PDC Project Theory

Charles Beauville *283103* Jeremy Di Dio *301453* Robin Jaccard *310682* Daniel Tavares *289061*

(a) (i)

$$\begin{aligned} Y &= e^{j\Theta}x + Z \\ &= e^{j\Theta}(x + Z'), Z' = e^{-j\Theta}Z \end{aligned}$$

From the first exercise set of the course, we know that $\arg(Z) \sim U[0, 2\pi[$. And with :

$$\left.\begin{aligned} &|Z'| = |Z| \\ &\arg(Z') = \arg(Z) - \Theta \mod (2\pi) \sim U[0, 2\pi[\end{aligned}\right\} \implies \text{Z' does not depend on } \Theta \implies Z' \perp\!\!\!\perp \Theta$$

We want to show that $|Y| \perp\!\!\!\perp \arg(Y)$. We have :

$$\begin{aligned} |Y| &= |x + Z'| \\ arg(Y) &= \Theta + \arg(x + Z') \mod (2\pi) \sim U[0, 2\pi[\end{aligned}$$

We know that $\Theta \sim U[0, 2\pi[$. Therefore, independently of the distribution of $\arg(x + Z'), \arg(Y) \mod (2\pi)$ will also be uniformly distributed over $[0,\ 2\pi[$. We observe that $|Y|$ does not depend on $\arg(Y)$ nor Θ, thus $|Y| \perp\!\!\!\perp \arg(Y)$

(ii) We also proved on the previous point that $\arg(Y) \sim U[0, 2\pi[$

(iii)

$$Y = \begin{cases} Y_1 = |x|cos(\Theta + \arg(x)) + \Re(Z) \\ Y_2 = |x|sin(\Theta + \arg(x)) + \Im(Z) \end{cases}$$

We can create a dummy random variable :

$$Z' = \begin{cases} Z'_1 = \Re(Z) - |x|cos(\Theta + \arg(x)) \\ Z'_2 = \Im(Z) - |x|sin(\Theta + \arg(x)) \end{cases}$$
$$Z' \sim \mathcal{N}((-|x|cos(\Theta + \arg(x)), -|x|sin(\Theta + \arg(x))^T, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$$

$|Y| < \rho \leftrightarrow Z$ brings $e^{j\Theta}x$ inside the circle centred in (0,0) of radius ρ.
We observe that it is equivalent to $|Z'| < \rho$

$$Pr(U < \rho) = \int_{Z'_1}\int_{Z'_2} f_{Z'} dz'_1 dz'_2$$

We compute the gaussian bivariate :

$$\begin{aligned} P(U < \rho) &= \int_{Z'_1}\int_{Z'_2} f_{Z'} dz'_1 dz'_2 \\ &= \int_{Z'_1}\int_{Z'_2} \frac{1}{2\pi} \exp{(-\frac{1}{2}(\boldsymbol{z} - \boldsymbol{\mu})^T(\boldsymbol{z} - \boldsymbol{\mu}))} dz'_1 dz'_2 \end{aligned}$$
$$\begin{aligned} (\boldsymbol{z} - \boldsymbol{\mu})^T(\boldsymbol{z} - \boldsymbol{\mu}) &= (-|x|\cos(\arg(x) + \Theta) + |z|\cos(\arg(z)))^2 + (-|x|\sin(\arg(x) + \Theta) + |z|\sin(\arg(z)))^2 \\ &= |x|^2 + |z|^2 - 2|x||z|\cos(\arg(x) + \Theta)\cos(\arg(z)) - 2|x||z|\sin(\arg(x) + \Theta)\sin(\arg(z)) \\ &= |x|^2 + |z|^2 - 2|x||z|\cos(\arg(x) + \Theta - \arg(z)) \end{aligned}$$

Because we will integrate over $[0, 2\pi[$ for $\arg(z)$ and cos is even, 2π periodic, $\arg(x) + \Theta$ can be omitted and $-\arg(z)$ replaced by $\arg(z)$.
We use the following change of variables :

$$\begin{aligned} |Z| &= r \\ \theta &= \arg(z) \end{aligned}$$

Thus we have $|\mathbb{J}| = r$. And therefore :

$$Pr(U < \rho) = (2\pi)^{-}1 \int_0^{2\pi}\int_0^{\rho} \exp{(-\frac{1}{2}\left[r^2 + |x|^2 - 2r|x|\cos(\theta)\right])} r dr d\theta$$

Finally we can conclude that for any $u \geq 0$:

$$\begin{aligned}
f_{U|X}(u|x) &= \frac{d}{du} \Pr(U < u|X = x) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{du} \left[\int_0^u \exp(-\frac{1}{2} \left[r^2 + |x|^2 - 2r|x|\cos(\Theta) \right]) r dr \right] d\Theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \exp(-\frac{1}{2} \left[u^2 + |x|^2 - 2u|x|\cos(\Theta) \right]) u d\Theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \exp\left(-\frac{u^2 + |x|^2}{2} \right) \exp(-2u|x|\cos(\Theta)) u d\Theta \\
&= u \exp\left(-\frac{u^2 + |x|^2}{2} \right) \frac{1}{2\pi} \int_0^{2\pi} \exp(-2u|x|\cos(\Theta)) d\Theta \\
f_{U|X}(u|x) &= u \exp\left(-\frac{u^2 + |x|^2}{2} \right) I_0(u|x|)
\end{aligned}$$

(b) To show that $U = T(Y) = |Y|$ is a sufficient statistic, we will show that

$$H \to T(Y) \to Y \iff P(Y|T(Y), H) = P(Y|T(Y))$$

or in other word, we want to show that the probability of Y knowing $T(Y) = |Y|$ is independent of H.

Using the formula $Y = |Y|e^{j\arg(Y)}$, we see than we can show the above equality by showing that H is independent from $\arg(Y)$.

Indeed, if we know $|Y|$, the only part we don't know about Y is $e^{j\arg Y}$.

Thus, to show that $P(Y|T(Y), H) = P(Y|T(Y))$ we simply need to show that H is independent from $e^{j\arg(Y)}$ or equivalently, to show that H is independent from $\arg(Y)$.

Finally, this independence is a direct result of part ii) of exercise a) where we showed that, knowing x (equivalent of knowing H), the argument of Y ($\arg(Y)$) is uniformly distributed on $[0, 2\pi[$.

Thus, $U = |Y|$ is a sufficient statistic.

(c) The output of the channel Y will have the same statistic for $|c_j| = |c_l|$, for $j \neq l$ because the multiplication by $\exp(j\Theta)$ shifts the phase by Θ. Θ is uniformly distributed between 0 and 2π. Hence, we lose any information about $\arg(X)$, only the magnitude matters. It is impossible to distinguish 2 hypotheses that have the same magnitude.

(d) To show that replacing c_j by $|c_j|$ has no impact on the performance of an ML decoder, we will show that the distribution of Y is not affected by this modification. Let also note that replacing c_j by $|c_j|$ is equivalent to setting $\arg c_j$ to 0.

Let $W = \theta + \arg c_j \mod 2\pi$. We can then rewrite Y as follows

$$Y = |c_j|e^{j(\theta + \arg c_j)} + Z = |c_j|e^{jW} + Z$$

We can clearly see that the only modified random variable in this new system is W as only $\arg c_j$ is modified. Thus, in order to show that the distribution of Y does not change in this new system, we can simply show that the distribution of W does not change.

Since $\theta \sim \mathcal{U}(0, 2\pi)$, we also have that $W \sim \mathcal{U}(0, 2\pi)$. This result hold because W is uniform on the unit circle in the complex plane. Thus, we can use the 2π modulation as shown above without modifying the distribution. Finally, in the case of $\arg c_j = 0$, we have $W = \theta \mod 2\pi$ which is again uniformly distributed between 0 and 2π.

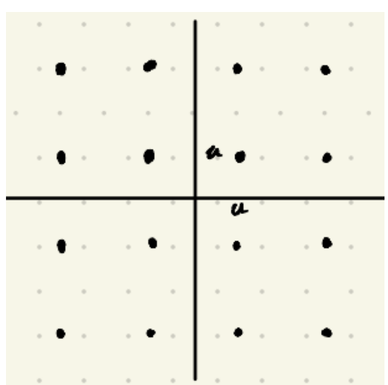
To conclude, we have shown that replacing c_j by $|c_j|$ does not affect the distribution of Y, thus an ML decoder on this new system will achieve the same performance as on the old system.

(e) $U = |Y|$ is a sufficient statistic. Hence, the error probability when observing U is the same as when observing Y. Here we have $||x| - u|$ is small compared to x because the only difference between $|x|$ and u is due to Z and we know that $a >> 1$.
Our system is then similar to a m-PAM with an AWGN channel. So we have that $P_e = \left(2 - \frac{2}{2^k}\right) Q\left(\frac{d}{2\sigma}\right) =$ with $d = 2a$ and $\sigma = 1$ we therefore have $\left(2 - \frac{2}{2^k}\right) Q\left(\frac{d}{2\sigma}\right) \approx 2Q(a)$ for k big enough.

(f)

$$
\begin{aligned}
\mathcal{E} &= \sum_{i=0}^{m-1} P_H(i)||c_i||^2 = \frac{1}{m} \sum_{i=0}^{m-1} \left((2i+1)a\right)^2 \\
&= \frac{1}{m} a^2 \sum_{i=0}^{m-1} (2i+1)^2 = \frac{a^2}{m} \frac{m(4m^2 - 1}{3} \\
&= \frac{a^2(4 \cdot 2^{2k} - 1)}{3} = \frac{a^2(4^{k+1} - 1)}{3} \\
\mathcal{E}_b &= \frac{\mathcal{E}}{k} = \frac{a^2(4^{k+1} - 1)}{3k} \approx \frac{a^2(4^{k+1})}{3k}
\end{aligned}
$$

(g) With the knowledge of Θ, we can use the QAM constellation. The c_j's will have the form $\pm(2p-1)a \pm i(2q-1)a$ with p,q in $1, ..., 2^{k-1}$. This gives us 2^{2k} different c_j (enough to transmit $2k$ bits).



Now we show that the energy per bit is $\frac{a^2(4^{k+1})}{3k}$:
By symmetry we only consider the first quadrant and multiply by 4:

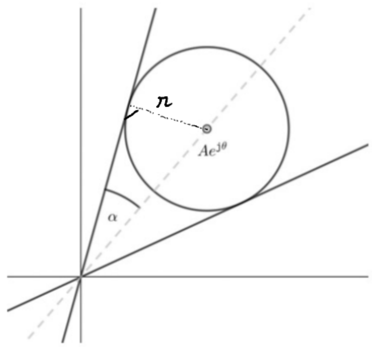
$$
\begin{aligned}
\mathcal{E} &= 4\sum_i P_H(i)||c_i||^2 = 4\frac{1}{2^{2k}}\sum_{p=1}^{2^{k-1}}\sum_{q=1}^{2^{k-1}}\left((2p+1)a\right)^2 + \left((2p+1)a\right)^2 \\
&= \frac{a^2}{2^{2k-2}}\sum_{p=1}^{2^{k-1}} 2^{k-1}(2p-1)^2 + \frac{2^{k-1}(4\left(2^{k-1}\right)^2-1)}{3} \\
&= \frac{a^2}{2^{2k-2}}2\frac{(2^{k-1})^2(4\cdot 2^{2k-2}-1)}{3} = \frac{2a^2\cdot 2^{2k-2}(2^{2k}-1)}{2^{k-2}3} = \frac{2a^2\left(4^k-1\right)}{3} \\
\mathcal{E}_b &= \frac{\mathcal{E}}{2k} = \frac{2a^2\left(4^k-1\right)}{3\cdot 2k} = \frac{a^2\left(4^k-1\right)}{3k}
\end{aligned}
$$

We still need to show that the error probability is less than $4Q(a)$. Recall : the real and imaginary parts of Z are i.i.d $\mathcal{N}(0,1)$
P_c = probability of being correct.
For the four corner parts, we have $P_c(corner) = \Pr\{(Z_R \leq a)(Z_C \geq -a)\}$
$= \Pr\{(Z_R \leq a)\}\Pr\{(Z_C \geq -a)\} = (1-Q(a))Q(-a) = (1-Q(a))^2$
For the points on the edges, we find:
$P_c(edge) = \Pr\{-a \leq Z_R \leq a)\}\Pr\{(Z_C \geq -a)\} = (1-2Q(a))(1-Q(-a))$
For all the points in the middle :
$P_c(middle) = \Pr\{-a \leq Z_R \leq a)\}\Pr\{-a \leq Z_C \leq a)\} = (1-2Q(a))^2$

Overall,

$$
\begin{aligned}
P_C &= \sum_{i=1}^{2^{2k}} P_H(i)P_C(i) \\
&= \frac{1}{2^{2k}}\left[4(1-Q(a))^2 + 4(2^k-2)(1-Q(a))(1-2Q(a)) + (2^k-2)^2(1-2Q(a))^2\right] \\
&= \frac{1}{2^{2k}}[4(Q(a)^2-2Q(a)+1) + 4(2^k-2)(1-3Q(a)+2Q(a)^2) \\
&\qquad + (2^{2k}-2^{k+2}+4)^2(1-4Q(a)+4Q(a)^2)] \\
&= 1 - \frac{8+12(2^k-2)+4(2^{2k}-2^{k+2}+4)}{2^{2k}}Q(a) + \frac{4+8(2^k-2)+4(2^{2k}-2^{k+2}+4)}{2^{2k}}Q(a)^2 \\
\implies P_e &= \frac{4\cdot 2^{2k}-4\cdot 2^k}{2^{2k}}Q(a) - \frac{4\cdot 2^{2k}-8\cdot 2^k+4}{2^{2k}}Q(a)^2 \\
&= \left(4-\frac{1}{2^{k-2}}\right)Q(a) - \left(4-\frac{1}{2^{k-3}}+\frac{4}{2^{2k}}\right)Q(a)^2 \\
&\leq 4Q(a)
\end{aligned}
$$

(h) The method to estimate Θ is simply to take $\arg(Y_1)$ as an estimate : $\hat{\Theta} = \arg(Y_1)$.
The probability that Y_1 falls outside of a circle of radius r centered around $Ae^{j\Theta}$ is $\exp(-r^2/2)$ as Z_1 is a 0 mean complex Gaussian (the real and imaginary parts are i.d.d $\mathcal{N}(0,1)$. If we want that the circle to inside the sector that contains all values that have an argument Θ' s.t. $|\Theta' - \Theta| < \alpha$, then $r = A\sin(\alpha)$.
So we have that
$\Pr(|\hat{\Theta} - \Theta| > \alpha)$ (i.e. the probability that Y_1 is outside of our sector) $\leq \exp(-A^2 \sin^2(\alpha)/2)$ (i.e. the probability that Y_1 is outside of our circle)



(i) We compute $Y = (Y_1 + Y_2 + ... + Y_{n_0})/\sqrt{n_0}$ and we estimate Θ with the plane of Y : $\hat{\Theta} = \arg(Y)$. It is the same situation as in (h) except this time our circle is centered around $\frac{n_0 A}{\sqrt{n_0}} e^{j\Theta} = \sqrt{n_0} A \cdot e^{j\Theta}$.
So we have that
$\Pr(|\hat{\Theta} - \Theta| > \alpha)$ (i.e. the probability that Y_1 is outside of our sector) $\leq \exp(-n_0 A^2 \sin^2(\alpha)/2)$ (i.e. the probability that Y_1 is outside of our circle)