Task 1: $P_X(x) = Pr(X \leq x)$, $Y = P_X(X)$

$$P_Y(y) = Pr(Y \leq y) = Pr(\underbrace{P_X(X) \leq y})$$

$$P_X(X) \leq y$$

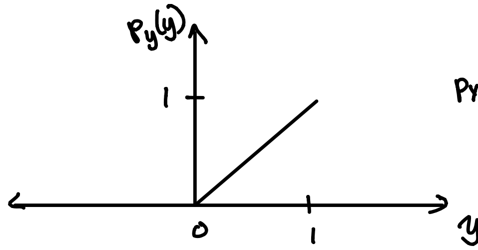
Due to the monotonicity of $P(X)$, it is invertible $\leftarrow$ $P_X^{-1}(P_X(X)) \leq P_X^{-1}(y)$

$$X \leq P_X^{-1}(y)$$

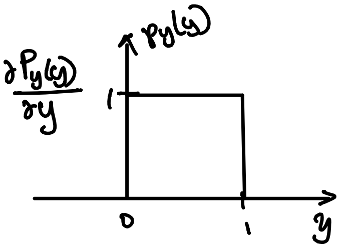
$$Pr(X \leq P_X^{-1}(y))$$

$$= P_X(P_X^{-1}(y))$$

$\underline{P_Y(y) = y}$ $\in [0,1]$, because cdf can only be between zero and one.



$p_Y(y) = \dfrac{\partial P_Y(y)}{\partial y}$ $\rightarrow$



Task 2

(a) $\boxed{p(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}}$ $G(t) = E_X[t^X] = \sum_{n=-\infty}^{\infty} p(x_n)\, t^{x_n}$

$$G(t) = \sum_{x_n=-\infty}^{\infty} p(x_n)\, t^{x_n}$$

$$= e^{-\lambda} \underbrace{\sum_{x_n=0}^{\infty} \frac{(\lambda t)^{x_n}}{x_n!}}_{e^{\lambda t} \text{ (Hint*)}} = e^{-\lambda} e^{\lambda t} = \underline{\underline{e^{\lambda(t-1)}}} \quad \square$$

poisson random variable has countable outcome space $\{0,1,2...\}$

(b) $p(x) = \binom{n}{x} r^x (1-r)^{n-x}$

$$G(t) = \sum_{x_n=0}^{\infty} \binom{n}{x_n} r^{x_n}(1-r)^{n-x_n} t^{x_n}$$

same for binomial

$$= \sum_{x_n=0}^{\infty} \binom{n}{x_n} (rt)^{x_n} (1-r)^{n-x_n}$$

Hint*

$$= ((rt) + (1-r))^n$$

$$= \underline{\underline{(1-r+rt)^n}}$$

c) $r = \frac{\lambda}{n}$, $n \to \infty$

$$G(t) = (1-r+rt)^n$$

$$\lim_{n\to\infty} G(t) = \left(1 - \frac{\lambda}{n} + \frac{\lambda t}{n}\right)^n$$

$$= \left(1 + \frac{\lambda(t-1)}{n}\right)^n$$

hint

$$= \underline{\underline{e^{\lambda(t-1)}}}$$

, this is the generating function of a Poisson distributed random variable

d) PMF: point mass function
GF: generating function

$$PMF(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$GF(t) = e^{\lambda(t-1)}$$

$N = N_1 + N_2$

Using GF: $\rightarrow$

$$G_{N_1}(t) = e^{\lambda_1(t-1)}$$

$$G_{N_2}(t) = e^{\lambda_2(t-1)}$$

$$G_N(t) = G_{N_1}(t)\, G_{N_2}(t)$$

$$= e^{(\lambda_1+\lambda_2)(t-1)}$$

$N = k$

$$p(k) = \frac{1}{k!} \frac{d^k}{dt^k} G_N(t) \Big|$$

$$k!\quad dt^k \qquad t=0$$

$$= \frac{1}{k!}\left[(\lambda_1+\lambda_2)^k e^{(\lambda_1+\lambda_2)(t-1)}\right]\Big|_{t=0}$$

$$p(k) = \frac{e^{-(\lambda_1+\lambda_2)}}{k!}(\lambda_1+\lambda_2)^k$$

Using PMF:

$$P(N_1+N_2=k) = \sum_{i=0}^{k} P(N_1+N_2=k, N_1=i)$$

$$= \sum_{i=0}^{k} P(N_2=k-i, N_1=i)$$

$$= \sum_{i=0}^{k} P(N_2=k-i)\underbrace{\ P(N_1=i)}_{\text{independent}}$$

$$= \sum_{i=0}^{k} \frac{e^{-\lambda_1}\lambda_1^{k-i}}{(k-i)!}\;\frac{e^{-\lambda_2}\lambda_2^{i}}{i!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}\sum_{i=0}^{k}\frac{\lambda_1^{k-i}\lambda_2^{i}}{i(k-i)!}\cdot k!$$

hint

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}(\lambda_1+\lambda_2)^k$$

GF method is faster.

Task 3: Exponential distribution, λ

$$P(t_i-t_{i-1}) = \lambda e^{-\lambda(t_i-t_{i-1})}$$

$P_D \in (0,1)$, $P_r(n_D|n) = \binom{n}{n_D} P_D^{n_D}(1-P_D)^{n-n_D}$

$= P_r(n_D, n-n_D|n)$

$$P(T=t_1 \mid t_1 \geq t_0) = \begin{cases}\lambda e^{-\lambda(t_1-t_0)}, & t_1 \geq t_0\end{cases}$$

0 , otherwise

a)

$$\boxed{\begin{aligned} p(t_1) &= Pr(T=t_1) \\ P(t_1) &= Pr(T\leq t_1) \end{aligned}}$$

$X = t_1 - t_0$

$$Pr(T=t_1 \mid t_1 \geq t_0) = \frac{Pr(t_1 \geq t_0 \mid T=t_1)\, Pr(T=t_1)}{Pr(t_1 \geq t_0)}$$

$t_1 \geq t_0$ $\qquad$ $t_1 < t_0$

$$= \frac{1 \cdot Pr(T=t_1)}{1 - Pr(t_1 < t_0)} \qquad \frac{0 \cdot Pr(T=t_1)}{Pr(t_1 \geq t_0)} = 0$$

$$= \frac{\lambda e^{-\lambda t_1}}{1-(1-Pr(t_0 \leq t_1))}$$

$$= \frac{\lambda e^{-\lambda t_1}}{1-(1-e^{-\lambda t_0})} = \underline{\underline{\lambda e^{-\lambda(t_1 - t_0)}}}$$

b) $t_n - t_0 = \sum_{i=1}^{n} t_i - t_{i-1}$, $t_1 \geq t_0$ $\qquad$ $t_i - t_{i-1} = X_i \sim$ exponential (λ)

MGF : $M_X(s) = E[e^{sX}] = \frac{\lambda}{\lambda - s}$, $s<\lambda$

$= \sum_{i=1}^{n} X_i$ $\qquad$ $M_Y(s) = \left(\frac{\lambda}{\lambda - s}\right)^n$

$$p_Y(y) = \underline{\underline{y^n e^{-\lambda y}}} \sim \text{Gamma}\left(\frac{1}{\lambda}\right)$$

$$p(t_n - t_0) = \underline{\frac{\lambda^n (t_n - t_0)^{n-1}}{(n-1)!} e^{-\lambda(t_n - t_0)}}$$

c) $Pr(t_{n+1} > T \mid t_n) = ?$, $t_n \leq T$

$= Pr(t_{n+1} - t_n > T - t_n \mid t_n) = 1 - Pr(t_{n+1} - t_n \leq T - t_n \mid t_n)$, $t_n \leq T$

$$= 1-(1-e^{-\lambda(T-t_n)}) = \underline{e^{-\lambda(T-t_n)}}, \quad t_n \le T$$

d) $Pr(n) = Pr(t_n \le T, t_{n+1} > T \mid t_1 \ge t_0)$ is poison

Marginalize $p(t_n, t_{n+1} > T)$ over t_n

$$P(t_n, t_{n+1} > T) = P(t_{n+1} > T \mid t_n)\, P(t_n \mid t_1 \ge t_0)$$

$$= P(t_{n+1} - t_n > T - t_n \mid t_n)\, P(t_n)$$

$$= e^{-\lambda(T-t_n)}\, P(t_n)$$

$$= e^{-\lambda(T-t_n)} \frac{\lambda^n (t_n)^{n-1}}{(n-1)!} e^{-\lambda(t_n)}$$

$$= e^{-\lambda(T)} \frac{\cdot \lambda^n (t_n)^{n-1}}{(n-1)!}$$

cdf ↓

$$P(t_n \le T, t_{n+1} > T \mid t_1 \ge t_0) =$$

$$= e^{-\lambda T} \int_{t_n = t_0}^{T} \frac{\lambda^n (t_n)^{n-1}}{(n-1)!} dt_n$$

$$= \frac{e^{-\lambda T} \lambda^n}{(n-1)!} \int_{t_n = t_0}^{T} (t_n)^{n-1} dt_n$$

$$= \frac{e^{-\lambda T} \lambda^n}{(n-1)!} \left[\frac{t_n^n}{n}\right]\Bigg|_{t_0}^{T}$$

$$Pr(n) = \frac{e^{-\lambda T} \lambda^n}{(n-1)!} \frac{T^n}{n} = \underline{\underline{\frac{e^{-\lambda T} (\lambda T)^n}{n!}}} \sim \text{Poisson}(\lambda T)$$

D

e) no boats at t_0

$\rightarrow P(n_D|n) = \binom{n}{n_D} P_D^{n_D}(1-P_D)^{n-n_D}$

$P(n_D, n) = P(n_D|n)\, P(n)$

$P(n_D, n_u) = P(n_D|n_u)\, P(n_u)$

$\rightarrow P(n_D|n_u) \overset{?}{=} P(n_D)$?

$\rightarrow P(n_D|n-n_D)$

$P_{n-n_D} = (1-P_{n_D})$

$= \binom{n}{n-n_D} P_{n-n_D}^{n-n_D} \underbrace{(1-P_{n-n_D})}_{P_D}^{n_D}$

$= \underline{P(n_D)}$

$P(n_D, n_u) = P(n_D|n_u)\, P(n_u)$

$= \underline{\underline{P(n_D)\, P(n_u)}}$

f) $P(n_D|m) = \frac{P(n_D, m)}{P(m)} = \frac{P(n_D)\cancel{P(m)}}{\cancel{P(m)}} = \underline{\underline{P_{n_D}}}$

g) $\hat{n}_{D\,MMSE} = E[n_D|m] = \int n_D\, p(m|n_D)\, dn_D$

$\hat{n}_{D\,MAP} = \underline{\arg\max P_{m|n_D}(m|n_D)\, P_{n_D}(n_D)}$

e) $P(n_D, n) = P(n_D|n)\, P(n)$

$$= p(n_D \mid n_u + n_D)\, p(n_D + n_u)$$

$$p(n_D \mid n_D + n_u) = \binom{n_D + n_u}{n_D} p_D^{n_D} (1 - p_D)^{n_D + n_u - n_D}$$

$$= \frac{(n_D + n_u)!}{n_u!\, n_D!} p_D^{n_D} (1 - p_D)^{n_u}$$

$$\frac{e^{-\lambda T} (\lambda T)^{n_D + n_u}}{(n_D + n_u)!}$$

$$= p(n_D \mid n_D + n_u)\, p(n_D + n_u)$$

$$= \frac{\cancel{(n_D + n_u)!}}{n_u!\, n_D!} p_D^{n_D} (1 - p_D)^{n_u}\, e^{-\lambda T} \frac{(\lambda T)^{n_D + n_u}}{\cancel{(n_D + n_u)!}}$$

$$= \frac{p_D^{n_D} p_u^{n_u} (\lambda T)^{n_D + n_u}}{n_u!\, n_D!}$$

$$p(n_D, n_u) = \underline{\underline{p(n_D)\, p(n_u)}}$$

Task 4: $x \in \mathbb{R}^n$ be $N(\mu, \Sigma)$

(a) $z = \Sigma^{-\frac{1}{2}} (x - \mu)$, $\Sigma^{\frac{1}{2}} \Sigma^{\frac{T}{2}} = \Sigma = \begin{bmatrix} \sigma^2 & & \\ & \sigma^2 & \\ & & \sigma^2 \\ & & & \ddots \end{bmatrix}$

$$x = \Sigma^{\frac{1}{2}} z + \mu$$

$$E\left(\Sigma^{-\frac{1}{2}} (x - \mu) \right) = \Sigma^{-\frac{1}{2}} \left(E(x) - E(\mu) \right)$$

$$= 0$$

$$\mathrm{Var}\left(\Sigma^{-\frac{1}{2}}(x-\mu)\right)$$
$$= \left(\Sigma^{-\frac{1}{2}}\right)^2 \left(\mathrm{Var}(x) + \underbrace{\cancel{\mathrm{Var}(\mu)}}_{=0}\right)$$
$$= \Sigma^{-1}\Sigma = \underline{\underline{I}}$$

$$\underline{\underline{z \sim N(0, I)}}.$$

(b) $y_i = z_i^2$, $z \sim N(0,1)$ $\quad y = f(z) = z_i^2$

$$h(y) = \sum_i N(f_i^{-1}(y), 0, 1)\left|\det\left(F_i^{-1}(y)\right)\right|.$$

$$= N(\sqrt{y_i}, 0, 1)\left|\det\left(\frac{1}{2\sqrt{y_i}}\right)\right| + N(-\sqrt{y_i}, 0, 1)\left|\det\left(\frac{-1}{2\sqrt{y_i}}\right)\right|$$

$$h(y_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i} \frac{1}{2\sqrt{y_i}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y_i} \frac{1}{2\sqrt{y_i}}$$

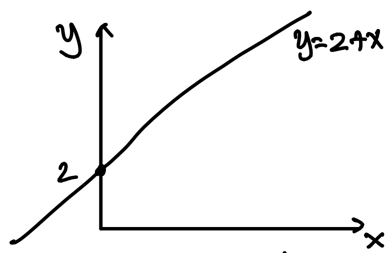
$$\underline{\underline{h(y_i) = \frac{1}{\sqrt{2\pi y_i}} e^{-\frac{1}{2}y_i}}} \sim \chi_1^2$$

c) $y = (x-\mu)^T \Sigma^{-1}(x-\mu) = z^T z = \sum z_i^2 = \sum y_i$

$y_i \sim \chi_1^2 \implies \underline{\underline{y \sim \chi_n^2}}$ (see MGF of χ_1^2)

Task 5)

$y = x + 2$



camera
radar

R^c, R^r - measurement noise

P - state

x: position

$x \sim \mathcal{N}(x; \bar{x}, P)$

measurements $\begin{cases} z^c = H^c x + v^c = x + v^c & , v^c \sim \mathcal{N}(0, R^c) \\ z^r = H^r x + v^r = x + v^r & , v^r \sim \mathcal{N}(0, R^r) \end{cases}$ (v^c: noise)

$x^+ = Fx + w \quad , w \sim \mathcal{N}(0, Q)$

a) $p(z^c | x) = \mathcal{N}(z_c; H_c x, R_c)$

$\mu_{z^c|x} = H_c x + 0 \quad , \quad P_{z^c|x} = P_{H_c x} + P_{z_c} = 0 + R$

b) $p(x, z_c) = p(z_c | x)\, p(x) = \mathcal{N}(z_c; H_c x, R_c)\, \mathcal{N}(x; H_c \bar{x}, P)$

from a)

c) $p(z^c) = \mathcal{N}(z_c; H_c \bar{x}, H_c P H_c^T + R)$, $\mu_{z_c} = \mu_{Hx} + \mu_{v_c} = H_c \bar{x} + 0$

$P_{z_c} = P_{Hx} + P_v = H_c P H_c^T + R$

$$p(x|z_c) = \frac{p(x, z_c)}{p(z_c)} = \frac{\mathcal{N}(z_c; H_c x, R_c)\,\mathcal{N}(x; H_c\bar{x}\ P)}{p(z_c; H_c\bar{x}, H_c P H_c^T + R)}$$

d) Yes, since x^+ is a linear transformation of x. (similar to $p(z_c)$)

$$p(x^+) = \mathcal{N}(x^+; F\bar{x}; FPF^T + Q)$$

$p(x|z_r)$ = similar to from c) $p(x|z_c)$, set $c = r$

e) $\hat{x}_{MMSE} = E[x|z_c] = \int x\, p(x|z_c)\, dx$

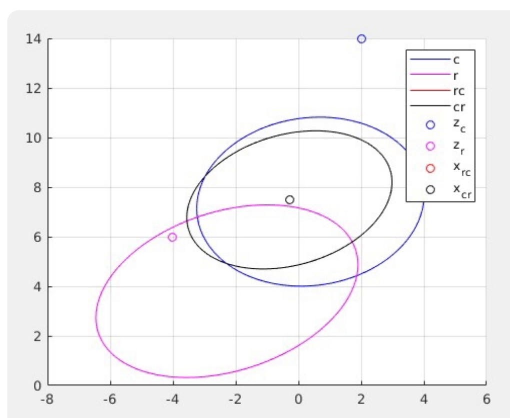
$$\hat{x} = \arg\max_x p_{x|z_c}(x|z_c) = \arg\max_x p_{z_c|x}(z_c|x)\, p_x(x)$$

f) see matlab

g) Yes, the distributions are the same. Since they are independent, no

f) g)

```
% FILL IN THE DOTS ...
condition_mean = @(x_bar, z, P, H, R) x_bar+P*H'*inv(H*P*H'+R)*(z-H*x_bar);
condition_cov = @(P, H, R) P-P*H'*inv(H*P*H'+R)*H*P';
%%
% task 5 (f)
% FILL IN FOR THE DOTS ...

% condition on camera
x_bar_c = condition_mean(x_bar,z_c,P,H_c,R_c);
P_c = condition_cov(P,H_c,R_c);

% condition on radar
x_bar_r = condition_mean(x_bar,z_r,P,H_r,R_r);
P_r = condition_cov(P,H_r,R_r);

% Plot 1 sigma ellipses
figure(2); clf; hold on; grid on;
data = x_bar + chol(P)' * circle;
plot( data(1,:), data(2, :) , 'DisplayName','prior')

data = x_bar_c + chol(P_c)' * circle;
plot( data(1,:), data(2, :) , 'DisplayName', 'c')

data = x_bar_r + chol(P_r)' * circle;
plot( data(1,:), data(2, :) , 'DisplayName', 'r')

% measurements
scatter(z_c(1), z_c(2), 'DisplayName', 'z_c')
scatter(z_r(1), z_r(2), 'DisplayName', 'z_r')
legend()

%%
% task 5 (g)

% condition the already camera conditioned on the radar
x_bar_cr = condition_mean(x_bar_c,z_r,P_c,H_r,R_r);
P_cr = condition_cov(P_c,H_r,R_r);

% P_cr: 2x2 double =
%
%   10.7043    2.3261
%    2.3261    7.7676
% radar conditioned on the camera
x_ an(x_bar_r,z_c,P_r,H_c,R_c);
P_ r,H_c,R_c);
```



Task 6)

$$z = Hx + w, \quad x \sim \mathcal{N}(\bar{x}, P), \quad w \sim \mathcal{N}(0, R)$$

$$(Q.6) \quad \hat{P} = \underbrace{\bar{P}}_{A} - \underbrace{\bar{P}H^T}_{B} \underbrace{(H\bar{P}H^T + R)^{-1}}_{D} \underbrace{H\bar{P}}_{C}$$

Matrix inversion lemma

$$\hat{P}^{-1} = \bar{P}^{-1} + \bar{P}^{-1}\bar{P}H^T\left(H\bar{P}H^T + R - H\bar{P}\bar{P}^{-1}\bar{P}H^T\right)^{-1}H\bar{P}\bar{P}^{-1}$$

$$= \underline{\bar{P}^{-1} + H^T\left(R^{-1}\right)H}$$