

TTK4250 Sensor Fusion

Assignment 5

Hand in: *Friday 25. October 16.00* on Blackboard (as single PDF-file) or to teaching assistants in the exercise class.

Tasks are to be solved on paper if you are not told otherwise, and you are supposed to show how you got to a particular answer. It is, however, encouraged to use MATLAB, Maple, etc. to verify your answers. Rottmann's mathematical formula collection is allowed at both the exercises and the exam.

Task 1: *Number of association events in JPDA*

Take the data association example in figure 8.1. We are wondering how many data association events there are here. There are 3 targets and 4 measurements, where no target gates all the measurements.

- (a) If we disregard the validation gates, there is a formula for calculating the number of association events in JPDA. If we have N targets that is going to be associated to m out of a total of M measurements ($m \leq \min(N, M)$), it becomes the number of ways to pick m out of N targets unordered and m out of M measurements unordered before assigning these m targets to the m measurements. This number is given by $\binom{N}{m} \binom{M}{m} m! = \frac{N!M!}{m!(N-m)!(M-m)!}$. However, m can be any number between 0 and $\min(N, M)$ so we have to sum over this range to find the total number of association events:

$$\sum_{m=0}^{\min(N, M)} \frac{N!M!}{m!(N-m)!(M-m)!} \quad (1)$$

What is this number for the example in figure 8.1?

- (b) We now include the validation gates. There is no simple formula for the number of events in this case. The simplest option is to enumerate the possibilities and count them. What is the number of association events in figure 8.1 considering the validation gates? How many percent of the non gated events is this? Would you say gating is helpful for data association?
- (c) How many of the events you counted does not have any misdetections?
- (d) If the unassociated measurements could also be new targets, there are 2 possible events for each of these. When m out of M measurements are known to be associated to already established targets, it is therefore also 2^{M-m} events for new targets or false alarms in addition to the possible target to measurement associations. The total number of events when not considering validation gates then becomes

$$\sum_{m=0}^{\min(N, M)} \frac{2^{M-m} N!M!}{m!(N-m)!(M-m)!} \quad (3)$$

What is this number for the example? Is there a lot to save by not having to consider these hypotheses (do not consider the validation gate)?

Task 2: *Finding the most probable associations in JPDA*

You are here to investigate the auction algorithm (Algorithm 5) and Murty's method (Algorithm 6) on

the association problem in figure 8.1. A reward matrix (log likelihood ratio matrix) for this association problem is given in equation (8.21). You can use $\epsilon = 0.01$ for this task.

- (a) Go through the auction algorithm (by hand) for the problem to show that the most likely association is $a(1) = 3$, $a(2) = 1$ and $a(3) = 2$, where $a(t) = j$ indicates that target t is associated to measurement j .
- (b) The last part basically solved lines 2-4 of Murty's method. Go through lines 5-19 of Murty's method to find the second best association. State the solution and reward for each solution. You do not need to go through more than one more run of the auction algorithm as you should be able to see the solutions straight away in this problem.

As there are some minor flaws in Murty's algorithm in the book, we restate how to perform it here.

1. Solve the initial problem and store the problem and results in a list, L .
2. Find the solution with highest reward in L and save it as the next best solution. Call the corresponding problem P and remove it from L .
3. Partition P into two subproblems:
 - (a) one with the first track's measurement assignment in P made impossible (corresponding entry in reward matrix set to $-\infty$). Solve this problem and put it in L if it gives a valid solution (eg. finite reward).
 - (b) one with the first track's measurement assignment in P enforced (corresponding to removing the track and measurement from the problem P (column and row in the reward matrix, respectively)). Set P to be this problem.
4. Repeat the partitioning process (step 3) until it is not possible to partition more. This happens when all tracks have their assignments enforced.
5. Go back to finding the next best solution (step 2) in L until L is empty or N solutions have been found.

In our case, the first partition will make the first subproblem by setting $R(3,1) = -\infty$ and solve that problem, and the second subproblem by removing column 1 and row 3 from R , making R_1 . The second partition will then take this second subproblem with reduced reward matrix size and set $R_1(1,1) = -\infty$ corresponding to setting $R(1,2) = -\infty$ in the original reward matrix, for creating the first subproblem of this problem partitioning. The second subproblem of this partitioning is then created by removing row 1 and column 1 in R_1 to create R_2 , which is the removal of column 1 and 2 and rows 1 and 3 in the original R .

- (c) Say that the best solution to the last problem was found to come from the solution of making the association $a(2) = 1$ impossible. What would the reward matrix look like for the selected problem-solution pair in line 6?

Task 3: *Pure quaternion powers and exponential*

For simplicity, we let the quaternion $[\eta \ \epsilon^T]^T$ be equivalently written as $\eta + \epsilon$ in this task. Here ϵ is of course the imaginary (hyper imaginary, if you prefer) part of the quaternion and η is the real part. All the rules for complex numbers can be shown to hold for quaternions, except commutativity of the imaginary parts in the product of two quaternions. For instance, we have that the quaternion product can be written as $(\eta_1 + \epsilon_1)(\eta_2 + \epsilon_2) = \eta_1\eta_2 + \eta_1\epsilon_2 + \eta_2\epsilon_1 + \epsilon_1\epsilon_2$ using the current notation. The imaginary product, $\epsilon_1\epsilon_2$, written out in terms of vector products gives $\epsilon_1\epsilon_2 = -\epsilon_1^T\epsilon_2 + \epsilon_1 \times \epsilon_2$, where the first term is real while the latter term is imaginary. Since the cross product is anti commutative ($v \times w = -w \times v$), the imaginary product is not commutative unless $\epsilon_1 \times \epsilon_2 = 0$. $v \times w = 0$ only holds when the vectors are parallel or at least one is zero.

We state two handy facts of imaginary numbers (complex numbers with zero real part).

- The powers of the imaginary numbers can be written as

$$\begin{aligned}(\alpha i)^{2n+1} &= \alpha^{2n+1} i^{2n} i = \alpha^{2n+1} (-1)^n i, & (\text{odd powers}) \\ (\alpha i)^{2n} &= \alpha^{2n} i^{2n} = \alpha^{2n} (-1)^n. & (\text{even powers})\end{aligned}$$

- The exponential function relates imaginary numbers to rotations through

$$\begin{aligned}e^{\alpha i} &= \sum_{n=0}^{\infty} \frac{(\alpha i)^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{(\alpha i)^{2n}}{(2n)!} \right] + \left[\sum_{n=0}^{\infty} \frac{(\alpha i)^{2n+1}}{(2n+1)!} \right] = \left[\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} \right] + \left[\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!} \right] i \\ &= \cos(\alpha) + \sin(\alpha)i.\end{aligned}$$

- (a) Show that the powers of a pure quaternion (zero real part), αv with $|v| = 1$, acts like the powers of imaginary numbers, only with v substituted for i . That is, show that $(\alpha v)^{2n+1} = \alpha^{2n+1} (-1)^n v$ and $(\alpha v)^{2n} = \alpha^{2n} (-1)^n$.

Hint: Write out the powers up to 2 and see if you can write the higher powers in terms of these, and note that $q^0 = 1$. Equation (10.21) and (10.34) can be useful.

- (b) Show that similarly to the imaginary numbers, the relationship $e^{\alpha v} = \cos(\alpha) + \sin(\alpha)v$ also holds for a pure quaternion αv , with $|v| = 1$ and α real. Briefly discuss how this relates to theorem 10.1.2.

Task 4: Quaternion kinematics

We are here going to derive equation (10.43) while showing that the first order approximation done in the book is actually valid. (10.43) states that $\dot{q} = \frac{1}{2}q\omega$, where the factor $\frac{1}{2}$ can be attributed to the quaternion only representing half the rotation, whereas ω is the full rotational rate. Similarly to the book we are going to use

$$q(t + \Delta t) = q(t)\Delta q(t, t + \Delta t).$$

However, we are going to use the result of the previous task to state that

$$\Delta q(t, t + \Delta t) = \cos\left(\frac{\Delta\alpha(t, t + \Delta t)}{2}\right) + \sin\left(\frac{\Delta\alpha(t, t + \Delta t)}{2}\right) \Delta v(t, t + \Delta t) = e^{\frac{\Delta\theta(t, t + \Delta t)}{2}}$$

for the pure quaternion $\Delta\theta(t, t + \Delta t) = \Delta\alpha(t, t + \Delta t)\Delta v(t, t + \Delta t)$. Here $\alpha(t, t + \Delta t)$ is the angle of rotation between time t and $t + \Delta t$, and $v(t, t + \Delta t)$ is the pure quaternion of unit length (ie. a unit vector) which specifies the axis this rotation is about. We also use (10.42) in the sense that $\lim_{\Delta t \rightarrow 0} \frac{\Delta\theta(t, t + \Delta t)}{\Delta t} = \omega(t)$.

- (a) What is the interpretation of $\omega(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta(t, t + \Delta t)}{\Delta t} = \omega(t)$, as it is stated here and in equation (10.42) in the book? Specifically mention which coordinate systems it relates and is specified in.
- (b) Show that the quaternion time derivative $\dot{q}(t) = \lim_{\Delta t \rightarrow 0} \frac{q(t + \Delta t) - q(t)}{\Delta t}$ can be written as $q(t) \lim_{\Delta t \rightarrow 0} \frac{e^{\frac{\Delta\theta(t, t + \Delta t)}{2}} - 1}{\Delta t}$, and that it gives equation (10.43).

Hint: The quaternions are distributive ($a(b + c) = ab + bc$). Constants (here a quaternion) in terms of the limiting variable (here Δt) can always be taken outside the limit. Also, using the series representation of the exponential function one can write

$$\begin{aligned}\Delta q(t, t + \Delta t) &= e^{\frac{\Delta\theta(t, t + \Delta t)}{2}} = 1 + \frac{\Delta\theta(t, t + \Delta t)}{2} + \sum_{n=2}^{\infty} \frac{\left(\frac{\Delta\theta(t, t + \Delta t)}{2}\right)^n}{n!} \\ &= 1 + \frac{\Delta\theta(t, t + \Delta t)}{2} + \frac{\Delta\theta(t, t + \Delta t)}{2} \sum_{n=2}^{\infty} \frac{\left(\frac{\Delta\theta(t, t + \Delta t)}{2}\right)^{n-1}}{n!} \\ &= 1 + \frac{\Delta\theta(t, t + \Delta t)}{2} [1 + O(\Delta\theta(t, t + \Delta t))],\end{aligned}$$

where $O(\Delta\theta(t, t + \Delta t)) \xrightarrow{\Delta t \rightarrow 0} 0$.

Task 5: *The error state dynamics*

In this task we are going to take a closer look at the proof of theorem 10.3.1 for the position, velocity and orientation error dynamics.

- (a) Which, if any, approximations are made to arrive at the linearized position error state dynamics $\delta\dot{\rho} = \dot{\rho}_t - \dot{\rho} = \delta v$?
- (b) Derive the linearized velocity error state dynamics $\delta\dot{v} = \dot{v}_t - \dot{v} = -R(q)S(a_m - a_b)\delta\theta - R(q)\delta a_b - R(q)a_n$ by using $R(\delta q) = R(e^{\frac{\delta\theta}{2}}) = e^{S(\delta\theta)}$ and its series. Which, if any, approximations are made?
- (c) Which, if any, approximations are made in order to derive the linearized orientation error state dynamics $\delta\dot{\theta} = -S(\omega_m - \omega_b)\delta\theta - \delta\omega_b - \omega_n$?