Galilean Laws of Motion

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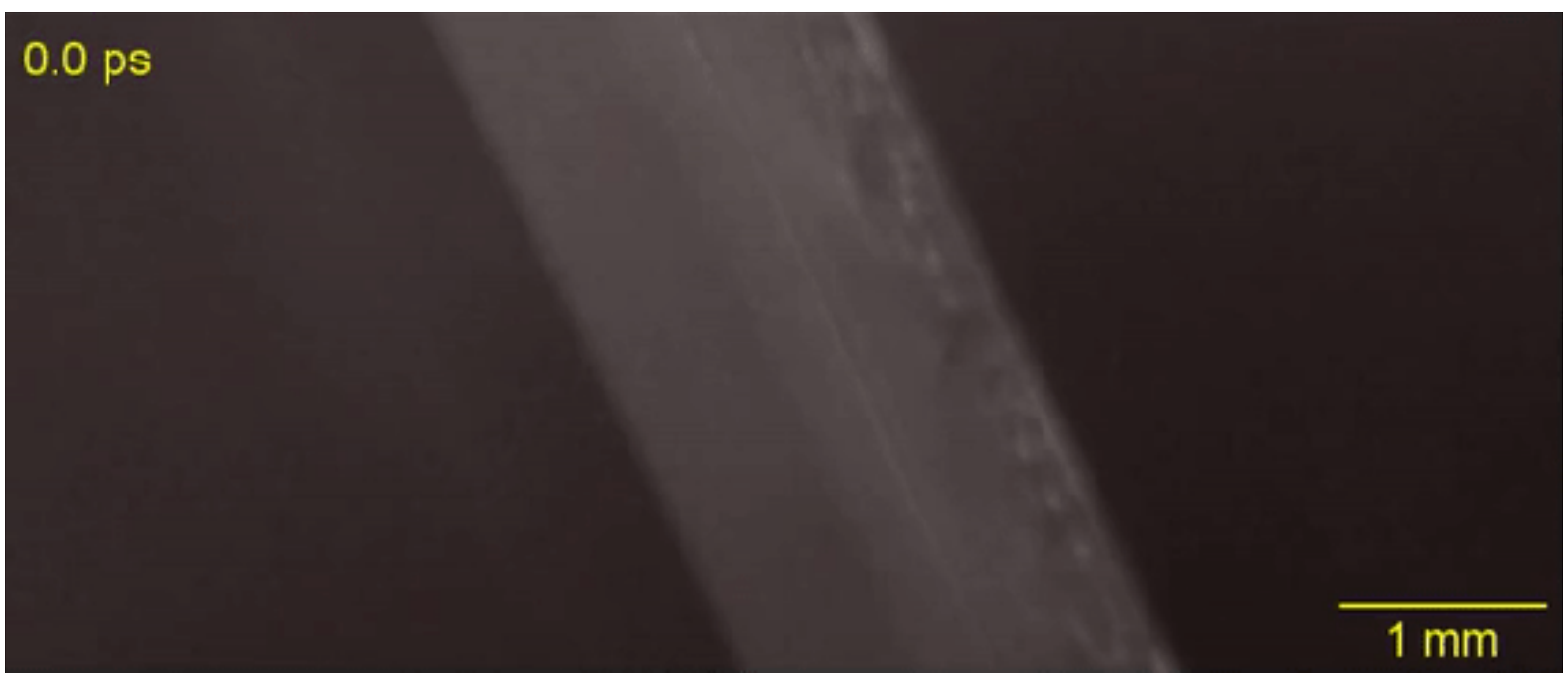
Motion

- What is motion?
- What causes motion?
- How do you describe it?



Motion is the change of **position** of an **object** with **time**

Motion

Motion is the change of **position** of an **object** with **time**

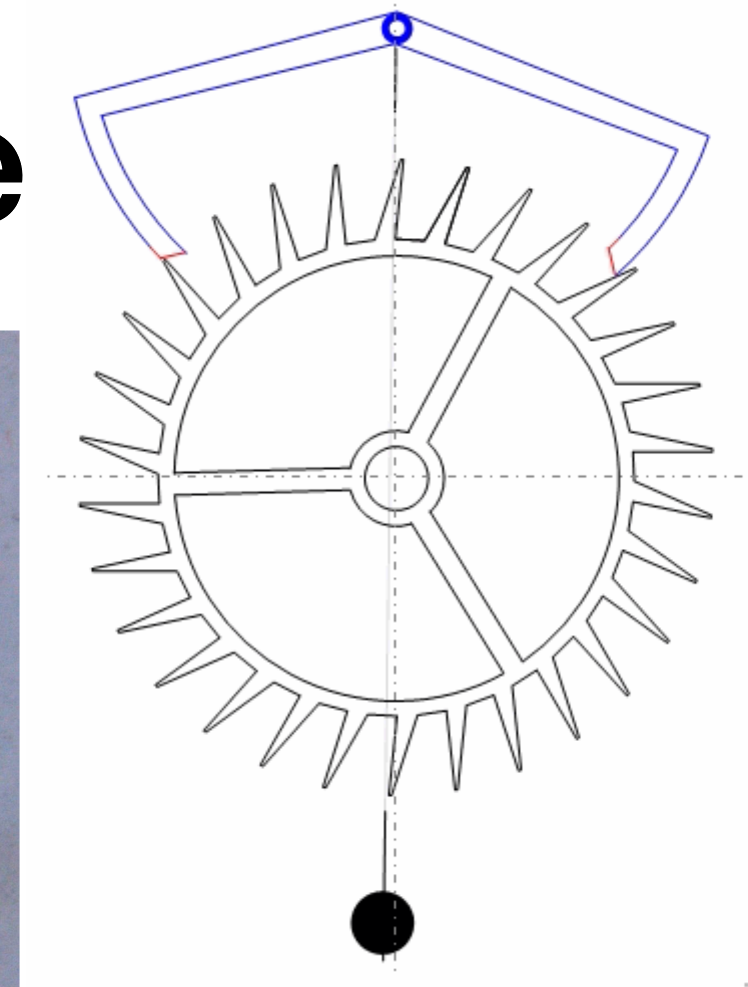
- **Object** - (Everything is made up of particles!!!)
  — Collection of atoms

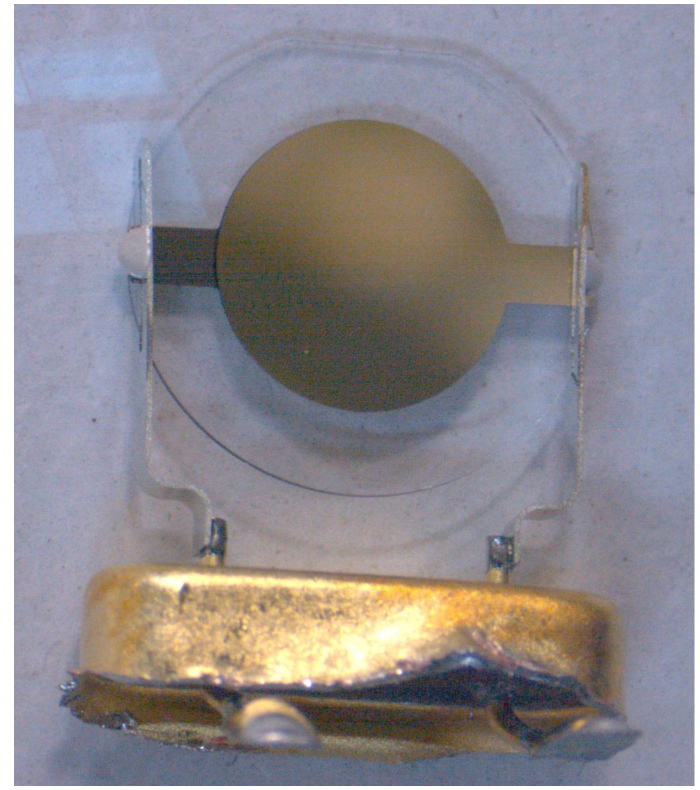
- **Position?**

- **Time?**

Our Experience of Time

- One dimensional
- Universal — the same for all observers
- Independent of space
- Continuous
- Homogeneous
- Isotropic





Quartz Crystal

Our Experience of Space

- Three dimensional
- Independent of time
- Continuous
- Homogeneous
- Isotropic

Galilean Space-Time Assumptions

There exists a class of observers called **inertial observers** such that any two inertial observers see that

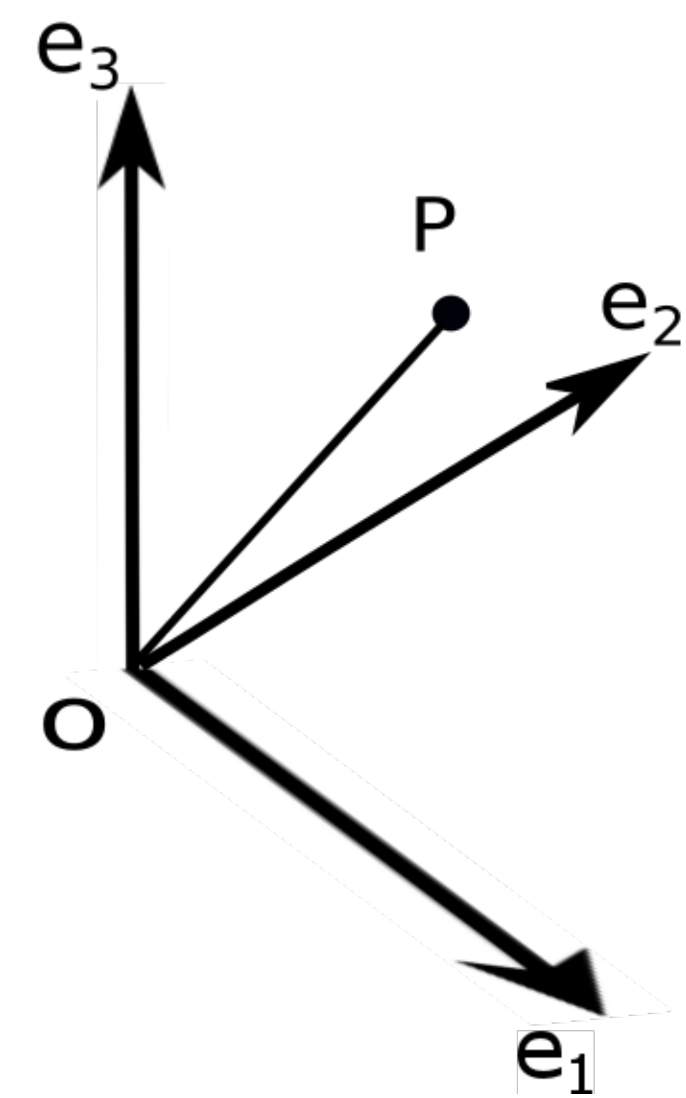
- Time is **independent of space**,
- Time is one dimensional, continuous, isotropic, homogeneous, and the difference in time between any two particular ‘instants’ is the same,
- Space is three dimensional, continuous, isotropic, homogeneous, and the distance between any two particular ‘points’ in space is the same.

Implications of the Galilean Space-Time Assumptions

- There exists a **Universal Clock** (accurate only up to an additive constant)

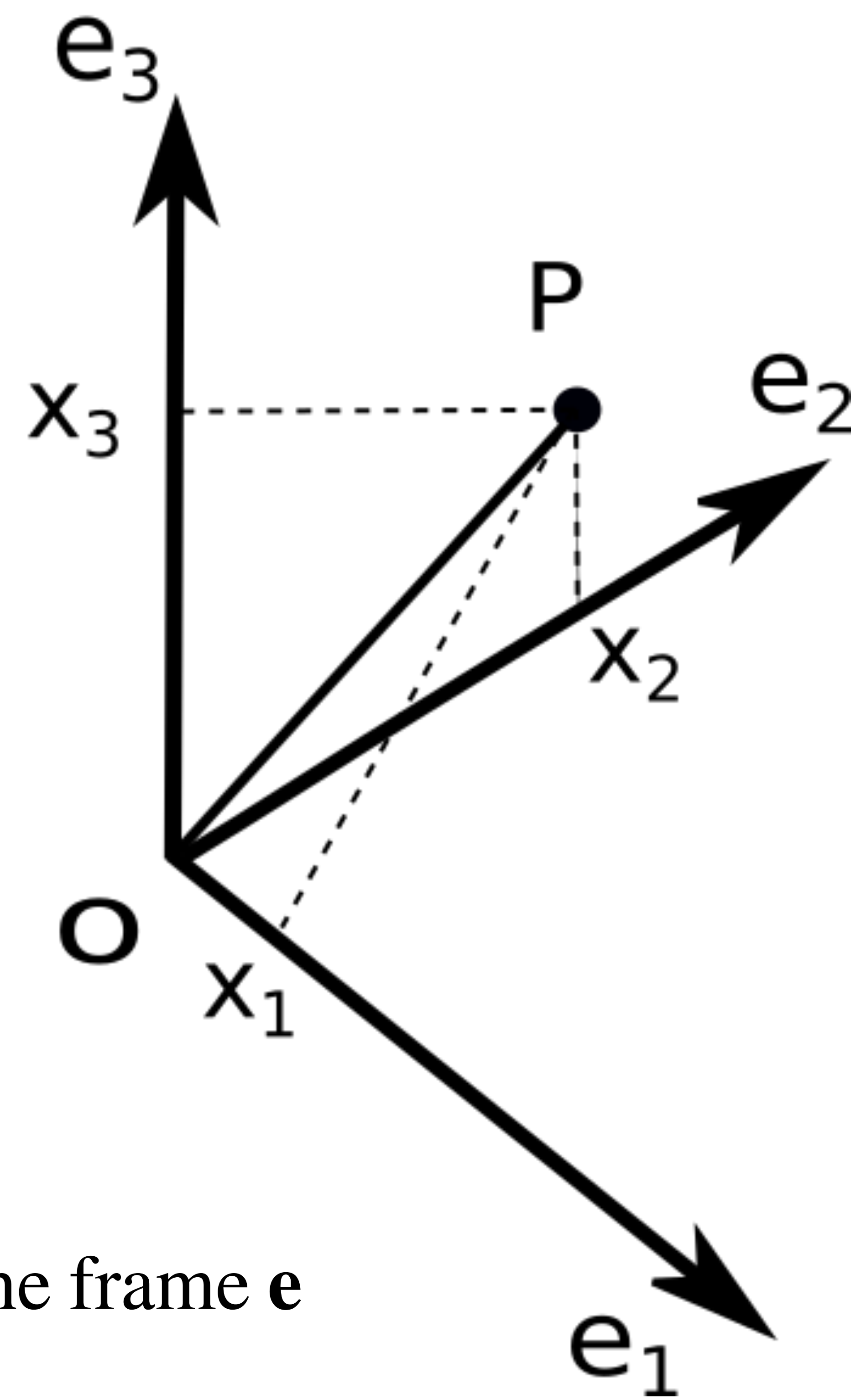
- Distance, Straight lines, Parallelism, Perpendicularity are universal notions - **Space is Euclidean**

    - Allows us to freely pick **orthonormal frames**



Representation of Points

$$
\begin{aligned}
OP &\triangleq x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \\
&= \underbrace{[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]}_{\mathbf{e}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} \\
&= \mathbf{e}\, x\,.
\end{aligned}
$$



$x \quad - \quad$ euclidean representation of P in the frame $\mathbf{e}$

Euclidean Geometry

$$OP \longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longleftrightarrow (x_1, x_2, x_3) \in \mathbb{R}^3 \qquad OQ \longleftrightarrow y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \longleftrightarrow (y_1, y_2, y_3) \in \mathbb{R}^3$$

$$\text{inner product in } \mathbb{R}^3 \ \rightarrow << x\,, y >> \triangleq x_1y_1 + x_2y_2 + x_3y_3$$

$$\text{norm in } \mathbb{R}^3 \ \rightarrow ||x|| \triangleq \sqrt{<< x\,, x >>} \triangleq \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\underbrace{d(P, Q)}_{\text{Distance Between } P \ \& \ Q} \qquad \triangleq ||y - x|| = \sqrt{<< (y - x)\,, (y - x) >>} = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2}$$

$$\text{Angle Between } OP \ \& \ OQ \rightarrow \theta \triangleq \cos^{-1}\left(\frac{<< x\,, y >>}{||x|| \cdot ||y||} \right)$$

Relationship Between Inertial Observers

Let $\mathbf{e}$ and $\mathbf{e}'$ two inertial observers

Representation of the space time event $\mathscr{A}$ by $\mathbf{e} \rightarrow (t, x) \in \mathbb{R} \times \mathbb{R}^3$

Representation of the space time event $\mathscr{A}$ by $\mathbf{e}' \rightarrow (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^3$

How are the two coordinates (t, x) & (τ, ξ) related to each other?

Space Time Assumptions $\longrightarrow$
$$\tau = T + t$$
$$\xi = a + R(x - v\,t)$$

$T \in \mathbb{R}, a \in \mathbb{R}^3\, R \in SO(3)$ are constant

$$\xrightarrow{T = 0,\, a = 0,\, R = I_{3\times3}} \quad \begin{aligned} \tau &= t \\ \xi &= x - v\,t \end{aligned}$$

Quick Summary

Galilean Space Time implies that there exists a special class of observers called inertial observers who see that

- time is a universal quantity
- a special class of spatial coordinates called Euclidean coordinates for 3D-space exists such that the distance between any two points in space is given by the Euclidean norm in an inertial observer independent manner.
- any two such observers must necessarily be moving with constant relative velocity with respect to each other without rotation.

Inertial Observer $\mathbf{e} \to (t, x)$

Inertial Observer $\mathbf{e}' \to (\tau, \xi)$

$$\boxed{\begin{aligned} \tau &= t \\ \xi &= x - v\,t \end{aligned}}$$

Description of Motion

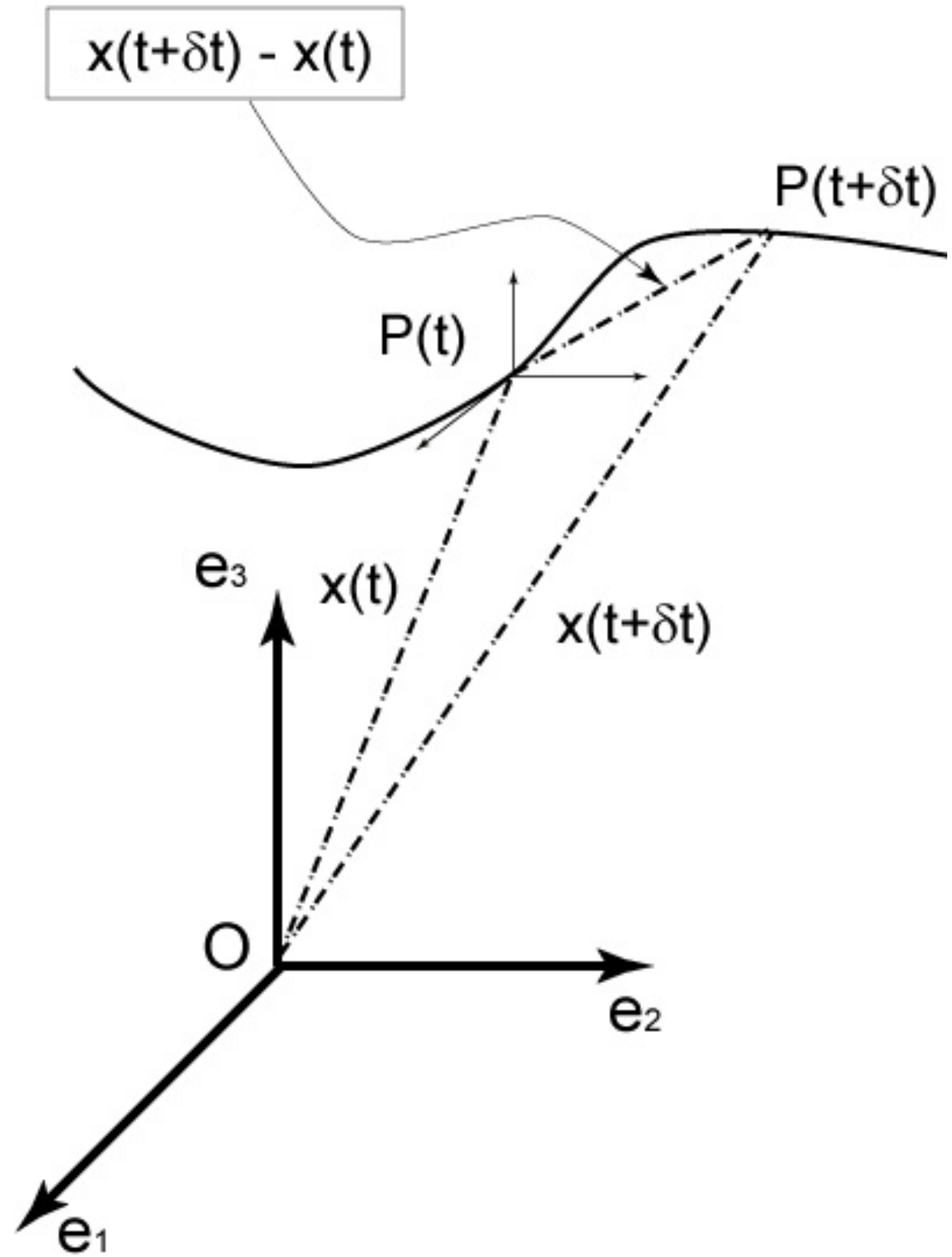
Velocity

$$\dot{x}(t) \triangleq \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$v \triangleq ||\dot{x}|| = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}$$

Acceleration

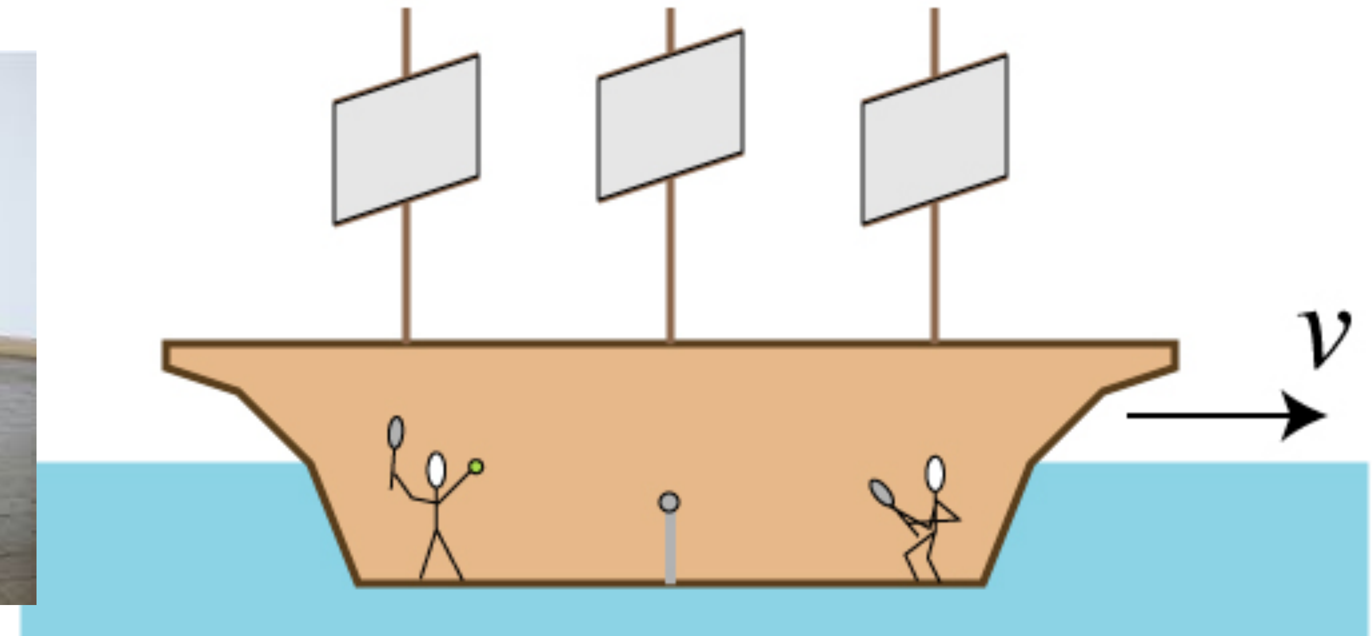
$$\ddot{x}(t) \triangleq \lim_{\delta t \to 0} \frac{\dot{x}(t+\delta t) - \dot{x}(t)}{\delta t} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}$$

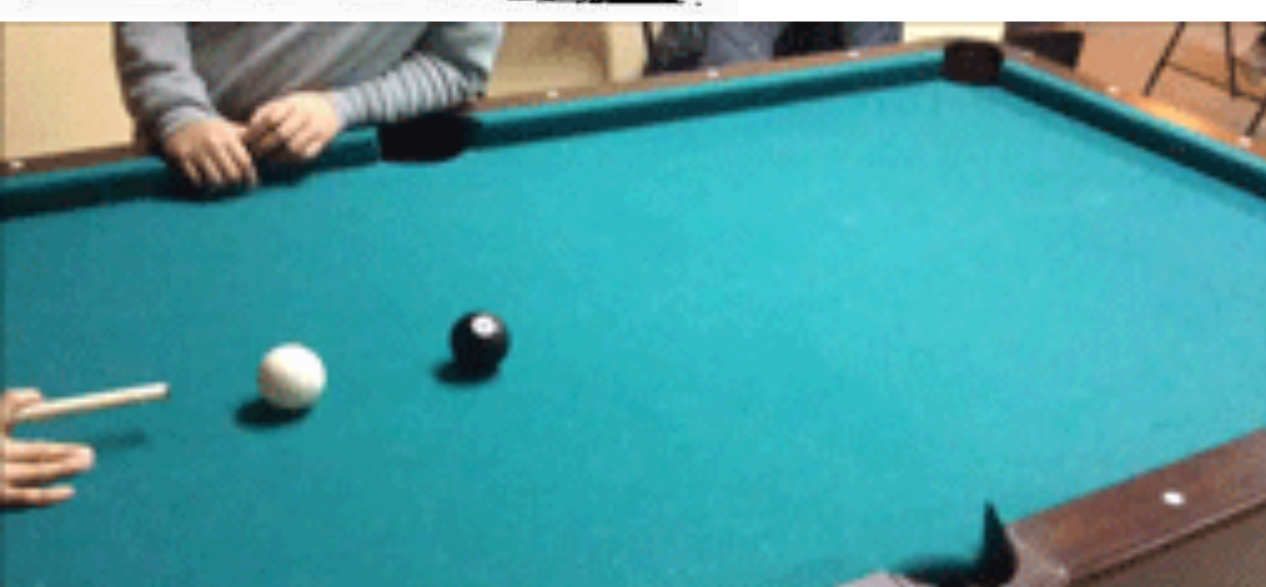


Position and velocity are not inertial observer invariant but acceleration is !

Laws Governing Motion

Must be the same for all inertial observers !





Galilean Laws of Mechanics

- Everything is made of particles
- Each particle has a unique property called mass that is the same for all inertial observers

Inertial Observers

$$\tau = t$$
$$\xi = x - v\,t$$

All particles in the Universe interact with each other in such a way that the total sum of linear momentum of all the particles remains constant as observed in any inertial frame.

$$\sum_i p_i = \text{constant}$$

Linear Momentum

$$p \triangleq m\dot{x}$$

Is not an inertial observer invariant quantity

Consequence of Conservation of Linear Momentum

$$\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} m_i \dot{x}_i = \text{constant} \longleftarrow$$

The constant is not an inertial observer invariant quantity

$$\sum_{i=1}^{N} \dot{p}_i = \sum_{i=1}^{N} m_i \ddot{x}_i = 0 \qquad \longleftarrow$$

All inertial observers agree that this quantity is zero

$$\Rightarrow \quad m_i \ddot{x}_i = -\sum_{j \neq i}^{N} m_j \ddot{x}_j \triangleq f_i$$

$f_i \triangleq$ force acting on the i^{th} particle $\longrightarrow$ Newton's 2^{nd} Law

$$N = 1 \quad \longrightarrow \quad m_i \ddot{x}_i = 0 \longrightarrow \text{Newton's } 1^{\text{st}} \text{ Law}$$

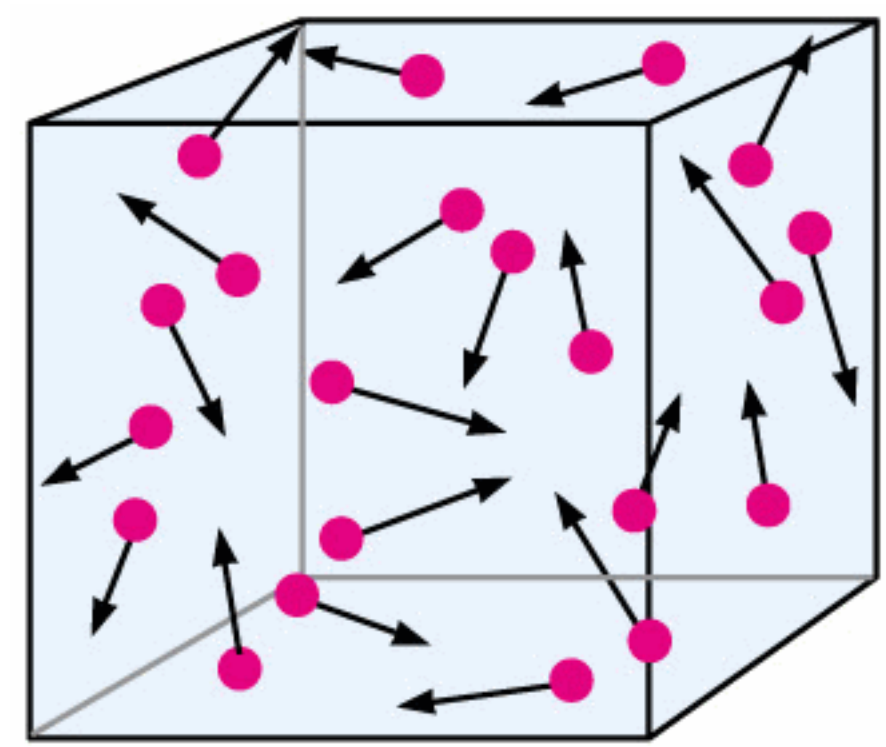
$$N = 2 \quad \longrightarrow \quad m_i \ddot{x}_i = -m_j \ddot{x}_j \longrightarrow \text{Newton's } 3^{\text{rd}} \text{ Law}$$

Force

- Arises **ONLY** due to particle interactions
- Force is an inertial observer invariant property
- There are only four types of known forces (interactions)
    - Gravity, Electromagnetic, Weak, Strong nuclear

$$m\ddot{x} = \dot{p} = f$$

Consequence of Conservation of Linear Momentum



$$\bar{x} \triangleq \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \longrightarrow \dot{\bar{x}} \triangleq \frac{\sum_{i=1}^{n} m_i \dot{x}_i}{\sum_{i=1}^{n} m_i} = \text{constant}$$

All inertial observers agree that the center of mass of an isolated but mutually interacting set of particles is moving at a constant speed !

Water Balls in Space

$$M\ddot{\ddot{x}} = 0$$

Water Balls in Space

$$M\ddot{x} = 0$$

Linear Momentum of a Set of Particles

Consider a set of externally and internally interacting particles

$$\dot{p}_i = f_i = f_i^e + \sum_{j \neq i}^{n} f_{ij}$$

Particle interactions are equal and opposite $\Rightarrow$ $f_{ji} = -f_{ij}$

$$\dot{p} = \sum_{i=1}^{n} \dot{p}_i = \sum_{i=1}^{n} f_i^e + \sum_{i=1}^{n} \sum_{j \neq i}^{n} f_{ij} = \sum_{i=1}^{n} f_i^e \triangleq f^e$$

f^e is the resultant of the external forces

$$\bar{x} \triangleq \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad M \triangleq \sum_{i=1}^{n} m_i \quad \Rightarrow \quad M\dot{\bar{x}} = p \quad \Rightarrow \quad M\ddot{\bar{x}} = f^e$$

Angular Momentum

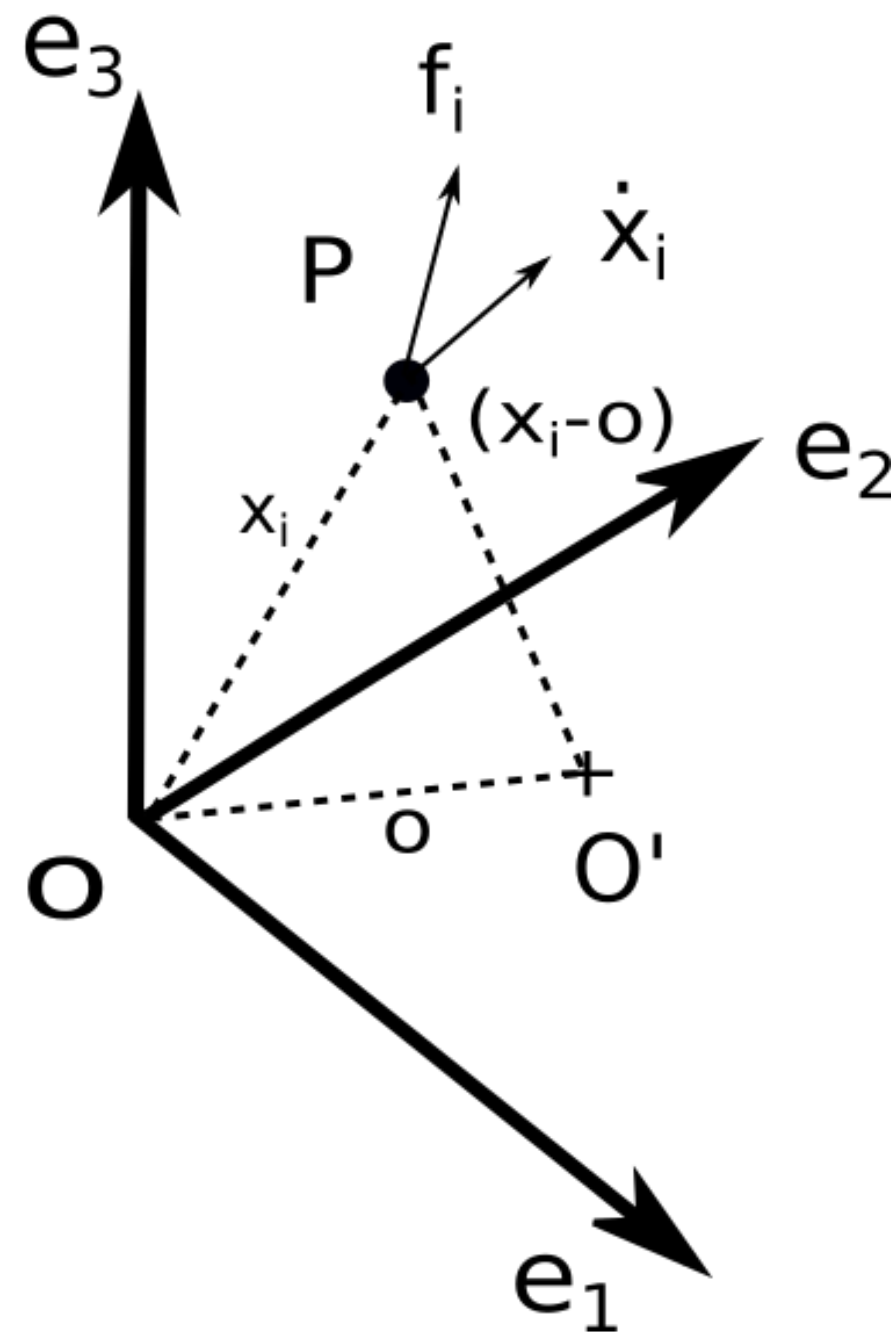
$$\pi_i \triangleq (x_i - o) \times p_i = (x_i - o) \times (m_i \dot{x}_i)$$

$$\dot{\pi}_i = -\dot{o} \times m_i \dot{x}_i + (x_i - o) \times f_i$$

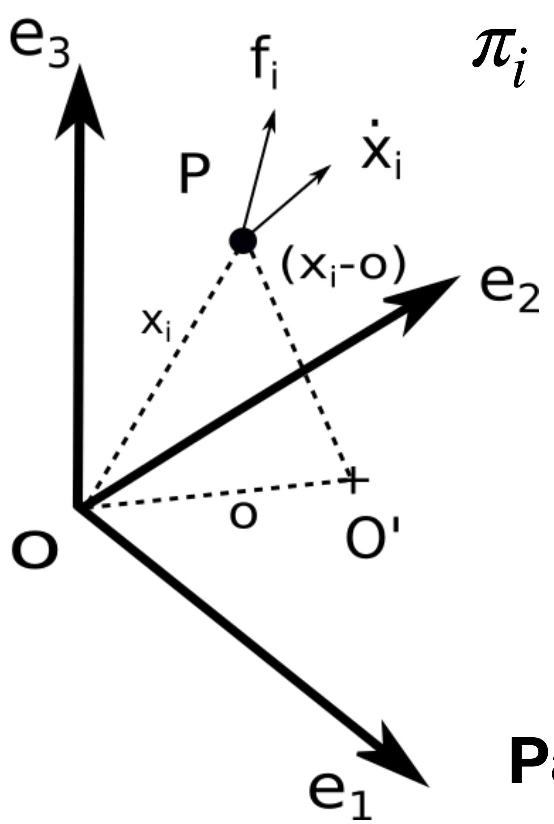
$$\tau_i \triangleq (x_i - o) \times f_i$$

$\tau_i =$ Moment of the force f_i about O′

If O′ is O $\longrightarrow \dot{\pi}_i = \tau_i$



Angular Momentum of a Set of Particles



$$\pi_i \triangleq (x_i - o) \times m_i \dot{x}_i \qquad \dot{\pi}_i = -\dot{o} \times m_i \dot{x}_i + \tau_i \qquad \tau_i \triangleq (x_i - o) \times f_i$$

$$\dot{\pi} = \sum_{i=1}^{n} \dot{\pi}_i = -\dot{o} \times \sum_{i=1}^{n} m_i \dot{x}_i + \sum_{i=1}^{n} (x_i - o) \times f_i$$

$$\sum_{i=1}^{n} (x_i - o) \times f_i = \sum_{i=1}^{n} \sum_{j \neq i}^{n} (x_i - o) \times f_{ij} + \sum_{i=1}^{n} (x_i - o) \times f_i^e$$

Particle interactions are equal and opposite ➡ $f_{ji} = -f_{ij}$

Assumption: Pairwise Particle Interactions Lie along the Line Joining the Particles

$$\sum_{i=1}^{n} \sum_{j \neq i}^{n} (x_i - o) \times f_{ij} = 0$$

$$\dot{\pi} = -\dot{o} \times \sum_{i=1}^{n} m_i \dot{x}_i + \tau_e = -M\dot{o} \times \dot{\bar{x}} + \tau^e .$$

$\tau^e = \sum_{i=1}^{n} (x_i - o) \times f_i^e$ is the resultant moment of the external forces

O is the center of mass ➡ $\dot{\pi} = \tau^e$

Motion of a Set of Particles

$$\boxed{\begin{aligned} M &\triangleq \sum_{i=1}^{n} m_i \\ \bar{x} &\triangleq \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \end{aligned}}$$

$$f^e \triangleq \sum_{i=1}^{n} f_i^e \qquad \tau^e \triangleq \sum_{i=1}^{n} (x_i - o) \times f_i^e$$

$$p \triangleq \sum_{i=1}^{n} m_i \dot{x}_i \qquad \pi \triangleq \sum_{i=1}^{n} (x_i - o) \times m_i \dot{x}_i$$

$\dot{p} = f^e$ $\Rightarrow$ **Rate of change of total linear momentum is equal to the resultant of the external forces**

$\Downarrow$

$M\ddot{\bar{x}} = f^e$ $\Rightarrow$ **The center of mass moves like a particle of mass M under the influence of the resultant external forces**

$\dot{\pi} = \tau_e$ $\Rightarrow$ **Rate of change of total angular momentum about the center of mass is equal to the resultant of the moments due to the external forces**

Conservation of Momentum

$$p \triangleq \sum_{i=1}^{n} m_i \dot{x}_i \qquad \pi \triangleq \sum_{i=1}^{n} (x_i - o) \times m_i \dot{x}_i$$

$$f^e \triangleq \sum_{i=1}^{n} f_i^e \qquad \tau^e \triangleq \sum_{i=1}^{n} (x_i - o) \times f_i^e$$

For a set of isolated but mutually interacting set of particle

$$f^e = 0 \qquad \tau^e = 0$$

$\dot{p} = f^e = 0$ → **Re-statement of conservation of the empirical fact that all inertial observers agree that total linear momentum of the Universe is conserved**

→ **Particle interactions are equal and opposite** → $f_{ji} = -f_{ij}$

Assumption: Pairwise Particle Interactions Lie along the Line Joining the Particles

→ $\dot{\pi} = \tau_e = 0$ **Thus every inertial observer will agree that total angular momentum of the Universe is conserved!**

Kinetic Energy

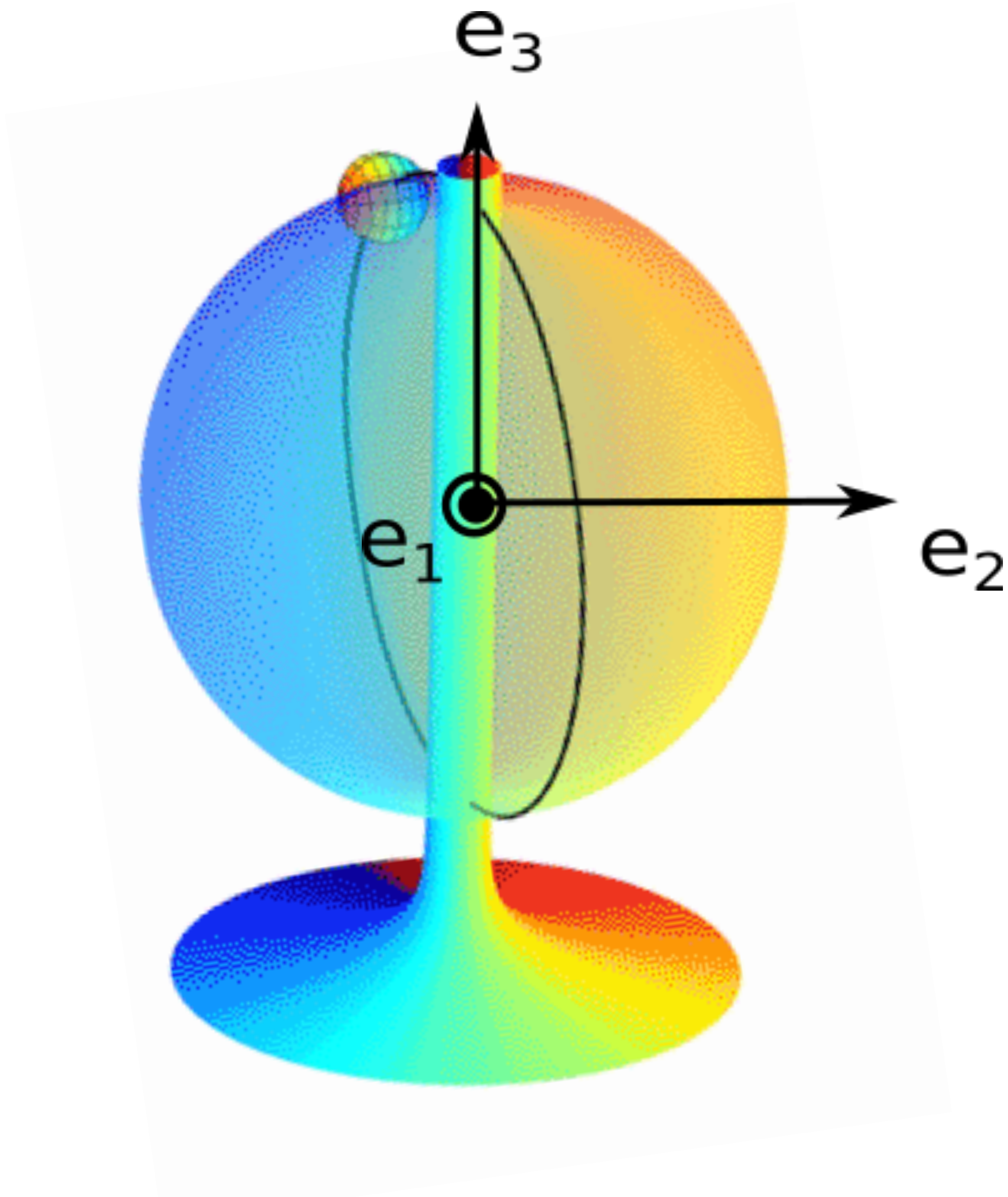
$$\mathrm{KE}_i \triangleq \frac{1}{2} m_i \langle\langle \dot{x}_i(t), \dot{x}_i(t) \rangle\rangle = \frac{1}{2} m_i || \dot{x}_i(t) ||^2$$

Is not an inertial observer invariant quantity!

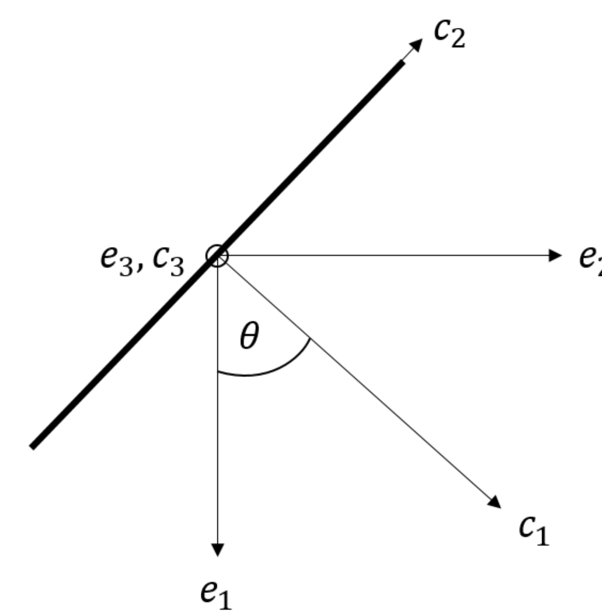
$$\frac{d}{dt}\mathrm{KE}_i = m_i \langle\langle \ddot{x}_i(t), \dot{x}_i(t) \rangle\rangle = \langle\langle f_i(t), \dot{x}_i(t) \rangle\rangle$$

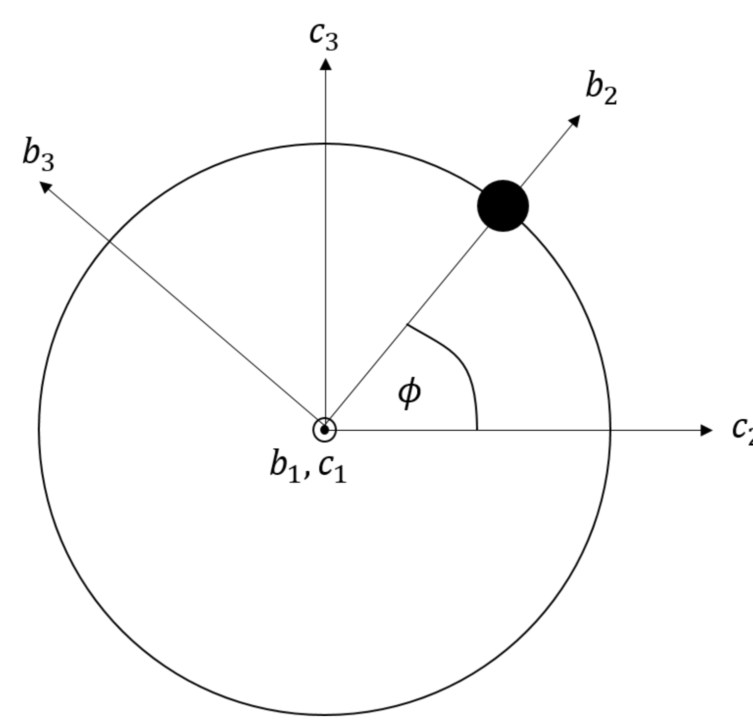
Work done by the force $\rightarrow$ $W(t_1, t_2) = \int_{t_1}^{t_2} \langle\langle f_i(t), \dot{x}_i(t) \rangle\rangle \, dt$

How to Describe a Bead on a Hoop?

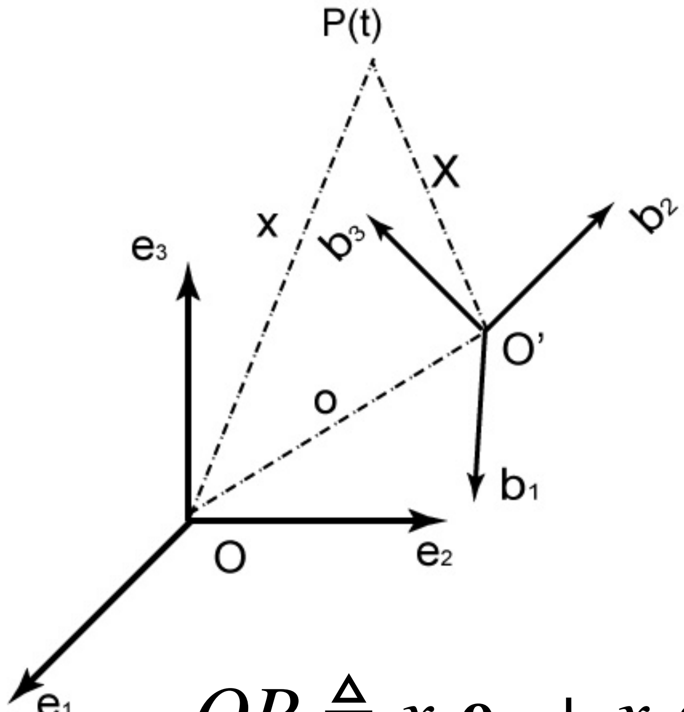


Difficult to write $m\ddot{x} = f$





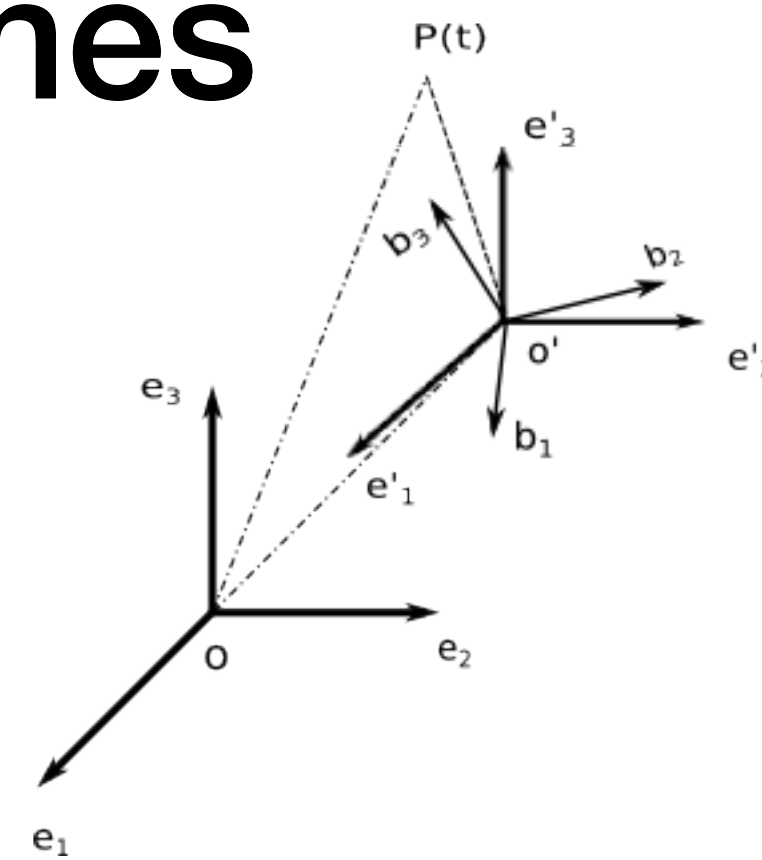
Representation of Position in Moving Frames



$$\mathbf{e} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

$$\mathbf{b} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]$$

$$\mathbf{e}' = [\mathbf{e}'_1 \quad \mathbf{e}'_2 \quad \mathbf{e}'_3]$$



$$OP \triangleq x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \mathbf{e}\,x$$

$$OO' \triangleq o_1\mathbf{e}_1 + o_2\mathbf{e}_2 + o_3\mathbf{e}_3 = \mathbf{e}\,o$$

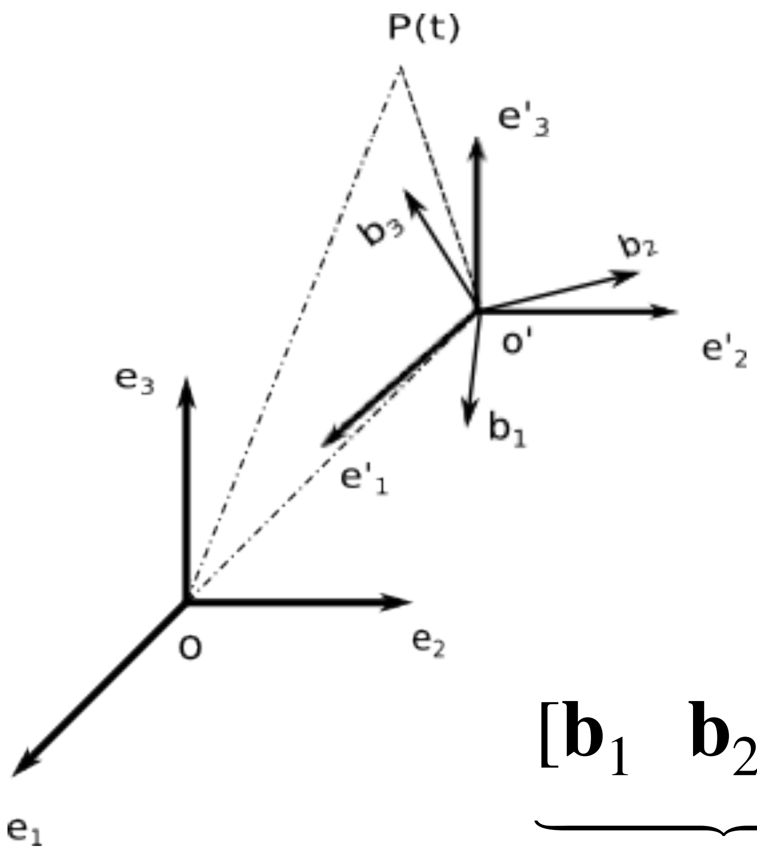
$$O'P \triangleq x'_1\mathbf{e}'_1 + x'_2\mathbf{e}'_2 + x'_3\mathbf{x}'_3 = \mathbf{e}'\,x'$$

$$O'P \triangleq X_1\mathbf{b}_1 + X_2\mathbf{b}_2 + X_3\mathbf{b}_3 = \mathbf{b}\,X$$

We need to know how x is related to X

$$\mathbf{e} \,//\, \mathbf{e}' \quad \rightarrow \quad x = o + x' \qquad\qquad x' \,?\, X$$

Relationship Between Orthonormal Frames



$$\mathbf{e} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

$$\mathbf{b} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]$$

$$\mathbf{e}' = [\mathbf{e}'_1 \quad \mathbf{e}'_2 \quad \mathbf{e}'_3]$$

$$\mathbf{b}_1 = r_{11}\mathbf{e}'_1 + r_{21}\mathbf{e}'_2 + r_{31}\mathbf{e}'_3$$

$$\mathbf{b}_2 = r_{12}\mathbf{e}'_1 + r_{22}\mathbf{e}'_2 + r_{32}\mathbf{e}'_3$$

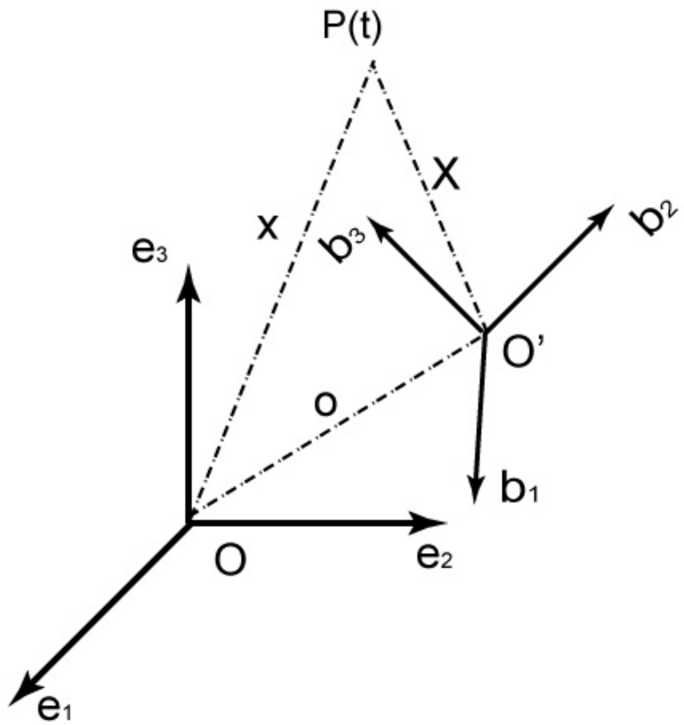
$$\mathbf{b}_3 = r_{13}\mathbf{e}'_1 + r_{23}\mathbf{e}'_2 + r_{33}\mathbf{e}'_3$$

$$\underbrace{[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]}_{\mathbf{b}} = \underbrace{[\mathbf{e}'_1 \quad \mathbf{e}'_2 \quad \mathbf{e}'_3]}_{\mathbf{e}'} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{R} \rightarrow \mathbf{b} = \mathbf{e}' R$$

$$O'P = \mathbf{e}'x' = \mathbf{b}X = \mathbf{e}'RX \quad \rightarrow \quad x' = RX$$

$$x' = RX \rightarrow \mathbf{e}' \text{ frame representation of } P$$

Back to Representation of Position in Moving Frames



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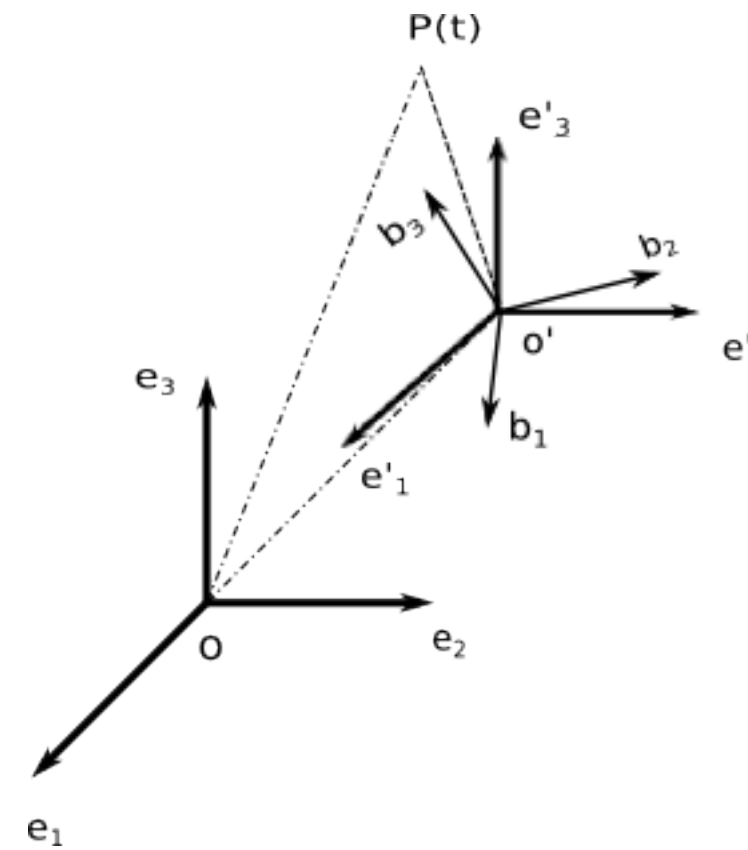


$$OP \triangleq x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \mathbf{e}\,x$$

$$OO' \triangleq o_1\mathbf{e}_1 + o_2\mathbf{e}_2 + o_3\mathbf{e}_3 = \mathbf{e}\,o$$

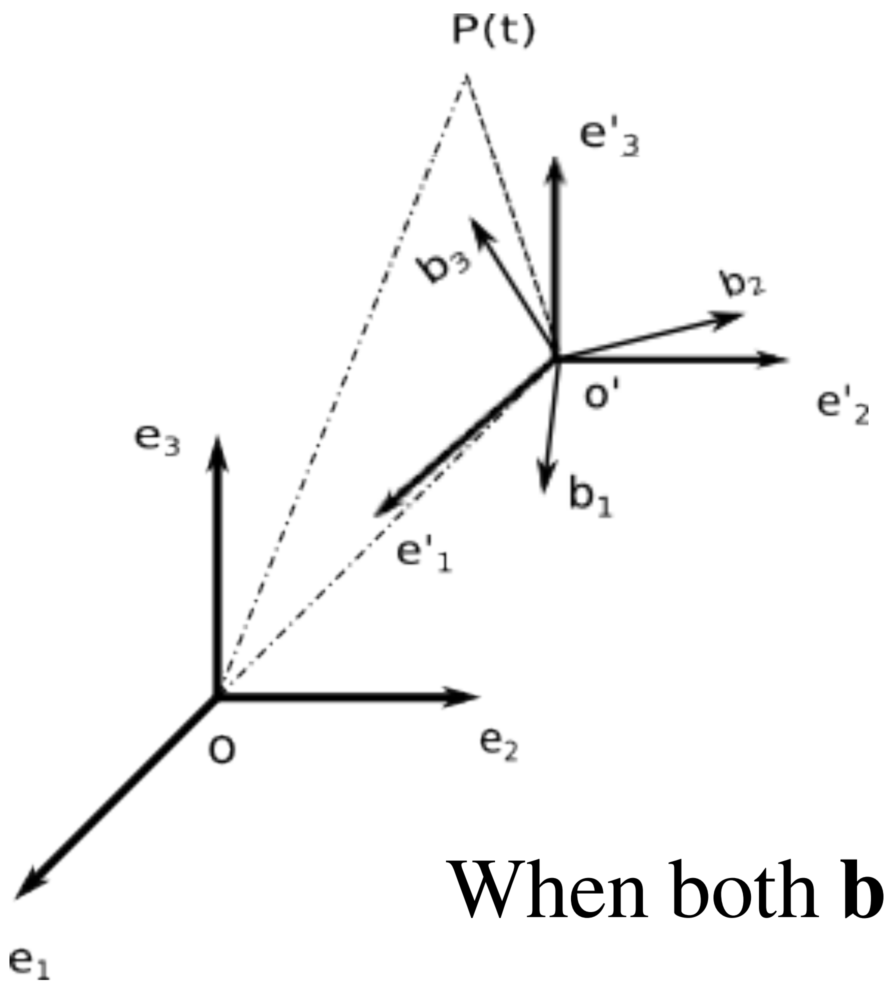
$$O'P \triangleq x'_1\mathbf{e}'_1 + x'_2\mathbf{e}'_2 + x'_3\mathbf{x}'_3 = \mathbf{e}'\,x'$$

$$O'P \triangleq X_1\mathbf{b}_1 + X_2\mathbf{b}_2 + X_3\mathbf{b}_3 = \mathbf{b}\,X$$

$$\mathbf{e} \,//\, \mathbf{e}' \quad \rightarrow \; x = o + x'$$

$$x' = RX \;\rightarrow\; \boxed{x = o + RX}$$

Relationship Between Moving Frames



$$O'P \triangleq \mathbf{b}\, X$$

$$O'P \triangleq \mathbf{e}'\, x'$$

$$\mathbf{b} = \mathbf{e}'\, R \;\rightarrow\; x' = RX$$

$$d(O'P) = ||x'|| = ||X|| \quad \forall\; P$$

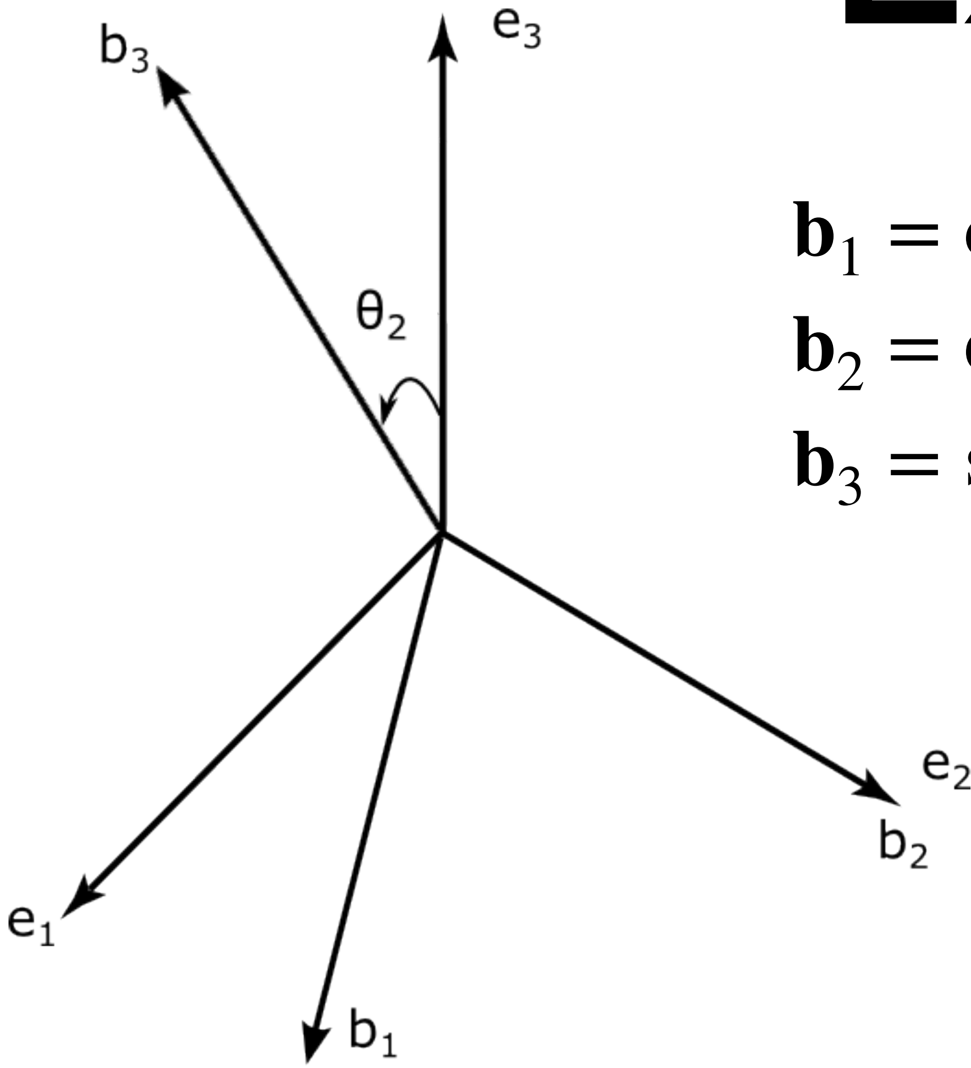
$$||x'|| = ||RX|| = ||X|| \quad \forall\; X$$

$$\rightarrow\; R^T R = RR^T = I_{3\times 3}$$

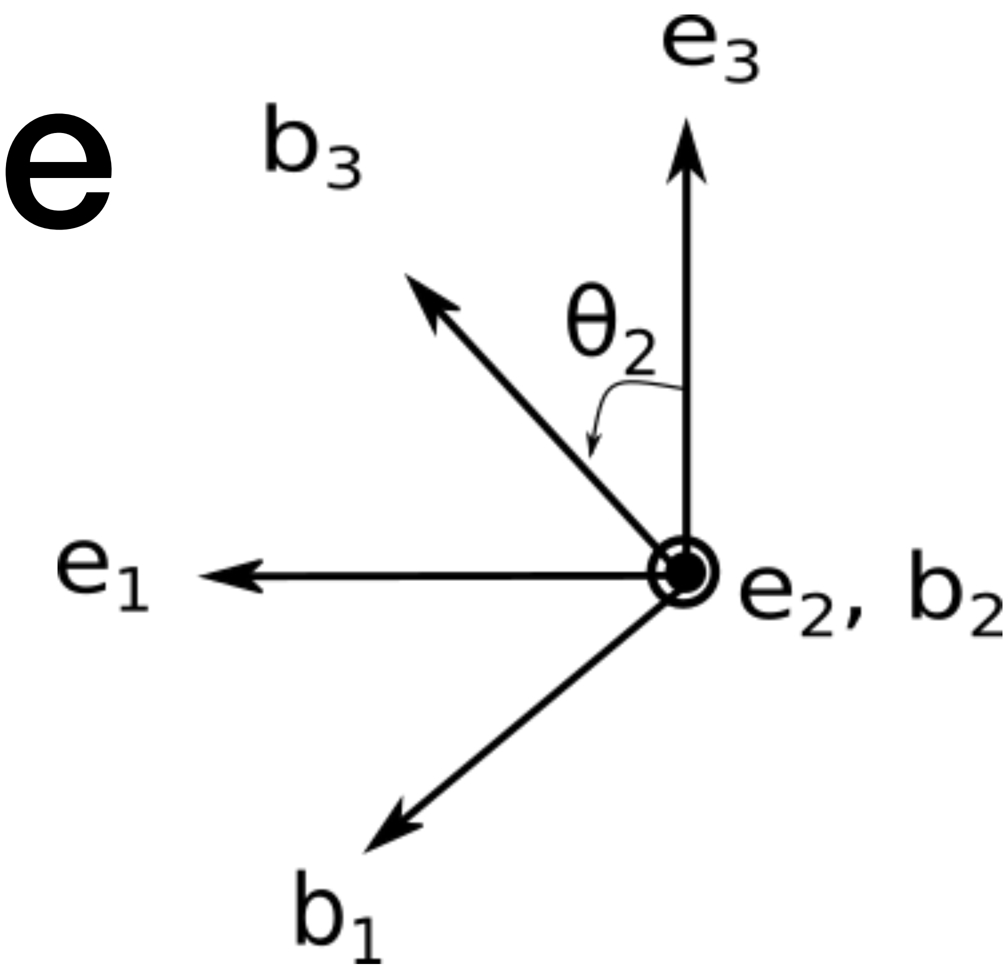
When both **b** & $\mathbf{e}'$ are right hand oriented $\rightarrow$ $\det(R) = 1$

$$\boxed{\begin{array}{c} RR^T = R^T R = I_{3\times 3} \quad \textbf{and} \quad \det(R) = 1 \\ \text{space of such R} \;=\; SO(3) \end{array}}$$

Example

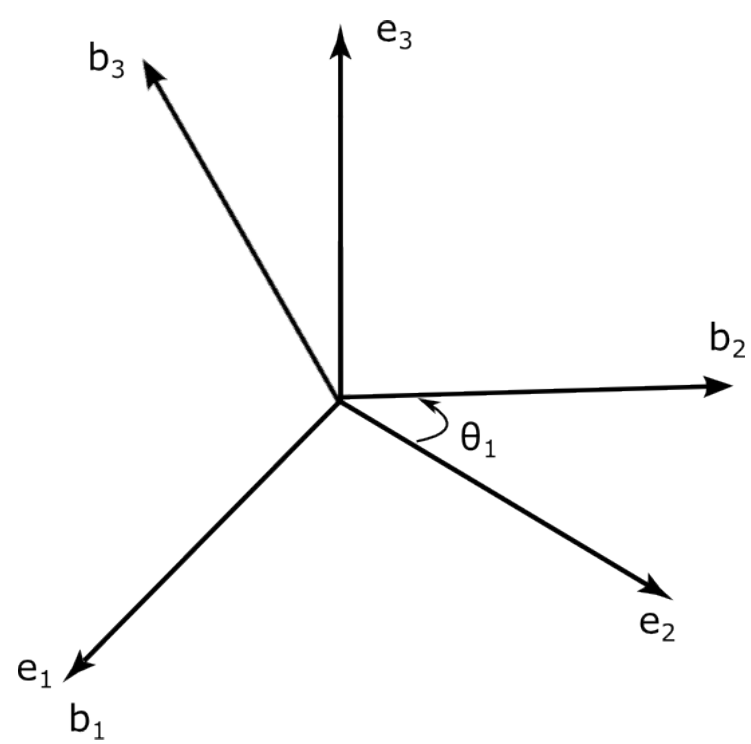


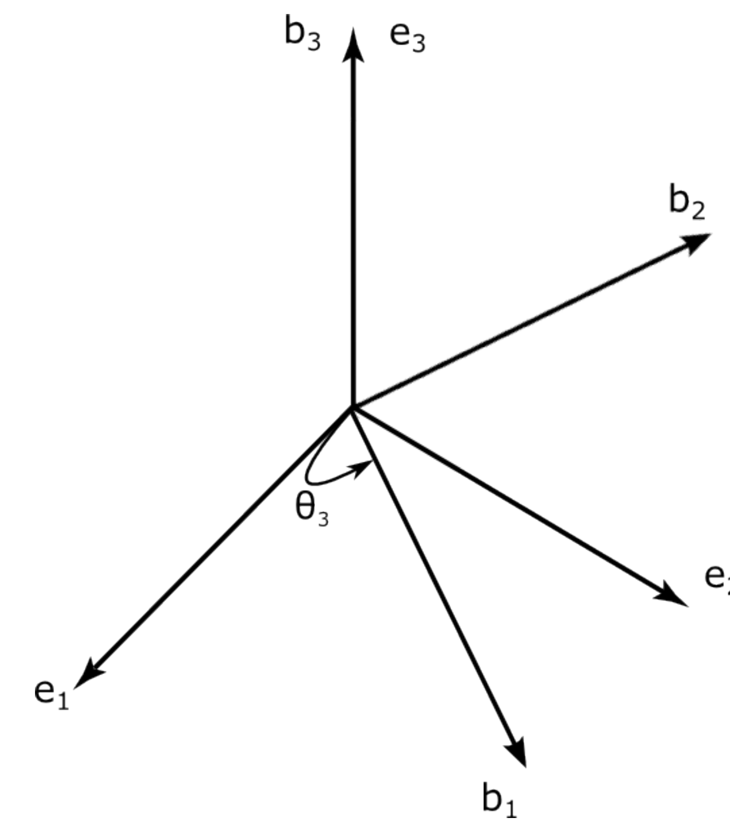
$$\mathbf{b}_1 = \cos\theta_2\,\mathbf{e}_1 - \sin\theta_2\,\mathbf{e}_3$$
$$\mathbf{b}_2 = \mathbf{e}_2$$
$$\mathbf{b}_3 = \sin\theta_2\,\mathbf{e}_1 + \cos\theta_2\,\mathbf{e}_3$$



$$\underbrace{[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]}_{\mathbf{b}} = \underbrace{[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]}_{\mathbf{e}} \underbrace{\begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}}_{R_2(\theta_2)}$$

2 More Simple Examples





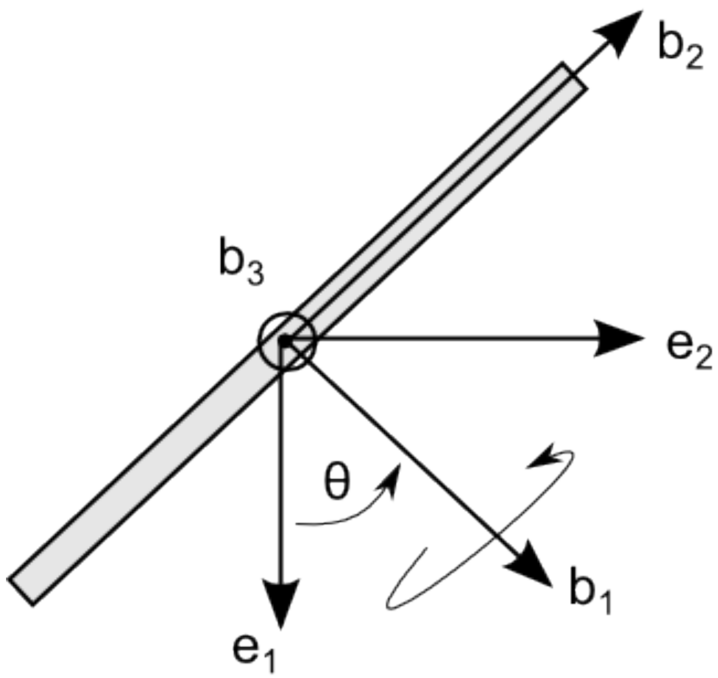
$$\mathbf{b} = \mathbf{e}\,R_1(\theta_1)$$

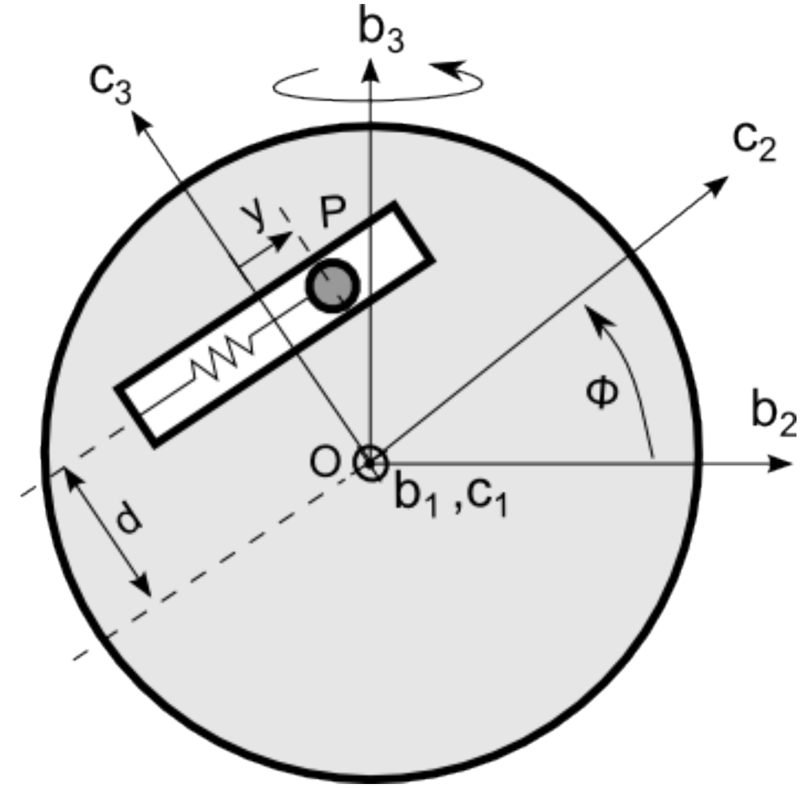
$$\mathbf{b} = \mathbf{e}\,R_3(\theta_3)$$

$$R_1(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$R_3(\theta_3) = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Another Example



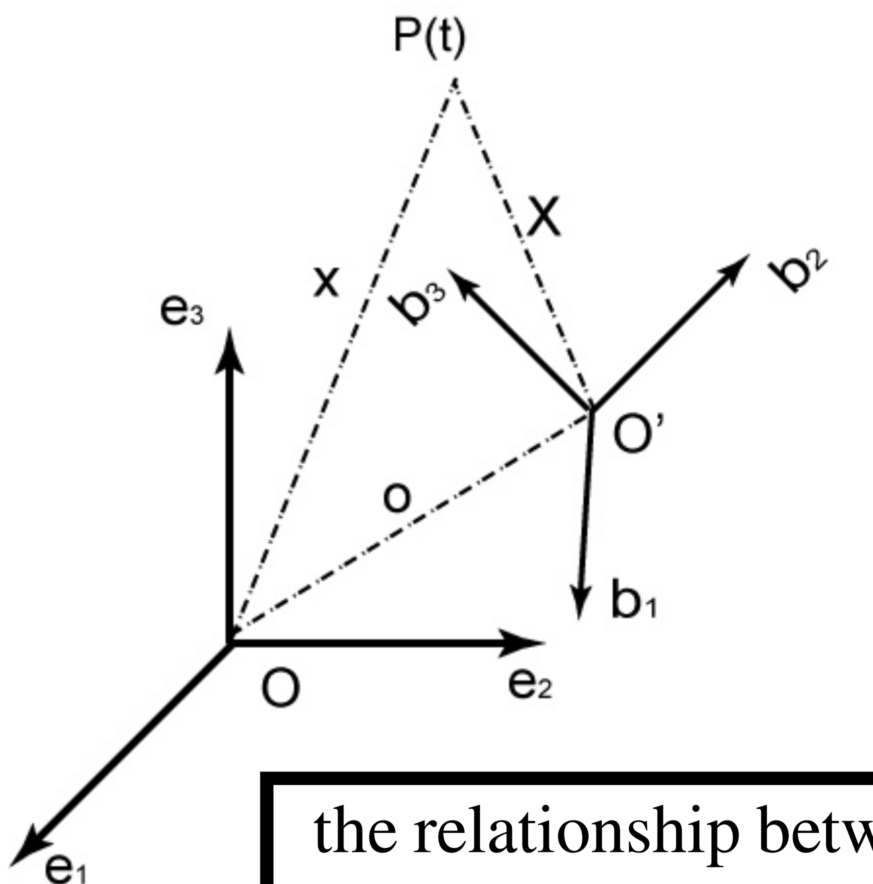


$$\mathbf{b} = \mathbf{e}\, R_3(\theta) \qquad \mathbf{c} = \mathbf{b}\, R_1(\phi)$$

$$\mathbf{c} = \mathbf{b}\, R_1(\phi) = \mathbf{e}\, \underbrace{R_3(\theta) R_1(\phi)}_{R}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & \sin\theta\sin\phi \\ \sin\theta & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Relationship Between Orthonormal Frames



$$\mathbf{e} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \qquad \mathbf{b} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]$$

$OO' = \mathbf{e}\, o$ **where** $o \in \mathbb{R}^3$

$$\mathbf{b} = \mathbf{e}'\, R = \mathbf{e}\, R \qquad\qquad (\mathbf{e} \,//\, \mathbf{e}')$$

$RR^T = R^T R = I_{3\times 3}$ **and** $\det(R) = 1$

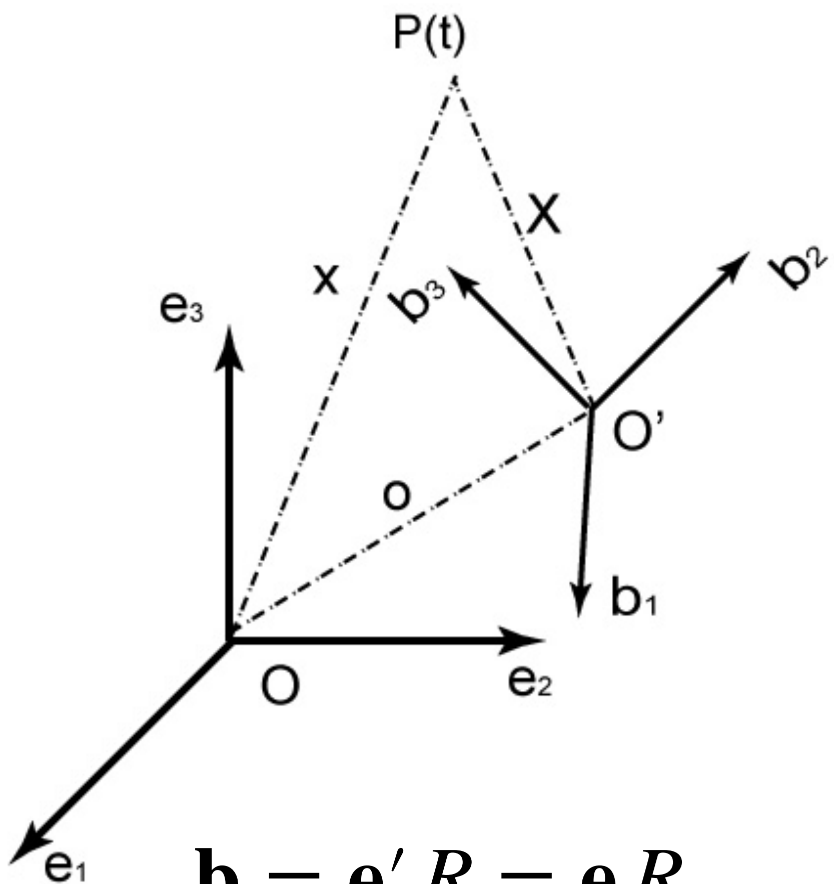
space of such R $= SO(3)$

the relationship between the two right hand oriented ortononormal frames $\mathbf{e}$ & $\mathbf{b}$ is uniquely given by the pair $(o, R) \in \mathbb{R}^3 \times SO(3)$

conversely the pair $(o, R) \in \mathbb{R}^3 \times SO(3)$ defines the relationship between two right hand oriented orthonomal frames $\mathbf{e}$ & $\mathbf{b}$

$$(o, R) \sim g \triangleq \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3) \longrightarrow \text{Euclidean motion group}$$

Description of Motion in Moving Frames



$$\mathbf{b} = \mathbf{e}'R = \mathbf{e}\,R$$

Position $\longrightarrow x = o + RX$

Velocity $\longrightarrow \dot{x} = \dot{o} + \dot{R}X + R\dot{X}$

How to compute $\dot{R}$?

Recall that $\quad R^TR = I_{3\times 3}$

Hence $\quad \dot{R}^TR + R^T\dot{R} = 0_{3\times 3}$

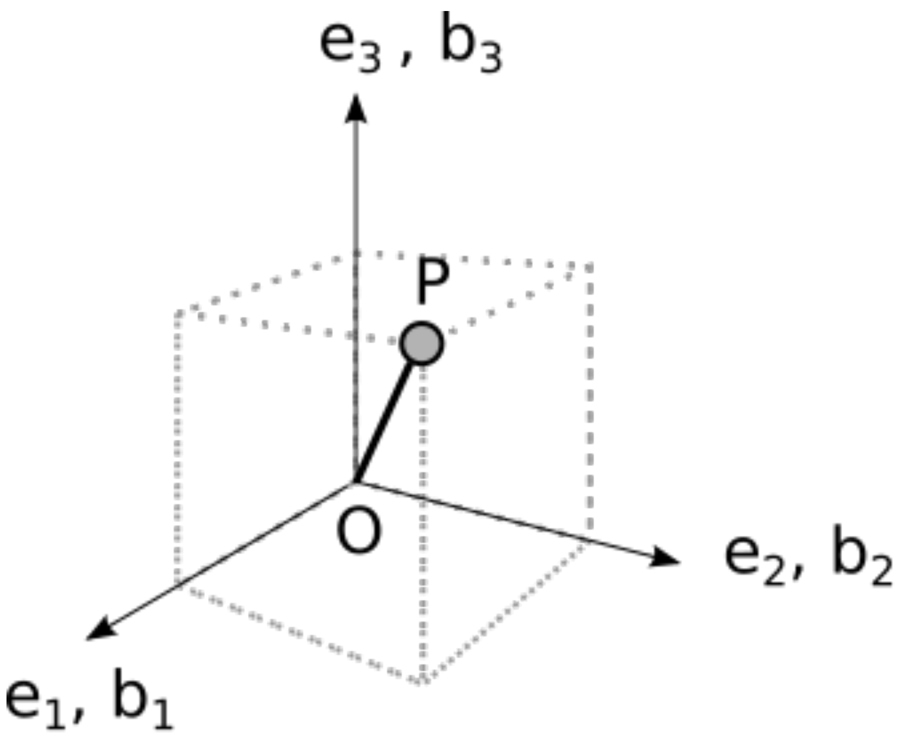
$$R^T\dot{R} = -\left(R^T\dot{R}\right)^T \triangleq \widehat{\Omega} \longrightarrow \text{Skew Symmetric}$$

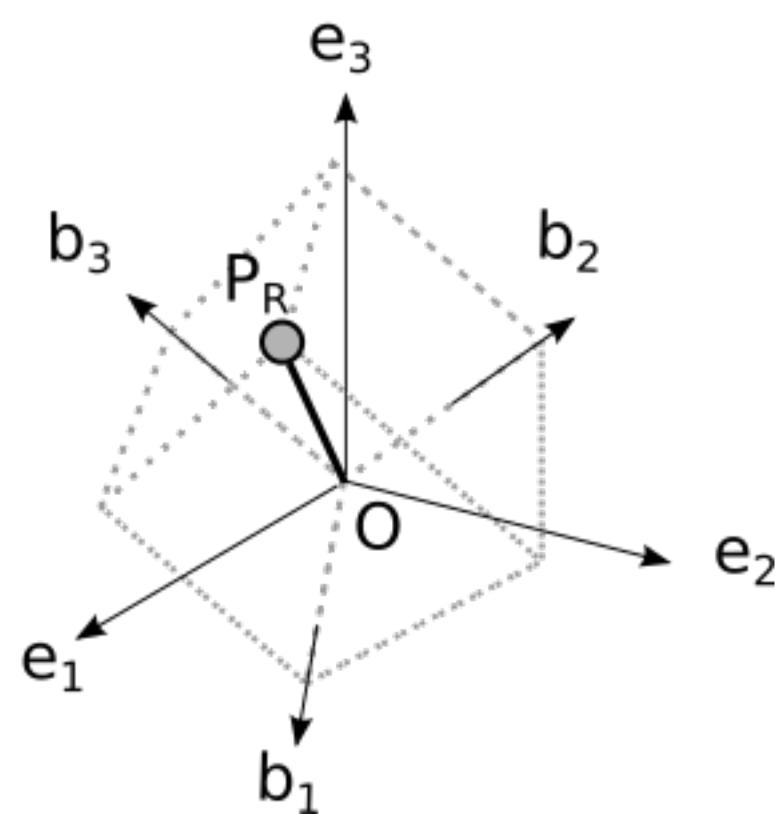
Velocity $\longrightarrow \dot{x} = \dot{o} + R\left(\widehat{\Omega}X + \dot{X}\right)$

Acceleration $\longrightarrow \ddot{x} = \ddot{o} + R\left(\widehat{\Omega}^2X + 2\widehat{\Omega}\dot{X} + \dot{\widehat{\Omega}}X + \ddot{X}\right)$

The Meaning of Ω

$R \in SO(3)$ as an action on $\mathbb{R}^3$





$$\mathbf{b} = \mathbf{e}\,R$$
$$R \in SO(3)$$

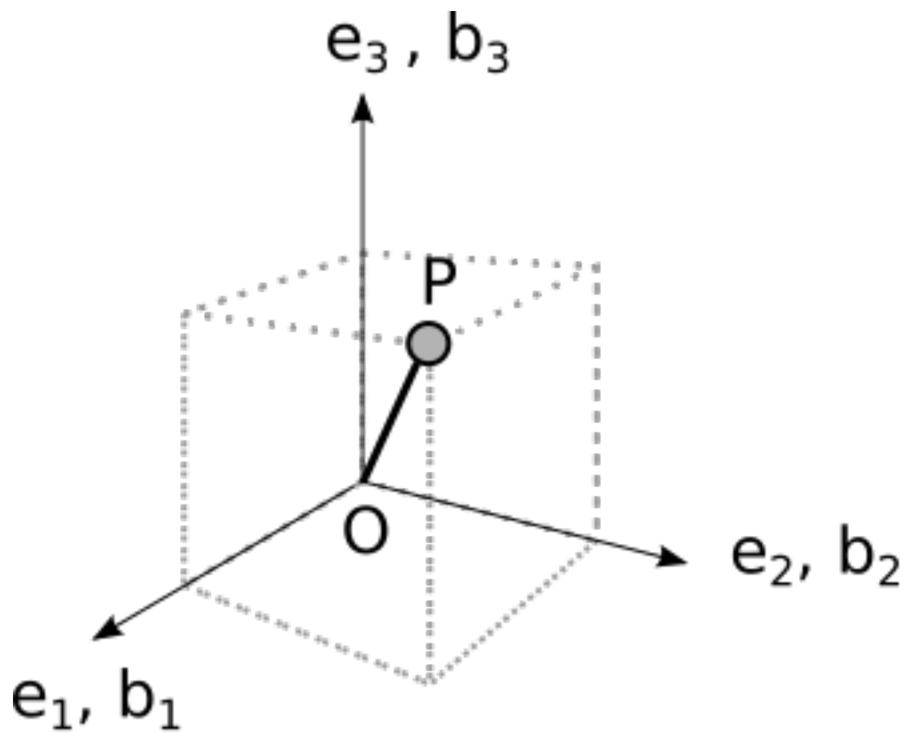
$$OP = \mathbf{e}\,x$$
$$OP = \mathbf{b}\,x$$

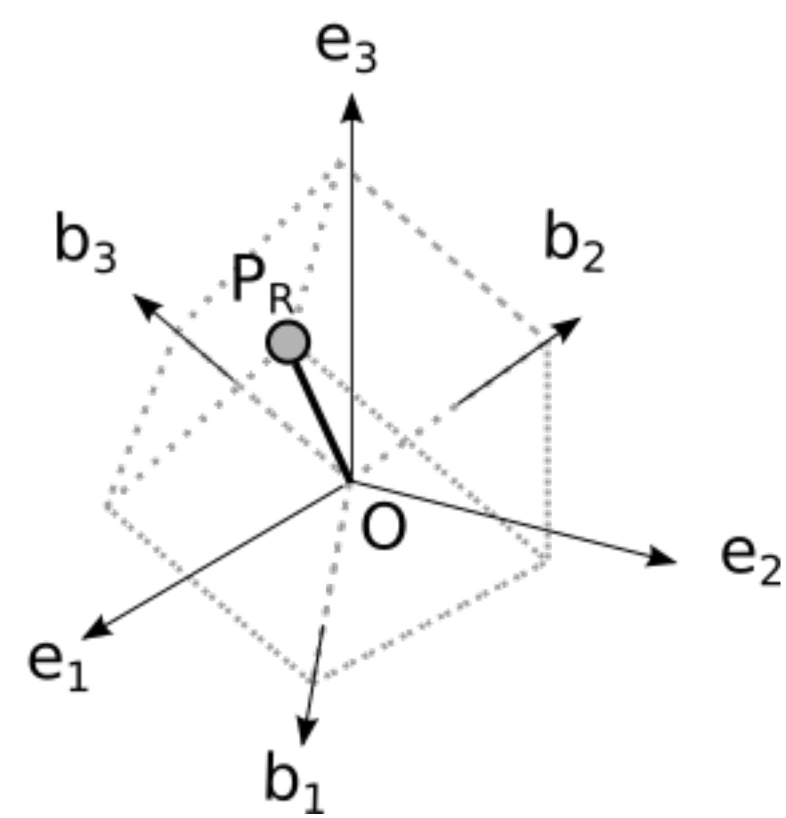
$$OP_R = \mathbf{b}x$$
$$OP_R = \mathbf{e}\,x_R$$

$$OP_R = \mathbf{e}\,x_R = \mathbf{b}x = (\mathbf{e}\,R)x = \mathbf{e}\,(Rx) \quad \rightarrow \quad x_R = Rx$$

thus R maps $x \rightarrow x_R = Rx$

$R \in SO(3)$ as an action on $\mathbb{R}^3$





$$\mathbf{b} = \mathbf{e}\,R$$
$$R \in SO(3)$$

$$OP = \mathbf{e}\,X$$
$$OP = \mathbf{b}\,X$$

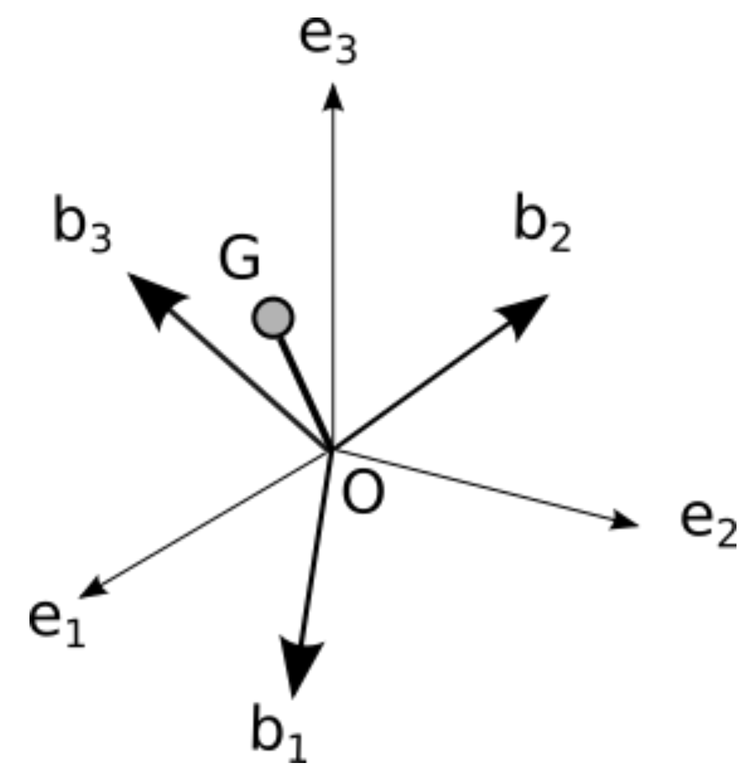
$$OP_R = \mathbf{b}X$$
$$OP_R = \mathbf{e}\,x_R$$

$$x_R = RX$$
$$R\ : \mathbb{R}^3 \to \mathbb{R}^3$$

$$R^T R = I\ \to ||RX - RY|| = ||X - Y|| \quad \text{and} \quad \langle\langle RX, RY\rangle\rangle = \langle\langle X, Y\rangle\rangle$$

thus R can be thought of as a mapping that takes P to P_R by rigid rotations

$R \in SO(3)$ as a coordinate transformation



$$OG = \mathbf{e}\gamma = \mathbf{b}\Gamma = \mathbf{e}R\Gamma \ \rightarrow \ \gamma = R\Gamma$$

$\gamma \ \rightarrow$ **e** frame representations of the quantity represnmted by OG

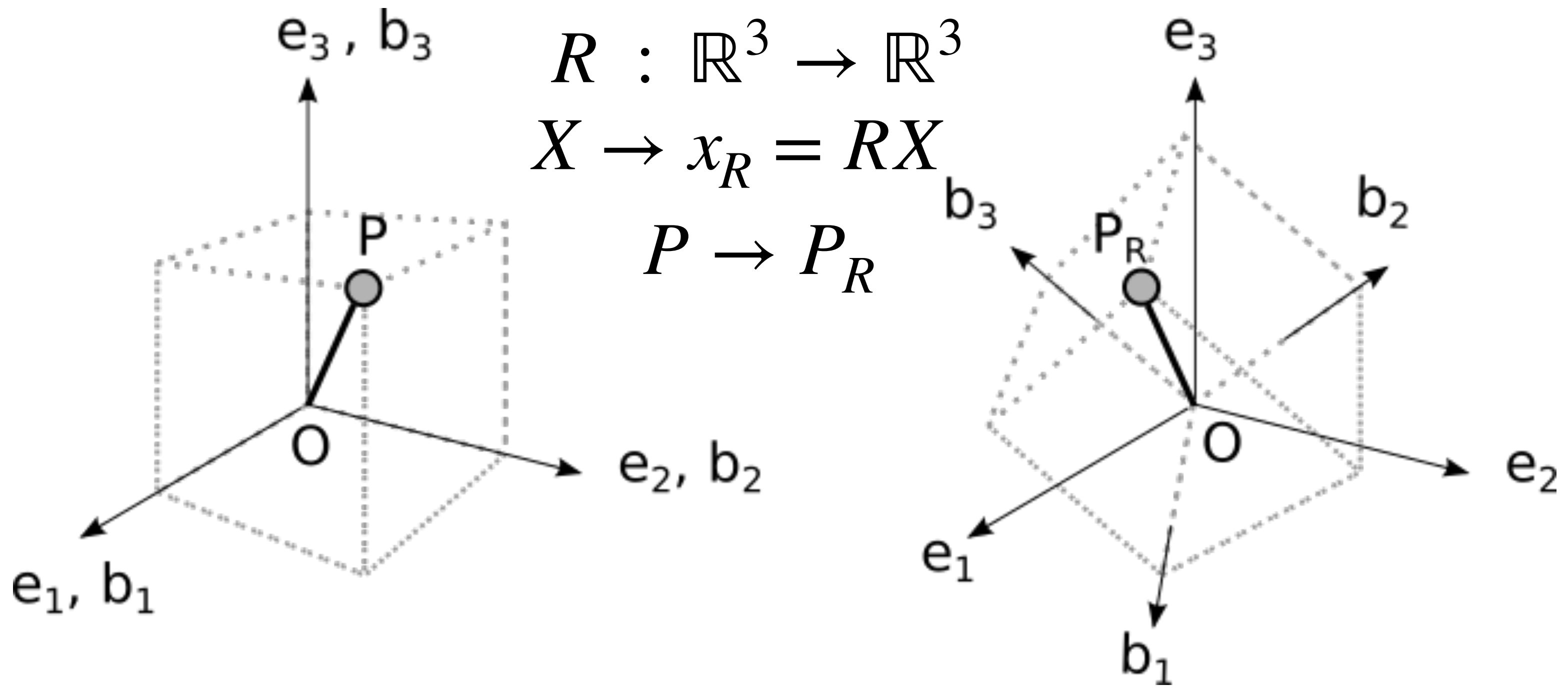
$\Gamma \ \rightarrow$ **b** frame representations of the quantity represnmted by OG

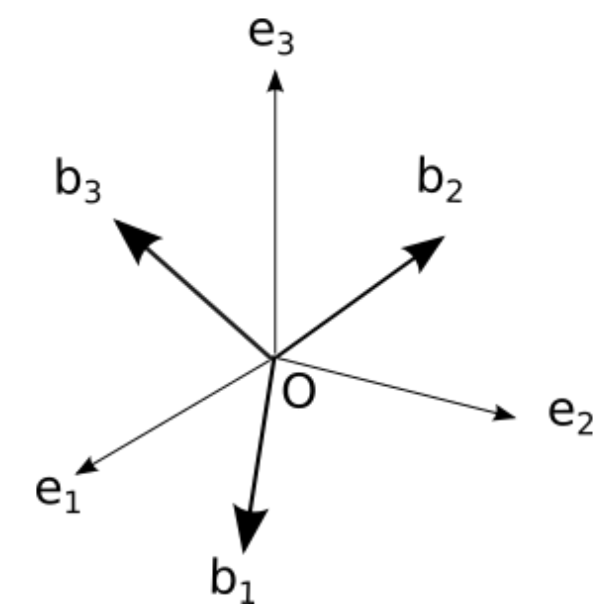
As a coordinate transformation between **b** and **e** quantities

The Meaning of $R \in SO(3)$

As an action of rigid rotation of points in space

$$R : \mathbb{R}^3 \to \mathbb{R}^3$$
$$X \to x_R = RX$$
$$P \to P_R$$

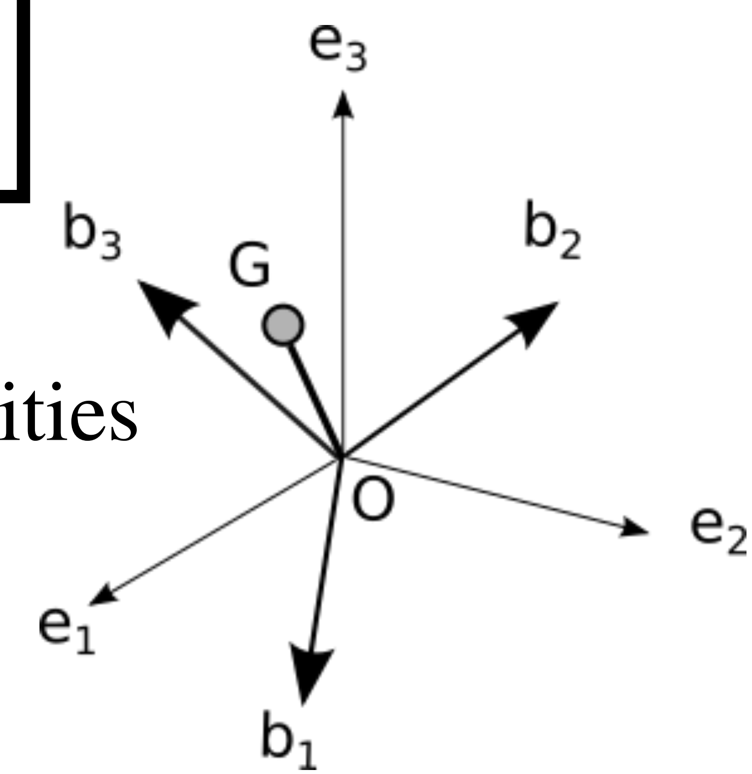




$\mathbf{b} = \mathbf{e}R$

As a coordinate transformation between **b** and **e** quantities

$$OG = \mathbf{e}\gamma = \mathbf{b}\Gamma = \mathbf{e}R\Gamma \;\to\; \gamma = R\Gamma$$



Infinitesimal Rotations

$$R^T \dot{R} = -(R^T \dot{R})^T = \widehat{\Omega} \longrightarrow \dot{R} = R\,\widehat{\Omega}$$

$$\widehat{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$

space of such $\widehat{\Omega} = so(3)$

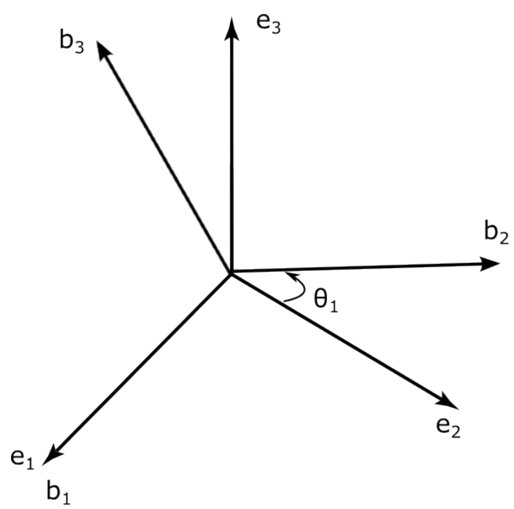
$$\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$$

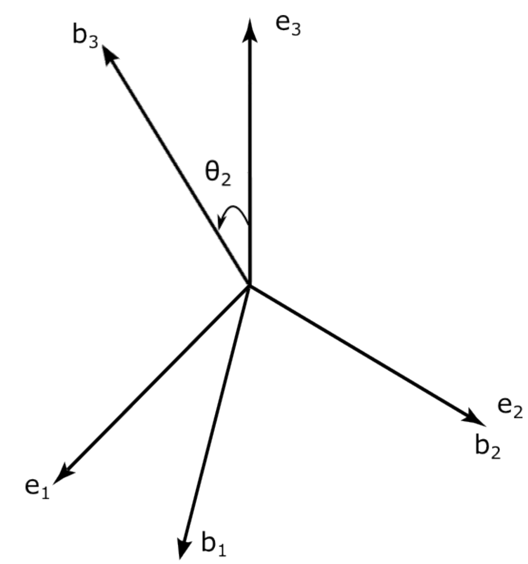
$$\Omega \in \mathbb{R}^3 \to \widehat{\Omega} \in so(3) \quad \Longrightarrow \quad \widehat{\ } : \mathbb{R}^3 \to so(3)$$

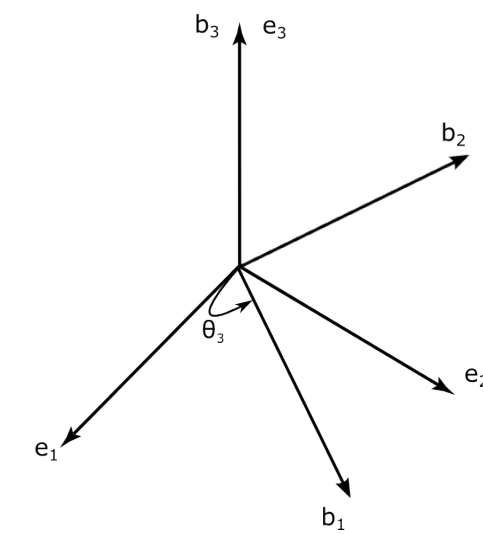
$\widehat{\ } : \mathbb{R}^3 \to so(3)$ is a linear $1-1$ and onto map

Ω — **What is the physical meaning of this quantity?**

Basic Examples







$$\mathbf{b} = \mathbf{e}\,R_1(\theta_1) \qquad \mathbf{b} = \mathbf{e}\,R_2(\theta_2) \qquad \mathbf{b} = \mathbf{e}\,R_3(\theta_3)$$

$$R_1(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \qquad R_2(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \qquad R_3(\theta_3) = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1^T\dot{R}_1 = \widehat{\Omega}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}_1 \\ 0 & \dot{\theta}_1 & 0 \end{bmatrix} \qquad R_2^T\dot{R}_2 = \widehat{\Omega}_2 = \begin{bmatrix} 0 & 0 & \dot{\theta}_2 \\ 0 & 0 & 0 \\ -\dot{\theta}_2 & 0 & 0 \end{bmatrix} \qquad R_3^T\dot{R}_3 = \widehat{\Omega}_3 = \begin{bmatrix} 0 & -\dot{\theta}_3 & 0 \\ \dot{\theta}_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Omega_1 = (\dot{\theta}_1, 0, 0) \qquad \Omega_2 = (0, \dot{\theta}_2, 0) \qquad \Omega_3 = (0, 0, \dot{\theta}_3)$$

Properties of the hat map

for any $X, Y, \Omega \in \mathbb{R}^3$

$$\widehat{\Omega}\, X = \Omega \times X,$$

$$\langle\langle X, Y \rangle\rangle = -\frac{1}{2}\mathrm{trace}(\widehat{X}\,\widehat{Y}),$$

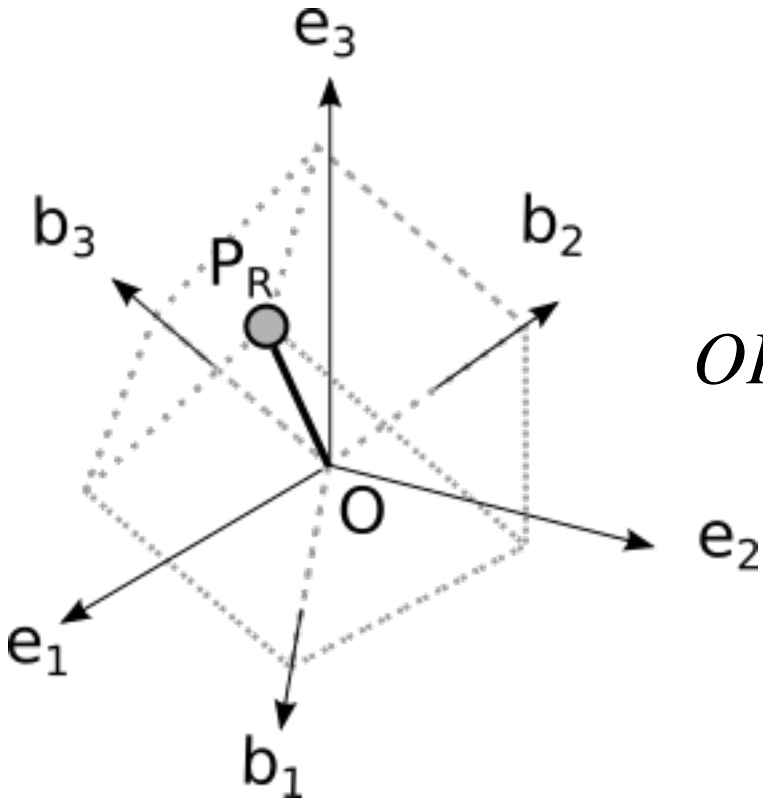
$$R\,(X \times Y) = (RX) \times (RY)$$

$$\widehat{RX} = R\,\widehat{X}\,R^T$$

$$\widehat{X}^2 = XX^T - ||X||^2 I_{3\times 3}$$

$$\widehat{X}^3 = -\,||X||^2\,\widehat{X}, \qquad \widehat{X}^4 = -\,||X||^2\,\widehat{X}^2, \qquad \widehat{X}^5 = ||X||^4\,\widehat{X}, \qquad \cdots$$

Infinitesimal Rotations



Particle fixed in a rotating frame $\mathbf{b}(t)$

$$\mathbf{b}(t) = \mathbf{e}\,R(t) \qquad \widehat{\Omega} = R^T \dot{R}$$

$$OP_R = \mathbf{e}\,x(t) \quad \& \quad OP_R = \mathbf{b}(t)\,X = (\mathbf{e}\,R(t))\,X \;\rightarrow x(t) = R(t)\,X$$

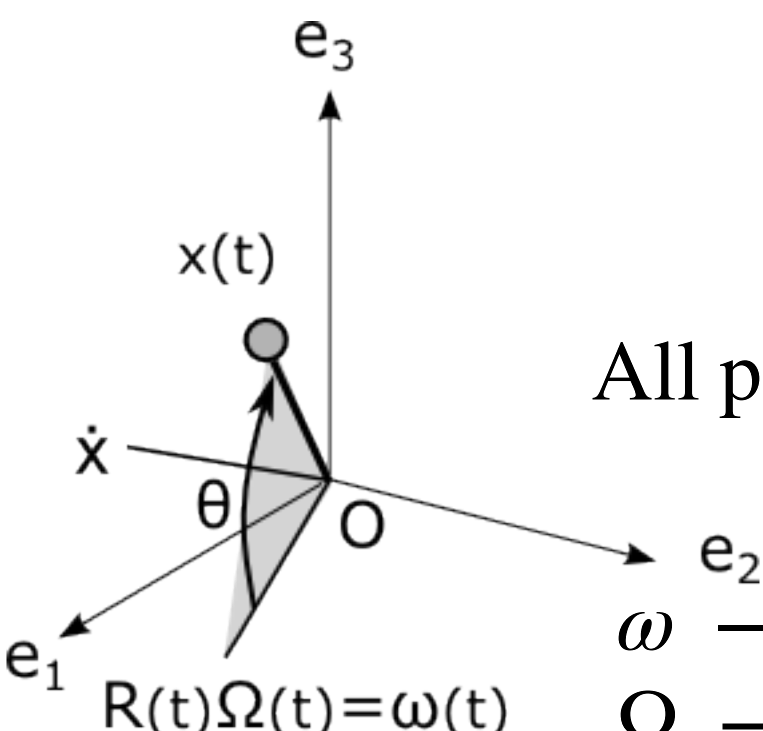
$$\dot{x} = \dot{R}X = R\,\widehat{\Omega}\,X = R(\Omega \times X)$$
$$= (R\Omega) \times (RX) = (R\Omega) \times x$$

e – frame representation of $\Omega \;\rightarrow \omega \triangleq R\Omega$

$$\dot{x} = (R\Omega) \times x = \omega \times x$$

$$\rightarrow \quad \dot{x} \perp x \quad \& \quad \dot{x} \perp \omega$$

All points on an axis in the direction of ω have zero velocity

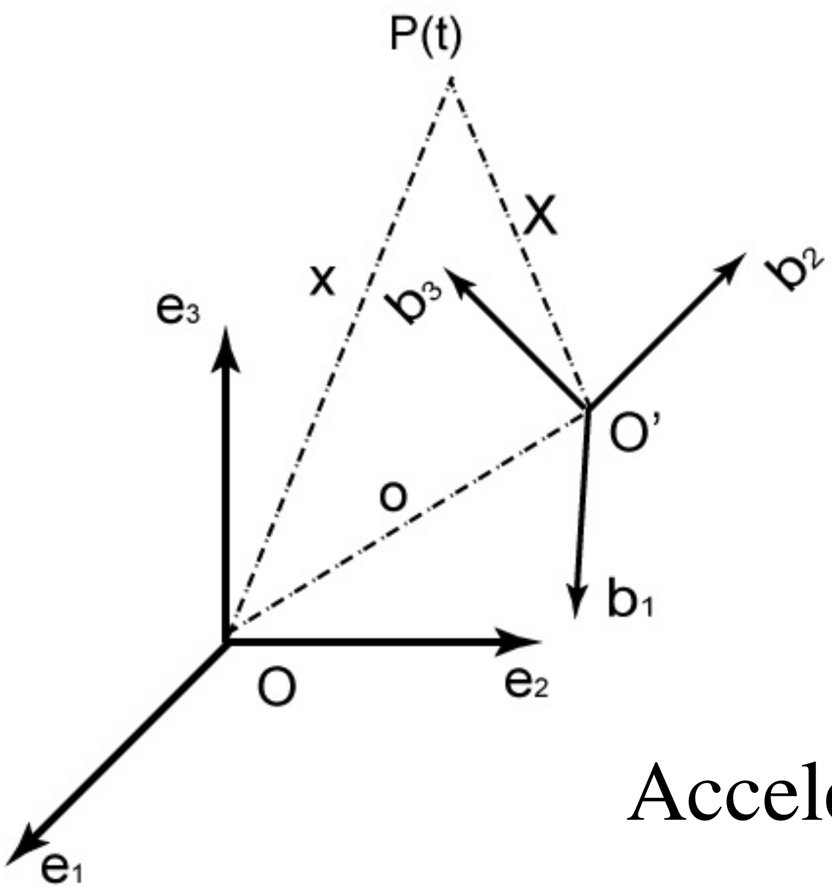


ω – is the anugular velocity of **b** expressed in the **e** frame

Ω – is the anugular velocity of **b** expressed in the **b** frame

Description of Motion in Accelerating Frames

Newton's Equation in Moving Frames



$$\mathbf{b} = \mathbf{e}'\, R = \mathbf{e}\, R \qquad \dot{R} = R\,\widehat{\Omega}$$

$$\text{Position} \longrightarrow x = o + RX$$

$$\text{Velocity} \longrightarrow \dot{x} = \dot{o} + R\left(\widehat{\Omega}\, X + \dot{X}\right)$$

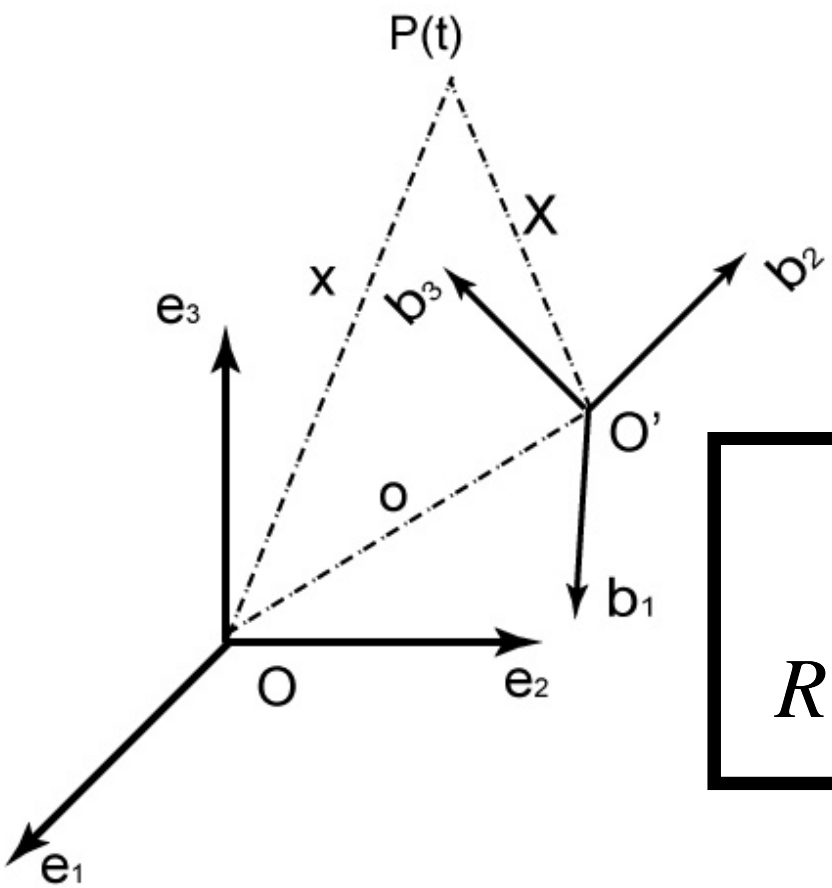
$$\text{Acceleration} \longrightarrow \ddot{x} = \ddot{o} + R\left(\widehat{\Omega}^{\,2} X + 2\,\widehat{\Omega}\,\dot{X} + \dot{\widehat{\Omega}}\, X + \ddot{X}\right)$$

Newton's equations for P in the Inertial frame $\mathbf{e}$ $\rightarrow$ $m\ddot{x} = f$

$$\boxed{\begin{array}{c}\text{Newton's equations for P in the Moving Frame } \mathbf{b} \\ R^T f \triangleq F = mR^T\ddot{o} + m\,\widehat{\Omega}^{\,2} X + 2m\,\widehat{\Omega}\,\dot{X} + m\,\dot{\widehat{\Omega}}\, X + m\ddot{X}\end{array}}$$

$\mathbf{b}$ frame representation of the force $\rightarrow F = R^T f$

The Source of Apparent Forces



$$\mathbf{b} = \mathbf{e}'\,R = \mathbf{e}\,R \qquad \dot{R} = R\,\widehat{\Omega}$$

$$\text{Newton's equations} \;\rightarrow\; m\ddot{x} = f$$

$$\boxed{\begin{array}{c}\text{Newton's equations for P in the Moving Frame } \mathbf{b} \\ R^T f \triangleq F = mR^T\ddot{o} + m\,\widehat{\Omega}^{\,2}X + 2m\,\widehat{\Omega}\,\dot{X} + m\,\dot{\widehat{\Omega}}\,X + m\ddot{X}\end{array}}$$

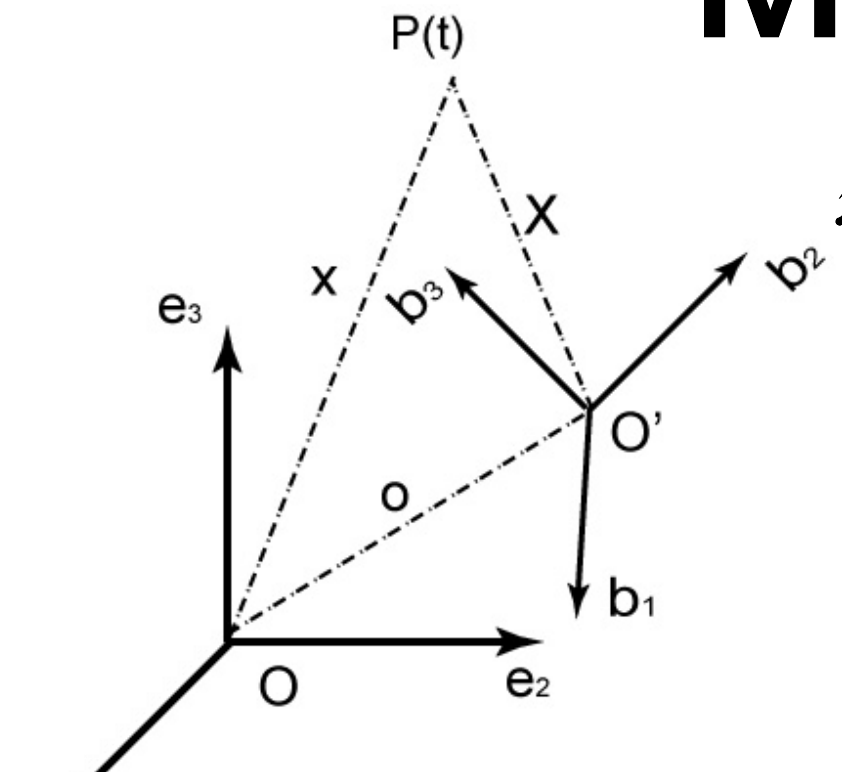
$\mathbf{e}$ is inertial

$$\mathbf{b} \text{ frame representation of the force on } P \;= F = R^T f$$

$$m\ddot{X} = F - \underbrace{\left(mR^T\ddot{o} + m\,\widehat{\Omega}^{\,2}X + 2m\,\widehat{\Omega}\,\dot{X} + m\,\dot{\widehat{\Omega}}\,X\right)}_{F_{app}}$$

$$F_{app} \triangleq \underbrace{-m\;R^T(t)\ddot{o}(t)}_{\textbf{Einstein}}\;\underbrace{-m\;\widehat{\Omega}^{\,2}(t)X(t)}_{\textbf{Centrifugal}}\;\underbrace{-\,2m\;\widehat{\Omega}\,(t)\dot{X}(t)}_{\textbf{Coriolis}}\;\underbrace{-\,m\;\dot{\widehat{\Omega}}\,(t)X(t)}_{\textbf{Euler}}$$

Angular Momentum in Moving Frames



$$\mathbf{b} = \mathbf{e}\, R$$

$$x = o + RX$$

$$\dot{x} = \dot{o} + R\left(\widehat{\Omega}\, X + \dot{X}\right)$$

$$\pi = m_p (x - o) \times \dot{x} = m_p\, R\left(X \times (\Omega \times X + \dot{X} + R^T \dot{o})\right)$$

$$= R\left(-m_p\, \widehat{X}^{\,2} \Omega + m_p X \times (R^T \dot{o} + \dot{X})\right)$$

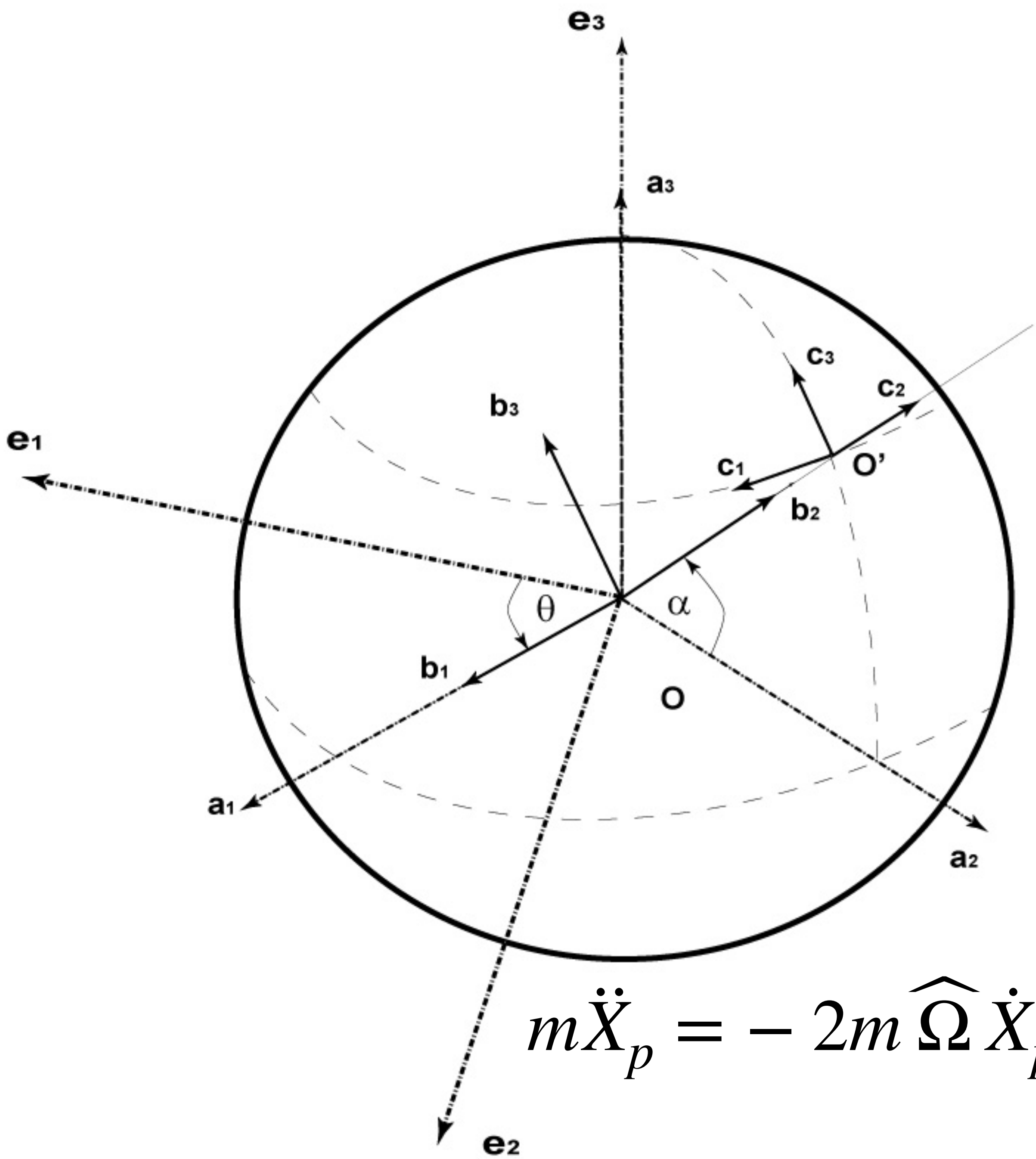
$$-m_p\, \widehat{X}^{\,2} = m_p\left(||X||^2 I_{3\times 3} - XX^T\right) \triangleq \mathbb{I}_p$$

$\mathbb{I}_p \ \rightarrow\ $ Moment of Inertia of P about the point O'

$$\pi = R\underbrace{\left(\mathbb{I}_p \Omega + m_p X \times (R^T \dot{o} + \dot{X})\right)}_{\Pi} = R\,\Pi$$

$\Pi \ \rightarrow\ \mathbf{b}$ – frame representation of angular momentum of P about the point O'

Effects of Earth’s Rotation



$$\mathbf{a} = \mathbf{e}R_3(\theta) \quad \mathbf{b} = \mathbf{a}R_1(\alpha) \quad \mathbf{c} \mathbin{//} \mathbf{b}$$

$$\mathbf{c} = \mathbf{e}\,\underbrace{R_3(\theta)R_1(\alpha)}_{R}$$

$$O'P = \mathbf{c}X_p \quad OO' = \mathbf{b}\,o$$

$$o = r\chi \quad \text{where} \quad \chi = [0 \;\; 1 \;\; 0]^T$$

$$m\ddot{X}_p = -2m\,\widehat{\Omega}\,\dot{X}_p - m\,\widehat{\Omega}^{\,2}(X_p + o) - mg\frac{(X_p + o)}{||X_p + o||} + R^T f^e$$

Refer to lecture notes for further details

Effects of Earth's Rotation

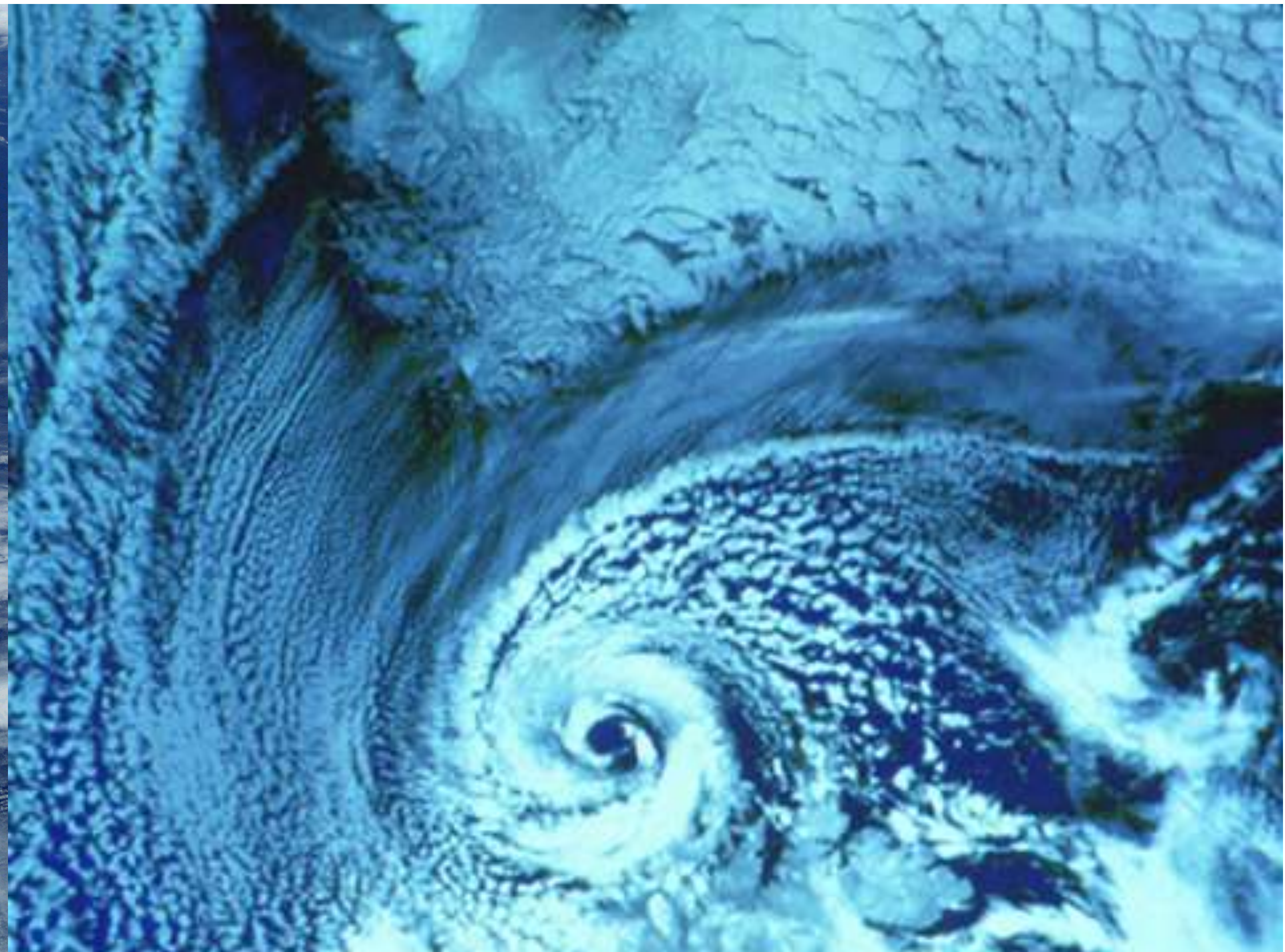
$$m\ddot{X}_p = -2m\widehat{\Omega}\dot{X}_p - m\widehat{\Omega}^2(X_p + o) - mg\frac{(X_p + o)}{||X_p + o||} + R^T f^e$$

$$(X_p + o)/||(X_p + o)|| \approx o = r\chi \;\rightarrow \qquad \ddot{X}_p = -2\widehat{\Omega}\dot{X}_p - \left(r\widehat{\Omega}^2 + gI_{3\times 3}\right)\chi + \frac{1}{m}R^T f^e$$

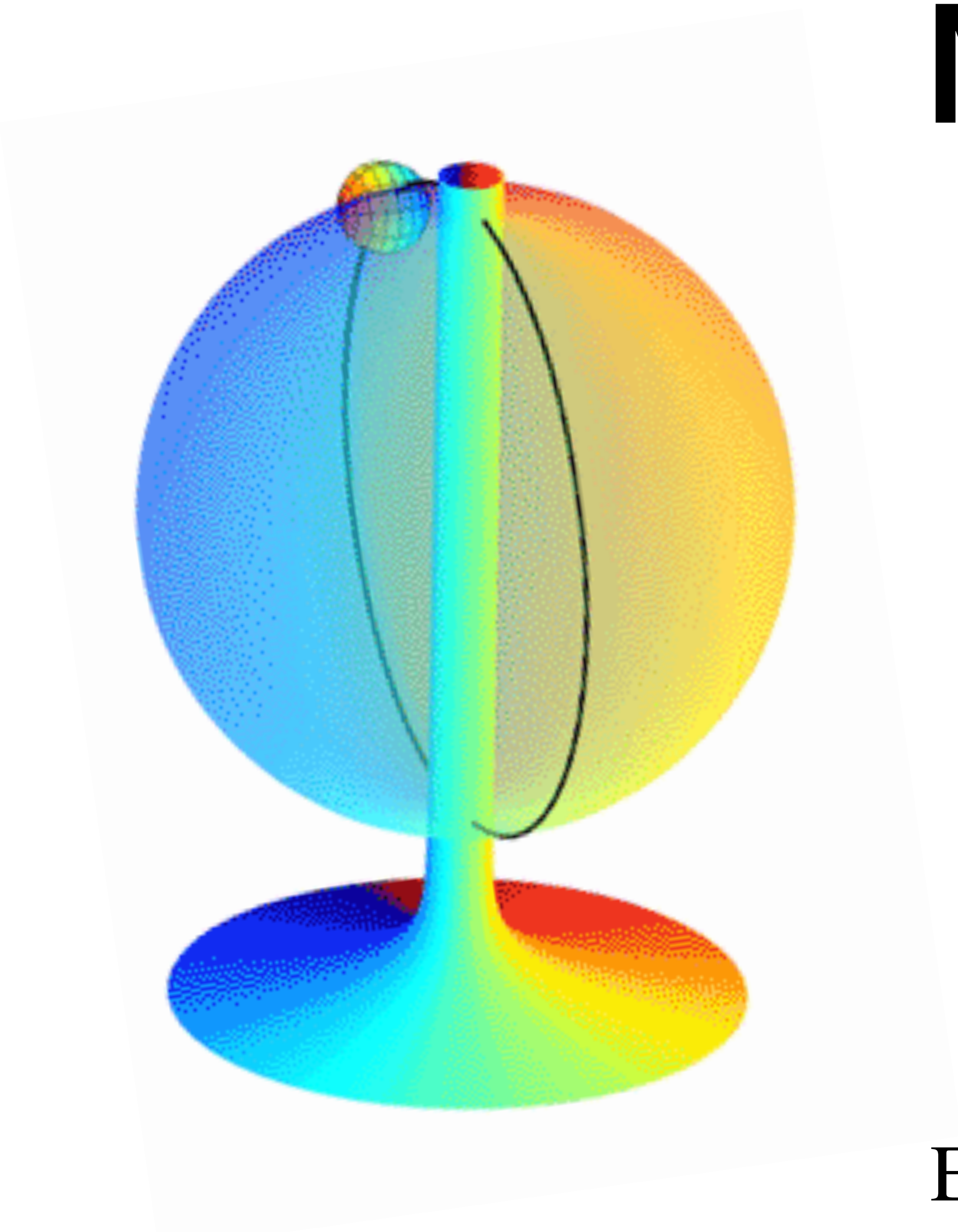
$$||\Omega|| = \frac{2\pi}{23h\ 56m\ 4s} = 7.292 \times 10^{-5}\ \mathrm{rad/s}$$

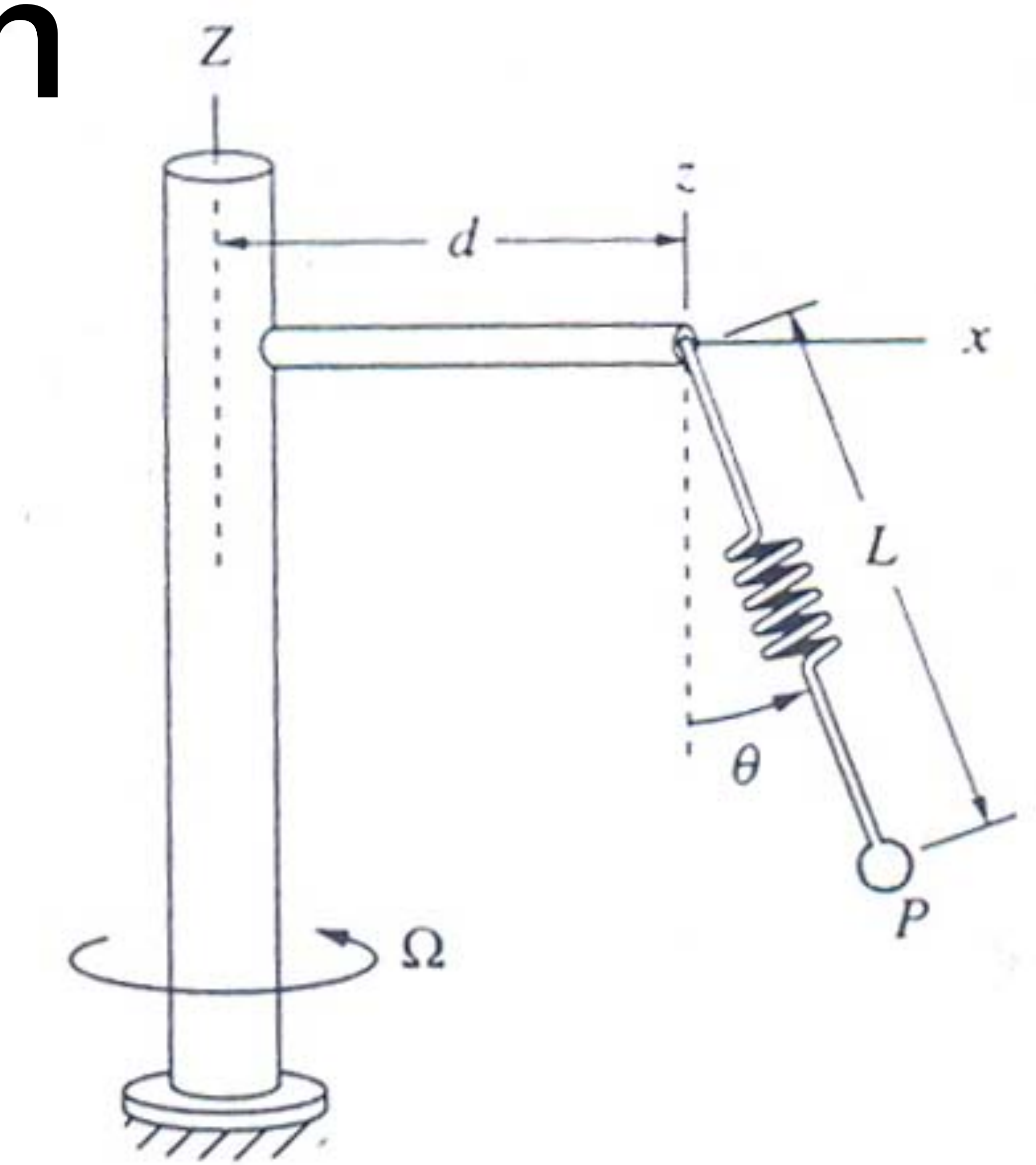
Cyclone Caterina

Northern polar cyclone



Write Down Equations of Motion





Easier to use

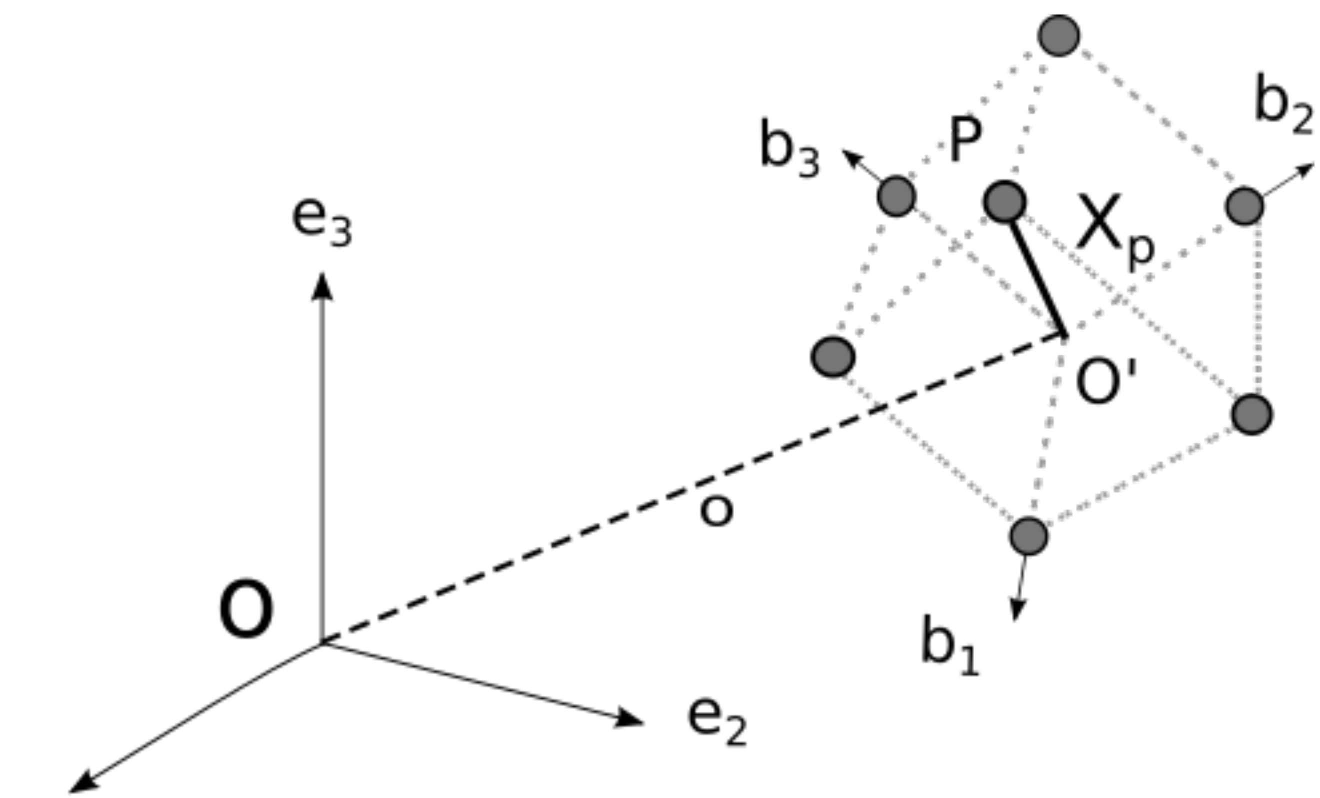
$$\dot{R} = R\widehat{\Omega} \qquad R^T f \triangleq F = mR^T\ddot{o} + m\widehat{\Omega}^2 X + 2m\,\widehat{\Omega}\,\dot{X} + m\dot{\widehat{\Omega}}\,X + m\ddot{X}$$

Instead of $m\ddot{x} = f$

Rigid Body Motion

Configuration of a Rigid Body

Rigid Body $\rightarrow$ Relative Distance Between Particles Remain Fixed



$$\mathbf{b} = \mathbf{e}'R = \mathbf{e}R$$

$$OO' = \mathbf{e}\, o$$

$$OP = \mathbf{e}x_p = \mathbf{b}X_p \;\rightarrow\; x_p = RX_p$$

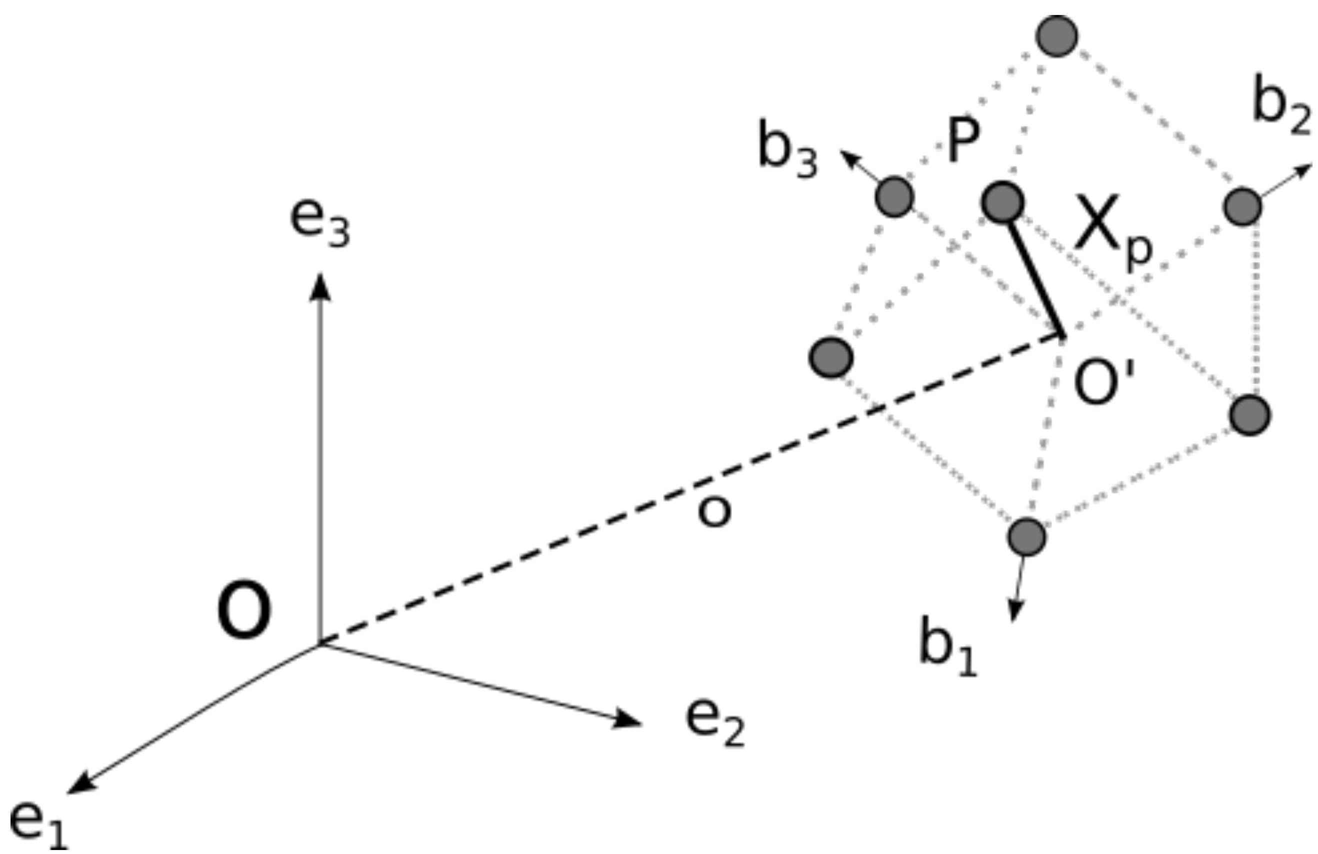
The pair $(o, R) \in \mathbb{R}^3 \times SO(3)$ uniquely determinse the configuration of the rigid body

$\mathbf{e}$ – An inertial frame

$\mathbf{b}$ – A frame fixed with respect to the particles

$$(o, R) \sim g \triangleq \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3) \;\longrightarrow\; \text{Euclidean motion group}$$

Rigid Body Kinematics



$$(o, R) \sim g \triangleq \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3)$$

$$\dot{g} = \begin{bmatrix} \dot{R} & \dot{o} \\ 0 & 0 \end{bmatrix}$$

$$V \triangleq R^T \dot{o}$$

$$\dot{R} = R\,\widehat{\Omega} \qquad \dot{o} = RV$$

$$\dot{g} = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{\Omega} & V \\ 0 & 0 \end{bmatrix} = g\,\zeta$$

Rigid Body Equations

$$\boxed{\begin{aligned} M &\triangleq \sum_{i=1}^{n} m_i \\ \bar{x} &\triangleq \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \end{aligned}}$$

$$f^e \triangleq \sum_{i=1}^{n} f_i^e \qquad \tau^e \triangleq \sum_{i=1}^{n} (x_i - o) \times f_i^e$$

$$p \triangleq \sum_{i=1}^{n} m_i \dot{x}_i \qquad \pi \triangleq \sum_{i=1}^{n} (x_i - o) \times m_i \dot{x}_i$$

$$\boxed{\begin{aligned} \dot{p} &= M\ddot{\bar{x}} = f^e \\ \dot{\pi} &= -M\dot{o} \times \dot{\bar{x}} + \tau^e \end{aligned}}$$

$$\bar{x} = o + R\bar{X}$$

$$\dot{\bar{x}} = \dot{o} + R\,\widehat{\Omega}\,\bar{X} + R\cancel{\dot{\bar{X}}}$$

$$\ddot{\bar{x}} = \ddot{o} + R(\widehat{\Omega}^2(t) + \dot{\widehat{\Omega}}(t))\bar{X}$$

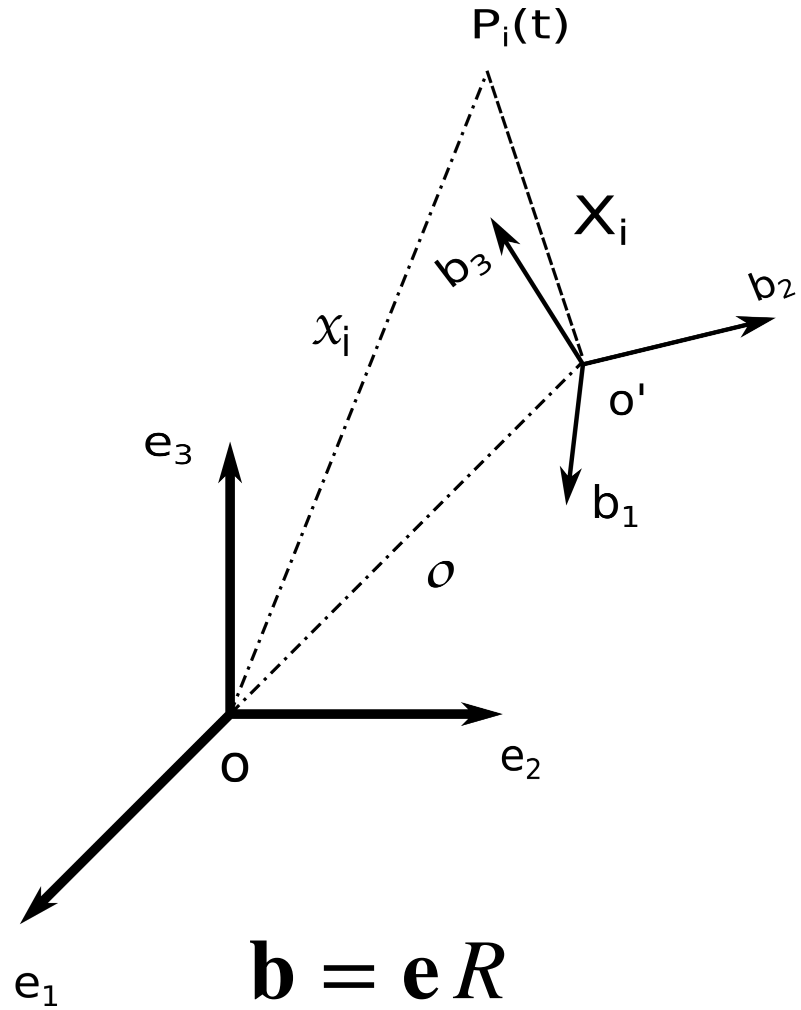
$$M\ddot{\bar{x}} = f^e$$

$$\boxed{MR^T\ddot{o} + M\left(\widehat{\Omega}^2(t) + \dot{\widehat{\Omega}}(t)\right)\bar{X} = R^T f^e \triangleq F^e}$$

$$\dot{\pi} = -M\dot{o} \times \dot{\bar{x}} + \tau^e$$

$$\boxed{\dot{\pi} = -M\dot{o} \times R(\Omega \times \bar{X}) + \tau^e}$$

Angular Momentum in the Body Frame



$$\mathbf{b} = \mathbf{e}\,R$$

$$x_i = o + RX_i$$

$$\dot{x}_i = \dot{o} + R\,\widehat{\Omega}\,X_i$$

$$\pi_i = m_i(x - o) \times \dot{x}_i = m_i\,R\left(X_i \times (\Omega \times X_i + R^T\dot{o})\right)$$

$$= R\left(-m_i\,\widehat{X}_i^2\Omega + m_i X_i \times R^T\dot{o}\right)$$

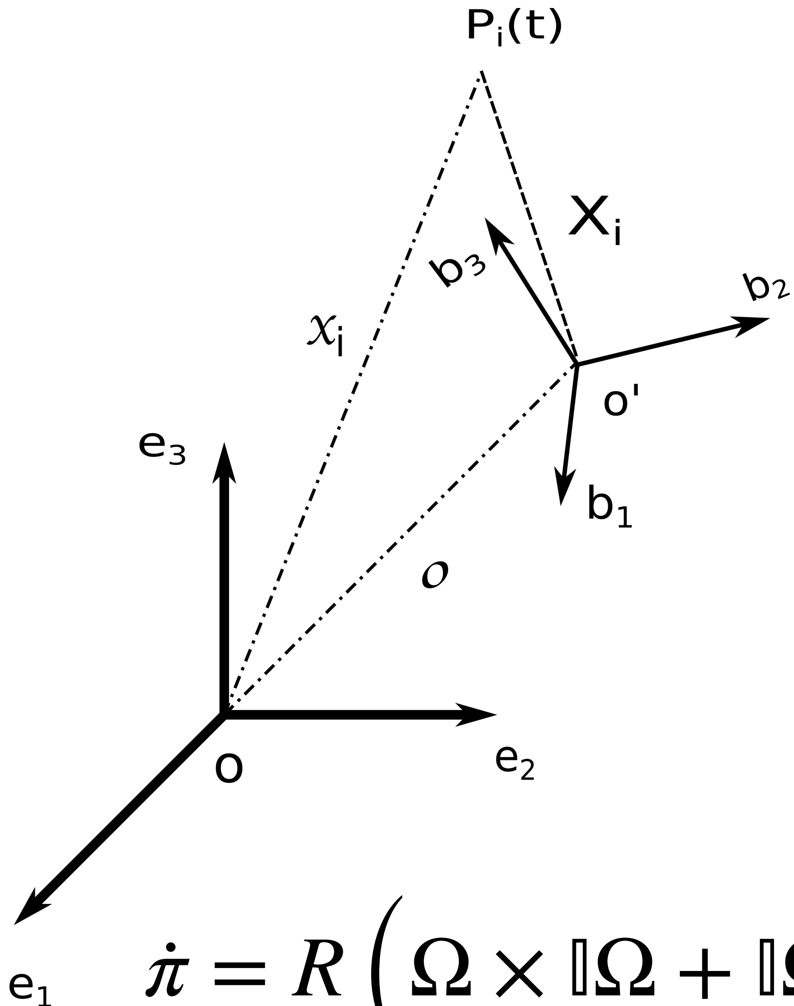
$$-m_i\,\widehat{X}_i^2 = m_i\left(||X_i||^2 I_{3\times 3} - X_i X_i^T\right) \triangleq \mathbb{I}_i$$

$\mathbb{I}_i \;\rightarrow\;$ Moment of Inertia of P about the point O'

$$\pi_i = R\,\underbrace{\left(\mathbb{I}_i\Omega + m_i X_i \times R^T\dot{o}\right)}_{\Pi_i} = R\,\Pi_i$$

$\Pi_i \;\rightarrow\;$ $\mathbf{b}$ – frame representation of angular momentum of P about the point O'

Rate of change of Angular Momentum



$$\pi_i = R\left(\mathbb{I}_i\Omega + m_i X_i \times R^T\dot{o}\right)$$

$$M \triangleq \sum_{i=1}^{n} m_i \qquad \mathbb{I} \triangleq \sum_{i=1}^{n} \mathbb{I}_i$$

$$\pi = \sum_{i=1}^{n} \pi_i = R\left(\mathbb{I}\Omega + M\bar{X} \times R^T\dot{o}\right)$$

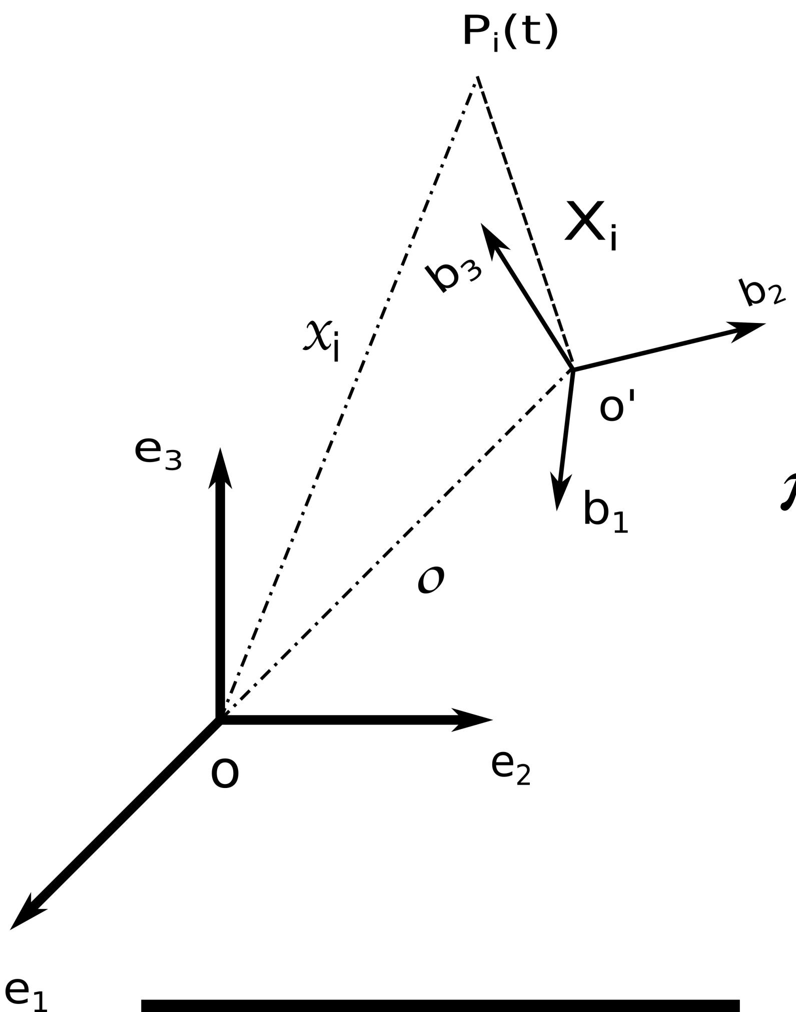
$$\dot{\pi} = R\left(\Omega \times \mathbb{I}\Omega + \mathbb{I}\dot{\Omega} + M\,\Omega \times \bar{X} \times R^T\dot{o} + M\bar{X} \times \left(R^T\ddot{o} - \Omega \times R^T\dot{o}\right)\right)$$

Using the Jacobi property $\quad A \times B \times C + B \times C \times A + C \times A \times B = 0$

$$\dot{\pi} = R\left(\Omega \times \mathbb{I}\Omega + \mathbb{I}\dot{\Omega} - M\,(R^T\dot{o}) \times \Omega \times \bar{X} + M\bar{X} \times R^T\ddot{o}\right)$$

Recall that we also showed $\quad \dot{\pi} = -M\dot{o} \times R(\Omega \times \bar{X}) + \tau^e$

Euler's Rigid Body Equation



$$M \triangleq \sum m_i$$

Total mass

$$\mathbb{I} \triangleq \sum \mathbb{I}_i = \sum m_i(||X_i||^2 I_{3\times 3} - X_i X_i^T)$$

Total moment of inertia about O

$$\dot{\pi} = R\left(\Omega \times \mathbb{I}\Omega + \mathbb{I}\dot{\Omega} - M\,(R^T\dot{o}) \times \Omega \times \bar{X} + M\bar{X} \times R^T\ddot{o}\right)$$

$$\dot{\pi} = -M\dot{o} \times R(\Omega \times \bar{X}) + \tau_e$$

$$\bar{X} \triangleq \frac{\sum m_i X_i}{\sum m_i}$$

Center of mass

$$\mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega - M\bar{X} \times R^T\ddot{o} + \underbrace{R^T\tau^e}_{T^e}$$

$$M\ddot{x} = f^e$$

$$M\ddot{o} + MR\left(\Omega \times \Omega \times \bar{X} + \dot{\Omega} \times \bar{X}\right) = f^e$$

Euler's Rigid Body Equation

$$\boxed{\mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega - M\bar{X} \times R^T\ddot{o} + \underbrace{R^T\tau^e}_{T^e}}$$

$$\boxed{M\ddot{o} + MR\left(\Omega \times \Omega \times \bar{X} + \dot{\Omega} \times \bar{X}\right) = f^e}$$

$$M\bar{X} \times R^T\ddot{o} = -M\left(\bar{X} \times \Omega \times \Omega \times \bar{X} - \bar{X} \times \bar{X} \times \dot{\Omega}\right) + \bar{X} \times R^T f^e$$

$$\widehat{A}^{\,2} = AA^T - ||A||^2 I_{3\times 3} \quad \rightarrow$$

$$\widehat{\bar{X}}\;\widehat{\Omega}^{\,2}\bar{X} = \widehat{\bar{X}}\;\Omega\Omega^T X = (\bar{X} \times \Omega)(\Omega \cdot \bar{X})$$

$$\widehat{\Omega}\;\widehat{\bar{X}}^{\,2}\Omega = \widehat{\Omega}\;\bar{X}\bar{X}^T\Omega = -(\bar{X} \times \Omega)(\Omega \cdot \bar{X})$$

$$M\bar{X} \times R^T\ddot{o} = -M\left(-\widehat{\Omega}\;\widehat{\bar{X}}^{\,2}\Omega - \widehat{\bar{X}}^{\,2}\dot{\Omega}\right) + \bar{X} \times R^T f^e$$

Euler's Rigid Body Equation

$$\boxed{\mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega - M\bar{X} \times R^T\ddot{o} + \underbrace{R^T\tau^e}_{T^e}}$$

$$\boxed{M\ddot{o} + MR\left(\Omega \times \Omega \times \bar{X} + \dot{\Omega} \times \bar{X}\right) = f^e}$$

$$M\bar{X} \times R^T\ddot{o} = -M\left(-\widehat{\Omega}\,\widehat{\bar{X}}^{\,2}\Omega - \widehat{\bar{X}}^{\,2}\dot{\Omega}\right) + \bar{X} \times R^T f^e$$

$$\mathbb{I}_X \triangleq -M\widehat{\bar{X}}^{\,2} \qquad \rightarrow \qquad M\bar{X} \times R^T\ddot{o} = -M\left(\Omega \times \mathbb{I}_X\Omega + \mathbb{I}_X\dot{\Omega}\right) + \bar{X} \times R^T f^e$$

$$\boxed{\mathbb{I} = \mathbb{I}_c + \mathbb{I}_x}$$

Parallel Axis Theorem

$$\boxed{\mathbb{I}_c\dot{\Omega} = \mathbb{I}_c\Omega \times \Omega \underbrace{- \bar{X} \times R^T f^e + R^T\tau^e}_{T_c^e}}$$

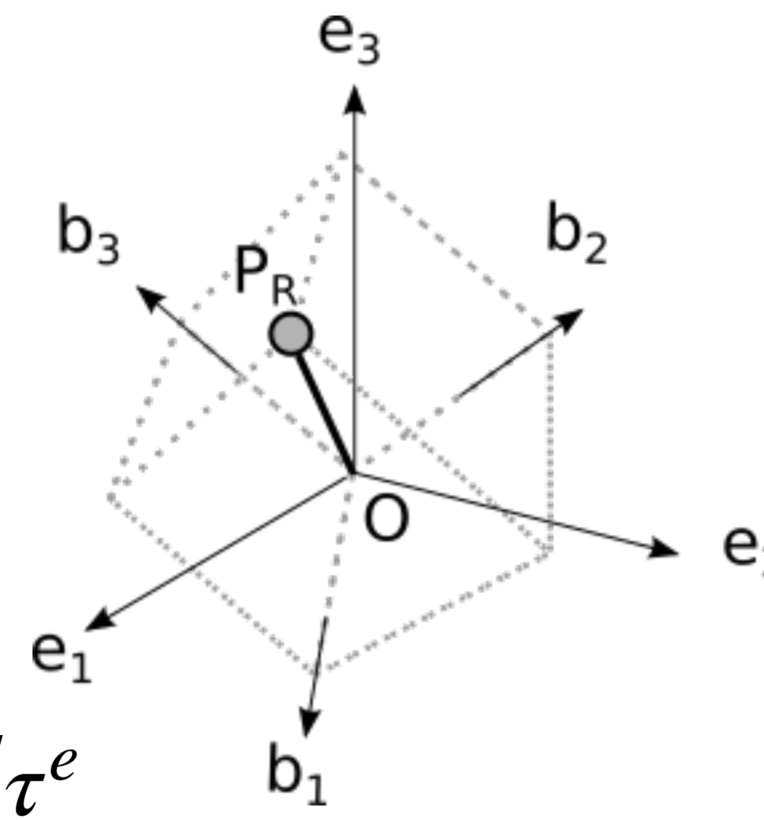
Euler's Rigid BodyEquations

Body frame at the center of mass

$$\bar{X} = 0$$

Kinematics

$$\dot{R} = R\,\widehat{\Omega}$$
$$\dot{o} = RV$$



Euler's Rigid Body Equations

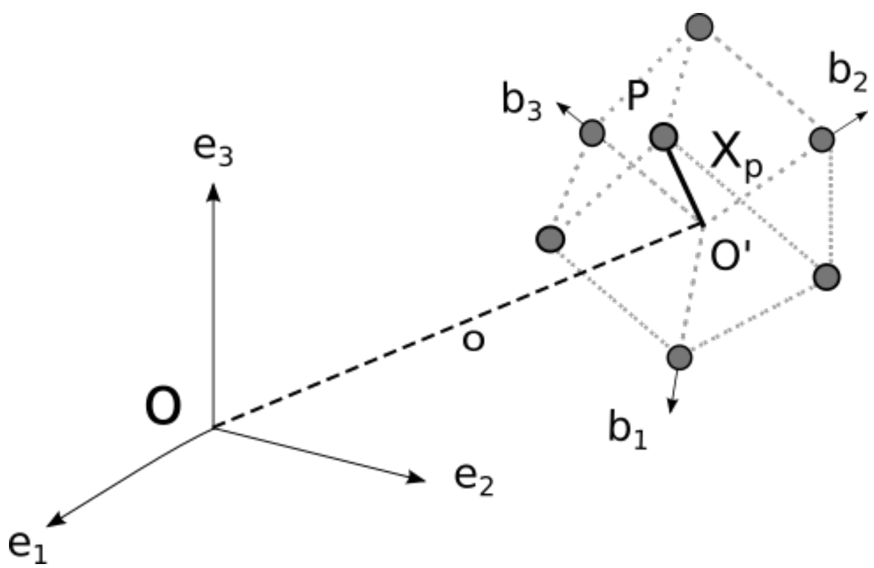
$$\mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega + \underbrace{R^T\tau^e}_{T^e}$$

$$M\dot{V} = -\,M\,\Omega \times V + \underbrace{R^T f^e}_{F^e}$$

The Last Equation can also be written in the particularly simple form

$$\ddot{o} = R(\dot{V} + \Omega \times V) \rightarrow M\ddot{o} = RF^e = f^e$$

Rigid Body Equations: Body Angular Momentum Version



$$\pi = R\left(\mathbb{I}\Omega + M\bar{X}\times(R^T\dot{o})\right)$$

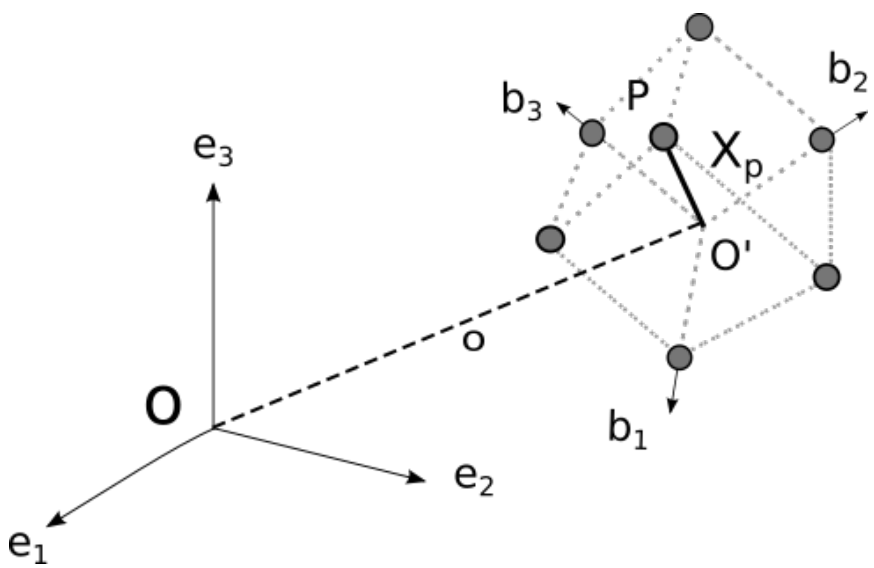
$$\boxed{\begin{array}{c}\textbf{Body frame at} \\ \textbf{the center of mass} \\ \bar{X} = 0\end{array}}$$

$$\pi = R\,\underbrace{\mathbb{I}\Omega}_{\Pi} = R\Pi$$

$$\begin{gathered}\dot{R} = R\,\widehat{\Omega} \\ \mathbb{I}\dot{\Omega} = \mathbb{I}\Omega\times\Omega + T^e \\ M\ddot{o} = f^e\end{gathered}$$

$$\boxed{\begin{gathered}\dot{R} = R\,\widehat{\mathbb{I}^{-1}\Pi} \\ \dot{\Pi} = \Pi\times\mathbb{I}^{-1}\Pi + T^e \\ M\ddot{o} = f^e\end{gathered}}$$

Rigid Body Equations: Spatial Angular Momentum Version



$$\pi = R\left(\mathbb{I}\Omega + M\bar{X} \times (R^T\dot{o})\right)$$

$$\boxed{\begin{array}{c}\textbf{Body frame at}\\ \textbf{the center of mass}\\ \bar{X} = 0\end{array}}$$

Spatial Angular Momentum

$$\pi = R\,\underbrace{\mathbb{I}\Omega}_{\Pi} = R\Pi$$

Body Angular Momentum

Spatial Angular Velocity

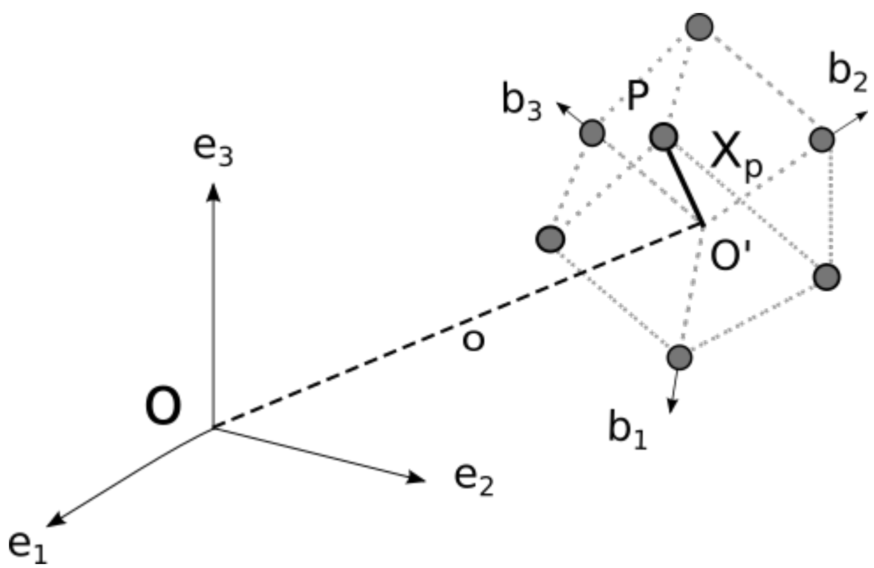
$$\omega = R\Omega$$

$$\pi = R\mathbb{I}\Omega = R\mathbb{I}R^T\omega = \mathbb{I}_R\omega$$

Spatial Inertia Tensor

$$\mathbb{I}_R \triangleq R\mathbb{I}R^T$$

Rigid Body Equations



$$
\begin{aligned}
\pi &= \mathbb{I}_R \omega \\
\dot{R} &= \widehat{\omega} R \\
\dot{p} &= f^e \\
\dot{\pi} &= \tau^e
\end{aligned}
$$

$$
\begin{gathered}
\dot{R} = R\,\widehat{\Omega} \\
\mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega + T^e \\
M\ddot{o} = f^e
\end{gathered}
$$

$$
\begin{gathered}
\dot{R} = R\,\widehat{\mathbb{I}^{-1}\Pi} \\
\dot{\Pi} = \Pi \times \mathbb{I}^{-1}\Pi + T^e \\
M\ddot{o} = f^e
\end{gathered}
$$

Rigid Body Equations in Principle Axis Body Frame

$$\boxed{\mathbb{J} \triangleq \sum \mathbb{J}_i = \sum m_i(||X_i||^2 I_{3\times 3} - X_i X_i^T)}$$

$\mathbb{J}$ is symmetric and positive definite $\quad \mathbb{J} = \mathrm{diag}\{\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3\}$

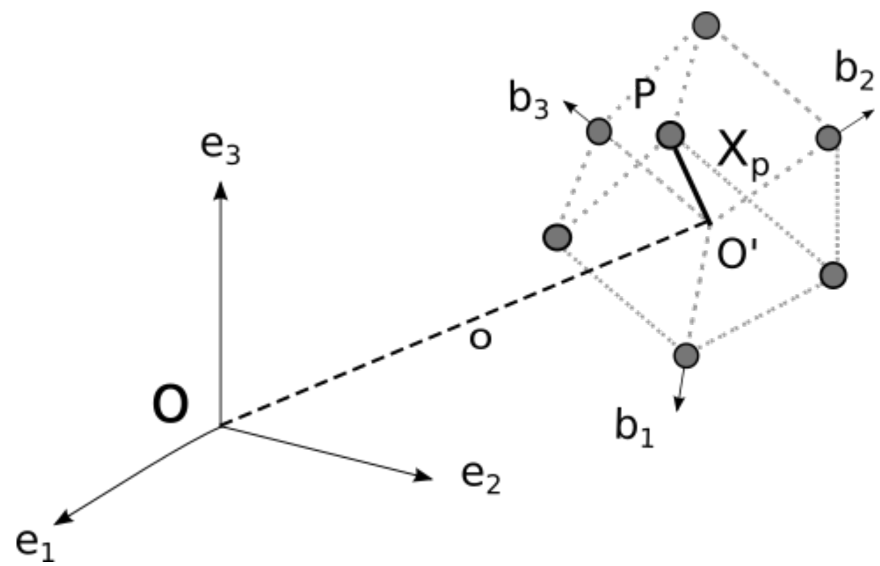
$$\boxed{\mathbb{J}\dot{\Omega} = \mathbb{J}\Omega \times \Omega + T^e}$$

$$\boxed{\begin{aligned}
\mathbb{J}_1\dot{\Omega}_1 &= (\mathbb{J}_2 - \mathbb{J}_3)\Omega_2\Omega_3 + T_1 \\
\mathbb{J}_2\dot{\Omega}_2 &= (\mathbb{J}_3 - \mathbb{J}_1)\Omega_3\Omega_1 + T_2 \\
\mathbb{J}_3\dot{\Omega}_3 &= (\mathbb{J}_1 - \mathbb{J}_2)\Omega_1\Omega_2 + T_3
\end{aligned}}$$

$$\boxed{\dot{\Pi} = \Pi \times \mathbb{J}^{-1}\Pi + T^e}$$

$$\boxed{\begin{aligned}
\dot{\Pi}_1 &= \frac{(\mathbb{J}_2 - \mathbb{J}_3)}{\mathbb{J}_2\mathbb{J}_3}\Pi_2\Pi_3 + T_1 \\
\dot{\Pi}_2 &= \frac{(\mathbb{J}_3 - \mathbb{J}_1)}{\mathbb{J}_3\mathbb{J}_1}\Pi_3\Pi_1 + T_2 \\
\dot{\Pi}_3 &= \frac{(\mathbb{J}_1 - \mathbb{J}_2)}{\mathbb{J}_1\mathbb{J}_2}\Pi_1\Pi_2 + T_3
\end{aligned}}$$

Kinetic Energy of a Rigid Body



$$\bar{X} = 0$$

$$\boxed{\begin{gathered} \dot{R} = R\,\widehat{\Omega} \\ \mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega + T^e \\ M\ddot{o} = f^e \end{gathered}}$$

$$\mathrm{KE} \triangleq \sum \frac{1}{2} m_i \,||\,\dot{x}_i(t)\,||^2 = \frac{1}{2} \sum m_i \,||\,\dot{o} + R\,\widehat{\Omega}\,X_i\,||^2 = \frac{1}{2} \sum m_i \,||\,R^T\dot{o} + \,\widehat{\Omega}\,X_i\,||^2$$

$$\mathrm{KE} = \frac{1}{2}\left(M\,||\,\dot{o}\,||^2 + \Omega^T \mathbb{I}\Omega\right)$$

$$\frac{d}{dt}\mathrm{KE} = \dot{o} \cdot f^e + \Omega \cdot T^e$$

Free Rigid Body Motion

Free Rigid Body Motion

$$\pi = \mathbb{I}_R \omega$$
$$\dot{R} = \widehat{\omega} R$$
$$\dot{p} = 0$$
$$\dot{\pi} = 0$$

$$\dot{R} = R\widehat{\mathbb{I}^{-1}\Pi}$$
$$\dot{\Pi} = \Pi \times \mathbb{I}^{-1}\Pi$$
$$M\ddot{o} = 0$$

$$\dot{\Pi} = \Pi \times \mathbb{I}^{-1}\Pi$$

$\mathbb{I}$ is symmetric and positive definite

$$\dot{\Pi}_1 = \frac{(\mathbb{I}_2 - \mathbb{I}_3)}{\mathbb{I}_2\mathbb{I}_3}\Pi_2\Pi_3$$
$$\dot{\Pi}_2 = \frac{(\mathbb{I}_3 - \mathbb{I}_1)}{\mathbb{I}_3\mathbb{I}_1}\Pi_3\Pi_1$$
$$\dot{\Pi}_3 = \frac{(\mathbb{I}_1 - \mathbb{I}_2)}{\mathbb{I}_1\mathbb{I}_2}\Pi_1\Pi_2$$

$$\mathrm{KE} = \frac{1}{2}\left(M||\dot{o}||^2 + \Omega^T\mathbb{I}\Omega\right)$$

$$\frac{d}{dt}\mathrm{KE} = \dot{o}\cdot f^e + \Omega \cdot T^e = 0$$

$$\frac{d}{dt}||\Pi(t)|| = \frac{\Pi \cdot T^e}{||\Pi||} = 0$$

Free Rigid Body Rotations

$$
\begin{aligned}
\dot{\Pi}_1 &= \frac{(\mathbb{I}_2 - \mathbb{I}_3)}{\mathbb{I}_2\mathbb{I}_3}\Pi_2\Pi_3, \\
\dot{\Pi}_2 &= \frac{(\mathbb{I}_3 - \mathbb{I}_1)}{\mathbb{I}_3\mathbb{I}_1}\Pi_3\Pi_1, \\
\dot{\Pi}_3 &= \frac{(\mathbb{I}_1 - \mathbb{I}_2)}{\mathbb{I}_1\mathbb{I}_2}\Pi_1\Pi_2\,.
\end{aligned}
$$

$$
\pi(t) = R(t)\begin{bmatrix} \mathbb{I}_1\Omega_1 \\ \mathbb{I}_2\Omega_2 \\ \mathbb{I}_3\Omega_3 \end{bmatrix} = R(t)\begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{bmatrix} = \pi = \text{constant}\;\; 3\times 1 \;\;\text{matrix},
$$

$$
\mathrm{KE} = \frac{1}{2}(\mathbb{I}_1\Omega_1^2 + \mathbb{I}_2\Omega_2^2 + \mathbb{I}_3\Omega_3^2) = \frac{1}{2}\left(\frac{\Pi_1^2}{\mathbb{I}_1} + \frac{\Pi_2^2}{\mathbb{I}_2} + \frac{\Pi_3^2}{\mathbb{I}_3}\right) = E = \text{constant},
$$

$$
||\,\Pi(t)\,||^2 = \mathbb{I}_1^2\Omega_1^2 + \mathbb{I}_2^2\Omega_2^2 + \mathbb{I}_3^2\Omega_3^2 = \Pi_1^2 + \Pi_2^2 + \Pi_3^2 = h^2 = \text{constant}\,.
$$

Consequences of Conservation

$$
\begin{aligned}
\dot{R} &= R\widehat{\mathbb{I}^{-1}\Pi} \\
\dot{\Pi}_1 &= \frac{(\mathbb{I}_2 - \mathbb{I}_3)}{\mathbb{I}_2\mathbb{I}_3}\Pi_2\Pi_3, \\
\dot{\Pi}_2 &= \frac{(\mathbb{I}_3 - \mathbb{I}_1)}{\mathbb{I}_3\mathbb{I}_1}\Pi_3\Pi_1, \\
\dot{\Pi}_3 &= \frac{(\mathbb{I}_1 - \mathbb{I}_2)}{\mathbb{I}_1\mathbb{I}_2}\Pi_1\Pi_2\,.
\end{aligned}
$$

$$
\pi(t) = R(t)\begin{bmatrix}\Pi_1 \\ \Pi_2 \\ \Pi_3\end{bmatrix} = \pi = \text{constant}\ \ 3\times 1\ \ \text{matrix},
$$

$$
\text{KE} = \frac{1}{2}\left(\frac{\Pi_1^2}{\mathbb{I}_1} + \frac{\Pi_2^2}{\mathbb{I}_2} + \frac{\Pi_3^2}{\mathbb{I}_3}\right) = E = \text{constant},
$$

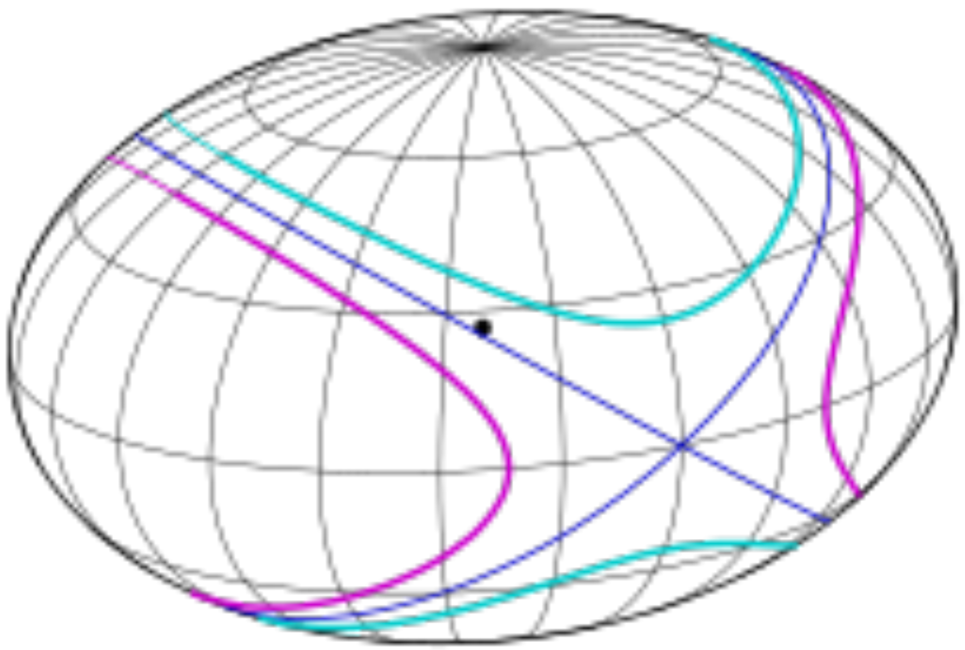
$$
||\Pi(t)||^2 = \Pi_1^2 + \Pi_2^2 + \Pi_3^2 = h^2 = \text{constant}\,.
$$

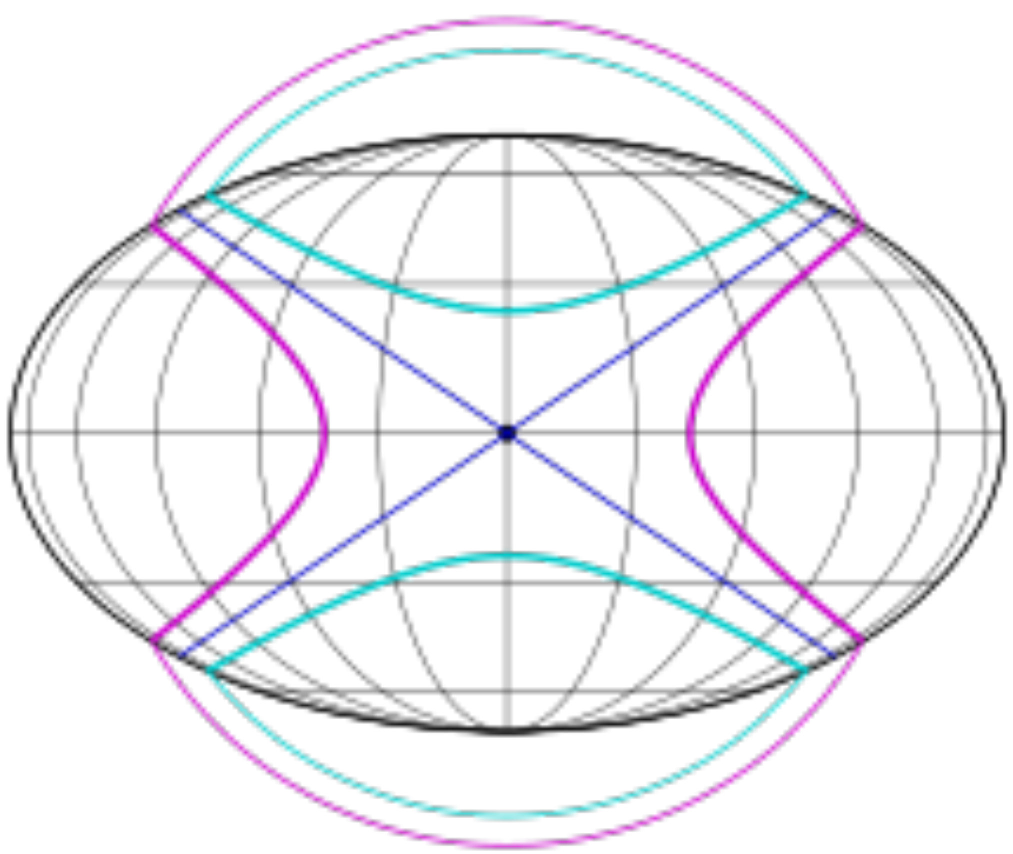
Solutions evolve on cosntant kinetic energy ellipsoids and cosntant $||\Pi||$ spheres

Trajectories in Body Angular Momentum Space

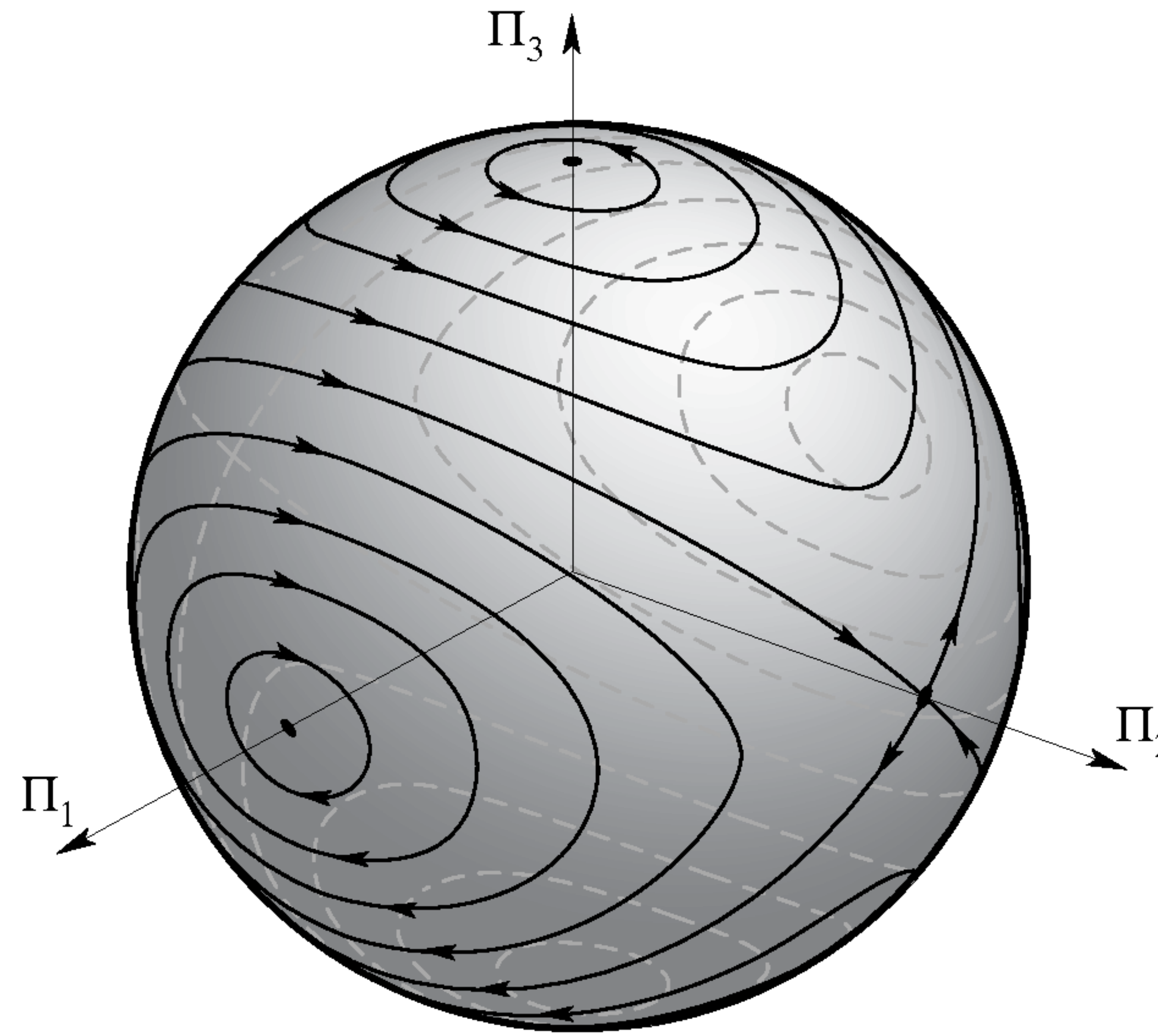
$$\frac{\Pi_1^2}{\mathbb{I}_1} + \frac{\Pi_2^2}{\mathbb{I}_2} + \frac{\Pi_3^2}{\mathbb{I}_3} = 2E$$

$$||\Pi(t)||^2 = \Pi_1^2 + \Pi_2^2 + \Pi_3^2 = h^2$$





$\mathbb{I}_1 > \mathbb{I}_2 > \mathbb{I}_3$



Trajectories in Body Angular Momentum Space

$$\dot{R} = R\widehat{\mathbb{I}^{-1}\Pi}$$

$$\dot{\Pi}_1 = \frac{(\mathbb{I}_2 - \mathbb{I}_3)}{\mathbb{I}_2\mathbb{I}_3}\Pi_2\Pi_3,$$

$$\dot{\Pi}_2 = \frac{(\mathbb{I}_3 - \mathbb{I}_1)}{\mathbb{I}_3\mathbb{I}_1}\Pi_3\Pi_1,$$

$$\dot{\Pi}_3 = \frac{(\mathbb{I}_1 - \mathbb{I}_2)}{\mathbb{I}_1\mathbb{I}_2}\Pi_1\Pi_2\,.$$

Relative Equilibria

$$\bar{\Pi}_1(t) \equiv \mu_1 \triangleq h[1\ \ 0\ \ 0]^T$$

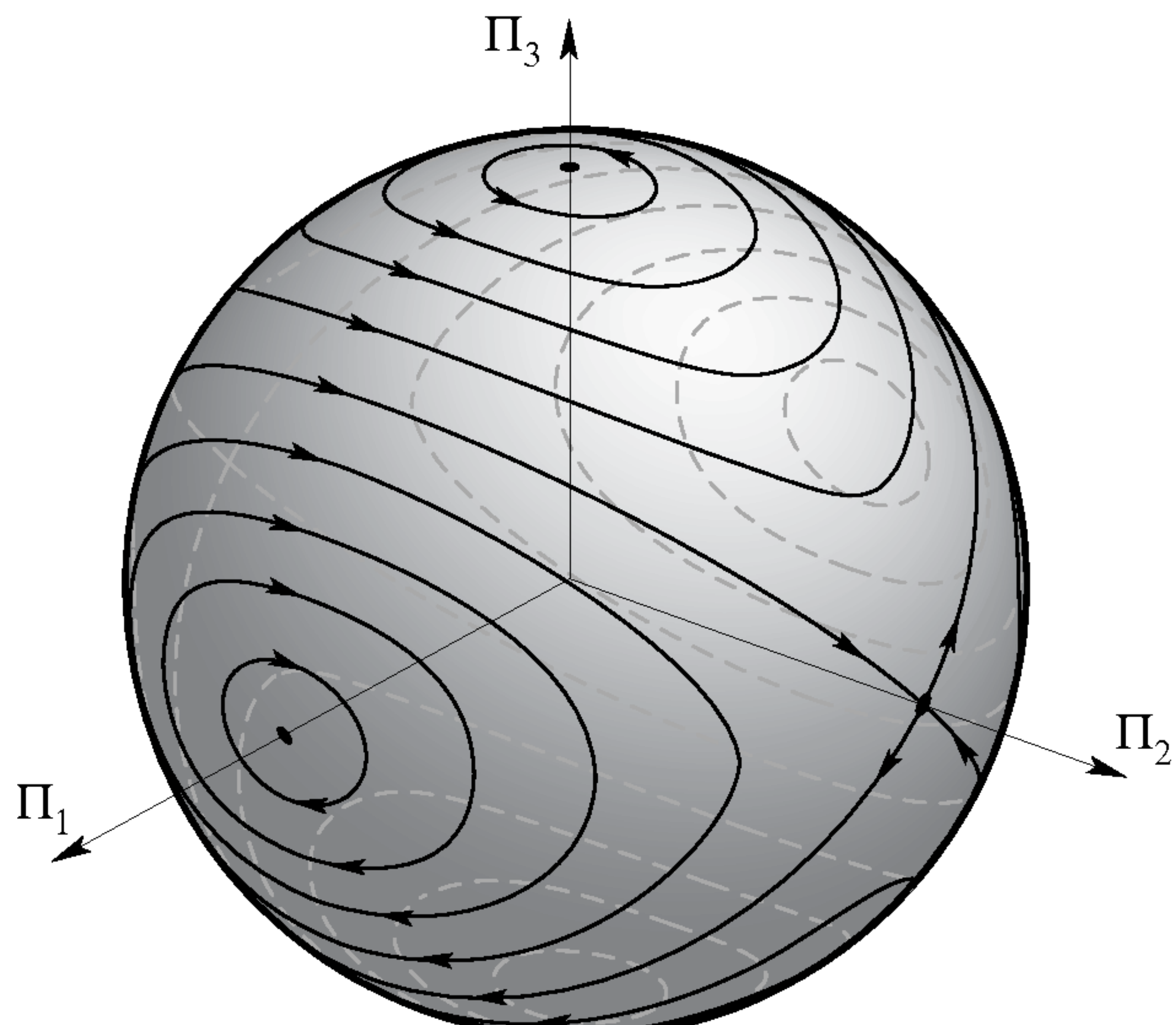
$$\bar{\Pi}_2(t) \equiv \mu_2 \triangleq h[0\ \ 1\ \ 0]^T$$

$$\bar{\Pi}_3(t) \equiv \mu_3 \triangleq h[0\ \ 0\ \ 1]^T$$

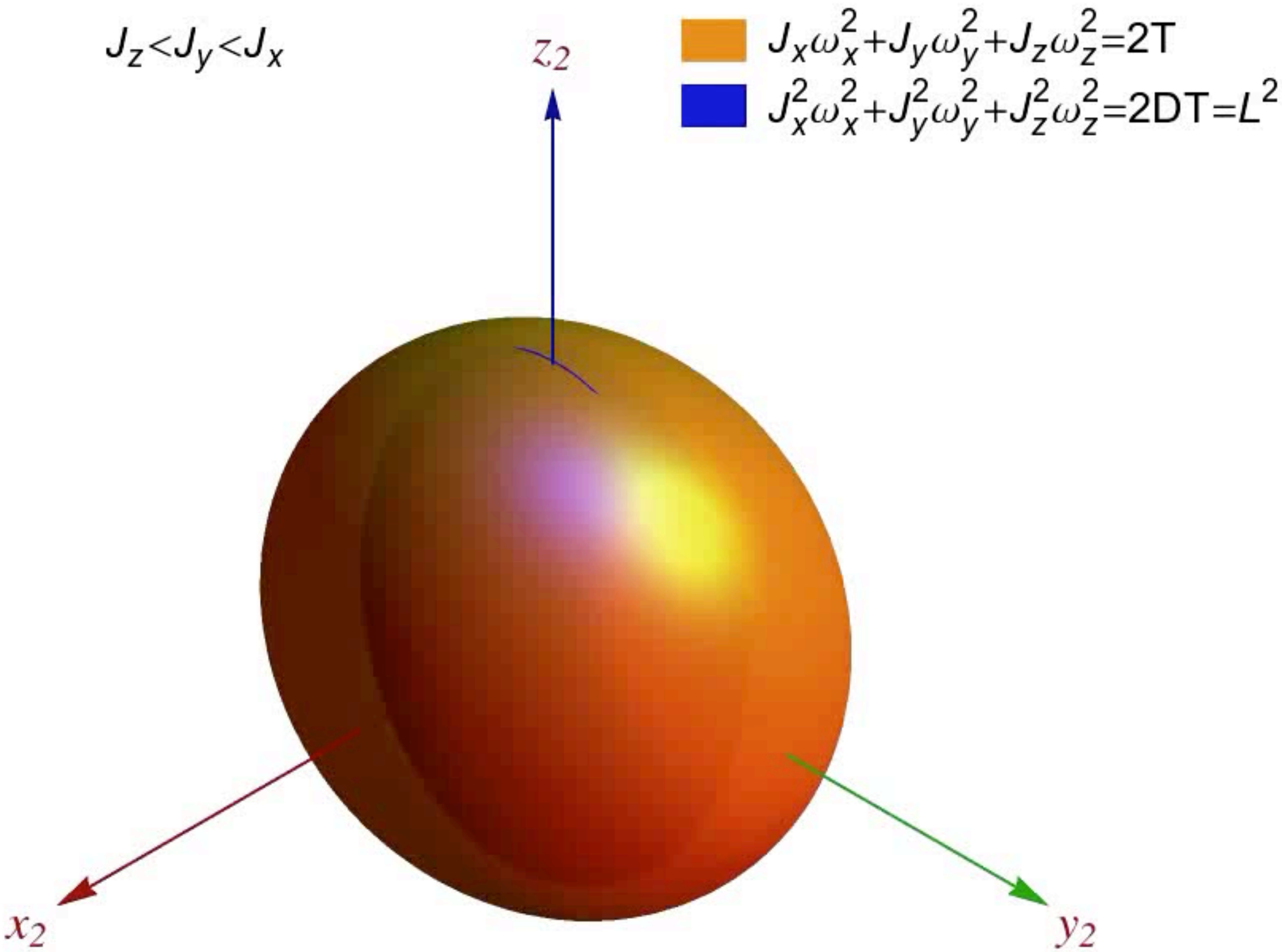
Almost all Solutions are periodic !

$\bar{\Pi}_1(t)$ & $\bar{\Pi}_3(t)$ are stable

$\bar{\Pi}_2(t)$ is unstable



A Nice Explanation by Vadim Yudintsev



$D=J_z$

D1-4
T-HANDLE
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GUIGNÉ
D1-5
LOLGIF.RU

The Space of Rotations

- **Is SO(3) a vector space?**
- **What is the dimension of SO(3)?**
- **Is SO(3) simply connected?**
- **How to parameterize SO(3)?**

Non-commutativity of Rotations

$$R_\alpha, R_\beta \in \mathrm{SO}(3)$$

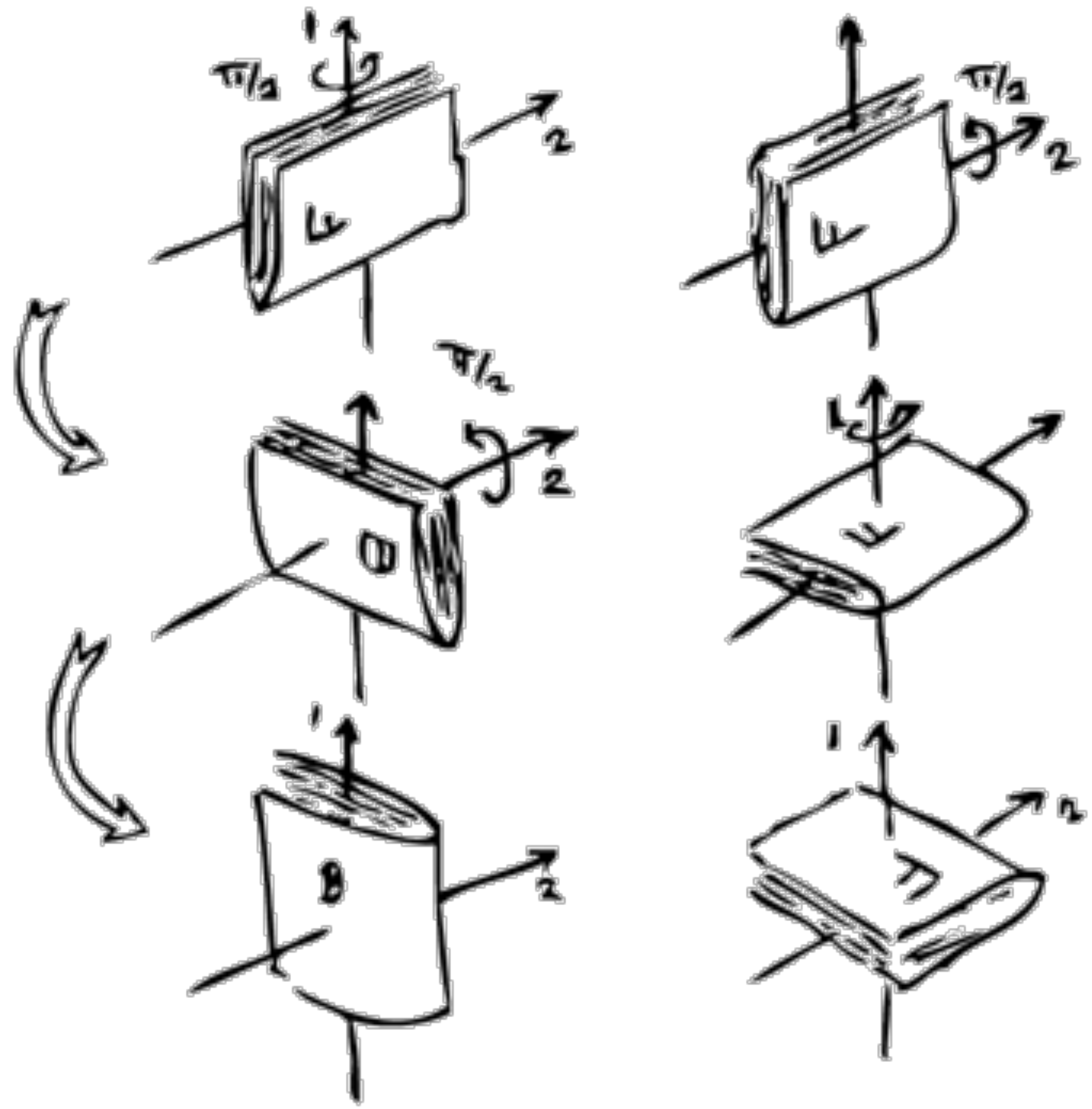
$$\mathbf{a} \triangleq \mathbf{e}R_\alpha$$
$$\mathbf{b} \triangleq \mathbf{a}R_\beta \quad \rightarrow \quad \mathbf{b} = \mathbf{e}R_\alpha R_\beta$$

$$\mathbf{a}' \triangleq \mathbf{e}R_\beta$$
$$\mathbf{b}' \triangleq \mathbf{a}'R_\alpha \quad \rightarrow \quad \mathbf{b}' = \mathbf{e}R_\beta R_\alpha$$

$$\mathbf{b} \; ? \; \mathbf{b}'$$

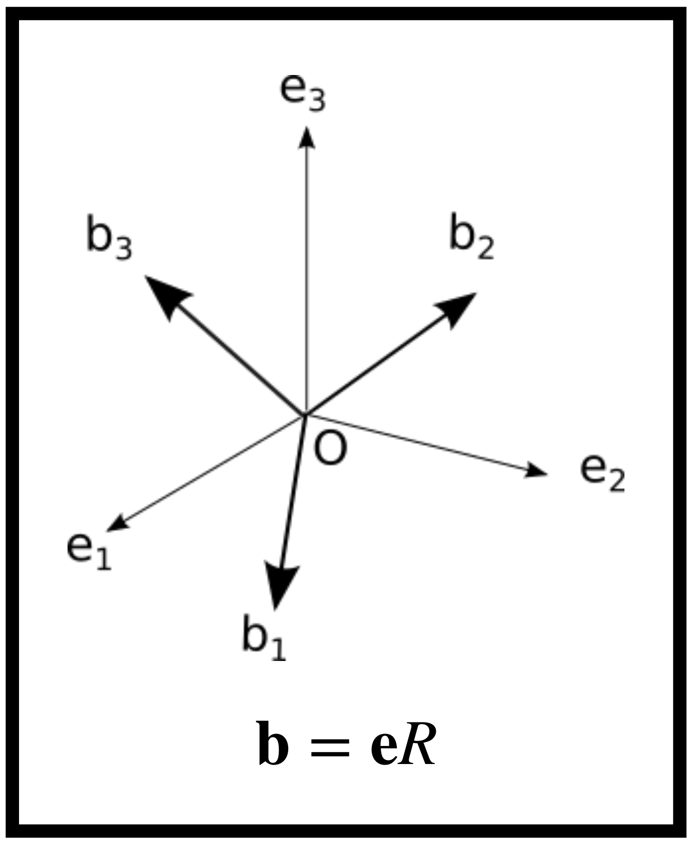
In general $R_\alpha R_\beta \neq R_\beta R_\alpha$

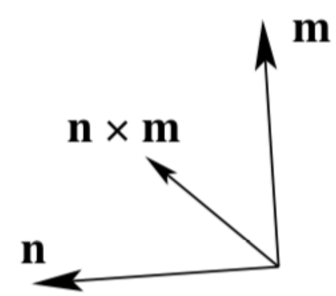
$$\mathbf{b} \neq \mathbf{b}'$$



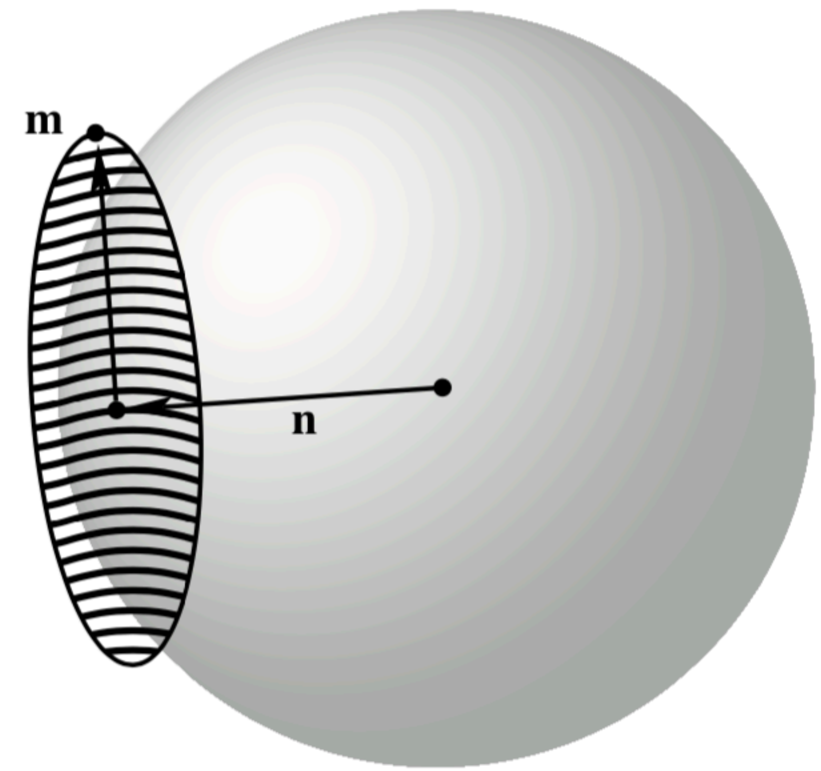
Poincare's Representation of SO(3)

$R \in SO(3)$ can be identified with an orthonormal frame





$$b_1 = n$$
$$b_2 = m$$
$$b_3 = n \times m$$



Every $R \in SO(3)$ can be represented by a unit tangent vector to the unit sphere and vice – versa

$$\rightarrow \quad SO(3) \simeq T_0\mathbb{S}^2$$

$$SO(3) \not\simeq \mathbb{S} \times \mathbb{S}^2$$

Rotations and the exp map

$$\dot{R} = R\,\widehat{\Omega} \qquad R(0) = I_{3\times 3} \quad \text{and} \quad \Omega = \text{constant}$$

solutions $R(t)$ correspond to a steady

rotation about the axis Ω at a rate $||\Omega||$

We can verify that the solution is

$$R(t) = \exp(t\,\widehat{\Omega}\,) \triangleq I + t\,\widehat{\Omega}\, + \frac{t^2}{2}\widehat{\Omega}^{\,2} + \frac{t^3}{3\,!}\widehat{\Omega}^{\,3} + \cdots$$

$R(1) = \exp(\,\widehat{\Omega}\,)$ is a rotation about the axis Ω by an angle $\theta \triangleq ||\Omega||$

$$\exp(\,\widehat{\Omega}\,) \in SO(3)$$

Note that $\quad \exp(\,\widehat{\Omega}\,)\,\Omega = \Omega$

Euler's Theorem

$$R \in SO(3) \quad \rightarrow \quad \lambda(R) = \{1, e^{i\theta}, e^{-i\theta}\}$$

$$\exists\, V \in \mathbb{R}^3 \quad \text{such that} \quad RV = V$$

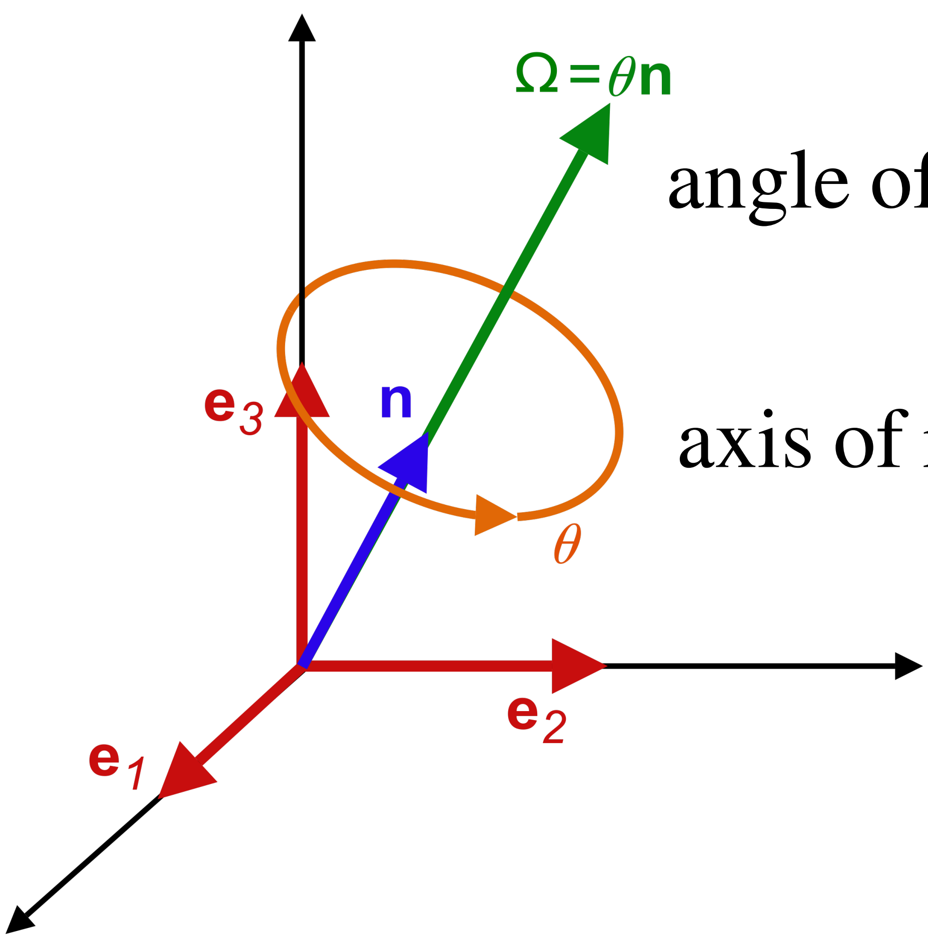
$R \in SO(3)$ is a rotation about some axis by some angle

Every $R \in SO(3)$ can be written down as $R = \exp(\widehat{\Omega})$ for some $\widehat{\Omega} \in so(3)$

Euler's Theorem and the Solid Sphere

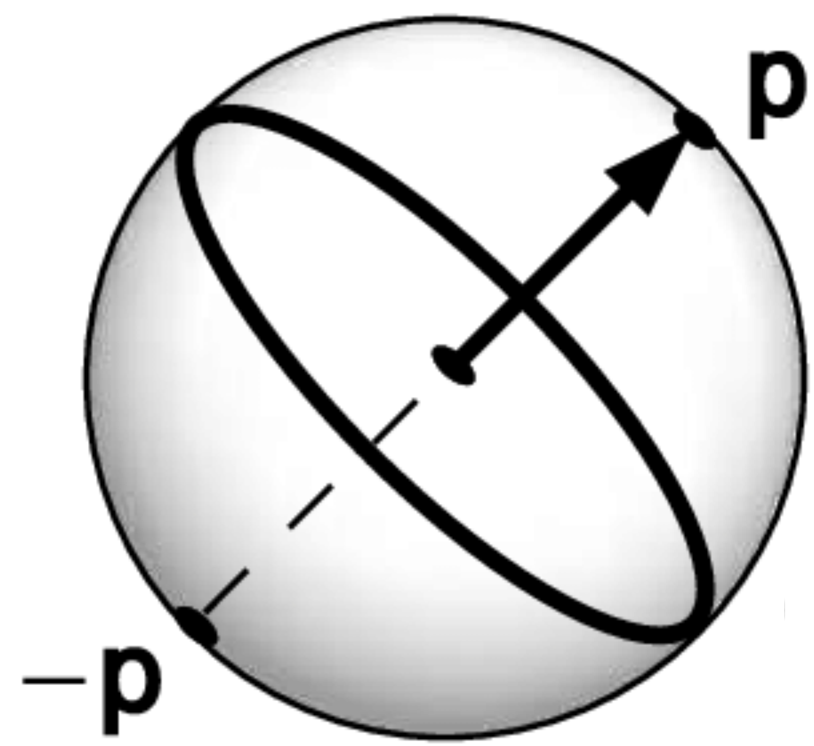
$R \in SO(3)$ is a rotation about some axis by some angle

i.e. $R = \exp(\widehat{\Omega})$ where $\widehat{\Omega} \in so(3)$



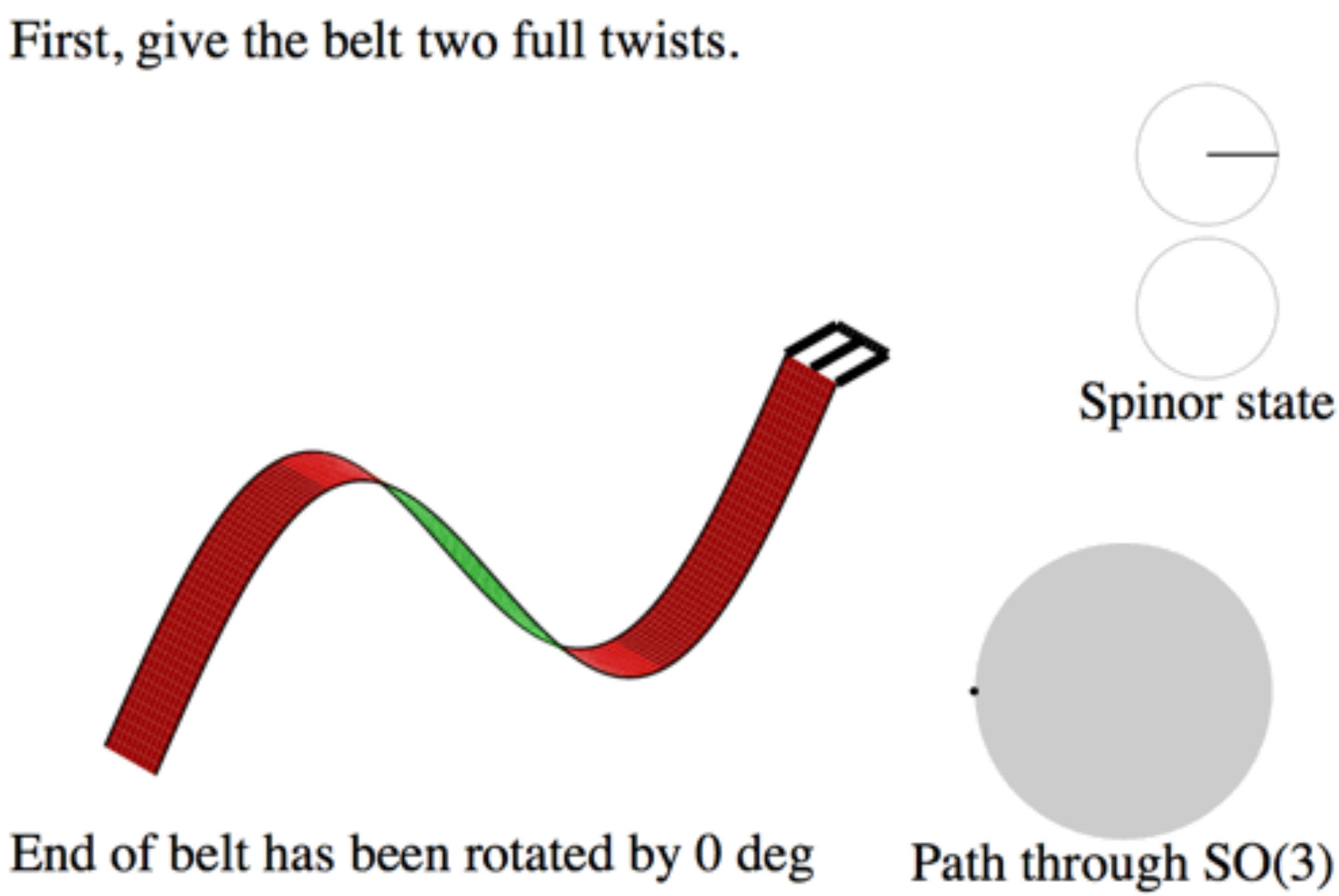
angle of rotation $\rightarrow \theta \triangleq ||\Omega||$

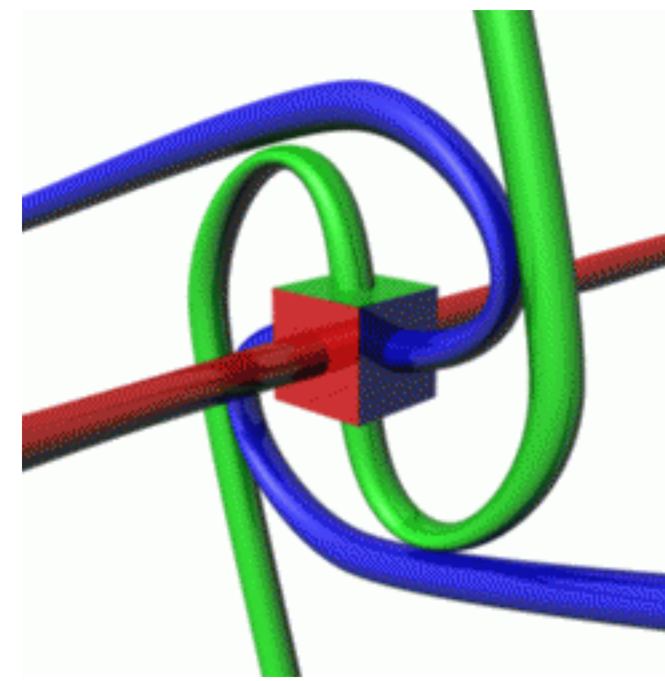
axis of rotation $\rightarrow n \triangleq \dfrac{\Omega}{||\Omega||}$



Every $R \in SO(3)$ can be represented by a point inside a solid sphere with antipodal points identified

Paul Dirac’s Belt Trick





Orientation Entanglement

SO(3) is not simply connected !

Quaternion Representation of Rotations

The map $\exp : so(3) \rightarrow SO(3)$ is onto

It can also be shown that it is also locally one − to − one

$$\exp(\widehat{\Omega}) = I + \widehat{\Omega} + \frac{1}{2}\widehat{\Omega}^2 + \frac{1}{3!}\widehat{\Omega}^3 + \cdots$$

$$\widehat{\Omega}^2 = (\Omega\Omega^T - ||\Omega||^2 I)$$

$$\widehat{\Omega}^3 = -||\Omega||^2\,\widehat{\Omega}, \qquad \widehat{\Omega}^4 = -||\Omega||^2\,\widehat{\Omega}^2, \qquad \widehat{\Omega}^5 = ||\Omega||^4\,\widehat{\Omega}, \qquad \cdots$$

Rodrigues Formula $\Rightarrow$ $$\exp\left(\widehat{\Omega}\right) = I + \frac{\sin||\Omega||}{||\Omega||}\widehat{\Omega} + \frac{1}{2}\left(\frac{\sin\frac{||\Omega||}{2}}{\frac{||\Omega||}{2}}\right)^2\widehat{\Omega}^2.$$

Quaternion Representation of Rotations

$$\exp\left(\widehat{\Omega}\right) = I + \frac{\sin||\Omega||}{||\Omega||}\widehat{\Omega} + \frac{1}{2}\left(\frac{\sin\frac{||\Omega||}{2}}{\frac{||\Omega||}{2}}\right)^2 \widehat{\Omega}^2.$$

Recall

$\exp\left(\widehat{\Omega}\right) \in SO(3)$ is a rotation about some axis by some angle

$$\text{angle of rotation} \rightarrow \theta \triangleq ||\Omega||$$

$$\text{axis of rotation} \rightarrow n \triangleq \frac{\Omega}{||\Omega||}$$

$$\exp\widehat{\Omega} = \exp\left(\theta\,\hat{n}\right) = I + \sin\theta\,\hat{n} + (1 - \cos\theta)\,\hat{n}^2$$

Quaternion Representation of Rotations

$$R = \exp\left(\theta\,\widehat{n}\right) = I + \sin\theta\,\widehat{n} + (1-\cos\theta)\,\widehat{n}^2$$

$$\text{were}\quad \theta = ||\Omega|| \quad \text{and} \quad n = \frac{\Omega}{||\Omega||}$$

$$\text{Let}\quad w \triangleq \sin\left(\frac{\theta}{2}\right)n \quad \text{and} \quad q_0 \triangleq \cos\left(\frac{\theta}{2}\right)$$

$$R = I + 2q_0\,\widehat{w} + 2\,\widehat{w}^2$$

$$\text{Note that}\quad q_0^2 + ||w||^2 = 1 \;\rightarrow\; (q_0, w) \in \mathbb{S}^3 = \{q \in \mathbb{R}^4 \;\;|\;\; ||q|| = 1\}$$

$$\text{Note that}\quad q = (q_0, w) \quad \text{and} \quad -q = (-q_0, -w) \rightarrow R$$

$$\mathrm{SO}(3) \simeq \mathbb{S}^3/\{1, -1\}$$

Quaternion Representation of Rotations

$$R = I + 2q_0 \widehat{w} + 2\widehat{w}^2$$

$$\text{where} \quad w \triangleq \sin\left(\frac{\theta}{2}\right)n \quad \text{and} \quad q_0 \triangleq \cos\left(\frac{\theta}{2}\right)$$

$$(q_0, w) \in \mathbb{S}^3 = \{q \in \mathbb{R}^4 \;\mid\; ||q|| = 1\} \qquad \mathrm{SO}(3) \simeq \mathbb{S}^3/\{1, -1\}$$

$$\mathrm{trace}(R) = -1 + 4q_0^2 = 2\cos\theta + 1,$$

$$R - R^T = 4q_0 \widehat{w} = 4\cos\left(\frac{\theta}{2}\right)\widehat{w}\ .$$

Quaternion Representation of Rotation Kinematics

$$\dot{R} = R\,\widehat{\Omega}$$

$$\text{Let} \quad \theta = ||\Omega|| \quad \text{and} \quad n = \frac{\Omega}{||\Omega||}$$

$$\begin{bmatrix} \dot{q}_0 \\ \dot{w} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\Omega \cdot w \\ q_0\Omega - \Omega \times w \end{bmatrix}$$

$$(q_0, w) \in \mathbb{S}^3 = \{q \in \mathbb{R}^4 \ \mid \ ||q|| = 1\}$$

SO(3) and SU(2)

$$\mathrm{SU}(2) = \{A \in \mathrm{GL}(2,\mathbb{C}) \ \ | \ \ \langle Ax, Ay \rangle = \langle x, y \rangle \ \ \text{and} \ \ \det(A) = 1\}$$

$su(2)$ **Space of traceless skew Hermitian matrices of dimension** 2×2

Pauli Spin Matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}], \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\} = \left\{ \frac{1}{2i}\sigma_1, \frac{1}{2i}\sigma_2, \frac{1}{2i}\sigma_3 \right\}$$ **Form a basis for** $su(2)$

$$\tilde{} \ : \mathbb{R}^3 \to \mathrm{su}(2)$$

$$\tilde{x} = \frac{1}{2i}(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) \triangleq \frac{1}{2i}(x \cdot \sigma) = \frac{1}{2}\begin{bmatrix} -ix_3 & -ix_1 - x_2 \\ -ix_1 + x_2 & ix_3 \end{bmatrix}$$

SO(3) and SU(2)

$$\mathrm{SU}(2) = \{A \in \mathrm{GL}(2,\mathbb{C}) \;\;|\;\; \langle Ax, Ay \rangle = \langle x, y \rangle \;\; \text{and} \;\; \det(A) = 1\}$$

$su(2)$ **Space of traceless skew Hermitian matrices of dimension** 2×2

$$\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\} = \left\{ \frac{1}{2i}\sigma_1, \frac{1}{2i}\sigma_2, \frac{1}{2i}\sigma_3 \right\}$$ **Form a basis for** $su(2)$

$$\tilde{e}_i \tilde{e}_i = \frac{1}{4} I_{2\times 2}$$

$$\tilde{e}_i \tilde{e}_j = \frac{1}{2}\tilde{e} \qquad \tilde{e}_j \tilde{e}_i = -\frac{1}{2}\tilde{e}_k \quad \text{where } (i, j, k) \text{ is a cyclic permutations of } (1,2,3)$$

$$\Omega \in \mathbb{R}^3 \qquad \theta = ||\Omega|| \qquad n = \Omega / ||\Omega||$$

$$\exp\left(\tilde{\Omega}\right) = \exp\left(\theta \frac{1}{2i}(n \cdot \sigma)\right) = \left(\cos\frac{\theta}{2}\right) I_{2\times 2} - i\left(\sin\frac{\theta}{2}\right)(n \cdot \sigma) = q_0 I_{2\times 2} - i(w \cdot \sigma)$$

SO(3) and SU(2)

$$\exp \; : \; su(2) \mapsto SU(2)$$

$$\Omega \in \mathbb{R}^3 \qquad \theta = ||\Omega|| \qquad n = \Omega/||\Omega||$$

$$\tilde{x} = \frac{1}{2i}(x \cdot \sigma)$$

$$\exp\left(\tilde{\Omega}\right) = \exp\left(\theta \frac{1}{2i}(n \cdot \sigma)\right) = q_0 \, I_{2\times 2} - i(w \cdot \sigma)$$

$$q = (q_0, w_1, w_2, w_3) \in \mathbb{S}^3 \to \begin{bmatrix} q_0 - iw_3 & -w_2 - iw_1 \\ w_2 - iw_1 & q_0 + iw_3 \end{bmatrix} \in \mathrm{SU}(2)\,.$$

$$SU(2) \simeq \mathbb{S}^3$$

Discretization of Rotational Kinematics

$$\dot{R} = R\,\widehat{\Omega} \qquad R(0) = R_0 \quad \text{and} \quad \Omega = \text{constant}$$

We can verify that the solution is $R(t) = R_0 \exp(t\,\widehat{\Omega}\,)$

$$R(t_{k+1}) = R(t_k)\exp(\Delta t\,\widehat{\Omega}\,)$$

What if $\Omega(t) \neq$ constant

$$R(t_{k+1}) \approx R(t_k)\exp(\Delta t\,\widehat{\Omega}_k)$$

Thank You