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SYSTEM MODELLING

INTRODUCTION

The problem of controlling the Two-Wheel Self Balancing Transporter (TWSBT) starts with the modeling of the system. In this chapter it will be introduced the kinematic model, the dynamic model (using Lagrange approach) and the decoupling of the system in balancing and steering subsystems. At the end of the chapter it will be presented some considerations about the results and the block-models derived from the equations.

KINEMATIC MODEL

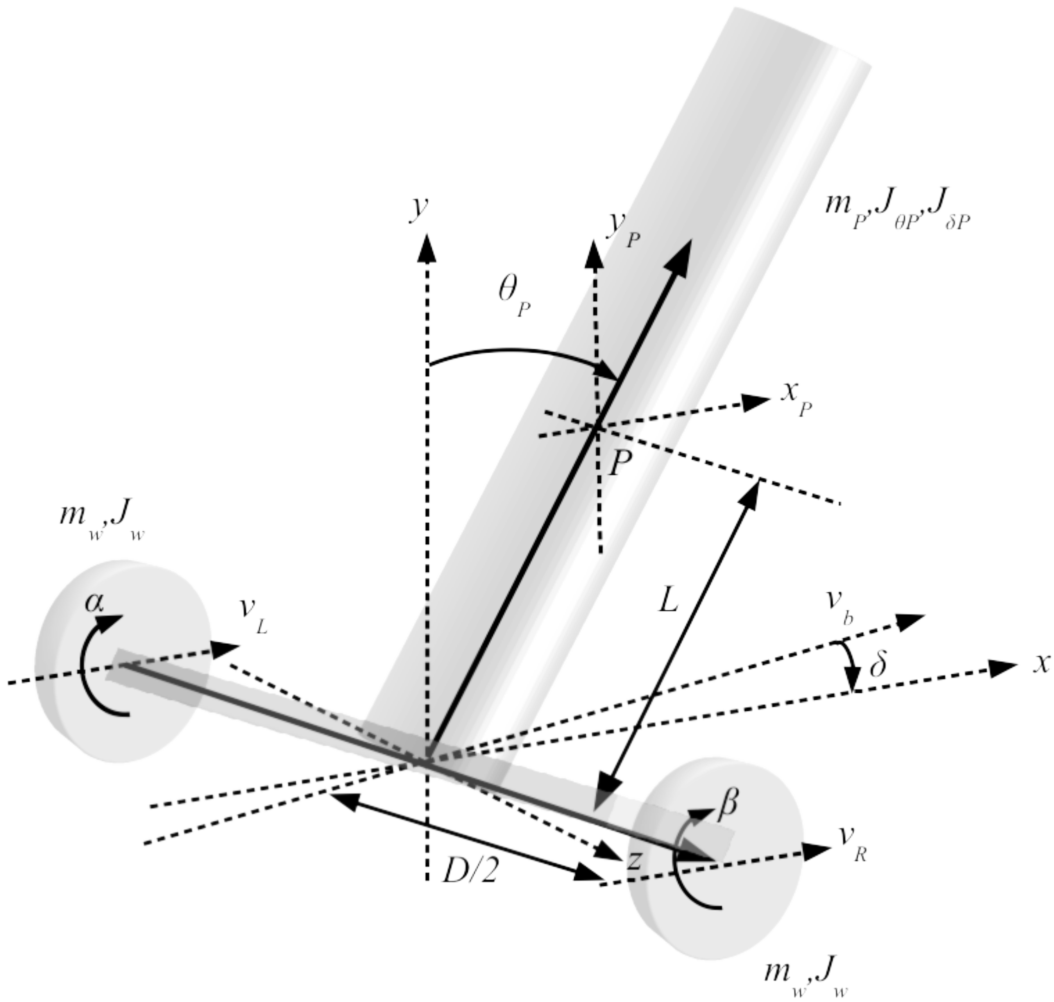

Figure 1: Graphical representation of the entities of the system.

The conventions used for this model are summarized in table 1.

m_P	mass of the rider	[kg]
m_w	mass of the wheel (identical in Left and Right)	[kg]
$J_{\theta P}$	inertia of the rider, referred to the pitch rotation	$[kgm^2]$
$J_{\delta P}$	inertia of the rider, referred to the yaw rotation	$[kgm^2]$
J_w	inertia of the wheel	$[kgm^2]$
α_m, β_m	angle of the (Left, Right) motor (referred to the base)	[rads]
α, β	angle of the (Left, Right) wheel (referred to the ground)	[rads]
θ_P	angle of the person (referred to the ground, where 0 is the vertical upper position)	[rads]
v_L, v_R	velocity of the (Left, Right) center of the wheel	[m/s]
x_b, v_b	horizontal position and velocity of the base center (origin)	[m],[m/s]
x_P, y_P, z_P	coordinates of the rider's center of mass	[m]
L	distance between base and center of mass of the rider	[m]
D	distance wheels	[m]
r	radius of the wheel	[m]
C_L, C_R	torques applied to the wheels, by the motor (after the gearbox)	[Nm]
ρ	reduction ratio between the motor rotation and wheel rotation	
τ_L, τ_R	torque of the (Left,Right) motor	[Nm]
ψ	viscous friction coefficient	

Table 1: Conventions

Assumptions

The following assumptions are used for the problem:

1. The friction is considered linear and proportional to the motor's rotation speed, even if different from reality
2. The rider is modeled as a rigid body (cylinder) of "2L" height
3. The efficiency of the gearbox is equal to 1
4. The reduction has no elasticity
5. The friction produced by the air, with the system's components, is neglected
6. The vertical coordinate of the base is taken as system's vertical origin

The relation between motor's position (referred to the rider's angle) and the wheel's angle (referred to the ground) is:

$$\begin{aligned} \alpha &= \theta_P + \rho\alpha_m \\ \beta &= \theta_P + \rho\beta_m \\ \dot{\alpha} &= \dot{\theta}_P + \rho\dot{\alpha}_m \\ \dot{\beta} &= \dot{\theta}_P + \rho\dot{\beta}_m \end{aligned} \tag{1}$$

The congruence equations for the wheels and base are:

$$\begin{aligned} v_L &= r\dot{\alpha} \\ v_R &= r\dot{\beta} \\ v_b &= \frac{v_L + v_R}{2} = r\frac{\dot{\alpha}+\dot{\beta}}{2} \\ \dot{\delta} &= \frac{v_L - v_R}{D} = r\frac{\dot{\alpha}-\dot{\beta}}{D} \end{aligned} \tag{2}$$

while, concerning the rider's center of mass:

$$\begin{aligned} x_P &= x_b + L\sin\theta_P\cos\delta \\ y_P &= L\cos\theta_P \\ z_P &= z_b + L\sin\theta_P\sin\delta \\ \dot{x}_P &= \dot{x}_b + L\dot{\theta}_P\cos\theta_P\cos\delta - L\dot{\delta}\sin\theta_P\sin\delta \\ \dot{y}_P &= L\dot{\theta}_P\sin\theta_P \\ \dot{z}_P &= \dot{z}_b + L\dot{\theta}_P\cos\theta_P\sin\delta + L\dot{\delta}\sin\theta_P\cos\delta \\ v_P^2 &= \dot{x}_P^2 + \dot{y}_P^2 + \dot{z}_P^2 = \\ &= v_b^2 + L^2\dot{\theta}_P^2 + 2L\dot{\theta}_P[\dot{x}_b\cos\theta_P\cos\delta + \dot{z}_b\cos\theta_P\sin\delta] + \\ &\quad +2L\dot{\delta}[-\dot{x}_b\sin\theta_P\sin\delta + \dot{z}_b\sin\theta_P\cos\delta] \end{aligned} \tag{3}$$

where:

$$\begin{aligned} \dot{x}_b &= v_b\cos\delta \\ \dot{z}_b &= v_b\sin\delta \end{aligned} \tag{4}$$

thus:

$$\begin{aligned} v_P^2 &= v_b^2 + L^2\dot{\theta}_P^2 + 2L\dot{\theta}_P[v_b\cos\theta_P\cos^2\delta + v_b\cos\theta_P\sin^2\delta] = \\ &= v_b^2 + L^2\dot{\theta}_P^2 + 2L\dot{\theta}_P v_b\cos\theta_P \end{aligned} \tag{5}$$

The resulting torque, after the gearbox of each motor, suffers of the viscous friction and it is modeled as:

$$\begin{array}{rcl} C_L & = & \frac{1}{\rho}(\tau_L - \psi\dot{\alpha}_m) = \frac{1}{\rho}\tau_L - \frac{\psi}{\rho^2}(\dot{\alpha} - \dot{\theta}_P) \\ C_R & = & \frac{1}{\rho}(\tau_R - \psi\dot{\beta}_m) = \frac{1}{\rho}\tau_R - \frac{\psi}{\rho^2}(\dot{\beta} - \dot{\theta}_P) \end{array} \tag{6}$$

kinetic energy of the wheels:

the following equations accounts for both translational and rotational components of the wheel motion. The kinetic energy associated with rotation of the wheel around its vertical axis is neglected.

$$\begin{array}{rcl} T_L & = & \frac{1}{2}m_w v_L^2 + \frac{1}{2}J_w\dot{\alpha}^2 = \frac{1}{2}(m_w r^2 + J_w)\dot{\alpha}^2 \\ T_R & = & \frac{1}{2}m_w v_R^2 + \frac{1}{2}J_w\dot{\beta}^2 = \frac{1}{2}(m_w r^2 + J_w)\dot{\beta}^2 \\ T_w & = & T_L + T_R = \frac{1}{2}(m_w r^2 + J_w)(\dot{\alpha} + \dot{\beta})^2 \end{array} \tag{7}$$

kinetic energy of the rider:

the kinetic energy of the rider is built-up with three components:
The translational kinetic energy is the following:

$$\begin{array}{rcl} T_{tP} & = & \frac{1}{2}m_P v_P^2 = \frac{1}{2}m_P\left(v_b^2 + L^2\dot{\theta}_P^2 + 2L\dot{\theta}_P v_b \cos\theta_P\right) \\ & = & \frac{1}{2}m_P\left[\left(r\frac{\dot{\alpha}+\dot{\beta}}{2}\right)^2 + L^2\dot{\theta}_P^2 + 2L\dot{\theta}_P r\frac{\dot{\alpha}+\dot{\beta}}{2}\cos\theta_P\right] \end{array} \tag{8}$$

The rotational kinetic energy due to the rotation of the rider around the wheel's centre (θ_P) is made-up by the following:

$$T_{\theta P} = \frac{1}{2}J_{\theta P}\dot{\theta}_P^2 = \frac{1}{2}m_P\left(\frac{r^2}{4} + \frac{h^2}{3}\right)\dot{\theta}_P^2 \tag{9}$$

The rotational kinetic energy due to the rotation around the vertical axis (y) is made-up by the following:

$$T_{yP} = \frac{1}{2}\left[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P}\right]\dot{\delta}^2 \tag{10}$$

The total kinetic energy of the rider is given by:

$$\begin{array}{rcl} T_P & = & T_{tP} + T_{\theta P} + T_{yP} = \\ & & \frac{1}{2}m_P\left[\left(r\frac{\dot{\alpha}+\dot{\beta}}{2}\right)^2 + L^2\dot{\theta}_P^2 + 2L\dot{\theta}_P r\frac{\dot{\alpha}+\dot{\beta}}{2}\cos\theta_P\right] + \\ & & \frac{1}{2}J_{\theta P}\dot{\theta}_P^2 + \\ & & \frac{1}{2}\left[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P}\right]\left(r\frac{\dot{\alpha}-\dot{\beta}}{D}\right)^2 \end{array} \tag{11}$$

kinetic energy of the motors:

The kinetic energy of the motors is:

$$\begin{array}{rcl} T_m & = & T_{mL} + T_{mR} = \frac{1}{2}J_m(\dot{\alpha}_m^2 + \dot{\beta}_m^2) = \\ & = & \frac{1}{2}\frac{J_m}{\rho^2}(\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\theta}_P^2 - 2\dot{\alpha}\dot{\theta}_P - 2\dot{\beta}\dot{\theta}_P) \end{array} \tag{12}$$

Total kinetic energy:
The overall kinetic energy of the system is:

$$\begin{aligned} T &= T_P + T_w + T_m = \\ & \frac{1}{2} m_P \left[\left(r \frac{\dot{\alpha}+\dot{\beta}}{2} \right)^2 + L^2 \dot{\theta}_P^2 + 2L\dot{\theta}_P r \frac{\dot{\alpha}+\dot{\beta}}{2} \cos\theta_P \right] + \\ & \frac{1}{2} J_{\theta P} \dot{\theta}_P^2 + \\ & \frac{1}{2} \left[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P} \right] \left(r \frac{\dot{\alpha}-\dot{\beta}}{D} \right)^2 + \\ & \frac{1}{2} (m_w r^2 + J_w)(\dot{\alpha}+\dot{\beta})^2 + \\ & \frac{1}{2} \frac{J_m}{\rho^2} (\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\theta}_P^2 - 2\dot{\alpha}\dot{\theta}_P - 2\dot{\beta}\dot{\theta}_P) \end{aligned} \tag{13}$$

potential energy (of the rider):

This contribute of potential energy is given by simply the vertical position of the rider's center of mass:

$$U = m_P g L \cos\theta_P \tag{14}$$

LAGRANGIAN

Given the previous, it is possible to write the expression of the Lagrangian for the system

$$L = T - U \tag{15}$$

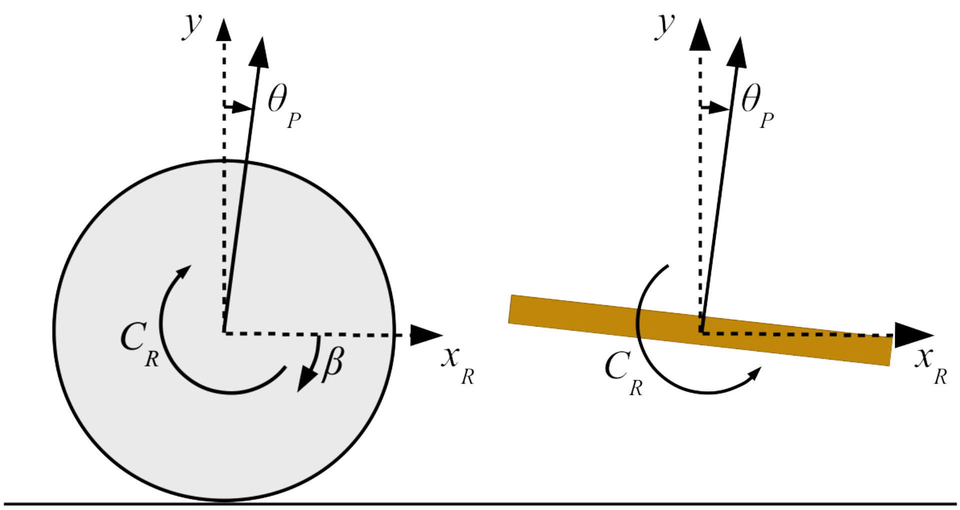

Figure 2: Detailed description of torques and references on right wheel. In the left, the wheel, while in the right the TWSBT base: note that the torque generated by the motor applies equal and opposite in the two bodies. The symmetric holds for the left wheel.

Having this, the Lagrange equations becomes:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) - \frac{\partial L}{\partial \alpha} &= C_L \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\beta}}\right) - \frac{\partial L}{\partial \beta} &= C_R \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_P}\right) - \frac{\partial L}{\partial \theta_P} &= -(C_L + C_R) \end{aligned} \tag{16}$$

The right-terms of the Lagrange equations represents the active contributes to the dynamics. The first and second equations' terms are quite obvious, while for the third equation, the active contribute is the sum of the torques of the two motors, as easily noticeable observing figure 2.

Which permits to write the complete form of the motion equations (31) this way:

$$
\begin{aligned}
\left[\frac{m_P r^2}{4} + \frac{r^2[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P}]}{D^2} + m_w r^2 + J_w + \frac{J_m}{\rho^2}\right]\ddot{\alpha} + \\
\left[\frac{m_P r^2}{4} - \frac{r^2[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P}]}{D^2} + m_w r^2 + J_w\right]\ddot{\beta} + \\
\left[\frac{m_P L r\cos\theta_P}{2} - \frac{J_m}{\rho^2}\right]\ddot{\theta}_P + \frac{\psi}{\rho^2}\dot{\alpha} - \frac{\psi}{\rho^2}\dot{\theta}_P + \\
-\frac{m_P L\dot{\theta}_P^2 r\sin\theta_P}{2} + \frac{[\sin\theta_P\cos\theta_P J_{\theta P} - \sin\theta_P\cos\theta_P J_{\delta P}]}{D^2} 2\dot{\theta}_P\left(\dot{\alpha} - \dot{\beta}\right) &= \frac{\tau_L}{\rho} \\
\left[\frac{m_P r^2}{4} - \frac{r^2[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P}]}{D^2} + m_w r^2 + J_w\right]\ddot{\alpha} + \\
\left[\frac{m_P r^2}{4} + \frac{r^2[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P}]}{D^2} + m_w r^2 + J_w + \frac{J_m}{\rho^2}\right]\ddot{\beta} + \\
\left[\frac{m_P L r\cos\theta_P}{2} - \frac{J_m}{\rho^2}\right]\ddot{\theta}_P + \frac{\psi}{\rho^2}\dot{\beta} - \frac{\psi}{\rho^2}\dot{\theta}_P + \\
-\frac{m_P L\dot{\theta}_P^2 r\sin\theta_P}{2} + \frac{[\sin\theta_P\cos\theta_P J_{\theta P} - \sin\theta_P\cos\theta_P J_{\delta P}]}{D^2} 2\dot{\theta}_P\left(\dot{\beta} - \dot{\alpha}\right) &= \frac{\tau_R}{\rho} \\
\left[\frac{m_P L r\cos\theta_P}{2} - \frac{J_m}{\rho^2}\right]\ddot{\alpha} + \left[\frac{m_P L r\cos\theta_P}{2} - \frac{J_m}{\rho^2}\right]\ddot{\beta} + \\
+\left[m_P L^2 + \frac{2J_m}{\rho^2} + J_{\theta_P}\right]\ddot{\theta}_P - m_P g L\sin\theta_P + \\
+\frac{2\psi}{\rho^2}\dot{\theta}_P - \frac{\psi}{\rho^2}\dot{\alpha} - \frac{\psi}{\rho^2}\dot{\beta} + \frac{r^2[\sin\theta_P\cos\theta_P J_{\theta P} - \sin\theta_P\cos\theta_P J_{\delta P}]}{D^2}(\dot{\alpha} - \dot{\beta})^2 &= -\frac{\tau_L}{\rho} - \frac{\tau_R}{\rho}
\end{aligned}
\tag{17}
$$

which are the motion equations for the system. Having this non-linear system, it is now possible to linearize it, in order to build a linear controller.

SYSTEM LINEARIZATION

The linearization of the system is performed around the vertical equilibrium condition ($\theta_P \approx 0$) and for small horizontal rotations ($\delta \approx 0$), which has as consequence:

$$
\begin{aligned}
\sin\theta_P &\approx \theta_P \\
\cos\theta_P &\approx 1 \\
\sin^2\theta_P &\approx 0 \\
\dot{\theta_P}(\dot{\alpha} - \dot{\beta}) &\approx 0 \\
(\dot{\alpha} - \dot{\beta})^2 &\approx 0 \\
\dot{\theta}_P^2 &\approx 0
\end{aligned}
\tag{18}
$$

The system (17), after linearization, becomes the following:

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{11} & m_{13} \\ m_{13} & m_{13} & m_{33} \end{bmatrix} \begin{Bmatrix} \ddot{\alpha} \\ \ddot{\beta} \\ \ddot{\theta}_P \end{Bmatrix} + \begin{bmatrix} c_{11} & 0 & -c_{11} \\ 0 & c_{11} & -c_{11} \\ -c_{11} & -c_{11} & 2c_{11} \end{bmatrix} \begin{Bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\theta}_P \end{Bmatrix} +$$
$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \\ \theta_P \end{Bmatrix} = \begin{Bmatrix} \frac{\tau_L}{\rho} \\ \frac{\tau_R}{\rho} \\ -(\frac{\tau_L}{\rho} + \frac{\tau_R}{\rho}) \end{Bmatrix}$$

where: (19)

$$\begin{aligned} m_{11} &= \left[\frac{m_P r^2}{4} + \frac{r^2 J_{\delta P}}{D^2} + m_w r^2 + J_w + \frac{J_m}{\rho^2}\right] \\ m_{12} &= \left[\frac{m_P r^2}{4} - \frac{r^2 J_{\delta P}}{D^2} + m_w r^2 + J_w\right] \\ m_{13} &= \left[\frac{m_P L r}{2} - \frac{J_m}{\rho^2}\right] \\ m_{33} &= \left[m_P L^2 + \frac{2J_m}{\rho^2} + J_{\theta_P}\right] \\ c_{11} &= \frac{\psi}{\rho^2} \\ k_{33} &= -m_P g L \end{aligned}$$

Which makes it possible to build-up the state-space system this way:

$$M\ddot{q} + C\dot{q} + Kq = u$$

$$q = \begin{Bmatrix} \alpha \\ \beta \\ \theta_P \end{Bmatrix}, \; x = \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} = \begin{Bmatrix} \alpha \\ \beta \\ \theta_P \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{\theta_P} \end{Bmatrix}, \; u = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{1}{\rho} \\ -\frac{1}{\rho} & -\frac{1}{\rho} \end{bmatrix} \begin{Bmatrix} \tau_L \\ \tau_R \end{Bmatrix}$$

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{11} & m_{13} \\ m_{13} & m_{13} & m_{33} \end{bmatrix} \quad (20)$$

$$C = \begin{bmatrix} c_{11} & 0 & -c_{11} \\ 0 & c_{11} & -c_{11} \\ -c_{11} & -c_{11} & 2c_{11} \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_{33} \end{bmatrix}$$

And finally the state-space representation:

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
x &= \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \\
\dot{x} &= Ax + Bu = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M^{-1}(-Kq - C\dot{q} + u) \end{bmatrix} = \\
&= \begin{bmatrix} 0_3 & I_3 \\ -M^{-1}K & -M^{-1}C \end{bmatrix} x + \begin{bmatrix} 0_3 \\ M^{-1} \end{bmatrix} u = \\
&\quad = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M^{-1}(-Kq - C\dot{q} + u) \end{bmatrix} = \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & a_{43} & a_{44} & 0 & a_{46} \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & a_{63} & a_{64} & 0 & a_{66} \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \\ \theta_P \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{\theta_P} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ b_{41} & b_{42} \\ b_{51} & b_{52} \\ b_{61} & b_{62} \end{bmatrix} \begin{Bmatrix} \tau_L \\ \tau_R \end{Bmatrix}
\end{aligned} \tag{21}
$$

The non-null terms in the [A] and [B] matrices are not reported. In the next section it will be clear why.

SYSTEM DECOUPLING

The system just introduced has the problem of a deep coupling between the *BALANCE* and the *STEERING* components. The scope of this section is to introduce a decoupling strategy for it, in a way it will be possible to design two separate controllers (one for the balance and one for the steering), presented in the successive chapter.

First of all, the state coordinates will be changed, using the expressions (2), so that it will be easier to talk about base velocity and steering angle, instead of the α and β quantities. The base-change matrix is:

$$
\begin{aligned}
q_s = \begin{Bmatrix} X_b \\ \delta \\ \theta_P \end{Bmatrix} &= \begin{bmatrix} r/2 & r/2 & 0 \\ r/D & -r/D & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \\ \theta_P \end{Bmatrix} = \begin{bmatrix} S \end{bmatrix} q \\
&\text{so that:} \\
\begin{Bmatrix} \alpha \\ \beta \\ \theta_P \end{Bmatrix} &= \begin{bmatrix} S \end{bmatrix}^{-1} \begin{Bmatrix} X_b \\ \delta \\ \theta_P \end{Bmatrix} = \begin{bmatrix} S \end{bmatrix}^{-1} q_s \\
&\text{thus:} \\
&\begin{bmatrix} M \end{bmatrix} \ddot{q} + \begin{bmatrix} C \end{bmatrix} \dot{q} + \begin{bmatrix} K \end{bmatrix} q = u \\
&\text{becomes:} \\
&\begin{bmatrix} M \end{bmatrix} \begin{bmatrix} S \end{bmatrix}^{-1} \ddot{q}_s + \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} S \end{bmatrix}^{-1} \dot{q}_s + \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} S \end{bmatrix}^{-1} q_s = u \\
&= \begin{bmatrix} M_s \end{bmatrix} \ddot{q}_s + \begin{bmatrix} C_s \end{bmatrix} \dot{q}_s + \begin{bmatrix} K_s \end{bmatrix} q_s
\end{aligned} \tag{22}
$$

Following the procedure, like in the formulas (24), the new system matrices will be:

$$
\begin{aligned}
\dot{x}_s &= \begin{bmatrix} q_s \\ \dot{q}_s \end{bmatrix} = A_s x_s + B_s u = \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & as_{43} & as_{44} & 0 & as_{46} \\ 0 & 0 & 0 & 0 & as_{55} & 0 \\ 0 & 0 & as_{63} & as_{64} & 0 & as_{66} \end{bmatrix} \begin{Bmatrix} X_b \\ \delta \\ \theta_P \\ \nu_b \\ \dot{\delta} \\ \dot{\theta}_P \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ bs_{41} & bs_{42} \\ bs_{51} & bs_{52} \\ bs_{61} & bs_{62} \end{bmatrix} \begin{Bmatrix} \tau_L \\ \tau_R \end{Bmatrix}
\end{aligned} \tag{23}
$$

At this point the important decoupling scheme. Since the "common-mode" torque oft the two motors contributes for the base movement and the "differential-mode" contributes for the base rotation, two new entities are introduced: the balancing and the rotation torques:

$$
\begin{aligned}
\begin{Bmatrix} \tau_\theta \\ \tau_\delta \end{Bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} \tau_L \\ \tau_R \end{Bmatrix} = \begin{bmatrix} D \end{bmatrix} \begin{Bmatrix} \tau_L \\ \tau_R \end{Bmatrix} \\
&\text{so that:} \\
\begin{Bmatrix} \tau_L \\ \tau_R \end{Bmatrix} &= \begin{bmatrix} D \end{bmatrix}^{-1} \begin{Bmatrix} \tau_\theta \\ \tau_\delta \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{Bmatrix} \tau_\theta \\ \tau_\delta \end{Bmatrix}
\end{aligned} \tag{24}
$$

Appliying the previous to the state-space causes the decoupling of the problem, but in order to manage better the problem, a more-convenient view of the state-space system comes with a re-arrangement of the state variables order. The final result is the following:

$$
\begin{aligned}
\dot{x}_N &= A_N x_N + B_N u_N = \\
\begin{Bmatrix} \dot{X}_b \\ \dot{\theta}_P \\ \ddot{X}_b \\ \ddot{\theta}_P \\ \dot{\delta} \\ \ddot{\delta} \end{Bmatrix} &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & an_{32} & an_{33} & an_{34} & 0 & 0 \\ 0 & an_{42} & an_{43} & an_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & an_{66} \end{bmatrix} \begin{Bmatrix} X_b \\ \theta_P \\ \dot{X}_b \\ \dot{\theta}_P \\ \delta \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ bn_{31} & 0 \\ bn_{41} & 0 \\ 0 & 0 \\ 0 & bn_{62} \end{bmatrix} \begin{Bmatrix} \tau_\theta \\ \tau_\delta \end{Bmatrix}
\end{aligned} \tag{25}
$$

In this final system it is clearly visible that the problem can be divided in the following two separate systems:

$$
\begin{aligned}
& \text{Equilibrium subsystem:} \\
\dot{x}_\theta &= A_\theta x_\theta + B_\theta \tau_\theta = \\
\begin{Bmatrix} \dot{X}_b \\ \dot{\theta}_P \\ \ddot{X}_b \\ \ddot{\theta}_P \end{Bmatrix} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & an_{32} & an_{33} & an_{34} \\ 0 & an_{42} & an_{43} & an_{44} \end{bmatrix} \begin{Bmatrix} X_b \\ \theta_P \\ \dot{X}_b \\ \dot{\theta}_P \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ bn_{31} \\ bn_{41} \end{bmatrix} \tau_\theta \\
& \text{and Steering subsystem:} \\
\dot{x}_\delta &= A_\delta x_\delta + B_\delta \tau_\delta = \\
\begin{Bmatrix} \dot{\delta} \\ \ddot{\delta} \end{Bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & an_{66} \end{bmatrix} \begin{Bmatrix} \delta \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} 0 \\ bn_{62} \end{bmatrix} \tau_\delta
\end{aligned}
\tag{26}
$$

STABILITY ANALISYS

The linearized, decoupled system has two subsystems that are interesting to analyze. In the following rows, some considerations about the linearized models (27), considering these parameters : a rider of 80kg and 1.80m height.

Concerning the *equilibrium* subsystem, it is a controllable system, even if underactuated. It owns the following poles and it is possible to note the presence of, as expected, an unstable (positive real) pole:

$$\text{poles of equilibrium} = 0\ ,\ 2.3267\ ,\ -2.3315\ ,\ -0.0167 \tag{27}$$

Represented in Gauss' plane in figure3.

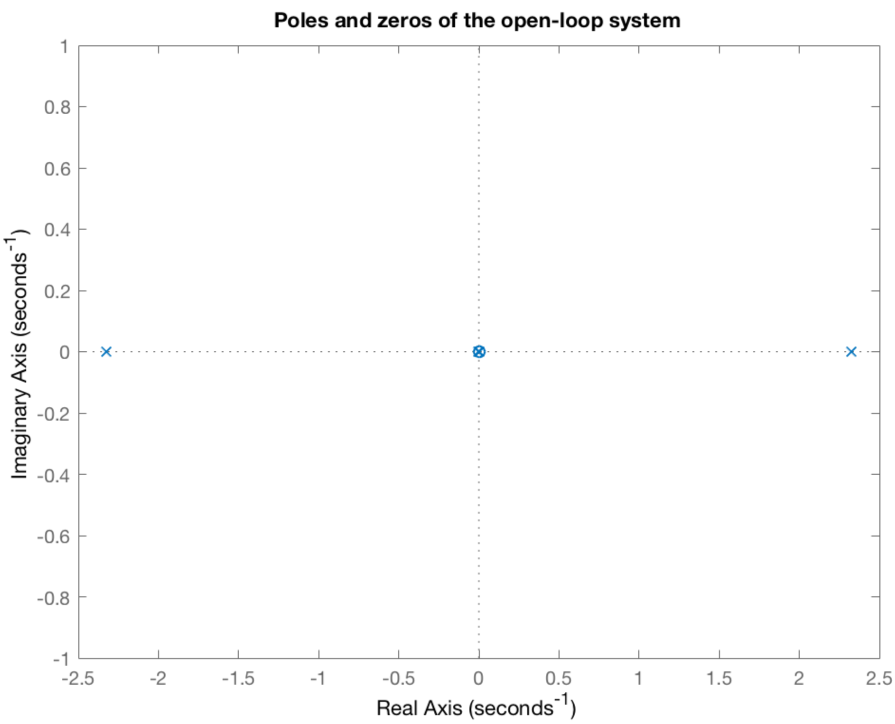

Figure 3: Poles and zero-es of equilibrium subsystem (open-loop unstable poles).

The unstable nature of the system is also confirmed by the nonlinear model, by plotting a free evolution obtained by applying a small perturbation to the tilt angle (5 degrees) and by holding $\tau_\theta = 0$. Those values, applied to the nonlinear model **??** gives the evolution visible in 4.

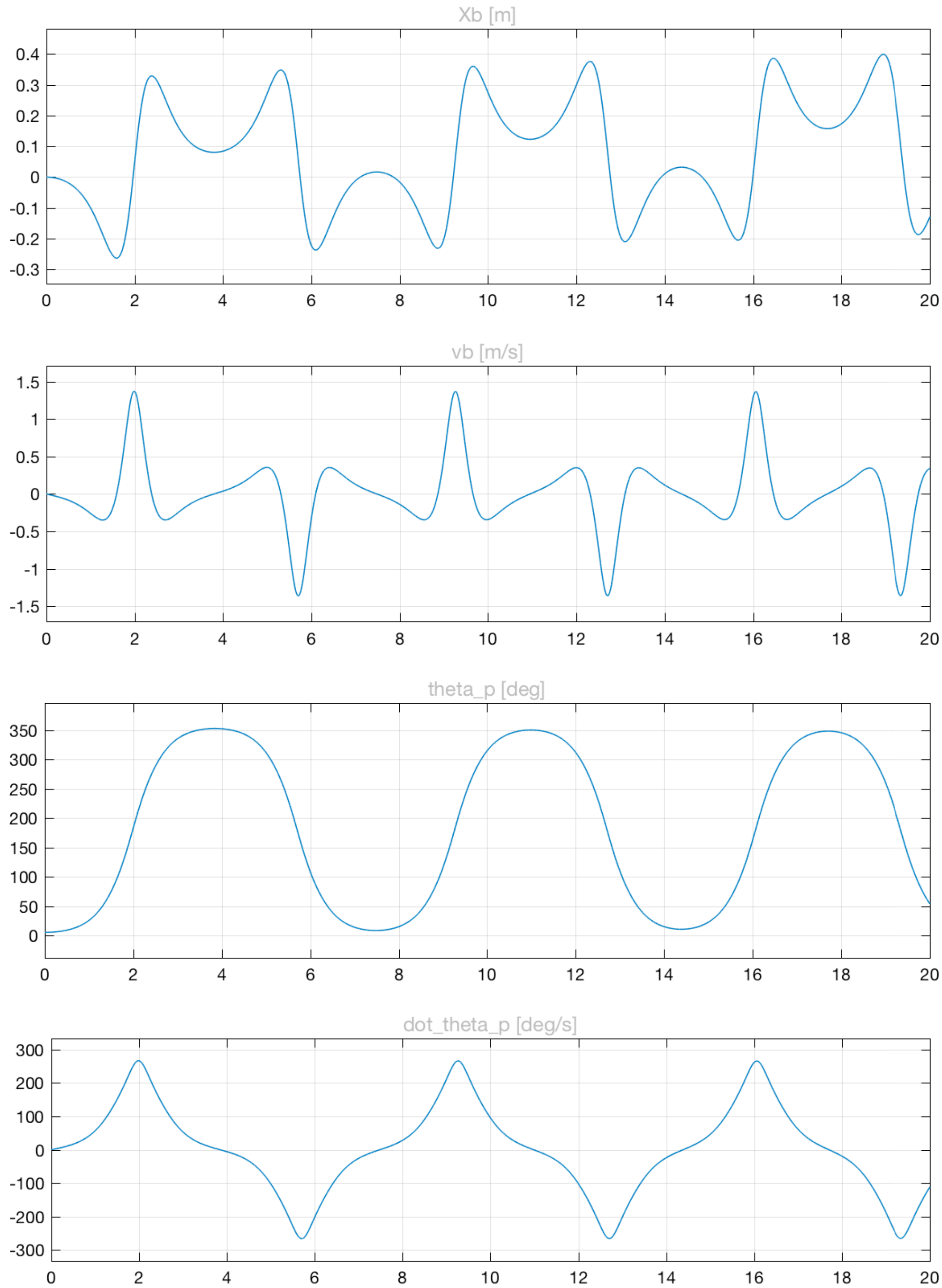

Figure 4: Free evolution of the model **??** with $\theta_P(0) = 5°$ and $\tau_\theta = 0$.

Concerning the *steering* subsystem, the poles are two: one (weakly) stable and one at the origin. The transfer function of the steering angle, against torque is:

$$W_\delta(s) = \frac{\delta}{\tau_\delta} \quad = \quad \frac{1}{s}\frac{7.2}{s+0.09861} \tag{28}$$

The root-locus and the Bode plot for the system are visible in figure 5.

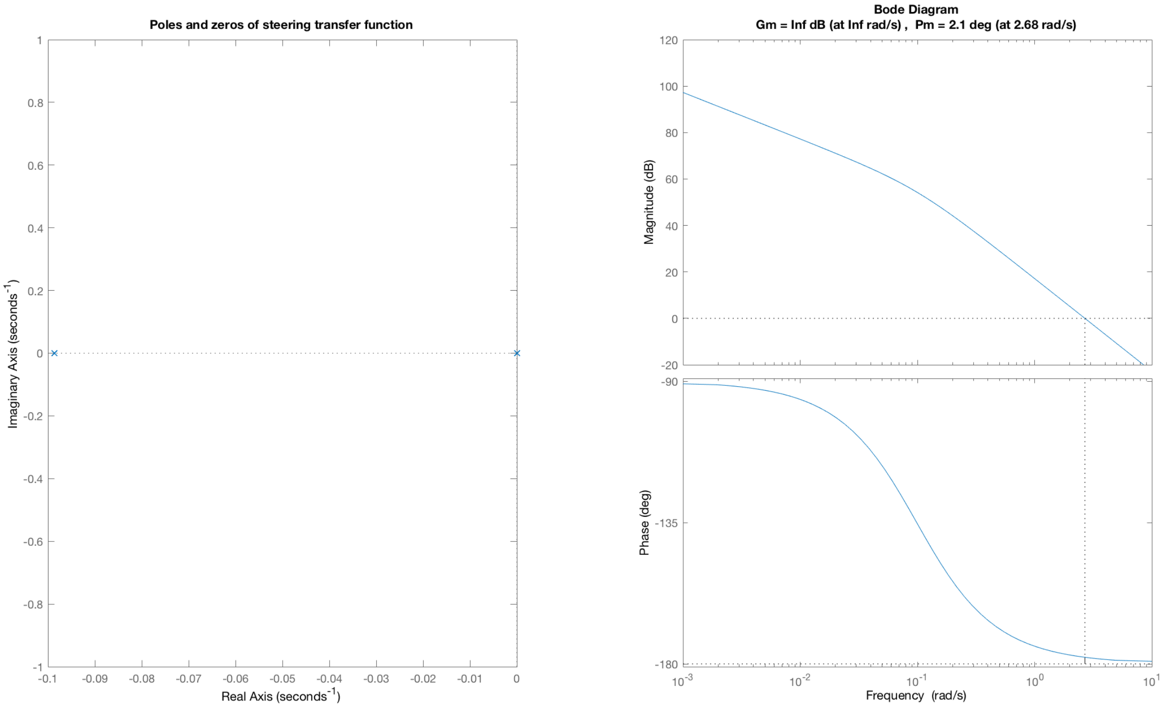

Figure 5: Root locus and Bode plot with stability margins of the $\frac{\delta}{\tau_\delta}$ transter function (29).

CONCLUSIONS

In this chapter it was introduced the simplest model for the study of the self-balancing inverse-pendulum system. This model was validated with simulations, using the simulink block-model. The properties of the system's subsystems (such as poles, zeros and stability) was also analyzed making this model usable in order to design a suitable controller.
In the following chapter a further step into the system modeling, by introducing a model extension, making it more realistic and interesting for study.

APPENDIX

A

APPENDIX-A: LAGRANGIAN DERIVATION STEP-BY-STEP

Even having the possibility to derive the expression of motion equations with automated tools, such as Wolfram Mathematica, in this appendix, the mathematical steps of the derivation of the motion equations. The final equations were verified also with the mentioned software.

LAGRANGIAN: DERIVATION FOR THE FIRST MODEL

Here the extended expression of the Lagrangian for the system

$$\begin{aligned} L \;=\;& T - U = \\ & = \frac{1}{2} m_P \left[r^2 \frac{\dot{\alpha}^2}{4} + r^2 \frac{\dot{\beta}^2}{4} + r^2 \frac{\dot{\alpha}\dot{\beta}}{2} + L^2 \dot{\theta}_P^2 + L \dot{\theta}_P r (\dot{\alpha} + \dot{\beta}) \cos\theta_P \right] + \\ & \frac{1}{2} J_{\theta P} \dot{\theta}_P^2 + \\ & \frac{1}{2} \left[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P} \right] \left[r^2 \frac{\dot{\alpha}^2}{D^2} + r^2 \frac{\dot{\beta}^2}{D^2} - 2 r^2 \frac{\dot{\alpha}\dot{\beta}}{D^2} \right] + \\ & \frac{1}{2} (m_w r^2 + J_w)(\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta}) + \\ & \frac{1}{2} \frac{J_m}{\rho^2} (\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\theta}_P^2 - 2\dot{\alpha}\dot{\theta}_P - 2\dot{\beta}\dot{\theta}_P) + \\ & - m_P g L \cos\theta_P \end{aligned} \tag{29}$$

Having this, the Lagrange equations becomes:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) - \frac{\partial L}{\partial \alpha} &= C_L \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\beta}}\right) - \frac{\partial L}{\partial \beta} &= C_R \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_P}\right) - \frac{\partial L}{\partial \theta_P} &= -(C_L + C_R) \end{aligned} \tag{30}$$

The right-terms of the Lagrange equations represents the active contributes to the dynamics. The first and second equations' terms are quite obvious, while for the third equation, the active contribute is the sum of the torques of the two motors, as easily noticeable observing figure 2.

Executing the partial derivatives just presented, the equations' becomes this non-linear system:

$$
\begin{aligned}
-\frac{\partial L}{\partial \alpha} &= 0 \\
-\frac{\partial L}{\partial \beta} &= 0 \\
-\frac{\partial L}{\partial \theta_P} &= m_P L\dot{\theta}_P r\frac{\dot{\alpha}+\dot{\beta}}{2}\sin\theta_P - m_P gL\sin\theta_P + \\
&\quad [\sin\theta_P\cos\theta_P J_{\theta P} - \sin\theta_P\cos\theta_P J_{\delta P}]\left[r^2\frac{\dot{\alpha}^2}{D^2} + r^2\frac{\dot{\beta}^2}{D^2} - 2r^2\frac{\dot{\alpha}\dot{\beta}}{D^2}\right] \\
\\
\frac{\partial L}{\partial \dot{\alpha}} &= \frac{1}{2}m_P\left[r^2\frac{\dot{\alpha}}{2} + r^2\frac{\dot{\beta}}{2} + L\dot{\theta}_P r\cos\theta_P\right] + \\
&\quad \left[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P}\right]\left[r^2\frac{\dot{\alpha}}{D^2} - r^2\frac{\dot{\beta}}{D^2}\right] + \\
&\quad (m_w r^2 + J_w)[\dot{\alpha} + \dot{\beta}] + \frac{J_m}{\rho^2}(\dot{\alpha} - \dot{\theta}_P) \\
\frac{\partial L}{\partial \dot{\beta}} &= \frac{1}{2}m_P\left[r^2\frac{\dot{\alpha}}{2} + r^2\frac{\dot{\beta}}{2} + L\dot{\theta}_P r\cos\theta_P\right] + \\
&\quad \left[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P}\right]\left[r^2\frac{\dot{\beta}}{D^2} - r^2\frac{\dot{\alpha}}{D^2}\right] + \\
&\quad (m_w r^2 + J_w)[\dot{\alpha} + \dot{\beta}] + \frac{J_m}{\rho^2}(\dot{\beta} - \dot{\theta}_P) \\
\frac{\partial L}{\partial \dot{\theta}_P} &= m_P\left[L^2\dot{\theta}_P + Lr\frac{\dot{\alpha}+\dot{\beta}}{2}\cos\theta_P\right] + J_{\theta P}\dot{\theta}_P + \frac{J_m}{\rho^2}(2\dot{\theta}_P - \dot{\alpha} - \dot{\beta}) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) &= \frac{1}{2}m_P\left[r^2\frac{\ddot{\alpha}}{2} + r^2\frac{\ddot{\beta}}{2} + L\ddot{\theta}_P r\cos\theta_P - L\dot{\theta}_P^2 r\sin\theta_P\right] + \\
&\quad \left[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P}\right]\left[r^2\frac{\ddot{\alpha}}{D^2} - r^2\frac{\ddot{\beta}}{D^2}\right] + \\
&\quad \left[2\dot{\theta}_P\sin\theta_P\cos\theta_P J_{\theta P} - 2\dot{\theta}_P\sin\theta_P\cos\theta_P J_{\delta P}\right]\left[r^2\frac{\dot{\alpha}}{D^2} - r^2\frac{\dot{\beta}}{D^2}\right] + \\
&\quad (m_w r^2 + J_w)[\ddot{\alpha} + \ddot{\beta}] + \frac{J_m}{\rho^2}(\ddot{\alpha} - \ddot{\theta}_P) \\
\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\beta}}\right) &= \frac{1}{2}m_P\left[r^2\frac{\ddot{\alpha}}{2} + r^2\frac{\ddot{\beta}}{2} + L\ddot{\theta}_P r\cos\theta_P - L\dot{\theta}_P^2 r\sin\theta_P\right] + \\
&\quad \left[\sin^2\theta_P J_{\theta P} + \cos^2\theta_P J_{\delta_P}\right]\left[r^2\frac{\ddot{\beta}}{D^2} - r^2\frac{\ddot{\alpha}}{D^2}\right] + \\
&\quad \left[2\dot{\theta}_P\sin\theta_P\cos\theta_P J_{\theta P} - 2\dot{\theta}_P\sin\theta_P\cos\theta_P J_{\delta P}\right]\left[r^2\frac{\dot{\beta}}{D^2} - r^2\frac{\dot{\alpha}}{D^2}\right] + \\
&\quad (m_w r^2 + J_w)[\ddot{\alpha} + \ddot{\beta}] + \frac{J_m}{\rho^2}(\ddot{\beta} - \ddot{\theta}_P) \\
\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_P}\right) &= m_P\left[L^2\ddot{\theta}_P + Lr\frac{\ddot{\alpha}+\ddot{\beta}}{2}\cos\theta_P - Lr\frac{\dot{\alpha}+\dot{\beta}}{2}\dot{\theta}_P\sin\theta_P\right] + \\
&\quad J_{\theta P}\ddot{\theta}_P + \frac{J_m}{\rho^2}(2\ddot{\theta}_P - \ddot{\alpha} - \ddot{\beta})
\end{aligned}
\tag{31}
$$

Which permits to write the complete form of the equations (31) this way:

$$
\begin{aligned}
\tfrac{1}{2}m_P\left[r^2\tfrac{\ddot{\alpha}}{2}+r^2\tfrac{\ddot{\beta}}{2}+L\ddot{\theta}_P r\cos\theta_P-L\dot{\theta}_P^2 r\sin\theta_P\right]+ & \\
\left[\sin^2\theta_P J_{\theta P}+\cos^2\theta_P J_{\delta_P}\right]\left[r^2\tfrac{\ddot{\alpha}}{D^2}-r^2\tfrac{\ddot{\beta}}{D^2}\right]+ & \\
\left[2\dot{\theta}_P\sin\theta_P\cos\theta_P J_{\theta P}-2\dot{\theta}_P\sin\theta_P\cos\theta_P J_{\delta P}\right]\left[r^2\tfrac{\dot{\alpha}}{D^2}-r^2\tfrac{\dot{\beta}}{D^2}\right]+ & \\
(m_w r^2+J_w)[\ddot{\alpha}+\ddot{\beta}]+\tfrac{J_m}{\rho^2}(\ddot{\alpha}-\ddot{\theta}_P) &= \tfrac{\tau_L}{\rho}-\tfrac{\psi}{\rho^2}(\dot{\alpha}-\dot{\theta}_P) \\[2ex]
\tfrac{1}{2}m_P\left[r^2\tfrac{\ddot{\alpha}}{2}+r^2\tfrac{\ddot{\beta}}{2}+L\ddot{\theta}_P r\cos\theta_P-L\dot{\theta}_P^2 r\sin\theta_P\right]+ & \\
\left[\sin^2\theta_P J_{\theta P}+\cos^2\theta_P J_{\delta_P}\right]\left[r^2\tfrac{\ddot{\beta}}{D^2}-r^2\tfrac{\ddot{\alpha}}{D^2}\right]+ & \\
\left[2\dot{\theta}_P\sin\theta_P\cos\theta_P J_{\theta P}-2\dot{\theta}_P\sin\theta_P\cos\theta_P J_{\delta P}\right]\left[r^2\tfrac{\dot{\beta}}{D^2}-r^2\tfrac{\dot{\alpha}}{D^2}\right]+ & \\
(m_w r^2+J_w)[\ddot{\alpha}+\ddot{\beta}]+\tfrac{J_m}{\rho^2}(\ddot{\beta}-\ddot{\theta}_P) &= \tfrac{\tau_R}{\rho}-\tfrac{\psi}{\rho^2}(\dot{\beta}-\dot{\theta}_P) \\[2ex]
m_P\left[L^2\ddot{\theta}_P+Lr\tfrac{\ddot{\alpha}+\ddot{\beta}}{2}\cos\theta_P\right]+ & \\
J_{\theta P}\ddot{\theta}_P+\tfrac{J_m}{\rho^2}(2\ddot{\theta}_P-\ddot{\alpha}-\ddot{\beta})+ & \\
-m_P g L\sin\theta_P+ & \\
[\sin\theta_P\cos\theta_P J_{\theta P}-\sin\theta_P\cos\theta_P J_{\delta P}]\left[r^2\tfrac{\dot{\alpha}^2}{D^2}+r^2\tfrac{\dot{\beta}^2}{D^2}-2r^2\tfrac{\dot{\alpha}\dot{\beta}}{D^2}\right] &= -\tfrac{\tau_L}{\rho}-\tfrac{\tau_R}{\rho}-\tfrac{2\psi}{\rho^2}\dot{\theta}_P+\tfrac{\psi}{\rho^2}\dot{\alpha}+\tfrac{\psi}{\rho^2}\dot{\beta}
\end{aligned}
\tag{32}
$$

By re-arranging the terms, it is easy to come back to the form of (17).