Lecture 09: Instrumental variables II

PPHA 34600
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From last time: introduction to IV

Recall that we want to split $D_i = B_i\varepsilon_i + C_i$ into the C_i and other parts

An instrumental variable...:

...Generates variation in C_i but is uncorrelated with ε_i

Z_i is a valid instrument for D_i when the following are satisfied:

1. **First stage**: $Cov(Z_i, D_i) \neq 0$
   - Z_i and D_i are related
   - Without this, you're capturing nothing
   - This is actually testable!
2. **Exclusion restriction**: $Cov(Z_i, \varepsilon_i) = 0$
   - Z_i and ε_i are **not** related
   - Z_i only affects Y_i through D_i
   - Fundamentally untestable! ☠

What makes IV so useful?

IV can be used in many ways:

- Causal inference (see last time)
- (Omitted variable bias)
- Measurement error

An omitted variable bias refresher

Suppose the true data generating process is:

$$Y_i = \alpha + \tau D_i + \beta X_i + \varepsilon_i$$

We'll assume:

- D_i and X_i are uncorrelated with ε_i
- D_i and X_i are correlated with each other
  - $\rightarrow$ $Cov(D_i, X_i) \neq 0$
- We don't observe X_i (☠)
  - $\rightarrow$ Now we have to run:

    $$Y_i = \alpha + \tau D_i + \nu_i$$

    where

    $$\nu_i = \varepsilon_i + \beta X_i$$

An omitted variable bias refresher

$$\hat{\tau} = \frac{Cov(Y_i, D_i)}{Var(D_i)}$$
$$= \underbrace{\frac{Cov(\alpha + \tau D_i + \beta X_i + \varepsilon_i, D_i)}{Var(D_i)}}_{\text{plug in for } Y_i}$$

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&= \underbrace{\tau + \beta \frac{Cov(D_i, X_i)}{Var(D_i)}}_{\text{simplify}}
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With omitted variable bias, we instead have

$$\hat{\tau} = \tau + \beta \frac{Cov(D_i, X_i)}{Var(D_i)} \neq \tau$$

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- Suppose we have an instrument, Z_i
- Our instrument, Z_i, moves D_i, but is uncorrelated with the error term

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  - $\rightarrow$ In other words, $Cov(Z_i, D_i) \neq 0$ and $Cov(Z_i, \nu_i) = 0$

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$$\hat{\tau}^{2SLS} = \tau + \beta \frac{Cov(\hat{D}_i, X_i)}{Var(\hat{D}_i)}$$

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$$\hat{\tau}^{2SLS} = \tau + \underbrace{0}_{\text{exclusion restriction}}$$

Measurement error

We often worry about measurement error:

- What happens if we don't perfectly observe D_i or Y_i?
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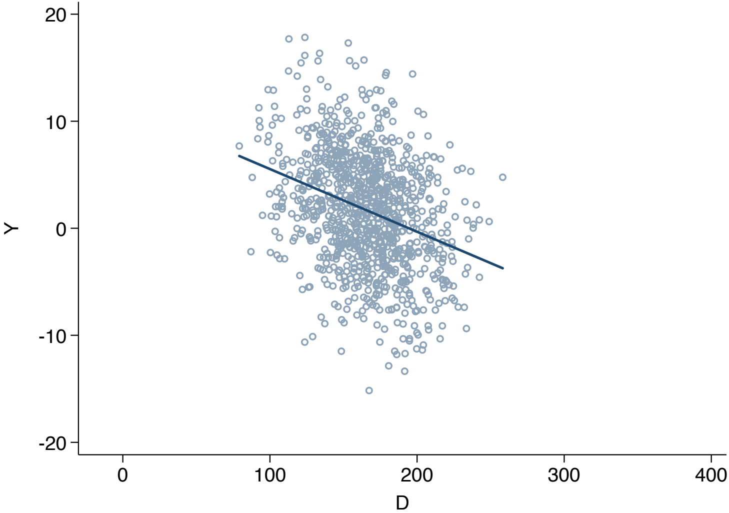
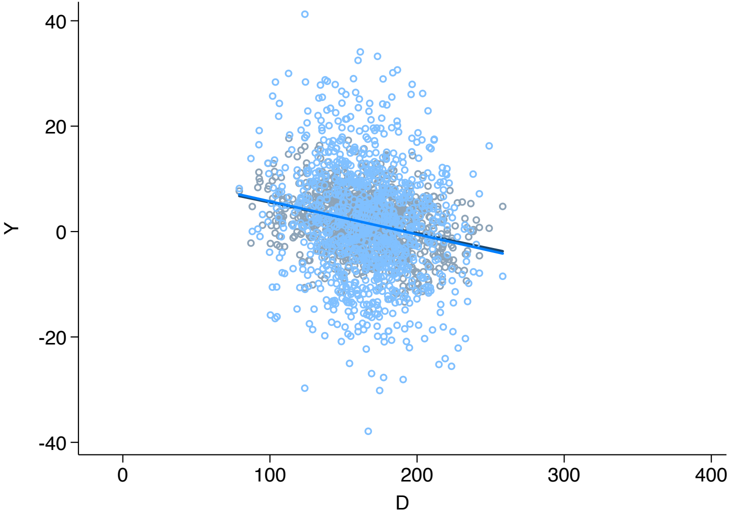
- The answer is...
- It depends!

Consider a true relationship



True relationship: $\tau = -0.059$

Measurement error in Y is fine



Estimated relationship: $\hat{\tau} = -0.061$

Why does this work?

We don't observe Y_i, but rather $\tilde{Y}_i = Y_i + \gamma_i$

$\rightarrow$ Assume $Cov(\gamma_i, \varepsilon_i) = 0$ and $Cov(\gamma_i, D_i) = 0$

If we run:

$$\tilde{Y}_i = \alpha + \tau D_i + \varepsilon_i$$

We'll estimate:

$$\hat{\tau} = \underbrace{\frac{Cov(\tilde{Y}_i, D_i)}{Var(D_i)}}_{\text{def'n of OLS}}$$

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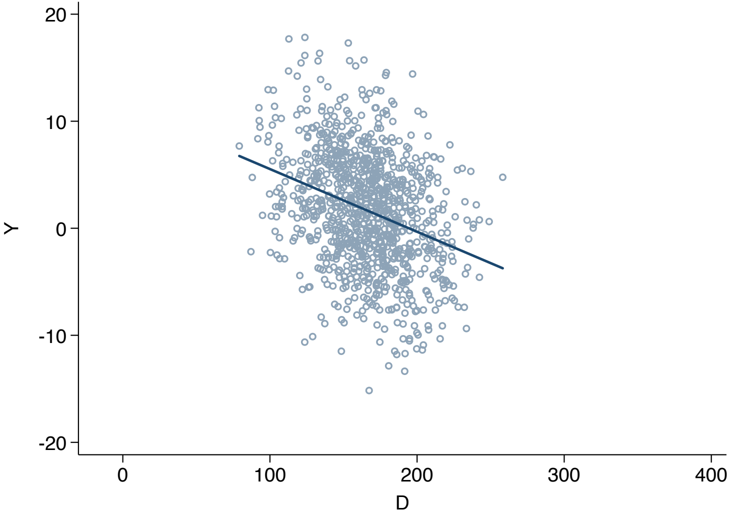
$$= \tau$$

Success!

Classical measurement error in D_i is bad



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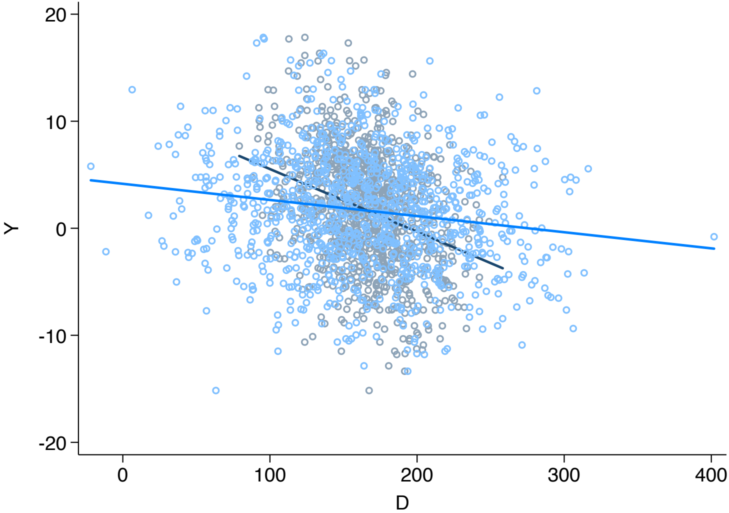
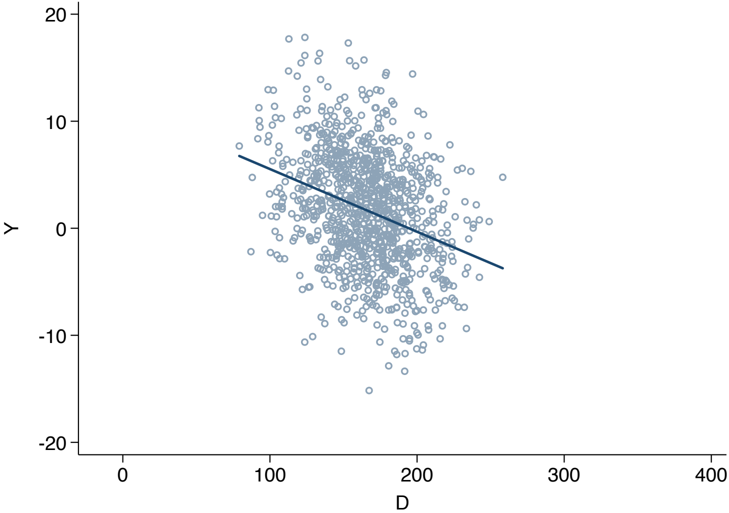


Estimated relationship: $\hat{\tau} = -0.015$

Non-classical measurement error in D_i is bad



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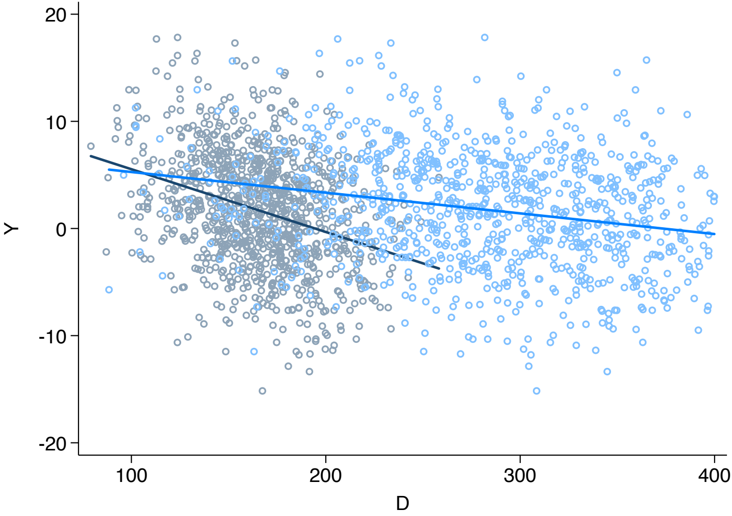

Estimated relationship: $\hat{\tau} = -0.019$

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We want to estimate:

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$$\tilde{D}_i = D_i + \gamma_i$$

Assume:

- $Cov(D_i, \varepsilon_i) = 0$: Treatment is (as good as) random
- $Cov(\gamma_i, D_i) = 0$: Measurement error is uncorrelated with treatment
- $Cov(\gamma_i, \varepsilon_i) = 0$: Measurement error is not in our original error term

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If we run:

$$Y_i = \alpha + \tau \tilde{D}_i + \varepsilon_i$$

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What's going wrong?

$$\hat{\tau} = \tau \left(\frac{Var(D_i)}{Var(D_i) + Var(\gamma_i)} \right)$$

This is the classic **attenuation bias**

- $\hat{\tau}$ is biased towards zero
- Note we assumed the most innocuous form of measurement error
- If measurement error is correlated with treatment, we get OVB

A second trip to the instrument store

To solve the measurement error problem, we'll use a clever instrument:

- We will instrument for $\tilde{D}_i$ with Z_i, a different noisy measure of D_i:

$$Z_i \equiv \mathring{D}_i = D_i + \zeta_i$$

Assume:

- $Cov(\zeta_i, D_i) = 0$: Measurement error is uncorrelated with treatment
- $Cov(\zeta_i, \gamma_i) = 0$: Measurement error in Z_i is uncorrelated w error in $\tilde{D}_i$
- $Cov(\zeta_i, \varepsilon_i) = 0$: Measurement error is uncorrelated with original error

Does this meet our two assumptions?

1. **First stage:** Yes! $Cov(Z_i, \tilde{D}_i) \neq 0$
2. **Exclusion restriction:** Yes! $Cov(Z_i, \varepsilon_i) = 0$

What do these assumptions buy us?

Remember that:

$$\hat{\tau}^{IV} = \frac{Cov(Y_i, Z_i)}{Cov(\tilde{D}_i, Z_i)}$$

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Success!

IV solves measurement error

<u>What's the intuition?</u>

- $\tilde{D}_i = D_i + \gamma_i$
- $Z_i = \mathring{D}_i = D_i + \zeta_i$

$\rightarrow$ Z_i and $\tilde{D}_i$ only have the **true** D_i in common

$\rightarrow$ We've assumed that $Cov(\gamma_i, \zeta_i) = 0$

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The first stage is:

$$\tilde{D}_i = \alpha + \pi \mathring{D}_i + \epsilon_i$$

$\rightarrow$ We're only using the variation from D_i (not from ζ_i or γ_i)!

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$\rightarrow$ **Important caveat:** This does not work with binary D_i!

- If true $D_i = 1$, measurement error can only be -1 or 0
- If true $D_i = 0$, measurement error can only be 0 or 1

$\rightarrow$ Measurement error in $\tilde{D}_i$ and $\mathring{D}_i$ will be correlated

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Assume:

- $Cov(D_i, \varepsilon_i) = 0$: Treatment is (as good as) random
- $Cov(\gamma_i, \varepsilon_i) = 0$: Measurement error is not in our original error term

Relax the orthogonality assumption:

- Allow $Cov(D_i, \gamma_i) \neq 0$: Measurement error can be correlated with treatment

IV with non-classical measurement error

Again, we'll now have:

$$\hat{\tau} = \frac{Cov(Y_i, D_i + \gamma_i)}{Var(D_i + \gamma_i)}$$

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IV with non-classical measurement error

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$$+ \frac{\tau Cov(D_i, \gamma_i) + Cov(D_i, \varepsilon_i) + Cov(\gamma_i, \varepsilon_i)}{Var(D_i) + Var(\gamma_i) + 2Cov(D_i, \gamma_i)}$$

$$= \underbrace{\tau \left(\frac{Var(D_i) + Cov(D_i, \gamma_i)}{Var(D_i) + Var(\gamma_i) + 2Cov(D_i, \gamma_i)} \right)}_{\text{rearrange}}$$

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$$= \underbrace{\tau \left(\frac{Var(D_i) + Cov(D_i, \gamma_i)}{Var(D_i) + Var(\gamma_i) + 2Cov(D_i, \gamma_i)} \right)}_{\text{rearrange}}$$

$\rightarrow$ Again, we get **bias**

$\rightarrow$ Note that this **need not attenuate** $\hat{\tau}$

$\rightarrow$ This can actually **flip the sign** of $\hat{\tau}$ relative to τ ☠

$\rightarrow$ (This depends on the sign of $Cov(D_i, \gamma_i)$)

What about IV?

Just like before...:

$$\begin{aligned}\hat{\tau}^{IV} &= \frac{Cov(Y_i, Z_i)}{Cov(\tilde{D}_i, Z_i)} \\ &= \underbrace{\frac{Cov(\tau D_i + \varepsilon_i, Z_i)}{Cov(D_i + \gamma_i, Z_i)}}_{\text{definition of } Y_i,\ \tilde{D}_i} \\ &= \underbrace{\frac{\tau Cov(D_i, Z_i) + Cov(\varepsilon_i, Z_i)}{Cov(D_i, Z_i) + Cov(\gamma_i, Z_i)}}_{\text{variance rules}} \\ &= \underbrace{\tau \left(\frac{Cov(D_i, Z_i)}{Cov(D_i, Z_i)}\right)}_{\text{assumptions}} \\ &= \tau\end{aligned}$$

Success!

Actual estimation is straightforward

All we need to do is run:

$$D_i = \gamma Z_i + \eta_i$$

and

$$Y_i = \tau \hat{D}_i + \varepsilon_i$$

where Z_i is our instrument (or other noisy measure of D_i)

$\rightarrow$ IV solves our measurement error issue!

Recap

TL;DR:

1. Instrumental variables are very powerful
2. With the right assumptions...
3. ...we can handle OVB and ME (and simultaneity)