vector are:
$\mathtt{ind} = [4k-3, 4k-2, 4k-1, 4k+1, 4k+2, 4k+3]$

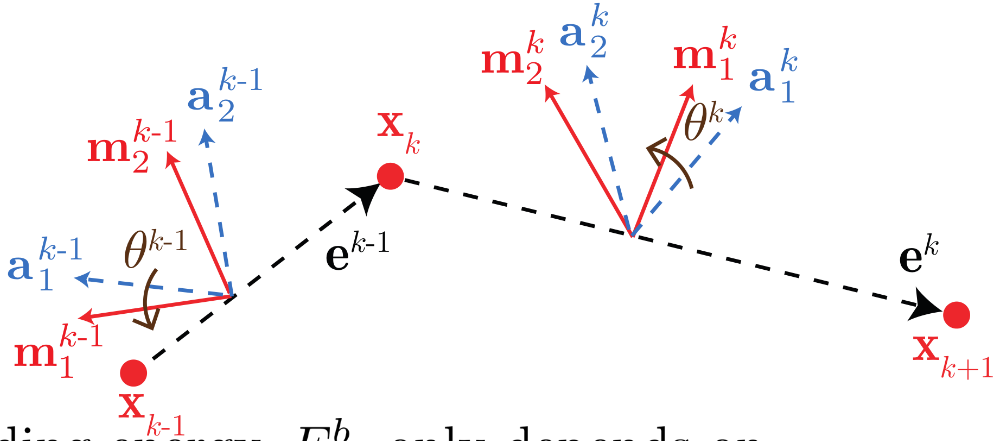
$\frac{\partial}{\partial q_i} E_k^s$ has 6 non-zero elements.
See Appendix for codes to compute non-zero "small" gradient, `dF`.



`F (ind) = F(ind) + dF`
Repeat for $k = 1, \ldots, N-1$

Gradient of Elastic Energy Computation

$$\frac{\partial}{\partial q_i} E_{\text{elastic}} = \sum_{k=1}^{N-1} \frac{\partial}{\partial q_i} E_k^s + \sum_{k=2}^{N-1} \boxed{\frac{\partial}{\partial q_i} E_k^b} + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^t$$

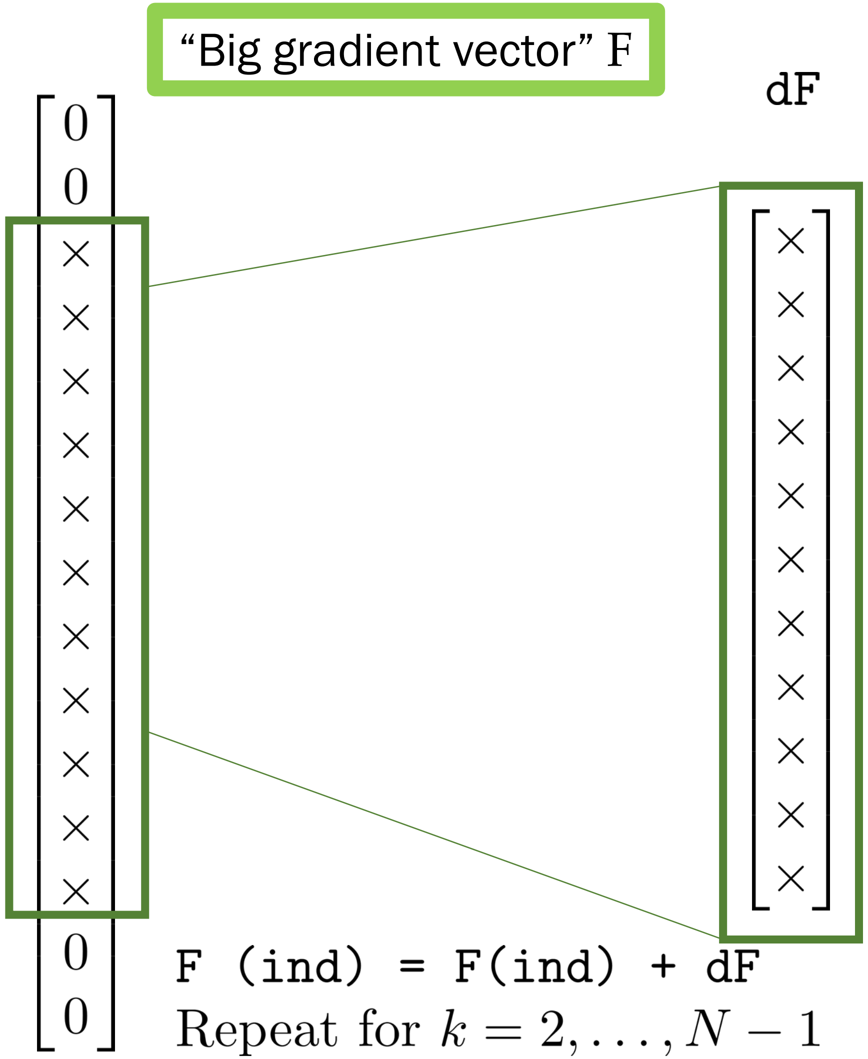


Bending energy, E_k^b, only depends on $\mathbf{x}_{k-1}, \theta^{k-1}, \mathbf{x}_k, \theta^k$, and $\mathbf{x}_{k+1}$.

The corresponding indices in the DOF vector are:
$\texttt{ind} = [4k-7, 4k-6, 4k-5, 4k-4, 4k-3, 4k-2, 4k-1, 4k, 4k+1, 4k+2, 4k+3]$

$\frac{\partial}{\partial q_i} E_k^b$ has 11 non-zero elements.
See Appendix for codes to compute non-zero "small" gradient, dF.

$\texttt{F} \equiv \frac{\partial}{\partial q_i} \rightarrow (4N-1)$ sized vector




```
F (ind) = F(ind) + dF
```
Repeat for $k = 2, \ldots, N-1$

Gradient of Elastic Energy Computation

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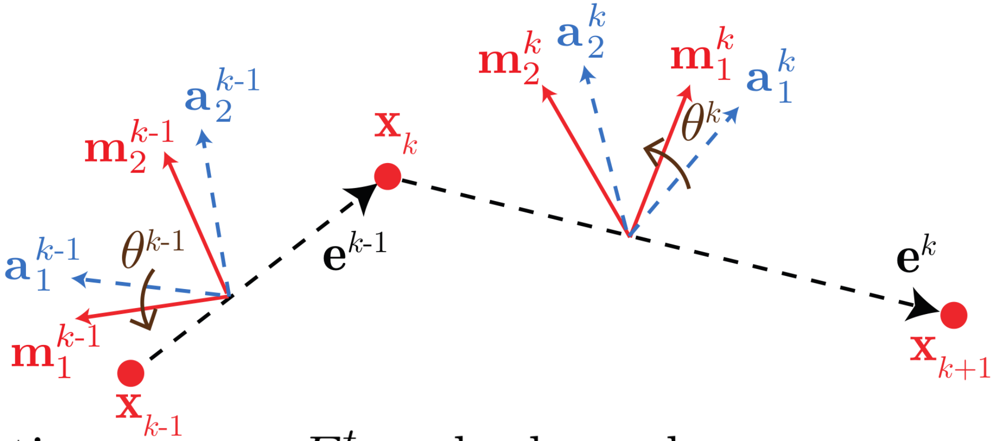
$$\frac{\partial}{\partial q_i} E_{\text{elastic}} = \sum_{k=1}^{N-1} \frac{\partial}{\partial q_i} E_k^s + \sum_{k=2}^{N-1} \frac{\partial}{\partial q_i} E_k^b + \sum_{k=2}^{N-1} \boxed{\frac{\partial}{\partial q_i} E_k^t}$$



Twisting energy, E_k^t, only depends on $\mathbf{x}_{k-1}, \theta^{k-1}, \mathbf{x}_k, \theta^k$, and $\mathbf{x}_{k+1}$.

The corresponding indices in the DOF vector are:
$\texttt{ind} = [4k-7, 4k-6, 4k-5, 4k-4,$
$4k-3, 4k-2, 4k-1, 4k, 4k+1, 4k+2, 4k+3]$

$\frac{\partial}{\partial q_i} E_k^t$ has 11 non-zero elements.
See Appendix for codes to compute non-zero "small" gradient, dF.

$\texttt{F} \equiv \frac{\partial}{\partial q_i} \rightarrow (4N-1)$ sized vector

"Big gradient vector" F

dF

$$\begin{bmatrix} 0 \\ 0 \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}$$

```
F (ind) = F(ind) + dF
```

Repeat for $k = 2, \ldots, N-1$

Hessian of Elastic Energy Computation

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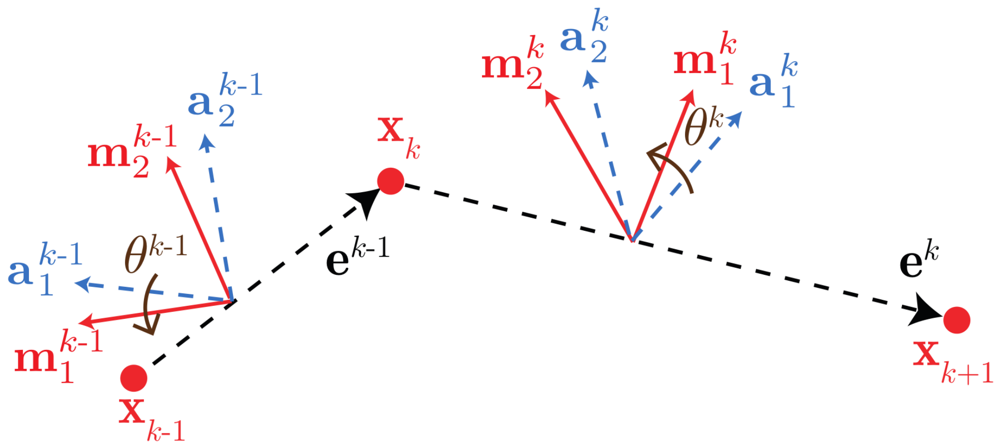
$$\frac{\partial^2}{\partial q_i \partial q_j} E_{\text{elastic}} = \sum_{k=1}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^s + \sum_{k=2}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^b + \sum_{k=2}^{N-1} \frac{\partial^2}{\partial q_i \partial q_j} E_k^t$$



$$\mathrm{J} \equiv \frac{\partial}{\partial q_i \partial q_j} E_{\text{elastic}} \rightarrow (4N-1) \times (4N-1) \text{ sized matrix}$$

"Big Hessian matrix" J

Non-zero component of Hessian of discrete energies
Code in Appendix of the Course Notes

```
F (ind) = F(ind) + dF
```

Repeat for $k = 1, \ldots, N-1$ (stretching)
Repeat for $k = 2, \ldots, N-1$ (bending and twisting)

```
J (ind, ind) = J(ind, ind) + dJ
```

Repeat for $k = 1, \ldots, N-1$ (stretching)
Repeat for $k = 2, \ldots, N-1$ (bending and twisting)

Pseudocode to compute gradient and Hessian

STRUCTURES-COMPUTER INTERACTION LABORATORY

Algorithm 1 Gradient and Hessian of Elastic Energy in a Rod

Require: $\mathbf{q} = \left[\mathbf{x}_0, \theta^0, \mathbf{x}_1, \theta^1, \ldots, \theta^{N-1}, \mathbf{x}_N\right]^T$ ▷ Degrees of Freedom
Require: EA, EI, GJ
Require: $\bar{l}_k \ (1 \leq k \leq N)$ ▷ Undeformed Voronoi length
Require: $\bar{\mathbf{e}}^k (1 \leq k < N)$ ▷ Undeformed edge length
Require: $\bar{\boldsymbol{\kappa}}_k^0 (1 < k < N)$ ▷ Natural curvature
Require: Reference frame, $\left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$, where $1 \leq k < N$
Require: Reference twist, $\Delta m_{k,\mathrm{ref}} \ (1 \leq k < N)$
Require: Material frame, $\left(\mathbf{m}_1^k(t_{j+1}), \mathbf{m}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$, where $1 \leq k < N$
Ensure: F ▷ $4N - 1$ sized elastic gradient vector, $\frac{\partial E_{\mathrm{elastic}}}{\partial \mathbf{q}}$
Ensure: J ▷ $(4N-1) \times (4N-1)$ sized elastic Hessian matrix, $\mathbb{J}^{\mathrm{elastic}}$

Output of the function

function GRAD_HESS_ELASTIC_ROD($\mathbf{q}$)

  `F` $\leftarrow$ `zeros(3N,1)`

  `J` $\leftarrow$ `zeros(3N,3N)`

  for $k \leftarrow 1$ to $N-1$ **do**

    `ind` $\leftarrow$ locations of $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$

    `[dF, dJ]` $\leftarrow$ `gradEs_hessEs`$(\mathbf{x}_k, \mathbf{x}_{k+1})$

    `F (ind)` $=$ `F (ind)` $+$ `dF`

    `J (ind)` $=$ `F (ind, ind)` $+$ `dJ`

  end for

Stretching Energy

Bending and Twisting Energy

  for $k \leftarrow 2$ to $N-1$ **do**

    `ind` $\leftarrow$ locations of $\mathbf{x}_{k-1}, \theta^{k-1}, \mathbf{x}_k, \theta^k$, and $\mathbf{x}_{k+1}$

    `[dF, dJ]` $\leftarrow$ `gradEb_hessEb`$(\mathbf{x}_{k-1}, \theta^{k-1}, \mathbf{x}_k, \theta^k, \mathbf{x}_{k+1})$

    `F (ind)` $=$ `F (ind)` $+$ `dF`

    `J (ind)` $=$ `J (ind, ind)` $+$ `dJ`

    `[dF, dJ]` $\leftarrow$ `gradEt_hessEt`$(\mathbf{x}_{k-1}, \theta^{k-1}, \mathbf{x}_k, \theta^k, \mathbf{x}_{k+1})$

    `F (ind)` $=$ `F (ind)` $+$ `dF`

    `J (ind)` $=$ `J (ind, ind)` $+$ `dJ`

  end for

  return `F` and `J`

end function

Algorithm 1 Gradient and Hessian of Elastic Energy in a Rod

Require: $\mathbf{q} = \left[\mathbf{x}_0, \theta^0, \mathbf{x}_1, \theta^1, \ldots, \theta^{N-1}, \mathbf{x}_N\right]^T$ ▷ Degrees of Freedom
Require: EA, EI, GJ
Require: $\bar{l}_k \ (1 \leq k \leq N)$ ▷ Undeformed Voronoi length
Require: $\bar{\mathbf{e}}^k (1 \leq k < N)$ ▷ Undeformed edge length
Require: $\bar{\boldsymbol{\kappa}}_k^0 (1 < k < N)$ ▷ Natural curvature
Require: Reference frame, $\left(\mathbf{a}_1^k(t_{j+1}), \mathbf{a}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$, where $1 \leq k < N$
Require: Reference twist, $\Delta m_{k,\text{ref}} \ (1 \leq k < N)$
Require: Material frame, $\left(\mathbf{m}_1^k(t_{j+1}), \mathbf{m}_2^k(t_{j+1}), \mathbf{t}^k(t_{j+1})\right)$, where $1 \leq k < N$
Ensure: `F` ▷ $4N-1$ sized elastic gradient vector, $\frac{\partial E_{\text{elastic}}}{\partial \mathbf{q}}$
Ensure: `J` ▷ $(4N-1) \times (4N-1)$ sized elastic Hessian matrix, $\mathbb{J}^{\text{elastic}}$

function GRAD_HESS_ELASTIC_ROD($\mathbf{q}$)
    `F` ← `zeros(3N,1)`
    `J` ← `zeros(3N,3N)`
    for $k \leftarrow 1$ to $N-1$ **do**
        `ind` ← locations of $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$
        `[dF, dJ]` ← `gradEs_hessEs`($\mathbf{x}_k, \mathbf{x}_{k+1}$)
        `F (ind)` = `F (ind)` + `dF`
        `J (ind)` = `F (ind, ind)` + `dJ`
    end for

    for $k \leftarrow 2$ to $N-1$ **do**
        `ind` ← locations of $\mathbf{x}_{k-1}, \theta^{k-1}, \mathbf{x}_k, \theta^k$, and $\mathbf{x}_{k+1}$
        `[dF, dJ]` ← `gradEb_hessEb`($\mathbf{x}_{k-1}, \theta^{k-1}, \mathbf{x}_k, \theta^k, \mathbf{x}_{k+1}$)
        `F (ind)` = `F (ind)` + `dF`
        `J (ind)` = `J (ind, ind)` + `dJ`
        `[dF, dJ]` ← `gradEt_hessEt`($\mathbf{x}_{k-1}, \theta^{k-1}, \mathbf{x}_k, \theta^k, \mathbf{x}_{k+1}$)
        `F (ind)` = `F (ind)` + `dF`
        `J (ind)` = `J (ind, ind)` + `dJ`
    end for
    return `F` and `J`
end function

Assignments

STRUCTURES-COMPUTER INTERACTION LABORATORY

- Complete the assignments in Chapter 6: *Discrete Twist* and Chapter 7: *Discrete Elastic Rods Algorithm*