Tutorial IV. Cosmic Expansion, Computer Assignment

Question 1. Computer assignment: numerical solutions to general FRWL universes

In general it is not possible to find analytical expressions for the expansion history $a(t)$ of the Universe. You will need to solve numerically, and the help of a computer is almost imperative. Let's first have a look at how to find a solution. Turn to the equation for the Hubble parameter $H(t)$,

$$H(t) \;=\; H_0 \sqrt{\frac{\Omega_{r,0}}{a^4} \,+\, \frac{\Omega_{m,0}}{a^3} \,+\, \frac{(1-\Omega_0)}{a^2} \,+\, \Omega_{\Lambda,0}} \tag{1}$$

a) Derive the expression for the time t for this generic situation,

$$H_0\,(t - t_{min}) \;=\; \int_{a_{min}}^{a} \frac{dx}{\sqrt{\Omega_{r,0}x^{-2} \,+\, \Omega_{m,0}x^{-1} \,+\, (1-\Omega_0) \,+\, \Omega_{\Lambda,0}x^2}} \tag{2}$$

with $a_{min} = 0$ for nearly all solutions, ie. the "normal" ones with a Big Bang. a_{min} may have some finite value in the case of the exceptional cases of a *Big Bounce* universe.

We are going to discard the contribution by radiation: we are going to study the behaviour of generic Universes filled with matter and a cosmological constant. Also, we also investigate case in which the universe is not generically flat.

To find a solution for a given Universe with matter density Ω_{m0}, cosmological constant contribution $\Omega_{\Lambda,0}$ and curvature dictated by $\Omega_0 = \Omega_{m0} + \Omega_{\Lambda,0}$, you have to numerically integrate the above integral for a range of values $a = [0,1]$ (or even further, e.g. $a = [0,10]$). You then obtain a long list of numbers $(a_j, H_0 t_j)$ (j=1,N). Subsequently, invert this relation to $(H_0 t_j, a_j)$ and make a plot of $a(t)$ vs. t.

Make sure that always your solutions have today's cosmic parameters, ie. today $a = 1$ and $H = H_0$, in the plots of $a(t)$ vs. t, take today as the "origin", ie. if you plot two different models on top of each other, they should intersect at $t = t_0$ (note $t = 0$ is a different time ago for different universes).

b) Solve numerically the above equation for a range of Universes, and plot a figure of expansion factor $a(t)$ vs. time $H_0 t$ (notice that time is plotted in terms of its dimensionless value $H_0 t$). In addition plot a figure of the age of the Universe ($H_0 t$ versus redshift z,

$$z = \frac{1}{a} - 1\,. \tag{3}$$

Do this for the following configurations:

- **Flat matter-dominated Universe**:

  $\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$.

  Compare this to the theoretically derived $a(t)$ for an Einstein-de Sitter Universe.

- **Flat Lambda dominated Universe**:

  $\Omega_{m,0} = 0$, $\Omega_{\Lambda,0} = 1$.

  Compare this to the theoretically derived $a(t)$ for a Lambda-dominated Universe.

- **Generic flat matter+Lambda Universes**

  $\Omega_0 = \Omega_{m,0} + \Omega_{\Lambda,0} = 1$:

  $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$:
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.1, 0.9)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.27, 0.73)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.5, 0.5)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.75, 0.25)$
  $(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.9, 0.1)$

  Compare these to the theoretically derived $a(t)$ for flat matter+Lambda Universes:

$$H_0\, t \;=\; \frac{2}{3\sqrt{1-\Omega_{m,0}}}\,\ln\{\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} \;+\; \sqrt{1+\left(\frac{a}{a_{m\Lambda}}\right)^{3}}\} \tag{4}$$

with

$$a_{m\Lambda} \;\equiv\; \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda 0}}\right)^{1/3} \;=\; \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3} \tag{5}$$

c) **Generic non-flat Universe**:
$(\Omega_{m,0}, \Omega_{\Lambda,0}) = (1.0, 0.3)$
$(\Omega_{m,0}, \Omega_{\Lambda,0}) = (1.0, 0.5)$
$(\Omega_{m,0}, \Omega_{\Lambda,0}) = (1.0, 1.0)$
$(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.3, 0.15)$
$(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.3, 0.3)$
$(\Omega_{m,0}, \Omega_{\Lambda,0}) = (0.3, 1.0)$