TTK4250 - Sensor Fusion Assignment 2

Martin Eek Gerhardsen

11. September 2019

Department of Engineering Cybernetics

Contents

1 Task 1: Bayesian estimation of an existence variable

1.1 a)

Defining $x_k \to$ measure the boat, and $z_k \to$ decide the boat was measured. Also worth noting that the there are two possible choices for the state, measured or not measured. Then:

$$\begin{aligned}
r_{k+1|k} &= p(x_{k+1}|z_{1:k}) = \int p(x_{k+1}, x_k|z_{1:k})dx_k \\
&= \int p(x_{k+1}|x_k)p(x_k|z_{1:k})dx_k \\
&= \sum_{x_k\{boat,no-boat\}} p(x_{k+1}|x_k)p(x_k|z_{1:k}) \\
&= P_S r_k + P_E(1 - r_k) = P_E + (P_S - P_E)r_k
\end{aligned}$$

1.2 b)

Then, using the update step:

$$\begin{aligned}
r_{k+1} &= \frac{p(z_{k+1}|x_{k+1})p(x_{k+1}|z_{1:k})}{p(z_{k+1}|z_{1:k})} \\
&= (P_D + (1 - P_D)P_{FA})r_{k+1|k}
\end{aligned}$$

2 Task 2: KF initialization of CV model without prior knowledge

Defining:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2.1 a)

Defining the CV-model:

$$\begin{aligned} \dot{x} &= Ax + Gn \\ n &\sim \mathcal{N}(0, D\delta(t-\tau)) \end{aligned}$$

With:

$$\begin{aligned} A &= \begin{bmatrix} O & I \\ O & O \end{bmatrix} \qquad G = \begin{bmatrix} O \\ I \end{bmatrix} \\ D &= \sigma_a^2 I \end{aligned}$$

Then we discretize (I'll also simplify things like $t_k - t_{k-1} = T$):

$$\begin{aligned} x_k &= Fx_{k-1} + v_k \\ F &= e^{AT} = I_{4x4} + AT + 0 \\ &= \begin{bmatrix} I & TI \\ O & I \end{bmatrix} \\ v_k &= \int_{t_{k-1}}^{t_k} e^{AT} Gn(\tau) d\tau \end{aligned}$$

For future use, the inverse of F is simply:

$$F^{-1} = \begin{bmatrix} I & -TI \\ O & I \end{bmatrix}$$

Further defining:

$$\begin{aligned} \hat{x}_1 &= \begin{bmatrix} \hat{p}_1 \\ \hat{u}_1 \end{bmatrix} = \begin{bmatrix} K_{p_1} & K_{p_0} \\ K_{u_1} & K_{u_0} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ z_k &= \begin{bmatrix} I & O \end{bmatrix} x_k + w_k = p_k + w_k \\ x_k &= \begin{bmatrix} p_k^\top & u_k^\top \end{bmatrix}^\top \\ w_k &\sim \mathcal{N}(0, R) \end{aligned}$$

Inserting:

$$
\begin{aligned}
z_k &= \begin{bmatrix} I & O \end{bmatrix} x_k + w_k \\
x_1 &= F x_0 + v_0, & x_0 = F^{-1}(x_1 - v_0) \\
z_0 &= \begin{bmatrix} I & O \end{bmatrix} F^{-1}(x_1 - v_0) + w_0 \\
&= \begin{bmatrix} I & -TI \end{bmatrix} (x_1 - v_0) + w_0 \\
&= p_1 - T u_1 - \begin{bmatrix} I & -TI \end{bmatrix} v_0 + w_0 \\
z_1 &= \begin{bmatrix} I & O \end{bmatrix} x_1 + w_1 \\
&= p_1 + w_1
\end{aligned}
$$

Note that z_0 is the position at $k = 1$ minus the timestep multiplied with the velocity, or the distance travelled in that timestep.

2.2 b)

Simplifying the estimate as:

$$
\begin{aligned}
\hat{x}_1 &= \begin{bmatrix} K_{p_1} & K_{p_0} \\ K_{u_1} & K_{u_0} \end{bmatrix} \begin{bmatrix} z_1 \\ z_0 \end{bmatrix} && (1) \\
&= \begin{bmatrix} K_{p_1} & K_{p_0} \\ K_{u_1} & K_{u_0} \end{bmatrix} \begin{bmatrix} p_1 + w_1 \\ p_1 - T u_1 - \begin{bmatrix} I & -TI \end{bmatrix} v_0 + w_0 \end{bmatrix} && (2)
\end{aligned}
$$

Using the fact that finding the expected value can be applied linearly, we simplify away all noise (as their expected value is assumed zero):

$$
\begin{aligned}
E[\hat{x}_1] &= \begin{bmatrix} E[K_{p_1} p_1 + K_{p_0}(p_1 - T u_1)] \\ E[K_{u_1} p_1 + K_{u_0}(p_1 - T u_1)] \end{bmatrix} \\
&= \begin{bmatrix} (K_{p_1} + K_{p_0}) p_1 - T K_{p_0} u_1 \\ (K_{u_1} + K_{u_0}) p_1 - T K_{u_0} u_1 \end{bmatrix} = \begin{bmatrix} p_1 \\ u_1 \end{bmatrix}
\end{aligned}
$$

Therefore, we may conclude that:

$$
\begin{aligned}
K_{p_1} &= I_2 & K_{p_0} &= O_2 \\
K_{u_1} &= \frac{1}{T} I_2 & K_{u_0} &= -\frac{1}{T} I_2
\end{aligned}
$$

or

$$
K = \begin{bmatrix} I_2 & O_2 \\ \frac{1}{T} I_2 & -\frac{1}{T} I_2 \end{bmatrix} \tag{3}
$$

will give an unbiased estimate.

2.3 c)

Finding the Q matrix as defined in Theorem 4.5.1 in the textbook, [1, page 60]:

$$
\begin{aligned}
Q = E[v_k v_k^\top] &= \int_0^T e^{(T-\tau)A} G D G^\top e^{(T-\tau)A^\top} \, d\tau \\
&= \int_0^T \begin{bmatrix} I & (T-\tau)I \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} \sigma_a^2 I \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ (T-\tau)I & I \end{bmatrix} d\tau \\
&= \int_0^T \begin{bmatrix} (T-\tau)I \\ I \end{bmatrix} \sigma_a^2 I \begin{bmatrix} (T-\tau)I & I \end{bmatrix} d\tau \\
&= \int_0^T \sigma_a^2 \begin{bmatrix} (\tau^2 - 2T\tau + T^2)I & (T-\tau)I \\ (T-\tau)I & I \end{bmatrix} d\tau \\
&= \sigma_a^a \begin{bmatrix} \frac{T^3}{3} I & \frac{T^2}{2} I \\ \frac{T^2}{2} I & TI \end{bmatrix}
\end{aligned}
$$

Then simplifying eq. (2) by removing the constants and inserting eq. (3), and using the rules for linear combinations of covariance matrices we find:

$$
\begin{aligned}
Var[\hat{x}_1] &= \begin{bmatrix} Var[w_1] & Cov(z_0, z_1) \\ Cov(z_0, z_1) & Var[\frac{1}{T} w_1 - \frac{1}{T} w_0 + u_1 + \frac{1}{T} \begin{bmatrix} I & -TI \end{bmatrix} v_0] \end{bmatrix} \\
&= \begin{bmatrix} R & Cov(z_0, z_1) \\ Cov(z_0, z_1) & \frac{2}{T^2} R + \frac{1}{T^2} \begin{bmatrix} I & -TI \end{bmatrix} Q \begin{bmatrix} I \\ -TI \end{bmatrix} \end{bmatrix}
\end{aligned}
$$

Calculating the result from indroducing the Q matrix, as calculated above:

$$
\begin{aligned}
\frac{1}{T^2} \begin{bmatrix} I & -TI \end{bmatrix} Q \begin{bmatrix} I \\ -TI \end{bmatrix} &= \frac{\sigma_a^2}{T^2} \begin{bmatrix} I & -TI \end{bmatrix} \begin{bmatrix} \frac{T^3}{3} I & \frac{T^2}{2} I \\ \frac{T^2}{2} I & TI \end{bmatrix} \begin{bmatrix} I \\ -TI \end{bmatrix} \\
&= \frac{\sigma_a^2}{T^2} \begin{bmatrix} I & -TI \end{bmatrix} \begin{bmatrix} \frac{T^3}{3} I - \frac{T^3}{2} I \\ \frac{T^2}{2} I - T^2 I \end{bmatrix} \\
&= \frac{\sigma_a^2}{T^2} (-\frac{T^3}{6} + \frac{T^3}{2} I) = \frac{\sigma_a^2 T}{3} I
\end{aligned}
$$

Noting that $Cov(a, b) = 0$ for $a, b \in \{w_k, v_k\}$, we calculate $Con(z_0, z_1)$:

$$
\begin{aligned}
Con(z_0, z_1) &= Con(z_1, z_0) = E[(z_0 - E[z_0])(z_1 - E[z_1])] \\
&= E[(p_1 + w_1 - E[p_1 + w_1]) \\
&(\frac{1}{T}w_1 - \frac{1}{T}w_0 + u_1 + \frac{1}{T}\begin{bmatrix} I & -TI \end{bmatrix} v_0 \\
&- E[\frac{1}{T}w_1 - \frac{1}{T}w_0 + u_1 + \frac{1}{T}\begin{bmatrix} I & -TI \end{bmatrix} v_0])] \\
&= E[w_1(\frac{1}{T}w_1 - \frac{1}{T}w_0 + \frac{1}{T}\begin{bmatrix} I & -TI \end{bmatrix} v_0)] \\
&= \frac{1}{T}E[w_1^2] - \frac{1}{T}[w_0 w_1] + \frac{1}{T}\begin{bmatrix} I & -TI \end{bmatrix} E[v_0 w_1] \\
&= \frac{1}{T}(Var[w_1] - E[w_1]^2) - \frac{1}{T}(Cov(w_0, w_1) + E[w_0]E[w_1]) \\
&+ \frac{1}{T}\begin{bmatrix} I & -TI \end{bmatrix} (Cov(v_0, w_1) + E[v_0]E[w_1]) \\
&= \frac{1}{T}R
\end{aligned}
$$

Then, we may find the covariance matrix of the estimate as:

$$
\begin{bmatrix} R & \frac{1}{T}R \\ \frac{1}{T}R & \frac{2}{T^2}R + \frac{\sigma_a^2 T}{3} I \end{bmatrix}
$$

2.4 d)

Solving for x_1

$$
\begin{aligned}
\hat{x}_1 &= \begin{bmatrix} I_2 & 0_2 \\ \frac{1}{T}I_2 & -\frac{1}{T}I_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_0 \end{bmatrix} = \begin{bmatrix} p_1 + w_1 \\ p_1 - Tu_1 - \begin{bmatrix} I_2 & -TI_2 \end{bmatrix} v_0 + w_0 \end{bmatrix} \\
&= \begin{bmatrix} I_2 & O_2 \\ I_2 & -TI_2 \end{bmatrix} x_1 + \begin{bmatrix} I_2 \\ O_2 \end{bmatrix} w_1 + \begin{bmatrix} O_2 \\ I_2 \end{bmatrix} w_0 - \begin{bmatrix} I_2 & O_2 \\ I_2 & -TI_2 \end{bmatrix} v_0 \\
&\begin{bmatrix} I_2 & O_2 \\ I_2 & -TI_2 \end{bmatrix}^{-1} = \begin{bmatrix} I_2 & O_2 \\ \frac{1}{T}I_2 & -\frac{1}{T}I_2 \end{bmatrix} \\
x_1 &= \begin{bmatrix} I_2 & O_2 \\ \frac{1}{T}I_2 & -\frac{1}{T}I_2 \end{bmatrix} \hat{x}_1 - \begin{bmatrix} I_2 & O_2 \\ \frac{1}{T}I_2 & -\frac{1}{T}I_2 \end{bmatrix} \begin{bmatrix} I_2 \\ O_2 \end{bmatrix} w_1 - \begin{bmatrix} I_2 & O_2 \\ \frac{1}{T}I_2 & -\frac{1}{T}I_2 \end{bmatrix} \begin{bmatrix} O_2 \\ I_2 \end{bmatrix} w_0 \\
&+ \begin{bmatrix} I_2 & O_2 \\ \frac{1}{T}I_2 & -\frac{1}{T}I_2 \end{bmatrix} \begin{bmatrix} I_2 & O_2 \\ I_2 & -TI_2 \end{bmatrix} v_0 \\
&= \begin{bmatrix} I_2 & O_2 \\ \frac{1}{T}I_2 & -\frac{1}{T}I_2 \end{bmatrix} \hat{x}_1 - \begin{bmatrix} I_2 \\ \frac{1}{T}I_2 \end{bmatrix} w_1 + \begin{bmatrix} O_2 \\ \frac{1}{T}I_2 \end{bmatrix} w_0 + \begin{bmatrix} O_2 & O_2 \\ -\frac{1}{T}I_2 & I_2 \end{bmatrix} v_0
\end{aligned}
$$

Knowing that a sum of gaussians is also gaussian, we may conclude that x_1 also is gaussian. Then, we can find:

$$E[x_1] = \begin{bmatrix} I_2 & O_2 \\ \frac{1}{T}I_2 & -\frac{1}{T}I_2 \end{bmatrix} \hat{x}_1$$

$$Var[x_1] = \begin{bmatrix} I_2 \\ \frac{1}{T}I_2 \end{bmatrix} R \begin{bmatrix} I_2 & \frac{1}{T} \end{bmatrix} + \begin{bmatrix} O_2 \\ \frac{1}{T} \end{bmatrix} R \begin{bmatrix} O_2 & \frac{1}{T} \end{bmatrix} + \begin{bmatrix} O_2 & O_2 \\ -\frac{1}{T}I_2 & I_2 \end{bmatrix} Q \begin{bmatrix} O_2 & -\frac{1}{T}I_2 \\ O_2 & I_2 \end{bmatrix}$$

Then, calculating the products:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{T} & 0 \\ 0 & \frac{1}{T} \end{bmatrix} \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{T} & 0 \\ 0 & 1 & 0 & \frac{1}{T} \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \\ \frac{1}{T}r_1 & \frac{1}{T}r_2 \\ \frac{1}{T}r_3 & \frac{1}{T}r_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{T} & 0 \\ 0 & 1 & 0 & \frac{1}{T} \end{bmatrix}$$

$$= \begin{bmatrix} r_1 & r_2 & \frac{1}{T}r_1 & \frac{1}{T}r_2 \\ r_3 & r_4 & \frac{1}{T}r_3 & \frac{1}{T}r_4 \\ \frac{1}{T}r_1 & \frac{1}{T}r_2 & \frac{1}{T^2}r_1 & \frac{1}{T^2}r_2 \\ \frac{1}{T}r_3 & \frac{1}{T}r_4 & \frac{1}{T^2}r_3 & \frac{1}{T^2}r_4 \end{bmatrix} = \begin{bmatrix} R & \frac{1}{T}R \\ \frac{1}{T}R & \frac{1}{T^2}R \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{T} & 0 \\ 0 & \frac{1}{T} \end{bmatrix} \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{T} & 0 \\ 0 & 0 & 0 & \frac{1}{T} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{T}r_1 & \frac{1}{T}r_2 \\ \frac{1}{T}r_3 & \frac{1}{T}r_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{T} & 0 \\ 0 & 0 & 0 & \frac{1}{T} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{T^2}r_1 & \frac{1}{T^2}r_2 \\ 0 & 0 & \frac{1}{T^2}r_3 & \frac{1}{T^2}r_4 \end{bmatrix} = \begin{bmatrix} O_2 & O_2 \\ O_2 & \frac{1}{T^2}R \end{bmatrix}$$

$$\sigma_a^2 \begin{bmatrix} O_2 & O_2 \\ -\frac{1}{T}I_2 & I_2 \end{bmatrix} \begin{bmatrix} \frac{T^3}{3}I_2 & \frac{T^2}{2}I_2 \\ \frac{T^2}{2}I_2 & TI_2 \end{bmatrix} \begin{bmatrix} O_2 & -\frac{1}{T}I_2 \\ O_2 & I_2 \end{bmatrix} = \sigma_a^2 \begin{bmatrix} O_2 & O_2 \\ -\frac{1}{T}I_2 & I_2 \end{bmatrix} \begin{bmatrix} O_2 & T^2(\frac{1}{2} - \frac{1}{3})I_2 \\ O_2 & T(1 - \frac{1}{2})I_2 \end{bmatrix}$$

$$= \begin{bmatrix} O_2 & O_2 \\ O_2 & T(\frac{1}{2} - \frac{1}{6})I_2 \end{bmatrix} = \begin{bmatrix} O_2 & O_2 \\ O_2 & \frac{T}{3}I_2 \end{bmatrix}$$

Such that:

$$Var[x_1] = \begin{bmatrix} R & \frac{1}{T}R \\ \frac{1}{T}R & \frac{1}{T^2}R \end{bmatrix} + \begin{bmatrix} O_2 & O_2 \\ O_2 & \frac{1}{T^2}R \end{bmatrix} + \begin{bmatrix} O_2 & O_2 \\ O_2 & \frac{T}{3}I_2 \end{bmatrix}$$

$$= \begin{bmatrix} R & \frac{1}{T}R \\ \frac{1}{T}R & \frac{2}{T^2}R + \frac{T}{3}I_2 \end{bmatrix}$$

And to sum up, the true state is distributed as a gaussian, with expected value $E[x_1]$ and covariance matrix $Var[x_1]$ as stated above.

2.5 e)

3 Task 3: Implement EKF in MATLAB

This task was implemented according to **Algorithm 2** The extended Kalman filter, as stated in the textbook, [1, page 73]. The code is added with the report.

4 Task 4: Make CV model to use with the EKF class

5 Task 5: Tuning of KF

6 Task 6: Implement a SIR particle filter for a pendulum

6.1 a)

The filter is not really preforming too well, but the primary reason here is that the filter is missing which side of $\theta = 0$ the pendulum is at. This is obviously due to the fact that the measurement device is placed directly bellow this point, meaninig that there is no difference in the measurement wether the pendulum is on one side or the other.

The code is added

6.2 b)

Placing the measurement device further to the left meant that the filter predicted the values much better. This makes sense, as there is now a measurable difference between $\theta < 0$ and $\theta > 0$.

For fun I also calculated and recorded the degeneracy of the filter over time, and it can be noted that degeneracy quickly became a problem several times.

6.3 c)

References

[1] Edmund Brekke. *Fundamentals of Sensor Fusion*. 2019.