Diofant Documentation

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Diofant Development Team

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CHAPTER ONE

INSTALLATION

The Diofant can be installed on any computer with Python 3.5 or above. You can install latest release with pip:

```
$ pip install diofant
```

or to install also extra dependencies:

```
$ pip install diofant[gmpy,plot]
```

Tip: Use venv to create isolated Python environment first, instead of installing everything system-wide.

1.1 From Sources

If you are a developer or like to get the latest updates as they come, be sure to install from git:

```
$ git clone git://github.com/diofant/diofant.git
$ cd diofant
$ pip install -e .[develop,docs]
```

Note: Diofant requires setuptools for installation from sources.

1.2 Run Diofant

To verify that your freshly-installed Diofant works, please start up the Python interpreter and execute some simple statements like the ones below:

```
>>> from diofant.abc import x
>>> ((1 + x)**(1/x)).limit(x, 0)
E
```

Tip: Use IPython for interactive work. Please refer to the documentation of module *`diofant.interactive`* (page 743) for details of available configuration settings.

For a starter guide on using Diofant, refer to the *Tutorial* (page 3).

1.3 Feedback

If you think there's a bug, you have a question or you would like to request a feature, please *open an issue ticket* (page 1191).

CHAPTER TWO

TUTORIAL

Warning: It assumed that the reader already knows the Python programming language. If you do not, please start from the official Python tutorial.

This tutorial aims to give an introduction to Diofant for someone who has not used the library before. Many features will be introduced in this tutorial, but they will not be exhaustive. In fact, virtually every functionality shown here will have more options or capabilities than what will be shown. The rest of documentation serves as API documentation, which extensively lists every feature and option of each function.

2.1 Introduction

Symbolic computation deals with the computation of mathematical objects symbolically. This means that the mathematical objects are represented exactly, not approximately, and mathematical expressions with unevaluated variables are left in symbolic form.

Let's take an example. Say we wanted to use the built-in Python functions to compute square roots. We might do something like this

```
>>> import math
>>> math.sqrt(9)
3.0
```

Here we got the exact answer — 9 is a perfect square — but usually it will be an approximate result

```
>>> math.sqrt(8)
2.8284271247461903
```

This is where symbolic computation first comes in: with a symbolic computation system like Diofant, square roots of numbers that are not perfect squares are left unevaluated by default

```
>>> import diofant
>>> diofant.sqrt(3)
sqrt(3)
```

Furthermore — and this is where we start to see the real power of symbolic computation — results can be symbolically simplified.

```
>>> diofant.sqrt(8)
2*sqrt(2)
```

Yet we can also approximate this number with any precision

```
>>> _.evalf(20)
2.8284271247461900976
```

The above example starts to show how we can manipulate irrational numbers exactly using Diofant. Now we introduce symbols.

Let us define a symbolic expression, representing the mathematical expression $x + 2y$.

```
>>> x, y = diofant.symbols('x y')
>>> expr = x + 2*y
>>> expr
x + 2*y
```

Note: Unlike many symbolic manipulation systems you may have used, in Diofant, symbols are not defined automatically. To define symbols, we must use *symbols()* (page 81), that takes a string of symbol names separated by spaces or commas, and creates *Symbol* (page 79) instances out of them.

Note that we wrote `x + 2*y`, using Python's mathematical syntax, just as we would if `x` and `y` were ordinary Python variables. But in this case, instead of evaluating to something, the expression remains as just `x + 2*y`. Now let us play around with it:

```
>>> expr + 1
x + 2*y + 1
>>> expr - x
2*y
```

Notice something in the above example. When we typed `expr - x`, we did not get `x + 2*y - x`, but rather just `2*y`. The `x` and the `-x` automatically canceled one another. This is similar to how `sqrt(8)` automatically turned into `2*sqrt(2)` above.

Tip: Use *evaluate()* (page 54) context or `evaluate` flag to prevent automatic evaluation, for example:

```
>>> diofant.sqrt(8, evaluate=False)
sqrt(8)
>>> _.doit()
2*sqrt(2)
```

This isn't always the case in Diofant, however:

```
>>> x*expr
x*(x + 2*y)
```

Here, we might have expected $x(x+2y)$ to transform into x^2+2xy, but instead we see that the expression was left alone. This is a common theme in Diofant. Aside from obvious simplifications like $x - x = 0$ and $\sqrt{8} = 2\sqrt{2}$, most simplifications are not performed automatically. This is because we might prefer the factored form $x(x+2y)$, or we might prefer the expanded form

$x^2 + 2xy$ — both forms are useful in different circumstances. In Diofant, there are functions to go from one form to the other

```
>>> diofant.expand(x*expr)
x**2 + 2*x*y
>>> diofant.factor(_)
x*(x + 2*y)
```

The real power of a symbolic computation system (which by the way, are also often called computer algebra systems, or just CASs) such as Diofant is the ability to do all sorts of computations symbolically: simplify expressions, compute derivatives, integrals, and limits, solve equations, work with matrices, and much more. Diofant includes modules for plotting, printing (like 2D pretty printed output of math formulas, or LaTeX), code generation, statistics, combinatorics, number theory, logic, and more. Here is a small sampling of the sort of symbolic power Diofant is capable of, to whet your appetite.

Note: From here on in this tutorial we assume that these statements were executed:

```
>>> from diofant import *
>>> x, y, z = symbols('x y z')
>>> init_printing(pretty_print=True, use_unicode=True)
```

Last one will make all further examples pretty print with unicode characters.

`import *` has been used here to aid the readability of the tutorial, but is best to avoid such wildcard import statements in production code, as they make it unclear which names are present in the namespace.

Take the derivative of $\sin(x)e^x$.

```
>>> diff(sin(x)*exp(x), x)
 x           x
e ⋅sin(x) + e ⋅cos(x)
```

Compute $\int (e^x \sin(x) + e^x \cos(x))\, dx$.

```
>>> integrate(exp(x)*sin(x) + exp(x)*cos(x), x)
 x
e ⋅sin(x)
```

Compute $\int_{-\infty}^{\infty} \sin(x^2)\, dx$.

```
>>> integrate(sin(x**2), (x, -oo, oo))
  ___   ___
╲╱ 2 ⋅╲╱ π
───────────
     2
```

Find $\lim_{x\to 0^+} \frac{\sin(x)}{x}$.

```
>>> limit(sin(x)/x, x, 0)
1
```

Solve $x^2 - 2 = 0$.

```
>>> solve(x**2 - 2, x)
⎡⎧      ___⎫  ⎧     ___⎫⎤
⎢⎨x: -╲╱ 2 ⎬, ⎨x: ╲╱ 2 ⎬⎥
⎣⎩         ⎭  ⎩        ⎭⎦
```

Solve the differential equation $f'' - f = e^x$.

```
>>> f = symbols('f', cls=Function)
>>> dsolve(Eq(f(x).diff(x, 2) - f(x), exp(x)), f(x))
        x ⎛      x⎞    -x
f(x) = e ⋅⎜C₂ + ─⎟ + e  ⋅C₁
          ⎝      2⎠
```

Find the eigenvalues of $\left[\begin{smallmatrix}1 & 2\\ 2 & 2\end{smallmatrix}\right]$.

```
>>> Matrix([[1, 2], [2, 2]]).eigenvals()
⎧                                  ⎫
⎪3   ╲╱ 17          ╲╱ 17    3     ⎪
⎨─ + ──────: 1, -  ────── + ─: 1⎬
⎪2     2              2      2     ⎪
⎩                                  ⎭
```

Rewrite the Bessel function $J_n(z)$ in terms of the spherical Bessel function $j_n(z)$.

```
>>> n = symbols('n')
>>> besselj(n, z).rewrite(jn)
  ___   ___
╲╱ 2 ⋅╲╱ z ⋅jn(n - 1/2, z)
──────────────────────────
            ___
          ╲╱ π
```

Print $\int_0^{\pi} \cos^2(x)\, dx$ using LaTeX.

```
>>> latex(Integral(cos(x)**2, (x, 0, pi)))
'\\int_{0}^{\\pi} \\cos^{2}{\\left (x \\right )}\\, dx'
```

2.2 Basics

Here we discuss some of the most basic aspects of expression manipulation in Diofant.

2.2.1 Assumptions

The assumptions system allows users to declare certain mathematical properties on symbols, such as being positive, imaginary or integer.

By default, all symbols are complex valued. This assumption makes it easier to treat mathematical problems in full generality.

```
>>> sqrt(x**2)
   ____
  ╱  2
╲╱  x
```

Yet obviously we can simplify above expression if some additional mathematical properties on x are assumed. This is where assumptions system come into play.

Assumptions are set on *Symbol* (page 79) objects when they are created. For instance, we can create a symbol that is assumed to be positive.

```
>>> p = symbols('p', positive=True)
```

And then, certain simplifications will be possible:

```
>>> sqrt(p**2)
p
```

The assumptions system additionally has deductive capabilities. You might check assumptions on any expression with `is_assumption` attributes, like *is_positive* (page 73).

```
>>> p.is_positive
True
>>> (1 + p).is_positive
True
>>> (-p).is_positive
False
```

Note: `False` is returned also if certain assumption doesn't make sense for given object.

In a three-valued logic, used by system, `None` represents the "unknown" case.

```
>>> (p - 1).is_positive is None
True
```

2.2.2 Substitution

One of the most common things you might want to do with a mathematical expression is substitution with *subs()* (page 49) method. It replaces all instances of something in an expression with something else.

```
>>> expr = cos(x) + 1
>>> expr.subs(x, y)
cos(y) + 1
>>> expr
cos(x) + 1
```

We see that performing substitution leaves original expression `expr` unchanged.

Note: Almost all Diofant expressions are immutable. No function (or method) will change them in-place.

To perform several substitutions in one shot, you can provide `Iterable` sequence of pairs.

```
>>> x**y
 y
x
>>> _.subs(((y, x**y), (y, x**x)))
```

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```
 ⎛ ⎛ x⎞⎞
 ⎜ ⎝x ⎠⎟
 ⎝x    ⎠
x
```

Use flag `simultaneous` to do all substitutions at once.

```
>>> (x - y).subs(((x, y), (y, x)))
0
>>> (x - y).subs(((x, y), (y, x)), simultaneous=True)
-x + y
```

2.2.3 Numerics

To evaluate a numerical expression into a floating point number with arbitrary precision, use *evalf()* (page 145). By default, 15 digits of precision are used.

```
>>> expr = sqrt(8)
>>> expr.evalf()
2.82842712474619
```

But you can change that. Let's compute the first 70 digits of π.

```
>>> pi.evalf(70)
3.141592653589793238462643383279502884197169399375105820974944592307816
```

Sometimes there are roundoff errors smaller than the desired precision that remain after an expression is evaluated. Such numbers can be removed by setting the `chop` flag.

```
>>> one = cos(1)**2 + sin(1)**2
>>> (one - 1).evalf(strict=False)
-0.e-146
>>> (one - 1).evalf(chop=True)
0
```

Discussed above method is not effective enough if you intend to evaluate an expression at many points, there are better ways, especially if you only care about machine precision.

The easiest way to convert a Diofant expression to an expression that can be numerically evaluated with libraries like `numpy` — use the *lambdify()* (page 1002) function. It acts like a `lambda` form, except it converts the Diofant names to the names of the given numerical library.

```
>>> import numpy
>>> a = numpy.arange(10)
>>> expr = sin(x)
>>> f = lambdify(x, expr, "numpy")
>>> f(a)
[ 0.          0.84147098  0.90929743  0.14112001 -0.7568025  -0.95892427
 -0.2794155   0.6569866   0.98935825  0.41211849]
```

You can use other libraries than NumPy. For example, the standard library `math` module.

```
>>> f = lambdify(x, expr, "math")
>>> f(0.1)
0.09983341664682815
```

2.3 Gotchas

Lets recall again, that Diofant is nothing more than a Python library, like `numpy` or even the Python standard library module `sys`. What this means is that Diofant does not add anything to the Python language. Limitations that are inherent in the language are also inherent in Diofant.

In this section we are trying to collect some things that could surprise newcomers.

2.3.1 Numbers

To begin with, it should be clear for you, that if you type a numeric literal — it will create a Python number of type `int` or `float`.

Diofant uses its own classes for numbers, for example *Integer* (page 89) instead of `int`. In most cases, Python numeric types will be correctly coersed to Diofant numbers during expression construction.

```
>>> 3 + x**2
 2
x  + 3
>>> type(_ - x**2)
<class 'diofant.core.numbers.Integer'>
```

But if you use some arithmetic operators between two numerical literals, Python will evaluate such expression before Diofant has a chance to get to them.

```
>>> x**(3/2)
 1.5
x
```

Tip: While working in the IPython console, you could use *IntegerDivisionWrapper* (page 744) AST transformer to wrap all integer divisions with *Rational* (page 88) automatically.

The universal solution is using correct Diofant numeric class to construct numbers explicitly. For example, *Rational* (page 88) in the above example

```
>>> x**Rational(3, 2)
 3/2
x
```

2.3.2 Equality

You may think that ==, which is used for equality testing in Python, is used for Diofant to test mathematical equality. This is not quite correct either. Let us see what happens when we use

==.

```
>>> (x + 1)**2 == x**2 + 2*x + 1
False
```

But, $(x+1)^2$ *does* equal x^2+2x+1. What is going on here?

In Diofant, == represents structural equality testing and $(x+1)^2$ and x^2+2x+1 are not the same in this sense. One is the power and the other is the addition of three terms.

There is a separate class, called *Eq* (page 109), which can be used to create a symbolic equation

```
>>> Eq((x + 1)**2 - x**2, 2*x + 1)
   2          2
- x  + (x + 1)  = 2⋅x + 1
```

It is not always return a `bool` object, like ==, but you may use some simplification methods to prove (or disprove) equation.

```
>>> expand(_)
true
```

2.3.3 Naming of Functions

Diofant uses different names for some mathematical functions than most computer algebra systems. In particular, the inverse trigonometric functions use the python names of *asin()* (page 304), *acos()* (page 305) and so on instead of `arcsin` and `arccos`.

2.4 Printing

As we have already seen, Diofant can pretty print its output using Unicode characters. This is a short introduction to the most common printing options available in Diofant. The most common ones are

- *Str* (page 11)
- *Repr* (page 11)
- *2D Pretty Printer* (page 11) (Unicode or ASCII)
- *LaTeX* (page 12)
- *Dot* (page 12)

In addition to these, there are also "printers" that can output Diofant objects to code, such as C, Fortran, or Mathematica.

Best printer is enabled automatically for interactive session (i.e. LaTeX in the IPython notebooks, pretty printer in the IPython console or str printer in the Python console). If you want manually configure pretty printing, please use the *init_printing()* (page 743) function.

Lets take this simple expression

```
>>> expr = Integral(sqrt(1/x), x)
```

and try several available printers.

2.4.1 Str

To get a string form of an expression, use `str`. This is also the form that is produced by `print()`. String forms are designed to be easy to read and mostly to be in a form that is a correct Python syntax so that it can be copied and pasted.

```
>>> str(expr)
'Integral(sqrt(1/x), x)'
>>> print(expr)
Integral(sqrt(1/x), x)
```

2.4.2 Repr

The repr form of an expression is designed to show the exact form of an expression, it would yield an object with the same value when passed to `eval()`. To get it, use `repr()`.

```
>>> repr(expr)
"Integral(Pow(Pow(Symbol('x'), Integer(-1)), Rational(1, 2)), Tuple(Symbol('x')))"
```

The repr form is mostly useful for understanding how an expression is built internally.

2.4.3 2D Pretty Printer

A two-dimensional (2D) textual representation of the expression can be obtained with *pretty()* (page 725).

```
>>> pretty(expr)
'⌠           \n⎮     ___   \n⎮    ╱ 1    \n⎮   ╱  ─  dx\n⎮ ╲╱   x    \n⌡           '
>>> print(_)
⌠
⎮     ___
⎮    ╱ 1
⎮   ╱  ─  dx
⎮ ╲╱   x
⌡
```

You can pass `use_unicode=False` to use ASCII symbols.

```
>>> print(pretty(expr, use_unicode=False))
  /
 |
 |     ___
 |    / 1
 |   /  -  dx
 | \/   x
 |
/
```

pprint() (page 725) prints the output to the screen.

```
>>> pprint(expr)
⌠
⎮     ___
```

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```
|     / 1
|    /  ─  dx
|  ╲/   x
⌡
```

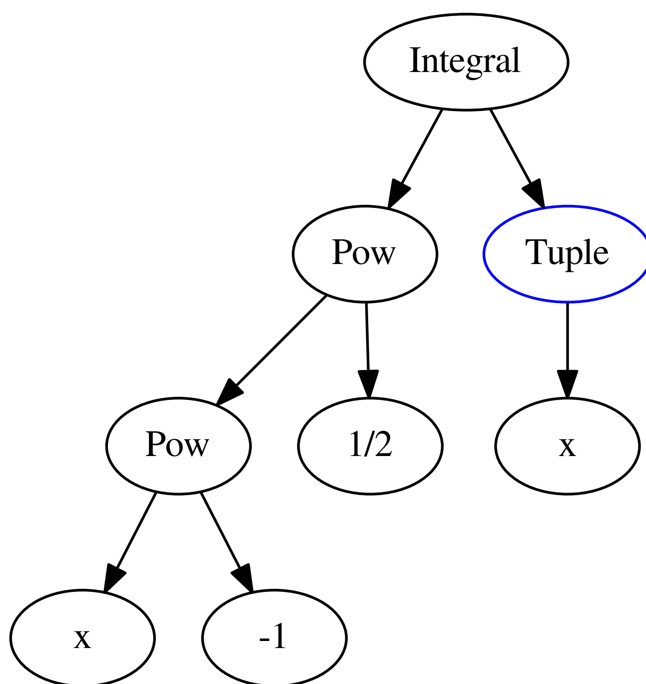
2.4.4 LaTeX

To get the LaTeX form of an expression, use *latex()* (page 733).

```
>>> print(latex(expr))
\int \sqrt{\frac{1}{x}}\, dx
```

2.4.5 Dot

dotprint() (page 742) function prints output to dot format, which can be rendered with Graphviz:



2.5 Simplification

The generic way to do *nontrivial* simplifications of expressions is calling *simplify()* (page 776) function.

```
>>> simplify(sin(x)**2 + cos(x)**2)
1
>>> simplify((x**3 + x**2 - x - 1)/(x**2 + 2*x + 1))
x - 1
>>> simplify(gamma(x)/gamma(x - 2))
(x - 2)⋅(x - 1)
```

There are also more directed simplification functions. These apply very specific rules to the input expression and are typically able to make guarantees about the output. For instance, the *factor()* (page 668) function, given a polynomial with rational coefficients in several variables, is guaranteed to produce a factorization into irreducible factors.

The *simplify()* (page 776) function applies almost all available in Diofant such specific simplification rules in some heuristics sequence to produce the simplest result.

Tip: The optional `measure` keyword argument for *simplify()* (page 776) lets the user specify the Python function used to determine how "simple" an expression is. The default is *count_ops()* (page 140), which returns the total number of operations in the expression.

That is why it is usually slow. But more important pitfail is that sometimes *simplify()* (page 776) doesn't "simplify" how you might expect, if, for example, it miss some transformation or apply it too early or too late. Lets look on an example

```
>>> simplify(x**2 + 2*x + 1)
 2
x  + 2⋅x + 1
>>> factor(_)
       2
(x + 1)
```

Obviously, the factored form is more "simple", as it has less arithmetic operations.

The function *simplify()* (page 776) is best when used interactively, when you just want to whittle down an expression to a simpler form. You may then choose to apply specific functions once you see what *simplify()* (page 776) returns, to get a more precise result. It is also useful when you have no idea what form an expression will take, and you need a catchall function to simplify it.

2.5.1 Rational Functions

expand() (page 135) is one of the most common simplification functions in Diofant. Although it has a lot of scopes, for now, we will consider its function in expanding polynomial expressions.

```
>>> expand((x + 1)**2)
 2
x  + 2⋅x + 1
>>> expand((x + 2)*(x - 3))
```

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```
 2
x  - x - 6
```

Given a polynomial, `expand()` (page 135) will put it into a canonical form of a sum of monomials with help of more directed expansion methods, namely `expand_multinomial()` (page 143) and `expand_mul()` (page 141).

`expand()` (page 135) may not sound like a simplification function. After all, by its very name, it makes expressions bigger, not smaller. Usually this is the case, but often an expression will become smaller upon calling `expand()` (page 135) on it due to cancellation.

```
>>> expand((x + 1)*(x - 2) - (x - 1)*x)
-2
```

Function `factor()` (page 668) takes a multivariate polynomial with rational coefficients and factors it into irreducible factors.

```
>>> factor(x**3 - x**2 + x - 1)
        ⎛ 2    ⎞
(x - 1)⋅⎝x  + 1⎠
>>> factor(x**2*z + 4*x*y*z + 4*y**2*z)
           2
z⋅(x + 2⋅y)
```

For polynomials, `factor()` (page 668) is the opposite of `expand()` (page 135).

Note: The input to `factor()` (page 668) and `expand()` (page 135) need not be polynomials in the strict sense. They will intelligently factor or expand any kind of expression (though, for example, the factors may not be irreducible if the input is no longer a polynomial over the rationals).

```
>>> expand((cos(x) + sin(x))**2)
   2                           2
sin (x) + 2⋅sin(x)⋅cos(x) + cos (x)
>>> factor(_)
                 2
(sin(x) + cos(x))
```

`collect()` (page 784) collects common powers of a term in an expression.

```
>>> x*y + x - 3 + 2*x**2 - z*x**2 + x**3
 3    2        2
x  - x ⋅z + 2⋅x  + x⋅y + x - 3
>>> collect(_, x)
 3    2
x  + x ⋅(-z + 2) + x⋅(y + 1) - 3
```

`collect()` (page 784) is particularly useful in conjunction with the `coeff()` (page 63) method.

```
>>> _.coeff(x, 2)
-z + 2
```

`cancel()` (page 671) will take any rational function and put it into the standard canonical form, p/q, where p and q are expanded polynomials with no common factors.

```
>>> 1/x + (3*x/2 - 2)/(x - 4)
3⋅x
─── - 2
 2        1
─────── + ─
 x - 4    x
>>> cancel(_)
   2
3⋅x  - 2⋅x - 8
──────────────
     2
  2⋅x  - 8⋅x
```

```
>>> expr = (x*y**2 - 2*x*y*z + x*z**2 + y**2 - 2*y*z + z**2)/(x**2 - 1)
>>> expr
   2                2    2            2
x⋅y  - 2⋅x⋅y⋅z + x⋅z  + y  - 2⋅y⋅z + z
───────────────────────────────────────
                  2
                 x  - 1
>>> cancel(_)
 2            2
y  - 2⋅y⋅z + z
───────────────
     x - 1
```

Note: Since *factor()* (page 668) will completely factorize both the numerator and the denominator of an expression, it can also be used to do the same thing:

```
>>> factor(expr)
       2
(y - z)
────────
 x - 1
```

However, it's less efficient if you are only interested in making sure that the expression is in canceled form.

apart() (page 716) performs a partial fraction decomposition on a rational function.

```
>>> (4*x**3 + 21*x**2 + 10*x + 12)/(x**4 + 5*x**3 + 5*x**2 + 4*x)
   3       2
4⋅x  + 21⋅x  + 10⋅x + 12
────────────────────────
  4      3      2
 x  + 5⋅x  + 5⋅x  + 4⋅x
>>> apart(_)
 2⋅x - 1       1     3
────────── - ───── + ─
 2           x + 4   x
x  + x + 1
```

2.5.2 Trigonometric Functions

To simplify expressions using trigonometric identities, use *trigsimp()* (page 789) function.

```
>>> trigsimp(sin(x)**2 + cos(x)**2)
1
>>> trigsimp(sin(x)**4 - 2*cos(x)**2*sin(x)**2 + cos(x)**4)
cos(4⋅x)   1
──────── + ─
   2       2
>>> trigsimp(sin(x)*tan(x)/sec(x))
   2
sin (x)
```

It also works with hyperbolic functions.

```
>>> trigsimp(cosh(x)**2 + sinh(x)**2)
cosh(2⋅x)
>>> trigsimp(sinh(x)/tanh(x))
cosh(x)
```

Much like *simplify()* (page 776) function, *trigsimp()* (page 789) applies various trigonometric identities to the input expression, and then uses a heuristic to return the "best" one.

To expand trigonometric functions, that is, apply the sum or double angle identities, use *expand_trig()* (page 142) function.

```
>>> expand_trig(sin(x + y))
sin(x)⋅cos(y) + sin(y)⋅cos(x)
>>> expand_trig(tan(2*x))
   2⋅tan(x)
─────────────
     2
- tan (x) + 1
```

2.5.3 Powers and Logarithms

powdenest() (page 793) function applies identity $(x^a)^b = x^{ab}$, from left to right, if assumptions allow.

```
>>> a, b = symbols('a b', extended_real=True)
>>> p = symbols('p', positive=True)
>>> powdenest((p**a)**b)
 a⋅b
p
```

powsimp() (page 792) function reduces expression by combining powers with similar bases and exponent.

```
>>> powsimp(z**x*z**y)
  x + y
 z
```

Again, as for *powdenest()* (page 793) above, for the identity $x^a y^a = (xy)^a$, that combine bases, we should be careful about assumptions.

```
>>> q = symbols('q', positive=True)
>>> powsimp(p**a*q**a)
     a
(p⋅q)
```

In general, this identity doesn't hold. For example, if $x = y = -1$ and $a = 1/2$.

expand_power_exp() (page 143) and *expand_power_base()* (page 143) functions do reverse of *powsimp()* (page 792).

```
>>> expand_power_exp(x**(y + z))
 y  z
x ⋅x
>>> expand_power_base((p*q)**a)
 a  a
p ⋅q
```

Logarithms have similar issues as powers. There are two main identities

1. $\log(xy) = \log(x) + \log(y)$
2. $\log(x^n) = n\log(x)$

Neither identity is true for arbitrary complex x and y, due to the branch cut in the complex plane for the complex logarithm.

To apply above identities from left to right, use *expand_log()* (page 142). As for powers, the identities will not be applied unless they are valid with given set of assumptions for symbols.

```
>>> expand_log(log(p*q))
log(p) + log(q)
>>> expand_log(log(p/q))
log(p) - log(q)
>>> expand_log(log(p**2))
2⋅log(p)
>>> expand_log(log(p**a))
a⋅log(p)
>>> expand_log(log(x*y))
log(x⋅y)
```

To apply identities from right to left, i.e. do reverse of *expand_log()* (page 142), use *logcombine()* (page 782) function.

```
>>> logcombine(log(p) + log(q))
log(p⋅q)
>>> logcombine(a*log(p))
   ⎛ a⎞
log⎝p ⎠
>>> logcombine(a*log(z))
a⋅log(z)
```

2.5.4 Special Functions

Diofant implements dozens of *special functions* (page 290), ranging from functions in combinatorics to mathematical physics.

To expand special functions in terms of some identities, use *expand_func()* (page 142). For example the gamma function *gamma* (page 350) can be expanded as

```
>>> expand_func(gamma(x + 3))
x⋅(x + 1)⋅(x + 2)⋅Γ(x)
```

This method also can help if you would like to rewrite the generalized hypergeometric function *hyper* (page 396) or the Meijer G-function *meijerg* (page 398) in terms of more standard functions.

```
>>> expand_func(hyper([1, 1], [2], z))
-log(-z + 1)
─────────────
      z
>>> meijerg([[1], [1]], [[1], []], -z)
┌─┐1, 1 ⎛1  1 │   ⎞
│╶┐     ⎜     │ -z⎟
└─┘2, 1 ⎝1    │   ⎠
>>> expand_func(_)
z ___
╲╱ e
```

To simplify combinatorial expressions, involving *factorial* (page 336), *binomial* (page 332) or *gamma* (page 350) — use *combsimp()* (page 795) function.

```
>>> combsimp(factorial(x)/factorial(x - 3))
x⋅(x - 2)⋅(x - 1)
>>> combsimp(binomial(x + 1, y + 1)/binomial(x, y))
x + 1
─────
y + 1
>>> combsimp(gamma(x)*gamma(1 - x))
   π
────────
sin(π⋅x)
```

2.6 Calculus

This section covers how to do basic calculus tasks such as derivatives, integrals, limits, and series expansions in Diofant.

2.6.1 Derivatives

To take derivatives, use the *diff()* (page 131) function.

```
>>> diff(cos(x), x)
-sin(x)
>>> diff(exp(x**2), x)
   ⎛ 2⎞
   ⎝x ⎠
2⋅e    ⋅x
```

diff() (page 131) can take multiple derivatives at once. To take multiple derivatives, pass the variable as many times as you wish to differentiate, or pass a number after the variable. For example, both of the following find the third derivative of x^4.

```
>>> diff(x**4, x, x, x)
24⋅x
>>> diff(x**4, x, 3)
24⋅x
```

You can also take derivatives with respect to many variables at once. Just pass each derivative in order, using the same syntax as for single variable derivatives. For example, each of the following will compute $\frac{\partial^7}{\partial x \partial y^2 \partial z^4} e^{xyz}$.

```
>>> expr = exp(x*y*z)
>>> diff(expr, x, y, y, z, z, z, z)
 x⋅y⋅z  3  2 ⎛ 3  3  3       2  2  2                ⎞
e     ⋅x ⋅y ⋅⎝x ⋅y ⋅z  + 14⋅x ⋅y ⋅z  + 52⋅x⋅y⋅z + 48⎠
>>> diff(expr, x, y, 2, z, 4)
 x⋅y⋅z  3  2 ⎛ 3  3  3       2  2  2                ⎞
e     ⋅x ⋅y ⋅⎝x ⋅y ⋅z  + 14⋅x ⋅y ⋅z  + 52⋅x⋅y⋅z + 48⎠
>>> diff(expr, x, y, y, z, 4)
 x⋅y⋅z  3  2 ⎛ 3  3  3       2  2  2                ⎞
e     ⋅x ⋅y ⋅⎝x ⋅y ⋅z  + 14⋅x ⋅y ⋅z  + 52⋅x⋅y⋅z + 48⎠
```

diff() (page 131) can also be called as a method *diff()* (page 65). The two ways of calling *diff()* (page 131) are exactly the same, and are provided only for convenience.

```
>>> expr.diff(x, y, y, z, 4)
 x⋅y⋅z  3  2 ⎛ 3  3  3       2  2  2                ⎞
e     ⋅x ⋅y ⋅⎝x ⋅y ⋅z  + 14⋅x ⋅y ⋅z  + 52⋅x⋅y⋅z + 48⎠
```

To create an unevaluated derivative, use the *Derivative* (page 128) class. It has the same syntax as *diff()* (page 131).

```
>>> Derivative(expr, x, y, y, z, 4)
     7
    ∂      ⎛ x⋅y⋅z⎞
──────────⎜e     ⎟
  4   2
∂z  ∂y  ∂x
```

Such unevaluated objects also used when Diofant does not know how to compute the derivative of an expression.

To evaluate an unevaluated derivative, use the *doit()* (page 44) method.

```
>>> _.doit()
 x⋅y⋅z  3  2 ⎛ 3  3  3       2  2  2                ⎞
e     ⋅x ⋅y ⋅⎝x ⋅y ⋅z  + 14⋅x ⋅y ⋅z  + 52⋅x⋅y⋅z + 48⎠
```

2.6.2 Integrals

To compute an integral, use the *integrate()* (page 522) function. There are two kinds of integrals, definite and indefinite. To compute an indefinite integral, do

```
>>> integrate(cos(x), x)
sin(x)
```

Note: For indefinite integrals, Diofant does not include the constant of integration.

For example, to compute a definite integral

$$\int_0^\infty e^{-x}\,dx,$$

we would do

```
>>> integrate(exp(-x), (x, 0, oo))
1
```

Tip: ∞ in Diofant is `oo` (that's the lowercase letter "oh" twice).

As with indefinite integrals, you can pass multiple limit tuples to perform a multiple integral. For example, to compute

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy,$$

do

```
>>> integrate(exp(-x**2 - y**2), (x, -oo, oo), (y, -oo, oo))
π
```

If *integrate()* (page 522) is unable to compute an integral, it returns an unevaluated *Integral* (page 529) object.

```
>>> integrate(x**x, x)
⌠
⎮  x
⎮ x  dx
⌡
>>> print(_)
Integral(x**x, x)
```

As with *Derivative* (page 128), you can create an unevaluated integral directly. To later evaluate this integral, call *doit()* (page 530).

```
>>> Integral(log(x)**2, x)
⌠
⎮    2
⎮ log (x) dx
⌡
>>> _.doit()
         2
x⋅log (x) - 2⋅x⋅log(x) + 2⋅x
```

integrate() (page 522) uses powerful algorithms that are always improving to compute both definite and indefinite integrals, including a partial implementation of the Risch algorithm

```
>>> Integral((x**4 + x**2*exp(x) - x**2 - 2*x*exp(x) - 2*x -
...           exp(x))*exp(x)/((x - 1)**2*(x + 1)**2*(exp(x) + 1)), x)
```

(continues on next page)

(continued from previous page)

```
⌠
⎮  x ⎛ x  2      x        x    4    2      ⎞
⎮ e ⋅⎝e ⋅x  - 2⋅e ⋅x - e  + x  - x  - 2⋅x⎠
⎮ ──────────────────────────────────────── dx
⎮        ⎛ x    ⎞        2        2
⎮        ⎝e  + 1⎠⋅(x - 1) ⋅(x + 1)
⌡
>>> _.doit()
    x
   e         ⎛ x    ⎞
────── + log⎝e  + 1⎠
 2
x  - 1
```

and an algorithm using Meijer G-functions that is useful for computing integrals in terms of special functions, especially definite integrals

```
>>> Integral(sin(x**2), x)
⌠
⎮    ⎛ 2⎞
⎮ sin⎝x ⎠ dx
⌡
>>> _.doit()
                                ⎛  ___  ⎞
                                ⎜╲╱ 2 ⋅x⎟
3⋅╲╱ 2 ⋅╲╱ π ⋅fresnels⎜───────⎟⋅Γ(3/4)
                                ⎜  ___  ⎟
                                ⎝╲╱ π   ⎠
──────────────────────────────────────
               8⋅Γ(7/4)
```

```
>>> Integral(x**y*exp(-x), (x, 0, oo))
∞
⌠
⎮  -x  y
⎮ e  ⋅x  dx
⌡
0
>>> _.doit()
⎧ Γ(y + 1)    for -re(y) < 1
⎪
⎪∞
⎪⌠
⎨⎮  -x  y
⎪⎮ e  ⋅x  dx    otherwise
⎪⌡
⎪0
⎩
```

This last example returned a *Piecewise* (page 323) expression because the integral does not converge unless $\Re(y) > 1$.

2.6.3 Sums and Products

Much like integrals, there are *summation()* (page 287) and *product()* (page 287) to compute sums and products respectively.

```
>>> summation(2**x, (x, 0, y - 1))
 y
2  - 1
>>> product(z, (x, 1, y))
 y
z
```

Unevaluated sums and products are represented by *Sum* (page 273) and *Product* (page 277) classes.

```
>>> Sum(1, (x, 1, z))
 z
___
\
 \    1
 /
/__
x = 1
>>> _.doit()
z
```

2.6.4 Limits

Diofant can compute symbolic limits with the *limit()* (page 759) function. To compute a directional limit

$$\lim_{x \to 0^+} \frac{\sin x}{x},$$

do

```
>>> limit(sin(x)/x, x, 0)
1
```

limit() (page 759) should be used instead of *subs()* (page 49) whenever the point of evaluation is a singularity. Even though Diofant has objects to represent ∞, using them for evaluation is not reliable because they do not keep track of things like rate of growth. Also, things like $\infty - \infty$ and $\frac{\infty}{\infty}$ return nan (not-a-number). For example

```
>>> expr = x**2/exp(x)
>>> expr.subs(x, oo)
nan
>>> limit(expr, x, oo)
0
```

Like *Derivative* (page 128) and *Integral* (page 529), *limit()* (page 759) has an unevaluated counterpart, *Limit* (page 758). To evaluate it, use *doit()* (page 759).

```
>>> Limit((cos(x) - 1)/x, x, 0)
     cos(x) - 1
 lim ──────────
x─→0⁺    x
>>> _.doit()
0
```

To change side, pass `'-'` as a third argument to *limit()* (page 759). For example, to compute

$$\lim_{x \to 0^-} \frac{1}{x},$$

do

```
>>> limit(1/x, x, 0, dir='-')
-∞
```

You can also evaluate bidirectional limit

```
>>> limit(sin(x)/x, x, 0, dir='real')
1
>>> limit(1/x, x, 0, dir='real')
Traceback (most recent call last):
...
PoleError: left and right limits for expression 1/x at point x=0 seems to be not equal
```

2.6.5 Series Expansion

Diofant can compute asymptotic series expansions of functions around a point.

```
>>> series(exp(sin(x)), x, 0, 4)
         2
        x     ⎛ 4⎞
1 + x + ── + O⎝x ⎠
        2
```

The $O\left(x^4\right)$ term, an instance of *O* (page 759) at the end represents the Landau order term at $x = 0$ (not to be confused with big O notation used in computer science, which generally represents the Landau order term at $x = \infty$). Order terms can be created and manipulated outside of *series()* (page 759).

```
>>> x + x**3 + x**6 + O(x**4)
     3    ⎛ 4⎞
x + x  + O⎝x ⎠
>>> x*O(1)
O(x)
```

If you do not want the order term, use the *removeO()* (page 77) method.

```
>>> series(exp(x), x, 0, 3).removeO()
 2
x
── + x + 1
2
```

The *O* (page 759) notation supports arbitrary limit points (other than 0):

```
>>> series(exp(x - 1), x, x0=1)
       2          3          4          5
(x - 1)    (x - 1)    (x - 1)    (x - 1)            ⎛       6        ⎞
──────── + ──────── + ──────── + ──────── + x + O⎝(x - 1) ; x → 1⎠
   2          6          24        120
```

2.7 Solvers

This section covers equations solving.

Note: Any expression in input, that not in an *Eq* (page 109) is automatically assumed to be equal to 0 by the solving functions.

2.7.1 Algebraic Equations

The main function for solving algebraic equations is *solve()* (page 839).

When solving a single equation, the output is a list of the solutions.

```
>>> solve(x**2 - x, x)
[{x: 0}, {x: 1}]
```

If no solutions are found, an empty list is returned.

```
>>> solve(exp(x), x)
[]
```

solve() (page 839) can also solve systems of equations.

```
>>> solve([x - y + 2, x + y - 3], [x, y])
[{x: 1/2, y: 5/2}]
>>> solve([x*y - 7, x + y - 6], [x, y])
⎡⎧       ___         ___    ⎫  ⎧     ___             ___    ⎫⎤
⎢⎨x: - ╲╱ 2  + 3, y: ╲╱ 2  + 3⎬, ⎨x: ╲╱ 2  + 3, y: - ╲╱ 2  + 3⎬⎥
⎣⎩                          ⎭  ⎩                            ⎭⎦
```

solve() (page 839) reports each solution only once.

```
>>> solve(x**3 - 6*x**2 + 9*x, x)
[{x: 0}, {x: 3}]
```

To get the solutions of a polynomial including multiplicity use *roots()* (page 713).

```
>>> roots(x**3 - 6*x**2 + 9*x, x)
{0: 1, 3: 2}
```

2.7.2 Differential Equations

To solve differential equations, use *dsolve()* (page 868). First, create an undefined function by passing `cls=Function` to the *symbols()* (page 81) function.

```
>>> f, g = symbols('f g', cls=Function)
```

`f` and `g` are now undefined functions. We can call `f(x)`, and it will represent an unknown function application. Derivatives of `f(x)` are unevaluated.

```
>>> f(x).diff(x)
d
──(f(x))
dx
```

To represent the differential equation $f''(x) - 2f'(x) + f(x) = \sin(x)$, we would thus use

```
>>> Eq(f(x).diff(x, x) - 2*f(x).diff(x) + f(x), sin(x))
                      2
         d           d
f(x) - 2⋅──(f(x)) + ───(f(x)) = sin(x)
         dx           2
                    dx
```

To solve the ODE, pass it and the function to solve for to *dsolve()* (page 868).

```
>>> dsolve(_, f(x))
        x               cos(x)
f(x) = e ⋅(C₁ + C₂⋅x) + ──────
                          2
```

dsolve() (page 868) returns an instance of *Eq* (page 109). This is because in general, solutions to differential equations cannot be solved explicitly for the function.

```
>>> dsolve(f(x).diff(x)*(1 - sin(f(x))), f(x))
f(x) + cos(f(x)) = C₁
```

The arbitrary constants in the solutions from dsolve are symbols of the form `C1`, `C2`, `C3`, and so on.

dsolve() (page 868) can also solve systems of equations, like *solve()* (page 839).

```
>>> dsolve([f(x).diff(x) - g(x), g(x).diff(x) - f(x)], [f(x), g(x)])
⎡           ⎛ x     -x⎞      ⎛ x     -x⎞              ⎛ x     -x⎞      ⎛ x     -x⎞⎤
⎢           ⎜e     e  ⎟      ⎜e     e  ⎟              ⎜e     e  ⎟      ⎜e     e  ⎟⎥
⎢f(x) = C₁⋅⎜── + ───⎟ + C₂⋅⎜── - ───⎟, g(x) = C₁⋅⎜── - ───⎟ + C₂⋅⎜── + ───⎟⎥
⎣           ⎝2      2 ⎠      ⎝2      2 ⎠              ⎝2      2 ⎠      ⎝2      2 ⎠⎦
```

2.8 Matrices

To make a matrix in Diofant, use the *Matrix* (page 560) object. A matrix is constructed by providing a list of row vectors that make up the matrix. For example, to construct the matrix

$$\begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & 2 \end{bmatrix}$$

use

```
>>> Matrix([[1, -1], [3, 4], [0, 2]])
⎡1  -1⎤
⎢     ⎥
⎢3  4 ⎥
⎢     ⎥
⎣0  2 ⎦
```

To make it easy to make column vectors, a list of elements is considered to be a column vector.

```
>>> Matrix([1, 2, 3])
⎡1⎤
⎢ ⎥
⎢2⎥
⎢ ⎥
⎣3⎦
```

One important thing to note about Diofant matrices is that, unlike every other object in Diofant, they are mutable. This means that they can be modified in place, as we will see below. Use *ImmutableMatrix* (page 635) in places that require immutability, such as inside other Diofant expressions or as keys to dictionaries.

2.8.1 Indexing

Diofant matrices support subscription of matrix elements with pair of integers or `slice` instances. In last case, new *Matrix* (page 560) instances will be returned.

```
>>> M = Matrix([[1, 2, 3], [4, 5, 6]])
>>> M[0, 1]
2
>>> M[1, 1]
5
>>> M[:, 1]
⎡2⎤
⎢ ⎥
⎣5⎦
>>> M[1, :-1]
[4  5]
```

To get an individual row or column of a matrix, you could also use methods *row()* (page 614) or *col()* (page 613).

```
>>> M.row(0)
[1  2  3]
>>> M.col(-1)
⎡3⎤
⎢ ⎥
⎣6⎦
```

It's possible to modify matrix elements.

```
>>> M[0, 0] = 0
>>> M
⎡0  2  3⎤
⎢       ⎥
```

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```
⎣4  5  6⎦
>>> M[1, 1:] = Matrix([[0, 0]])
>>> M
⎡0  2  3⎤
⎢       ⎥
⎣4  0  0⎦
```

2.8.2 Reshape and Rearrange

To get the shape of a matrix use *shape* (page 599) property

```
>>> M = Matrix([[1, 2, 3], [-2, 0, 4]])
>>> M
⎡1   2  3⎤
⎢        ⎥
⎣-2  0  4⎦
>>> M.shape
(2, 3)
```

To delete a row or column, use methods *row_del()* (page 617) or *col_del()* (page 615).

```
>>> M.col_del(0)
>>> M
⎡2  3⎤
⎢    ⎥
⎣0  4⎦
>>> M.row_del(1)
>>> M
[2  3]
```

Note: You can see, that these methods will modify the Matrix **in place**. In general, as a rule, such methods will return None.

To insert rows or columns, use methods *row_insert()* (page 598) or *col_insert()* (page 576).

```
>>> M
[2  3]
>>> M = M.row_insert(1, Matrix([[0, 4]]))
>>> M
⎡2  3⎤
⎢    ⎥
⎣0  4⎦
>>> M = M.col_insert(0, Matrix([1, -2]))
>>> M
⎡1   2  3⎤
⎢        ⎥
⎣-2  0  4⎦
```

To swap two given rows or columns, use methods *row_swap()* (page 618) or *col_swap()* (page 616).

```
>>> M.row_swap(0, 1)
>>> M
⎡-2  0  4⎤
⎢        ⎥
⎣1   2  3⎦
>>> M.col_swap(1, 2)
>>> M
⎡-2  4  0⎤
⎢        ⎥
⎣1   3  2⎦
```

To take the transpose of a Matrix, use *T* (page 572) property.

```
>>> M.T
⎡-2  1⎤
⎢     ⎥
⎢4   3⎥
⎢     ⎥
⎣0   2⎦
```

2.8.3 Algebraic Operations

Simple operations like addition and multiplication are done just by using +, *, and **. To find the inverse of a matrix, just raise it to the -1 power.

```
>>> M = Matrix([[1, 3], [-2, 3]])
>>> N = Matrix([[0, 3], [0, 7]])
>>> M + N
⎡1   6 ⎤
⎢      ⎥
⎣-2  10⎦
>>> M*N
⎡0  24⎤
⎢     ⎥
⎣0  15⎦
>>> 3*M
⎡3   9⎤
⎢     ⎥
⎣-6  9⎦
>>> M**2
⎡-5  12⎤
⎢      ⎥
⎣-8  3 ⎦
>>> M**-1
⎡1/3  -1/3⎤
⎢         ⎥
⎣2/9  1/9 ⎦
>>> N**-1
Traceback (most recent call last):
...
ValueError: Matrix det == 0; not invertible.
```

2.8.4 Special Matrices

Several constructors exist for creating common matrices. To create an identity matrix, use *eye()* (page 604) function.

```
>>> eye(3)
⎡1  0  0⎤
⎢       ⎥
⎢0  1  0⎥
⎢       ⎥
⎣0  0  1⎦
>>> eye(4)
⎡1  0  0  0⎤
⎢          ⎥
⎢0  1  0  0⎥
⎢          ⎥
⎢0  0  1  0⎥
⎢          ⎥
⎣0  0  0  1⎦
```

To create a matrix of all zeros, use *zeros()* (page 604) function.

```
>>> zeros(2, 3)
⎡0  0  0⎤
⎢       ⎥
⎣0  0  0⎦
```

Similarly, function *ones()* (page 604) creates a matrix of ones.

```
>>> ones(3, 2)
⎡1  1⎤
⎢    ⎥
⎢1  1⎥
⎢    ⎥
⎣1  1⎦
```

To create diagonal matrices, use function *diag()* (page 604). Its arguments can be either numbers or matrices. A number is interpreted as a 1×1 matrix. The matrices are stacked diagonally.

```
>>> diag(1, 2, 3)
⎡1  0  0⎤
⎢       ⎥
⎢0  2  0⎥
⎢       ⎥
⎣0  0  3⎦
>>> diag(-1, ones(2, 2), Matrix([5, 7, 5]))
⎡-1  0  0  0⎤
⎢           ⎥
⎢0   1  1  0⎥
⎢           ⎥
⎢0   1  1  0⎥
⎢           ⎥
⎢0   0  0  5⎥
⎢           ⎥
```

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```
⎢0   0  0  7⎥
⎢           ⎥
⎣0   0  0  5⎦
```

2.8.5 Advanced Methods

To compute the determinant of a matrix, use *det()* (page 577) method.

```
>>> M = Matrix([[1, 0, 1], [2, -1, 3], [4, 3, 2]])
>>> M
⎡1  0   1⎤
⎢        ⎥
⎢2  -1  3⎥
⎢        ⎥
⎣4  3   2⎦
>>> det(M)
-1
```

To put a matrix into reduced row echelon form, use method *rref()* (page 599). It returns a tuple of two elements. The first is the reduced row echelon form, and the second is a list of indices of the pivot columns.

```
>>> M = Matrix([[1, 0, 1, 3], [2, 3, 4, 7], [-1, -3, -3, -4]])
>>> M
⎡1   0   1   3 ⎤
⎢              ⎥
⎢2   3   4   7 ⎥
⎢              ⎥
⎣-1  -3  -3  -4⎦
>>> M.rref()
⎛⎡1  0   1    3 ⎤, [0, 1]⎞
⎜⎢              ⎥        ⎟
⎜⎢0  1  2/3  1/3⎥        ⎟
⎜⎢              ⎥        ⎟
⎝⎣0  0   0    0 ⎦        ⎠
```

To find the nullspace of a matrix, use method *nullspace()* (page 595). It returns a list of column vectors that span the nullspace of the matrix.

```
>>> M = Matrix([[1, 2, 3, 0, 0], [4, 10, 0, 0, 1]])
>>> M
⎡1  2   3  0  0⎤
⎢              ⎥
⎣4  10  0  0  1⎦
>>> M.nullspace()
⎡⎡-15⎤, ⎡0⎤, ⎡ 1  ⎤⎤
⎢⎢   ⎥  ⎢ ⎥  ⎢    ⎥⎥
⎢⎢ 6 ⎥  ⎢0⎥  ⎢-1/2⎥⎥
⎢⎢   ⎥  ⎢ ⎥  ⎢    ⎥⎥
⎢⎢ 1 ⎥  ⎢0⎥  ⎢ 0  ⎥⎥
⎢⎢   ⎥  ⎢ ⎥  ⎢    ⎥⎥
⎢⎢ 0 ⎥  ⎢1⎥  ⎢ 0  ⎥⎥
```

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```
││    │  │ │  │      ││
⎣⎣ 0 ⎦  ⎣0⎦  ⎣ 1   ⎦⎦
```

To find the eigenvalues of a matrix, use method *eigenvals()* (page 580). It returns a dictionary of roots including its multiplicity (similar to the output of *roots()* (page 713) function).

```
>>> M = Matrix([[3, -2,  4, -2], [5,  3, -3, -2],
...             [5, -2,  2, -2], [5, -2, -3,  3]])
>>> M
⎡3  -2  4   -2⎤
⎢             ⎥
⎢5  3   -3  -2⎥
⎢             ⎥
⎢5  -2  2   -2⎥
⎢             ⎥
⎣5  -2  -3  3 ⎦
>>> M.eigenvals()
{-2: 1, 3: 1, 5: 2}
```

This means that M has eigenvalues -2, 3, and 5, and that the eigenvalues -2 and 3 have algebraic multiplicity 1 and that the eigenvalue 5 has algebraic multiplicity 2.

Matrices can have symbolic elements.

```
>>> Matrix([[x, x + y], [y, x]])
⎡x  x + y⎤
⎢        ⎥
⎣y    x  ⎦
>>> _.eigenvals()
⎧         ___________          ___________   ⎫
⎨x - ╲╱ y⋅(x + y) : 1, x + ╲╱ y⋅(x + y) : 1⎬
⎩                                            ⎭
```

To find the eigenvectors of a matrix, use method *eigenvects()* (page 580).

```
>>> M.eigenvects()
⎡⎛-2, 1, ⎡⎡0⎤⎤⎞, ⎛3, 1, ⎡⎡1⎤⎤⎞, ⎛5, 2, ⎡⎡1⎤, ⎡0 ⎤⎤⎞⎤
⎢⎜       ⎢⎢ ⎥⎥⎟  ⎜      ⎢⎢ ⎥⎥⎟  ⎜      ⎢⎢ ⎥  ⎢  ⎥⎥⎟⎥
⎢⎜       ⎢⎢1⎥⎥⎟  ⎜      ⎢⎢1⎥⎥⎟  ⎜      ⎢⎢1⎥  ⎢-1⎥⎥⎟⎥
⎢⎜       ⎢⎢ ⎥⎥⎟  ⎜      ⎢⎢ ⎥⎥⎟  ⎜      ⎢⎢ ⎥  ⎢  ⎥⎥⎟⎥
⎢⎜       ⎢⎢1⎥⎥⎟  ⎜      ⎢⎢1⎥⎥⎟  ⎜      ⎢⎢1⎥  ⎢0 ⎥⎥⎟⎥
⎢⎜       ⎢⎢ ⎥⎥⎟  ⎜      ⎢⎢ ⎥⎥⎟  ⎜      ⎢⎢ ⎥  ⎢  ⎥⎥⎟⎥
⎣⎝       ⎣⎣1⎦⎦⎠  ⎝      ⎣⎣1⎦⎦⎠  ⎝      ⎣⎣0⎦  ⎣1 ⎦⎦⎠⎦
```

This shows us that, for example, the eigenvalue 5 also has geometric multiplicity 2, because it has two eigenvectors. Because the algebraic and geometric multiplicities are the same for all the eigenvalues, M is diagonalizable.

To diagonalize a matrix, use method *diagonalize()* (page 578). It returns a tuple (P, D), where D is diagonal and $M = PDP^{-1}$.

```
>>> P, D = M.diagonalize()
>>> P
⎡0  1  1  0 ⎤
⎢           ⎥
```

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```
⎢1  1  1  -1⎥
⎢           ⎥
⎢1  1  1  0 ⎥
⎢           ⎥
⎣1  1  0  1 ⎦
>>> D
⎡-2  0  0  0⎤
⎢           ⎥
⎢0   3  0  0⎥
⎢           ⎥
⎢0   0  5  0⎥
⎢           ⎥
⎣0   0  0  5⎦
>>> P*D*P**-1 == M
True
```

If all you want is the characteristic polynomial, use method *charpoly()* (page 575). This is more efficient than *eigenvals()* (page 580) method, because sometimes symbolic roots can be expensive to calculate.

```
>>> p = M.charpoly(x)
>>> factor(p)
       2
(x - 5) ⋅(x - 3)⋅(x + 2)
```

To compute Jordan canonical form J for matrix M and its similarity transformation P (i.e. such that $J = PMP^{-1}$), use method *jordan_form()* (page 592).

```
>>> M = Matrix([[-2, 4], [1, 3]])
>>> P, J = M.jordan_form()
>>> J
⎡                         ⎤
⎢1     ____               ⎥
⎢─ + ╲╱ 41                ⎥
⎢    ──────        0      ⎥
⎢2     2                  ⎥
⎢                         ⎥
⎢                         ⎥
⎢               ____      ⎥
⎢             ╲╱ 41     1 ⎥
⎢    0      - ────── + ─  ⎥
⎣               2       2 ⎦
```

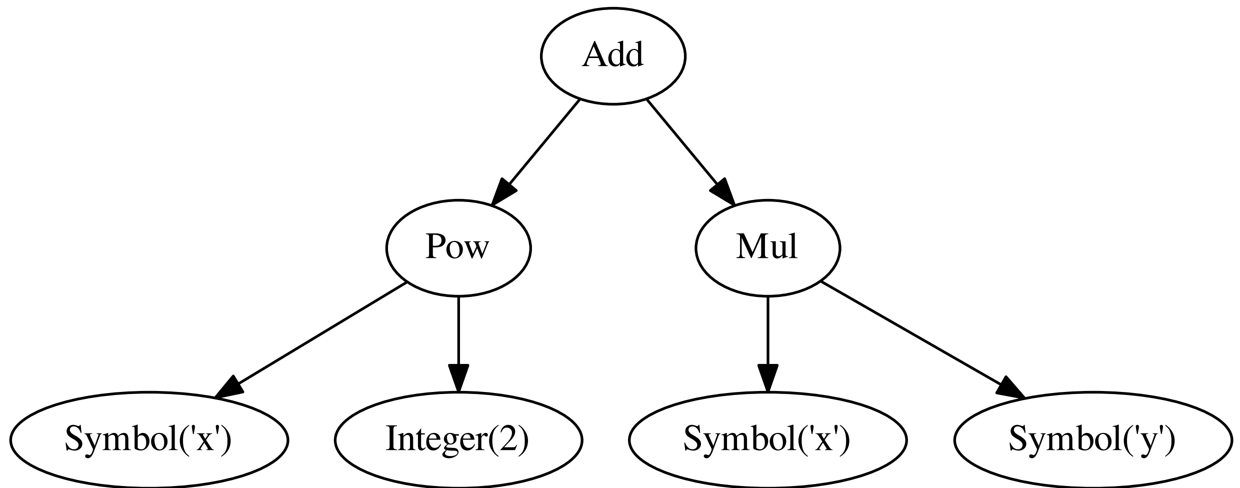
2.9 Expression Trees

In this section, we discuss some ways that we can perform advanced manipulation of expressions.

Most generic interface to represent a mathematical expression in Diofant is a tree. Let us take the expression $xy + x^2$. We can see what this expression looks like internally by using `repr()`

```
>>> repr(x*y + x**2)
"Add(Pow(Symbol('x'), Integer(2)), Mul(Symbol('x'), Symbol('y')))"
```

The easiest way to tear this apart is to look at a diagram of the expression tree:



First, let's look at the leaves of this tree. We got here instances of the *Symbol* (page 79) class and the Diofant version of integers, instance of the *Integer* (page 89) class, even technically we input integer literal 2.

What about `x*y`? As we might expect, this is the multiplication of `x` and `y`. The Diofant class for multiplication is *Mul* (page 103).

```
>>> repr(x*y)
"Mul(Symbol('x'), Symbol('y'))"
```

Thus, we could have created the same object by writing

```
>>> Mul(x, y)
x⋅y
```

When we write `x**2`, this creates a *Pow* (page 100) class instance.

```
>>> repr(x**2)
"Pow(Symbol('x'), Integer(2))"
```

We could have created the same object by calling

```
>>> Pow(x, 2)
 2
x
```

Now we get to our final expression, `x*y + x**2`. This is the addition of our last two objects. The Diofant class for addition is *Add* (page 106), so, as you might expect, to create this object, we could use

```
>>> Add(Pow(x, 2), Mul(x, y))
 2
```

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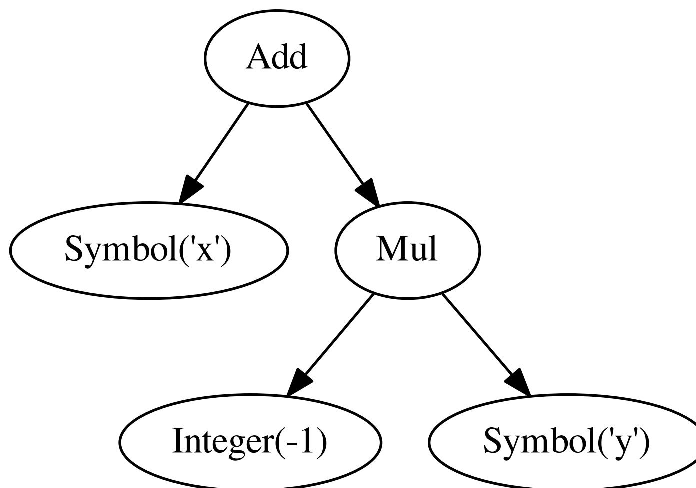
```
x  + x⋅y
>>> x*y + x**2
 2
x  + x⋅y
```

Note: You may have noticed that the order we entered our expression and the order that it came out from printers like `repr()` or in the graph were different. This because the arguments of *Add* (page 106) and the commutative arguments of *Mul* (page 103) are stored in an arbitrary (but consistent!) order, which is independent of the order inputted.

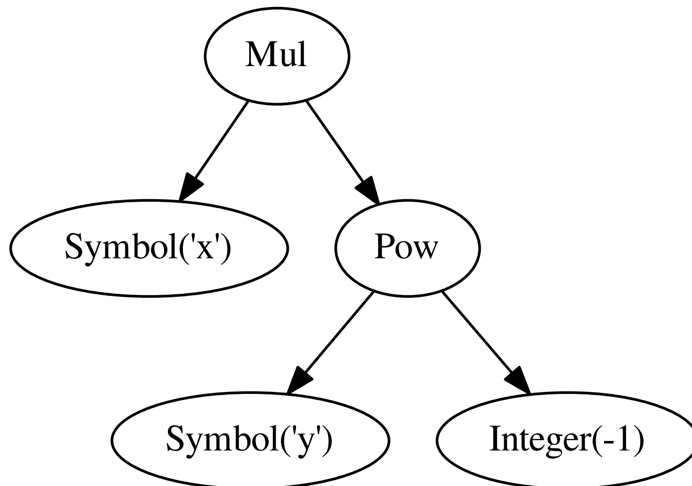
There is no subtraction class. `x - y` is represented as `x + (-1)*y`

```
>>> repr(x - y)
"Add(Symbol('x'), Mul(Integer(-1), Symbol('y')))"
```



Similarly to subtraction, there is no division class.

```
>>> repr(x/y)
"Mul(Symbol('x'), Pow(Symbol('y'), Integer(-1)))"
```



We see that `x/y` is represented as `x*y**(-1)`.

But what about `x/2`? Following the pattern of the previous example, we might expect to see `Mul(x, Pow(Integer(2), -1))`. But instead, we have

```
>>> repr(x/2)
"Mul(Rational(1, 2), Symbol('x'))"
```

Rational numbers are always combined into a single term in a multiplication, so that when we divide by 2, it is represented as multiplying by 1/2.

2.9.1 Walking the Tree

Let's look at how to dig our way through an expression tree. For this every object in Diofant has a very generic interface — two important attributes, *func* (page 44), and *args* (page 42).

The head of the object is encoded in the *func* (page 44) attribute. Usually it is the same as the class of the object, but not always.

```
>>> expr = 2 + x*y
>>> expr
x·y + 2
>>> expr.func
<class 'diofant.core.add.Add'>
>>> type(expr)
<class 'diofant.core.add.Add'>
```

The class of an object need not be the same as the one used to create it.

```
>>> Add(x, x)
2·x
>>> _.func
<class 'diofant.core.mul.Mul'>
```

Note: Diofant classes make heavy use of the `__new__()` class constructor, which, unlike `__init__()`, allows a different class to be returned from the constructor.

The children of a node in the tree are held in the *args* (page 42) attribute.

```
>>> expr.args
(2, x⋅y)
```

From this, we can see `expr` can be completely reconstructed from its *func* (page 44) and its *args* (page 42).

```
>>> expr.func(*expr.args)
x⋅y + 2
```

Note: Every well-formed expression must either have empty *args* (page 42) or satisfy invariant

```
>>> expr == expr.func(*expr.args)
True
```

Empty *args* (page 42) signal that we have hit a leaf of the expression tree.

```
>>> x.args
()
>>> Integer(2).args
()
```

This interface allows us to write recursive generators that walk expression trees either in post-order or pre-order fashion.

```
>>> expr = x*y + 2
>>> for term in preorder_traversal(expr):
...     print(term)
x*y + 2
2
x*y
x
y
```

CHAPTER

THREE

MODULES REFERENCE

Because every feature of Diofant must have a test case, when you are not sure how to use something, just look into the `tests/` directories, find that feature and read the tests for it, that will tell you everything you need to know.

Most of the things are already documented though in this document, that is automatically generated using Diofant's docstrings.

Click the "modules" (modindex) link in the top right corner to easily access any Diofant module, or use this contents:

3.1 Core

3.1.1 sympify

sympify

`diofant.core.sympify.`**`sympify`**(*a*, *locals=None*, *convert_xor=True*, *strict=False*, *rational=False*, *evaluate=None*)

Converts an arbitrary expression to a type that can be used inside Diofant.

For example, it will convert Python ints into instance of diofant.Rational, floats into instances of diofant.Float, etc. It is also able to coerce symbolic expressions which inherit from Basic. This can be useful in cooperation with SAGE.

It currently accepts as arguments:

- any object defined in diofant
- standard numeric python types: int, long, float, Decimal
- strings (like "0.09" or "2e-19")
- booleans, including `None` (will leave `None` unchanged)
- lists, sets or tuples containing any of the above

If the argument is already a type that Diofant understands, it will do nothing but return that value. This can be used at the beginning of a function to ensure you are working with the correct type.

```
>>> sympify(2).is_integer
True
>>> sympify(2).is_extended_real
True
```

```
>>> sympify(2.0).is_extended_real
True
>>> sympify("2.0").is_extended_real
True
>>> sympify("2e-45").is_extended_real
True
```

If the expression could not be converted, a SympifyError is raised.

```
>>> sympify("x***2")
Traceback (most recent call last):
...
SympifyError: SympifyError: "could not parse u'x***2'"
```

Locals

The sympification happens with access to everything that is loaded by `from diofant import *`; anything used in a string that is not defined by that import will be converted to a symbol. In the following, the `bitcount` function is treated as a symbol and the `O` is interpreted as the Order object (used with series) and it raises an error when used improperly:

```
>>> s = 'bitcount(42)'
>>> sympify(s)
bitcount(42)
>>> sympify("O(x)")
O(x)
>>> sympify("O + 1")
Traceback (most recent call last):
...
TypeError: unbound method...
```

In order to have `bitcount` be recognized it can be imported into a namespace dictionary and passed as locals:

```
>>> ns = {}
>>> exec('from diofant.core.evalf import bitcount', ns)
>>> sympify(s, locals=ns)
6
```

In order to have the `O` interpreted as a Symbol, identify it as such in the namespace dictionary. This can be done in a variety of ways; all three of the following are possibilities:

```
>>> ns["O"] = Symbol("O")  # method 1
>>> exec('from diofant.abc import O', ns)  # method 2
>>> ns.update(dict(O=Symbol("O")))  # method 3
>>> sympify("O + 1", locals=ns)
O + 1
```

If you want *all* single-letter and Greek-letter variables to be symbols then you can use the clashing-symbols dictionaries that have been defined there as private variables: _clash1 (single-letter variables), _clash2 (the multi-letter Greek names) or _clash (both single and multi-letter names that are defined in abc).

```
>>> from diofant.abc import _clash1
>>> _clash1
{'E': E, 'I': I, 'N': N, 'O': O, 'S': S}
```

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```
>>> sympify('E & O', _clash1)
And(E, O)
```

Strict

If the option `strict` is set to `True`, only the types for which an explicit conversion has been defined are converted. In the other cases, a SympifyError is raised.

```
>>> print(sympify(None))
None
>>> sympify(None, strict=True)
Traceback (most recent call last):
...
SympifyError: SympifyError: None
```

Evaluation

If the option `evaluate` is set to `False`, then arithmetic and operators will be converted into their Diofant equivalents and the `evaluate=False` option will be added. Nested `Add` or `Mul` will be denested first. This is done via an AST transformation that replaces operators with their Diofant equivalents, so if an operand redefines any of those operations, the redefined operators will not be used.

```
>>> sympify('2**2 / 3 + 5')
19/3
>>> sympify('2**2 / 3 + 5', evaluate=False)
2**2/3 + 5
```

Sometimes autosimplification during sympification results in expressions that are very different in structure than what was entered. Below you can see how an expression reduces to -1 by autosimplification, but does not do so when `evaluate` option is used.

```
>>> -2*(-(-x + 1/x)/(x*(x - 1/x)**2) - 1/(x*(x - 1/x))) - 1
-1
>>> s = '-2*(-(-x + 1/x)/(x*(x - 1/x)**2) - 1/(x*(x - 1/x))) - 1'
>>> sympify(s)
-1
>>> sympify(s, evaluate=False)
-2*((x - 1/x)/(x*(x - 1/x)**2) - 1/(x*(x - 1/x))) - 1
```

Extending

To extend `sympify` to convert custom objects (not derived from `Basic`), just define a `_diofant_` method to your class. You can do that even to classes that you do not own by subclassing or adding the method at runtime.

```
>>> class MyList1:
...     def __iter__(self):
...         yield 1
...         yield 2
...         return
...     def __getitem__(self, i): return list(self)[i]
...     def _diofant_(self): return Matrix(self)
>>> sympify(MyList1())
Matrix([
[1],
[2]])
```

If you do not have control over the class definition you could also use the `converter` global dictionary. The key is the class and the value is a function that takes a single argument and returns the desired Diofant object, e.g. `converter[MyList] = lambda x: Matrix(x)`.

```
>>> class MyList2:   # XXX Do not do this if you control the class!
...     def __iter__(self):  #     Use _diofant_!
...         yield 1
...         yield 2
...         return
...     def __getitem__(self, i): return list(self)[i]
>>> converter[MyList2] = lambda x: Matrix(x)
>>> sympify(MyList2())
Matrix([
[1],
[2]])
```

3.1.2 assumptions

This module contains the machinery handling assumptions.

All symbolic objects have assumption attributes that can be accessed via .is_<assumption name> attribute, i.e. *is_integer* (page 71). Full set of defined assumption names are accessible as `Expr.default_assumptions.rules.defined_facts` attribute.

Assumptions determine certain properties of symbolic objects and can have 3 possible values: True, False, None. True is returned if the object has the property and False is returned if it doesn't or can't (i.e. doesn't make sense):

```
>>> I.is_algebraic
True
>>> I.is_real
False
>>> I.is_prime
False
```

When the property cannot be determined (or when a method is not implemented) None will be returned, e.g. a generic symbol, x, may or may not be positive so a value of None is returned for x.is_positive.

By default, all symbolic values are in the largest set in the given context without specifying the property. For example, a symbol that has a property being integer, is also real, complex, etc.

See Also

See also:

ImaginaryUnit (page 98) *is_algebraic* (page 67) *is_real* (page 74) *is_prime* (page 73)

Notes

Assumption values are stored in obj._assumptions dictionary or are returned by getter methods (with property decorators) or are attributes of objects/classes.

class diofant.core.assumptions.**ManagedProperties**(**args*, ***kws*)
Metaclass for classes with old-style assumptions

class diofant.core.assumptions.**StdFactKB**(*facts=None*)
A FactKB specialised for the built-in rules

This is the only kind of FactKB that Basic objects should use.

copy() → a shallow copy of D

diofant.core.assumptions.**as_property**(*fact*)
Convert a fact name to the name of the corresponding property

diofant.core.assumptions.**check_assumptions**(*expr*, ***assumptions*)
Checks if expression $expr$ satisfies all assumptions.

Parameters **expr** : Expr

Expression to test.

****assumptions** : dict

Keyword arguments to specify assumptions to test.

Returns True, False or None (if can't conclude).

Examples

```
>>> check_assumptions(-5, integer=True)
True
>>> x = Symbol('x', positive=True)
>>> check_assumptions(2*x + 1, negative=True)
False
>>> check_assumptions(z, real=True) is None
True
```

diofant.core.assumptions.**make_property**(*fact*)
Create the automagic property corresponding to a fact.

3.1.3 cache

cacheit

diofant.core.cache.**cacheit**(*f*)
Caching decorator.

The result of cached function must be *immutable*.

Examples

```
>>> @cacheit
... def f(a, b):
...     print(a, b)
...     return a + b
```

```
>>> f(x, y)
x y
x + y
>>> f(x, y)
x + y
```

3.1.4 basic

Base class for all the objects in Diofant

class diofant.core.basic.**Atom**
A parent class for atomic things.

An atom is an expression with no subexpressions, for example Symbol, Number, Rational or Integer, but not Add, Mul, Pow.

classmethod class_key()
Nice order of classes.

doit(***hints*)
Evaluate objects that are not evaluated by default.

See also:

Basic.doit (page 44)

sort_key(*order=None*)
Return a sort key.

class diofant.core.basic.**Basic**
Base class for all objects in Diofant.

Always use `args` property, when accessing parameters of some instance.

args
Returns a tuple of arguments of 'self'.

Examples

```
>>> cot(x).args
(x,)
>>> (x*y).args
(x, y)
```

atoms(**types*)
Returns the atoms that form the current object.

By default, only objects that are truly atomic and can't be divided into smaller pieces are returned: symbols, numbers, and number symbols like I and pi. It is possible to request atoms of any type, however, as demonstrated below.

Examples

```
>>> e = 1 + x + 2*sin(y + I*pi)
>>> e.atoms()
{1, 2, I, pi, x, y}
```

If one or more types are given, the results will contain only those types of atoms.

```
>>> e.atoms(Symbol)
{x, y}
```

```
>>> e.atoms(Number)
{1, 2}
```

```
>>> e.atoms(Number, NumberSymbol)
{1, 2, pi}
```

```
>>> e.atoms(Number, NumberSymbol, I)
{1, 2, I, pi}
```

Note that I (imaginary unit) and zoo (complex infinity) are special types of number symbols and are not part of the NumberSymbol class.

The type can be given implicitly, too:

```
>>> e.atoms(x)
{x, y}
```

Be careful to check your assumptions when using the implicit option since `Integer(1).is_Integer = True` but `type(Integer(1))` is `One`, a special type of diofant atom, while `type(Integer(2))` is type `Integer` and will find all integers in an expression:

```
>>> e.atoms(Integer(1))
{1}
```

```
>>> e.atoms(Integer(2))
{1, 2}
```

Finally, arguments to atoms() can select more than atomic atoms: any diofant type can be listed as an argument and those types of "atoms" as found in scanning the arguments of the expression recursively:

```
>>> from diofant.core.function import AppliedUndef
```

```
>>> (1 + x + 2*sin(y + I*pi)).atoms(Mul)
{I*pi, 2*sin(y + I*pi)}
```

```
>>> f = Function('f')
>>> e = 1 + f(x) + 2*sin(y + I*pi)
>>> e.atoms(Function)
{f(x), sin(y + I*pi)}
>>> (1 + f(x) + 2*sin(y + I*pi)).atoms(AppliedUndef)
{f(x)}
```

classmethod class_key()

Nice order of classes.

copy()

Return swallow copy of self.

count(*query*)

Count the number of matching subexpressions.

count_ops(*visual=None*)
wrapper for count_ops that returns the operation count.

doit(***hints*)
Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

dummy_eq(*other*, *symbol=None*)
Compare two expressions and handle dummy symbols.

Examples

```
>>> u = Dummy('u')
```

```
>>> (u**2 + 1).dummy_eq(x**2 + 1)
True
>>> (u**2 + 1) == (x**2 + 1)
False
```

```
>>> (u**2 + y).dummy_eq(x**2 + y, x)
True
>>> (u**2 + y).dummy_eq(x**2 + y, y)
False
```

find(*query*)
Find all subexpressions matching a query.

free_symbols
Return from the atoms of self those which are free symbols.

For most expressions, all symbols are free symbols. For some classes this is not true. e.g. Integrals use Symbols for the dummy variables which are bound variables, so Integral has a method to return all symbols except those. Derivative keeps track of symbols with respect to which it will perform a derivative; those are bound variables, too, so it has its own free_symbols method.

Any other method that uses bound variables should implement a free_symbols method.

func

The top-level function in an expression.

The following should hold for all objects:

```
x == x.func(*x.args)
```

Examples

```
>>> a = 2*x
>>> a.func
<class 'diofant.core.mul.Mul'>
>>> a.args
(2, x)
>>> a.func(*a.args)
2*x
>>> a == a.func(*a.args)
True
```

has(**patterns*)

Test if any subexpression matches any of the patterns.

Parameters ***patterns** : tuple of Expr

List of expressions to search for match.

Returns bool

False if there is no match or patterns list is empty, else True.

Examples

```
>>> e = x**2 + sin(x*y)
>>> e.has(z)
False
>>> e.has(x, y, z)
True
>>> x.has()
False
```

match(*pattern*)

Pattern matching.

Wild symbols match all.

Parameters **pattern** : Expr

An expression that may contain Wild symbols.

Returns dict or None

If pattern match self, return a dictionary of replacement rules, such that:

```
pattern.xreplace(self.match(pattern)) == self
```

See also:

xreplace (page 50), *diofant.core.symbol.Wild* (page 80)

Examples

```
>>> p = Wild("p")
>>> q = Wild("q")
>>> e = (x + y)**(x + y)
>>> e.match(p**p)
{p_: x + y}
>>> e.match(p**q)
{p_: x + y, q_: x + y}
>>> (p**q).xreplace(_)
(x + y)**(x + y)
```

rcall(**args*)

Apply on the argument recursively through the expression tree.

This method is used to simulate a common abuse of notation for operators. For instance in Diofant the the following will not work:

`(x+Lambda(y, 2*y))(z) == x+2*z`,

however you can use

```
>>> (x + Lambda(y, 2*y)).rcall(z)
x + 2*z
```

replace(*query*, *value*, *exact=False*)

Replace matching subexpressions of `self` with `value`.

Traverses an expression tree and performs replacement of matching subexpressions from the bottom to the top of the tree in a simultaneous fashion so changes made are targeted only once. In addition, if an expression containing more than one Wild symbol is being used to match subexpressions and the `exact` flag is True, then the match will only succeed if non-zero values are received for each Wild that appears in the match pattern.

The list of possible combinations of queries and replacement values is listed below:

See also:

subs **(page 49)** substitution of subexpressions as defined by the objects themselves.

xreplace **(page 50)** exact node replacement in expr tree; also capable of using matching rules

Examples

Initial setup

```
>>> f = log(sin(x)) + tan(sin(x**2))
```

1.1. type -> type obj.replace(type, newtype)

When object of type `type` is found, replace it with the result of passing its argument(s) to `newtype`.

```
>>> f.replace(sin, cos)
log(cos(x)) + tan(cos(x**2))
>>> (x*y).replace(Mul, Add)
x + y
```

1.2. type -> func obj.replace(type, func)

When object of type `type` is found, apply `func` to its argument(s). `func` must be written to handle the number of arguments of `type`.

```
>>> f.replace(sin, lambda arg: sin(2*arg))
log(sin(2*x)) + tan(sin(2*x**2))
>>> (x*y).replace(Mul, lambda *args: sin(2*Mul(*args)))
sin(2*x*y)
```

2.1. pattern -> expr obj.replace(pattern(wild), expr(wild))

Replace subexpressions matching `pattern` with the expression written in terms of the Wild symbols in `pattern`.

```
>>> a = Wild('a')
>>> f.replace(sin(a), tan(a))
log(tan(x)) + tan(tan(x**2))
>>> f.replace(sin(a), tan(a/2))
log(tan(x/2)) + tan(tan(x**2/2))
>>> f.replace(sin(a), a)
log(x) + tan(x**2)
>>> (x*y).replace(a*x, a)
y
```

When the default value of False is used with patterns that have more than one Wild symbol, non-intuitive results may be obtained:

```
>>> b = Wild('b')
>>> (2*x).replace(a*x + b, b - a)
2/x
```

For this reason, the `exact` option can be used to make the replacement only when the match gives non-zero values for all Wild symbols:

```
>>> (2*x + y).replace(a*x + b, b - a, exact=True)
y - 2
>>> (2*x).replace(a*x + b, b - a, exact=True)
2*x
```

2.2. pattern -> func obj.replace(pattern(wild), lambda wild: expr(wild))

All behavior is the same as in 2.1 but now a function in terms of pattern variables is used rather than an expression:

```
>>> f.replace(sin(a), lambda a: sin(2*a))
log(sin(2*x)) + tan(sin(2*x**2))
```

3.1. func -> func obj.replace(filter, func)

Replace subexpression `e` with `func(e)` if `filter(e)` is True.

```
>>> g = 2*sin(x**3)
>>> g.replace(lambda expr: expr.is_Number, lambda expr: expr**2)
4*sin(x**9)
```

The expression itself is also targeted by the query but is done in such a fashion that changes are not made twice.

```
>>> e = x*(x*y + 1)
>>> e.replace(lambda x: x.is_Mul, lambda x: 2*x)
2*x*(2*x*y + 1)
```

rewrite(**args*, ***hints*)
Rewrite functions in terms of other functions.

Rewrites expression containing applications of functions of one kind in terms of functions of different kind. For example you can rewrite trigonometric functions as complex exponentials or combinatorial functions as gamma function.

As a pattern this function accepts a list of functions to to rewrite (instances of DefinedFunction class). As rule you can use string or a destination function instance (in this case rewrite() will use the str() function).

There is also the possibility to pass hints on how to rewrite the given expressions. For now there is only one such hint defined called 'deep'. When 'deep' is set to False it will forbid functions to rewrite their contents.

Examples

Unspecified pattern:

```
>>> sin(x).rewrite(exp)
-I*(E**(I*x) - E**(-I*x))/2
```

Pattern as a single function:

```
>>> sin(x).rewrite(sin, exp)
-I*(E**(I*x) - E**(-I*x))/2
```

Pattern as a list of functions:

```
>>> sin(x).rewrite([sin], exp)
-I*(E**(I*x) - E**(-I*x))/2
```

sort_key(*order=None*)
Return a sort key.

Examples

```
>>> sorted([Rational(1, 2), I, -I], key=lambda x: x.sort_key())
[1/2, -I, I]
```

```
>>> [x, 1/x, 1/x**2, x**2, x**S.Half, x**Rational(1, 4), x**Rational(3, 2)]
[x, 1/x, x**(-2), x**2, sqrt(x), x**(1/4), x**(3/2)]
>>> sorted(_, key=lambda x: x.sort_key())
[x**(-2), 1/x, x**(1/4), sqrt(x), x, x**(3/2), x**2]
```

subs(**args*, ***kwargs*)

Substitutes old for new in an expression after sympifying args.

$args$ **is either:**

- two arguments, e.g. foo.subs(old, new)
- **one iterable argument, e.g. foo.subs(iterable). The iterable may be**
  - **o an iterable container with (old, new) pairs. In this case the** replacements are processed in the order given with successive patterns possibly affecting replacements already made.
  - **o a dict or set whose key/value items correspond to old/new pairs.** In this case the old/new pairs will be sorted by op count and in case of a tie, by number of args and the default_sort_key. The resulting sorted list is then processed as an iterable container (see previous).

If the keyword `simultaneous` is True, the subexpressions will not be evaluated until all the substitutions have been made.

See also:

replace **(page 46)** replacement capable of doing wildcard-like matching, parsing of match, and conditional replacements

xreplace **(page 50)** exact node replacement in expr tree; also capable of using matching rules

diofant.core.evalf.EvalfMixin.evalf **(page 145)** calculates the given formula to a desired level of precision

Examples

```
>>> (1 + x*y).subs(x, pi)
pi*y + 1
>>> (1 + x*y).subs({x: pi, y: 2})
1 + 2*pi
>>> (1 + x*y).subs([(x, pi), (y, 2)])
1 + 2*pi
>>> reps = [(y, x**2), (x, 2)]
>>> (x + y).subs(reps)
6
>>> (x + y).subs(reversed(reps))
x**2 + 2
```

```
>>> (x**2 + x**4).subs(x**2, y)
y**2 + y
```

To replace only the x**2 but not the x**4, use xreplace:

```
>>> (x**2 + x**4).xreplace({x**2: y})
x**4 + y
```

To delay evaluation until all substitutions have been made, set the keyword `simultaneous` to True:

```
>>> (x/y).subs([(x, 0), (y, 0)])
0
>>> (x/y).subs([(x, 0), (y, 0)], simultaneous=True)
nan
```

This has the added feature of not allowing subsequent substitutions to affect those already made:

```
>>> ((x + y)/y).subs({x + y: y, y: x + y})
1
>>> ((x + y)/y).subs({x + y: y, y: x + y}, simultaneous=True)
y/(x + y)
```

In order to obtain a canonical result, unordered iterables are sorted by count_op length, number of arguments and by the default_sort_key to break any ties. All other iterables are left unsorted.

```
>>> from diofant.abc import a, b, c, d, e
```

```
>>> expr = sqrt(sin(2*x))*sin(exp(x)*x)*cos(2*x) + sin(2*x)
```

```
>>> expr.subs({sqrt(sin(2*x)): a, sin(2*x): b,
...            cos(2*x): c, x: d, exp(x): e})
a*c*sin(d*e) + b
```

The resulting expression represents a literal replacement of the old arguments with the new arguments. This may not reflect the limiting behavior of the expression:

```
>>> (x**3 - 3*x).subs({x: oo})
nan
```

```
>>> limit(x**3 - 3*x, x, oo)
oo
```

If the substitution will be followed by numerical evaluation, it is better to pass the substitution to evalf as

```
>>> (1/x).evalf(21, subs={x: 3.0}, strict=False)
0.333333333333333333333
```

rather than

```
>>> (1/x).subs({x: 3.0}).evalf(21, strict=False)
0.333333333333333
```

as the former will ensure that the desired level of precision is obtained.

xreplace(*rule*)

Replace occurrences of objects within the expression.

Parameters rule : dict-like

Expresses a replacement rule

Returns xreplace : the result of the replacement

See also:

replace **(page 46)** replacement capable of doing wildcard-like matching, parsing of match, and conditional replacements

subs **(page 49)** substitution of subexpressions as defined by the objects themselves.

Examples

```
>>> x, y, z = symbols('x y z')
>>> (1 + x*y).xreplace({x: pi})
pi*y + 1
>>> (1 + x*y).xreplace({x: pi, y: 2})
1 + 2*pi
```

Replacements occur only if an entire node in the expression tree is matched:

```
>>> (x*y + z).xreplace({x*y: pi})
z + pi
>>> (x*y*z).xreplace({x*y: pi})
x*y*z
>>> (2*x).xreplace({2*x: y, x: z})
y
>>> (2*2*x).xreplace({2*x: y, x: z})
4*z
>>> (x + y + 2).xreplace({x + y: 2})
x + y + 2
>>> (x + 2 + exp(x + 2)).xreplace({x + 2: y})
E**y + x + 2
```

xreplace doesn't differentiate between free and bound symbols. In the following, subs(x, y) would not change x since it is a bound symbol, but xreplace does:

```
>>> Integral(x, (x, 1, 2*x)).xreplace({x: y})
Integral(y, (y, 1, 2*y))
```

Trying to replace x with an expression raises an error:

```
>>> Integral(x, (x, 1, 2*x)).xreplace({x: 2*y})
Traceback (most recent call last):
...
ValueError: Invalid limits given: ((2*y, 1, 4*y),)
```

class diofant.core.basic.**preorder_traversal**(*node*, *keys=None*)
Do a pre-order traversal of a tree.

This iterator recursively yields nodes that it has visited in a pre-order fashion. That is, it yields the current node then descends through the tree breadth-first to yield all of a node's children's pre-order traversal.

For an expression, the order of the traversal depends on the order of .args, which in many cases can be arbitrary.

Parameters node : diofant expression

The expression to traverse.

keys : (default None) sort key(s)

The key(s) used to sort args of Basic objects. When None, args of Basic objects are processed in arbitrary order. If key is defined, it will be passed along to ordered() as the only key(s) to use to sort the arguments; if key is simply True then the default keys of ordered will be used.

Yields subtree : diofant expression

All of the subtrees in the tree.

Examples

```
>>> x, y, z = symbols('x y z')
```

The nodes are returned in the order that they are encountered unless key is given; simply passing key=True will guarantee that the traversal is unique.

```
>>> list(preorder_traversal((x + y)*z, keys=True))
[z*(x + y), z, x + y, x, y]
```

skip()

Skip yielding current node's (last yielded node's) subtrees.

Examples

```
>>> pt = preorder_traversal((x+y*z)*z)
>>> for i in pt:
...     print(i)
...     if i == x + y*z:
...         pt.skip()
z*(x + y*z)
z
x + y*z
```

3.1.5 core

3.1.6 singleton

Singleton mechanism

`diofant.core.singleton.S = S`

Alias for instance of *SingletonRegistry* (page 53).

class `diofant.core.singleton.Singleton`

Metaclass for singleton classes.

A singleton class has only one instance which is returned every time the class is instantiated. Additionally, this instance can be accessed through the global registry object *S* (page 52) as `S.<class_name>`.

Notes

Instance creation is delayed until the first time the value is accessed.

This metaclass is a subclass of ManagedProperties because that is the metaclass of many classes that need to be Singletons (Python does not allow subclasses to have a different metaclass than the superclass, except the subclass may use a subclassed metaclass).

Examples

```
>>> class MySingleton(Basic, metaclass=Singleton):
...     pass
>>> Basic() is Basic()
False
>>> MySingleton() is MySingleton()
True
>>> S.MySingleton is MySingleton()
True
```

class diofant.core.singleton.**SingletonRegistry**

The registry for the singleton classes.

Several classes in Diofant appear so often that they are singletonized, that is, using some metaprogramming they are made so that they can only be instantiated once (see the *diofant.core.singleton.Singleton* (page 52) class for details). For instance, every time you create `Integer(0)`, this will return the same instance, *diofant.core.numbers.Zero* (page 94).

```
>>> a = Integer(0)
>>> a is S.Zero
True
```

All singleton instances are attributes of the *S* (page 52) object, so `Integer(0)` can also be accessed as `S.Zero`.

Notes

For the most part, the fact that certain objects are singletonized is an implementation detail that users shouldn't need to worry about. In Diofant library code, `is` comparison is often used for performance purposes. The primary advantage of *S* (page 52) for end users is the convenient access to certain instances that are otherwise difficult to type, like `S.Half` (instead of `Rational(1, 2)`).

When using `is` comparison, make sure the argument is a *Basic* (page 42) instance. For example,

```
>>> int(0) is S.Zero
False
```

class diofant.core.singleton.**SingletonWithManagedProperties**(**args*, ***kws*)

Metaclass for singleton classes with managed properties.

3.1.7 evaluate

diofant.core.evaluate.**evaluate**(*x*)

Control automatic evaluation

This context managers controls whether or not all Diofant functions evaluate by default.

Note that much of Diofant expects evaluated expressions. This functionality is experimental and is unlikely to function as intended on large expressions.

Examples

```
>>> x + x
2*x
>>> with evaluate(False):
...     x + x
x + x
```

3.1.8 expr

3.1.9 Expr

class diofant.core.expr.**Expr**

Base class for algebraic expressions.

Everything that requires arithmetic operations to be defined should subclass this class, instead of Basic (which should be used only for argument storage and expression manipulation, i.e. pattern matching, substitutions, etc).

See also:

diofant.core.basic.Basic (page 42)

adjoint()

Compute conjugate transpose or Hermite conjugation.

See also:

diofant.functions.elementary.complexes.adjoint (page 294)

apart(*x=None*, ***args*)

See the apart function in diofant.polys

args_cnc(*cset=False*, *warn=True*, *split_1=True*)

Return [commutative factors, non-commutative factors] of self.

self is treated as a Mul and the ordering of the factors is maintained. If `cset` is True the commutative factors will be returned in a set. If there were repeated factors (as may happen with an unevaluated Mul) then an error will be raised unless it is explicitly suppressed by setting `warn` to False.

Note: -1 is always separated from a Number unless split_1 is False.

```
>>> A, B = symbols('A B', commutative=0)
>>> x, y = symbols('x y')
>>> (-2*x*y).args_cnc()
[[-1, 2, x, y], []]
```

(continues on next page)

(continued from previous page)

```
>>> (-2.5*x).args_cnc()
[[-1, 2.5, x], []]
>>> (-2*x*A*B*y).args_cnc()
[[-1, 2, x, y], [A, B]]
>>> (-2*x*A*B*y).args_cnc(split_1=False)
[[-2, x, y], [A, B]]
>>> (-2*x*y).args_cnc(cset=True)
[{-1, 2, x, y}, []]
```

The arg is always treated as a Mul:

```
>>> (-2 + x + A).args_cnc()
[[], [x - 2 + A]]
>>> (-oo).args_cnc() # -oo is a singleton
[[-1, oo], []]
```

as_base_exp()
Return base and exp of self.

See also:

diofant.core.power.Pow.as_base_exp (page 101)

as_coeff_Add(*rational=False*)
Efficiently extract the coefficient of a summation.

as_coeff_Mul(*rational=False*)
Efficiently extract the coefficient of a product.

as_coeff_add(**deps*)
Return the tuple (c, args) where self is written as an Add, a.

c should be a Rational added to any terms of the Add that are independent of deps.

args should be a tuple of all other terms of a; args is empty if self is a Number or if self is independent of deps (when given).

This should be used when you don't know if self is an Add or not but you want to treat self as an Add or if you want to process the individual arguments of the tail of self as an Add.

- if you know self is an Add and want only the head, use self.args[0];
- if you don't want to process the arguments of the tail but need the tail then use self.as_two_terms() which gives the head and tail.
- if you want to split self into an independent and dependent parts use `self.as_independent(*deps)`

```
>>> (Integer(3)).as_coeff_add()
(3, ())
>>> (3 + x).as_coeff_add()
(3, (x,))
>>> (3 + x + y).as_coeff_add(x)
(y + 3, (x,))
>>> (3 + y).as_coeff_add(x)
(y + 3, ())
```

as_coeff_exponent(*x*)
`c*x**e -> c,e` where x can be any symbolic expression.

as_coeff_mul(**deps*, ***kwargs*)

Return the tuple (c, args) where self is written as a Mul, m.

c should be a Rational multiplied by any terms of the Mul that are independent of deps.

args should be a tuple of all other terms of m; args is empty if self is a Number or if self is independent of deps (when given).

This should be used when you don't know if self is a Mul or not but you want to treat self as a Mul or if you want to process the individual arguments of the tail of self as a Mul.

- if you know self is a Mul and want only the head, use self.args[0];
- if you don't want to process the arguments of the tail but need the tail then use self.as_two_terms() which gives the head and tail;
- if you want to split self into an independent and dependent parts use `self.as_independent(*deps)`

```
>>> (Integer(3)).as_coeff_mul()
(3, ())
>>> (3*x*y).as_coeff_mul()
(3, (x, y))
>>> (3*x*y).as_coeff_mul(x)
(3*y, (x,))
>>> (3*y).as_coeff_mul(x)
(3*y, ())
```

as_coefficient(*expr*)

Extracts symbolic coefficient at the given expression.

In other words, this functions separates 'self' into the product of 'expr' and 'expr'-free coefficient. If such separation is not possible it will return None.

See also:

coeff **(page 63)** return sum of terms have a given factor

as_coeff_Add **(page 55)** separate the additive constant from an expression

as_coeff_Mul **(page 55)** separate the multiplicative constant from an expression

as_independent **(page 58)** separate x-dependent terms/factors from others

diofant.polys.polytools.Poly.coeff_monomial **(page 678)** efficiently find the single coefficient of a monomial in Poly

diofant.polys.polytools.Poly.nth **(page 695)** like coeff_monomial but powers of monomial terms are used

Examples

```
>>> E.as_coefficient(E)
1
>>> (2*E).as_coefficient(E)
2
>>> (2*sin(E)*E).as_coefficient(E)
```

Two terms have E in them so a sum is returned. (If one were desiring the coefficient of the term exactly matching E then the constant from the returned expression could be selected. Or, for greater precision, a method of Poly can be used to indicate the desired term from which the coefficient is desired.)

```
>>> (2*E + x*E).as_coefficient(E)
x + 2
>>> _.args[0]  # just want the exact match
2
>>> p = Poly(2*E + x*E); p
Poly(x*E + 2*E, x, E, domain='ZZ')
>>> p.coeff_monomial(E)
2
>>> p.nth(0, 1)
2
```

Since the following cannot be written as a product containing E as a factor, None is returned. (If the coefficient `2*x` is desired then the `coeff` method should be used.)

```
>>> (2*E*x + x).as_coefficient(E)
>>> (2*E*x + x).coeff(E)
2*x
```

```
>>> (E*(x + 1) + x).as_coefficient(E)
```

```
>>> (2*pi*I).as_coefficient(pi*I)
2
>>> (2*I).as_coefficient(pi*I)
```

as_coefficients_dict()

Return a dictionary mapping terms to their Rational coefficient. Since the dictionary is a defaultdict, inquiries about terms which were not present will return a coefficient of 0. If an expression is not an Add it is considered to have a single term.

Examples

```
>>> from diofant.abc import a
>>> (3*x + a*x + 4).as_coefficients_dict()
{1: 4, x: 3, a*x: 1}
>>> _[a]
0
>>> (3*a*x).as_coefficients_dict()
{a*x: 3}
```

as_content_primitive(*radical=False*)

This method should recursively remove a Rational from all arguments and return that (content) and the new self (primitive). The content should always be positive and `Mul(*foo.as_content_primitive()) == foo`. The primitive need no be in canonical form and should try to preserve the underlying structure if possible (i.e. expand_mul should not be applied to self).

Examples

```
>>> eq = 2 + 2*x + 2*y*(3 + 3*y)
```

The as_content_primitive function is recursive and retains structure:

```
>>> eq.as_content_primitive()
(2, x + 3*y*(y + 1) + 1)
```

Integer powers will have Rationals extracted from the base:

```
>>> ((2 + 6*x)**2).as_content_primitive()
(4, (3*x + 1)**2)
>>> ((2 + 6*x)**(2*y)).as_content_primitive()
(1, (2*(3*x + 1))**(2*y))
```

Terms may end up joining once their as_content_primitives are added:

```
>>> ((5*(x*(1 + y)) + 2*x*(3 + 3*y))).as_content_primitive()
(11, x*(y + 1))
>>> ((3*(x*(1 + y)) + 2*x*(3 + 3*y))).as_content_primitive()
(9, x*(y + 1))
>>> ((3*(z*(1 + y)) + 2.0*x*(3 + 3*y))).as_content_primitive()
(1, 6.0*x*(y + 1) + 3*z*(y + 1))
>>> ((5*(x*(1 + y)) + 2*x*(3 + 3*y))**2).as_content_primitive()
(121, x**2*(y + 1)**2)
>>> ((5*(x*(1 + y)) + 2.0*x*(3 + 3*y))**2).as_content_primitive()
(1, 121.0*x**2*(y + 1)**2)
```

Radical content can also be factored out of the primitive:

```
>>> (2*sqrt(2) + 4*sqrt(10)).as_content_primitive(radical=True)
(2, sqrt(2)*(1 + 2*sqrt(5)))
```

as_expr(**gens*)

Convert a polynomial to a Diofant expression.

Examples

```
>>> f = (x**2 + x*y).as_poly(x, y)
>>> f.as_expr()
x**2 + x*y
```

```
>>> sin(x).as_expr()
sin(x)
```

as_independent(**deps, **hint*)

A mostly naive separation of a Mul or Add into arguments that are not are dependent on deps. To obtain as complete a separation of variables as possible, use a separation method first, e.g.:

- separatevars() to change Mul, Add and Pow (including exp) into Mul
- .expand(mul=True) to change Add or Mul into Add
- .expand(log=True) to change log expr into an Add

The only non-naive thing that is done here is to respect noncommutative ordering of variables.

The returned tuple (i, d) has the following interpretation:

- i will has no variable that appears in deps
- d will be 1 or else have terms that contain variables that are in deps
- if self is an Add then self = i + d
- if self is a Mul then self = i*d
- if self is anything else, either tuple (self, S.One) or (S.One, self) is returned.

To force the expression to be treated as an Add, use the hint as_Add=True

See also:

diofant.simplify.simplify.separatevars (page 778), *expand* (page 65), *diofant.core.add.Add.as_two_terms* (page 107), *diofant.core.mul.Mul.as_two_terms* (page 104), *as_coeff_add* (page 55), *as_coeff_mul* (page 55)

Examples

– self is an Add

```
>>> (x + x*y).as_independent(x)
(0, x*y + x)
>>> (x + x*y).as_independent(y)
(x, x*y)
>>> (2*x*sin(x) + y + x + z).as_independent(x)
(y + z, 2*x*sin(x) + x)
>>> (2*x*sin(x) + y + x + z).as_independent(x, y)
(z, 2*x*sin(x) + x + y)
```

– self is a Mul

```
>>> (x*sin(x)*cos(y)).as_independent(x)
(cos(y), x*sin(x))
```

non-commutative terms cannot always be separated out when self is a Mul

```
>>> n1, n2, n3 = symbols('n1 n2 n3', commutative=False)
>>> (n1 + n1*n2).as_independent(n2)
(n1, n1*n2)
>>> (n2*n1 + n1*n2).as_independent(n2)
(0, n1*n2 + n2*n1)
>>> (n1*n2*n3).as_independent(n1)
(1, n1*n2*n3)
>>> (n1*n2*n3).as_independent(n2)
(n1, n2*n3)
>>> ((x-n1)*(x-y)).as_independent(x)
(1, (x - y)*(x - n1))
```

– self is anything else:

```
>>> (sin(x)).as_independent(x)
(1, sin(x))
```

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```
>>> (sin(x)).as_independent(y)
(sin(x), 1)
>>> exp(x+y).as_independent(x)
(1, E**(x + y))
```

– force self to be treated as an Add:

```
>>> (3*x).as_independent(x, as_Add=True)
(0, 3*x)
```

– force self to be treated as a Mul:

```
>>> (3+x).as_independent(x, as_Add=False)
(1, x + 3)
>>> (-3+x).as_independent(x, as_Add=False)
(1, x - 3)
```

Note how the below differs from the above in making the constant on the dep term positive.

```
>>> (y*(-3+x)).as_independent(x)
(y, x - 3)
```

– use .as_independent() for true independence testing instead of .has(). The former considers only symbols in the free symbols while the latter considers all symbols

```
>>> I = Integral(x, (x, 1, 2))
>>> I.has(x)
True
>>> x in I.free_symbols
False
>>> I.as_independent(x) == (I, 1)
True
>>> (I + x).as_independent(x) == (I, x)
True
```

Note: when trying to get independent terms, a separation method might need to be used first. In this case, it is important to keep track of what you send to this routine so you know how to interpret the returned values

```
>>> separatevars(exp(x+y)).as_independent(x)
(E**y, E**x)
>>> (x + x*y).as_independent(y)
(x, x*y)
>>> separatevars(x + x*y).as_independent(y)
(x, y + 1)
>>> (x*(1 + y)).as_independent(y)
(x, y + 1)
>>> (x*(1 + y)).expand(mul=True).as_independent(y)
(x, x*y)
>>> a, b=symbols('a b', positive=True)
>>> (log(a*b).expand(log=True)).as_independent(b)
(log(a), log(b))
```

as_leading_term(**symbols*)

Returns the leading (nonzero) term of the series expansion of self.

The _eval_as_leading_term routines are used to do this, and they must always return a non-zero value.

Examples

```
>>> (1 + x + x**2).as_leading_term(x)
1
>>> (1/x**2 + x + x**2).as_leading_term(x)
x**(-2)
```

as_numer_denom()

expression -> a/b -> a, b

This is just a stub that should be defined by an object's class methods to get anything else.

See also:

normal **(page 75)** return a/b instead of a, b

as_ordered_factors(*order=None*)

Return list of ordered factors (if Mul) else [self].

as_ordered_terms(*order=None, data=False*)

Transform an expression to an ordered list of terms.

Examples

```
>>> (sin(x)**2*cos(x) + sin(x)**2 + 1).as_ordered_terms()
[sin(x)**2*cos(x), sin(x)**2, 1]
```

as_poly(**gens, **args*)

Converts `self` to a polynomial or returns `None`.

Examples

```
>>> (x**2 + x*y).as_poly()
Poly(x**2 + x*y, x, y, domain='ZZ')
```

```
>>> (x**2 + x*y).as_poly(x, y)
Poly(x**2 + x*y, x, y, domain='ZZ')
```

```
>>> (x**2 + sin(y)).as_poly(x, y) is None
True
```

as_powers_dict()

Return self as a dictionary of factors with each factor being treated as a power. The keys are the bases of the factors and the values, the corresponding exponents. The resulting dictionary should be used with caution if the expression is a Mul and

contains non- commutative factors since the order that they appeared will be lost in the dictionary.

as_real_imag(*deep=True*, ***hints*)

Performs complex expansion on 'self' and returns a tuple containing collected both real and imaginary parts. This method can't be confused with re() and im() functions, which does not perform complex expansion at evaluation.

However it is possible to expand both re() and im() functions and get exactly the same results as with a single call to this function.

```
>>> x, y = symbols('x y', extended_real=True)
```

```
>>> (x + y*I).as_real_imag()
(x, y)
```

```
>>> from diofant.abc import w
```

```
>>> (z + w*I).as_real_imag()
(re(z) - im(w), re(w) + im(z))
```

as_terms()

Transform an expression to a list of terms.

aseries(*x*, *n=6*, *bound=0*, *hir=False*)

Returns asymptotic expansion for "self". See *[R77]* (page 1247)

This is equivalent to `self.series(x, oo, n)`

Use the `hir` parameter to produce hierarchical series. It stops the recursion at an early level and may provide nicer and more useful results.

If the most rapidly varying subexpression of a given expression f is f itself, the algorithm tries to find a normalized representation of the mrv set and rewrites f using this normalized representation. Use the `bound` parameter to give limit on rewriting coefficients in its normalized form.

If the expansion contains an order term, it will be either `O(x**(-n))` or `O(w**(-n))` where `w` belongs to the most rapidly varying expression of `self`.

Notes

This algorithm is directly induced from the limit computational algorithm provided by Gruntz. It majorly uses the mrv and rewrite sub-routines. The overall idea of this algorithm is first to look for the most rapidly varying subexpression w of a given expression f and then expands f in a series in w. Then same thing is recursively done on the leading coefficient till we get constant coefficients.

References

[R75] (page 1247), *[R76]* (page 1247), *[R77]* (page 1247)

Examples

```
>>> e = sin(1/x + exp(-x)) - sin(1/x)
>>> e.aseries(x)
E**(-x)*(1/(24*x**4) - 1/(2*x**2) + 1 + O(x**(-6), (x, oo)))
>>> e.aseries(x, n=3, hir=True)
-E**(-2*x)*sin(1/x)/2 + E**(-x)*cos(1/x) + O(E**(-3*x), (x, oo))
```

```
>>> e = exp(exp(x)/(1 - 1/x))
>>> e.aseries(x, bound=3)
E**(E**x)*E**(E**x/x**2)*E**(E**x/x)*E**(-E**x + E**x/(1 - 1/x) - E**x/x -
↪E**x/x**2)
>>> e.aseries(x)
E**(E**x/(1 - 1/x))
```

cancel(**gens, **args*)
See the cancel function in diofant.polys

canonical_variables
Return a dictionary mapping any variable defined in `self.variables` as underscore-suffixed numbers corresponding to their position in `self.variables`. Enough underscores are added to ensure that there will be no clash with existing free symbols.

Examples

```
>>> Lambda(x, 2*x).canonical_variables
{x: 0_}
```

coeff(*x, n=1, right=False*)
Returns the coefficient from the term(s) containing `x**n` or None. If `n` is zero then all terms independent of `x` will be returned.

When x is noncommutative, the coeff to the left (default) or right of x can be returned. The keyword 'right' is ignored when x is commutative.

See also:

diofant.core.expr.Expr.as_coefficient (page 56), *diofant.core.expr.Expr.as_coeff_Add* (page 55), *diofant.core.expr.Expr.as_coeff_Mul* (page 55), *diofant.core.expr.Expr.as_independent* (page 58), *diofant.polys.polytools.Poly.coeff_monomial* (page 678), *diofant.polys.polytools.Poly.nth* (page 695)

Examples

You can select terms that have an explicit negative in front of them:

```
>>> (-x + 2*y).coeff(-1)
x
>>> (x - 2*y).coeff(-1)
2*y
```

You can select terms with no Rational coefficient:

```
>>> (x + 2*y).coeff(1)
x
>>> (3 + 2*x + 4*x**2).coeff(1)
0
```

You can select terms independent of x by making n=0; in this case expr.as_independent(x)[0] is returned (and 0 will be returned instead of None):

```
>>> (3 + 2*x + 4*x**2).coeff(x, 0)
3
>>> eq = ((x + 1)**3).expand() + 1
>>> eq
x**3 + 3*x**2 + 3*x + 2
>>> [eq.coeff(x, i) for i in reversed(range(4))]
[1, 3, 3, 2]
>>> eq -= 2
>>> [eq.coeff(x, i) for i in reversed(range(4))]
[1, 3, 3, 0]
```

You can select terms that have a numerical term in front of them:

```
>>> (-x - 2*y).coeff(2)
-y
>>> (x + sqrt(2)*x).coeff(sqrt(2))
x
```

The matching is exact:

```
>>> (3 + 2*x + 4*x**2).coeff(x)
2
>>> (3 + 2*x + 4*x**2).coeff(x**2)
4
>>> (3 + 2*x + 4*x**2).coeff(x**3)
0
>>> (z*(x + y)**2).coeff((x + y)**2)
z
>>> (z*(x + y)**2).coeff(x + y)
0
```

In addition, no factoring is done, so 1 + z*(1 + y) is not obtained from the following:

```
>>> (x + z*(x + x*y)).coeff(x)
1
```

If such factoring is desired, factor_terms can be used first:

```
>>> factor_terms(x + z*(x + x*y)).coeff(x)
z*(y + 1) + 1
```

```
>>> n, m, o = symbols('n m o', commutative=False)
>>> n.coeff(n)
1
>>> (3*n).coeff(n)
3
>>> (n*m + m*n*m).coeff(n) # = (1 + m)*n*m
1 + m
```

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```
>>> (n*m + m*n*m).coeff(n, right=True) # = (1 + m)*n*m
m
```

If there is more than one possible coefficient 0 is returned:

```
>>> (n*m + m*n).coeff(n)
0
```

If there is only one possible coefficient, it is returned:

```
>>> (n*m + x*m*n).coeff(m*n)
x
>>> (n*m + x*m*n).coeff(m*n, right=1)
1
```

collect(*syms*, *func=None*, *evaluate=True*, *exact=False*, *distribute_order_term=True*)
See the collect function in diofant.simplify

combsimp()
See the combsimp function in diofant.simplify

compute_leading_term(*x*, *logx=None*)
as_leading_term is only allowed for results of .series() This is a wrapper to compute a series first.

conjugate()
Returns the complex conjugate of self.

See also:

`diofant.functions.elementary.complexes.conjugate` (page 295)

could_extract_minus_sign()
Canonical way to choose an element in the set {e, -e} where e is any expression. If the canonical element is e, we have e.could_extract_minus_sign() == True, else e.could_extract_minus_sign() == False.

For any expression, the set `{e.could_extract_minus_sign(), (-e).could_extract_minus_sign()}` must be `{True, False}`.

```
>>> (x-y).could_extract_minus_sign() != (y-x).could_extract_minus_sign()
True
```

count_ops(*visual=None*)
wrapper for count_ops that returns the operation count.

diff(**symbols*, ***kwargs*)
Alias for `diff()` (page 131).

equals(*other*, *failing_expression=False*)
Return True if self == other, False if it doesn't, or None. If failing_expression is True then the expression which did not simplify to a 0 will be returned instead of None.

If `self` is a Number (or complex number) that is not zero, then the result is False.

If `self` is a number and has not evaluated to zero, evalf will be used to test whether the expression evaluates to zero. If it does so and the result has significance (i.e. the precision is either -1, for a Rational result, or is greater than 1) then the evalf value will be used to return True or False.

expand(*deep=True*, *modulus=None*, *power_base=True*, *power_exp=True*, *mul=True*, *log=True*, *multinomial=True*, *basic=True*, ***hints*)

Expand an expression using hints.

See also:

diofant.core.function.expand (page 135)

extract_additively(*c*)

Return self - c if it's possible to subtract c from self and make all matching coefficients move towards zero, else return None.

See also:

extract_multiplicatively (page 67), *coeff* (page 63), *as_coefficient* (page 56)

Examples

```
>>> e = 2*x + 3
>>> e.extract_additively(x + 1)
x + 2
>>> e.extract_additively(3*x)
>>> e.extract_additively(4)
>>> (y*(x + 1)).extract_additively(x + 1)
>>> ((x + 1)*(x + 2*y + 1) + 3).extract_additively(x + 1)
(x + 1)*(x + 2*y) + 3
```

Sometimes auto-expansion will return a less simplified result than desired; gcd_terms might be used in such cases:

```
>>> (4*x*(y + 1) + y).extract_additively(x)
4*x*(y + 1) + x*(4*y + 3) - x*(4*y + 4) + y
>>> gcd_terms(_)
x*(4*y + 3) + y
```

extract_branch_factor(*allow_half=False*)

Try to write self as `exp_polar(2*pi*I*n)*z` in a nice way. Return (z, n).

```
>>> exp_polar(I*pi).extract_branch_factor()
(exp_polar(I*pi), 0)
>>> exp_polar(2*I*pi).extract_branch_factor()
(1, 1)
>>> exp_polar(-pi*I).extract_branch_factor()
(exp_polar(I*pi), -1)
>>> exp_polar(3*pi*I + x).extract_branch_factor()
(exp_polar(x + I*pi), 1)
>>> (y*exp_polar(-5*pi*I)*exp_polar(3*pi*I + 2*pi*x)).extract_branch_factor()
(y*exp_polar(2*pi*x), -1)
>>> exp_polar(-I*pi/2).extract_branch_factor()
(exp_polar(-I*pi/2), 0)
```

If allow_half is True, also extract exp_polar(I*pi):

```
>>> exp_polar(I*pi).extract_branch_factor(allow_half=True)
(1, 1/2)
>>> exp_polar(2*I*pi).extract_branch_factor(allow_half=True)
(1, 1)
```

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```
>>> exp_polar(3*I*pi).extract_branch_factor(allow_half=True)
(1, 3/2)
>>> exp_polar(-I*pi).extract_branch_factor(allow_half=True)
(1, -1/2)
```

extract_multiplicatively(*c*)

Return None if it's not possible to make self in the form c * something in a nice way, i.e. preserving the properties of arguments of self.

```
>>> x, y = symbols('x y', extended_real=True)
```

```
>>> ((x*y)**3).extract_multiplicatively(x**2 * y)
x*y**2
```

```
>>> ((x*y)**3).extract_multiplicatively(x**4 * y)
```

```
>>> (2*x).extract_multiplicatively(2)
x
```

```
>>> (2*x).extract_multiplicatively(3)
```

```
>>> (Rational(1, 2)*x).extract_multiplicatively(3)
x/6
```

factor(**gens, **args*)

See the factor() function in diofant.polys.polytools

getO()

Returns the additive O(..) symbol if there is one, else None.

getn()

Returns the order of the expression.

The order is determined either from the O(...) term. If there is no O(...) term, it returns None.

Examples

```
>>> (1 + x + O(x**2)).getn()
2
>>> (1 + x).getn()
```

integrate(**args, **kwargs*)

See the integrate function in diofant.integrals

invert(*g, *gens, **args*)

Return the multiplicative inverse of `self` mod `g` where `self` (and `g`) may be symbolic expressions).

See also:

diofant.core.numbers.mod_inverse (page 93), *diofant.polys.polytools.invert* (page 661)

is_algebraic

Test if self can have only values from the set of algebraic numbers *[R78]* (page 1247).

References

[R78] (page 1247)

is_algebraic_expr(**syms*)

This tests whether a given expression is algebraic or not, in the given symbols, syms. When syms is not given, all free symbols will be used. The rational function does not have to be in expanded or in any kind of canonical form.

This function returns False for expressions that are "algebraic expressions" *[R79]* (page 1247) with symbolic exponents. This is a simple extension to the is_rational_function, including rational exponentiation.

See also:

is_rational_function (page 74)

References

[R79] (page 1247)

Examples

```
>>> x = Symbol('x', extended_real=True)
>>> sqrt(1 + x).is_rational_function()
False
>>> sqrt(1 + x).is_algebraic_expr()
True
```

This function does not attempt any nontrivial simplifications that may result in an expression that does not appear to be an algebraic expression to become one.

```
>>> a = sqrt(exp(x)**2 + 2*exp(x) + 1)/(exp(x) + 1)
>>> a.is_algebraic_expr(x)
False
>>> factor(a).is_algebraic_expr()
True
```

is_antihermitian

Test if self belongs to the field of antihermitian operators.

is_commutative

Test if self commutes with any other object wrt multiplication operation.

is_comparable

Test if self can be computed to a real number with precision.

Examples

```
>>> (I*exp_polar(I*pi/2)).is_comparable
True
>>> (I*exp_polar(I*pi*2)).is_comparable
False
```

is_complex
Test if self can have only values from the set of complex numbers.

See also:

is_real (page 74)

is_composite
Test if self is a positive integer that has at least one positive divisor other than 1 or the number itself. See *[R80]* (page 1247).

References

[R80] (page 1247)

is_constant(**wrt*, ***flags*)
Return True if self is constant, False if not, or None if the constancy could not be determined conclusively.

If an expression has no free symbols then it is a constant. If there are free symbols it is possible that the expression is a constant, perhaps (but not necessarily) zero. To test such expressions, two strategies are tried:

1) numerical evaluation at two random points. If two such evaluations give two different values and the values have a precision greater than 1 then self is not constant. If the evaluations agree or could not be obtained with any precision, no decision is made. The numerical testing is done only if `wrt` is different than the free symbols.

2) differentiation with respect to variables in 'wrt' (or all free symbols if omitted) to see if the expression is constant or not. This will not always lead to an expression that is zero even though an expression is constant (see added test in test_expr.py). If all derivatives are zero then self is constant with respect to the given symbols.

If neither evaluation nor differentiation can prove the expression is constant, None is returned unless two numerical values happened to be the same and the flag `failing_number` is True – in that case the numerical value will be returned.

If flag simplify=False is passed, self will not be simplified; the default is True since self should be simplified before testing.

Examples

```
>>> from diofant.abc import a
```

```
>>> x.is_constant()
False
>>> Integer(2).is_constant()
True
```

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```
>>> Sum(x, (x, 1, 10)).is_constant()
True
>>> Sum(x, (x, 1, n)).is_constant()
False
>>> Sum(x, (x, 1, n)).is_constant(y)
True
>>> Sum(x, (x, 1, n)).is_constant(n)
False
>>> Sum(x, (x, 1, n)).is_constant(x)
True
>>> eq = a*cos(x)**2 + a*sin(x)**2 - a
>>> eq.is_constant()
True
>>> eq.subs({x: pi, a: 2}) == eq.subs({x: pi, a: 3}) == 0
True
```

```
>>> (0**x).is_constant()
False
>>> x.is_constant()
False
>>> (x**x).is_constant()
False
>>> one = cos(x)**2 + sin(x)**2
>>> one.is_constant()
True
>>> ((one - 1)**(x + 1)).is_constant() in (True, False) # could be 0 or 1
True
```

is_even

Test if self can have only values from the set of even integers *[R81]* (page 1247).

See also:

is_odd (page 72)

References

[R81] (page 1247)

is_extended_real

Test if self can have only values on the extended real number line *[R82]* (page 1247).

See also:

is_real (page 74)

References

[R82] (page 1247)

is_finite

Test if self absolute value is bounded. See *[R83]* (page 1247).

References

[R83] (page 1247)

is_hermitian
Test if self belongs to the field of hermitian operators.

is_hypergeometric(*k*)
Test if self is a hypergeometric term in k.

See also:

diofant.simplify.simplify.hypersimp (page 779)

is_imaginary
Test if self is an imaginary number [R84] (page 1247).

I.e. that it can be written as a real number multiplied by the imaginary unit `I`.

References

[R84] (page 1247)

is_infinite
Test if self absolute value can be arbitrarily large. See [R85] (page 1247), [R86] (page 1247).

References

[R85] (page 1247), [R86] (page 1247)

is_integer
Test if self can have only values from the set of integers.

is_irrational
Test if self value cannot be represented exactly by Rational, see [R87] (page 1247).

References

[R87] (page 1247)

is_negative
Test if self can have only negative values [R88] (page 1247).

References

[R88] (page 1247)

is_noninteger
Test if self can have only values from the subset of real numbers, that aren't integers.

is_nonnegative
Test if self can have only nonnegative values [R89] (page 1247).

See also:

is_negative (page 71)

References

[R89] (page 1247)

is_nonpositive
Test if self can have only nonpositive values.

is_nonzero
Test if self is nonzero.

See also:

is_zero (page 75)

is_number
Returns True if 'self' has no free symbols.

It will be faster than `if not self.free_symbols`, however, since `is_number` will fail as soon as it hits a free symbol.

Examples

```
>>> x.is_number
False
>>> (2*x).is_number
False
>>> (2 + log(2)).is_number
True
>>> (2 + Integral(2, x)).is_number
False
>>> (2 + Integral(2, (x, 1, 2))).is_number
True
```

is_odd
Test if self can have only values from the set of odd integers [R90] (page 1247).

See also:

is_even (page 70)

References

[R90] (page 1247)

is_polar
Test if self can have values from the Riemann surface of the logarithm.

See also:

diofant.functions.elementary.complexes.polar_lift (page 295), *diofant.functions.elementary.complexes.principal_branch* (page 296), *diofant.functions.elementary.exponential.exp_polar* (page 321)

is_polynomial(**syms*)
Return True if self is a polynomial in syms and False otherwise.

This checks if self is an exact polynomial in syms. This function returns False for expressions that are "polynomials" with symbolic exponents. Thus, you should be able to apply polynomial algorithms to expressions for which this returns True, and

Poly(expr, *syms) should work if and only if expr.is_polynomial(*syms) returns True. The polynomial does not have to be in expanded form. If no symbols are given, all free symbols in the expression will be used.

This is not part of the assumptions system. You cannot do Symbol('z', polynomial=True).

See also:

is_rational_function (page 74)

Examples

```
>>> ((x**2 + 1)**4).is_polynomial(x)
True
>>> ((x**2 + 1)**4).is_polynomial()
True
>>> (2**x + 1).is_polynomial(x)
False
```

```
>>> n = Symbol('n', nonnegative=True, integer=True)
>>> (x**n + 1).is_polynomial(x)
False
```

This function does not attempt any nontrivial simplifications that may result in an expression that does not appear to be a polynomial to become one.

```
>>> y = Symbol('y', positive=True)
>>> a = sqrt(y**2 + 2*y + 1)
>>> a.is_polynomial(y)
False
>>> factor(a)
y + 1
>>> factor(a).is_polynomial(y)
True
```

```
>>> b = (y**2 + 2*y + 1)/(y + 1)
>>> b.is_polynomial(y)
False
>>> cancel(b)
y + 1
>>> cancel(b).is_polynomial(y)
True
```

is_positive
Test if self can have only positive values.

is_prime
Test if self is a natural number greater than 1 that has no positive divisors other than 1 and itself. See *[R91]* (page 1247).

References

[R91] (page 1247)

is_rational
Test if self can have only values from the set of rationals.

is_rational_function(**syms*)
Test whether function is a ratio of two polynomials in the given symbols, syms. When syms is not given, all free symbols will be used. The rational function does not have to be in expanded or in any kind of canonical form.

This function returns False for expressions that are "rational functions" with symbolic exponents. Thus, you should be able to call .as_numer_denom() and apply polynomial algorithms to the result for expressions for which this returns True.

This is not part of the assumptions system. You cannot do Symbol('z', rational_function=True).

See also:

is_algebraic_expr (page 68)

Examples

```
>>> (x/y).is_rational_function()
True
```

```
>>> (x**2).is_rational_function()
True
```

```
>>> (x/sin(y)).is_rational_function(y)
False
```

```
>>> n = Symbol('n', integer=True)
>>> (x**n + 1).is_rational_function(x)
False
```

This function does not attempt any nontrivial simplifications that may result in an expression that does not appear to be a rational function to become one.

```
>>> y = Symbol('y', positive=True)
>>> a = sqrt(y**2 + 2*y + 1)/y
>>> a.is_rational_function(y)
False
>>> factor(a)
(y + 1)/y
>>> factor(a).is_rational_function(y)
True
```

is_real
Test if self can have only values from the set of real numbers *[R92]* (page 1247).

See also:

is_complex (page 69)

References

[R92] (page 1247)

is_transcendental
Test if self can have only values from the set of transcendental numbers *[R93]* (page 1247).

References

[R93] (page 1247)

is_zero
Test if self is zero.

See also:

is_nonzero (page 72)

limit(*x, xlim, dir='+'*)
Compute limit x->xlim.

lseries(*x=None, x0=0, dir='+', logx=None*)
Wrapper for series yielding an iterator of the terms of the series.

Note: an infinite series will yield an infinite iterator. The following, for exaxmple, will never terminate. It will just keep printing terms of the sin(x) series:

```
for term in sin(x).lseries(x):
    print term
```

The advantage of lseries() over nseries() is that many times you are just interested in the next term in the series (i.e. the first term for example), but you don't know how many you should ask for in nseries() using the "n" parameter.

See also:

nseries (page 75)

normal()
canonicalize ratio, i.e. return numerator if denominator is 1

nseries(*x, n=6, logx=None*)
Calculate "n" terms of series in x around 0

This calculates n terms of series in the innermost expressions and then builds up the final series just by "cross-multiplying" everything out.

Advantage – it's fast, because we don't have to determine how many terms we need to calculate in advance.

Disadvantage – you may end up with less terms than you may have expected, but the O(x**n) term appended will always be correct and so the result, though perhaps shorter, will also be correct.

Parameters x : Symbol

variable for series expansion (positive and finite symbol)

n : Integer, optional

number of terms to calculate. Default is 6.

logx : Symbol, optional

This can be used to replace any log(x) in the returned series with a symbolic value to avoid evaluating log(x) at 0.

See also:

series (page 78), *lseries* (page 75)

Notes

This method call the helper method _eval_nseries. Such methods should be implemented in subclasses.

The series expansion code is an important part of the gruntz algorithm for determining limits. _eval_nseries has to return a generalized power series with coefficients in C(log(x), log):

```
c_0*x**e_0 + ... (finitely many terms)
```

where e_i are numbers (not necessarily integers) and c_i involve only numbers, the function log, and log(x). (This also means it must not contain log(x(1 + p)), this *has* to be expanded to log(x) + log(1 + p) if p.is_positive.)

Examples

```
>>> sin(x).nseries(x)
x - x**3/6 + x**5/120 + O(x**7)
>>> log(x + 1).nseries(x, 5)
x - x**2/2 + x**3/3 - x**4/4 + O(x**5)
```

Handling of the `logx` parameter — in the following example the expansion fails since `sin` does not have an asymptotic expansion at -oo (the limit of log(x) as x approaches 0).

```
>>> e = sin(log(x))
>>> e.nseries(x)
Traceback (most recent call last):
...
PoleError: ...
>>> logx = Symbol('logx')
>>> e.nseries(x, logx=logx)
sin(logx)
```

nsimplify(*constants=[], tolerance=None, full=False*)
See the nsimplify function in diofant.simplify

powsimp(***args*)
See the powsimp function in diofant.simplify

primitive()
Return the positive Rational that can be extracted non-recursively from every term of self (i.e., self is treated like an Add). This is like the as_coeff_Mul() method but primitive always extracts a positive Rational (never a negative or a Float).

Examples

```
>>> (3*(x + 1)**2).primitive()
(3, (x + 1)**2)
>>> a = (6*x + 2); a.primitive()
(2, 3*x + 1)
>>> b = (x/2 + 3); b.primitive()
(1/2, x + 6)
>>> (a*b).primitive()
(1, (x/2 + 3)*(6*x + 2))
```

radsimp(***kwargs*)
See the radsimp function in diofant.simplify

ratsimp()
See the ratsimp function in diofant.simplify

removeO()
Removes the additive O(..) symbol if there is one

round(*p=0*)
Return x rounded to the given decimal place.

If a complex number would results, apply round to the real and imaginary components of the number.

Notes

Do not confuse the Python builtin function, round, with the Diofant method of the same name. The former always returns a float (or raises an error if applied to a complex value) while the latter returns either a Number or a complex number:

```
>>> isinstance(round(Integer(123), -2), Number)
False
>>> isinstance(Integer(123).round(-2), Number)
True
>>> isinstance((3*I).round(), Mul)
True
>>> isinstance((1 + 3*I).round(), Add)
True
```

Examples

```
>>> Float(10.5).round()
11.
>>> pi.round()
3.
>>> pi.round(2)
3.14
>>> (2*pi + E*I).round()
6.0 + 3.0*I
```

The round method has a chopping effect:

```
>>> (2*pi + I/10).round()
6.
```

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```
>>> (pi/10 + 2*I).round()
2.0*I
>>> (pi/10 + E*I).round(2)
0.31 + 2.72*I
```

series(*x=None*, *x0=0*, *n=6*, *dir='+'*, *logx=None*)

Series expansion of "self" around `x = x0` yielding either terms of the series one by one (the lazy series given when n=None), else all the terms at once when n != None.

Returns the series expansion of "self" around the point `x = x0` with respect to `x` up to `O((x - x0)**n, x, x0)` (default n is 6).

If `x=None` and `self` is univariate, the univariate symbol will be supplied, otherwise an error will be raised.

```
>>> cos(x).series()
1 - x**2/2 + x**4/24 + O(x**6)
>>> cos(x).series(n=4)
1 - x**2/2 + O(x**4)
>>> cos(x).series(x, x0=1, n=2)
cos(1) - (x - 1)*sin(1) + O((x - 1)**2, (x, 1))
>>> e = cos(x + exp(y))
>>> e.series(y, n=2)
cos(x + 1) - y*sin(x + 1) + O(y**2)
>>> e.series(x, n=2)
cos(E**y) - x*sin(E**y) + O(x**2)
```

If `n=None` then a generator of the series terms will be returned.

```
>>> term = cos(x).series(n=None)
>>> [next(term) for i in range(2)]
[1, -x**2/2]
```

For `dir=+` (default) the series is calculated from the right and for `dir=-` the series from the left. For smooth functions this flag will not alter the results.

```
>>> abs(x).series(dir="+")
x
>>> abs(x).series(dir="-")
-x
```

For rational expressions this method may return original expression.

```
>>> (1/x).series(x, n=8)
1/x
```

simplify(*ratio=1.7*, *measure=None*)

See the simplify function in diofant.simplify

sort_key(*order=None*)

Return a sort key.

taylor_term(*n*, *x*, **previous_terms*)

General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

together(**args*, ***kwargs*)
See the together function in diofant.polys

transpose()
Transpose self.

See also:

diofant.functions.elementary.complexes.transpose (page 297)

trigsimp(***args*)
See the trigsimp function in diofant.simplify

3.1.10 AtomicExpr

class diofant.core.expr.**AtomicExpr**
A parent class for object which are both atoms and Exprs.

For example: Symbol, Number, Rational, Integer, … But not: Add, Mul, Pow, …

3.1.11 symbol

Symbol

class diofant.core.symbol.**Symbol**
Symbol is a placeholder for atomic symbolic expression.

It has a name and a set of assumptions.

Parameters name : str

The name for Symbol.

****assumptions** : dict

Keyword arguments to specify assumptions for Symbol. Default assumption is commutative=True.

See also:

diofant.core.assumptions (page 40), *Dummy* (page 81), *Wild* (page 80)

Examples

```
>>> a, b = symbols('a b')
>>> bool(a*b == b*a)
True
```

You can override default assumptions:

```
>>> A, B = symbols('A B', commutative = False)
>>> bool(A*B != B*A)
True
>>> bool(A*B*2 == 2*A*B) == True # multiplication by scalars is commutative
True
```

Wild

class diofant.core.symbol.**Wild**

A Wild symbol matches anything, whatever is not explicitly excluded.

See also:

Symbol (page 79)

Notes

When using Wild, be sure to use the exclude keyword to make the pattern more precise. Without the exclude pattern, you may get matches that are technically correct, but not what you wanted. For example, using the above without exclude:

```
>>> a, b = symbols('a b', cls=Wild)
>>> (2 + 3*y).match(a*x + b*y)
{a_: 2/x, b_: 3}
```

This is technically correct, because (2/x)*x + 3*y == 2 + 3*y, but you probably wanted it to not match at all. The issue is that you really didn't want a and b to include x and y, and the exclude parameter lets you specify exactly this. With the exclude parameter, the pattern will not match.

```
>>> a = Wild('a', exclude=[x, y])
>>> b = Wild('b', exclude=[x, y])
>>> (2 + 3*y).match(a*x + b*y)
```

Exclude also helps remove ambiguity from matches.

```
>>> E = 2*x**3*y*z
>>> a, b = symbols('a b', cls=Wild)
>>> E.match(a*b)
{a_: 2*y*z, b_: x**3}
>>> a = Wild('a', exclude=[x, y])
>>> E.match(a*b)
{a_: z, b_: 2*x**3*y}
>>> a = Wild('a', exclude=[x, y, z])
>>> E.match(a*b)
{a_: 2, b_: x**3*y*z}
```

Examples

```
>>> a = Wild('a')
>>> x.match(a)
{a_: x}
>>> pi.match(a)
{a_: pi}
>>> (3*x**2).match(a*x)
{a_: 3*x}
>>> cos(x).match(a)
{a_: cos(x)}
>>> b = Wild('b', exclude=[x])
>>> (3*x**2).match(b*x)
```

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```
>>> b.match(a)
{a_: b_}
>>> A = WildFunction('A')
>>> A.match(a)
{a_: A_}
```

Dummy

class diofant.core.symbol.**Dummy**

Dummy symbols are each unique, identified by an internal count index:

```
>>> bool(Dummy("x") == Dummy("x")) == True
False
```

If a name is not supplied then a string value of the count index will be used. This is useful when a temporary variable is needed and the name of the variable used in the expression is not important.

```
>>> Dummy()
_Dummy_10
```

See also:

Symbol (page 79)

classmethod class_key()

Nice order of classes.

sort_key(*order=None*)

Return a sort key.

symbols

diofant.core.symbol.**symbols**(*names*, ***args*)

Transform strings into instances of *Symbol* (page 79) class.

symbols() (page 81) function returns a sequence of symbols with names taken from `names` argument, which can be a comma or whitespace delimited string, or a sequence of strings:

```
>>> a, b, c = symbols('a b c')
```

The type of output is dependent on the properties of input arguments:

```
>>> symbols('x')
x
>>> symbols('x,')
(x,)
>>> symbols('x,y')
(x, y)
>>> symbols(('a', 'b', 'c'))
(a, b, c)
>>> symbols(['a', 'b', 'c'])
[a, b, c]
```

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```
>>> symbols({'a', 'b', 'c'})
{a, b, c}
```

If an iterable container is needed for a single symbol, set the `seq` argument to `True` or terminate the symbol name with a comma:

```
>>> symbols('x', seq=True)
(x,)
```

To reduce typing, range syntax is supported to create indexed symbols. Ranges are indicated by a colon and the type of range is determined by the character to the right of the colon. If the character is a digit then all contiguous digits to the left are taken as the nonnegative starting value (or 0 if there is no digit left of the colon) and all contiguous digits to the right are taken as 1 greater than the ending value:

```
>>> symbols('x:10')
(x0, x1, x2, x3, x4, x5, x6, x7, x8, x9)

>>> symbols('x5:10')
(x5, x6, x7, x8, x9)
>>> symbols('x5(:2)')
(x50, x51)

>>> symbols('x5:10 y:5')
(x5, x6, x7, x8, x9, y0, y1, y2, y3, y4)

>>> symbols(('x5:10', 'y:5'))
((x5, x6, x7, x8, x9), (y0, y1, y2, y3, y4))
```

If the character to the right of the colon is a letter, then the single letter to the left (or 'a' if there is none) is taken as the start and all characters in the lexicographic range *through* the letter to the right are used as the range:

```
>>> symbols('x:z')
(x, y, z)
>>> symbols('x:c')  # null range
()
>>> symbols('x(:c)')
(xa, xb, xc)

>>> symbols(':c')
(a, b, c)

>>> symbols('a:d, x:z')
(a, b, c, d, x, y, z)

>>> symbols(('a:d', 'x:z'))
((a, b, c, d), (x, y, z))
```

Multiple ranges are supported; contiguous numerical ranges should be separated by parentheses to disambiguate the ending number of one range from the starting number of the next:

```
>>> symbols('x:2(1:3)')
(x01, x02, x11, x12)
```

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```
>>> symbols(':3:2')  # parsing is from left to right
(00, 01, 10, 11, 20, 21)
```

Only one pair of parentheses surrounding ranges are removed, so to include parentheses around ranges, double them. And to include spaces, commas, or colons, escape them with a backslash:

```
>>> symbols('x((a:b))')
(x(a), x(b))
>>> symbols(r'x(:1\,:2)')  # or 'x((:1)\,(:2))'
(x(0,0), x(0,1))
```

All newly created symbols have assumptions set according to `args`:

```
>>> a = symbols('a', integer=True)
>>> a.is_integer
True

>>> x, y, z = symbols('x y z', extended_real=True)
>>> x.is_extended_real and y.is_extended_real and z.is_extended_real
True
```

Despite its name, *symbols()* (page 81) can create symbol-like objects like instances of Function or Wild classes. To achieve this, set `cls` keyword argument to the desired type:

```
>>> symbols('f g h', cls=Function)
(f, g, h)

>>> type(_[0])
<class 'diofant.core.function.UndefinedFunction'>
```

var

`diofant.core.symbol.var`(*names*, ***args*)

Create symbols and inject them into the global namespace.

This calls *symbols()* (page 81) with the same arguments and puts the results into the *global* namespace. It's recommended not to use *var()* (page 83) in library code, where *symbols()* (page 81) has to be used.

See also:

symbols (page 81)

Examples

```
>>> var('x')
x
>>> x
x
```

```
>>> var('a ab abc')
(a, ab, abc)
```

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```
>>> abc
abc
```

```
>>> var('x y', extended_real=True)
(x, y)
>>> x.is_extended_real and y.is_extended_real
True
```

3.1.12 numbers

Number

class diofant.core.numbers.**Number**

Represents any kind of number in diofant.

Floating point numbers are represented by the Float class. Integer numbers (of any size), together with rational numbers (again, there is no limit on their size) are represented by the Rational class.

If you want to represent, for example, `1+sqrt(2)`, then you need to do:

```
Rational(1) + sqrt(Rational(2))
```

as_coeff_Add(*rational=False*)

Efficiently extract the coefficient of a summation.

as_coeff_Mul(*rational=False*)

Efficiently extract the coefficient of a product.

as_coeff_add(**deps*)

Return the tuple (c, args) where self is written as an Add.

See also:

`diofant.core.expr.Expr.as_coeff_add` (page 55)

as_coeff_mul(**deps, **kwargs*)

Return the tuple (c, args) where self is written as a Mul.

See also:

`diofant.core.expr.Expr.as_coeff_mul` (page 55)

classmethod class_key()

Nice order of classes.

cofactors(*other*)

Compute GCD and cofactors of $self$ and $other$.

gcd(*other*)

Compute GCD of $self$ and $other$.

invert(*other, *gens, **args*)

Return the multiplicative inverse of `self` mod `g` where `self` (and `g`) may be symbolic expressions).

See also:

diofant.core.numbers.mod_inverse (page 93), *diofant.polys.polytools.invert* (page 661)

is_constant(**wrt*, ***flags*)
Return True if self is constant.

See also:

diofant.core.expr.Expr.is_constant (page 69)

lcm(*other*)
Compute LCM of $self$ and $other$.

sort_key(*order=None*)
Return a sort key.

Float

class diofant.core.numbers.**Float**
Represent a floating-point number of arbitrary precision.

Notes

Floats are inexact by their nature unless their value is a binary-exact value.

```
>>> approx, exact = Float(.1, 1), Float(.125, 1)
```

For calculation purposes, you can change the precision of Float, but this will not increase the accuracy of the inexact value. The following is the most accurate 5-digit approximation of a value of 0.1 that had only 1 digit of precision:

```
>>> Float(approx, 5)
0.099609
```

Please note that you can't increase precision with evalf:

```
>>> approx.evalf(5)
Traceback (most recent call last):
...
PrecisionExhausted: ...
```

By contrast, 0.125 is exact in binary (as it is in base 10) and so it can be passed to Float constructor to obtain an arbitrary precision with matching accuracy:

```
>>> Float(exact, 5)
0.12500
>>> Float(exact, 20)
0.12500000000000000000
```

Trying to make a high-precision Float from a float is not disallowed, but one must keep in mind that the *underlying float* (not the apparent decimal value) is being obtained with high precision. For example, 0.3 does not have a finite binary representation. The closest rational is the fraction 5404319552844595/2**54. So if you try to obtain a Float of 0.3 to 20 digits of precision you will not see the same thing as 0.3 followed by 19 zeros:

```
>>> Float(0.3, 20)
0.29999999999999998890
```

If you want a 20-digit value of the decimal 0.3 (not the floating point approximation of 0.3) you should send the 0.3 as a string. The underlying representation is still binary but a higher precision than Python's float is used:

```
>>> Float('0.3', 20)
0.30000000000000000000
```

Although you can increase the precision of an existing Float using Float it will not increase the accuracy – the underlying value is not changed:

```
>>> def show(f): # binary rep of Float
...     from diofant import Mul, Pow
...     s, m, e, b = f._mpf_
...     v = Mul(int(m), Pow(2, int(e), evaluate=False), evaluate=False)
...     print('%s at prec=%s' % (v, f._prec))
...
>>> t = Float('0.3', 3)
>>> show(t)
4915/2**14 at prec=13
>>> show(Float(t, 20)) # higher prec, not higher accuracy
4915/2**14 at prec=70
>>> show(Float(t, 2)) # lower prec
307/2**10 at prec=10
```

Finally, Floats can be instantiated with an mpf tuple (n, c, p) to produce the number (-1)**n*c*2**p:

```
>>> n, c, p = 1, 5, 0
>>> (-1)**n*c*2**p
-5
>>> Float((1, 5, 0))
-5.00000000000000
```

An actual mpf tuple also contains the number of bits in c as the last element of the tuple:

```
>>> _._mpf_
(1, 5, 0, 3)
```

This is not needed for instantiation and is not the same thing as the precision. The mpf tuple and the precision are two separate quantities that Float tracks.

Examples

```
>>> Float(3.5)
3.50000000000000
>>> Float(3)
3.00000000000000
```

Creating Floats from strings (and Python `int` type) will give a minimum precision of 15 digits, but the precision will automatically increase to capture all digits entered.

```
>>> Float(1)
1.00000000000000
>>> Float(10**20)
100000000000000000000.
```

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```
>>> Float('1e20')
100000000000000000000.
```

However, *floating-point* numbers (Python `float` types) retain only 15 digits of precision:

```
>>> Float(1e20)
1.00000000000000e+20
>>> Float(1.23456789123456789)
1.23456789123457
```

It may be preferable to enter high-precision decimal numbers as strings:

Float('1.23456789123456789') 1.23456789123456789

The desired number of digits can also be specified:

```
>>> Float('1e-3', 3)
0.00100
>>> Float(100, 4)
100.0
```

Float can automatically count significant figures if a null string is sent for the precision; space are also allowed in the string. (Auto- counting is only allowed for strings and ints).

```
>>> Float('123 456 789 . 123 456', '')
123456789.123456
>>> Float('12e-3', '')
0.012
>>> Float(3, '')
3.
```

If a number is written in scientific notation, only the digits before the exponent are considered significant if a decimal appears, otherwise the "e" signifies only how to move the decimal:

```
>>> Float('60.e2', '')  # 2 digits significant
6.0e+3
>>> Float('60e2', '')  # 4 digits significant
6000.
>>> Float('600e-2', '')  # 3 digits significant
6.00
```

ceiling()
Compute ceiling of self.

epsilon_eq(*other*, *epsilon='1e-15'*)
Test approximate equality.

floor()
Compute floor of self.

num
Return mpmath representation of self.

Rational

class diofant.core.numbers.**Rational**

Represents integers and rational numbers (p/q) of any size.

See also:

diofant.core.sympify.sympify (page 37), *diofant.simplify.simplify.nsimplify* (page 780)

Examples

```
>>> Rational(3)
3
>>> Rational(1, 2)
1/2
```

Rational is unprejudiced in accepting input. If a float is passed, the underlying value of the binary representation will be returned:

```
>>> Rational(.5)
1/2
>>> Rational(.2)
3602879701896397/18014398509481984
```

If the simpler representation of the float is desired then consider limiting the denominator to the desired value or convert the float to a string (which is roughly equivalent to limiting the denominator to 10**12):

```
>>> Rational(str(.2))
1/5
>>> Rational(.2).limit_denominator(10**12)
1/5
```

An arbitrarily precise Rational is obtained when a string literal is passed:

```
>>> Rational("1.23")
123/100
>>> Rational('1e-2')
1/100
>>> Rational(".1")
1/10
```

The conversion of floats to expressions or simple fractions can be handled with nsimplify:

```
>>> nsimplify(.3)  # numbers that have a simple form
3/10
```

But if the input does not reduce to a literal Rational, an error will be raised:

```
>>> Rational(pi)
Traceback (most recent call last):
...
TypeError: invalid input: pi
```

Low-level access numerator and denominator as .p and .q:

```
>>> r = Rational(3, 4)
>>> r
3/4
>>> r.p
3
>>> r.q
4
```

Note that p and q return integers (not Diofant Integers) so some care is needed when using them in expressions:

```
>>> r.p/r.q
0.75
```

as_content_primitive(*radical=False*)
Return the tuple (R, self/R) where R is the positive Rational extracted from self.

See also:

diofant.core.expr.Expr.as_content_primitive (page 57)

Examples

```
>>> Rational(-3, 2).as_content_primitive()
(3/2, -1)
```

factors(*limit=None, use_trial=True, use_rho=False, use_pm1=False, verbose=False, visual=False*)
A wrapper to factorint which return factors of self that are smaller than limit (or cheap to compute). Special methods of factoring are disabled by default so that only trial division is used.

gcd(*other*)
Compute GCD of $self$ and $other$.

lcm(*other*)
Compute LCM of $self$ and $other$.

limit_denominator(*max_denominator=1000000*)
Closest Rational to self with denominator at most max_denominator.

```
>>> Rational('3.141592653589793').limit_denominator(10)
22/7
>>> Rational('3.141592653589793').limit_denominator(100)
311/99
```

Integer

class diofant.core.numbers.**Integer**

is_composite
Test if self is a positive integer that has at least one positive divisor other than 1 or the number itself. See *[R94]* (page 1247).

References

[R94] (page 1247)

is_even

Test if self can have only values from the set of even integers *[R95]* (page 1247).

See also:

is_odd (page 90)

References

[R95] (page 1247)

is_imaginary

Test if self is an imaginary number *[R96]* (page 1247).

I.e. that it can be written as a real number multiplied by the imaginary unit `I`.

References

[R96] (page 1247)

is_nonzero

Test if self is nonzero.

See also:

is_zero (page 90)

is_odd

Test if self can have only values from the set of odd integers *[R97]* (page 1247).

See also:

is_even (page 90)

References

[R97] (page 1247)

is_prime

Test if self is a natural number greater than 1 that has no positive divisors other than 1 and itself. See *[R98]* (page 1247).

References

[R98] (page 1247)

is_zero

Test if self is zero.

See also:

is_nonzero (page 90)

AlgebraicNumber

class diofant.core.numbers.**AlgebraicNumber**

Class for algebraic numbers in Diofant.

Represents the algebraic number in the field $\mathbb{Q}[\theta]$ given by

$$c_n\theta^n + c_{n-1}\theta^{n-1} + \dots + c_0$$

Parameters expr : Expr

A generator θ for the algebraic number.

coeffs : tuple, optional

A tuple of rational coefficients $(c_n, c_{n-1}, \dots, c_0)$. The default is `(1, 0)`.

alias : Symbol, optional

Alias to denote the generator θ.

See also:

diofant.polys.rootoftools.RootOf (page 711)

Examples

```
>>> a = AlgebraicNumber(sqrt(3), alias='a')
```

Numbers in the same field are automatically combined by arithmetic operations.

```
>>> a + 1
a + 1
>>> _ + a
2*a + 1
```

Powers with integer exponents also automatically evaluated and the coefficient list reduced accordingly to the degree of the minimal polynomial of θ.

```
>>> _**3
30*a + 37
>>> 1/a
a/3
```

The generator θ can be any algebraic number, represented in terms of radicals or RootOf objects.

```
>>> b = AlgebraicNumber(RootOf(x**7 - x + 1, 1), (1, 2, -1), 'b')
>>> b
b**2 + 2*b - 1
>>> b**7
490*b**6 - 119*b**5 - 196*b**4 - 203*b**3 - 265*b**2 + 637*b - 198
```

as_expr(*x=None*)

Create a Basic expression from `self`.

as_poly(*x=None*)

Create a Poly instance from `self`.

coeffs()
Returns all Diofant coefficients of an algebraic number.

free_symbols
Return from the atoms of self those which are free symbols.

For most expressions, all symbols are free symbols. For some classes this is not true. e.g. Integrals use Symbols for the dummy variables which are bound variables, so Integral has a method to return all symbols except those. Derivative keeps track of symbols with respect to which it will perform a derivative; those are bound variables, too, so it has its own free_symbols method.

Any other method that uses bound variables should implement a free_symbols method.

is_aliased
Returns `True` if `alias` was set.

native_coeffs()
Returns all native coefficients of an algebraic number.

to_algebraic_integer()
Convert `self` to an algebraic integer.

NumberSymbol

class diofant.core.numbers.**NumberSymbol**

approximation_interval(*number_cls*)
Return an interval with number_cls endpoints that contains the value of NumberSymbol. If not implemented, then return None.

RealNumber

diofant.core.numbers.**RealNumber**
alias of *diofant.core.numbers.Float* (page 85)

igcd

diofant.core.numbers.**igcd**(**args*)
Computes positive integer greatest common divisor.

Examples

```
>>> igcd(2, 4)
2
>>> igcd(5, 10, 15)
5
```

ilcm

diofant.core.numbers.**ilcm**(**args*)
Computes integer least common multiple.

Examples

```
>>> ilcm(5, 10)
10
>>> ilcm(7, 3)
21
>>> ilcm(5, 10, 15)
30
```

mod_inverse

`diofant.core.numbers.mod_inverse(a, m)`

Return the number c such that, (a * c) % m == 1 where c has the same sign as a. If no such value exists, a ValueError is raised.

References

[R99] (page 1247), *[R100]* (page 1247)

Examples

Suppose we wish to find multiplicative inverse x of 3 modulo 11. This is the same as finding x such that 3 * x = 1 (mod 11). One value of x that satisfies this congruence is 4. Because 3 * 4 = 12 and 12 = 1 mod(11). This is the value return by mod_inverse:

```
>>> mod_inverse(3, 11)
4
>>> mod_inverse(-3, 11)
-4
```

When there is a common factor between the numerators of `a` and `m` the inverse does not exist:

```
>>> mod_inverse(2, 4)
Traceback (most recent call last):
...
ValueError: inverse of 2 mod 4 does not exist
```

```
>>> mod_inverse(Integer(2)/7, Integer(5)/2)
7/2
```

seterr

`diofant.core.numbers.seterr(divide=False)`

Should diofant raise an exception on 0/0 or return a nan?

divide == True .... raise an exception divide == False ... return nan

Zero

class diofant.core.numbers.**Zero**

The number zero.

Zero is a singleton, and can be accessed by `S.Zero`

References

[R101] (page 1247)

Examples

```
>>> Integer(0) is S.Zero
True
>>> 1/S.Zero
zoo
```

One

class diofant.core.numbers.**One**

The number one.

One is a singleton, and can be accessed by `S.One`.

References

[R102] (page 1248)

Examples

```
>>> Integer(1) is S.One
True
```

factors(*limit=None, use_trial=True, use_rho=False, use_pm1=False, verbose=False, visual=False*)

A wrapper to factorint which return factors of self that are smaller than limit (or cheap to compute). Special methods of factoring are disabled by default so that only trial division is used.

NegativeOne

class diofant.core.numbers.**NegativeOne**

The number negative one.

NegativeOne is a singleton, and can be accessed by `S.NegativeOne`.

See also:

One (page 94)

References

[R103] (page 1248)

Examples

```
>>> Integer(-1) is S.NegativeOne
True
```

Half

class diofant.core.numbers.**Half**
The rational number 1/2.

Half is a singleton, and can be accessed by `S.Half`.

References

[R104] (page 1248)

Examples

```
>>> Rational(1, 2) is S.Half
True
```

NaN

class diofant.core.numbers.**NaN**
Not a Number.

This serves as a place holder for numeric values that are indeterminate. Most operations on NaN, produce another NaN. Most indeterminate forms, such as `0/0` or `oo - oo` produce NaN. Two exceptions are ``0**0 and oo**0``, which all produce `1` (this is consistent with Python's float).

NaN is loosely related to floating point nan, which is defined in the IEEE 754 floating point standard, and corresponds to the Python `float('nan')`. Differences are noted below.

NaN is mathematically not equal to anything else, even NaN itself. This explains the initially counter-intuitive results with `Eq` and `==` in the examples below.

NaN is not comparable so inequalities raise a TypeError. This is in constrast with floating point nan where all inequalities are false.

NaN is a singleton, and can be accessed by `nan`.

References

[R105] (page 1248)

Examples

```
>>> nan is nan
True
>>> oo - oo
nan
>>> nan + 1
nan
>>> Eq(nan, nan)   # mathematical equality
false
>>> nan == nan     # structural equality
True
```

Infinity

class diofant.core.numbers.**Infinity**

Positive infinite quantity.

In real analysis the symbol ∞ denotes an unbounded limit: $x \to \infty$ means that x grows without bound.

Infinity is often used not only to define a limit but as a value in the affinely extended real number system. Points labeled $+\infty$ and $-\infty$ can be added to the topological space of the real numbers, producing the two-point compactification of the real numbers. Adding algebraic properties to this gives us the extended real numbers.

Infinity is a singleton, and can be accessed by `oo`, or can be imported as `oo`.

See also:

NegativeInfinity (page 96), *NaN* (page 95)

References

[R106] (page 1248)

Examples

```
>>> 1 + oo
oo
>>> 42/oo
0
>>> x = Symbol('x')
>>> limit(exp(x), x, oo)
oo
```

NegativeInfinity

class diofant.core.numbers.**NegativeInfinity**

Negative infinite quantity.

NegativeInfinity is a singleton, and can be accessed by `-oo`.

See also:

Infinity (page 96)

ComplexInfinity

class diofant.core.numbers.**ComplexInfinity**

Complex infinity.

In complex analysis the symbol $\tilde\infty$, called "complex infinity", represents a quantity with infinite magnitude, but undetermined complex phase.

ComplexInfinity is a singleton, and can be accessed by as `zoo`.

See also:

Infinity (page 96)

Examples

```
>>> zoo + 42
zoo
>>> 42/zoo
0
>>> zoo + zoo
nan
>>> zoo*zoo
zoo
```

Exp1

class diofant.core.numbers.**Exp1**

The e constant.

The transcendental number $e = 2.718281828\ldots$ is the base of the natural logarithm and of the exponential function, $e = \exp(1)$. Sometimes called Euler's number or Napier's constant.

Exp1 is a singleton, and can be imported as `E`.

References

[R107] (page 1248)

Examples

```
>>> E is exp(1)
True
>>> log(E)
1
```

approximation_interval(*number_cls*)

Return an interval with number_cls endpoints that contains the value of NumberSymbol. If not implemented, then return None.

ImaginaryUnit

class diofant.core.numbers.**ImaginaryUnit**

The imaginary unit, $i = \sqrt{-1}$.

I is a singleton, and can be imported as `I`.

References

[R108] (page 1248)

Examples

```
>>> sqrt(-1)
I
>>> I*I
-1
>>> 1/I
-I
```

as_base_exp()

Return base and exp of self.

See also:

diofant.core.power.Pow.as_base_exp (page 101)

Pi

class diofant.core.numbers.**Pi**

The π constant.

The transcendental number $\pi = 3.141592654\ldots$ represents the ratio of a circle's circumference to its diameter, the area of the unit circle, the half-period of trigonometric functions, and many other things in mathematics.

Pi is a singleton, and can be imported as `pi`.

References

[R109] (page 1248)

Examples

```
>>> pi > 3
true
>>> pi.is_irrational
True
>>> x = Symbol('x')
>>> sin(x + 2*pi)
sin(x)
```

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```
>>> integrate(exp(-x**2), (x, -oo, oo))
sqrt(pi)
```

approximation_interval(*number_cls*)
Return an interval with number_cls endpoints that contains the value of NumberSymbol. If not implemented, then return None.

EulerGamma

class diofant.core.numbers.**EulerGamma**
The Euler-Mascheroni constant.

$\gamma = 0.5772157\ldots$ (also called Euler's constant) is a mathematical constant recurring in analysis and number theory. It is defined as the limiting difference between the harmonic series and the natural logarithm:

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n\right)$$

References

[R110] (page 1248)

Examples

```
>>> EulerGamma.is_irrational
>>> EulerGamma > 0
true
>>> EulerGamma > 1
false
```

approximation_interval(*number_cls*)
Return an interval with number_cls endpoints that contains the value of NumberSymbol. If not implemented, then return None.

Catalan

class diofant.core.numbers.**Catalan**
Catalan's constant.

$K = 0.91596559\ldots$ is given by the infinite series

$$K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

References

[R111] (page 1248)

Examples

```
>>> Catalan.is_irrational
>>> Catalan > 0
true
>>> Catalan > 1
false
```

approximation_interval(*number_cls*)
Return an interval with number_cls endpoints that contains the value of NumberSymbol. If not implemented, then return None.

GoldenRatio

class diofant.core.numbers.**GoldenRatio**
The golden ratio, ϕ.

$\phi = \frac{1+\sqrt{5}}{2}$ is algebraic number. Two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities, i.e. their maximum.

References

[R112] (page 1248)

Examples

```
>>> GoldenRatio > 1
true
>>> GoldenRatio.expand(func=True)
1/2 + sqrt(5)/2
>>> GoldenRatio.is_irrational
True
```

approximation_interval(*number_cls*)
Return an interval with number_cls endpoints that contains the value of NumberSymbol. If not implemented, then return None.

3.1.13 power

Pow

class diofant.core.power.**Pow**
Defines the expression x**y as "x raised to a power y".

For complex numbers x and y, Pow gives the principal value of $exp(y * log(x))$.

Singleton definitions involving (0, 1, -1, oo, -oo, I, -I):

expr	value	reason
z**0	1	Although arguments over 0**0 exist, see [2].
z**1	z	
(-oo)**(-1)	0	
(-1)**-1	-1	
S.Zero**-1	zoo	This is not strictly true, as 0**-1 may be undefined, but is convenient in some contexts where the base is assumed to be positive.
1**-1	1	
oo**-1	0	
0**oo	0	Because for all complex numbers z near 0, z**oo -> 0.
0**-oo	zoo	This is not strictly true, as 0**oo may be oscillating between positive and negative values or rotating in the complex plane. It is convenient, however, when the base is positive.
1**oo 1**-oo	nan	Because there are various cases where lim(x(t),t)=1, lim(y(t),t)=oo (or -oo), but lim(x(t)**y(t), t) != 1. See [3].
z**zoo	nan	No limit for z**t for t -> zoo.
(-1)**oo (-1)**(-oo)	nan	Because of oscillations in the limit.
oo**oo	oo	
oo**-oo	0	
(-oo)**oo (-oo)**-oo	nan	
oo**I (-oo)**I	nan	oo**e could probably be best thought of as the limit of x**e for real x as x tends to oo. If e is I, then the limit does not exist and nan is used to indicate that.
oo**(1+I) (-oo)**(1+I)	zoo	If the real part of e is positive, then the limit of abs(x**e) is oo. So the limit value is zoo.
oo**(-1+I) -oo**(-1+I)	0	If the real part of e is negative, then the limit is 0.

Because symbolic computations are more flexible that floating point calculations and we prefer to never return an incorrect answer, we choose not to conform to all IEEE 754 conventions. This helps us avoid extra test-case code in the calculation of limits.

See also:

diofant.core.numbers.Infinity (page 96), *diofant.core.numbers.NegativeInfinity* (page 96), *diofant.core.numbers.NaN* (page 95)

References

[R113] (page 1248), [R114] (page 1248), [R115] (page 1248)

as_base_exp()

Return base and exp of self.

If base is 1/Integer, then return Integer, -exp. If this extra processing is not needed, the base and exp properties will give the raw arguments

Examples

```
>>> p = Pow(S.Half, 2, evaluate=False)
>>> p.as_base_exp()
(2, -2)
>>> p.args
(1/2, 2)
```

as_content_primitive(*radical=False*)

Return the tuple (R, self/R) where R is the positive Rational extracted from self.

See also:

diofant.core.expr.Expr.as_content_primitive (page 57)

Examples

```
>>> sqrt(4 + 4*sqrt(2)).as_content_primitive()
(2, sqrt(1 + sqrt(2)))
>>> sqrt(3 + 3*sqrt(2)).as_content_primitive()
(1, sqrt(3)*sqrt(1 + sqrt(2)))
```

```
>>> ((2*x + 2)**2).as_content_primitive()
(4, (x + 1)**2)
>>> (4**((1 + y)/2)).as_content_primitive()
(2, 4**(y/2))
>>> (3**((1 + y)/2)).as_content_primitive()
(1, 3**((y + 1)/2))
>>> (3**((5 + y)/2)).as_content_primitive()
(9, 3**((y + 1)/2))
>>> eq = 3**(2 + 2*x)
>>> powsimp(eq) == eq
True
>>> eq.as_content_primitive()
(9, 3**(2*x))
>>> powsimp(Mul(*_))
3**(2*x + 2)
```

```
>>> eq = (2 + 2*x)**y
>>> s = expand_power_base(eq); s.is_Mul, s
(False, (2*x + 2)**y)
>>> eq.as_content_primitive()
(1, (2*(x + 1))**y)
>>> s = expand_power_base(_[1]); s.is_Mul, s
(True, 2**y*(x + 1)**y)
```

as_real_imag(*deep=True, **hints*)

Returns real and imaginary parts of self

See also:

diofant.core.expr.Expr.as_real_imag (page 62)

base
Returns base of the power expression.

classmethod class_key()
Nice order of classes.

exp
Returns exponent of the power expression.

integer_nthroot

diofant.core.power.**integer_nthroot**(*y*, *n*)
Return a tuple containing x = floor(y**(1/n)) and a boolean indicating whether the result is exact (that is, whether x**n == y).

```
>>> integer_nthroot(16, 2)
(4, True)
>>> integer_nthroot(26, 2)
(5, False)
```

3.1.14 mul

Mul

class diofant.core.mul.**Mul**

as_base_exp()
Return base and exp of self.

See also:

diofant.core.expr.Expr.as_base_exp (page 55)

as_coeff_Mul(*rational=False*)
Efficiently extract the coefficient of a product.

as_coeff_mul(**deps*, ***kwargs*)
Return the tuple (c, args) where self is written as a Mul.

See also:

diofant.core.expr.Expr.as_coeff_mul (page 55)

as_content_primitive(*radical=False*)
Return the tuple (R, self/R) where R is the positive Rational extracted from self.

See also:

diofant.core.expr.Expr.as_content_primitive (page 57)

Examples

```
>>> (-3*sqrt(2)*(2 - 2*sqrt(2))).as_content_primitive()
(6, -sqrt(2)*(-sqrt(2) + 1))
```

as_ordered_factors(*order=None*)
Transform an expression into an ordered list of factors.

Examples

```
>>> (2*x*y*sin(x)*cos(x)).as_ordered_factors()
[2, x, y, sin(x), cos(x)]
```

as_powers_dict()
Return self as a dictionary of factors with each factor being treated as a power.

See also:

diofant.core.expr.Expr.as_powers_dict (page 61)

as_real_imag(*deep=True, **hints*)
Returns real and imaginary parts of self

See also:

diofant.core.expr.Expr.as_real_imag (page 62)

as_two_terms()
Return head and tail of self.

This is the most efficient way to get the head and tail of an expression.

- if you want only the head, use self.args[0];
- if you want to process the arguments of the tail then use self.as_coef_mul() which gives the head and a tuple containing the arguments of the tail when treated as a Mul.
- if you want the coefficient when self is treated as an Add then use self.as_coeff_add()[0]

```
>>> (3*x*y).as_two_terms()
(3, x*y)
```

classmethod class_key()
Nice order of classes.

classmethod flatten(*seq*)
Return commutative, noncommutative and order arguments by combining related terms.

Notes

- In an expression like `a*b*c`, python process this through diofant as `Mul(Mul(a, b), c)`. This can have undesirable consequences.
  - Sometimes terms are not combined as one would like: {c.f. https://github.com/sympy/sympy/issues/4596}

```
>>> 2*(x + 1) # this is the 2-arg Mul behavior
2*x + 2
>>> y*(x + 1)*2
```

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```
2*y*(x + 1)
>>> 2*(x + 1)*y # 2-arg result will be obtained first
y*(2*x + 2)
>>> Mul(2, x + 1, y) # all 3 args simultaneously processed
2*y*(x + 1)
>>> 2*((x + 1)*y) # parentheses can control this behavior
2*y*(x + 1)
```

Powers with compound bases may not find a single base to combine with unless all arguments are processed at once. Post-processing may be necessary in such cases. {c.f. https://github.com/sympy/sympy/issues/5728}

```
>>> a = sqrt(x*sqrt(y))
>>> a**3
(x*sqrt(y))**(3/2)
>>> Mul(a, a, a)
(x*sqrt(y))**(3/2)
>>> a*a*a
x*sqrt(y)*sqrt(x*sqrt(y))
>>> _.subs(a.base, z).subs(z, a.base)
(x*sqrt(y))**(3/2)
```

- If more than two terms are being multiplied then all the previous terms will be re-processed for each new argument. So if each of `a`, `b` and `c` were *Mul* (page 103) expression, then `a*b*c` (or building up the product with `*=`) will process all the arguments of `a` and `b` twice: once when `a*b` is computed and again when `c` is multiplied.

  Using `Mul(a, b, c)` will process all arguments once.

- The results of Mul are cached according to arguments, so flatten will only be called once for `Mul(a, b, c)`. If you can structure a calculation so the arguments are most likely to be repeats then this can save time in computing the answer. For example, say you had a Mul, M, that you wished to divide by `d[i]` and multiply by `n[i]` and you suspect there are many repeats in `n`. It would be better to compute `M*n[i]/d[i]` rather than `M/d[i]*n[i]` since every time n[i] is a repeat, the product, `M*n[i]` will be returned without flattening – the cached value will be returned. If you divide by the `d[i]` first (and those are more unique than the `n[i]`) then that will create a new Mul, `M/d[i]` the args of which will be traversed again when it is multiplied by `n[i]`.

  {c.f. https://github.com/sympy/sympy/issues/5706}

  This consideration is moot if the cache is turned off.

The validity of the above notes depends on the implementation details of Mul and flatten which may change at any time. Therefore, you should only consider them when your code is highly performance sensitive.

prod

`diofant.core.mul.`**`prod`**`(`*a*, *start=1*`)`

Return product of elements of a. Start with int 1 so if only ints are included then an int result is returned.

Examples

```
>>> prod(range(3))
0
>>> type(_) is int
True
>>> prod([Integer(2), 3])
6
>>> _.is_Integer
True
```

You can start the product at something other than 1:

```
>>> prod([1, 2], 3)
6
```

3.1.15 add

Add

class diofant.core.add.**Add**

as_coeff_Add(*rational=False*)
Efficiently extract the coefficient of a summation.

as_coeff_add(**deps*)
Returns a tuple (coeff, args) where self is treated as an Add and coeff is the Number term and args is a tuple of all other terms.

Examples

```
>>> (7 + 3*x).as_coeff_add()
(7, (3*x,))
>>> (7*x).as_coeff_add()
(0, (7*x,))
```

as_coefficients_dict()
Return a dictionary mapping terms to their Rational coefficient.

Since the dictionary is a defaultdict, inquiries about terms which were not present will return a coefficient of 0. If an expression is not an Add it is considered to have a single term.

Examples

```
>>> (3*x + x*y + 4).as_coefficients_dict()
{1: 4, x: 3, x*y: 1}
>>> _[y]
0
>>> (3*y*x).as_coefficients_dict()
{x*y: 3}
```

as_content_primitive(*radical=False*)

Return the tuple (R, self/R) where R is the positive Rational extracted from self. If radical is True (default is False) then common radicals will be removed and included as a factor of the primitive expression.

See also:

diofant.core.expr.Expr.as_content_primitive (page 57)

Examples

```
>>> (3 + 3*sqrt(2)).as_content_primitive()
(3, 1 + sqrt(2))
```

Radical content can also be factored out of the primitive:

```
>>> (2*sqrt(2) + 4*sqrt(10)).as_content_primitive(radical=True)
(2, sqrt(2)*(1 + 2*sqrt(5)))
```

as_real_imag(*deep=True*, ***hints*)

returns a tuple representing a complex number

Examples

```
>>> (7 + 9*I).as_real_imag()
(7, 9)
>>> ((1 + I)/(1 - I)).as_real_imag()
(0, 1)
>>> ((1 + 2*I)*(1 + 3*I)).as_real_imag()
(-5, 5)
```

as_two_terms()

Return head and tail of self.

This is the most efficient way to get the head and tail of an expression.

- if you want only the head, use self.args[0];
- if you want to process the arguments of the tail then use self.as_coef_add() which gives the head and a tuple containing the arguments of the tail when treated as an Add.
- if you want the coefficient when self is treated as a Mul then use self.as_coeff_mul()[0]

```
>>> (3*x*y).as_two_terms()
(3, x*y)
```

classmethod class_key()

Nice order of classes

extract_leading_order(*symbols*)

Returns the leading term and its order.

Examples

```
>>> (x + 1 + 1/x**5).extract_leading_order(x)
((x**(-5), O(x**(-5))),)
>>> (1 + x).extract_leading_order(x)
((1, O(1)),)
>>> (x + x**2).extract_leading_order(x)
((x, O(x)),)
```

classmethod flatten(*seq*)

Takes the sequence "seq" of nested Adds and returns a flatten list.

Returns: (commutative_part, noncommutative_part, order_symbols)

Applies associativity, all terms are commutable with respect to addition.

See also:

diofant.core.mul.Mul.flatten (page 104)

getO()

Returns the additive O(..) symbol.

See also:

diofant.core.expr.Expr.getO (page 67)

primitive()

Return `(R, self/R)` where `R`` is the Rational GCD of `self``.

`R` is collected only from the leading coefficient of each term.

See also:

diofant.polys.polytools.primitive (page 666)

Examples

```
>>> (2*x + 4*y).primitive()
(2, x + 2*y)
```

```
>>> (2*x/3 + 4*y/9).primitive()
(2/9, 3*x + 2*y)
```

```
>>> (2*x/3 + 4.2*y).primitive()
(1/3, 2*x + 12.6*y)
```

No subprocessing of term factors is performed:

```
>>> ((2 + 2*x)*x + 2).primitive()
(1, x*(2*x + 2) + 2)
```

Recursive subprocessing can be done with the as_content_primitive() method:

```
>>> ((2 + 2*x)*x + 2).as_content_primitive()
(2, x*(x + 1) + 1)
```

removeO()

Removes the additive O(..) symbol.

See also:

diofant.core.expr.Expr.removeO (page 77)

3.1.16 mod

Mod

class diofant.core.mod.**Mod**

Represents a modulo operation on symbolic expressions.

Receives two arguments, dividend p and divisor q.

The convention used is the same as Python's: the remainder always has the same sign as the divisor.

Examples

```
>>> x**2 % y
x**2%y
>>> _.subs({x: 5, y: 6})
1
```

classmethod eval(*p*, *q*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

3.1.17 relational

Rel

diofant.core.relational.**Rel**

alias of *diofant.core.relational.Relational* (page 110)

Eq

diofant.core.relational.**Eq**

alias of *diofant.core.relational.Equality* (page 111)

Ne

diofant.core.relational.**Ne**

alias of *diofant.core.relational.Unequality* (page 119)

Lt

`diofant.core.relational.Lt`
alias of *diofant.core.relational.StrictLessThan* (page 122)

Le

`diofant.core.relational.Le`
alias of *diofant.core.relational.LessThan* (page 115)

Gt

`diofant.core.relational.Gt`
alias of *diofant.core.relational.StrictGreaterThan* (page 119)

Ge

`diofant.core.relational.Ge`
alias of *diofant.core.relational.GreaterThan* (page 112)

Relational

class `diofant.core.relational.Relational`
Base class for all relation types.

Subclasses of Relational should generally be instantiated directly, but Relational can be instantiated with a valid rop value to dispatch to the appropriate subclass.

Parameters rop : str or None

Indicates what subclass to instantiate. Valid values can be found in the keys of Relational.ValidRelationalOperator.

Examples

```
>>> Rel(y, x+x**2, '==')
Eq(y, x**2 + x)
```

as_set()
Rewrites univariate inequality in terms of real sets

Examples

```
>>> x = Symbol('x', extended_real=True)
>>> (x > 0).as_set()
(0, oo)
>>> Eq(x, 0).as_set()
{0}
```

canonical
Return a canonical form of the relational.

The rules for the canonical form, in order of decreasing priority are:

1. Number on right if left is not a Number;
2. Symbol on the left;
3. Gt/Ge changed to Lt/Le;
4. Lt/Le are unchanged;
5. Eq and Ne get ordered args.

equals(*other*, *failing_expression=False*)
Return True if the sides of the relationship are mathematically identical and the type of relationship is the same. If failing_expression is True, return the expression whose truth value was unknown.

lhs
The left-hand side of the relation.

reversed
Return the relationship with sides (and sign) reversed.

Examples

```
>>> Eq(x, 1)
Eq(x, 1)
>>> _.reversed
Eq(1, x)
>>> x < 1
x < 1
>>> _.reversed
1 > x
```

rhs
The right-hand side of the relation.

Equality

class diofant.core.relational.**Equality**
An equal relation between two objects.

Represents that two objects are equal. If they can be easily shown to be definitively equal (or unequal), this will reduce to True (or False). Otherwise, the relation is maintained as an unevaluated Equality object. Use the `simplify` function on this object for more nontrivial evaluation of the equality relation.

As usual, the keyword argument `evaluate=False` can be used to prevent any evaluation.

See also:

diofant.logic.boolalg.Equivalent **(page 547)** for representing equality between two boolean expressions

Notes

This class is not the same as the == operator. The == operator tests for exact structural equality between two expressions; this class compares expressions mathematically.

If either object defines an $_eval_Eq$ method, it can be used in place of the default algorithm. If $lhs._eval_Eq(rhs)$ or $rhs._eval_Eq(lhs)$ returns anything other than None, that return value will be substituted for the Equality. If None is returned by $_eval_Eq$, an Equality object will be created as usual.

Examples

```
>>> Eq(y, x + x**2)
Eq(y, x**2 + x)
>>> Eq(2, 5)
false
>>> Eq(2, 5, evaluate=False)
Eq(2, 5)
>>> _.doit()
false
>>> Eq(exp(x), exp(x).rewrite(cos))
Eq(E**x, sinh(x) + cosh(x))
>>> simplify(_)
true
```

GreaterThan

class diofant.core.relational.**GreaterThan**

Class representations of inequalities.

The *Than classes represent unequal relationships, where the left-hand side is generally bigger or smaller than the right-hand side. For example, the GreaterThan class represents an unequal relationship where the left-hand side is at least as big as the right side, if not bigger. In mathematical notation:

lhs >= rhs

In total, there are four *Than classes, to represent the four inequalities:

Class Name	Symbol
GreaterThan	(>=)
LessThan	(<=)
StrictGreaterThan	(>)
StrictLessThan	(<)

All classes take two arguments, lhs and rhs.

Signature Example	Math equivalent
GreaterThan(lhs, rhs)	lhs >= rhs
LessThan(lhs, rhs)	lhs <= rhs
StrictGreaterThan(lhs, rhs)	lhs > rhs
StrictLessThan(lhs, rhs)	lhs < rhs

In addition to the normal .lhs and .rhs of Relations, *Than inequality objects also have the .lts and .gts properties, which represent the "less than side" and "greater than side" of the operator. Use of .lts and .gts in an algorithm rather than .lhs and .rhs as an assumption of inequality direction will make more explicit the intent of a certain section of code, and will make it similarly more robust to client code changes:

```
>>> e = GreaterThan(x, 1)
>>> e
x >= 1
>>> '%s >= %s is the same as %s <= %s' % (e.gts, e.lts, e.lts, e.gts)
'x >= 1 is the same as 1 <= x'
```

Notes

There are a couple of "gotchas" when using Python's operators.

The first enters the mix when comparing against a literal number as the lhs argument. Due to the order that Python decides to parse a statement, it may not immediately find two objects comparable. For example, to evaluate the statement (1 < x), Python will first recognize the number 1 as a native number, and then that x is *not* a native number. At this point, because a native Python number does not know how to compare itself with a Diofant object Python will try the reflective operation, (x > 1). Unfortunately, there is no way available to Diofant to recognize this has happened, so the statement (1 < x) will turn silently into (x > 1).

```
>>> e1 = x >  1
>>> e2 = x >= 1
>>> e3 = x <  1
>>> e4 = x <= 1
>>> e5 = 1 >  x
>>> e6 = 1 >= x
>>> e7 = 1 <  x
>>> e8 = 1 <= x
>>> print("%s     %s\n"*4 % (e1, e2, e3, e4, e5, e6, e7, e8))
x > 1     x >= 1
x < 1     x <= 1
x < 1     x <= 1
x > 1     x >= 1
```

If the order of the statement is important (for visual output to the console, perhaps), one can work around this annoyance in a couple ways: (1) "sympify" the literal before comparison, (2) use one of the wrappers, or (3) use the less succinct methods described above:

```
>>> e1 = Integer(1) >  x
>>> e2 = Integer(1) >= x
>>> e3 = Integer(1) <  x
>>> e4 = Integer(1) <= x
>>> e5 = Gt(1, x)
>>> e6 = Ge(1, x)
>>> e7 = Lt(1, x)
>>> e8 = Le(1, x)
>>> print("%s     %s\n"*4 % (e1, e2, e3, e4, e5, e6, e7, e8))
1 > x     1 >= x
1 < x     1 <= x
```

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```
1 > x     1 >= x
1 < x     1 <= x
```

The other gotcha is with chained inequalities. Occasionally, one may be tempted to write statements like:

```
>>> e = x < y < z
Traceback (most recent call last):
...
TypeError: symbolic boolean expression has no truth value.
```

Due to an implementation detail or decision of Python *[R116]* (page 1248), there is no way for Diofant to reliably create that as a chained inequality. To create a chained inequality, the only method currently available is to make use of And:

```
>>> e = And(x < y, y < z)
>>> type(e)
And
>>> e
And(x < y, y < z)
```

Note that this is different than chaining an equality directly via use of parenthesis (this is currently an open bug in Diofant *[R117]* (page 1248)):

```
>>> e = (x < y) < z
>>> type(e)
<class 'diofant.core.relational.StrictLessThan'>
>>> e
(x < y) < z
```

Any code that explicitly relies on this latter functionality will not be robust as this behaviour is completely wrong and will be corrected at some point. For the time being (circa Jan 2012), use And to create chained inequalities.

References

[R116] (page 1248), *[R117]* (page 1248)

Examples

One generally does not instantiate these classes directly, but uses various convenience methods:

```
>>> e1 = Ge(x, 2)  # Ge is a convenience wrapper
>>> print(e1)
x >= 2
```

```
>>> rels = Ge(x, 2), Gt(x, 2), Le(x, 2), Lt(x, 2)
>>> print('%s\n%s\n%s\n%s' % rels)
x >= 2
x > 2
x <= 2
x < 2
```

Another option is to use the Python inequality operators (>=, >, <=, <) directly. Their main advantage over the Ge, Gt, Le, and Lt counterparts, is that one can write a more “mathematical looking” statement rather than littering the math with oddball function calls. However there are certain (minor) caveats of which to be aware (search for ‘gotcha’, below).

```
>>> e2 = x >= 2
>>> print(e2)
x >= 2
>>> print("e1: %s,    e2: %s" % (e1, e2))
e1: x >= 2,    e2: x >= 2
>>> e1 == e2
True
```

However, it is also perfectly valid to instantiate a *Than class less succinctly and less conveniently:

```
>>> rels = Rel(x, 1, '>='), Relational(x, 1, '>='), GreaterThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x >= 1
x >= 1
x >= 1
```

```
>>> rels = Rel(x, 1, '>'), Relational(x, 1, '>'), StrictGreaterThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x > 1
x > 1
x > 1
```

```
>>> rels = Rel(x, 1, '<='), Relational(x, 1, '<='), LessThan(x, 1)
>>> print("%s\n%s\n%s" % rels)
x <= 1
x <= 1
x <= 1
```

```
>>> rels = Rel(x, 1, '<'), Relational(x, 1, '<'), StrictLessThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x < 1
x < 1
x < 1
```

LessThan

class diofant.core.relational.**LessThan**

Class representations of inequalities.

The *Than classes represent unequal relationships, where the left-hand side is generally bigger or smaller than the right-hand side. For example, the GreaterThan class represents an unequal relationship where the left-hand side is at least as big as the right side, if not bigger. In mathematical notation:

lhs >= rhs

In total, there are four *Than classes, to represent the four inequalities:

Class Name	Symbol
GreaterThan	(>=)
LessThan	(<=)
StrictGreaterThan	(>)
StrictLessThan	(<)

All classes take two arguments, lhs and rhs.

Signature Example	Math equivalent
GreaterThan(lhs, rhs)	lhs >= rhs
LessThan(lhs, rhs)	lhs <= rhs
StrictGreaterThan(lhs, rhs)	lhs > rhs
StrictLessThan(lhs, rhs)	lhs < rhs

In addition to the normal .lhs and .rhs of Relations, `*Than` inequality objects also have the .lts and .gts properties, which represent the "less than side" and "greater than side" of the operator. Use of .lts and .gts in an algorithm rather than .lhs and .rhs as an assumption of inequality direction will make more explicit the intent of a certain section of code, and will make it similarly more robust to client code changes:

```
>>> e = GreaterThan(x, 1)
>>> e
x >= 1
>>> '%s >= %s is the same as %s <= %s' % (e.gts, e.lts, e.lts, e.gts)
'x >= 1 is the same as 1 <= x'
```

Notes

There are a couple of "gotchas" when using Python's operators.

The first enters the mix when comparing against a literal number as the lhs argument. Due to the order that Python decides to parse a statement, it may not immediately find two objects comparable. For example, to evaluate the statement (1 < x), Python will first recognize the number 1 as a native number, and then that x is *not* a native number. At this point, because a native Python number does not know how to compare itself with a Diofant object Python will try the reflective operation, (x > 1). Unfortunately, there is no way available to Diofant to recognize this has happened, so the statement (1 < x) will turn silently into (x > 1).

```
>>> e1 = x >  1
>>> e2 = x >= 1
>>> e3 = x <  1
>>> e4 = x <= 1
>>> e5 = 1 >  x
>>> e6 = 1 >= x
>>> e7 = 1 <  x
>>> e8 = 1 <= x
>>> print("%s     %s\n"*4 % (e1, e2, e3, e4, e5, e6, e7, e8))
x > 1     x >= 1
x < 1     x <= 1
x < 1     x <= 1
x > 1     x >= 1
```

If the order of the statement is important (for visual output to the console, perhaps), one can work around this annoyance in a couple ways: (1) "sympify" the literal before comparison, (2) use one of the wrappers, or (3) use the less succinct methods described above:

```
>>> e1 = Integer(1) >  x
>>> e2 = Integer(1) >= x
>>> e3 = Integer(1) <  x
>>> e4 = Integer(1) <= x
>>> e5 = Gt(1, x)
>>> e6 = Ge(1, x)
>>> e7 = Lt(1, x)
>>> e8 = Le(1, x)
>>> print("%s     %s\n"*4 % (e1, e2, e3, e4, e5, e6, e7, e8))
1 > x     1 >= x
1 < x     1 <= x
1 > x     1 >= x
1 < x     1 <= x
```

The other gotcha is with chained inequalities. Occasionally, one may be tempted to write statements like:

```
>>> e = x < y < z
Traceback (most recent call last):
...
TypeError: symbolic boolean expression has no truth value.
```

Due to an implementation detail or decision of Python *[R118]* (page 1248), there is no way for Diofant to reliably create that as a chained inequality. To create a chained inequality, the only method currently available is to make use of And:

```
>>> e = And(x < y, y < z)
>>> type(e)
And
>>> e
And(x < y, y < z)
```

Note that this is different than chaining an equality directly via use of parenthesis (this is currently an open bug in Diofant *[R119]* (page 1249)):

```
>>> e = (x < y) < z
>>> type(e)
<class 'diofant.core.relational.StrictLessThan'>
>>> e
(x < y) < z
```

Any code that explicitly relies on this latter functionality will not be robust as this behaviour is completely wrong and will be corrected at some point. For the time being (circa Jan 2012), use And to create chained inequalities.

References

[R118] (page 1248), *[R119]* (page 1249)

Examples

One generally does not instantiate these classes directly, but uses various convenience methods:

```
>>> e1 = Ge(x, 2)  # Ge is a convenience wrapper
>>> print(e1)
x >= 2
```

```
>>> rels = Ge(x, 2), Gt(x, 2), Le(x, 2), Lt(x, 2)
>>> print('%s\n%s\n%s\n%s' % rels)
x >= 2
x > 2
x <= 2
x < 2
```

Another option is to use the Python inequality operators (>=, >, <=, <) directly. Their main advantage over the Ge, Gt, Le, and Lt counterparts, is that one can write a more "mathematical looking" statement rather than littering the math with oddball function calls. However there are certain (minor) caveats of which to be aware (search for 'gotcha', below).

```
>>> e2 = x >= 2
>>> print(e2)
x >= 2
>>> print("e1: %s,    e2: %s" % (e1, e2))
e1: x >= 2,    e2: x >= 2
>>> e1 == e2
True
```

However, it is also perfectly valid to instantiate a *Than class less succinctly and less conveniently:

```
>>> rels = Rel(x, 1, '>='), Relational(x, 1, '>='), GreaterThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x >= 1
x >= 1
x >= 1
```

```
>>> rels = Rel(x, 1, '>'), Relational(x, 1, '>'), StrictGreaterThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x > 1
x > 1
x > 1
```

```
>>> rels = Rel(x, 1, '<='), Relational(x, 1, '<='), LessThan(x, 1)
>>> print("%s\n%s\n%s" % rels)
x <= 1
x <= 1
x <= 1
```

```
>>> rels = Rel(x, 1, '<'), Relational(x, 1, '<'), StrictLessThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x < 1
x < 1
x < 1
```

Unequality

class diofant.core.relational.**Unequality**
An unequal relation between two objects.

Represents that two objects are not equal. If they can be shown to be definitively equal, this will reduce to False; if definitively unequal, this will reduce to True. Otherwise, the relation is maintained as an Unequality object.

See also:

Equality (page 111)

Notes

This class is not the same as the != operator. The != operator tests for exact structural equality between two expressions; this class compares expressions mathematically.

This class is effectively the inverse of Equality. As such, it uses the same algorithms, including any available $_eval_Eq$ methods.

Examples

```
>>> Ne(y, x+x**2)
Ne(y, x**2 + x)
```

StrictGreaterThan

class diofant.core.relational.**StrictGreaterThan**
Class representations of inequalities.

The *Than classes represent unequal relationships, where the left-hand side is generally bigger or smaller than the right-hand side. For example, the GreaterThan class represents an unequal relationship where the left-hand side is at least as big as the right side, if not bigger. In mathematical notation:

lhs >= rhs

In total, there are four *Than classes, to represent the four inequalities:

Class Name	Symbol
GreaterThan	(>=)
LessThan	(<=)
StrictGreaterThan	(>)
StrictLessThan	(<)

All classes take two arguments, lhs and rhs.

Signature Example	Math equivalent
GreaterThan(lhs, rhs)	lhs >= rhs
LessThan(lhs, rhs)	lhs <= rhs
StrictGreaterThan(lhs, rhs)	lhs > rhs
StrictLessThan(lhs, rhs)	lhs < rhs

In addition to the normal .lhs and .rhs of Relations, *Than inequality objects also have the .lts and .gts properties, which represent the "less than side" and "greater than side" of the operator. Use of .lts and .gts in an algorithm rather than .lhs and .rhs as an assumption of inequality direction will make more explicit the intent of a certain section of code, and will make it similarly more robust to client code changes:

```
>>> e = GreaterThan(x, 1)
>>> e
x >= 1
>>> '%s >= %s is the same as %s <= %s' % (e.gts, e.lts, e.lts, e.gts)
'x >= 1 is the same as 1 <= x'
```

Notes

There are a couple of "gotchas" when using Python's operators.

The first enters the mix when comparing against a literal number as the lhs argument. Due to the order that Python decides to parse a statement, it may not immediately find two objects comparable. For example, to evaluate the statement (1 < x), Python will first recognize the number 1 as a native number, and then that x is *not* a native number. At this point, because a native Python number does not know how to compare itself with a Diofant object Python will try the reflective operation, (x > 1). Unfortunately, there is no way available to Diofant to recognize this has happened, so the statement (1 < x) will turn silently into (x > 1).

```
>>> e1 = x >  1
>>> e2 = x >= 1
>>> e3 = x <  1
>>> e4 = x <= 1
>>> e5 = 1 >  x
>>> e6 = 1 >= x
>>> e7 = 1 <  x
>>> e8 = 1 <= x
>>> print("%s     %s\n"*4 % (e1, e2, e3, e4, e5, e6, e7, e8))
x > 1     x >= 1
x < 1     x <= 1
x < 1     x <= 1
x > 1     x >= 1
```

If the order of the statement is important (for visual output to the console, perhaps), one can work around this annoyance in a couple ways: (1) "sympify" the literal before comparison, (2) use one of the wrappers, or (3) use the less succinct methods described above:

```
>>> e1 = Integer(1) >  x
>>> e2 = Integer(1) >= x
>>> e3 = Integer(1) <  x
>>> e4 = Integer(1) <= x
>>> e5 = Gt(1, x)
>>> e6 = Ge(1, x)
>>> e7 = Lt(1, x)
>>> e8 = Le(1, x)
>>> print("%s     %s\n"*4 % (e1, e2, e3, e4, e5, e6, e7, e8))
1 > x     1 >= x
1 < x     1 <= x
```

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```
1 > x     1 >= x
1 < x     1 <= x
```

The other gotcha is with chained inequalities. Occasionally, one may be tempted to write statements like:

```
>>> e = x < y < z
Traceback (most recent call last):
...
TypeError: symbolic boolean expression has no truth value.
```

Due to an implementation detail or decision of Python *[R120]* (page 1249), there is no way for Diofant to reliably create that as a chained inequality. To create a chained inequality, the only method currently available is to make use of And:

```
>>> e = And(x < y, y < z)
>>> type(e)
And
>>> e
And(x < y, y < z)
```

Note that this is different than chaining an equality directly via use of parenthesis (this is currently an open bug in Diofant *[R121]* (page 1249)):

```
>>> e = (x < y) < z
>>> type(e)
<class 'diofant.core.relational.StrictLessThan'>
>>> e
(x < y) < z
```

Any code that explicitly relies on this latter functionality will not be robust as this behaviour is completely wrong and will be corrected at some point. For the time being (circa Jan 2012), use And to create chained inequalities.

References

[R120] (page 1249), *[R121]* (page 1249)

Examples

One generally does not instantiate these classes directly, but uses various convenience methods:

```
>>> e1 = Ge(x, 2)  # Ge is a convenience wrapper
>>> print(e1)
x >= 2
```

```
>>> rels = Ge(x, 2), Gt(x, 2), Le(x, 2), Lt(x, 2)
>>> print('%s\n%s\n%s\n%s' % rels)
x >= 2
x > 2
x <= 2
x < 2
```

Another option is to use the Python inequality operators (>=, >, <=, <) directly. Their main advantage over the Ge, Gt, Le, and Lt counterparts, is that one can write a more "mathematical looking" statement rather than littering the math with oddball function calls. However there are certain (minor) caveats of which to be aware (search for 'gotcha', below).

```
>>> e2 = x >= 2
>>> print(e2)
x >= 2
>>> print("e1: %s,    e2: %s" % (e1, e2))
e1: x >= 2,    e2: x >= 2
>>> e1 == e2
True
```

However, it is also perfectly valid to instantiate a *Than class less succinctly and less conveniently:

```
>>> rels = Rel(x, 1, '>='), Relational(x, 1, '>='), GreaterThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x >= 1
x >= 1
x >= 1
```

```
>>> rels = Rel(x, 1, '>'), Relational(x, 1, '>'), StrictGreaterThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x > 1
x > 1
x > 1
```

```
>>> rels = Rel(x, 1, '<='), Relational(x, 1, '<='), LessThan(x, 1)
>>> print("%s\n%s\n%s" % rels)
x <= 1
x <= 1
x <= 1
```

```
>>> rels = Rel(x, 1, '<'), Relational(x, 1, '<'), StrictLessThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x < 1
x < 1
x < 1
```

StrictLessThan

class diofant.core.relational.**StrictLessThan**

Class representations of inequalities.

The *Than classes represent unequal relationships, where the left-hand side is generally bigger or smaller than the right-hand side. For example, the GreaterThan class represents an unequal relationship where the left-hand side is at least as big as the right side, if not bigger. In mathematical notation:

lhs >= rhs

In total, there are four *Than classes, to represent the four inequalities:

Class Name	Symbol
GreaterThan	(>=)
LessThan	(<=)
StrictGreaterThan	(>)
StrictLessThan	(<)

All classes take two arguments, lhs and rhs.

Signature Example	Math equivalent
GreaterThan(lhs, rhs)	lhs >= rhs
LessThan(lhs, rhs)	lhs <= rhs
StrictGreaterThan(lhs, rhs)	lhs > rhs
StrictLessThan(lhs, rhs)	lhs < rhs

In addition to the normal .lhs and .rhs of Relations, `*Than` inequality objects also have the .lts and .gts properties, which represent the "less than side" and "greater than side" of the operator. Use of .lts and .gts in an algorithm rather than .lhs and .rhs as an assumption of inequality direction will make more explicit the intent of a certain section of code, and will make it similarly more robust to client code changes:

```
>>> e = GreaterThan(x, 1)
>>> e
x >= 1
>>> '%s >= %s is the same as %s <= %s' % (e.gts, e.lts, e.lts, e.gts)
'x >= 1 is the same as 1 <= x'
```

Notes

There are a couple of "gotchas" when using Python's operators.

The first enters the mix when comparing against a literal number as the lhs argument. Due to the order that Python decides to parse a statement, it may not immediately find two objects comparable. For example, to evaluate the statement (1 < x), Python will first recognize the number 1 as a native number, and then that x is *not* a native number. At this point, because a native Python number does not know how to compare itself with a Diofant object Python will try the reflective operation, (x > 1). Unfortunately, there is no way available to Diofant to recognize this has happened, so the statement (1 < x) will turn silently into (x > 1).

```
>>> e1 = x >  1
>>> e2 = x >= 1
>>> e3 = x <  1
>>> e4 = x <= 1
>>> e5 = 1 >  x
>>> e6 = 1 >= x
>>> e7 = 1 <  x
>>> e8 = 1 <= x
>>> print("%s     %s\n"*4 % (e1, e2, e3, e4, e5, e6, e7, e8))
x > 1     x >= 1
x < 1     x <= 1
x < 1     x <= 1
x > 1     x >= 1
```

If the order of the statement is important (for visual output to the console, perhaps), one can work around this annoyance in a couple ways: (1) "sympify" the literal before comparison, (2) use one of the wrappers, or (3) use the less succinct methods described above:

```
>>> e1 = Integer(1) >  x
>>> e2 = Integer(1) >= x
>>> e3 = Integer(1) <  x
>>> e4 = Integer(1) <= x
>>> e5 = Gt(1, x)
>>> e6 = Ge(1, x)
>>> e7 = Lt(1, x)
>>> e8 = Le(1, x)
>>> print("%s     %s\n"*4 % (e1, e2, e3, e4, e5, e6, e7, e8))
1 > x     1 >= x
1 < x     1 <= x
1 > x     1 >= x
1 < x     1 <= x
```

The other gotcha is with chained inequalities. Occasionally, one may be tempted to write statements like:

```
>>> e = x < y < z
Traceback (most recent call last):
...
TypeError: symbolic boolean expression has no truth value.
```

Due to an implementation detail or decision of Python *[R122]* (page 1250), there is no way for Diofant to reliably create that as a chained inequality. To create a chained inequality, the only method currently available is to make use of And:

```
>>> e = And(x < y, y < z)
>>> type(e)
And
>>> e
And(x < y, y < z)
```

Note that this is different than chaining an equality directly via use of parenthesis (this is currently an open bug in Diofant *[R123]* (page 1250)):

```
>>> e = (x < y) < z
>>> type(e)
<class 'diofant.core.relational.StrictLessThan'>
>>> e
(x < y) < z
```

Any code that explicitly relies on this latter functionality will not be robust as this behaviour is completely wrong and will be corrected at some point. For the time being (circa Jan 2012), use And to create chained inequalities.

References

[R122] (page 1250), *[R123]* (page 1250)

Examples

One generally does not instantiate these classes directly, but uses various convenience methods:

```
>>> e1 = Ge(x, 2)  # Ge is a convenience wrapper
>>> print(e1)
x >= 2
```

```
>>> rels = Ge(x, 2), Gt(x, 2), Le(x, 2), Lt(x, 2)
>>> print('%s\n%s\n%s\n%s' % rels)
x >= 2
x > 2
x <= 2
x < 2
```

Another option is to use the Python inequality operators (>=, >, <=, <) directly. Their main advantage over the Ge, Gt, Le, and Lt counterparts, is that one can write a more "mathematical looking" statement rather than littering the math with oddball function calls. However there are certain (minor) caveats of which to be aware (search for 'gotcha', below).

```
>>> e2 = x >= 2
>>> print(e2)
x >= 2
>>> print("e1: %s,    e2: %s" % (e1, e2))
e1: x >= 2,    e2: x >= 2
>>> e1 == e2
True
```

However, it is also perfectly valid to instantiate a *Than class less succinctly and less conveniently:

```
>>> rels = Rel(x, 1, '>='), Relational(x, 1, '>='), GreaterThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x >= 1
x >= 1
x >= 1
```

```
>>> rels = Rel(x, 1, '>'), Relational(x, 1, '>'), StrictGreaterThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x > 1
x > 1
x > 1
```

```
>>> rels = Rel(x, 1, '<='), Relational(x, 1, '<='), LessThan(x, 1)
>>> print("%s\n%s\n%s" % rels)
x <= 1
x <= 1
x <= 1
```

```
>>> rels = Rel(x, 1, '<'), Relational(x, 1, '<'), StrictLessThan(x, 1)
>>> print('%s\n%s\n%s' % rels)
x < 1
x < 1
x < 1
```

3.1.18 multidimensional

vectorize

class diofant.core.multidimensional.**vectorize**(**mdargs*)

Generalizes a function taking scalars to accept multidimensional arguments.

Examples

```
>>> @vectorize(0)
... def vsin(x):
...     return sin(x)
```

```
>>> vsin([1, x, y])
[sin(1), sin(x), sin(y)]
```

```
>>> @vectorize(0, 1)
... def vdiff(f, y):
...     return diff(f, y)
```

```
>>> vdiff([f(x, y), g(x, y)], [x, y])
[[Derivative(f(x, y), x), Derivative(f(x, y), y)],
 [Derivative(g(x, y), x), Derivative(g(x, y), y)]]
```

3.1.19 function

Lambda

class diofant.core.function.**Lambda**

Lambda(x, expr) represents a lambda function similar to Python's 'lambda x: expr'. A function of several variables is written as Lambda((x, y, ...), expr).

A simple example:

```
>>> f = Lambda(x, x**2)
>>> f(4)
16
```

For multivariate functions, use:

```
>>> from diofant.abc import t
>>> f2 = Lambda((x, y, z, t), x + y**z + t**z)
>>> f2(1, 2, 3, 4)
73
```

A handy shortcut for lots of arguments:

```
>>> p = x, y, z
>>> f = Lambda(p, x + y*z)
>>> f(*p)
x + y*z
```

expr
The return value of the function

free_symbols
Return from the atoms of self those which are free symbols.

See also:

diofant.core.basic.Basic.free_symbols (page 44)

variables
The variables used in the internal representation of the function

WildFunction

class diofant.core.function.**WildFunction**(*name*, ***assumptions*)
A WildFunction function matches any function (with its arguments).

Examples

```
>>> F = WildFunction('F')
>>> F.nargs
Naturals0()
>>> x.match(F)
>>> F.match(F)
{F_: F_}
>>> f(x).match(F)
{F_: f(x)}
>>> cos(x).match(F)
{F_: cos(x)}
>>> f(x, y).match(F)
{F_: f(x, y)}
```

To match functions with a given number of arguments, set `nargs` to the desired value at instantiation:

```
>>> F = WildFunction('F', nargs=2)
>>> F.nargs
{2}
>>> f(x).match(F)
>>> f(x, y).match(F)
{F_: f(x, y)}
```

To match functions with a range of arguments, set `nargs` to a tuple containing the desired number of arguments, e.g. if `nargs = (1, 2)` then functions with 1 or 2 arguments will be matched.

```
>>> F = WildFunction('F', nargs=(1, 2))
>>> F.nargs
{1, 2}
>>> f(x).match(F)
{F_: f(x)}
>>> f(x, y).match(F)
{F_: f(x, y)}
>>> f(x, y, 1).match(F)
```

Derivative

class diofant.core.function.**Derivative**

Carries out differentiation of the given expression with respect to symbols.

expr must define ._eval_derivative(symbol) method that returns the differentiation result. This function only needs to consider the non-trivial case where expr contains symbol and it should call the diff() method internally (not _eval_derivative); Derivative should be the only one to call _eval_derivative.

Simplification of high-order derivatives:

Because there can be a significant amount of simplification that can be done when multiple differentiations are performed, results will be automatically simplified in a fairly conservative fashion unless the keyword `simplify` is set to False.

```
>>> e = sqrt((x + 1)**2 + x)
>>> diff(e, x, 5, simplify=False).count_ops()
136
>>> diff(e, x, 5).count_ops()
30
```

Ordering of variables:

If evaluate is set to True and the expression can not be evaluated, the list of differentiation symbols will be sorted, that is, the expression is assumed to have continuous derivatives up to the order asked. This sorting assumes that derivatives wrt Symbols commute, derivatives wrt non-Symbols commute, but Symbol and non-Symbol derivatives don't commute with each other.

Derivative wrt non-Symbols:

This class also allows derivatives wrt non-Symbols that have _diff_wrt set to True, such as Function and Derivative. When a derivative wrt a non- Symbol is attempted, the non-Symbol is temporarily converted to a Symbol while the differentiation is performed.

Note that this may seem strange, that Derivative allows things like f(g(x)).diff(g(x)), or even f(cos(x)).diff(cos(x)). The motivation for allowing this syntax is to make it easier to work with variational calculus (i.e., the Euler-Lagrange method). The best way to understand this is that the action of derivative with respect to a non-Symbol is defined by the above description: the object is substituted for a Symbol and the derivative is taken with respect to that. This action is only allowed for objects for which this can be done unambiguously, for example Function and Derivative objects. Note that this leads to what may appear to be mathematically inconsistent results. For example:

```
>>> (2*cos(x)).diff(cos(x))
2
>>> (2*sqrt(1 - sin(x)**2)).diff(cos(x))
0
```

This appears wrong because in fact 2*cos(x) and 2*sqrt(1 - sin(x)**2) are identically equal. However this is the wrong way to think of this. Think of it instead as if we have something like this:

```
>>> from diofant.abc import c, s
>>> def F(u):
...     return 2*u
...
```

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```
>>> def G(u):
...     return 2*sqrt(1 - u**2)
...
>>> F(cos(x))
2*cos(x)
>>> G(sin(x))
2*sqrt(-sin(x)**2 + 1)
>>> F(c).diff(c)
2
>>> F(c).diff(c)
2
>>> G(s).diff(c)
0
>>> G(sin(x)).diff(cos(x))
0
```

Here, the Symbols c and s act just like the functions cos(x) and sin(x), respectively. Think of 2*cos(x) as f(c).subs(c, cos(x)) (or f(c) *at* c = cos(x)) and 2*sqrt(1 - sin(x)**2) as g(s).subs(s, sin(x)) (or g(s) *at* s = sin(x)), where f(u) == 2*u and g(u) == 2*sqrt(1 - u**2). Here, we define the function first and evaluate it at the function, but we can actually unambiguously do this in reverse in Diofant, because expr.subs(Function, Symbol) is well-defined: just structurally replace the function everywhere it appears in the expression.

This is the same notational convenience used in the Euler-Lagrange method when one says F(t, f(t), f'(t)).diff(f(t)). What is actually meant is that the expression in question is represented by some F(t, u, v) at u = f(t) and v = f'(t), and F(t, f(t), f'(t)).diff(f(t)) simply means F(t, u, v).diff(u) at u = f(t).

We do not allow derivatives to be taken with respect to expressions where this is not so well defined. For example, we do not allow expr.diff(x*y) because there are multiple ways of structurally defining where x*y appears in an expression, some of which may surprise the reader (for example, a very strict definition would have that (x*y*z).diff(x*y) == 0).

```
>>> (x*y*z).diff(x*y)
Traceback (most recent call last):
...
ValueError: Can't differentiate wrt the variable: x*y, 1
```

Note that this definition also fits in nicely with the definition of the chain rule. Note how the chain rule in Diofant is defined using unevaluated Subs objects:

```
>>> f(2*g(x)).diff(x)
2*Derivative(g(x), x)*Subs(Derivative(f(_xi_1), _xi_1),
                               (_xi_1,), (2*g(x),))
>>> f(g(x)).diff(x)
Derivative(g(x), x)*Subs(Derivative(f(_xi_1), _xi_1),
                             (_xi_1,), (g(x),))
```

Finally, note that, to be consistent with variational calculus, and to ensure that the definition of substituting a Function for a Symbol in an expression is well-defined, derivatives of functions are assumed to not be related to the function. In other words, we have:

```
>>> diff(f(x), x).diff(f(x))
0
```

The same is true for derivatives of different orders:

```
>>> diff(f(x), x, 2).diff(diff(f(x), x, 1))
0
>>> diff(f(x), x, 1).diff(diff(f(x), x, 2))
0
```

Note, any class can allow derivatives to be taken with respect to itself.

Examples

Some basic examples:

```
>>> Derivative(x**2, x, evaluate=True)
2*x
>>> Derivative(Derivative(f(x, y), x), y)
Derivative(f(x, y), x, y)
>>> Derivative(f(x), x, 3)
Derivative(f(x), x, x, x)
>>> Derivative(f(x, y), y, x, evaluate=True)
Derivative(f(x, y), x, y)
```

Now some derivatives wrt functions:

```
>>> Derivative(f(x)**2, f(x), evaluate=True)
2*f(x)
>>> Derivative(f(g(x)), x, evaluate=True)
Derivative(g(x), x)*Subs(Derivative(f(_xi_1), _xi_1),
                            (_xi_1,), (g(x),))
```

doit(***hints*)
Evaluate objects that are not evaluated by default.

See also:

diofant.core.basic.Basic.doit (page 44)

doit_numerically(*z0*)
Evaluate the derivative at z numerically.

When we can represent derivatives at a point, this should be folded into the normal evalf. For now, we need a special method.

expr
Return expression

free_symbols
Return from the atoms of self those which are free symbols.

See also:

diofant.core.basic.Basic.free_symbols (page 44)

variables
Return tuple of symbols, wrt derivative is taken.

diff

diofant.core.function.**diff**(*f*, **symbols*, ***kwargs*)
Differentiate f with respect to symbols.

This is just a wrapper to unify .diff() and the Derivative class; its interface is similar to that of integrate(). You can use the same shortcuts for multiple variables as with Derivative. For example, diff(f(x), x, x, x) and diff(f(x), x, 3) both return the third derivative of f(x).

You can pass evaluate=False to get an unevaluated Derivative class. Note that if there are 0 symbols (such as diff(f(x), x, 0), then the result will be the function (the zeroth derivative), even if evaluate=False.

See also:

Derivative (page 128)

diofant.geometry.util.idiff **(page 428)** computes the derivative implicitly

References

[R124] (page 1250)

Examples

```
>>> diff(sin(x), x)
cos(x)
>>> diff(f(x), x, x, x)
Derivative(f(x), x, x, x)
>>> diff(f(x), x, 3)
Derivative(f(x), x, x, x)
>>> diff(sin(x)*cos(y), x, 2, y, 2)
sin(x)*cos(y)
```

```
>>> type(diff(sin(x), x))
cos
>>> type(diff(sin(x), x, evaluate=False))
<class 'diofant.core.function.Derivative'>
>>> type(diff(sin(x), x, 0))
sin
>>> type(diff(sin(x), x, 0, evaluate=False))
sin
```

```
>>> diff(sin(x))
cos(x)
>>> diff(sin(x*y))
Traceback (most recent call last):
...
ValueError: specify differentiation variables to differentiate sin(x*y)
```

Note that `diff(sin(x))` syntax is meant only for convenience in interactive sessions and should be avoided in library code.

FunctionClass

class diofant.core.function.**FunctionClass**(**args*, ***kwargs*)

Base class for function classes. FunctionClass is a subclass of type.

Use Function('<function name>' [, signature]) to create undefined function classes.

nargs

Return a set of the allowed number of arguments for the function.

Examples

If the function can take any number of arguments, the set of whole numbers is returned:

```
>>> Function('f').nargs
Naturals0()
```

If the function was initialized to accept one or more arguments, a corresponding set will be returned:

```
>>> Function('f', nargs=1).nargs
{1}
>>> Function('f', nargs=(2, 1)).nargs
{1, 2}
```

The undefined function, after application, also has the nargs attribute; the actual number of arguments is always available by checking the `args` attribute:

```
>>> f(1).nargs
Naturals0()
>>> len(f(1).args)
1
```

Function

class diofant.core.function.**Function**

Base class for applied mathematical functions.

It also serves as a constructor for undefined function classes.

Examples

First example shows how to use Function as a constructor for undefined function classes:

```
>>> g = g(x)
>>> f
f
>>> f(x)
f(x)
>>> g
g(x)
>>> f(x).diff(x)
Derivative(f(x), x)
```

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```
>>> g.diff(x)
Derivative(g(x), x)
```

In the following example Function is used as a base class for `my_func` that represents a mathematical function *my_func*. Suppose that it is well known, that *my_func(0)* is *1* and *my_func* at infinity goes to *0*, so we want those two simplifications to occur automatically. Suppose also that *my_func(x)* is real exactly when *x* is real. Here is an implementation that honours those requirements:

```
>>> class my_func(Function):
...
...     @classmethod
...     def eval(cls, x):
...         if x.is_Number:
...             if x is S.Zero:
...                 return S.One
...             elif x is oo:
...                 return S.Zero
...
...     def _eval_is_real(self):
...         return self.args[0].is_real
...
>>> x = Symbol('x')
>>> my_func(0) + sin(0)
1
>>> my_func(oo)
0
>>> my_func(3.54).n() # Not yet implemented for my_func.
my_func(3.54)
>>> my_func(I).is_real
False
```

In order for `my_func` to become useful, several other methods would need to be implemented. See source code of some of the already implemented functions for more complete examples.

Also, if the function can take more than one argument, then `nargs` must be defined, e.g. if `my_func` can take one or two arguments then,

```
>>> class my_func(Function):
...     nargs = (1, 2)
...
>>>
```

as_base_exp()
Returns the method as the 2-tuple (base, exponent).

classmethod class_key()
Nice order of classes.

fdiff(*argindex=1*)
Returns the first derivative of the function.

Note: Not all functions are the same

Diofant defines many functions (like `cos` and `factorial`). It also allows the user to create

generic functions which act as argument holders. Such functions are created just like symbols:

```
>>> f = Function('f')
>>> f(2) + f(x)
f(2) + f(x)
```

If you want to see which functions appear in an expression you can use the atoms method:

```
>>> e = (f(x) + cos(x) + 2)
>>> e.atoms(Function)
{f(x), cos(x)}
```

If you just want the function you defined, not Diofant functions, the thing to search for is AppliedUndef:

```
>>> from diofant.core.function import AppliedUndef
>>> e.atoms(AppliedUndef)
{f(x)}
```

Subs

class diofant.core.function.**Subs**

Represents unevaluated substitutions of an expression.

`Subs(expr, x, x0)` receives 3 arguments: an expression, a variable or list of distinct variables and a point or list of evaluation points corresponding to those variables.

`Subs` objects are generally useful to represent unevaluated derivatives calculated at a point.

The variables may be expressions, but they are subjected to the limitations of subs(), so it is usually a good practice to use only symbols for variables, since in that case there can be no ambiguity.

There's no automatic expansion - use the method .doit() to effect all possible substitutions of the object and also of objects inside the expression.

When evaluating derivatives at a point that is not a symbol, a Subs object is returned. One is also able to calculate derivatives of Subs objects - in this case the expression is always expanded (for the unevaluated form, use Derivative()).

A simple example:

```
>>> e = Subs(f(x).diff(x), x, y)
>>> e.subs(y, 0)
Subs(Derivative(f(x), x), (x,), (0,))
>>> e.subs(f, sin).doit()
cos(y)
```

An example with several variables:

```
>>> Subs(f(x)*sin(y) + z, (x, y), (0, 1))
Subs(z + f(x)*sin(y), (x, y), (0, 1))
>>> _.doit()
z + f(0)*sin(1)
```

doit(***hints*)

Evaluate objects that are not evaluated by default.

See also:

diofant.core.basic.Basic.doit (page 44)

evalf(*dps=15*, ***options*)

Evaluate the given formula to an accuracy of dps decimal digits.

See also:

diofant.core.evalf.EvalfMixin.evalf (page 145)

expr

The expression on which the substitution operates

free_symbols

Return from the atoms of self those which are free symbols.

See also:

diofant.core.basic.Basic.free_symbols (page 44)

n(*dps=15*, ***options*)

point

The values for which the variables are to be substituted

variables

The variables to be evaluated

expand

diofant.core.function.**expand**(*e*, *deep=True*, *modulus=None*, *power_base=True*, *power_exp=True*, *mul=True*, *log=True*, *multinomial=True*, *basic=True*, ***hints*)

Expand an expression using methods given as hints.

Hints evaluated unless explicitly set to False are: `basic`, `log`, `multinomial`, `mul`, `power_base`, and `power_exp`. The following hints are supported but not applied unless set to True: `complex`, `func`, and `trig`. In addition, the following meta-hints are supported by some or all of the other hints: `frac`, `numer`, `denom`, `modulus`, and `force`. `deep` is supported by all hints. Additionally, subclasses of Expr may define their own hints or meta-hints.

Parameters basic : boolean, optional

This hint is used for any special rewriting of an object that should be done automatically (along with the other hints like `mul`) when expand is called. This is a catch-all hint to handle any sort of expansion that may not be described by the existing hint names.

deep : boolean, optional

If `deep` is set to `True` (the default), things like arguments of functions are recursively expanded. Use `deep=False` to only expand on the top level.

mul : boolean, optional

Distributes multiplication over addition (‘‘):

```
>>> (y*(x + z)).expand(mul=True)
x*y + y*z
```

multinomial : boolean, optional

Expand (x + y + …)**n where n is a positive integer.

```
>>> ((x + y + z)**2).expand(multinomial=True)
x**2 + 2*x*y + 2*x*z + y**2 + 2*y*z + z**2
```

power_exp : boolean, optional

Expand addition in exponents into multiplied bases.

```
>>> exp(x + y).expand(power_exp=True)
E**x*E**y
>>> (2**(x + y)).expand(power_exp=True)
2**x*2**y
```

power_base : boolean, optional

Split powers of multiplied bases.

This only happens by default if assumptions allow, or if the `force` meta-hint is used:

```
>>> ((x*y)**z).expand(power_base=True)
(x*y)**z
>>> ((x*y)**z).expand(power_base=True, force=True)
x**z*y**z
>>> ((2*y)**z).expand(power_base=True)
2**z*y**z
```

Note that in some cases where this expansion always holds, Diofant performs it automatically:

```
>>> (x*y)**2
x**2*y**2
```

log : boolean, optional

Pull out power of an argument as a coefficient and split logs products into sums of logs.

Note that these only work if the arguments of the log function have the proper assumptions–the arguments must be positive and the exponents must be real–or else the `force` hint must be True:

```
>>> log(x**2*y).expand(log=True)
log(x**2*y)
>>> log(x**2*y).expand(log=True, force=True)
2*log(x) + log(y)
>>> x, y = symbols('x y', positive=True)
>>> log(x**2*y).expand(log=True)
2*log(x) + log(y)
```

complex : boolean, optional

Split an expression into real and imaginary parts.

```
>>> x, y = symbols('x y')
>>> (x + y).expand(complex=True)
re(x) + re(y) + I*im(x) + I*im(y)
>>> cos(x).expand(complex=True)
-I*sin(re(x))*sinh(im(x)) + cos(re(x))*cosh(im(x))
```

Note that this is just a wrapper around `as_real_imag()`. Most objects that wish to redefine `_eval_expand_complex()` should consider redefining `as_real_imag()` instead.

func : boolean

Expand other functions.

```
>>> gamma(x + 1).expand(func=True)
x*gamma(x)
```

trig : boolean, optional

Do trigonometric expansions.

```
>>> cos(x + y).expand(trig=True)
-sin(x)*sin(y) + cos(x)*cos(y)
>>> sin(2*x).expand(trig=True)
2*sin(x)*cos(x)
```

Note that the forms of `sin(n*x)` and `cos(n*x)` in terms of `sin(x)` and `cos(x)` are not unique, due to the identity $\sin^2(x) + \cos^2(x) = 1$. The current implementation uses the form obtained from Chebyshev polynomials, but this may change. See *[R125]* (page 1250) for more information.

force : boolean, optional

If the `force` hint is used, assumptions about variables will be ignored in making the expansion.

See also:

expand_log (page 142), *expand_mul* (page 141), *expand_multinomial* (page 143), *expand_complex* (page 142), *expand_trig* (page 142), *expand_power_base* (page 143), *expand_power_exp* (page 143), *expand_func* (page 142), *diofant.simplify.hyperexpand.hyperexpand* (page 799)

Notes

- You can shut off unwanted methods:

```
>>> (exp(x + y)*(x + y)).expand()
E**x*E**y*x + E**x*E**y*y
>>> (exp(x + y)*(x + y)).expand(power_exp=False)
E**(x + y)*x + E**(x + y)*y
>>> (exp(x + y)*(x + y)).expand(mul=False)
E**x*E**y*(x + y)
```

- Use deep=False to only expand on the top level:

```
>>> exp(x + exp(x + y)).expand()
E**x*E**(E**x*E**y)
>>> exp(x + exp(x + y)).expand(deep=False)
E**(E**(x + y))*E**x
```

- Hints are applied in an arbitrary, but consistent order (in the current implementation, they are applied in alphabetical order, except multinomial comes before mul, but this may change). Because of this, some hints may prevent expansion by other hints if they are applied first. For example, `mul` may distribute multiplications and prevent `log` and `power_base` from expanding them. Also, if `mul` is applied before `multinomial`, the expression might not be fully distributed. The solution is to use the various `expand_hint` helper functions or to use `hint=False` to this function to finely control which hints are applied. Here are some examples:

```
>>> x, y, z = symbols('x y z', positive=True)

>>> expand(log(x*(y + z)))
log(x) + log(y + z)
```

  Here, we see that `log` was applied before `mul`. To get the mul expanded form, either of the following will work:

```
>>> expand_mul(log(x*(y + z)))
log(x*y + x*z)
>>> expand(log(x*(y + z)), log=False)
log(x*y + x*z)
```

  A similar thing can happen with the `power_base` hint:

```
>>> expand((x*(y + z))**x)
(x*y + x*z)**x
```

  To get the `power_base` expanded form, either of the following will work:

```
>>> expand((x*(y + z))**x, mul=False)
x**x*(y + z)**x
>>> expand_power_base((x*(y + z))**x)
x**x*(y + z)**x

>>> expand((x + y)*y/x)
y + y**2/x
```

  The parts of a rational expression can be targeted:

```
>>> expand((x + y)*y/x/(x + 1), frac=True)
(x*y + y**2)/(x**2 + x)
>>> expand((x + y)*y/x/(x + 1), numer=True)
(x*y + y**2)/(x*(x + 1))
>>> expand((x + y)*y/x/(x + 1), denom=True)
y*(x + y)/(x**2 + x)
```

- The `modulus` meta-hint can be used to reduce the coefficients of an expression post-expansion:

```
>>> expand((3*x + 1)**2)
9*x**2 + 6*x + 1
```

(continues on next page)

(continued from previous page)

```
>>> expand((3*x + 1)**2, modulus=5)
4*x**2 + x + 1
```

- Either expand() the function or .expand() the method can be used. Both are equivalent:

```
>>> expand((x + 1)**2)
x**2 + 2*x + 1
>>> ((x + 1)**2).expand()
x**2 + 2*x + 1
```

- Objects can define their own expand hints by defining _eval_expand_hint(). The function should take the form:

```
def _eval_expand_hint(self, **hints):
    # Only apply the method to the top-level expression
    ...
```

See also the example below. Objects should define _eval_expand_hint() methods only if hint applies to that specific object. The generic _eval_expand_hint() method defined in Expr will handle the no-op case.

Each hint should be responsible for expanding that hint only. Furthermore, the expansion should be applied to the top-level expression only. expand() takes care of the recursion that happens when deep=True.

You should only call _eval_expand_hint() methods directly if you are 100% sure that the object has the method, as otherwise you are liable to get unexpected AttributeError's. Note, again, that you do not need to recursively apply the hint to args of your object: this is handled automatically by expand(). _eval_expand_hint() should generally not be used at all outside of an _eval_expand_hint() method. If you want to apply a specific expansion from within another method, use the public expand() function, method, or expand_hint() functions.

In order for expand to work, objects must be rebuildable by their args, i.e., obj.func(*obj.args) == obj must hold.

Expand methods are passed **hints so that expand hints may use 'metahints'–hints that control how different expand methods are applied. For example, the force=True hint described above that causes expand(log=True) to ignore assumptions is such a metahint. The deep meta-hint is handled exclusively by expand() and is not passed to _eval_expand_hint() methods.

Note that expansion hints should generally be methods that perform some kind of 'expansion'. For hints that simply rewrite an expression, use the .rewrite() API.

References

[R125] (page 1250)

Examples

```
>>> class MyClass(Expr):
...     def __new__(cls, *args):
...         args = sympify(args)
...         return Expr.__new__(cls, *args)
...
...     def _eval_expand_double(self, **hints):
...         '''
...         Doubles the args of MyClass.
...
...         If there more than four args, doubling is not performed,
...         unless force=True is also used (False by default).
...         '''
...         force = hints.pop('force', False)
...         if not force and len(self.args) > 4:
...             return self
...         return self.func(*(self.args + self.args))
...
>>> a = MyClass(1, 2, MyClass(3, 4))
>>> a
MyClass(1, 2, MyClass(3, 4))
>>> a.expand(double=True)
MyClass(1, 2, MyClass(3, 4, 3, 4), 1, 2, MyClass(3, 4, 3, 4))
>>> a.expand(double=True, deep=False)
MyClass(1, 2, MyClass(3, 4), 1, 2, MyClass(3, 4))
```

```
>>> b = MyClass(1, 2, 3, 4, 5)
>>> b.expand(double=True)
MyClass(1, 2, 3, 4, 5)
>>> b.expand(double=True, force=True)
MyClass(1, 2, 3, 4, 5, 1, 2, 3, 4, 5)
```

PoleError

class diofant.core.function.**PoleError**

count_ops

diofant.core.function.**count_ops**(*expr*, *visual=False*)

Return a representation (integer or expression) of the operations in expr.

If visual is False (default) then the sum of the coefficients of the visual expression will be returned.

If visual is True then the number of each type of operation is shown with the core class types (or their virtual equivalent) multiplied by the number of times they occur.

If expr is an iterable, the sum of the op counts of the items will be returned.

Examples

```
>>> from diofant.abc import a, b
```

Although there isn't a SUB object, minus signs are interpreted as either negations or subtractions:

```
>>> (x - y).count_ops(visual=True)
SUB
>>> (-x).count_ops(visual=True)
NEG
```

Here, there are two Adds and a Pow:

```
>>> (1 + a + b**2).count_ops(visual=True)
2*ADD + POW
```

In the following, an Add, Mul, Pow and two functions:

```
>>> (sin(x)*x + sin(x)**2).count_ops(visual=True)
ADD + MUL + POW + 2*SIN
```

for a total of 5:

```
>>> (sin(x)*x + sin(x)**2).count_ops(visual=False)
5
```

Note that "what you type" is not always what you get. The expression 1/x/y is translated by diofant into 1/(x*y) so it gives a DIV and MUL rather than two DIVs:

```
>>> (1/x/y).count_ops(visual=True)
DIV + MUL
```

The visual option can be used to demonstrate the difference in operations for expressions in different forms. Here, the Horner representation is compared with the expanded form of a polynomial:

```
>>> eq = x*(1 + x*(2 + x*(3 + x)))
>>> count_ops(eq.expand(), visual=True) - count_ops(eq, visual=True)
-MUL + 3*POW
```

The count_ops function also handles iterables:

```
>>> count_ops([x, sin(x), None, True, x + 2], visual=False)
2
>>> count_ops([x, sin(x), None, True, x + 2], visual=True)
ADD + SIN
>>> count_ops({x: sin(x), x + 2: y + 1}, visual=True)
2*ADD + SIN
```

expand_mul

`diofant.core.function.expand_mul(expr, deep=True)`
Wrapper around expand that only uses the mul hint. See the expand docstring for more information.

Examples

```
>>> x, y = symbols('x y', positive=True)
>>> expand_mul(exp(x+y)*(x+y)*log(x*y**2))
E**(x + y)*x*log(x*y**2) + E**(x + y)*y*log(x*y**2)
```

expand_log

diofant.core.function.**expand_log**(*expr*, *deep=True*, *force=False*)

Wrapper around expand that only uses the log hint. See the expand docstring for more information.

Examples

```
>>> x, y = symbols('x y', positive=True)
>>> expand_log(exp(x+y)*(x+y)*log(x*y**2))
E**(x + y)*(x + y)*(log(x) + 2*log(y))
```

expand_func

diofant.core.function.**expand_func**(*expr*, *deep=True*)

Wrapper around expand that only uses the func hint. See the expand docstring for more information.

Examples

```
>>> expand_func(gamma(x + 2))
x*(x + 1)*gamma(x)
```

expand_trig

diofant.core.function.**expand_trig**(*expr*, *deep=True*)

Wrapper around expand that only uses the trig hint. See the expand docstring for more information.

Examples

```
>>> expand_trig(sin(x+y)*(x+y))
(x + y)*(sin(x)*cos(y) + sin(y)*cos(x))
```

expand_complex

diofant.core.function.**expand_complex**(*expr*, *deep=True*)

Wrapper around expand that only uses the complex hint. See the expand docstring for more information.

See also:

diofant.core.expr.Expr.as_real_imag (page 62)

Examples

```
>>> expand_complex(exp(z))
E**re(z)*I*sin(im(z)) + E**re(z)*cos(im(z))
>>> expand_complex(sqrt(I))
sqrt(2)/2 + sqrt(2)*I/2
```

expand_multinomial

diofant.core.function.**expand_multinomial**(*expr*, *deep=True*)

Wrapper around expand that only uses the multinomial hint. See the expand docstring for more information.

Examples

```
>>> x, y = symbols('x y', positive=True)
>>> expand_multinomial((x + exp(x + 1))**2)
2*E**(x + 1)*x + E**(2*x + 2) + x**2
```

expand_power_exp

diofant.core.function.**expand_power_exp**(*expr*, *deep=True*)

Wrapper around expand that only uses the power_exp hint.

See also:

expand (page 135)

Examples

```
>>> expand_power_exp(x**(y + 2))
x**2*x**y
```

expand_power_base

diofant.core.function.**expand_power_base**(*expr*, *deep=True*, *force=False*)

Wrapper around expand that only uses the power_base hint.

A wrapper to expand(power_base=True) which separates a power with a base that is a Mul into a product of powers, without performing any other expansions, provided that assumptions about the power's base and exponent allow.

deep=False (default is True) will only apply to the top-level expression.

force=True (default is False) will cause the expansion to ignore assumptions about the base and exponent. When False, the expansion will only happen if the base is non-negative or the exponent is an integer.

```
>>> (x*y)**2
x**2*y**2
```

```
>>> (2*x)**y
(2*x)**y
>>> expand_power_base(_)
2**y*x**y
```

```
>>> expand_power_base((x*y)**z)
(x*y)**z
>>> expand_power_base((x*y)**z, force=True)
x**z*y**z
>>> expand_power_base(sin((x*y)**z), deep=False)
sin((x*y)**z)
>>> expand_power_base(sin((x*y)**z), force=True)
sin(x**z*y**z)
```

```
>>> expand_power_base((2*sin(x))**y + (2*cos(x))**y)
2**y*sin(x)**y + 2**y*cos(x)**y
```

```
>>> expand_power_base((2*exp(y))**x)
2**x*(E**y)**x
```

```
>>> expand_power_base((2*cos(x))**y)
2**y*cos(x)**y
```

Notice that sums are left untouched. If this is not the desired behavior, apply full `expand()` to the expression:

```
>>> expand_power_base(((x+y)*z)**2)
z**2*(x + y)**2
>>> (((x+y)*z)**2).expand()
x**2*z**2 + 2*x*y*z**2 + y**2*z**2
```

```
>>> expand_power_base((2*y)**(1+z))
2**(z + 1)*y**(z + 1)
>>> ((2*y)**(1+z)).expand()
2*2**z*y*y**z
```

See also:

expand (page 135)

nfloat

`diofant.core.function.nfloat(expr, n=15, exponent=False)`

Make all Rationals in expr Floats except those in exponents (unless the exponents flag is set to True).

Examples

```
>>> nfloat(x**4 + x/2 + cos(pi/3) + 1 + sqrt(y))
x**4 + 0.5*x + sqrt(y) + 1.5
>>> nfloat(x**4 + sqrt(y), exponent=True)
x**4.0 + y**0.5
```

3.1.20 evalf

class diofant.core.evalf.**EvalfMixin**

Mixin class adding evalf capability.

evalf(*dps=15*, *subs=None*, *maxn=110*, *chop=False*, *strict=True*, *quad=None*)

Evaluate the given formula to an accuracy of dps decimal digits. Optional keyword arguments:

subs=<dict> Substitute numerical values for symbols, e.g. subs={x:3, y:1+pi}. The substitutions must be given as a dictionary.

maxn=<integer> Allow a maximum temporary working precision of maxn digits (default=110)

chop=<bool> Replace tiny real or imaginary parts in subresults by exact zeros (default=False)

strict=<bool> Raise PrecisionExhausted if any subresult fails to evaluate to full accuracy, given the available maxprec (default=True)

quad=<str> Choose algorithm for numerical quadrature. By default, tanh-sinh quadrature is used. For oscillatory integrals on an infinite interval, try quad='osc'.

n(*dps=15*, *subs=None*, *maxn=110*, *chop=False*, *strict=True*, *quad=None*)

Evaluate the given formula to an accuracy of dps decimal digits. Optional keyword arguments:

subs=<dict> Substitute numerical values for symbols, e.g. subs={x:3, y:1+pi}. The substitutions must be given as a dictionary.

maxn=<integer> Allow a maximum temporary working precision of maxn digits (default=110)

chop=<bool> Replace tiny real or imaginary parts in subresults by exact zeros (default=False)

strict=<bool> Raise PrecisionExhausted if any subresult fails to evaluate to full accuracy, given the available maxprec (default=True)

quad=<str> Choose algorithm for numerical quadrature. By default, tanh-sinh quadrature is used. For oscillatory integrals on an infinite interval, try quad='osc'.

PrecisionExhausted

class diofant.core.evalf.**PrecisionExhausted**

N

diofant.core.evalf.**N**(*x*, *dps=15*, ***options*)

Calls x.evalf(dps, **options).

See also:

diofant.core.evalf.EvalfMixin.evalf (page 145)

Examples

```
>>> Sum(1/k**k, (k, 1, oo))
Sum(k**(-k), (k, 1, oo))
>>> N(_, 4)
1.291
```

3.1.21 containers

Tuple

class diofant.core.containers.**Tuple**

Wrapper around the builtin tuple object

The Tuple is a subclass of Basic, so that it works well in the Diofant framework. The wrapped tuple is available as self.args, but you can also access elements or slices with [:] syntax.

```
>>> a, b, c, d = symbols('a b c d')
>>> Tuple(a, b, c)[1:]
(b, c)
>>> Tuple(a, b, c).subs(a, d)
(d, b, c)
```

index(*value*[, *start*[, *stop*]]) → integer – return first index of value.

Raises ValueError if the value is not present.

tuple_count(*value*)

T.count(value) -> integer – return number of occurrences of value

Dict

class diofant.core.containers.**Dict**

Wrapper around the builtin dict object

The Dict is a subclass of Basic, so that it works well in the Diofant framework. Because it is immutable, it may be included in sets, but its values must all be given at instantiation and cannot be changed afterwards. Otherwise it behaves identically to the Python dict.

```
>>> D = Dict({1: 'one', 2: 'two'})
>>> for key in D:
...     if key == 1:
...         print('%s %s' % (key, D[key]))
1 one
```

The args are sympified so the 1 and 2 are Integers and the values are Symbols. Queries automatically sympify args so the following work:

```
>>> 1 in D
True
>>> D.has('one') # searches keys and values
True
>>> 'one' in D # not in the keys
False
>>> D[1]
one
```

args
Returns a tuple of arguments of 'self'.

See also:

diofant.core.basic.Basic.args (page 42)

get(*k*[, *d*]) → D[k] if k in D, else d. d defaults to None.

items() → list of D's (key, value) pairs, as 2-tuples

keys() → list of D's keys

values() → list of D's values

3.1.22 compatibility

Reimplementations of constructs introduced in later versions of Python than we support. Also some functions that are needed Diofant-wide and are located here for easy import.

class diofant.core.compatibility.**NotIterable**
Use this as mixin when creating a class which is not supposed to return true when iterable() is called on its instances. I.e. avoid infinite loop when calling e.g. list() on the instance

diofant.core.compatibility.**as_int**(*n*)
Convert the argument to a builtin integer.

The return value is guaranteed to be equal to the input. ValueError is raised if the input has a non-integral value.

Examples

```
>>> 3.0
3.0
>>> as_int(3.0)  # convert to int and test for equality
3
>>> int(sqrt(10))
3
>>> as_int(sqrt(10))
Traceback (most recent call last):
...
ValueError: ... is not an integer
```

diofant.core.compatibility.**default_sort_key**(*item*, *order=None*)
Return a key that can be used for sorting.

The key has the structure:

(class_key, (len(args), args), exponent.sort_key(), coefficient)

This key is supplied by the sort_key routine of Basic objects when `item` is a Basic object or an object (other than a string) that sympifies to a Basic object. Otherwise, this function produces the key.

The `order` argument is passed along to the sort_key routine and is used to determine how the terms *within* an expression are ordered. (See examples below) `order` options are: 'lex', 'grlex', 'grevlex', and reversed values of the same (e.g. 'rev-lex'). The default order value is None (which translates to 'lex').

See also:

ordered (page 150), *diofant.core.expr.Expr.as_ordered_factors* (page 61), *diofant.core.expr.Expr.as_ordered_terms* (page 61)

Notes

The key returned is useful for getting items into a canonical order that will be the same across platforms. It is not directly useful for sorting lists of expressions:

```
>>> a, b = x, 1/x
```

Since `a` has only 1 term, its value of sort_key is unaffected by `order`:

```
>>> a.sort_key() == a.sort_key('rev-lex')
True
```

If `a` and `b` are combined then the key will differ because there are terms that can be ordered:

```
>>> eq = a + b
>>> eq.sort_key() == eq.sort_key('rev-lex')
False
>>> eq.as_ordered_terms()
[x, 1/x]
>>> eq.as_ordered_terms('rev-lex')
[1/x, x]
```

But since the keys for each of these terms are independent of `order`'s value, they don't sort differently when they appear separately in a list:

```
>>> sorted(eq.args, key=default_sort_key)
[1/x, x]
>>> sorted(eq.args, key=lambda i: default_sort_key(i, order='rev-lex'))
[1/x, x]
```

The order of terms obtained when using these keys is the order that would be obtained if those terms were *factors* in a product.

Although it is useful for quickly putting expressions in canonical order, it does not sort expressions based on their complexity defined by the number of operations, power of variables and others:

```
>>> sorted([sin(x)*cos(x), sin(x)], key=default_sort_key)
[sin(x)*cos(x), sin(x)]
>>> sorted([x, x**2, sqrt(x), x**3], key=default_sort_key)
[sqrt(x), x, x**2, x**3]
```

Examples

```
>>> from diofant.core.function import UndefinedFunction
```

The following are equivalent ways of getting the key for an object:

```
>>> x.sort_key() == default_sort_key(x)
True
```

Here are some examples of the key that is produced:

```
>>> default_sort_key(UndefinedFunction('f'))
((0, 0, 'UndefinedFunction'), (1, ('f',)), ((1, 0, 'Number'),
    (0, ()), (), 1), 1)
>>> default_sort_key('1')
((0, 0, 'str'), (1, ('1',)), ((1, 0, 'Number'), (0, ()), (), 1), 1)
>>> default_sort_key(S.One)
((1, 0, 'Number'), (0, ()), (), 1)
>>> default_sort_key(2)
((1, 0, 'Number'), (0, ()), (), 2)
```

While sort_key is a method only defined for Diofant objects, default_sort_key will accept anything as an argument so it is more robust as a sorting key. For the following, using key= lambda i: i.sort_key() would fail because 2 doesn't have a sort_key method; that's why default_sort_key is used. Note, that it also handles sympification of non-string items likes ints:

```
>>> a = [2, I, -I]
>>> sorted(a, key=default_sort_key)
[2, -I, I]
```

The returned key can be used anywhere that a key can be specified for a function, e.g. sort, min, max, etc...:

```
>>> a.sort(key=default_sort_key); a[0]
2
>>> min(a, key=default_sort_key)
2
```

diofant.core.compatibility.**is_sequence**(*i*, *include=None*)

Return a boolean indicating whether `i` is a sequence in the Diofant sense. If anything that fails the test below should be included as being a sequence for your application, set 'include' to that object's type; multiple types should be passed as a tuple of types.

Note: although generators can generate a sequence, they often need special handling to make sure their elements are captured before the generator is exhausted, so these are not included by default in the definition of a sequence.

See also:

iterable (page 150)

Examples

```
>>> from types import GeneratorType
>>> is_sequence([])
True
>>> is_sequence(set())
False
>>> is_sequence('abc')
False
>>> is_sequence('abc', include=str)
True
>>> generator = (c for c in 'abc')
>>> is_sequence(generator)
False
>>> is_sequence(generator, include=(str, GeneratorType))
True
```

`diofant.core.compatibility.`**`iterable`**(*i, exclude=(<class 'str'>, <class 'dict'>, <class 'diofant.core.compatibility.NotIterable'>)*)

Return a boolean indicating whether i is Diofant iterable. True also indicates that the iterator is finite, i.e. you e.g. call list(...) on the instance.

When Diofant is working with iterables, it is almost always assuming that the iterable is not a string or a mapping, so those are excluded by default. If you want a pure Python definition, make exclude=None. To exclude multiple items, pass them as a tuple.

See also:

is_sequence (page 149)

Examples

```
>>> things = [[1], (1,), {1}, Tuple(1), (j for j in [1, 2]), {1: 2}, '1', 1]
>>> for i in things:
...     print('%s %s' % (iterable(i), type(i)))
True <... 'list'>
True <... 'tuple'>
True <... 'set'>
True <class 'diofant.core.containers.Tuple'>
True <... 'generator'>
False <... 'dict'>
False <... 'str'>
False <... 'int'>
```

```
>>> iterable({}, exclude=None)
True
>>> iterable({}, exclude=str)
True
>>> iterable("no", exclude=str)
False
```

`diofant.core.compatibility.`**`ordered`**(*seq, keys=None, default=True, warn=False*)

Return an iterator of the seq where keys are used to break ties in a conservative fashion: if, after applying a key, there are no ties then no other keys will be computed.

Two default keys will be applied if 1) keys are not provided or 2) the given keys don't resolve all ties (but only if $default$ is True). The two keys are $_nodes$ (which places smaller expressions before large) and $default_sort_key$ which (if the $sort_key$ for an object is defined properly) should resolve any ties.

If `warn` is True then an error will be raised if there were no keys remaining to break ties. This can be used if it was expected that there should be no ties between items that are not identical.

Notes

The decorated sort is one of the fastest ways to sort a sequence for which special item comparison is desired: the sequence is decorated, sorted on the basis of the decoration (e.g. making all letters lower case) and then undecorated. If one wants to break ties for items that have the same decorated value, a second key can be used. But if the second key is expensive to compute then it is inefficient to decorate all items with both keys: only those items having identical first key values need to be decorated. This function applies keys successively only when needed to break ties. By yielding an iterator, use of the tie-breaker is delayed as long as possible.

This function is best used in cases when use of the first key is expected to be a good hashing function; if there are no unique hashes from application of a key then that key should not have been used. The exception, however, is that even if there are many collisions, if the first group is small and one does not need to process all items in the list then time will not be wasted sorting what one was not interested in. For example, if one were looking for the minimum in a list and there were several criteria used to define the sort order, then this function would be good at returning that quickly if the first group of candidates is small relative to the number of items being processed.

Examples

The count_ops is not sufficient to break ties in this list and the first two items appear in their original order (i.e. the sorting is stable):

```
>>> list(ordered([y + 2, x + 2, x**2 + y + 3],
...              count_ops, default=False, warn=False))
[y + 2, x + 2, x**2 + y + 3]
```

The default_sort_key allows the tie to be broken:

```
>>> list(ordered([y + 2, x + 2, x**2 + y + 3]))
[x + 2, y + 2, x**2 + y + 3]
```

Here, sequences are sorted by length, then sum:

```
>>> seq, keys = [[[1, 2, 1], [0, 3, 1], [1, 1, 3], [2], [1]],
...              [lambda x: len(x), lambda x: sum(x)]]
>>> list(ordered(seq, keys, default=False, warn=False))
[[1], [2], [1, 2, 1], [0, 3, 1], [1, 1, 3]]
```

If `warn` is True, an error will be raised if there were not enough keys to break ties:

```
>>> list(ordered(seq, keys, default=False, warn=True))
Traceback (most recent call last):
...
ValueError: not enough keys to break ties
```

iterable

`diofant.core.compatibility.iterable`(*i*, *exclude=(<class 'str'>, <class 'dict'>, <class 'diofant.core.compatibility.NotIterable'>)*)

Return a boolean indicating whether `i` is Diofant iterable. True also indicates that the iterator is finite, i.e. you e.g. call list(...) on the instance.

When Diofant is working with iterables, it is almost always assuming that the iterable is not a string or a mapping, so those are excluded by default. If you want a pure Python definition, make exclude=None. To exclude multiple items, pass them as a tuple.

See also:

is_sequence (page 149)

Examples

```
>>> things = [[1], (1,), {1}, Tuple(1), (j for j in [1, 2]), {1: 2}, '1', 1]
>>> for i in things:
...     print('%s %s' % (iterable(i), type(i)))
True <... 'list'>
True <... 'tuple'>
True <... 'set'>
True <class 'diofant.core.containers.Tuple'>
True <... 'generator'>
False <... 'dict'>
False <... 'str'>
False <... 'int'>
```

```
>>> iterable({}, exclude=None)
True
>>> iterable({}, exclude=str)
True
>>> iterable("no", exclude=str)
False
```

is_sequence

`diofant.core.compatibility.is_sequence`(*i*, *include=None*)

Return a boolean indicating whether `i` is a sequence in the Diofant sense. If anything that fails the test below should be included as being a sequence for your application, set 'include' to that object's type; multiple types should be passed as a tuple of types.

Note: although generators can generate a sequence, they often need special handling to make sure their elements are captured before the generator is exhausted, so these are not included by default in the definition of a sequence.

See also:

iterable (page 150)

Examples

```
>>> from types import GeneratorType
>>> is_sequence([])
True
>>> is_sequence(set())
False
>>> is_sequence('abc')
False
>>> is_sequence('abc', include=str)
True
>>> generator = (c for c in 'abc')
>>> is_sequence(generator)
False
>>> is_sequence(generator, include=(str, GeneratorType))
True
```

as_int

diofant.core.compatibility.**as_int**(*n*)

Convert the argument to a builtin integer.

The return value is guaranteed to be equal to the input. ValueError is raised if the input has a non-integral value.

Examples

```
>>> 3.0
3.0
>>> as_int(3.0)  # convert to int and test for equality
3
>>> int(sqrt(10))
3
>>> as_int(sqrt(10))
Traceback (most recent call last):
...
ValueError: ... is not an integer
```

3.1.23 exprtools

gcd_terms

diofant.core.exprtools.**gcd_terms**(*terms*, *isprimitive=False*, *clear=True*, *fraction=True*)

Compute the GCD of `terms` and put them together.

`terms` can be an expression or a non-Basic sequence of expressions which will be handled as though they are terms from a sum.

If `isprimitive` is True the _gcd_terms will not run the primitive method on the terms.

clear controls the removal of integers from the denominator of an Add expression. When True (default), all numerical denominator will be cleared; when False the denominators will be cleared only if all terms had numerical denominators other than 1.

fraction, when True (default), will put the expression over a common denominator.

See also:

factor_terms (page 154), *diofant.polys.polytools.terms_gcd* (page 664)

Examples

```
>>> gcd_terms((x + 1)**2*y + (x + 1)*y**2)
y*(x + 1)*(x + y + 1)
>>> gcd_terms(x/2 + 1)
(x + 2)/2
>>> gcd_terms(x/2 + 1, clear=False)
x/2 + 1
>>> gcd_terms(x/2 + y/2, clear=False)
(x + y)/2
>>> gcd_terms(x/2 + 1/x)
(x**2 + 2)/(2*x)
>>> gcd_terms(x/2 + 1/x, fraction=False)
(x + 2/x)/2
>>> gcd_terms(x/2 + 1/x, fraction=False, clear=False)
x/2 + 1/x
```

```
>>> gcd_terms(x/2/y + 1/x/y)
(x**2 + 2)/(2*x*y)
>>> gcd_terms(x/2/y + 1/x/y, fraction=False, clear=False)
(x + 2/x)/(2*y)
```

The clear flag was ignored in this case because the returned expression was a rational expression, not a simple sum.

factor_terms

diofant.core.exprtools.**factor_terms**(*expr*, *radical=False*, *clear=False*, *fraction=False*, *sign=True*)

Remove common factors from terms in all arguments without changing the underlying structure of the expr. No expansion or simplification (and no processing of noncommutatives) is performed.

If radical=True then a radical common to all terms will be factored out of any Add subexpressions of the expr.

If clear=False (default) then coefficients will not be separated from a single Add if they can be distributed to leave one or more terms with integer coefficients.

If fraction=True (default is False) then a common denominator will be constructed for the expression.

If sign=True (default) then even if the only factor in common is a -1, it will be factored out of the expression.

See also:

gcd_terms (page 153), *diofant.polys.polytools.terms_gcd* (page 664)

Examples

```
>>> factor_terms(x + x*(2 + 4*y)**3)
x*(8*(2*y + 1)**3 + 1)
>>> A = Symbol('A', commutative=False)
>>> factor_terms(x*A + x*A + x*y*A)
x*(y*A + 2*A)
```

When `clear` is False, a rational will only be factored out of an Add expression if all terms of the Add have coefficients that are fractions:

```
>>> factor_terms(x/2 + 1, clear=False)
x/2 + 1
>>> factor_terms(x/2 + 1, clear=True)
(x + 2)/2
```

This only applies when there is a single Add that the coefficient multiplies:

```
>>> factor_terms(x*y/2 + y, clear=True)
y*(x + 2)/2
>>> factor_terms(x*y/2 + y, clear=False) == _
True
```

If a -1 is all that can be factored out, to *not* factor it out, the flag `sign` must be False:

```
>>> factor_terms(-x - y)
-(x + y)
>>> factor_terms(-x - y, sign=False)
-x - y
>>> factor_terms(-2*x - 2*y, sign=False)
-2*(x + y)
```

3.2 Combinatorics

3.2.1 Partitions

class diofant.combinatorics.partitions.**Partition**

This class represents an abstract partition.

A partition is a set of disjoint sets whose union equals a given set.

See also:

diofant.utilities.iterables.partitions (page 994), *diofant.utilities.iterables.multiset_partitions* (page 990)

RGS

Returns the “restricted growth string” of the partition.

The RGS is returned as a list of indices, L, where L[i] indicates the block in which element i appears. For example, in a partition of 3 elements (a, b, c) into 2 blocks ([c], [a, b]) the RGS is [1, 1, 0]: “a” is in block 1, “b” is in block 1 and “c” is in block 0.

Examples

```
>>> a = Partition([1, 2], [3], [4, 5])
>>> a.members
(1, 2, 3, 4, 5)
>>> a.RGS
(0, 0, 1, 2, 2)
>>> a + 1
{{3}, {4}, {5}, {1, 2}}
>>> _.RGS
(0, 0, 1, 2, 3)
```

classmethod from_rgs(*rgs, elements*)
Creates a set partition from a restricted growth string.

The indices given in rgs are assumed to be the index of the element as given in elements *as provided* (the elements are not sorted by this routine). Block numbering starts from 0. If any block was not referenced in `rgs` an error will be raised.

Examples

```
>>> Partition.from_rgs([0, 1, 2, 0, 1], list('abcde'))
{{c}, {a, d}, {b, e}}
>>> Partition.from_rgs([0, 1, 2, 0, 1], list('cbead'))
{{e}, {a, c}, {b, d}}
>>> a = Partition([1, 4], [2], [3, 5])
>>> Partition.from_rgs(a.RGS, a.members)
{{2}, {1, 4}, {3, 5}}
```

partition
Return partition as a sorted list of lists.

Examples

```
>>> Partition([1], [2, 3]).partition
[[1], [2, 3]]
```

rank
Gets the rank of a partition.

Examples

```
>>> a = Partition([1, 2], [3], [4, 5])
>>> a.rank
13
```

sort_key(*order=None*)
Return a canonical key that can be used for sorting.

Ordering is based on the size and sorted elements of the partition and ties are broken with the rank.

Examples

```
>>> a = Partition([1, 2])
>>> b = Partition([3, 4])
>>> c = Partition([1, x])
>>> d = Partition(list(range(4)))
>>> l = [d, b, a + 1, a, c]
>>> l.sort(key=default_sort_key); l
[{{1, 2}}, {{1}, {2}}, {{1, x}}, {{3, 4}}, {{0, 1, 2, 3}}]
```

class `diofant.combinatorics.partitions.`**`IntegerPartition`**

This class represents an integer partition.

In number theory and combinatorics, a partition of a positive integer, `n`, also called an integer partition, is a way of writing `n` as a list of positive integers that sum to n. Two partitions that differ only in the order of summands are considered to be the same partition; if order matters then the partitions are referred to as compositions. For example, 4 has five partitions: [4], [3, 1], [2, 2], [2, 1, 1], and [1, 1, 1, 1]; the compositions [1, 2, 1] and [1, 1, 2] are the same as partition [2, 1, 1].

See also:

diofant.utilities.iterables.partitions (page 994), *diofant.utilities.iterables.multiset_partitions* (page 990)

References

[R52] (page 1250)

`as_dict`()

Return the partition as a dictionary whose keys are the partition integers and the values are the multiplicity of that integer.

Examples

```
>>> IntegerPartition([1]*3 + [2] + [3]*4).as_dict()
{1: 3, 2: 1, 3: 4}
```

`as_ferrers`(*char='#'*)

Prints the ferrer diagram of a partition.

Examples

```
>>> print(IntegerPartition([1, 1, 5]).as_ferrers())
#####
#
#
```

`conjugate`

Computes the conjugate partition of itself.

Examples

```
>>> a = IntegerPartition([6, 3, 3, 2, 1])
>>> a.conjugate
[5, 4, 3, 1, 1, 1]
```

next_lex()

Return the next partition of the integer, n, in lexical order, wrapping around to [n] if the partition is [1, …, 1].

Examples

```
>>> p = IntegerPartition([3, 1])
>>> print(p.next_lex())
[4]
>>> p.partition < p.next_lex().partition
True
```

prev_lex()

Return the previous partition of the integer, n, in lexical order, wrapping around to [1, …, 1] if the partition is [n].

Examples

```
>>> p = IntegerPartition([4])
>>> print(p.prev_lex())
[3, 1]
>>> p.partition > p.prev_lex().partition
True
```

diofant.combinatorics.partitions.**random_integer_partition**(*n*, *seed=None*)

Generates a random integer partition summing to `n` as a list of reverse-sorted integers.

Examples

For the following, a seed is given so a known value can be shown; in practice, the seed would not be given.

```
>>> random_integer_partition(100, seed=[1, 1, 12, 1, 2, 1, 85, 1])
[85, 12, 2, 1]
>>> random_integer_partition(10, seed=[1, 2, 3, 1, 5, 1])
[5, 3, 1, 1]
>>> random_integer_partition(1)
[1]
```

diofant.combinatorics.partitions.**RGS_generalized**(*m*)

Computes the m + 1 generalized unrestricted growth strings and returns them as rows in matrix.

Examples

```
>>> RGS_generalized(6)
Matrix([
[  1,   1,   1,  1,  1, 1, 1],
[  1,   2,   3,  4,  5, 6, 0],
[  2,   5,  10, 17, 26, 0, 0],
[  5,  15,  37, 77,  0, 0, 0],
[ 15,  52, 151,  0,  0, 0, 0],
[ 52, 203,   0,  0,  0, 0, 0],
[203,   0,   0,  0,  0, 0, 0]])
```

diofant.combinatorics.partitions.**RGS_enum**(*m*)
RGS_enum computes the total number of restricted growth strings possible for a superset of size m.

Examples

```
>>> RGS_enum(4)
15
>>> RGS_enum(5)
52
>>> RGS_enum(6)
203
```

We can check that the enumeration is correct by actually generating the partitions. Here, the 15 partitions of 4 items are generated:

```
>>> a = Partition(list(range(4)))
>>> s = set()
>>> for i in range(20):
...     s.add(a)
...     a += 1
...
>>> assert len(s) == 15
```

diofant.combinatorics.partitions.**RGS_unrank**(*rank, m*)
Gives the unranked restricted growth string for a given superset size.

Examples

```
>>> RGS_unrank(14, 4)
[0, 1, 2, 3]
>>> RGS_unrank(0, 4)
[0, 0, 0, 0]
```

diofant.combinatorics.partitions.**RGS_rank**(*rgs*)
Computes the rank of a restricted growth string.

Examples

```
>>> RGS_rank([0, 1, 2, 1, 3])
42
>>> RGS_rank(RGS_unrank(4, 7))
4
```

3.2.2 Permutations

class diofant.combinatorics.permutations.**Permutation**

A permutation, alternatively known as an 'arrangement number' or 'ordering' is an arrangement of the elements of an ordered list into a one-to-one mapping with itself. The permutation of a given arrangement is given by indicating the positions of the elements after re-arrangement *[R55]* (page 1250). For example, if one started with elements [x, y, a, b] (in that order) and they were reordered as [x, y, b, a] then the permutation would be [0, 1, 3, 2]. Notice that (in Diofant) the first element is always referred to as 0 and the permutation uses the indices of the elements in the original ordering, not the elements (a, b, etc...) themselves.

```
>>> Permutation.print_cyclic = False
```

See also:

Cycle (page 181)

Notes

Permutations Notation

Permutations are commonly represented in disjoint cycle or array forms.

Array Notation and 2-line Form

In the 2-line form, the elements and their final positions are shown as a matrix with 2 rows:

[0 1 2 ... n-1] [p(0) p(1) p(2) ... p(n-1)]

Since the first line is always range(n), where n is the size of p, it is sufficient to represent the permutation by the second line, referred to as the "array form" of the permutation. This is entered in brackets as the argument to the Permutation class:

```
>>> p = Permutation([0, 2, 1]); p
Permutation([0, 2, 1])
```

Given i in range(p.size), the permutation maps i to i^p

```
>>> [i^p for i in range(p.size)]
[0, 2, 1]
```

The composite of two permutations p*q means first apply p, then q, so i^(p*q) = (i^p)^q which is i^p^q according to Python precedence rules:

```
>>> q = Permutation([2, 1, 0])
>>> [i^p^q for i in range(3)]
[2, 0, 1]
>>> [i^(p*q) for i in range(3)]
[2, 0, 1]
```

One can use also the notation p(i) = i^p, but then the composition rule is (p*q)(i) = q(p(i)), not p(q(i)):

```
>>> [(p*q)(i) for i in range(p.size)]
[2, 0, 1]
>>> [q(p(i)) for i in range(p.size)]
[2, 0, 1]
>>> [p(q(i)) for i in range(p.size)]
[1, 2, 0]
```

Disjoint Cycle Notation

In disjoint cycle notation, only the elements that have shifted are indicated. In the above case, the 2 and 1 switched places. This can be entered in two ways:

```
>>> Permutation(1, 2) == Permutation([[1, 2]]) == p
True
```

Only the relative ordering of elements in a cycle matter:

```
>>> Permutation(1, 2, 3) == Permutation(2, 3, 1) == Permutation(3, 1, 2)
True
```

The disjoint cycle notation is convenient when representing permutations that have several cycles in them:

```
>>> Permutation(1, 2)(3, 5) == Permutation([[1, 2], [3, 5]])
True
```

It also provides some economy in entry when computing products of permutations that are written in disjoint cycle notation:

```
>>> Permutation(1, 2)(1, 3)(2, 3)
Permutation([0, 3, 2, 1])
>>> _ == Permutation([[1, 2]])*Permutation([[1, 3]])*Permutation([[2, 3]])
True
```

Entering a singleton in a permutation is a way to indicate the size of the permutation. The `size` keyword can also be used.

Array-form entry:

```
>>> Permutation([[1, 2], [9]])
Permutation([0, 2, 1], size=10)
>>> Permutation([[1, 2]], size=10)
Permutation([0, 2, 1], size=10)
```

Cyclic-form entry:

```
>>> Permutation(1, 2, size=10)
Permutation([0, 2, 1], size=10)
```

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(continued from previous page)

```
>>> Permutation(9)(1, 2)
Permutation([0, 2, 1], size=10)
```

Caution: no singleton containing an element larger than the largest in any previous cycle can be entered. This is an important difference in how Permutation and Cycle handle the __call__ syntax. A singleton argument at the start of a Permutation performs instantiation of the Permutation and is permitted:

```
>>> Permutation(5)
Permutation([], size=6)
```

A singleton entered after instantiation is a call to the permutation – a function call – and if the argument is out of range it will trigger an error. For this reason, it is better to start the cycle with the singleton:

The following fails because there is is no element 3:

```
>>> Permutation(1, 2)(3)
Traceback (most recent call last):
...
IndexError: list index out of range
```

This is ok: only the call to an out of range singleton is prohibited; otherwise the permutation autosizes:

```
>>> Permutation(3)(1, 2)
Permutation([0, 2, 1, 3])
>>> Permutation(1, 2)(3, 4) == Permutation(3, 4)(1, 2)
True
```

Equality testing

The array forms must be the same in order for permutations to be equal:

```
>>> Permutation([1, 0, 2, 3]) == Permutation([1, 0])
False
```

Identity Permutation

The identity permutation is a permutation in which no element is out of place. It can be entered in a variety of ways. All the following create an identity permutation of size 4:

```
>>> I = Permutation([0, 1, 2, 3])
>>> all(p == I for p in [
... Permutation(3),
... Permutation(range(4)),
... Permutation([], size=4),
... Permutation(size=4)])
True
```

Watch out for entering the range *inside* a set of brackets (which is cycle notation):

```
>>> I == Permutation([range(4)])
False
```

Permutation Printing

There are a few things to note about how Permutations are printed.

1) If you prefer one form (array or cycle) over another, you can set that with the print_cyclic flag.

```
>>> Permutation(1, 2)(4, 5)(3, 4)
Permutation([0, 2, 1, 4, 5, 3])
>>> p = _
```

```
>>> Permutation.print_cyclic = True
>>> p
Permutation(1, 2)(3, 4, 5)
>>> Permutation.print_cyclic = False
```

2) Regardless of the setting, a list of elements in the array for cyclic form can be obtained and either of those can be copied and supplied as the argument to Permutation:

```
>>> p.array_form
[0, 2, 1, 4, 5, 3]
>>> p.cyclic_form
[[1, 2], [3, 4, 5]]
>>> Permutation(_) == p
True
```

3) Printing is economical in that as little as possible is printed while retaining all information about the size of the permutation:

```
>>> Permutation([1, 0, 2, 3])
Permutation([1, 0, 2, 3])
>>> Permutation([1, 0, 2, 3], size=20)
Permutation([1, 0], size=20)
>>> Permutation([1, 0, 2, 4, 3, 5, 6], size=20)
Permutation([1, 0, 2, 4, 3], size=20)
```

```
>>> p = Permutation([1, 0, 2, 3])
>>> Permutation.print_cyclic = True
>>> p
Permutation(3)(0, 1)
>>> Permutation.print_cyclic = False
```

The 2 was not printed but it is still there as can be seen with the array_form and size methods:

```
>>> p.array_form
[1, 0, 2, 3]
>>> p.size
4
```

Short introduction to other methods

The permutation can act as a bijective function, telling what element is located at a given position

```
>>> q = Permutation([5, 2, 3, 4, 1, 0])
>>> q.array_form[1] # the hard way
2
>>> q(1) # the easy way
2
>>> {i: q(i) for i in range(q.size)}  # showing the bijection
{0: 5, 1: 2, 2: 3, 3: 4, 4: 1, 5: 0}
```

The full cyclic form (including singletons) can be obtained:

```
>>> p.full_cyclic_form
[[0, 1], [2], [3]]
```

Any permutation can be factored into transpositions of pairs of elements:

```
>>> Permutation([[1, 2], [3, 4, 5]]).transpositions()
[(1, 2), (3, 5), (3, 4)]
>>> Permutation.rmul(*[Permutation([ti], size=6) for ti in _]).cyclic_form
[[1, 2], [3, 4, 5]]
```

The number of permutations on a set of n elements is given by n! and is called the cardinality.

```
>>> p.size
4
>>> p.cardinality
24
```

A given permutation has a rank among all the possible permutations of the same elements, but what that rank is depends on how the permutations are enumerated. (There are a number of different methods of doing so.) The lexicographic rank is given by the rank method and this rank is used to increment a permutation with addition/subtraction:

```
>>> p.rank()
6
>>> p + 1
Permutation([1, 0, 3, 2])
>>> p.next_lex()
Permutation([1, 0, 3, 2])
>>> _.rank()
7
>>> p.unrank_lex(p.size, rank=7)
Permutation([1, 0, 3, 2])
```

The product of two permutations p and q is defined as their composition as functions, (p*q)(i) = q(p(i)) *[R59]* (page 1250).

```
>>> p = Permutation([1, 0, 2, 3])
>>> q = Permutation([2, 3, 1, 0])
>>> list(q*p)
[2, 3, 0, 1]
>>> list(p*q)
[3, 2, 1, 0]
>>> [q(p(i)) for i in range(p.size)]
[3, 2, 1, 0]
```

The permutation can be 'applied' to any list-like object, not only Permutations:

```
>>> p(['zero', 'one', 'four', 'two'])
 ['one', 'zero', 'four', 'two']
>>> p('zo42')
['o', 'z', '4', '2']
```

If you have a list of arbitrary elements, the corresponding permutation can be found with the from_sequence method:

```
>>> Permutation.from_sequence('SymPy')
Permutation([1, 3, 2, 0, 4])
```

References

[R54] (page 1250), [R55] (page 1250), [R56] (page 1250), [R57] (page 1250), [R58] (page 1250), [R59] (page 1250), [R60] (page 1250)

array_form

Return a copy of the attribute _array_form Examples ========

```
>>> Permutation.print_cyclic = False
>>> p = Permutation([[2, 0], [3, 1]])
>>> p.array_form
[2, 3, 0, 1]
>>> Permutation([[2, 0, 3, 1]]).array_form
[3, 2, 0, 1]
>>> Permutation([2, 0, 3, 1]).array_form
[2, 0, 3, 1]
>>> Permutation([[1, 2], [4, 5]]).array_form
[0, 2, 1, 3, 5, 4]
```

ascents()

Returns the positions of ascents in a permutation, ie, the location where p[i] < p[i+1]

See also:

`descents` (page 167), `inversions` (page 171), `min` (page 175), `max` (page 174)

Examples

```
>>> p = Permutation([4, 0, 1, 3, 2])
>>> p.ascents()
[1, 2]
```

atoms()

Returns all the elements of a permutation

Examples

```
>>> Permutation([0, 1, 2, 3, 4, 5]).atoms()
{0, 1, 2, 3, 4, 5}
>>> Permutation([[0, 1], [2, 3], [4, 5]]).atoms()
{0, 1, 2, 3, 4, 5}
```

cardinality

Returns the number of all possible permutations.

See also:

`length` (page 174), `order` (page 176), `rank` (page 177), `size` (page 179)

Examples

```
>>> p = Permutation([0, 1, 2, 3])
>>> p.cardinality
24
```

commutator(*x*)
Return the commutator of self and x: `~x*~self*x*self`

If f and g are part of a group, G, then the commutator of f and g is the group identity iff f and g commute, i.e. fg == gf.

References

https://en.wikipedia.org/wiki/Commutator

Examples

```
>>> Permutation.print_cyclic = False
>>> p = Permutation([0, 2, 3, 1])
>>> x = Permutation([2, 0, 3, 1])
>>> c = p.commutator(x); c
Permutation([2, 1, 3, 0])
>>> c == ~x*~p*x*p
True
```

```
>>> I = Permutation(3)
>>> p = [I + i for i in range(6)]
>>> for i in range(len(p)):
...     for j in range(len(p)):
...         c = p[i].commutator(p[j])
...         if p[i]*p[j] == p[j]*p[i]:
...             assert c == I
...         else:
...             assert c != I
...
```

commutes_with(*other*)
Checks if the elements are commuting.

Examples

```
>>> a = Permutation([1, 4, 3, 0, 2, 5])
>>> b = Permutation([0, 1, 2, 3, 4, 5])
>>> a.commutes_with(b)
True
>>> b = Permutation([2, 3, 5, 4, 1, 0])
>>> a.commutes_with(b)
False
```

cycle_structure
Return the cycle structure of the permutation as a dictionary indicating the multiplicity of each cycle length.

Examples

```
>>> Permutation.print_cyclic = True
>>> Permutation(3).cycle_structure
{1: 4}
>>> Permutation(0, 4, 3)(1, 2)(5, 6).cycle_structure
{2: 2, 3: 1}
```

cycles

Returns the number of cycles contained in the permutation (including singletons).

See also:

diofant.functions.combinatorial.numbers.stirling (page 344)

Examples

```
>>> Permutation([0, 1, 2]).cycles
3
>>> Permutation([0, 1, 2]).full_cyclic_form
[[0], [1], [2]]
>>> Permutation(0, 1)(2, 3).cycles
2
```

cyclic_form

This is used to convert to the cyclic notation from the canonical notation. Singletons are omitted.

See also:

array_form (page 165), *full_cyclic_form* (page 168)

Examples

```
>>> Permutation.print_cyclic = False
>>> p = Permutation([0, 3, 1, 2])
>>> p.cyclic_form
[[1, 3, 2]]
>>> Permutation([1, 0, 2, 4, 3, 5]).cyclic_form
[[0, 1], [3, 4]]
```

descents()

Returns the positions of descents in a permutation, ie, the location where p[i] > p[i+1]

See also:

ascents (page 165), *inversions* (page 171), *min* (page 175), *max* (page 174)

Examples

```
>>> p = Permutation([4, 0, 1, 3, 2])
>>> p.descents()
[0, 3]
```

classmethod from_inversion_vector(*inversion*)

Calculates the permutation from the inversion vector.

Examples

```
>>> Permutation.print_cyclic = False
>>> Permutation.from_inversion_vector([3, 2, 1, 0, 0])
Permutation([3, 2, 1, 0, 4, 5])
```

classmethod from_sequence(*i*, *key=None*)

Return the permutation needed to obtain i from the sorted elements of i. If custom sorting is desired, a key can be given.

Examples

```
>>> Permutation.print_cyclic = True
```

```
>>> Permutation.from_sequence('SymPy')
Permutation(4)(0, 1, 3)
>>> _(sorted("SymPy"))
['S', 'y', 'm', 'P', 'y']
>>> Permutation.from_sequence('SymPy', key=lambda x: x.lower())
Permutation(4)(0, 2)(1, 3)
```

full_cyclic_form

Return permutation in cyclic form including singletons.

Examples

```
>>> Permutation([0, 2, 1]).full_cyclic_form
[[0], [1, 2]]
```

get_adjacency_distance(*other*)

Computes the adjacency distance between two permutations.

This metric counts the number of times a pair i,j of jobs is adjacent in both p and p'. If n_adj is this quantity then the adjacency distance is n - n_adj - 1 [1]

[1] Reeves, Colin R. Landscapes, Operators and Heuristic search, Annals of Operational Research, 86, pp 473-490. (1999)

See also:

get_precedence_matrix (page 170), *get_precedence_distance* (page 169), *get_adjacency_matrix* (page 169)

Examples

```
>>> p = Permutation([0, 3, 1, 2, 4])
>>> q = Permutation.josephus(4, 5, 2)
>>> p.get_adjacency_distance(q)
```

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```
3
>>> r = Permutation([0, 2, 1, 4, 3])
>>> p.get_adjacency_distance(r)
4
```

get_adjacency_matrix()

Computes the adjacency matrix of a permutation.

If job i is adjacent to job j in a permutation p then we set m[i, j] = 1 where m is the adjacency matrix of p.

See also:

get_precedence_matrix (page 170), *get_precedence_distance* (page 169), *get_adjacency_distance* (page 168)

Examples

```
>>> p = Permutation.josephus(3, 6, 1)
>>> p.get_adjacency_matrix()
Matrix([
[0, 0, 0, 0, 0, 0],
[0, 0, 0, 0, 1, 0],
[0, 0, 0, 0, 0, 1],
[0, 1, 0, 0, 0, 0],
[1, 0, 0, 0, 0, 0],
[0, 0, 0, 1, 0, 0]])
>>> q = Permutation([0, 1, 2, 3])
>>> q.get_adjacency_matrix()
Matrix([
[0, 1, 0, 0],
[0, 0, 1, 0],
[0, 0, 0, 1],
[0, 0, 0, 0]])
```

get_positional_distance(*other*)

Computes the positional distance between two permutations.

See also:

get_precedence_distance (page 169), *get_adjacency_distance* (page 168)

Examples

```
>>> p = Permutation([0, 3, 1, 2, 4])
>>> q = Permutation.josephus(4, 5, 2)
>>> r = Permutation([3, 1, 4, 0, 2])
>>> p.get_positional_distance(q)
12
>>> p.get_positional_distance(r)
12
```

get_precedence_distance(*other*)

Computes the precedence distance between two permutations.

Suppose p and p' represent n jobs. The precedence metric counts the number of times a job j is preceded by job i in both p and p'. This metric is commutative.

See also:

get_precedence_matrix (page 170), *get_adjacency_matrix* (page 169), *get_adjacency_distance* (page 168)

Examples

```
>>> p = Permutation([2, 0, 4, 3, 1])
>>> q = Permutation([3, 1, 2, 4, 0])
>>> p.get_precedence_distance(q)
7
>>> q.get_precedence_distance(p)
7
```

get_precedence_matrix()

Gets the precedence matrix. This is used for computing the distance between two permutations.

See also:

get_precedence_distance (page 169), *get_adjacency_matrix* (page 169), *get_adjacency_distance* (page 168)

Examples

```
>>> p = Permutation.josephus(3, 6, 1)
>>> Permutation.print_cyclic = False
>>> p
Permutation([2, 5, 3, 1, 4, 0])
>>> p.get_precedence_matrix()
Matrix([
[0, 0, 0, 0, 0, 0],
[1, 0, 0, 0, 1, 0],
[1, 1, 0, 1, 1, 1],
[1, 1, 0, 0, 1, 0],
[1, 0, 0, 0, 0, 0],
[1, 1, 0, 1, 1, 0]])
```

index()

Returns the index of a permutation.

The index of a permutation is the sum of all subscripts j such that p[j] is greater than p[j+1].

Examples

```
>>> p = Permutation([3, 0, 2, 1, 4])
>>> p.index()
2
```

inversion_vector()

Return the inversion vector of the permutation.

The inversion vector consists of elements whose value indicates the number of elements in the permutation that are lesser than it and lie on its right hand side.

The inversion vector is the same as the Lehmer encoding of a permutation.

See also:

from_inversion_vector (page 167)

Examples

```
>>> p = Permutation([4, 8, 0, 7, 1, 5, 3, 6, 2])
>>> p.inversion_vector()
[4, 7, 0, 5, 0, 2, 1, 1]
>>> p = Permutation([3, 2, 1, 0])
>>> p.inversion_vector()
[3, 2, 1]
```

The inversion vector increases lexicographically with the rank of the permutation, the -ith element cycling through 0..i.

```
>>> p = Permutation(2)
>>> Permutation.print_cyclic = False
>>> while p:
...     print('%s %s %s' % (str(p), p.inversion_vector(), p.rank()))
...     p = p.next_lex()
...
Permutation([0, 1, 2]) [0, 0] 0
Permutation([0, 2, 1]) [0, 1] 1
Permutation([1, 0, 2]) [1, 0] 2
Permutation([1, 2, 0]) [1, 1] 3
Permutation([2, 0, 1]) [2, 0] 4
Permutation([2, 1, 0]) [2, 1] 5
```

inversions()

Computes the number of inversions of a permutation.

An inversion is where i > j but p[i] < p[j].

For small length of p, it iterates over all i and j values and calculates the number of inversions. For large length of p, it uses a variation of merge sort to calculate the number of inversions.

See also:

descents (page 167), *ascents* (page 165), *min* (page 175), *max* (page 174)

References

[1] https://www.cp.eng.chula.ac.th/~piak/teaching/algo/algo2008/count-inv.htm

Examples

```
>>> p = Permutation([0, 1, 2, 3, 4, 5])
>>> p.inversions()
0
>>> Permutation([3, 2, 1, 0]).inversions()
6
```

is_Empty

Checks to see if the permutation is a set with zero elements

See also:

is_Singleton (page 172)

Examples

```
>>> Permutation([]).is_Empty
True
>>> Permutation([0]).is_Empty
False
```

is_Identity

Returns True if the Permutation is an identity permutation.

See also:

order (page 176)

Examples

```
>>> p = Permutation([])
>>> p.is_Identity
True
>>> p = Permutation([[0], [1], [2]])
>>> p.is_Identity
True
>>> p = Permutation([0, 1, 2])
>>> p.is_Identity
True
>>> p = Permutation([0, 2, 1])
>>> p.is_Identity
False
```

is_Singleton

Checks to see if the permutation contains only one number and is thus the only possible permutation of this set of numbers

See also:

is_Empty (page 172)

Examples

```
>>> Permutation([0]).is_Singleton
True
>>> Permutation([0, 1]).is_Singleton
False
```

is_even

Checks if a permutation is even.

See also:

is_odd (page 173)

Examples

```
>>> p = Permutation([0, 1, 2, 3])
>>> p.is_even
True
>>> p = Permutation([3, 2, 1, 0])
>>> p.is_even
True
```

is_odd

Checks if a permutation is odd.

See also:

is_even (page 173)

Examples

```
>>> p = Permutation([0, 1, 2, 3])
>>> p.is_odd
False
>>> p = Permutation([3, 2, 0, 1])
>>> p.is_odd
True
```

classmethod josephus(*m, n, s=1*)

Return as a permutation the shuffling of range(n) using the Josephus scheme in which every m-th item is selected until all have been chosen. The returned permutation has elements listed by the order in which they were selected.

The parameter `s` stops the selection process when there are `s` items remaining and these are selected by continuing the selection, counting by 1 rather than by `m`.

Consider selecting every 3rd item from 6 until only 2 remain:

```
choices    chosen
========   ======
  012345
  01 345   2
  01 34    25
  01  4    253
```

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```
 0   4    2531
 0        25314
          253140
```

References

[R61] (page 1250), [R62] (page 1251)

Examples

```
>>> Permutation.josephus(3, 6, 2).array_form
[2, 5, 3, 1, 4, 0]
```

length()

Returns the number of integers moved by a permutation.

See also:

min (page 175), *max* (page 174), *support* (page 179), *cardinality* (page 165), *order* (page 176), *rank* (page 177), *size* (page 179)

Examples

```
>>> Permutation([0, 3, 2, 1]).length()
2
>>> Permutation([[0, 1], [2, 3]]).length()
4
```

list(*size=None*)

Return the permutation as an explicit list, possibly trimming unmoved elements if size is less than the maximum element in the permutation; if this is desired, setting `size=-1` will guarantee such trimming.

Examples

```
>>> Permutation.print_cyclic = False
>>> p = Permutation(2, 3)(4, 5)
>>> p.list()
[0, 1, 3, 2, 5, 4]
>>> p.list(10)
[0, 1, 3, 2, 5, 4, 6, 7, 8, 9]
```

Passing a length too small will trim trailing, unchanged elements in the permutation:

```
>>> Permutation(2, 4)(1, 2, 4).list(-1)
[0, 2, 1]
>>> Permutation(3).list(-1)
[]
```

max()

The maximum element moved by the permutation.

See also:

min (page 175), *descents* (page 167), *ascents* (page 165), *inversions* (page 171)

Examples

```
>>> p = Permutation([1, 0, 2, 3, 4])
>>> p.max()
1
```

min()

The minimum element moved by the permutation.

See also:

max (page 174), *descents* (page 167), *ascents* (page 165), *inversions* (page 171)

Examples

```
>>> p = Permutation([0, 1, 4, 3, 2])
>>> p.min()
2
```

mul_inv(*other*)

other*~self, self and other have _array_form

next_lex()

Returns the next permutation in lexicographical order. If self is the last permutation in lexicographical order it returns None. See [4] section 2.4.

See also:

rank (page 177), *unrank_lex* (page 180)

Examples

```
>>> p = Permutation([2, 3, 1, 0])
>>> p = Permutation([2, 3, 1, 0]); p.rank()
17
>>> p = p.next_lex(); p.rank()
18
```

next_nonlex()

Returns the next permutation in nonlex order [3]. If self is the last permutation in this order it returns None.

See also:

rank_nonlex (page 177), *unrank_nonlex* (page 180)

Examples

```
>>> Permutation.print_cyclic = False
>>> p = Permutation([2, 0, 3, 1]); p.rank_nonlex()
5
>>> p = p.next_nonlex(); p
Permutation([3, 0, 1, 2])
>>> p.rank_nonlex()
6
```

next_trotterjohnson()

Returns the next permutation in Trotter-Johnson order. If self is the last permutation it returns None. See [4] section 2.4. If it is desired to generate all such permutations, they can be generated in order more quickly with the `generate_bell` function.

See also:

rank_trotterjohnson (page 177), *unrank_trotterjohnson* (page 181), *diofant.utilities.iterables.generate_bell* (page 984)

Examples

```
>>> Permutation.print_cyclic = False
>>> p = Permutation([3, 0, 2, 1])
>>> p.rank_trotterjohnson()
4
>>> p = p.next_trotterjohnson(); p
Permutation([0, 3, 2, 1])
>>> p.rank_trotterjohnson()
5
```

order()

Computes the order of a permutation.

When the permutation is raised to the power of its order it equals the identity permutation.

See also:

is_Identity (page 172), *cardinality* (page 165), *length* (page 174), *rank* (page 177), *size* (page 179)

Examples

```
>>> Permutation.print_cyclic = False
>>> p = Permutation([3, 1, 5, 2, 4, 0])
>>> p.order()
4
>>> (p**(p.order()))
Permutation([], size=6)
```

parity()

Computes the parity of a permutation.

The parity of a permutation reflects the parity of the number of inversions in the permutation, i.e., the number of pairs of x and y such that `x > y` but `p[x] < p[y]`.

See also:

_af_parity (page 182)

Examples

```
>>> p = Permutation([0, 1, 2, 3])
>>> p.parity()
0
>>> p = Permutation([3, 2, 0, 1])
>>> p.parity()
1
```

classmethod random(*n*)

Generates a random permutation of length n.

Uses the underlying Python pseudo-random number generator.

Examples

```
>>> Permutation.random(2) in (Permutation([1, 0]), Permutation([0, 1]))
True
```

rank()

Returns the lexicographic rank of the permutation.

See also:

next_lex (page 175), *unrank_lex* (page 180), *cardinality* (page 165), *length* (page 174), *order* (page 176), *size* (page 179)

Examples

```
>>> p = Permutation([0, 1, 2, 3])
>>> p.rank()
0
>>> p = Permutation([3, 2, 1, 0])
>>> p.rank()
23
```

rank_nonlex(*inv_perm=None*)

This is a linear time ranking algorithm that does not enforce lexicographic order [3].

See also:

next_nonlex (page 175), *unrank_nonlex* (page 180)

Examples

```
>>> p = Permutation([0, 1, 2, 3])
>>> p.rank_nonlex()
23
```

rank_trotterjohnson()

Returns the Trotter Johnson rank, which we get from the minimal change algorithm. See [4] section 2.4.

See also:

unrank_trotterjohnson (page 181), *next_trotterjohnson* (page 176)

Examples

```
>>> p = Permutation([0, 1, 2, 3])
>>> p.rank_trotterjohnson()
0
>>> p = Permutation([0, 2, 1, 3])
>>> p.rank_trotterjohnson()
7
```

static rmul()

Return product of Permutations [a, b, c, …] as the Permutation whose ith value is a(b(c(i))).

a, b, c, … can be Permutation objects or tuples.

Notes

All items in the sequence will be parsed by Permutation as necessary as long as the first item is a Permutation:

```
>>> Permutation.rmul(a, [0, 2, 1]) == Permutation.rmul(a, b)
True
```

The reverse order of arguments will raise a TypeError.

Examples

```
>>> Permutation.print_cyclic = False
```

```
>>> a, b = [1, 0, 2], [0, 2, 1]
>>> a = Permutation(a); b = Permutation(b)
>>> list(Permutation.rmul(a, b))
[1, 2, 0]
>>> [a(b(i)) for i in range(3)]
[1, 2, 0]
```

This handles the operands in reverse order compared to the * operator:

```
>>> a = Permutation(a); b = Permutation(b)
>>> list(a*b)
[2, 0, 1]
>>> [b(a(i)) for i in range(3)]
[2, 0, 1]
```

static rmul_with_af()
same as rmul, but the elements of args are Permutation objects which have _array_form

runs()
Returns the runs of a permutation.

An ascending sequence in a permutation is called a run [5].

Examples

```
>>> p = Permutation([2, 5, 7, 3, 6, 0, 1, 4, 8])
>>> p.runs()
[[2, 5, 7], [3, 6], [0, 1, 4, 8]]
>>> q = Permutation([1, 3, 2, 0])
>>> q.runs()
[[1, 3], [2], [0]]
```

signature()
Gives the signature of the permutation needed to place the elements of the permutation in canonical order.

The signature is calculated as (-1)^<number of inversions>

See also:

inversions (page 171)

Examples

```
>>> p = Permutation([0, 1, 2])
>>> p.inversions()
0
>>> p.signature()
1
>>> q = Permutation([0, 2, 1])
>>> q.inversions()
1
>>> q.signature()
-1
```

size
Returns the number of elements in the permutation.

See also:

cardinality (page 165), *length* (page 174), *order* (page 176), *rank* (page 177)

Examples

```
>>> Permutation([[3, 2], [0, 1]]).size
4
```

support()
Return the elements in permutation, P, for which P[i] != i.

Examples

```
>>> p = Permutation([[3, 2], [0, 1], [4]])
>>> p.array_form
[1, 0, 3, 2, 4]
>>> p.support()
[0, 1, 2, 3]
```

transpositions()

Return the permutation decomposed into a list of transpositions.

It is always possible to express a permutation as the product of transpositions, see [1]

References

1. https://en.wikipedia.org/wiki/Transposition_%28mathematics%29#Properties

Examples

```
>>> p = Permutation([[1, 2, 3], [0, 4, 5, 6, 7]])
>>> t = p.transpositions()
>>> t
[(0, 7), (0, 6), (0, 5), (0, 4), (1, 3), (1, 2)]
>>> print(''.join(str(c) for c in t))
(0, 7)(0, 6)(0, 5)(0, 4)(1, 3)(1, 2)
>>> Permutation.rmul(*[Permutation([ti], size=p.size) for ti in t]) == p
True
```

classmethod unrank_lex(*size*, *rank*)

Lexicographic permutation unranking.

See also:

rank (page 177), *next_lex* (page 175)

Examples

```
>>> Permutation.print_cyclic = False
>>> a = Permutation.unrank_lex(5, 10)
>>> a.rank()
10
>>> a
Permutation([0, 2, 4, 1, 3])
```

classmethod unrank_nonlex(*n*, *r*)

This is a linear time unranking algorithm that does not respect lexicographic order [3].

See also:

next_nonlex (page 175), *rank_nonlex* (page 177)

Examples

```
>>> Permutation.print_cyclic = False
>>> Permutation.unrank_nonlex(4, 5)
Permutation([2, 0, 3, 1])
>>> Permutation.unrank_nonlex(4, -1)
Permutation([0, 1, 2, 3])
```

classmethod unrank_trotterjohnson(*size*, *rank*)

Trotter Johnson permutation unranking. See [4] section 2.4.

See also:

rank_trotterjohnson (page 177), *next_trotterjohnson* (page 176)

Examples

```
>>> Permutation.unrank_trotterjohnson(5, 10)
Permutation([0, 3, 1, 2, 4])
```

class diofant.combinatorics.permutations.**Cycle**(**args*)

Wrapper around dict which provides the functionality of a disjoint cycle.

A cycle shows the rule to use to move subsets of elements to obtain a permutation. The Cycle class is more flexible than Permutation in that 1) all elements need not be present in order to investigate how multiple cycles act in sequence and 2) it can contain singletons:

A Cycle will automatically parse a cycle given as a tuple on the rhs:

```
>>> Cycle(1, 2)(2, 3)
Cycle(1, 3, 2)
```

The identity cycle, Cycle(), can be used to start a product:

```
>>> Cycle()(1, 2)(2, 3)
Cycle(1, 3, 2)
```

The array form of a Cycle can be obtained by calling the list method (or passing it to the list function) and all elements from 0 will be shown:

```
>>> a = Cycle(1, 2)
>>> a.list()
[0, 2, 1]
>>> list(a)
[0, 2, 1]
```

If a larger (or smaller) range is desired use the list method and provide the desired size – but the Cycle cannot be truncated to a size smaller than the largest element that is out of place:

```
>>> b = Cycle(2, 4)(1, 2)(3, 1, 4)(1, 3)
>>> b.list()
[0, 2, 1, 3, 4]
>>> b.list(b.size + 1)
[0, 2, 1, 3, 4, 5]
>>> b.list(-1)
[0, 2, 1]
```

Singletons are not shown when printing with one exception: the largest element is always shown – as a singleton if necessary:

```
>>> Cycle(1, 4, 10)(4, 5)
Cycle(1, 5, 4, 10)
>>> Cycle(1, 2)(4)(5)(10)
Cycle(1, 2)(10)
```

The array form can be used to instantiate a Permutation so other properties of the permutation can be investigated:

```
>>> Perm(Cycle(1, 2)(3, 4).list()).transpositions()
[(1, 2), (3, 4)]
```

See also:

Permutation (page 160)

Notes

The underlying structure of the Cycle is a dictionary and although the __iter__ method has been redefined to give the array form of the cycle, the underlying dictionary items are still available with the such methods as items():

```
>>> list(Cycle(1, 2).items())
[(1, 2), (2, 1)]
```

copy() → a shallow copy of D

list(*size=None*)

Return the cycles as an explicit list starting from 0 up to the greater of the largest value in the cycles and size.

Truncation of trailing unmoved items will occur when size is less than the maximum element in the cycle; if this is desired, setting `size=-1` will guarantee such trimming.

Examples

```
>>> Permutation.print_cyclic = False
>>> p = Cycle(2, 3)(4, 5)
>>> p.list()
[0, 1, 3, 2, 5, 4]
>>> p.list(10)
[0, 1, 3, 2, 5, 4, 6, 7, 8, 9]
```

Passing a length too small will trim trailing, unchanged elements in the permutation:

```
>>> Cycle(2, 4)(1, 2, 4).list(-1)
[0, 2, 1]
```

diofant.combinatorics.permutations.**_af_parity**(*pi*)

Computes the parity of a permutation in array form.

The parity of a permutation reflects the parity of the number of inversions in the permutation, i.e., the number of pairs of x and y such that x > y but p[x] < p[y].

See also:

Permutation (page 160)

Examples

```
>>> _af_parity([0, 1, 2, 3])
0
>>> _af_parity([3, 2, 0, 1])
1
```

Generators

generators.**symmetric**()

Generates the symmetric group of order n, Sn.

Examples

```
>>> Permutation.print_cyclic = True
>>> list(symmetric(3))
[Permutation(2), Permutation(1, 2), Permutation(2)(0, 1),
 Permutation(0, 1, 2), Permutation(0, 2, 1), Permutation(0, 2)]
```

generators.**cyclic**()

Generates the cyclic group of order n, Cn.

See also:

dihedral (page 183)

Examples

```
>>> Permutation.print_cyclic = True
>>> list(cyclic(5))
[Permutation(4), Permutation(0, 1, 2, 3, 4), Permutation(0, 2, 4, 1, 3),
 Permutation(0, 3, 1, 4, 2), Permutation(0, 4, 3, 2, 1)]
```

generators.**alternating**()

Generates the alternating group of order n, An.

Examples

```
>>> Permutation.print_cyclic = True
>>> list(alternating(3))
[Permutation(2), Permutation(0, 1, 2), Permutation(0, 2, 1)]
```

generators.**dihedral**()

Generates the dihedral group of order 2n, Dn.

The result is given as a subgroup of Sn, except for the special cases n=1 (the group S2) and n=2 (the Klein 4-group) where that's not possible and embeddings in S2 and S4 respectively are given.

See also:

cyclic (page 183)

Examples

```
>>> Permutation.print_cyclic = True
>>> list(dihedral(3))
[Permutation(2), Permutation(0, 2), Permutation(0, 1, 2),
 Permutation(1, 2), Permutation(0, 2, 1), Permutation(2)(0, 1)]
```

3.2.3 Permutation Groups

class diofant.combinatorics.perm_groups.**PermutationGroup**

The class defining a Permutation group.

PermutationGroup([p1, p2, ..., pn]) returns the permutation group generated by the list of permutations. This group can be supplied to Polyhedron if one desires to decorate the elements to which the indices of the permutation refer.

See also:

diofant.combinatorics.polyhedron.Polyhedron (page 211), *diofant.combinatorics.permutations.Permutation* (page 160)

References

[1] Holt, D., Eick, B., O'Brien, E. "Handbook of Computational Group Theory"

[2] Seress, A. "Permutation Group Algorithms"

[3] https://en.wikipedia.org/wiki/Schreier_vector

[4] https://en.wikipedia.org/wiki/Nielsen_transformation #Product_replacement_algorithm

[5] Frank Celler, Charles R.Leedham-Green, Scott H.Murray, Alice C.Niemeyer, and E.A.O'Brien. "Generating Random Elements of a Finite Group"

[6] https://en.wikipedia.org/wiki/Block_%28permutation_group_theory%29

[7] https://web.archive.org/web/20170105021515/http://www.algorithmist.com:80/index.php/Union_Find

[8] https://en.wikipedia.org/wiki/Multiply_transitive_group#Multiply_transitive_groups

[9] https://en.wikipedia.org/wiki/Center_%28group_theory%29

[10] https://en.wikipedia.org/wiki/Centralizer_and_normalizer

[11] https://groupprops.subwiki.org/wiki/Derived_subgroup

[12] https://en.wikipedia.org/wiki/Nilpotent_group

[13] http://www.math.colostate.edu/~hulpke/CGT/cgtnotes.pdf

Examples

```
>>> Permutation.print_cyclic = True
```

The permutations corresponding to motion of the front, right and bottom face of a 2x2 Rubik's cube are defined:

```
>>> F = Permutation(2, 19, 21, 8)(3, 17, 20, 10)(4, 6, 7, 5)
>>> R = Permutation(1, 5, 21, 14)(3, 7, 23, 12)(8, 10, 11, 9)
>>> D = Permutation(6, 18, 14, 10)(7, 19, 15, 11)(20, 22, 23, 21)
```

These are passed as permutations to PermutationGroup:

```
>>> G = PermutationGroup(F, R, D)
>>> G.order()
3674160
```

The group can be supplied to a Polyhedron in order to track the objects being moved. An example involving the 2x2 Rubik's cube is given there, but here is a simple demonstration:

```
>>> a = Permutation(2, 1)
>>> b = Permutation(1, 0)
>>> G = PermutationGroup(a, b)
>>> P = Polyhedron(list('ABC'), pgroup=G)
>>> P.corners
(A, B, C)
>>> P.rotate(0)  # apply permutation 0
>>> P.corners
(A, C, B)
>>> P.reset()
>>> P.corners
(A, B, C)
```

Or one can make a permutation as a product of selected permutations and apply them to an iterable directly:

```
>>> P10 = G.make_perm([0, 1])
>>> P10('ABC')
['C', 'A', 'B']
```

__contains__(*i*)
Return True if i is contained in PermutationGroup.

Examples

```
>>> p = Permutation(1, 2, 3)
>>> Permutation(3) in PermutationGroup(p)
True
```

__eq__(*other*)
Return True if PermutationGroup generated by elements in the group are same i.e they represent the same PermutationGroup.

Examples

```
>>> p = Permutation(0, 1, 2, 3, 4, 5)
>>> G = PermutationGroup([p, p**2])
>>> H = PermutationGroup([p**2, p])
>>> G.generators == H.generators
False
>>> G == H
True
```

__hash__()
Return hash(self).

__mul__(*other*)
Return the direct product of two permutation groups as a permutation group.

This implementation realizes the direct product by shifting the index set for the generators of the second group: so if we have G acting on n1 points and H acting on n2 points, G*H acts on n1 + n2 points.

Examples

```
>>> G = CyclicGroup(5)
>>> H = G*G
>>> H
PermutationGroup([
    Permutation(9)(0, 1, 2, 3, 4),
    Permutation(5, 6, 7, 8, 9)])
>>> H.order()
25
```

static __new__(**args*, ***kwargs*)
The default constructor. Accepts Cycle and Permutation forms. Removes duplicates unless `dups` keyword is False.

_random_pr_init(*r*, *n*, *_random_prec_n=None*)
Initialize random generators for the product replacement algorithm.

The implementation uses a modification of the original product replacement algorithm due to Leedham-Green, as described in [1], pp. 69-71; also, see [2], pp. 27-29 for a detailed theoretical analysis of the original product replacement algorithm, and [4].

The product replacement algorithm is used for producing random, uniformly distributed elements of a group `G` with a set of generators `S`. For the initialization `_random_pr_init`, a list `R` of `\max\{r, |S|\}` group generators is created as the attribute `G._random_gens`, repeating elements of `S` if necessary, and the identity element of `G` is appended to `R` - we shall refer to this last element as the accumulator. Then the function `random_pr()` is called `n` times, randomizing the list `R` while preserving the generation of `G` by `R`. The function `random_pr()` itself takes two random elements `g, h` among all elements of `R` but the accumulator and replaces `g` with a randomly chosen element from `\{gh, g(~h), hg, (~h)g\}`. Then the accumulator is multiplied by whatever `g` was replaced by. The new value of the accumulator is then returned by `random_pr()`.

The elements returned will eventually (for n large enough) become uniformly distributed across G ([5]). For practical purposes however, the values n = 50, r = 11 are suggested in [1].

See also:

random_pr (page 206)

Notes

THIS FUNCTION HAS SIDE EFFECTS: it changes the attribute self._random_gens

_union_find_merge(*first*, *second*, *ranks*, *parents*, *not_rep*)

Merges two classes in a union-find data structure.

Used in the implementation of Atkinson's algorithm as suggested in [1], pp. 83-87. The class merging process uses union by rank as an optimization. ([7])

See also:

minimal_block (page 202), *_union_find_rep* (page 187)

Notes

THIS FUNCTION HAS SIDE EFFECTS: the list of class representatives, `parents`, the list of class sizes, `ranks`, and the list of elements that are not representatives, `not_rep`, are changed due to class merging.

References

[1] Holt, D., Eick, B., O'Brien, E. "Handbook of computational group theory"

[7] https://web.archive.org/web/20170105021515/http://www.algorithmist.com:80/index.php/Union_Find

_union_find_rep(*num*, *parents*)

Find representative of a class in a union-find data structure.

Used in the implementation of Atkinson's algorithm as suggested in [1], pp. 83-87. After the representative of the class to which `num` belongs is found, path compression is performed as an optimization ([7]).

See also:

minimal_block (page 202), *_union_find_merge* (page 187)

Notes

THIS FUNCTION HAS SIDE EFFECTS: the list of class representatives, `parents`, is altered due to path compression.

References

[1] Holt, D., Eick, B., O'Brien, E. "Handbook of computational group theory"

[7] https://web.archive.org/web/20170105021515/http://www.algorithmist.com:80/index.php/Union_Find

base

Return a base from the Schreier-Sims algorithm.

For a permutation group `G`, a base is a sequence of points `B = (b_1, b_2, ..., b_k)` such that no element of `G` apart from the identity fixes all the points in `B`. The concepts of a base and strong generating set and their applications are discussed in depth in [1], pp. 87-89 and [2], pp. 55-57.

An alternative way to think of `B` is that it gives the indices of the stabilizer cosets that contain more than the identity permutation.

See also:

strong_gens (page 209), *basic_transversals* (page 190), *basic_orbits* (page 189), *basic_stabilizers* (page 189)

Examples

```
>>> G = PermutationGroup([Permutation(0, 1, 3)(2, 4)])
>>> G.base
[0, 2]
```

baseswap(*base*, *strong_gens*, *pos*, *randomized=False*, *transversals=None*, *basic_orbits=None*, *strong_gens_distr=None*)

Swap two consecutive base points in base and strong generating set.

If a base for a group `G` is given by `(b_1, b_2, ..., b_k)`, this function returns a base `(b_1, b_2, ..., b_{i+1}, b_i, ..., b_k)`, where `i` is given by `pos`, and a strong generating set relative to that base. The original base and strong generating set are not modified.

The randomized version (default) is of Las Vegas type.

Parameters base, strong_gens

The base and strong generating set.

pos

The position at which swapping is performed.

randomized

A switch between randomized and deterministic version.

transversals

The transversals for the basic orbits, if known.

basic_orbits

The basic orbits, if known.

strong_gens_distr

The strong generators distributed by basic stabilizers, if known.

Returns (base, strong_gens)

`base` is the new base, and `strong_gens` is a generating set relative to it.

See also:

schreier_sims (page 206)

Notes

The deterministic version of the algorithm is discussed in [1], pp. 102-103; the randomized version is discussed in [1], p.103, and [2], p.98. It is of Las Vegas type. Notice that [1] contains a mistake in the pseudocode and discussion of BASESWAP: on line 3 of the pseudocode, `|\beta_{i+1}^{\left\langle T\right\rangle}|` should be replaced by `|\beta_{i}^{\left\langle T\right\rangle}|`, and the same for the discussion of the algorithm.

Examples

```
>>> from diofant.combinatorics.testutil import _verify_bsgs
>>> S = SymmetricGroup(4)
>>> S.schreier_sims()
>>> S.base
[0, 1, 2]
>>> base, gens = S.baseswap(S.base, S.strong_gens, 1, randomized=False)
>>> base, gens
([0, 2, 1],
[Permutation(0, 1, 2, 3), Permutation(3)(0, 1), Permutation(1, 3, 2),
 Permutation(2, 3), Permutation(1, 3)])
```

check that base, gens is a BSGS

```
>>> S1 = PermutationGroup(gens)
>>> _verify_bsgs(S1, base, gens)
True
```

basic_orbits

Return the basic orbits relative to a base and strong generating set.

If `(b_1, b_2, ..., b_k)` is a base for a group `G`, and `G^{(i)} = G_{b_1, b_2, ..., b_{i-1}}` is the `i-th` basic stabilizer (so that `G^{(1)} = G`), the `i-th` basic orbit relative to this base is the orbit of `b_i` under `G^{(i)}`. See [1], pp. 87-89 for more information.

See also:

base (page 188), *strong_gens* (page 209), *basic_transversals* (page 190), *basic_stabilizers* (page 189)

Examples

```
>>> S = SymmetricGroup(4)
>>> S.basic_orbits
[[0, 1, 2, 3], [1, 2, 3], [2, 3]]
```

basic_stabilizers

Return a chain of stabilizers relative to a base and strong generating set.

The i-th basic stabilizer `G^{(i)}` relative to a base `(b_1, b_2, ..., b_k)` is `G_{b_1, b_2, ..., b_{i-1}}`. For more information, see [1], pp. 87-89.

See also:

base (page 188), *strong_gens* (page 209), *basic_orbits* (page 189), *basic_transversals* (page 190)

Examples

```
>>> A = AlternatingGroup(4)
>>> A.schreier_sims()
>>> A.base
[0, 1]
>>> for g in A.basic_stabilizers:
...     print(g)
...
PermutationGroup([
    Permutation(3)(0, 1, 2),
    Permutation(1, 2, 3)])
PermutationGroup([
    Permutation(1, 2, 3)])
```

basic_transversals

Return basic transversals relative to a base and strong generating set.

The basic transversals are transversals of the basic orbits. They are provided as a list of dictionaries, each dictionary having keys - the elements of one of the basic orbits, and values - the corresponding transversal elements. See [1], pp. 87-89 for more information.

See also:

strong_gens (page 209), *base* (page 188), *basic_orbits* (page 189), *basic_stabilizers* (page 189)

Examples

```
>>> A = AlternatingGroup(4)
>>> A.basic_transversals
[{0: Permutation(3),
  1: Permutation(3)(0, 1, 2),
  2: Permutation(3)(0, 2, 1),
  3: Permutation(0, 3, 1)},
 {1: Permutation(3),
  2: Permutation(1, 2, 3),
  3: Permutation(1, 3, 2)}]
```

center()

Return the center of a permutation group.

The center for a group `G` is defined as `Z(G) = \{z\in G | \forall g\in G, zg = gz \}`, the set of elements of `G` that commute with all elements of `G`. It is equal to the centralizer of `G` inside `G`, and is naturally a subgroup of `G` ([9]).

See also:

centralizer (page 191)

Notes

This is a naive implementation that is a straightforward application of .centralizer()

Examples

```
>>> D = DihedralGroup(4)
>>> G = D.center()
>>> G.order()
2
```

centralizer(*other*)

Return the centralizer of a group/set/element.

The centralizer of a set of permutations S inside a group G is the set of elements of G that commute with all elements of S:

```
``C_G(S) = \{ g \in G | gs = sg \forall s \in S\}`` ([10])
```

Usually, S is a subset of G, but if G is a proper subgroup of the full symmetric group, we allow for S to have elements outside G.

It is naturally a subgroup of G; the centralizer of a permutation group is equal to the centralizer of any set of generators for that group, since any element commuting with the generators commutes with any product of the generators.

Parameters other

a permutation group/list of permutations/single permutation

See also:

subgroup_search (page 210)

Notes

The implementation is an application of `.subgroup_search()` with tests using a specific base for the group G.

Examples

```
>>> S = SymmetricGroup(6)
>>> C = CyclicGroup(6)
>>> H = S.centralizer(C)
>>> H.is_subgroup(C)
True
```

commutator(*G*, *H*)
Return the commutator of two subgroups.

For a permutation group `K` and subgroups `G`, `H`, the commutator of `G` and `H` is defined as the group generated by all the commutators `[g, h] = hgh^{-1}g^{-1}` for `g` in `G` and `h` in `H`. It is naturally a subgroup of `K` ([1], p.27).

See also:

derived_subgroup (page 195)

Notes

The commutator of two subgroups `H, G` is equal to the normal closure of the commutators of all the generators, i.e. `hgh^{-1}g^{-1}` for `h` a generator of `H` and `g` a generator of `G` ([1], p.28)

Examples

```
>>> S = SymmetricGroup(5)
>>> A = AlternatingGroup(5)
>>> G = S.commutator(S, A)
>>> G.is_subgroup(A)
True
```

contains(*g*, *strict=True*)
Test if permutation `g` belong to self, `G`.

If `g` is an element of `G` it can be written as a product of factors drawn from the cosets of `G`'s stabilizers. To see if `g` is one of the actual generators defining the group use `G.has(g)`.

If `strict` is not True, `g` will be resized, if necessary, to match the size of permutations in `self`.

See also:

coset_factor (page 193), *diofant.core.basic.Basic.has* (page 45)

Examples

```
>>> Permutation.print_cyclic = True
```

```
>>> a = Permutation(1, 2)
>>> b = Permutation(2, 3, 1)
>>> G = PermutationGroup(a, b, degree=5)
>>> G.contains(G[0])  # trivial check
True
>>> elem = Permutation([[2, 3]], size=5)
>>> G.contains(elem)
True
>>> G.contains(Permutation(4)(0, 1, 2, 3))
False
```

If strict is False, a permutation will be resized, if necessary:

```
>>> H = PermutationGroup(Permutation(5))
>>> H.contains(Permutation(3))
False
>>> H.contains(Permutation(3), strict=False)
True
```

To test if a given permutation is present in the group:

```
>>> elem in G.generators
False
>>> G.has(elem)
False
```

coset_factor(*g*, *factor_index=False*)

Return G's (self's) coset factorization of g

If g is an element of G then it can be written as the product of permutations drawn from the Schreier-Sims coset decomposition,

The permutations returned in f are those for which the product gives g: `g = f[n]*...f[1]*f[0]` where `n = len(B)` and `B = G.base`. f[i] is one of the permutations in `self._basic_orbits[i]`.

If factor_index==True, returns a tuple `[b[0],..,b[n]]`, where `b[i]` belongs to `self._basic_orbits[i]`

Examples

```
>>> Permutation.print_cyclic = True
>>> a = Permutation(0, 1, 3, 7, 6, 4)(2, 5)
>>> b = Permutation(0, 1, 3, 2)(4, 5, 7, 6)
>>> G = PermutationGroup([a, b])
```

Define g:

```
>>> g = Permutation(7)(1, 2, 4)(3, 6, 5)
```

Confirm that it is an element of G:

```
>>> G.contains(g)
True
```

Thus, it can be written as a product of factors (up to 3) drawn from u. See below that a factor from u1 and u2 and the Identity permutation have been used:

```
>>> f = G.coset_factor(g)
>>> f[2]*f[1]*f[0] == g
True
>>> f1 = G.coset_factor(g, True); f1
[0, 4, 4]
>>> tr = G.basic_transversals
>>> f[0] == tr[0][f1[0]]
True
```

If g is not an element of G then [] is returned:

```
>>> c = Permutation(5, 6, 7)
>>> G.coset_factor(c)
[]
```

see util._strip

coset_rank(*g*)
rank using Schreier-Sims representation

The coset rank of g is the ordering number in which it appears in the lexicographic listing according to the coset decomposition

The ordering is the same as in G.generate(method='coset'). If g does not belong to the group it returns None.

See also:

coset_factor (page 193)

Examples

```
>>> Permutation.print_cyclic = True
>>> a = Permutation(0, 1, 3, 7, 6, 4)(2, 5)
>>> b = Permutation(0, 1, 3, 2)(4, 5, 7, 6)
>>> G = PermutationGroup([a, b])
>>> c = Permutation(7)(2, 4)(3, 5)
>>> G.coset_rank(c)
16
>>> G.coset_unrank(16)
Permutation(7)(2, 4)(3, 5)
```

coset_unrank(*rank, af=False*)
unrank using Schreier-Sims representation

coset_unrank is the inverse operation of coset_rank if 0 <= rank < order; otherwise it returns None.

degree
Returns the size of the permutations in the group.

The number of permutations comprising the group is given by len(group); the number of permutations that can be generated by the group is given by group.order().

See also:

order (page 205)

Examples

```
>>> Permutation.print_cyclic = True
>>> a = Permutation([1, 0, 2])
>>> G = PermutationGroup([a])
>>> G.degree
3
>>> len(G)
1
>>> G.order()
```

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```
2
>>> list(G.generate())
[Permutation(2), Permutation(2)(0, 1)]
```

derived_series()

Return the derived series for the group.

The derived series for a group G is defined as G = G_0 > G_1 > G_2 > \ldots where G_i = [G_{i-1}, G_{i-1}], i.e. G_i is the derived subgroup of G_{i-1}, for i\in\mathbb{N}. When we have G_k = G_{k-1} for some k\in\mathbb{N}, the series terminates.

Returns A list of permutation groups containing the members of the derived series in the order G = G_0, G_1, G_2, \ldots.

See also:

derived_subgroup (page 195)

Examples

```
>>> A = AlternatingGroup(5)
>>> len(A.derived_series())
1
>>> S = SymmetricGroup(4)
>>> len(S.derived_series())
4
>>> S.derived_series()[1].is_subgroup(AlternatingGroup(4))
True
>>> S.derived_series()[2].is_subgroup(DihedralGroup(2))
True
```

derived_subgroup()

Compute the derived subgroup.

The derived subgroup, or commutator subgroup is the subgroup generated by all commutators [g, h] = hgh^{-1}g^{-1} for g, h\in G ; it is equal to the normal closure of the set of commutators of the generators ([1], p.28, [11]).

See also:

derived_series (page 195)

Examples

```
>>> Permutation.print_cyclic = True
>>> a = Permutation([1, 0, 2, 4, 3])
>>> b = Permutation([0, 1, 3, 2, 4])
>>> G = PermutationGroup([a, b])
>>> C = G.derived_subgroup()
>>> list(C.generate(af=True))
[[0, 1, 2, 3, 4], [0, 1, 3, 4, 2], [0, 1, 4, 2, 3]]
```

elements

Returns all the elements of the permutation group in a list

generate(*method='coset'*, *af=False*)

Return iterator to generate the elements of the group

Iteration is done with one of these methods:

```
method='coset'  using the Schreier-Sims coset representation
method='dimino' using the Dimino method
```

If af = True it yields the array form of the permutations

Examples

```
>>> Permutation.print_cyclic = True
```

The permutation group given in the tetrahedron object is also true groups:

```
>>> G = tetrahedron.pgroup
>>> G.is_group
True
```

Also the group generated by the permutations in the tetrahedron pgroup – even the first two – is a proper group:

```
>>> H = PermutationGroup(G[0], G[1])
>>> J = PermutationGroup(list(H.generate())); J
PermutationGroup([
    Permutation(0, 1)(2, 3),
    Permutation(3),
    Permutation(1, 2, 3),
    Permutation(1, 3, 2),
    Permutation(0, 3, 1),
    Permutation(0, 2, 3),
    Permutation(0, 3)(1, 2),
    Permutation(0, 1, 3),
    Permutation(3)(0, 2, 1),
    Permutation(0, 3, 2),
    Permutation(3)(0, 1, 2),
    Permutation(0, 2)(1, 3)])
>>> _.is_group
True
```

generate_dimino(*af=False*)

Yield group elements using Dimino's algorithm

If af == True it yields the array form of the permutations

References

[1] The Implementation of Various Algorithms for Permutation Groups in the Computer Algebra System: AXIOM, N.J. Doye, M.Sc. Thesis

Examples

```
>>> Permutation.print_cyclic = True
>>> a = Permutation([0, 2, 1, 3])
>>> b = Permutation([0, 2, 3, 1])
>>> g = PermutationGroup([a, b])
>>> list(g.generate_dimino(af=True))
[[0, 1, 2, 3], [0, 2, 1, 3], [0, 2, 3, 1],
 [0, 1, 3, 2], [0, 3, 2, 1], [0, 3, 1, 2]]
```

generate_schreier_sims(*af=False*)
Yield group elements using the Schreier-Sims representation in coset_rank order

If af = True it yields the array form of the permutations

Examples

```
>>> Permutation.print_cyclic = True
>>> a = Permutation([0, 2, 1, 3])
>>> b = Permutation([0, 2, 3, 1])
>>> g = PermutationGroup([a, b])
>>> list(g.generate_schreier_sims(af=True))
[[0, 1, 2, 3], [0, 2, 1, 3], [0, 3, 2, 1],
 [0, 1, 3, 2], [0, 2, 3, 1], [0, 3, 1, 2]]
```

generators
Returns the generators of the group.

Examples

```
>>> Permutation.print_cyclic = True
>>> a = Permutation([0, 2, 1])
>>> b = Permutation([1, 0, 2])
>>> G = PermutationGroup([a, b])
>>> G.generators
[Permutation(1, 2), Permutation(2)(0, 1)]
```

is_abelian
Test if the group is Abelian.

Examples

```
>>> Permutation.print_cyclic = True
>>> a = Permutation([0, 2, 1])
>>> b = Permutation([1, 0, 2])
>>> G = PermutationGroup([a, b])
>>> G.is_abelian
False
>>> a = Permutation([0, 2, 1])
>>> G = PermutationGroup([a])
>>> G.is_abelian
True
```

is_alt_sym(*eps=0.05*, *_random_prec=None*)

Monte Carlo test for the symmetric/alternating group for degrees >= 8.

More specifically, it is one-sided Monte Carlo with the answer True (i.e., G is symmetric/alternating) guaranteed to be correct, and the answer False being incorrect with probability eps.

See also:

diofant.combinatorics.util._check_cycles_alt_sym (page 231)

Notes

The algorithm itself uses some nontrivial results from group theory and number theory: 1) If a transitive group `G` of degree `n` contains an element with a cycle of length `n/2 < p < n-2` for `p` a prime, `G` is the symmetric or alternating group ([1], pp. 81-82) 2) The proportion of elements in the symmetric/alternating group having the property described in 1) is approximately `\log(2)/\log(n)` ([1], p.82; [2], pp. 226-227). The helper function `_check_cycles_alt_sym` is used to go over the cycles in a permutation and look for ones satisfying 1).

Examples

```
>>> D = DihedralGroup(10)
>>> D.is_alt_sym()
False
```

is_nilpotent

Test if the group is nilpotent.

A group `G` is nilpotent if it has a central series of finite length. Alternatively, `G` is nilpotent if its lower central series terminates with the trivial group. Every nilpotent group is also solvable ([1], p.29, [12]).

See also:

lower_central_series (page 201), *is_solvable* (page 199)

Examples

```
>>> C = CyclicGroup(6)
>>> C.is_nilpotent
True
>>> S = SymmetricGroup(5)
>>> S.is_nilpotent
False
```

is_normal(*gr*, *strict=True*)

Test if G=self is a normal subgroup of gr.

G is normal in gr if for each g2 in G, g1 in gr, g = g1*g2*g1**-1 belongs to G It is sufficient to check this for each g1 in gr.generator and g2 g2 in G.generator

Examples

```
>>> Permutation.print_cyclic = True
>>> a = Permutation([1, 2, 0])
>>> b = Permutation([1, 0, 2])
>>> G = PermutationGroup([a, b])
>>> G1 = PermutationGroup([a, Permutation([2, 0, 1])])
>>> G1.is_normal(G)
True
```

is_primitive(*randomized=True*)
Test if a group is primitive.

A permutation group G acting on a set S is called primitive if S contains no nontrivial block under the action of G (a block is nontrivial if its cardinality is more than 1).

See also:

minimal_block (page 202), *random_stab* (page 206)

Notes

The algorithm is described in [1], p.83, and uses the function minimal_block to search for blocks of the form `\{0, k\}` for k ranging over representatives for the orbits of `G_0`, the stabilizer of `0`. This algorithm has complexity `O(n^2)` where `n` is the degree of the group, and will perform badly if `G_0` is small.

There are two implementations offered: one finds `G_0` deterministically using the function `stabilizer`, and the other (default) produces random elements of `G_0` using `random_stab`, hoping that they generate a subgroup of `G_0` with not too many more orbits than G_0 (this is suggested in [1], p.83). Behavior is changed by the `randomized` flag.

Examples

```
>>> D = DihedralGroup(10)
>>> D.is_primitive()
False
```

is_solvable
Test if the group is solvable.

G is solvable if its derived series terminates with the trivial group ([1], p.29).

See also:

is_nilpotent (page 198), *derived_series* (page 195)

Examples

```
>>> S = SymmetricGroup(3)
>>> S.is_solvable
True
```

is_subgroup(*G*, *strict=True*)

Return True if all elements of self belong to G.

If `strict` is False then if `self`'s degree is smaller than G's, the elements will be resized to have the same degree.

Examples

Testing is strict by default: the degree of each group must be the same:

```
>>> p = Permutation(0, 1, 2, 3, 4, 5)
>>> G1 = PermutationGroup([Permutation(0, 1, 2), Permutation(0, 1)])
>>> G2 = PermutationGroup([Permutation(0, 2), Permutation(0, 1, 2)])
>>> G3 = PermutationGroup([p, p**2])
>>> assert G1.order() == G2.order() == G3.order() == 6
>>> G1.is_subgroup(G2)
True
>>> G1.is_subgroup(G3)
False
>>> G3.is_subgroup(PermutationGroup(G3[1]))
False
>>> G3.is_subgroup(PermutationGroup(G3[0]))
True
```

To ignore the size, set `strict` to False:

```
>>> S3 = SymmetricGroup(3)
>>> S5 = SymmetricGroup(5)
>>> S3.is_subgroup(S5, strict=False)
True
>>> C7 = CyclicGroup(7)
>>> G = S5*C7
>>> S5.is_subgroup(G, False)
True
>>> C7.is_subgroup(G, 0)
False
```

is_transitive(*strict=True*)

Test if the group is transitive.

A group is transitive if it has a single orbit.

If `strict` is False the group is transitive if it has a single orbit of length different from 1.

Examples

```
>>> a = Permutation([0, 2, 1, 3])
>>> b = Permutation([2, 0, 1, 3])
>>> G1 = PermutationGroup([a, b])
>>> G1.is_transitive()
False
>>> G1.is_transitive(strict=False)
True
>>> c = Permutation([2, 3, 0, 1])
```

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```
>>> G2 = PermutationGroup([a, c])
>>> G2.is_transitive()
True
>>> d = Permutation([1, 0, 2, 3])
>>> e = Permutation([0, 1, 3, 2])
>>> G3 = PermutationGroup([d, e])
>>> G3.is_transitive() or G3.is_transitive(strict=False)
False
```

is_trivial

Test if the group is the trivial group.

This is true if the group contains only the identity permutation.

Examples

```
>>> G = PermutationGroup([Permutation([0, 1, 2])])
>>> G.is_trivial
True
```

lower_central_series()

Return the lower central series for the group.

The lower central series for a group G is the series `G = G_0 > G_1 > G_2 > \ldots` where `G_k = [G, G_{k-1}]`, i.e. every term after the first is equal to the commutator of G and the previous term in G1 ([1], p.29).

Returns A list of permutation groups in the order

`G = G_0, G_1, G_2, \ldots`

See also:

commutator (page 191), *derived_series* (page 195)

Examples

```
>>> A = AlternatingGroup(4)
>>> len(A.lower_central_series())
2
>>> A.lower_central_series()[1].is_subgroup(DihedralGroup(2))
True
```

make_perm(*n*, *seed=None*)

Multiply n randomly selected permutations from pgroup together, starting with the identity permutation. If n is a list of integers, those integers will be used to select the permutations and they will be applied in L to R order: make_perm((A, B, C)) will give CBA(I) where I is the identity permutation.

seed is used to set the seed for the random selection of permutations from pgroup. If this is a list of integers, the corresponding permutations from pgroup will be selected in the order give. This is mainly used for testing purposes.

See also:

random (page 206)

Examples

```
>>> Permutation.print_cyclic = True
>>> a, b = [Permutation([1, 0, 3, 2]), Permutation([1, 3, 0, 2])]
>>> G = PermutationGroup([a, b])
>>> G.make_perm(1, [0])
Permutation(0, 1)(2, 3)
>>> G.make_perm(3, [0, 1, 0])
Permutation(0, 2, 3, 1)
>>> G.make_perm([0, 1, 0])
Permutation(0, 2, 3, 1)
```

max_div

Maximum proper divisor of the degree of a permutation group.

See also:

minimal_block (page 202), *_union_find_merge* (page 187)

Notes

Obviously, this is the degree divided by its minimal proper divisor (larger than 1, if one exists). As it is guaranteed to be prime, the `sieve` from `diofant.ntheory` is used. This function is also used as an optimization tool for the functions `minimal_block` and `_union_find_merge`.

Examples

```
>>> G = PermutationGroup([Permutation([0, 2, 1, 3])])
>>> G.max_div
2
```

minimal_block(*points*)

For a transitive group, finds the block system generated by `points`.

If a group `G` acts on a set `S`, a nonempty subset `B` of `S` is called a block under the action of `G` if for all `g` in `G` we have `gB = B` (`g` fixes `B`) or `gB` and `B` have no common points (`g` moves `B` entirely). ([1], p.23; [6]).

The distinct translates `gB` of a block `B` for `g` in `G` partition the set `S` and this set of translates is known as a block system. Moreover, we obviously have that all blocks in the partition have the same size, hence the block size divides `|S|` ([1], p.23). A `G`-congruence is an equivalence relation ~ on the set `S` such that `a ~ b` implies `g(a) ~ g(b)` for all `g` in `G`. For a transitive group, the equivalence classes of a `G`-congruence and the blocks of a block system are the same thing ([1], p.23).

The algorithm below checks the group for transitivity, and then finds the `G`-congruence generated by the pairs `(p_0, p_1), (p_0, p_2), ..., (p_0,p_{k-1})` which is the same as finding the maximal block system (i.e., the one with minimum block size) such that `p_0, ..., p_{k-1}` are in the same block ([1], p.83).

It is an implementation of Atkinson's algorithm, as suggested in [1], and manipulates an equivalence relation on the set `S` using a union-find data structure. The running time is just above `O(|points||S|)`. ([1], pp. 83-87; [7]).

See also:

_union_find_rep (page 187), *_union_find_merge* (page 187), *is_transitive* (page 200), *is_primitive* (page 199)

Examples

```
>>> D = DihedralGroup(10)
>>> D.minimal_block([0, 5])
[0, 6, 2, 8, 4, 0, 6, 2, 8, 4]
>>> D.minimal_block([0, 1])
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
```

normal_closure(*other*, *k=10*)
Return the normal closure of a subgroup/set of permutations.

If S is a subset of a group G, the normal closure of A in G is defined as the intersection of all normal subgroups of G that contain A ([1], p.14). Alternatively, it is the group generated by the conjugates x^{-1}yx for x a generator of G and y a generator of the subgroup \left\langle S\right\rangle generated by S (for some chosen generating set for \left\langle S\right\rangle) ([1], p.73).

Parameters other

a subgroup/list of permutations/single permutation

k

an implementation-specific parameter that determines the number of conjugates that are adjoined to other at once

See also:

commutator (page 191), *derived_subgroup* (page 195), *random_pr* (page 206)

Notes

The algorithm is described in [1], pp. 73-74; it makes use of the generation of random elements for permutation groups by the product replacement algorithm.

Examples

```
>>> S = SymmetricGroup(5)
>>> C = CyclicGroup(5)
>>> G = S.normal_closure(C)
>>> G.order()
60
>>> G.is_subgroup(AlternatingGroup(5))
True
```

orbit(*alpha*, *action='tuples'*)
Compute the orbit of alpha \{g(\alpha) | g \in G\} as a set.

The time complexity of the algorithm used here is O(|Orb|*r) where |Orb| is the size of the orbit and r is the number of generators of the group. For a more detailed analysis, see [1], p.78, [2], pp. 19-21. Here alpha can be a single point, or a list of points.

If alpha is a single point, the ordinary orbit is computed. if alpha is a list of points, there are three available options:

'union' - computes the union of the orbits of the points in the list 'tuples' - computes the orbit of the list interpreted as an ordered tuple under the group action (i.e., g((1,2,3)) = (g(1), g(2), g(3))) 'sets' - computes the orbit of the list interpreted as a sets

See also:

orbit_transversal (page 204)

Examples

```
>>> a = Permutation([1, 2, 0, 4, 5, 6, 3])
>>> G = PermutationGroup([a])
>>> G.orbit(0)
{0, 1, 2}
>>> G.orbit([0, 4], 'union')
{0, 1, 2, 3, 4, 5, 6}
```

orbit_rep(*alpha*, *beta*, *schreier_vector=None*)
Return a group element which sends `alpha` to `beta`.

If `beta` is not in the orbit of `alpha`, the function returns `False`. This implementation makes use of the schreier vector. For a proof of correctness, see [1], p.80

See also:

schreier_vector (page 208)

Examples

```
>>> Permutation.print_cyclic = True
>>> G = AlternatingGroup(5)
>>> G.orbit_rep(0, 4)
Permutation(0, 4, 1, 2, 3)
```

orbit_transversal(*alpha*, *pairs=False*)
Computes a transversal for the orbit of `alpha` as a set.

For a permutation group `G`, a transversal for the orbit `Orb = \{g(\alpha) | g \in G\}` is a set `\{g_\beta | g_\beta(\alpha) = \beta\}` for `\beta \in Orb`. Note that there may be more than one possible transversal. If `pairs` is set to `True`, it returns the list of pairs `(\beta, g_\beta)`. For a proof of correctness, see [1], p.79

See also:

orbit (page 203)

Examples

```
>>> Permutation.print_cyclic = True
>>> G = DihedralGroup(6)
>>> G.orbit_transversal(0)
```

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```
[Permutation(5),
 Permutation(0, 1, 2, 3, 4, 5),
 Permutation(0, 5)(1, 4)(2, 3),
 Permutation(0, 2, 4)(1, 3, 5),
 Permutation(5)(0, 4)(1, 3),
 Permutation(0, 3)(1, 4)(2, 5)]
```

orbits(*rep=False*)
Return the orbits of self, ordered according to lowest element in each orbit.

Examples

```
>>> a = Permutation(1, 5)(2, 3)(4, 0, 6)
>>> b = Permutation(1, 5)(3, 4)(2, 6, 0)
>>> G = PermutationGroup([a, b])
>>> G.orbits()
[{0, 2, 3, 4, 6}, {1, 5}]
```

order()
Return the order of the group: the number of permutations that can be generated from elements of the group.

The number of permutations comprising the group is given by len(group); the length of each permutation in the group is given by group.size.

See also:

degree (page 194)

Examples

```
>>> a = Permutation([1, 0, 2])
>>> G = PermutationGroup([a])
>>> G.degree
3
>>> len(G)
1
>>> G.order()
2
>>> list(G.generate())
[Permutation(2), Permutation(2)(0, 1)]
```

```
>>> a = Permutation([0, 2, 1])
>>> b = Permutation([1, 0, 2])
>>> G = PermutationGroup([a, b])
>>> G.order()
6
```

pointwise_stabilizer(*points, incremental=True*)
Return the pointwise stabilizer for a set of points.

For a permutation group `G` and a set of points `\{p_1, p_2,\ldots, p_k\}`, the pointwise stabilizer of `p_1, p_2, \ldots, p_k` is defined as `G_{p_1,\ldots, p_k}`

= \{g\in G | g(p_i) = p_i \forall i\in\{1, 2,\ldots,k\}\} ([1],p20). It is a subgroup of ``G.

See also:

stabilizer (page 209), *schreier_sims_incremental* (page 207)

Notes

When incremental == True, rather than the obvious implementation using successive calls to .stabilizer(), this uses the incremental Schreier-Sims algorithm to obtain a base with starting segment - the given points.

Examples

```
>>> S = SymmetricGroup(7)
>>> Stab = S.pointwise_stabilizer([2, 3, 5])
>>> Stab.is_subgroup(S.stabilizer(2).stabilizer(3).stabilizer(5))
True
```

random(*af=False*)
Return a random group element

random_pr(*gen_count=11*, *iterations=50*, *_random_prec=None*)
Return a random group element using product replacement.

For the details of the product replacement algorithm, see `_random_pr_init` In `random_pr` the actual 'product replacement' is performed. Notice that if the attribute `_random_gens` is empty, it needs to be initialized by `_random_pr_init`.

See also:

_random_pr_init (page 186)

random_stab(*alpha*, *schreier_vector=None*, *_random_prec=None*)
Random element from the stabilizer of `alpha`.

The schreier vector for `alpha` is an optional argument used for speeding up repeated calls. The algorithm is described in [1], p.81

See also:

random_pr (page 206), *orbit_rep* (page 204)

schreier_sims()
Schreier-Sims algorithm.

It computes the generators of the chain of stabilizers G > G_{b_1} > .. > G_{b1,..,b_r} > 1 in which G_{b_1,..,b_i} stabilizes b_1,..,b_i, and the corresponding s cosets. An element of the group can be written as the product h_1*..*h_s.

We use the incremental Schreier-Sims algorithm.

Examples

```
>>> a = Permutation([0, 2, 1])
>>> b = Permutation([1, 0, 2])
>>> G = PermutationGroup([a, b])
>>> G.schreier_sims()
>>> G.basic_transversals
[{0: Permutation(2)(0, 1), 1: Permutation(2), 2: Permutation(1, 2)},
 {0: Permutation(2), 2: Permutation(0, 2)}]
```

schreier_sims_incremental(*base=None*, *gens=None*)

Extend a sequence of points and generating set to a base and strong generating set.

Parameters base

The sequence of points to be extended to a base. Optional parameter with default value `[]`.

gens

The generating set to be extended to a strong generating set relative to the base obtained. Optional parameter with default value `self.generators`.

Returns (base, strong_gens)

`base` is the base obtained, and `strong_gens` is the strong generating set relative to it. The original parameters `base`, `gens` remain unchanged.

See also:

schreier_sims (page 206), *schreier_sims_random* (page 207)

Notes

This version of the Schreier-Sims algorithm runs in polynomial time. There are certain assumptions in the implementation - if the trivial group is provided, `base` and `gens` are returned immediately, as any sequence of points is a base for the trivial group. If the identity is present in the generators `gens`, it is removed as it is a redundant generator. The implementation is described in [1], pp. 90-93.

Examples

```
>>> from diofant.combinatorics.testutil import _verify_bsgs
>>> A = AlternatingGroup(7)
>>> base = [2, 3]
>>> seq = [2, 3]
>>> base, strong_gens = A.schreier_sims_incremental(base=seq)
>>> _verify_bsgs(A, base, strong_gens)
True
>>> base[:2]
[2, 3]
```

schreier_sims_random(*base=None*, *gens=None*, *consec_succ=10*, *_random_prec=None*)

Randomized Schreier-Sims algorithm.

The randomized Schreier-Sims algorithm takes the sequence `base` and the generating set `gens`, and extends `base` to a base, and `gens` to a strong generating set relative to that base with probability of a wrong answer at most `2^{-consec_succ}`, provided the random generators are sufficiently random.

Parameters base

The sequence to be extended to a base.

gens

The generating set to be extended to a strong generating set.

consec_succ

The parameter defining the probability of a wrong answer.

_random_prec

An internal parameter used for testing purposes.

Returns (base, strong_gens)

`base` is the base and `strong_gens` is the strong generating set relative to it.

See also:

schreier_sims (page 206)

Notes

The algorithm is described in detail in [1], pp. 97-98. It extends the orbits `orbs` and the permutation groups `stabs` to basic orbits and basic stabilizers for the base and strong generating set produced in the end. The idea of the extension process is to "sift" random group elements through the stabilizer chain and amend the stabilizers/orbits along the way when a sift is not successful. The helper function `_strip` is used to attempt to decompose a random group element according to the current state of the stabilizer chain and report whether the element was fully decomposed (successful sift) or not (unsuccessful sift). In the latter case, the level at which the sift failed is reported and used to amend `stabs`, `base`, `gens` and `orbs` accordingly. The halting condition is for `consec_succ` consecutive successful sifts to pass. This makes sure that the current `base` and `gens` form a BSGS with probability at least `1 - 1/\text{consec_succ}`.

Examples

```
>>> from diofant.combinatorics.testutil import _verify_bsgs
>>> S = SymmetricGroup(5)
>>> base, strong_gens = S.schreier_sims_random(consec_succ=5)
>>> _verify_bsgs(S, base, strong_gens)
True
```

schreier_vector(*alpha*)

Computes the schreier vector for `alpha`.

The Schreier vector efficiently stores information about the orbit of `alpha`. It can later be used to quickly obtain elements of the group that send `alpha` to a particular element in the orbit. Notice that the Schreier vector depends on the order in which

the group generators are listed. For a definition, see [3]. Since list indices start from zero, we adopt the convention to use "None" instead of 0 to signify that an element doesn't belong to the orbit. For the algorithm and its correctness, see [2], pp.78-80.

See also:

orbit (page 203)

Examples

```
>>> a = Permutation([2, 4, 6, 3, 1, 5, 0])
>>> b = Permutation([0, 1, 3, 5, 4, 6, 2])
>>> G = PermutationGroup([a, b])
>>> G.schreier_vector(0)
[-1, None, 0, 1, None, 1, 0]
```

stabilizer(*alpha*)

Return the stabilizer subgroup of `alpha`.

The stabilizer of `\alpha` is the group `G_\alpha = \{g \in G | g(\alpha) = \alpha\}`. For a proof of correctness, see [1], p.79.

See also:

orbit (page 203)

Examples

```
>>> Permutation.print_cyclic = True
>>> G = DihedralGroup(6)
>>> G.stabilizer(5)
PermutationGroup([
    Permutation(5)(0, 4)(1, 3),
    Permutation(5)])
```

strong_gens

Return a strong generating set from the Schreier-Sims algorithm.

A generating set `S = \{g_1, g_2, ..., g_t\}` for a permutation group `G` is a strong generating set relative to the sequence of points (referred to as a "base") `(b_1, b_2, ..., b_k)` if, for `1 \leq i \leq k` we have that the intersection of the pointwise stabilizer `G^{(i+1)} := G_{b_1, b_2, ..., b_i}` with `S` generates the pointwise stabilizer `G^{(i+1)}`. The concepts of a base and strong generating set and their applications are discussed in depth in [1], pp. 87-89 and [2], pp. 55-57.

See also:

base (page 188), *basic_transversals* (page 190), *basic_orbits* (page 189), *basic_stabilizers* (page 189)

Examples

```
>>> D = DihedralGroup(4)
>>> D.strong_gens
[Permutation(0, 1, 2, 3), Permutation(0, 3)(1, 2), Permutation(1, 3)]
>>> D.base
[0, 1]
```

subgroup_search(*prop*, *base=None*, *strong_gens=None*, *tests=None*, *init_subgroup=None*)

Find the subgroup of all elements satisfying the property `prop`.

This is done by a depth-first search with respect to base images that uses several tests to prune the search tree.

Parameters prop

The property to be used. Has to be callable on group elements and always return `True` or `False`. It is assumed that all group elements satisfying `prop` indeed form a subgroup.

base

A base for the supergroup.

strong_gens

A strong generating set for the supergroup.

tests

A list of callables of length equal to the length of `base`. These are used to rule out group elements by partial base images, so that `tests[l](g)` returns False if the element `g` is known not to satisfy prop base on where g sends the first `l + 1` base points.

init_subgroup

if a subgroup of the sought group is known in advance, it can be passed to the function as this parameter.

Returns res

The subgroup of all elements satisfying `prop`. The generating set for this group is guaranteed to be a strong generating set relative to the base `base`.

Notes

This function is extremely lenghty and complicated and will require some careful attention. The implementation is described in [1], pp. 114-117, and the comments for the code here follow the lines of the pseudocode in the book for clarity.

The complexity is exponential in general, since the search process by itself visits all members of the supergroup. However, there are a lot of tests which are used to prune the search tree, and users can define their own tests via the `tests` parameter, so in practice, and for some computations, it's not terrible.

A crucial part in the procedure is the frequent base change performed (this is line 11 in the pseudocode) in order to obtain a new basic stabilizer. The book mentiones that this can be done by using `.baseswap(...)`, however the current implementation

uses a more straightforward way to find the next basic stabilizer - calling the function `.stabilizer(...)` on the previous basic stabilizer.

Examples

```
>>> from diofant.combinatorics.testutil import _verify_bsgs
>>> S = SymmetricGroup(7)
>>> prop_even = lambda x: x.is_even
>>> base, strong_gens = S.schreier_sims_incremental()
>>> G = S.subgroup_search(prop_even, base=base, strong_gens=strong_gens)
>>> G.is_subgroup(AlternatingGroup(7))
True
>>> _verify_bsgs(G, base, G.generators)
True
```

transitivity_degree

Compute the degree of transitivity of the group.

A permutation group G acting on `\Omega = \{0, 1, ..., n-1\}` is k-fold transitive, if, for any k points `(a_1, a_2, ..., a_k)\in\Omega` and any k points `(b_1, b_2, ..., b_k)\in\Omega` there exists `g\in G` such that `g(a_1)=b_1, g(a_2)=b_2, ..., g(a_k)=b_k` The degree of transitivity of G is the maximum k such that G is k-fold transitive. ([8])

See also:

is_transitive (page 200), *orbit* (page 203)

Examples

```
>>> a = Permutation([1, 2, 0])
>>> b = Permutation([1, 0, 2])
>>> G = PermutationGroup([a, b])
>>> G.transitivity_degree
3
```

3.2.4 Polyhedron

class diofant.combinatorics.polyhedron.**Polyhedron**

Represents the polyhedral symmetry group (PSG).

The PSG is one of the symmetry groups of the Platonic solids. There are three polyhedral groups: the tetrahedral group of order 12, the octahedral group of order 24, and the icosahedral group of order 60.

All doctests have been given in the docstring of the constructor of the object.

References

http://mathworld.wolfram.com/PolyhedralGroup.html

array_form

Return the indices of the corners.

The indices are given relative to the original position of corners.

See also:

corners (page 212), *cyclic_form* (page 212)

Examples

```
>>> tetrahedron.array_form
[0, 1, 2, 3]
```

```
>>> tetrahedron.rotate(0)
>>> tetrahedron.array_form
[0, 2, 3, 1]
>>> tetrahedron.pgroup[0].array_form
[0, 2, 3, 1]
>>> tetrahedron.reset()
```

corners

Get the corners of the Polyhedron.

The method `vertices` is an alias for `corners`.

See also:

array_form (page 211), *cyclic_form* (page 212)

Examples

```
>>> from diofant.abc import a, b, c, d
>>> p = Polyhedron(list('abcd'))
>>> p.corners == p.vertices == (a, b, c, d)
True
```

cyclic_form

Return the indices of the corners in cyclic notation.

The indices are given relative to the original position of corners.

See also:

corners (page 212), *array_form* (page 211)

edges

Given the faces of the polyhedra we can get the edges.

Examples

```
>>> from diofant.abc import a, b, c
>>> corners = (a, b, c)
>>> faces = [(0, 1, 2)]
>>> Polyhedron(corners, faces).edges
{(0, 1), (0, 2), (1, 2)}
```

faces

Get the faces of the Polyhedron.

pgroup
Get the permutations of the Polyhedron.

reset()
Return corners to their original positions.

Examples

```
>>> tetrahedron.corners
(0, 1, 2, 3)
>>> tetrahedron.rotate(0)
>>> tetrahedron.corners
(0, 2, 3, 1)
>>> tetrahedron.reset()
>>> tetrahedron.corners
(0, 1, 2, 3)
```

rotate(*perm*)
Apply a permutation to the polyhedron *in place*. The permutation may be given as a Permutation instance or an integer indicating which permutation from pgroup of the Polyhedron should be applied.

This is an operation that is analogous to rotation about an axis by a fixed increment.

Notes

When a Permutation is applied, no check is done to see if that is a valid permutation for the Polyhedron. For example, a cube could be given a permutation which effectively swaps only 2 vertices. A valid permutation (that rotates the object in a physical way) will be obtained if one only uses permutations from the `pgroup` of the Polyhedron. On the other hand, allowing arbitrary rotations (applications of permutations) gives a way to follow named elements rather than indices since Polyhedron allows vertices to be named while Permutation works only with indices.

Examples

```
>>> cube.corners
(0, 1, 2, 3, 4, 5, 6, 7)
>>> cube.rotate(0)
>>> cube.corners
(1, 2, 3, 0, 5, 6, 7, 4)
```

A non-physical "rotation" that is not prohibited by this method:

```
>>> cube.reset()
>>> cube.rotate(Permutation([[1, 2]], size=8))
>>> cube.corners
(0, 2, 1, 3, 4, 5, 6, 7)
```

Polyhedron can be used to follow elements of set that are identified by letters instead of integers:

```
>>> shadow = h5 = Polyhedron(list('abcde'))
>>> p = Permutation([3, 0, 1, 2, 4])
>>> h5.rotate(p)
>>> h5.corners
(d, a, b, c, e)
>>> _ == shadow.corners
True
>>> copy = h5.copy()
>>> h5.rotate(p)
>>> h5.corners == copy.corners
False
```

size

Get the number of corners of the Polyhedron.

vertices

Get the corners of the Polyhedron.

The method `vertices` is an alias for `corners`.

See also:

array_form (page 211), *cyclic_form* (page 212)

Examples

```
>>> from diofant.abc import a, b, c, d
>>> p = Polyhedron(list('abcd'))
>>> p.corners == p.vertices == (a, b, c, d)
True
```

3.2.5 Prufer Sequences

class diofant.combinatorics.prufer.**Prufer**

The Prufer correspondence is an algorithm that describes the bijection between labeled trees and the Prufer code. A Prufer code of a labeled tree is unique up to isomorphism and has a length of n - 2.

Prufer sequences were first used by Heinz Prufer to give a proof of Cayley's formula.

References

[R63] (page 1251)

static edges()

Return a list of edges and the number of nodes from the given runs that connect nodes in an integer-labelled tree.

All node numbers will be shifted so that the minimum node is 0. It is not a problem if edges are repeated in the runs; only unique edges are returned. There is no assumption made about what the range of the node labels should be, but all nodes from the smallest through the largest must be present.

Examples

```
>>> Prufer.edges([1, 2, 3], [2, 4, 5]) # a T
([[0, 1], [1, 2], [1, 3], [3, 4]], 5)
```

Duplicate edges are removed:

```
>>> Prufer.edges([0, 1, 2, 3], [1, 4, 5], [1, 4, 6]) # a K
([[0, 1], [1, 2], [1, 4], [2, 3], [4, 5], [4, 6]], 7)
```

next(*delta=1*)
Generates the Prufer sequence that is delta beyond the current one.

See also:

prufer_rank (page 215), *rank* (page 216), *prev* (page 215), *size* (page 216)

Examples

```
>>> a = Prufer([[0, 1], [0, 2], [0, 3]])
>>> b = a.next(1) # == a.next()
>>> b.tree_repr
[[0, 2], [0, 1], [1, 3]]
>>> b.rank
1
```

nodes
Returns the number of nodes in the tree.

Examples

```
>>> Prufer([[0, 3], [1, 3], [2, 3], [3, 4], [4, 5]]).nodes
6
>>> Prufer([1, 0, 0]).nodes
5
```

prev(*delta=1*)
Generates the Prufer sequence that is -delta before the current one.

See also:

prufer_rank (page 215), *rank* (page 216), *next* (page 215), *size* (page 216)

Examples

```
>>> a = Prufer([[0, 1], [1, 2], [2, 3], [1, 4]])
>>> a.rank
36
>>> b = a.prev()
>>> b
Prufer((1, 2, 0))
>>> b.rank
35
```

prufer_rank()

Computes the rank of a Prufer sequence.

See also:

rank (page 216), *next* (page 215), *prev* (page 215), *size* (page 216)

Examples

```
>>> a = Prufer([[0, 1], [0, 2], [0, 3]])
>>> a.prufer_rank()
0
```

prufer_repr

Returns Prufer sequence for the Prufer object.

This sequence is found by removing the highest numbered vertex, recording the node it was attached to, and continuing until only two vertices remain. The Prufer sequence is the list of recorded nodes.

See also:

to_prufer (page 217)

Examples

```
>>> Prufer([[0, 3], [1, 3], [2, 3], [3, 4], [4, 5]]).prufer_repr
[3, 3, 3, 4]
>>> Prufer([1, 0, 0]).prufer_repr
[1, 0, 0]
```

rank

Returns the rank of the Prufer sequence.

See also:

prufer_rank (page 215), *next* (page 215), *prev* (page 215), *size* (page 216)

Examples

```
>>> p = Prufer([[0, 3], [1, 3], [2, 3], [3, 4], [4, 5]])
>>> p.rank
778
>>> p.next(1).rank
779
>>> p.prev().rank
777
```

size

Return the number of possible trees of this Prufer object.

See also:

prufer_rank (page 215), *rank* (page 216), *next* (page 215), *prev* (page 215)

Examples

```
>>> Prufer([0]*4).size == Prufer([6]*4).size == 1296
True
```

static to_prufer(*n*)

Return the Prufer sequence for a tree given as a list of edges where n is the number of nodes in the tree.

See also:

prufer_repr **(page 216)** returns Prufer sequence of a Prufer object.

Examples

```
>>> a = Prufer([[0, 1], [0, 2], [0, 3]])
>>> a.prufer_repr
[0, 0]
>>> Prufer.to_prufer([[0, 1], [0, 2], [0, 3]], 4)
[0, 0]
```

static to_tree()

Return the tree (as a list of edges) of the given Prufer sequence.

See also:

tree_repr **(page 217)** returns tree representation of a Prufer object.

References

- https://hamberg.no/erlend/posts/2010-11-06-prufer-sequence-compact-tree-representation.html

Examples

```
>>> a = Prufer([0, 2], 4)
>>> a.tree_repr
[[0, 1], [0, 2], [2, 3]]
>>> Prufer.to_tree([0, 2])
[[0, 1], [0, 2], [2, 3]]
```

tree_repr

Returns the tree representation of the Prufer object.

See also:

to_tree (page 217)

Examples

```
>>> Prufer([[0, 3], [1, 3], [2, 3], [3, 4], [4, 5]]).tree_repr
[[0, 3], [1, 3], [2, 3], [3, 4], [4, 5]]
>>> Prufer([1, 0, 0]).tree_repr
[[1, 2], [0, 1], [0, 3], [0, 4]]
```

classmethod unrank(*rank*, *n*)
Finds the unranked Prufer sequence.

Examples

```
>>> Prufer.unrank(0, 4)
Prufer((0, 0))
```

3.2.6 Subsets

class diofant.combinatorics.subsets.**Subset**
Represents a basic subset object.

We generate subsets using essentially two techniques, binary enumeration and lexicographic enumeration. The Subset class takes two arguments, the first one describes the initial subset to consider and the second describes the superset.

Examples

```
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.next_binary().subset
['b']
>>> a.prev_binary().subset
['c']
```

classmethod bitlist_from_subset(*subset*, *superset*)
Gets the bitlist corresponding to a subset.

See also:

subset_from_bitlist (page 222)

Examples

```
>>> Subset.bitlist_from_subset(['c', 'd'], ['a', 'b', 'c', 'd'])
'0011'
```

cardinality
Returns the number of all possible subsets.

See also:

subset (page 222), *superset* (page 223), *size* (page 222), *superset_size* (page 223)

Examples

```
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.cardinality
16
```

iterate_binary(k)

This is a helper function. It iterates over the binary subsets by k steps. This variable can be both positive or negative.

See also:

next_binary (page 219), *prev_binary* (page 220)

Examples

```
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.iterate_binary(-2).subset
['d']
>>> a = Subset(['a', 'b', 'c'], ['a', 'b', 'c', 'd'])
>>> a.iterate_binary(2).subset
[]
```

iterate_graycode(k)

Helper function used for prev_gray and next_gray. It performs k step overs to get the respective Gray codes.

See also:

next_gray (page 219), *prev_gray* (page 220)

Examples

```
>>> a = Subset([1, 2, 3], [1, 2, 3, 4])
>>> a.iterate_graycode(3).subset
[1, 4]
>>> a.iterate_graycode(-2).subset
[1, 2, 4]
```

next_binary()

Generates the next binary ordered subset.

See also:

prev_binary (page 220), *iterate_binary* (page 219)

Examples

```
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.next_binary().subset
['b']
>>> a = Subset(['a', 'b', 'c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.next_binary().subset
[]
```

next_gray()
Generates the next Gray code ordered subset.

See also:

iterate_graycode (page 219), *prev_gray* (page 220)

Examples

```
>>> a = Subset([1, 2, 3], [1, 2, 3, 4])
>>> a.next_gray().subset
[1, 3]
```

next_lexicographic()
Generates the next lexicographically ordered subset.

See also:

prev_lexicographic (page 221)

Examples

```
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.next_lexicographic().subset
['d']
>>> a = Subset(['d'], ['a', 'b', 'c', 'd'])
>>> a.next_lexicographic().subset
[]
```

prev_binary()
Generates the previous binary ordered subset.

See also:

next_binary (page 219), *iterate_binary* (page 219)

Examples

```
>>> a = Subset([], ['a', 'b', 'c', 'd'])
>>> a.prev_binary().subset
['a', 'b', 'c', 'd']
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.prev_binary().subset
['c']
```

prev_gray()
Generates the previous Gray code ordered subset.

See also:

iterate_graycode (page 219), *next_gray* (page 219)

Examples

```
>>> a = Subset([2, 3, 4], [1, 2, 3, 4, 5])
>>> a.prev_gray().subset
[2, 3, 4, 5]
```

prev_lexicographic()

Generates the previous lexicographically ordered subset.

See also:

next_lexicographic (page 220)

Examples

```
>>> a = Subset([], ['a', 'b', 'c', 'd'])
>>> a.prev_lexicographic().subset
['d']
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.prev_lexicographic().subset
['c']
```

rank_binary

Computes the binary ordered rank.

See also:

iterate_binary (page 219), *unrank_binary* (page 223)

Examples

```
>>> a = Subset([], ['a', 'b', 'c', 'd'])
>>> a.rank_binary
0
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.rank_binary
3
```

rank_gray

Computes the Gray code ranking of the subset.

See also:

iterate_graycode (page 219), *unrank_gray* (page 223)

Examples

```
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.rank_gray
2
>>> a = Subset([2, 4, 5], [1, 2, 3, 4, 5, 6])
>>> a.rank_gray
27
```

rank_lexicographic
Computes the lexicographic ranking of the subset.

Examples

```
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.rank_lexicographic
14
>>> a = Subset([2, 4, 5], [1, 2, 3, 4, 5, 6])
>>> a.rank_lexicographic
43
```

size
Gets the size of the subset.

See also:

subset (page 222), *superset* (page 223), *superset_size* (page 223), *cardinality* (page 218)

Examples

```
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.size
2
```

subset
Gets the subset represented by the current instance.

See also:

superset (page 223), *size* (page 222), *superset_size* (page 223), *cardinality* (page 218)

Examples

```
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.subset
['c', 'd']
```

classmethod subset_from_bitlist(*super_set*, *bitlist*)
Gets the subset defined by the bitlist.

See also:

bitlist_from_subset (page 218)

Examples

```
>>> Subset.subset_from_bitlist(['a', 'b', 'c', 'd'], '0011').subset
['c', 'd']
```

classmethod subset_indices(*subset, superset*)

Return indices of subset in superset in a list; the list is empty if all elements of subset are not in superset.

Examples

```
>>> superset = [1, 3, 2, 5, 4]
>>> Subset.subset_indices([3, 2, 1], superset)
[1, 2, 0]
>>> Subset.subset_indices([1, 6], superset)
[]
>>> Subset.subset_indices([], superset)
[]
```

superset

Gets the superset of the subset.

See also:

subset (page 222), *size* (page 222), *superset_size* (page 223), *cardinality* (page 218)

Examples

```
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.superset
['a', 'b', 'c', 'd']
```

superset_size

Returns the size of the superset.

See also:

subset (page 222), *superset* (page 223), *size* (page 222), *cardinality* (page 218)

Examples

```
>>> a = Subset(['c', 'd'], ['a', 'b', 'c', 'd'])
>>> a.superset_size
4
```

classmethod unrank_binary(*rank, superset*)

Gets the binary ordered subset of the specified rank.

See also:

iterate_binary (page 219), *rank_binary* (page 221)

Examples

```
>>> Subset.unrank_binary(4, ['a', 'b', 'c', 'd']).subset
['b']
```

classmethod unrank_gray(*rank*, *superset*)

Gets the Gray code ordered subset of the specified rank.

See also:

iterate_graycode (page 219), *rank_gray* (page 221)

Examples

```
>>> Subset.unrank_gray(4, ['a', 'b', 'c']).subset
['a', 'b']
>>> Subset.unrank_gray(0, ['a', 'b', 'c']).subset
[]
```

subsets.**ksubsets**(*k*)

Finds the subsets of size k in lexicographic order.

This uses the itertools generator.

See also:

Subset (page 218)

Examples

```
>>> list(ksubsets([1, 2, 3], 2))
[(1, 2), (1, 3), (2, 3)]
>>> list(ksubsets([1, 2, 3, 4, 5], 2))
[(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4),
 (2, 5), (3, 4), (3, 5), (4, 5)]
```

3.2.7 Gray Code

class diofant.combinatorics.graycode.**GrayCode**

A Gray code is essentially a Hamiltonian walk on a n-dimensional cube with edge length of one. The vertices of the cube are represented by vectors whose values are binary. The Hamilton walk visits each vertex exactly once. The Gray code for a 3d cube is [‘000’,‘100’,‘110’,‘010’,‘011’,‘111’,‘101’, ‘001’].

A Gray code solves the problem of sequentially generating all possible subsets of n objects in such a way that each subset is obtained from the previous one by either deleting or adding a single object. In the above example, 1 indicates that the object is present, and 0 indicates that its absent.

Gray codes have applications in statistics as well when we want to compute various statistics related to subsets in an efficient manner.

References

[R48] (page 1251), *[R49]* (page 1251)

Examples

```
>>> a = GrayCode(3)
>>> list(a.generate_gray())
['000', '001', '011', '010', '110', '111', '101', '100']
>>> a = GrayCode(4)
>>> list(a.generate_gray())
['0000', '0001', '0011', '0010', '0110', '0111', '0101', '0100',
 '1100', '1101', '1111', '1110', '1010', '1011', '1001', '1000']
```

current
Returns the currently referenced Gray code as a bit string.

Examples

```
>>> GrayCode(3, start='100').current
'100'
```

generate_gray(***hints*)
Generates the sequence of bit vectors of a Gray Code.

[1] Knuth, D. (2011). The Art of Computer Programming, Vol 4, Addison Wesley

See also:

skip (page 226)

Examples

```
>>> a = GrayCode(3)
>>> list(a.generate_gray())
['000', '001', '011', '010', '110', '111', '101', '100']
>>> list(a.generate_gray(start='011'))
['011', '010', '110', '111', '101', '100']
>>> list(a.generate_gray(rank=4))
['110', '111', '101', '100']
```

n
Returns the dimension of the Gray code.

Examples

```
>>> a = GrayCode(5)
>>> a.n
5
```

next(*delta=1*)
Returns the Gray code a distance `delta` (default = 1) from the current value in canonical order.

Examples

```
>>> a = GrayCode(3, start='110')
>>> a.next().current
'111'
>>> a.next(-1).current
'010'
```

rank

Ranks the Gray code.

A ranking algorithm determines the position (or rank) of a combinatorial object among all the objects w.r.t. a given order. For example, the 4 bit binary reflected Gray code (BRGC) '0101' has a rank of 6 as it appears in the 6th position in the canonical ordering of the family of 4 bit Gray codes.

See also:

unrank (page 227)

References

[R50] (page 1251)

Examples

```
>>> a = GrayCode(3)
>>> list(a.generate_gray())
['000', '001', '011', '010', '110', '111', '101', '100']
>>> GrayCode(3, start='100').rank
7
>>> GrayCode(3, rank=7).current
'100'
```

selections

Returns the number of bit vectors in the Gray code.

Examples

```
>>> a = GrayCode(3)
>>> a.selections
8
```

skip()

Skips the bit generation.

See also:

generate_gray (page 225)

Examples

```
>>> a = GrayCode(3)
>>> for i in a.generate_gray():
...     if i == '010':
...         a.skip()
...     print(i)
...
000
001
011
010
111
101
100
```

classmethod unrank(*n*, *rank*)

Unranks an n-bit sized Gray code of rank k. This method exists so that a derivative GrayCode class can define its own code of a given rank.

The string here is generated in reverse order to allow for tail-call optimization.

See also:

rank (page 226)

Examples

```
>>> GrayCode(5, rank=3).current
'00010'
>>> GrayCode.unrank(5, 3)
'00010'
```

graycode.**random_bitstring**()

Generates a random bitlist of length n.

Examples

```
>>> random_bitstring(3)
100
```

graycode.**gray_to_bin**()

Convert from Gray coding to binary coding.

We assume big endian encoding.

See also:

bin_to_gray (page 227)

Examples

```
>>> gray_to_bin('100')
'111'
```

graycode.**bin_to_gray**()

Convert from binary coding to gray coding.

We assume big endian encoding.

See also:

gray_to_bin (page 227)

Examples

```
>>> bin_to_gray('111')
'100'
```

graycode.**get_subset_from_bitstring**(*bitstring*)

Gets the subset defined by the bitstring.

See also:

graycode_subsets (page 228)

Examples

```
>>> get_subset_from_bitstring(['a', 'b', 'c', 'd'], '0011')
['c', 'd']
>>> get_subset_from_bitstring(['c', 'a', 'c', 'c'], '1100')
['c', 'a']
```

graycode.**graycode_subsets**()

Generates the subsets as enumerated by a Gray code.

See also:

get_subset_from_bitstring (page 228)

Examples

```
>>> list(graycode_subsets(['a', 'b', 'c']))
[[], ['c'], ['b', 'c'], ['b'], ['a', 'b'], ['a', 'b', 'c'],
 ['a', 'c'], ['a']]
>>> list(graycode_subsets(['a', 'b', 'c', 'c']))
[[], ['c'], ['c', 'c'], ['c'], ['b', 'c'], ['b', 'c', 'c'],
 ['b', 'c'], ['b'], ['a', 'b'], ['a', 'b', 'c'], ['a', 'b', 'c', 'c'],
 ['a', 'b', 'c'], ['a', 'c'], ['a', 'c', 'c'], ['a', 'c'], ['a']]
```

3.2.8 Named Groups

diofant.combinatorics.named_groups.**SymmetricGroup**(*n*)

Generates the symmetric group on `n` elements as a permutation group.

The generators taken are the `n`-cycle `(0 1 2 ... n-1)` and the transposition `(0 1)` (in cycle notation). (See [1]). After the group is generated, some of its basic properties are set.

See also:

CyclicGroup (page 229), *DihedralGroup* (page 229), *AlternatingGroup* (page 230)

References

[1] https://en.wikipedia.org/wiki/Symmetric_group#Generators_and_relations

Examples

```
>>> G = SymmetricGroup(4)
>>> G.is_group
True
>>> G.order()
24
>>> list(G.generate_schreier_sims(af=True))
[[0, 1, 2, 3], [1, 2, 3, 0], [2, 3, 0, 1], [3, 1, 2, 0], [0, 2, 3, 1],
[1, 3, 0, 2], [2, 0, 1, 3], [3, 2, 0, 1], [0, 3, 1, 2], [1, 0, 2, 3],
[2, 1, 3, 0], [3, 0, 1, 2], [0, 1, 3, 2], [1, 2, 0, 3], [2, 3, 1, 0],
[3, 1, 0, 2], [0, 2, 1, 3], [1, 3, 2, 0], [2, 0, 3, 1], [3, 2, 1, 0],
[0, 3, 2, 1], [1, 0, 3, 2], [2, 1, 0, 3], [3, 0, 2, 1]]
```

`diofant.combinatorics.named_groups.`**`CyclicGroup`**`(n)`

Generates the cyclic group of order `n` as a permutation group.

The generator taken is the `n`-cycle `(0 1 2 ... n-1)` (in cycle notation). After the group is generated, some of its basic properties are set.

See also:

SymmetricGroup (page 228), *DihedralGroup* (page 229), *AlternatingGroup* (page 230)

Examples

```
>>> G = CyclicGroup(6)
>>> G.is_group
True
>>> G.order()
6
>>> list(G.generate_schreier_sims(af=True))
[[0, 1, 2, 3, 4, 5], [1, 2, 3, 4, 5, 0], [2, 3, 4, 5, 0, 1],
[3, 4, 5, 0, 1, 2], [4, 5, 0, 1, 2, 3], [5, 0, 1, 2, 3, 4]]
```

`diofant.combinatorics.named_groups.`**`DihedralGroup`**`(n)`

Generates the dihedral group D_n as a permutation group.

The dihedral group D_n is the group of symmetries of the regular `n`-gon. The generators taken are the `n`-cycle `a = (0 1 2 ... n-1)` (a rotation of the `n`-gon) and `b = (0 n-1)(1 n-2)...` (a reflection of the `n`-gon) in cycle rotation. It is easy to see that these satisfy `a**n = b**2 = 1` and `bab = ~a` so they indeed generate D_n (See [1]). After the group is generated, some of its basic properties are set.

See also:

SymmetricGroup (page 228), *CyclicGroup* (page 229), *AlternatingGroup* (page 230)

References

[1] https://en.wikipedia.org/wiki/Dihedral_group

Examples

```
>>> G = DihedralGroup(5)
>>> G.is_group
True
>>> a = list(G.generate_dimino())
>>> [perm.cyclic_form for perm in a]
[[], [[0, 1, 2, 3, 4]], [[0, 2, 4, 1, 3]],
[[0, 3, 1, 4, 2]], [[0, 4, 3, 2, 1]], [[0, 4], [1, 3]],
[[1, 4], [2, 3]], [[0, 1], [2, 4]], [[0, 2], [3, 4]],
[[0, 3], [1, 2]]]
```

diofant.combinatorics.named_groups.**AlternatingGroup**(*n*)

Generates the alternating group on n elements as a permutation group.

For `n > 2`, the generators taken are `(0 1 2), (0 1 2 ... n-1)` for n odd and `(0 1 2), (1 2 ... n-1)` for n even (See [1], p.31, ex.6.9.). After the group is generated, some of its basic properties are set. The cases `n = 1, 2` are handled separately.

See also:

SymmetricGroup (page 228), *CyclicGroup* (page 229), *DihedralGroup* (page 229)

References

[1] Armstrong, M. “Groups and Symmetry”

Examples

```
>>> G = AlternatingGroup(4)
>>> G.is_group
True
>>> a = list(G.generate_dimino())
>>> len(a)
12
>>> all(perm.is_even for perm in a)
True
```

diofant.combinatorics.named_groups.**AbelianGroup**(**cyclic_orders*)

Returns the direct product of cyclic groups with the given orders.

According to the structure theorem for finite abelian groups ([1]), every finite abelian group can be written as the direct product of finitely many cyclic groups.

See also:

diofant.combinatorics.group_constructs.DirectProduct (page 236)

References

[R51] (page 1251)

Examples

```
>>> Permutation.print_cyclic = True
>>> AbelianGroup(3, 4)
PermutationGroup([
        Permutation(6)(0, 1, 2),
        Permutation(3, 4, 5, 6)])
>>> _.is_group
True
```

3.2.9 Utilities

diofant.combinatorics.util.**_base_ordering**(*base, degree*)

Order $\{0, 1, ..., n-1\}$ so that base points come first and in order.

Parameters ``base`` - the base

``degree`` - the degree of the associated permutation group

Returns A list `base_ordering` such that `base_ordering[point]` is the number of `point` in the ordering.

Notes

This is used in backtrack searches, when we define a relation $<<$ on the underlying set for a permutation group of degree n, $\{0, 1, ..., n-1\}$, so that if $(b_1, b_2, ..., b_k)$ is a base we have $b_i << b_j$ whenever $i < j$ and $b_i << a$ for all $i \in \{1, 2, ..., k\}$ and a is not in the base. The idea is developed and applied to backtracking algorithms in [1], pp.108-132. The points that are not in the base are taken in increasing order.

References

[R64] (page 1251)

Examples

```
>>> S = SymmetricGroup(4)
>>> S.schreier_sims()
>>> _base_ordering(S.base, S.degree)
[0, 1, 2, 3]
```

diofant.combinatorics.util.**_check_cycles_alt_sym**(*perm*)

Checks for cycles of prime length p with n/2 < p < n-2.

Here n is the degree of the permutation. This is a helper function for the function is_alt_sym from diofant.combinatorics.perm_groups.

See also:

diofant.combinatorics.perm_groups.PermutationGroup.is_alt_sym (page 197)

Examples

```
>>> a = Permutation([[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10], [11, 12]])
>>> _check_cycles_alt_sym(a)
False
>>> b = Permutation([[0, 1, 2, 3, 4, 5, 6], [7, 8, 9, 10]])
>>> _check_cycles_alt_sym(b)
True
```

diofant.combinatorics.util.**_distribute_gens_by_base**(*base*, *gens*)

Distribute the group elements gens by membership in basic stabilizers.

Notice that for a base $(b_1, b_2, ..., b_k)$, the basic stabilizers are defined as $G^{(i)} = G_{b_1,...,b_{i-1}}$ for $i \in \{1, 2, ..., k\}$.

Parameters ‘‘base‘‘ - a sequence of points in ‘{0, 1, ..., n-1}‘

‘‘gens‘‘ - a list of elements of a permutation group of degree ‘n‘.

Returns List of length k, where k is

the length of base. The i-th entry contains those elements in

gens which fix the first i elements of base (so that the

0-th entry is equal to gens itself). If no element fixes the first

i elements of base, the i-th element is set to a list containing

the identity element.

See also:

_strong_gens_from_distr (page 235), *_orbits_transversals_from_bsgs* (page 233), *_handle_precomputed_bsgs* (page 232)

Examples

```
>>> Permutation.print_cyclic = True
>>> D = DihedralGroup(3)
>>> D.schreier_sims()
>>> D.strong_gens
[Permutation(0, 1, 2), Permutation(0, 2), Permutation(1, 2)]
>>> D.base
[0, 1]
>>> _distribute_gens_by_base(D.base, D.strong_gens)
[[Permutation(0, 1, 2), Permutation(0, 2), Permutation(1, 2)],
 [Permutation(1, 2)]]
```

diofant.combinatorics.util.**_handle_precomputed_bsgs**(*base*, *strong_gens*, *transversals=None*, *basic_orbits=None*, *strong_gens_distr=None*)

Calculate BSGS-related structures from those present.

The base and strong generating set must be provided; if any of the transversals, basic orbits or distributed strong generators are not provided, they will be calculated from the base and strong generating set.

Parameters ``base`` - the base

``strong_gens`` - the strong generators

``transversals`` - basic transversals

``basic_orbits`` - basic orbits

``strong_gens_distr`` - strong generators distributed by membership in basic

stabilizers

Returns `(transversals, basic_orbits, strong_gens_distr)` where `transversals`

are the basic transversals, `basic_orbits` are the basic orbits, and

`strong_gens_distr` are the strong generators distributed by membership

in basic stabilizers.

See also:

_orbits_transversals_from_bsgs (page 233), *_distribute_gens_by_base* (page 232)

Examples

```
>>> Permutation.print_cyclic = True
>>> D = DihedralGroup(3)
>>> D.schreier_sims()
>>> _handle_precomputed_bsgs(D.base, D.strong_gens,
... basic_orbits=D.basic_orbits)
([{0: Permutation(2), 1: Permutation(0, 1, 2), 2: Permutation(0, 2)},
{1: Permutation(2), 2: Permutation(1, 2)}],
[[0, 1, 2], [1, 2]], [[Permutation(0, 1, 2),
                       Permutation(0, 2),
                       Permutation(1, 2)],
                      [Permutation(1, 2)]])
```

diofant.combinatorics.util.**_orbits_transversals_from_bsgs**(*base*, *strong_gens_distr*, *transversals_only=False*)

Compute basic orbits and transversals from a base and strong generating set.

The generators are provided as distributed across the basic stabilizers. If the optional argument `transversals_only` is set to True, only the transversals are returned.

Parameters ``base`` - the base

``strong_gens_distr`` - strong generators distributed by membership in basic

stabilizers

``transversals_only`` - a flag switching between returning only the

transversals/ both orbits and transversals

See also:

_distribute_gens_by_base (page 232), *_handle_precomputed_bsgs* (page 232)

Examples

```
>>> Permutation.print_cyclic = True
>>> S = SymmetricGroup(3)
>>> S.schreier_sims()
>>> strong_gens_distr = _distribute_gens_by_base(S.base, S.strong_gens)
>>> _orbits_transversals_from_bsgs(S.base, strong_gens_distr)
([[0, 1, 2], [1, 2]],
[{0: Permutation(2), 1: Permutation(0, 1, 2), 2: Permutation(0, 2, 1)},
{1: Permutation(2), 2: Permutation(1, 2)}])
```

diofant.combinatorics.util.**_remove_gens**(*base, strong_gens, basic_orbits=None, strong_gens_distr=None*)

Remove redundant generators from a strong generating set.

Parameters ‘‘base‘‘ - a base

‘‘strong_gens‘‘ - a strong generating set relative to ‘‘base‘‘

‘‘basic_orbits‘‘ - basic orbits

‘‘strong_gens_distr‘‘ - strong generators distributed by membership in basic

stabilizers

Returns A strong generating set with respect to `base` which is a subset of `strong_gens`.

Notes

This procedure is outlined in [1],p.95.

References

[1] Holt, D., Eick, B., O’Brien, E. “Handbook of computational group theory”

Examples

```
>>> from diofant.combinatorics.testutil import _verify_bsgs
>>> S = SymmetricGroup(15)
>>> base, strong_gens = S.schreier_sims_incremental()
>>> new_gens = _remove_gens(base, strong_gens)
>>> len(new_gens)
14
>>> _verify_bsgs(S, base, new_gens)
True
```

`diofant.combinatorics.util._strip(g, base, orbits, transversals)`

Attempt to decompose a permutation using a (possibly partial) BSGS structure.

This is done by treating the sequence `base` as an actual base, and the orbits `orbits` and transversals `transversals` as basic orbits and transversals relative to it.

This process is called "sifting". A sift is unsuccessful when a certain orbit element is not found or when after the sift the decomposition doesn't end with the identity element.

The argument `transversals` is a list of dictionaries that provides transversal elements for the orbits `orbits`.

Parameters ''g'' - permutation to be decomposed

''base'' - sequence of points

''orbits'' - a list in which the ''i''-th entry is an orbit of ''base[i]''

under some subgroup of the pointwise stabilizer of '

'base[0], base[1], ..., base[i - 1]''. The groups themselves are implicit

in this function since the only information we need is encoded in the orbits

and transversals

''transversals'' - a list of orbit transversals associated with the orbits

''orbits''.

See also:

diofant.combinatorics.perm_groups.PermutationGroup.schreier_sims (page 206), *diofant.combinatorics.perm_groups.PermutationGroup.schreier_sims_random* (page 207)

Notes

The algorithm is described in [1],pp.89-90. The reason for returning both the current state of the element being decomposed and the level at which the sifting ends is that they provide important information for the randomized version of the Schreier-Sims algorithm.

References

[1] Holt, D., Eick, B., O'Brien, E. "Handbook of computational group theory"

Examples

```
>>> Permutation.print_cyclic = True
>>> S = SymmetricGroup(5)
>>> S.schreier_sims()
>>> g = Permutation([0, 2, 3, 1, 4])
>>> _strip(g, S.base, S.basic_orbits, S.basic_transversals)
(Permutation(4), 5)
```

diofant.combinatorics.util.**_strong_gens_from_distr**(*strong_gens_distr*)
Retrieve strong generating set from generators of basic stabilizers.

This is just the union of the generators of the first and second basic stabilizers.

Parameters ‘‘strong_gens_distr‘‘ - strong generators distributed by membership in basic

stabilizers

See also:

_distribute_gens_by_base (page 232)

Examples

```
>>> Permutation.print_cyclic = True
>>> S = SymmetricGroup(3)
>>> S.schreier_sims()
>>> S.strong_gens
[Permutation(0, 1, 2), Permutation(2)(0, 1), Permutation(1, 2)]
>>> strong_gens_distr = _distribute_gens_by_base(S.base, S.strong_gens)
>>> _strong_gens_from_distr(strong_gens_distr)
[Permutation(0, 1, 2), Permutation(2)(0, 1), Permutation(1, 2)]
```

3.2.10 Group constructors

diofant.combinatorics.group_constructs.**DirectProduct**(**groups*)
Returns the direct product of several groups as a permutation group.

This is implemented much like the *__mul__()* (page 186) procedure for taking the direct product of two permutation groups, but the idea of shifting the generators is realized in the case of an arbitrary number of groups. A call to DirectProduct(G1, G2, ..., Gn) is generally expected to be faster than a call to G1*G2*...*Gn (and thus the need for this algorithm).

See also:

diofant.combinatorics.perm_groups.PermutationGroup.__mul__ (page 186)

Examples

```
>>> C = CyclicGroup(4)
>>> G = DirectProduct(C, C, C)
>>> G.order()
64
```

3.2.11 Test Utilities

diofant.combinatorics.testutil.**_cmp_perm_lists**(*first, second*)
Compare two lists of permutations as sets.

This is used for testing purposes. Since the array form of a permutation is currently a list, Permutation is not hashable and cannot be put into a set.

Examples

```
>>> a = Permutation([0, 2, 3, 4, 1])
>>> b = Permutation([1, 2, 0, 4, 3])
>>> c = Permutation([3, 4, 0, 1, 2])
>>> ls1 = [a, b, c]
>>> ls2 = [b, c, a]
>>> _cmp_perm_lists(ls1, ls2)
True
```

diofant.combinatorics.testutil.**_naive_list_centralizer**(*self*, *other*, *af=False*)

diofant.combinatorics.testutil.**_verify_bsgs**(*group*, *base*, *gens*)

Verify the correctness of a base and strong generating set.

This is a naive implementation using the definition of a base and a strong generating set relative to it. There are other procedures for verifying a base and strong generating set, but this one will serve for more robust testing.

See also:

diofant.combinatorics.perm_groups.PermutationGroup.schreier_sims (page 206)

Examples

```
>>> A = AlternatingGroup(4)
>>> A.schreier_sims()
>>> _verify_bsgs(A, A.base, A.strong_gens)
True
```

diofant.combinatorics.testutil.**_verify_centralizer**(*group*, *arg*, *centr=None*)

Verify the centralizer of a group/set/element inside another group.

This is used for testing `.centralizer()` from `diofant.combinatorics.perm_groups`

See also:

_naive_list_centralizer (page 237), *diofant.combinatorics.perm_groups.PermutationGroup.centralizer* (page 191), *_cmp_perm_lists* (page 236)

Examples

… AlternatingGroup) >>> S = SymmetricGroup(5) >>> A = AlternatingGroup(5) >>> centr = PermutationGroup([Permutation([0, 1, 2, 3, 4])]) >>> _verify_centralizer(S, A, centr) True

diofant.combinatorics.testutil.**_verify_normal_closure**(*group*, *arg*, *closure=None*)

Verify the normal closure of a subgroup/subset/element in a group.

This is used to test diofant.combinatorics.perm_groups.PermutationGroup.normal_closure

See also:

diofant.combinatorics.perm_groups.PermutationGroup.normal_closure (page 203)

Examples

```
>>> S = SymmetricGroup(3)
>>> A = AlternatingGroup(3)
>>> _verify_normal_closure(S, A, closure=A)
True
```

3.2.12 Tensor Canonicalization

diofant.combinatorics.tensor_can.**canonicalize**(*g*, *dummies*, *msym*, **v*)
canonicalize tensor formed by tensors

Parameters g : permutation representing the tensor

dummies : list representing the dummy indices

it can be a list of dummy indices of the same type or a list of lists of dummy indices, one list for each type of index; the dummy indices must come after the free indices, and put in order contravariant, covariant [d0, -d0, d1,-d1,...]

msym : symmetry of the metric(s)

it can be an integer or a list; in the first case it is the symmetry of the dummy index metric; in the second case it is the list of the symmetries of the index metric for each type

v : list, (base_i, gens_i, n_i, sym_i) for tensors of type i

base_i, gens_i : BSGS for tensors of this type.

The BSGS should have minimal base under lexicographic ordering; if not, an attempt is made do get the minimal BSGS; in case of failure, canonicalize_naive is used, which is much slower.

n_i : number of tensors of type i.

sym_i : symmetry under exchange of component tensors of type i.

Both for msym and sym_i the cases are

- None no symmetry
- 0 commuting
- 1 anticommuting

Returns 0 if the tensor is zero, else return the array form of

the permutation representing the canonical form of the tensor.

Notes

First one uses canonical_free to get the minimum tensor under lexicographic order, using only the slot symmetries. If the component tensors have not minimal BSGS, it is attempted to find it; if the attempt fails canonicalize_naive is used instead.

Compute the residual slot symmetry keeping fixed the free indices using tensor_gens(base, gens, list_free_indices, sym).

Reduce the problem eliminating the free indices.

Then use double_coset_can_rep and lift back the result reintroducing the free indices.

Examples

one type of index with commuting metric;

A_{ab} and B_{ab} antisymmetric and commuting

$T = A_{d0d1} * B^{d0}{}_{d2} * B^{d2d1}$

$ord = [d0, -d0, d1, -d1, d2, -d2]$ order of the indices

g = [1, 3, 0, 5, 4, 2, 6, 7]

$T_c = 0$

```
>>> base2a, gens2a = get_symmetric_group_sgs(2, 1)
>>> t0 = (base2a, gens2a, 1, 0)
>>> t1 = (base2a, gens2a, 2, 0)
>>> g = Permutation([1, 3, 0, 5, 4, 2, 6, 7])
>>> canonicalize(g, range(6), 0, t0, t1)
0
```

same as above, but with B_{ab} anticommuting

$T_c = -A^{d0d1} * B_{d0}{}^{d2} * B_{d1d2}$

can = [0,2,1,4,3,5,7,6]

```
>>> t1 = (base2a, gens2a, 2, 1)
>>> canonicalize(g, range(6), 0, t0, t1)
[0, 2, 1, 4, 3, 5, 7, 6]
```

two types of indices $[a, b, c, d, e, f]$ and $[m, n]$, in this order, both with commuting metric

f^{abc} antisymmetric, commuting

A_{ma} no symmetry, commuting

$T = f^c{}_{da} * f^f{}_{eb} * A_m{}^d * A^{mb} * A_n{}^a * A^{ne}$

ord = [c,f,a,-a,b,-b,d,-d,e,-e,m,-m,n,-n]

g = [0,7,3, 1,9,5, 11,6, 10,4, 13,2, 12,8, 14,15]

The canonical tensor is $T_c = -f^{cab} * f^{fde} * A^m{}_a * A_{md} * A^n{}_b * A_{ne}$

can = [0,2,4, 1,6,8, 10,3, 11,7, 12,5, 13,9, 15,14]

```
>>> base_f, gens_f = get_symmetric_group_sgs(3, 1)
>>> base1, gens1 = get_symmetric_group_sgs(1)
>>> base_A, gens_A = bsgs_direct_product(base1, gens1, base1, gens1)
>>> t0 = (base_f, gens_f, 2, 0)
>>> t1 = (base_A, gens_A, 4, 0)
>>> dummies = [range(2, 10), range(10, 14)]
>>> g = Permutation([0, 7, 3, 1, 9, 5, 11, 6, 10, 4, 13, 2, 12, 8, 14, 15])
>>> canonicalize(g, dummies, [0, 0], t0, t1)
[0, 2, 4, 1, 6, 8, 10, 3, 11, 7, 12, 5, 13, 9, 15, 14]
```

diofant.combinatorics.tensor_can.**double_coset_can_rep**(*dummies, sym, b_S, sgens, S_transversals, g*)

Butler-Portugal algorithm for tensor canonicalization with dummy indices

dummies list of lists of dummy indices, one list for each type of index; the dummy indices are put in order contravariant, covariant [d0, -d0, d1, -d1, …].

sym list of the symmetries of the index metric for each type.

possible symmetries of the metrics

- 0 symmetric
- 1 antisymmetric
- None no symmetry

b_S base of a minimal slot symmetry BSGS.

sgens generators of the slot symmetry BSGS.

S_transversals transversals for the slot BSGS.

g permutation representing the tensor.

Return 0 if the tensor is zero, else return the array form of the permutation representing the canonical form of the tensor.

A tensor with dummy indices can be represented in a number of equivalent ways which typically grows exponentially with the number of indices. To be able to establish if two tensors with many indices are equal becomes computationally very slow in absence of an efficient algorithm.

The Butler-Portugal algorithm [3] is an efficient algorithm to put tensors in canonical form, solving the above problem.

Portugal observed that a tensor can be represented by a permutation, and that the class of tensors equivalent to it under slot and dummy symmetries is equivalent to the double coset $D * g * S$ (Note: in this documentation we use the conventions for multiplication of permutations p, q with (p*q)(i) = p[q[i]] which is opposite to the one used in the Permutation class)

Using the algorithm by Butler to find a representative of the double coset one can find a canonical form for the tensor.

To see this correspondence, let g be a permutation in array form; a tensor with indices ind (the indices including both the contravariant and the covariant ones) can be written as

$t = T(ind[g[0], ..., ind[g[n-1]]),$

where $n = len(ind)$; g has size $n+2$, the last two indices for the sign of the tensor (trick introduced in [4]).

A slot symmetry transformation s is a permutation acting on the slots $t- > T(ind[(g*s)[0]], ..., ind[(g*s)[n-1]])$

A dummy symmetry transformation acts on ind $t- > T(ind[(d*g)[0]], ..., ind[(d*g)[n-1]])$

Being interested only in the transformations of the tensor under these symmetries, one can represent the tensor by g, which transforms as

$g- > d*g*s$, so it belongs to the coset $D*g*S$.

Let us explain the conventions by an example.

Given a tensor $T^{d3d2d1}{}_{d1d2d3}$ **with the slot symmetries** $T^{a0a1a2a3a4a5} = -T^{a2a1a0a3a4a5}$

$T^{a0a1a2a3a4a5} = -T^{a4a1a2a3a0a5}$

and symmetric metric, find the tensor equivalent to it which is the lowest under the ordering of indices: lexicographic ordering $d1, d2, d3$ then and contravariant index before covariant index; that is the canonical form of the tensor.

The canonical form is $-T^{d1d2d3}{}_{d1d2d3}$ obtained using $T^{a0a1a2a3a4a5} = -T^{a2a1a0a3a4a5}$.

To convert this problem in the input for this function, use the following labelling of the index names (- for covariant for short) $d1, -d1, d2, -d2, d3, -d3$

$T^{d3d2d1}{}_{d1d2d3}$ corresponds to $g = [4, 2, 0, 1, 3, 5, 6, 7]$ where the last two indices are for the sign

$sgens = [Permutation(0, 2)(6, 7), Permutation(0, 4)(6, 7)]$

sgens[0] is the slot symmetry $-(0, 2)$ $T^{a0a1a2a3a4a5} = -T^{a2a1a0a3a4a5}$

sgens[1] is the slot symmetry $-(0, 4)$ $T^{a0a1a2a3a4a5} = -T^{a4a1a2a3a0a5}$

The dummy symmetry group D is generated by the strong base generators $[(0, 1), (2, 3), (4, 5), (0, 1)(2, 3), (2, 3)(4, 5)]$

The dummy symmetry acts from the left $d = [1, 0, 2, 3, 4, 5, 6, 7]$ exchange $d1- > -d1$ $T^{d3d2d1}{}_{d1d2d3} == T^{d3d2}{}_{d1}{}^{d1}{}_{d2d3}$

$g = [4, 2, 0, 1, 3, 5, 6, 7]- > [4, 2, 1, 0, 3, 5, 6, 7] =_a f_r mul(d, g)$ which differs from ${}_a f_r mul(g, d)$.

The slot symmetry acts from the right $s = [2, 1, 0, 3, 4, 5, 7, 6]$ exchanges slots 0 and 2 and changes sign $T^{d3d2d1}{}_{d1d2d3} == -T^{d1d2d3}{}_{d1d2d3}$

$g = [4, 2, 0, 1, 3, 5, 6, 7]- > [0, 2, 4, 1, 3, 5, 7, 6] =_a f_r mul(g, s)$

Example in which the tensor is zero, same slot symmetries as above: $T^{d3}{}_{d1,d2}{}^{d1}{}_{d3}{}^{d2}$

$= -T^{d3}{}_{d1,d3}{}^{d1}{}_{d2}{}^{d2}$ under slot symmetry $-(2, 4)$;

$= T_{d3d1}{}^{d3d1}{}_{d2}{}^{d2}$ under slot symmetry $-(0, 2)$;

$= T^{d3}{}_{d1d3}{}^{d1}{}_{d2}{}^{d2}$ symmetric metric;

$= 0$ since two of these lines have tensors differ only for the sign.

The double coset D*g*S consists of permutations $h = d * g * s$ corresponding to equivalent tensors; if there are two h which are the same apart from the sign, return zero; otherwise choose as representative the tensor with indices ordered lexicographically according to $[d1, -d1, d2, -d2, d3, -d3]$ that is $rep = min(D * g * S) = min([d * g * sfordinDforsinS])$

The indices are fixed one by one; first choose the lowest index for slot 0, then the lowest remaining index for slot 1, etc. Doing this one obtains a chain of stabilizers

$S- > S_{b0}- > S_{b0,b1}- > ...$ and $D- > D_{p0}- > D_{p0,p1}- > ...$

where $[b0, b1, ...] = range(b)$ is a base of the symmetric group; the strong base b_S of S is an ordered sublist of it; therefore it is sufficient to compute once the strong base generators of S using the Schreier-Sims algorithm; the stabilizers of the strong base generators are the strong base generators of the stabilizer subgroup.

$dbase = [p0, p1, ...]$ is not in general in lexicographic order, so that one must recompute the strong base generators each time; however this is trivial, there is no need to use the Schreier-Sims algorithm for D.

The algorithm keeps a TAB of elements (s_i, d_i, h_i) where $h_i = d_i * g * s_i$ satisfying $h_i[j] = p_j$ for $0 <= j < i$ starting from $s_0 = id, d_0 = id, h_0 = g$.

The equations $h_0[0] = p_0, h_1[1] = p_1, \ldots$ are solved in this order, choosing each time the lowest possible value of p_i

For $j < i$ $d_i * g * s_i * S_{b_0,\ldots,b_{i-1}} * b_j = D_{p_0,\ldots,p_{i-1}} * p_j$ so that for dx in $D_{p_0,\ldots,p_{i-1}}$ and sx in $S_{base[0],\ldots,base[i-1]}$ one has $dx * d_i * g * s_i * sx * b_j = p_j$

Search for dx, sx such that this equation holds for $j = i$; it can be written as $s_i * sx * b_j = J, dx * d_i * g * J = p_j$ $sx * b_j = s_i ** -1 * J; sx = trace(s_i ** -1, S_{b_0,\ldots,b_{i-1}})$ $dx ** -1 * p_j = d_i * g * J; dx = trace(d_i * g * J, D_{p_0,\ldots,p_{i-1}})$

$s_{i+1} = s_i * trace(s_i ** -1 * J, S_{b_0,\ldots,b_{i-1}})$ $d_{i+1} = trace(d_i * g * J, D_{p_0,\ldots,p_{i-1}}) ** -1 * d_i$ $h_{i+1} * b_i = d_{i+1} * g * s_{i+1} * b_i = p_i$

$h_n * b_j = p_j$ for all j, so that h_n is the solution.

Add the found (s, d, h) to TAB1.

At the end of the iteration sort TAB1 with respect to the h; if there are two consecutive h in TAB1 which differ only for the sign, the tensor is zero, so return 0; if there are two consecutive h which are equal, keep only one.

Then stabilize the slot generators under i and the dummy generators under p_i.

Assign $TAB = TAB1$ at the end of the iteration step.

At the end TAB contains a unique (s, d, h), since all the slots of the tensor h have been fixed to have the minimum value according to the symmetries. The algorithm returns h.

It is important that the slot BSGS has lexicographic minimal base, otherwise there is an i which does not belong to the slot base for which p_i is fixed by the dummy symmetry only, while i is not invariant from the slot stabilizer, so p_i is not in general the minimal value.

This algorithm differs slightly from the original algorithm [3]: the canonical form is minimal lexicographically, and the BSGS has minimal base under lexicographic order. Equal tensors h are eliminated from TAB.

Examples

```
>>> gens = [Permutation(x) for x in [[2, 1, 0, 3, 4, 5, 7, 6], [4, 1, 2, 3, 0, 5,
↪7, 6]]]
>>> base = [0, 2]
>>> g = Permutation([4, 2, 0, 1, 3, 5, 6, 7])
>>> transversals = get_transversals(base, gens)
>>> double_coset_can_rep([list(range(6))], [0], base, gens, transversals, g)
[0, 1, 2, 3, 4, 5, 7, 6]
```

```
>>> g = Permutation([4, 1, 3, 0, 5, 2, 6, 7])
>>> double_coset_can_rep([list(range(6))], [0], base, gens, transversals, g)
0
```

diofant.combinatorics.tensor_can.**get_symmetric_group_sgs**(*n*, *antisym=False*)

Return base, gens of the minimal BSGS for (anti)symmetric tensor

`n` rank of the tensor

`antisym = False` symmetric tensor `antisym = True` antisymmetric tensor

Examples

```
>>> Permutation.print_cyclic = True
>>> get_symmetric_group_sgs(3)
([0, 1], [Permutation(4)(0, 1), Permutation(4)(1, 2)])
```

diofant.combinatorics.tensor_can.**bsgs_direct_product**(*base1*, *gens1*, *base2*, *gens2*, *signed=True*)

direct product of two BSGS

base1 base of the first BSGS.

gens1 strong generating sequence of the first BSGS.

base2, gens2 similarly for the second BSGS.

signed flag for signed permutations.

Examples

```
>>> Permutation.print_cyclic = True
>>> base1, gens1 = get_symmetric_group_sgs(1)
>>> base2, gens2 = get_symmetric_group_sgs(2)
>>> bsgs_direct_product(base1, gens1, base2, gens2)
([1], [Permutation(4)(1, 2)])
```

3.3 Number Theory

3.3.1 Ntheory Class Reference

class diofant.ntheory.generate.**Sieve**

An infinite list of prime numbers, implemented as a dynamically growing sieve of Eratosthenes. When a lookup is requested involving an odd number that has not been sieved, the sieve is automatically extended up to that number.

```
>>> from array import array # this line and next for doctest only
>>> sieve._list = array('l', [2, 3, 5, 7, 11, 13])
```

```
>>> 25 in sieve
False
>>> sieve._list
array('l', [2, 3, 5, 7, 11, 13, 17, 19, 23])
```

extend(*n*)

Grow the sieve to cover all primes <= n (a real number).

Examples

```
>>> from array import array # this line and next for doctest only
>>> sieve._list = array('l', [2, 3, 5, 7, 11, 13])
```

```
>>> sieve.extend(30)
>>> sieve[10] == 29
True
```

extend_to_no(*i*)
Extend to include the ith prime number.

i must be an integer.

The list is extended by 50% if it is too short, so it is likely that it will be longer than requested.

Examples

```
>>> from array import array # this line and next for doctest only
>>> sieve._list = array('l', [2, 3, 5, 7, 11, 13])
```

```
>>> sieve.extend_to_no(9)
>>> sieve._list
array('l', [2, 3, 5, 7, 11, 13, 17, 19, 23])
```

primerange(*a*, *b*)
Generate all prime numbers in the range [a, b).

Examples

```
>>> print([i for i in sieve.primerange(7, 18)])
[7, 11, 13, 17]
```

search(*n*)
Return the indices i, j of the primes that bound n.

If n is prime then i == j.

Although n can be an expression, if ceiling cannot convert it to an integer then an n error will be raised.

Examples

```
>>> sieve.search(25)
(9, 10)
>>> sieve.search(23)
(9, 9)
```

3.3.2 Ntheory Functions Reference

diofant.ntheory.generate.**prime**(*nth*)
Return the nth prime, with the primes indexed as prime(1) = 2, prime(2) = 3, etc.... The nth prime is approximately n*log(n) and can never be larger than 2**n.

See also:

diofant.ntheory.primetest.isprime **(page 263)** Test if n is prime

primerange **(page 246)** Generate all primes in a given range

primepi **(page 245)** Return the number of primes less than or equal to n

References

[R417] (page 1251)

Examples

```
>>> prime(10)
29
>>> prime(1)
2
```

diofant.ntheory.generate.**primepi**(*n*)

Return the value of the prime counting function pi(n) = the number of prime numbers less than or equal to n.

See also:

diofant.ntheory.primetest.isprime **(page 263)** Test if n is prime

primerange **(page 246)** Generate all primes in a given range

prime **(page 244)** Return the nth prime

Examples

```
>>> primepi(25)
9
```

diofant.ntheory.generate.**nextprime**(*n*, *ith=1*)

Return the ith prime greater than n.

i must be an integer.

See also:

prevprime **(page 245)** Return the largest prime smaller than n

primerange **(page 246)** Generate all primes in a given range

Notes

Potential primes are located at 6*j +/- 1. This property is used during searching.

```
>>> [(i, nextprime(i)) for i in range(10, 15)]
[(10, 11), (11, 13), (12, 13), (13, 17), (14, 17)]
>>> nextprime(2, ith=2)  # the 2nd prime after 2
5
```

diofant.ntheory.generate.**prevprime**(*n*)

Return the largest prime smaller than n.

See also:

nextprime **(page 245)** Return the ith prime greater than n

primerange **(page 246)** Generates all primes in a given range

Notes

Potential primes are located at 6*j +/- 1. This property is used during searching.

```
>>> [(i, prevprime(i)) for i in range(10, 15)]
[(10, 7), (11, 7), (12, 11), (13, 11), (14, 13)]
```

diofant.ntheory.generate.**primerange**(*a*, *b*)

Generate a list of all prime numbers in the range [a, b).

If the range exists in the default sieve, the values will be returned from there; otherwise values will be returned but will not modify the sieve.

See also:

nextprime **(page 245)** Return the ith prime greater than n

prevprime **(page 245)** Return the largest prime smaller than n

randprime **(page 247)** Returns a random prime in a given range

primorial **(page 247)** Returns the product of primes based on condition

Sieve.primerange **(page 244)** return range from already computed primes or extend the sieve to contain the requested range.

Notes

Some famous conjectures about the occurrence of primes in a given range are *[R418]* (page 1251):

- **Twin primes: though often not, the following will give 2 primes** an infinite number of times: primerange(6*n - 1, 6*n + 2)
- **Legendre's: the following always yields at least one prime** primerange(n**2, (n+1)**2+1)
- **Bertrand's (proven): there is always a prime in the range** primerange(n, 2*n)
- **Brocard's: there are at least four primes in the range** primerange(prime(n)**2, prime(n+1)**2)

The average gap between primes is log(n) *[R419]* (page 1251); the gap between primes can be arbitrarily large since sequences of composite numbers are arbitrarily large, e.g. the numbers in the sequence n! + 2, n! + 3 … n! + n are all composite.

References

[R418] (page 1251), [R419] (page 1251)

Examples

```
>>> print([i for i in primerange(1, 30)])
[2, 3, 5, 7, 11, 13, 17, 19, 23, 29]
```

The Sieve method, primerange, is generally faster but it will occupy more memory as the sieve stores values. The default instance of Sieve, named sieve, can be used:

```
>>> list(sieve.primerange(1, 30))
[2, 3, 5, 7, 11, 13, 17, 19, 23, 29]
```

`diofant.ntheory.generate.`**`randprime`**`(`*a*, *b*`)`

Return a random prime number in the range [a, b).

Bertrand's postulate assures that randprime(a, 2*a) will always succeed for a > 1.

See also:

primerange **(page 246)** Generate all primes in a given range

References

[R420] (page 1251)

Examples

```
>>> randprime(1, 30)
13
>>> isprime(randprime(1, 30))
True
```

`diofant.ntheory.generate.`**`primorial`**`(`*n*, *nth=True*`)`

Returns the product of the first n primes (default) or the primes less than or equal to n (when `nth=False`).

```
>>> primorial(4) # the first 4 primes are 2, 3, 5, 7
210
>>> primorial(4, nth=False) # primes <= 4 are 2 and 3
6
>>> primorial(1)
2
>>> primorial(1, nth=False)
1
>>> primorial(sqrt(101), nth=False)
210
```

One can argue that the primes are infinite since if you take a set of primes and multiply them together (e.g. the primorial) and then add or subtract 1, the result cannot be

divided by any of the original factors, hence either 1 or more new primes must divide this product of primes.

In this case, the number itself is a new prime:

```
>>> factorint(primorial(4) + 1)
{211: 1}
```

In this case two new primes are the factors:

```
>>> factorint(primorial(4) - 1)
{11: 1, 19: 1}
```

Here, some primes smaller and larger than the primes multiplied together are obtained:

```
>>> p = list(primerange(10, 20))
>>> sorted(set(primefactors(Mul(*p) + 1)).difference(set(p)))
[2, 5, 31, 149]
```

See also:

primerange (page 246) Generate all primes in a given range

diofant.ntheory.generate.**cycle_length**(*f*, *x0*, *nmax=None*, *values=False*)

For a given iterated sequence, return a generator that gives the length of the iterated cycle (lambda) and the length of terms before the cycle begins (mu); if `values` is True then the terms of the sequence will be returned instead. The sequence is started with value `x0`.

Note: more than the first lambda + mu terms may be returned and this is the cost of cycle detection with Brent's method; there are, however, generally less terms calculated than would have been calculated if the proper ending point were determined, e.g. by using Floyd's method.

This will yield successive values of i <– func(i):

```
>>> def iter(func, i):
...     while 1:
...         ii = func(i)
...         yield ii
...         i = ii
...
```

A function is defined:

```
>>> func = lambda i: (i**2 + 1) % 51
```

and given a seed of 4 and the mu and lambda terms calculated:

```
>>> next(cycle_length(func, 4))
(6, 2)
```

We can see what is meant by looking at the output:

```
>>> n = cycle_length(func, 4, values=True)
>>> list(ni for ni in n)
[17, 35, 2, 5, 26, 14, 44, 50, 2, 5, 26, 14]
```

There are 6 repeating values after the first 2.

If a sequence is suspected of being longer than you might wish, nmax can be used to exit early (and mu will be returned as None):

```
>>> next(cycle_length(func, 4, nmax = 4))
(4, None)
>>> [ni for ni in cycle_length(func, 4, nmax = 4, values=True)]
[17, 35, 2, 5]
```

References

[R421] (page 1251)

diofant.ntheory.factor_.**smoothness**(*n*)

Return the B-smooth and B-power smooth values of n.

The smoothness of n is the largest prime factor of n; the power- smoothness is the largest divisor raised to its multiplicity.

```
>>> smoothness(2**7*3**2)
(3, 128)
>>> smoothness(2**4*13)
(13, 16)
>>> smoothness(2)
(2, 2)
```

See also:

factorint (page 254), *smoothness_p* (page 249)

diofant.ntheory.factor_.**smoothness_p**(*n*, *m=-1*, *power=0*, *visual=None*)

Return a list of [m, (p, (M, sm(p + m), psm(p + m)))...] where:

1. p**M is the base-p divisor of n
2. sm(p + m) is the smoothness of p + m (m = -1 by default)
3. psm(p + m) is the power smoothness of p + m

The list is sorted according to smoothness (default) or by power smoothness if power=1.

The smoothness of the numbers to the left (m = -1) or right (m = 1) of a factor govern the results that are obtained from the p +/- 1 type factoring methods.

```
>>> smoothness_p(10431, m=1)
(1, [(3, (2, 2, 4)), (19, (1, 5, 5)), (61, (1, 31, 31))])
>>> smoothness_p(10431)
(-1, [(3, (2, 2, 2)), (19, (1, 3, 9)), (61, (1, 5, 5))])
>>> smoothness_p(10431, power=1)
(-1, [(3, (2, 2, 2)), (61, (1, 5, 5)), (19, (1, 3, 9))])
```

If visual=True then an annotated string will be returned:

```
>>> print(smoothness_p(21477639576571, visual=1))
p**i=4410317**1 has p-1 B=1787, B-pow=1787
p**i=4869863**1 has p-1 B=2434931, B-pow=2434931
```

This string can also be generated directly from a factorization dictionary and vice versa:

```
>>> f = factorint(17*9)
>>> f
{3: 2, 17: 1}
>>> smoothness_p(f)
'p**i=3**2 has p-1 B=2, B-pow=2\np**i=17**1 has p-1 B=2, B-pow=16'
>>> smoothness_p(_)
{3: 2, 17: 1}
```

The table of the output logic is:

	Visual		
Input	True	False	other
dict	str	tuple	str
str	str	tuple	dict
tuple	str	tuple	str
n	str	tuple	tuple
mul	str	tuple	tuple

See also:

factorint (page 254), *smoothness* (page 249)

diofant.ntheory.factor_.**trailing**(*n*)

Count the number of trailing zero digits in the binary representation of n, i.e. determine the largest power of 2 that divides n.

Examples

```
>>> trailing(128)
7
>>> trailing(63)
0
```

diofant.ntheory.factor_.**multiplicity**(*p*, *n*)

Find the greatest integer m such that p**m divides n.

Examples

```
>>> [multiplicity(5, n) for n in [8, 5, 25, 125, 250]]
[0, 1, 2, 3, 3]
>>> multiplicity(3, Rational(1, 9))
-2
```

diofant.ntheory.factor_.**perfect_power**(*n*, *candidates=None*, *big=True*, *factor=True*)

Return `(b, e)` such that `n == b**e` if `n` is a perfect power; otherwise return `False`.

By default, the base is recursively decomposed and the exponents collected so the largest possible `e` is sought. If `big=False` then the smallest possible `e` (thus prime) will be chosen.

If `candidates` for exponents are given, they are assumed to be sorted and the first one that is larger than the computed maximum will signal failure for the routine.

If `factor=True` then simultaneous factorization of n is attempted since finding a factor indicates the only possible root for n. This is True by default since only a few small factors will be tested in the course of searching for the perfect power.

Examples

```
>>> perfect_power(16)
(2, 4)
>>> perfect_power(16, big = False)
(4, 2)
```

diofant.ntheory.factor_.**pollard_rho**(*n, s=2, a=1, retries=5, seed=1234, max_steps=None, F=None*)

Use Pollard's rho method to try to extract a nontrivial factor of `n`. The returned factor may be a composite number. If no factor is found, `None` is returned.

The algorithm generates pseudo-random values of x with a generator function, replacing x with F(x). If F is not supplied then the function x**2 + `a` is used. The first value supplied to F(x) is `s`. Upon failure (if `retries` is > 0) a new `a` and `s` will be supplied; the `a` will be ignored if F was supplied.

The sequence of numbers generated by such functions generally have a a lead-up to some number and then loop around back to that number and begin to repeat the sequence, e.g. 1, 2, 3, 4, 5, 3, 4, 5 – this leader and loop look a bit like the Greek letter rho, and thus the name, 'rho'.

For a given function, very different leader-loop values can be obtained so it is a good idea to allow for retries:

```
>>> n = 16843009
>>> F = lambda x: (2048*pow(x, 2, n) + 32767)%n
>>> for s in range(5):
...     print('loop length = %4i; leader length = %3i' % next(cycle_length(F, s)))
...
loop length = 2489; leader length =  42
loop length =   78; leader length = 120
loop length = 1482; leader length =  99
loop length = 1482; leader length = 285
loop length = 1482; leader length = 100
```

Here is an explicit example where there is a two element leadup to a sequence of 3 numbers (11, 14, 4) that then repeat:

```
>>> x = 2
>>> for i in range(9):
...     x = (x**2 + 12)%17
...     print(x)
...
16
13
11
14
4
```

(continues on next page)

(continued from previous page)

```
11
14
4
11
>>> next(cycle_length(lambda x: (x**2+12)%17, 2))
(3, 2)
>>> list(cycle_length(lambda x: (x**2+12)%17, 2, values=True))
[16, 13, 11, 14, 4]
```

Instead of checking the differences of all generated values for a gcd with n, only the kth and 2*kth numbers are checked, e.g. 1st and 2nd, 2nd and 4th, 3rd and 6th until it has been detected that the loop has been traversed. Loops may be many thousands of steps long before rho finds a factor or reports failure. If `max_steps` is specified, the iteration is cancelled with a failure after the specified number of steps.

References

[R422] (page 1251)

Examples

```
>>> n = 16843009
>>> F = lambda x: (2048*pow(x, 2, n) + 32767) % n
>>> pollard_rho(n, F=F)
257
```

Use the default setting with a bad value of `a` and no retries:

```
>>> pollard_rho(n, a=n-2, retries=0)
```

If retries is > 0 then perhaps the problem will correct itself when new values are generated for a:

```
>>> pollard_rho(n, a=n-2, retries=1)
257
```

`diofant.ntheory.factor_.`**`pollard_pm1`**(*n*, *B=10*, *a=2*, *retries=0*, *seed=1234*)

Use Pollard's p-1 method to try to extract a nontrivial factor of `n`. Either a divisor (perhaps composite) or `None` is returned.

The value of `a` is the base that is used in the test gcd(a**M - 1, n). The default is 2. If `retries` > 0 then if no factor is found after the first attempt, a new `a` will be generated randomly (using the `seed`) and the process repeated.

Note: the value of M is lcm(1..B) = reduce(ilcm, range(2, B + 1)).

A search is made for factors next to even numbers having a power smoothness less than `B`. Choosing a larger B increases the likelihood of finding a larger factor but takes longer. Whether a factor of n is found or not depends on `a` and the power smoothness of the even mumber just less than the factor p (hence the name p - 1).

Although some discussion of what constitutes a good `a` some descriptions are hard to interpret. At the modular.math site referenced below it is stated that if gcd(a**M - 1, n) = N then a**M % q**r is 1 for every prime power divisor of N. But consider the following:

```
>>> n = 257*1009
>>> smoothness_p(n)
(-1, [(257, (1, 2, 256)), (1009, (1, 7, 16))])
```

So we should (and can) find a root with B=16:

```
>>> pollard_pm1(n, B=16, a=3)
1009
```

If we attempt to increase B to 256 we find that it doesn't work:

```
>>> pollard_pm1(n, B=256)
>>>
```

But if the value of a is changed we find that only multiples of 257 work, e.g.:

```
>>> pollard_pm1(n, B=256, a=257)
1009
```

Checking different a values shows that all the ones that didn't work had a gcd value not equal to n but equal to one of the factors:

```
>>> M = 1
>>> for i in range(2, 256):
...     M = ilcm(M, i)
...
>>> {igcd(pow(a, M, n) - 1, n) for a in range(2, 256) if
...  igcd(pow(a, M, n) - 1, n) != n}
{1009}
```

But does aM % d for every divisor of n give 1?

```
>>> aM = pow(255, M, n)
>>> [(d, aM%Pow(*d.args)) for d in factorint(n, visual=True).args]
[(257**1, 1), (1009**1, 1)]
```

No, only one of them. So perhaps the principle is that a root will be found for a given value of B provided that:

1. the power smoothness of the p - 1 value next to the root does not exceed B
2. a**M % p != 1 for any of the divisors of n.

By trying more than one a it is possible that one of them will yield a factor.

References

[R423] (page 1251), *[R424]* (page 1251), *[R425]* (page 1251)

Examples

With the default smoothness bound, this number can't be cracked:

```
>>> pollard_pm1(21477639576571)
```

Increasing the smoothness bound helps:

```
>>> pollard_pm1(21477639576571, B=2000)
4410317
```

Looking at the smoothness of the factors of this number we find:

```
>>> print(smoothness_p(21477639576571, visual=1))
p**i=4410317**1 has p-1 B=1787, B-pow=1787
p**i=4869863**1 has p-1 B=2434931, B-pow=2434931
```

The B and B-pow are the same for the p - 1 factorizations of the divisors because those factorizations had a very large prime factor:

```
>>> factorint(4410317 - 1)
{2: 2, 617: 1, 1787: 1}
>>> factorint(4869863-1)
{2: 1, 2434931: 1}
```

Note that until B reaches the B-pow value of 1787, the number is not cracked;

```
>>> pollard_pm1(21477639576571, B=1786)
>>> pollard_pm1(21477639576571, B=1787)
4410317
```

The B value has to do with the factors of the number next to the divisor, not the divisors themselves. A worst case scenario is that the number next to the factor p has a large prime divisisor or is a perfect power. If these conditions apply then the power-smoothness will be about p/2 or p. The more realistic is that there will be a large prime factor next to p requiring a B value on the order of p/2. Although primes may have been searched for up to this level, the p/2 is a factor of p - 1, something that we don't know. The modular.math reference below states that 15% of numbers in the range of 10**15 to 15**15 + 10**4 are 10**6 power smooth so a B of 10**6 will fail 85% of the time in that range. From 10**8 to 10**8 + 10**3 the percentages are nearly reversed…but in that range the simple trial division is quite fast.

`diofant.ntheory.factor_.`**`factorint`**(*n*, *limit=None*, *use_trial=True*, *use_rho=True*, *use_pm1=True*, *verbose=False*, *visual=None*)

Given a positive integer `n`, `factorint(n)` returns a dict containing the prime factors of `n` as keys and their respective multiplicities as values. For example:

```
>>> factorint(2000)    # 2000 = (2**4) * (5**3)
{2: 4, 5: 3}
>>> factorint(65537)   # This number is prime
{65537: 1}
```

For input less than 2, factorint behaves as follows:

- `factorint(1)` returns the empty factorization, `{}`
- `factorint(0)` returns `{0:1}`
- `factorint(-n)` adds `-1:1` to the factors and then factors `n`

Partial Factorization:

If `limit` (> 3) is specified, the search is stopped after performing trial division up to (and including) the limit (or taking a corresponding number of rho/p-1 steps). This is useful if one has a large number and only is interested in finding small factors (if any). Note that setting a limit does not prevent larger factors from being found early; it simply means

that the largest factor may be composite. Since checking for perfect power is relatively cheap, it is done regardless of the limit setting.

This number, for example, has two small factors and a huge semi-prime factor that cannot be reduced easily:

```
>>> a = 1407633717262338957430697921446883
>>> f = factorint(a, limit=10000)
>>> f
{7: 1, 991: 1, 202916782076162456022877024859: 1}
>>> isprime(max(f))
False
```

This number has a small factor and a residual perfect power whose base is greater than the limit:

```
>>> factorint(3*101**7, limit=5)
{3: 1, 101: 7}
```

Visual Factorization:

If `visual` is set to `True`, then it will return a visual factorization of the integer. For example:

```
>>> pprint(factorint(4200, visual=True), use_unicode=False)
 3  1  2  1
2 *3 *5 *7
```

Note that this is achieved by using the evaluate=False flag in Mul and Pow. If you do other manipulations with an expression where evaluate=False, it may evaluate. Therefore, you should use the visual option only for visualization, and use the normal dictionary returned by visual=False if you want to perform operations on the factors.

You can easily switch between the two forms by sending them back to factorint:

```
>>> regular = factorint(1764); regular
{2: 2, 3: 2, 7: 2}
>>> pprint(factorint(regular), use_unicode=False)
 2  2  2
2 *3 *7
```

```
>>> visual = factorint(1764, visual=True); pprint(visual, use_unicode=False)
 2  2  2
2 *3 *7
>>> print(factorint(visual))
{2: 2, 3: 2, 7: 2}
```

If you want to send a number to be factored in a partially factored form you can do so with a dictionary or unevaluated expression:

```
>>> factorint(factorint({4: 2, 12: 3})) # twice to toggle to dict form
{2: 10, 3: 3}
>>> factorint(Mul(4, 12, evaluate=False))
{2: 4, 3: 1}
```

The table of the output logic is:

Input	True	False	other
dict	mul	dict	mul
n	mul	dict	dict
mul	mul	dict	dict

See also:

smoothness (page 249), *smoothness_p* (page 249), *divisors* (page 257)

Notes

The function switches between multiple algorithms. Trial division quickly finds small factors (of the order 1-5 digits), and finds all large factors if given enough time. The Pollard rho and p-1 algorithms are used to find large factors ahead of time; they will often find factors of the order of 10 digits within a few seconds:

```
>>> factors = factorint(12345678910111213141516)
>>> for base, exp in sorted(factors.items()):
...     print('%s %s' % (base, exp))
...
2 2
2507191691 1
1231026625769 1
```

Any of these methods can optionally be disabled with the following boolean parameters:

- `use_trial`: Toggle use of trial division
- `use_rho`: Toggle use of Pollard's rho method
- `use_pm1`: Toggle use of Pollard's p-1 method

`factorint` also periodically checks if the remaining part is a prime number or a perfect power, and in those cases stops.

If `verbose` is set to `True`, detailed progress is printed.

`diofant.ntheory.factor_.`**`primefactors`**`(`*n*, *limit=None*, *verbose=False*`)`

Return a sorted list of n's prime factors, ignoring multiplicity and any composite factor that remains if the limit was set too low for complete factorization. Unlike factorint(), primefactors() does not return -1 or 0.

See also:

divisors (page 257)

Examples

```
>>> primefactors(6)
[2, 3]
>>> primefactors(-5)
[5]
```

```
>>> sorted(factorint(123456).items())
[(2, 6), (3, 1), (643, 1)]
>>> primefactors(123456)
[2, 3, 643]
```

```
>>> sorted(factorint(10000000001, limit=200).items())
[(101, 1), (99009901, 1)]
>>> isprime(99009901)
False
>>> primefactors(10000000001, limit=300)
[101]
```

`diofant.ntheory.factor_.`**`divisors`**(*n*, *generator=False*)

Return all divisors of n.

Divisors are sorted from 1..n by default. If generator is True an unordered generator is returned.

The number of divisors of n can be quite large if there are many prime factors (counting repeated factors). If only the number of factors is desired use divisor_count(n).

See also:

primefactors (page 256), *factorint* (page 254), *divisor_count* (page 257)

References

[R426] (page 1251)

Examples

```
>>> divisors(24)
[1, 2, 3, 4, 6, 8, 12, 24]
>>> divisor_count(24)
8
```

```
>>> list(divisors(120, generator=True))
[1, 2, 4, 8, 3, 6, 12, 24, 5, 10, 20, 40, 15, 30, 60, 120]
```

`diofant.ntheory.factor_.`**`divisor_count`**(*n*, *modulus=1*)

Return the number of divisors of `n`.

If `modulus` is not 1 then only those that are divisible by `modulus` are counted.

See also:

factorint (page 254), *divisors* (page 257), *totient* (page 258)

References

[R427] (page 1251)

Examples

```
>>> divisor_count(6)
4
```

diofant.ntheory.factor_.**totient**(*n*)

Calculate the Euler totient function phi(n)

```
>>> totient(1)
1
>>> totient(25)
20
```

See also:

divisor_count (page 257)

diofant.ntheory.factor_.**core**(*n*, *t=2*)

Calculate core(n,t) = $core_t(n)$ of a positive integer n.

`core_2(n)` is equal to the squarefree part of n

If n's prime factorization is:

$$n = \prod_{i=1}^{\omega} p_i^{m_i},$$

then

$$core_t(n) = \prod_{i=1}^{\omega} p_i^{m_i \mod t}.$$

Parameters t : core(n,t) calculates the t-th power free part of n

`core(n, 2)` is the squarefree part of `n` `core(n, 3)` is the cubefree part of `n`

Default for t is 2.

See also:

factorint (page 254), *diofant.solvers.diophantine.square_factor* (page 857)

References

[R428] (page 1251)

Examples

```
>>> from diofant.ntheory.factor_ import core
>>> core(24, 2)
6
>>> core(9424, 3)
1178
>>> core(379238)
379238
```

(continues on next page)

(continued from previous page)

```
>>> core(15**11, 10)
15
```

diofant.ntheory.modular.**symmetric_residue**(*a*, *m*)

Return the residual mod m such that it is within half of the modulus.

```
>>> symmetric_residue(1, 6)
1
>>> symmetric_residue(4, 6)
-2
```

diofant.ntheory.modular.**crt**(*m*, *v*, *symmetric=False*, *check=True*)

Chinese Remainder Theorem.

The moduli in m are assumed to be pairwise coprime. The output is then an integer f, such that f = v_i mod m_i for each pair out of v and m. If `symmetric` is False a positive integer will be returned, else |f| will be less than or equal to the LCM of the moduli, and thus f may be negative.

If the moduli are not co-prime the correct result will be returned if/when the test of the result is found to be incorrect. This result will be None if there is no solution.

The keyword `check` can be set to False if it is known that the moduli are coprime.

As an example consider a set of residues `U = [49, 76, 65]` and a set of moduli `M = [99, 97, 95]`. Then we have:

```
>>> crt([99, 97, 95], [49, 76, 65])
(639985, 912285)
```

This is the correct result because:

```
>>> [639985 % m for m in [99, 97, 95]]
[49, 76, 65]
```

If the moduli are not co-prime, you may receive an incorrect result if you use `check=False`:

```
>>> crt([12, 6, 17], [3, 4, 2], check=False)
(954, 1224)
>>> [954 % m for m in [12, 6, 17]]
[6, 0, 2]
>>> crt([12, 6, 17], [3, 4, 2]) is None
True
>>> crt([3, 6], [2, 5])
(5, 6)
```

Note: the order of gf_crt's arguments is reversed relative to crt, and that solve_congruence takes residue, modulus pairs.

Programmer's note: rather than checking that all pairs of moduli share no GCD (an O(n**2) test) and rather than factoring all moduli and seeing that there is no factor in common, a check that the result gives the indicated residuals is performed – an O(n) operation.

See also:

solve_congruence (page 260)

diofant.polys.galoistools.gf_crt **(page 1100)** low level crt routine used by this routine

diofant.ntheory.modular.**crt1**(*m*)
First part of Chinese Remainder Theorem, for multiple application.

Examples

```
>>> crt1([18, 42, 6])
(4536, [252, 108, 756], [0, 2, 0])
```

diofant.ntheory.modular.**crt2**(*m*, *v*, *mm*, *e*, *s*, *symmetric=False*)
Second part of Chinese Remainder Theorem, for multiple application.

Examples

```
>>> mm, e, s = crt1([18, 42, 6])
>>> crt2([18, 42, 6], [0, 0, 0], mm, e, s)
(0, 4536)
```

diofant.ntheory.modular.**solve_congruence**(**remainder_modulus_pairs*, ***hint*)
Compute the integer `n` that has the residual `ai` when it is divided by `mi` where the `ai` and `mi` are given as pairs to this function: ((a1, m1), (a2, m2), ...). If there is no solution, return. Otherwise return `n` and its modulus.

The `mi` values need not be co-prime. If it is known that the moduli are not co-prime then the hint `check` can be set to False (default=True) and the check for a quicker solution via crt() (valid when the moduli are co-prime) will be skipped.

If the hint `symmetric` is True (default is False), the value of `n` will be within 1/2 of the modulus, possibly negative.

See also:

crt **(page 259)** high level routine implementing the Chinese Remainder Theorem

Examples

What number is 2 mod 3, 3 mod 5 and 2 mod 7?

```
>>> solve_congruence((2, 3), (3, 5), (2, 7))
(23, 105)
>>> [23 % m for m in [3, 5, 7]]
[2, 3, 2]
```

If you prefer to work with all remainder in one list and all moduli in another, send the arguments like this:

```
>>> solve_congruence(*zip((2, 3, 2), (3, 5, 7)))
(23, 105)
```

The moduli need not be co-prime; in this case there may or may not be a solution:

```
>>> solve_congruence((2, 3), (4, 6)) is None
True
```

```
>>> solve_congruence((2, 3), (5, 6))
(5, 6)
```

The symmetric flag will make the result be within 1/2 of the modulus:

```
>>> solve_congruence((2, 3), (5, 6), symmetric=True)
(-1, 6)
```

diofant.ntheory.multinomial.**binomial_coefficients**(*n*)

Return a dictionary containing pairs $(k1, k2) : C_{kn}$ where C_{kn} are binomial coefficients and $n = k1 + k2$.

See also:

binomial_coefficients_list (page 261), *multinomial_coefficients* (page 261)

Examples

```
>>> binomial_coefficients(9)
{(0, 9): 1, (1, 8): 9, (2, 7): 36,
 (3, 6): 84, (4, 5): 126, (5, 4): 126, (6, 3): 84,
 (7, 2): 36, (8, 1): 9, (9, 0): 1}
```

diofant.ntheory.multinomial.**binomial_coefficients_list**(*n*)

Return a list of binomial coefficients as rows of the Pascal's triangle.

See also:

binomial_coefficients (page 261), *multinomial_coefficients* (page 261)

Examples

```
>>> binomial_coefficients_list(9)
[1, 9, 36, 84, 126, 126, 84, 36, 9, 1]
```

diofant.ntheory.multinomial.**multinomial_coefficients**(*m*, *n*)

Return a dictionary containing pairs `{(k1,k2,..,km) : C_kn}` where `C_kn` are multinomial coefficients such that `n=k1+k2+..+km`.

See also:

binomial_coefficients_list (page 261), *binomial_coefficients* (page 261)

Notes

The algorithm is based on the following result:

$$\binom{n}{k_1, \ldots, k_m} = \frac{k_1 + 1}{n - k_1} \sum_{i=2}^{m} \binom{n}{k_1 + 1, \ldots, k_i - 1, \ldots}$$

Examples

```
>>> multinomial_coefficients(2, 5)
{(0, 5): 1, (1, 4): 5,
 (2, 3): 10, (3, 2): 10, (4, 1): 5, (5, 0): 1}
```

diofant.ntheory.multinomial.**multinomial_coefficients_iterator**(*m*, *n*, *_tuple=<class 'tuple'>*)

multinomial coefficient iterator

This routine has been optimized for m large with respect to n by taking advantage of the fact that when the monomial tuples t are stripped of zeros, their coefficient is the same as that of the monomial tuples from `multinomial_coefficients(n, n)`. Therefore, the latter coefficients are precomputed to save memory and time.

```
>>> m53, m33 = multinomial_coefficients(5, 3), multinomial_coefficients(3, 3)
>>> m53[(0, 0, 0, 1, 2)] == m53[(0, 0, 1, 0, 2)] == m53[(1, 0, 2, 0, 0)] ==␣
↪m33[(0, 1, 2)]
True
```

Examples

```
>>> it = multinomial_coefficients_iterator(20, 3)
>>> next(it)
((3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), 1)
```

diofant.ntheory.partitions_.**npartitions**(*n*)

Calculate the partition function P(n), i.e. the number of ways that n can be written as a sum of positive integers.

P(n) is computed using the Hardy-Ramanujan-Rademacher formula *[R429]* (page 1251).

The correctness of this implementation has been tested for 10**n up to n = 8.

References

[R429] (page 1251)

Examples

```
>>> npartitions(25)
1958
```

diofant.ntheory.primetest.**is_square**(*n*, *prep=True*)

Return True if n == a * a for some integer a, else False.

If n is suspected of *not* being a square then this is a quick method of confirming that it is not.

See also:

diofant.core.power.integer_nthroot (page 103)

References

[R430] (page 1251)

`diofant.ntheory.primetest.mr(n, bases)`

Perform a Miller-Rabin strong pseudoprime test on n using a given list of bases/witnesses.

References

[R431] (page 1251), *[R432]* (page 1251)

Examples

```
>>> mr(1373651, [2, 3])
False
>>> mr(479001599, [31, 73])
True
```

`diofant.ntheory.primetest.isprime(n)`

Test if n is a prime number (True) or not (False). For n < 10**16 the answer is accurate; greater n values have a small probability of actually being pseudoprimes.

Negative primes (e.g. -2) are not considered prime.

The function first looks for trivial factors, and if none is found, performs a safe Miller-Rabin strong pseudoprime test with bases that are known to prove a number prime. Finally, a general Miller-Rabin test is done with the first k bases which will report a pseudoprime as a prime with an error of about 4**-k. The current value of k is 46 so the error is about 2 x 10**-28.

See also:

diofant.ntheory.generate.primerange **(page 246)** Generates all primes in a given range

diofant.ntheory.generate.primepi **(page 245)** Return the number of primes less than or equal to n

diofant.ntheory.generate.prime **(page 244)** Return the nth prime

Examples

```
>>> isprime(13)
True
>>> isprime(15)
False
```

`diofant.ntheory.residue_ntheory.n_order(a, n)`

Returns the order of `a` modulo `n`.

The order of `a` modulo `n` is the smallest integer `k` such that `a**k` leaves a remainder of 1 with `n`.

Examples

```
>>> n_order(3, 7)
6
>>> n_order(4, 7)
3
```

diofant.ntheory.residue_ntheory.**is_primitive_root**(*a*, *p*)
Returns True if a is a primitive root of p

a is said to be the primitive root of p if gcd(a, p) == 1 and totient(p) is the smallest positive number s.t.:

```
a**totient(p) cong 1 mod(p)
```

Examples

```
>>> is_primitive_root(3, 10)
True
>>> is_primitive_root(9, 10)
False
>>> n_order(3, 10) == totient(10)
True
>>> n_order(9, 10) == totient(10)
False
```

diofant.ntheory.residue_ntheory.**primitive_root**(*p*)
Returns the smallest primitive root or None.

Parameters p : positive integer

References

[R433] (page 1251), [R434] (page 1251)

Examples

```
>>> primitive_root(19)
2
```

diofant.ntheory.residue_ntheory.**sqrt_mod**(*a*, *p*, *all_roots=False*)
Find a root of `x**2 = a mod p`.

Parameters a : integer

p : positive integer

all_roots : if True the list of roots is returned or None

Notes

If there is no root it is returned None; else the returned root is less or equal to `p // 2`; in general is not the smallest one. It is returned `p // 2` only if it is the only root.

Use `all_roots` only when it is expected that all the roots fit in memory; otherwise use `sqrt_mod_iter`.

Examples

```
>>> sqrt_mod(11, 43)
21
>>> sqrt_mod(17, 32, True)
[7, 9, 23, 25]
```

`diofant.ntheory.residue_ntheory.`**`quadratic_residues`**(*p*)

Returns the list of quadratic residues.

Examples

```
>>> quadratic_residues(7)
[0, 1, 2, 4]
```

`diofant.ntheory.residue_ntheory.`**`nthroot_mod`**(*a, n, p, all_roots=False*)

Find the solutions to `x**n = a mod p`.

Parameters a : integer

n : positive integer

p : positive integer

all_roots : if False returns the smallest root, else the list of roots

Examples

```
>>> nthroot_mod(11, 4, 19)
8
>>> nthroot_mod(11, 4, 19, True)
[8, 11]
>>> nthroot_mod(68, 3, 109)
23
```

`diofant.ntheory.residue_ntheory.`**`is_nthpow_residue`**(*a, n, m*)

Returns True if `x**n == a (mod m)` has solutions.

References

[R435] (page 1251)

`diofant.ntheory.residue_ntheory.`**`is_quad_residue`**(*a, p*)

Returns True if a (mod p) is in the set of squares mod p, i.e a % p in {i**2 % p for i in range(p)}. If p is an odd prime, an iterative method is used to make the determination:

```
>>> sorted({i**2 % 7 for i in range(7)})
[0, 1, 2, 4]
>>> [j for j in range(7) if is_quad_residue(j, 7)]
[0, 1, 2, 4]
```

See also:

legendre_symbol (page 266), *jacobi_symbol* (page 266)

diofant.ntheory.residue_ntheory.**legendre_symbol**(*a*, *p*)

Returns the Legendre symbol (a/p).

For an integer a and an odd prime p, the Legendre symbol is defined as

$$\genfrac(){}{}{a}{p} = \begin{cases} 0 & \text{if } p \text{ divides } a \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p \end{cases}$$

Parameters **a** : integer

p : odd prime

See also:

is_quad_residue (page 265), *jacobi_symbol* (page 266)

References

[R436] (page 1251)

Examples

```
>>> [legendre_symbol(i, 7) for i in range(7)]
[0, 1, 1, -1, 1, -1, -1]
>>> sorted({i**2 % 7 for i in range(7)})
[0, 1, 2, 4]
```

diofant.ntheory.residue_ntheory.**jacobi_symbol**(*m*, *n*)

Returns the Jacobi symbol (m/n).

For any integer m and any positive odd integer n the Jacobi symbol is defined as the product of the Legendre symbols corresponding to the prime factors of n:

$$\genfrac(){}{}{m}{n} = \genfrac(){}{}{m}{p^1}^{\alpha_1} \genfrac(){}{}{m}{p^2}^{\alpha_2} ... \genfrac(){}{}{m}{p^k}^{\alpha_k} \text{ where } n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$$

Like the Legendre symbol, if the Jacobi symbol $\genfrac(){}{}{m}{n} = -1$ then m is a quadratic nonresidue modulo n.

But, unlike the Legendre symbol, if the Jacobi symbol $\genfrac(){}{}{m}{n} = 1$ then m may or may not be a quadratic residue modulo n.

Parameters **m** : integer

n : odd positive integer

See also:

is_quad_residue (page 265), *legendre_symbol* (page 266)

Examples

```
>>> jacobi_symbol(45, 77)
-1
>>> jacobi_symbol(60, 121)
1
```

The relationship between the `jacobi_symbol` and `legendre_symbol` can be demonstrated as follows:

```
>>> L = legendre_symbol
>>> Integer(45).factors()
{3: 2, 5: 1}
>>> jacobi_symbol(7, 45) == L(7, 3)**2 * L(7, 5)**1
True
```

`diofant.ntheory.continued_fraction.`**`continued_fraction_convergents`**(*cf*)

Return an iterator over the convergents of a continued fraction.

The parameter should be an iterable returning successive partial quotients of the continued fraction, such as might be returned by continued_fraction_iterator. In computing the convergents, the continued fraction need not be strictly in canonical form (all integers, all but the first positive). Rational and negative elements may be present in the expansion.

See also:

continued_fraction_iterator (page 267)

Examples

```
>>> list(continued_fraction_convergents([0, 2, 1, 2]))
[0, 1/2, 1/3, 3/8]
```

```
>>> list(continued_fraction_convergents([1, Rational(1, 2), -7, Rational(1, 4)]))
[1, 3, 19/5, 7]
```

```
>>> it = continued_fraction_convergents(continued_fraction_iterator(pi))
>>> for n in range(7):
...     print(next(it))
3
22/7
333/106
355/113
103993/33102
104348/33215
208341/66317
```

`diofant.ntheory.continued_fraction.`**`continued_fraction_iterator`**(*x*)

Return continued fraction expansion of x as iterator.

References

[R437] (page 1251)

Examples

```
>>> list(continued_fraction_iterator(Rational(3, 8)))
[0, 2, 1, 2]
>>> list(continued_fraction_iterator(Rational(-3, 8)))
[-1, 1, 1, 1, 2]
```

```
>>> for i, v in enumerate(continued_fraction_iterator(pi)):
...     if i > 7:
...         break
...     print(v)
3
7
15
1
292
1
1
1
```

diofant.ntheory.continued_fraction.**continued_fraction_periodic**(*p*, *q*, *d=0*)
Find the periodic continued fraction *[R438]* (page 1251) expansion.

Compute the continued fraction expansion of a rational or a quadratic surd, i.e. $\frac{p+\sqrt{d}}{q}$, where p, q and $d \geq 0$ are integers.

Parameters p : int

the rational part of the number's numerator

q : int

the denominator of the number

d : int, optional

the irrational part (discriminator) of the number's numerator

Returns list

the continued fraction representation (canonical form) as a list of integers, optionally ending (for quadratic irrationals) with repeating block as the last term of this list.

See also:

continued_fraction_iterator (page 267), *continued_fraction_reduce* (page 269)

References

[R438] (page 1251), *[R439]* (page 1251)

Examples

```
>>> continued_fraction_periodic(3, 2, 7)
[2, [1, 4, 1, 1]]
```

Golden ratio has the simplest continued fraction expansion:

```
>>> continued_fraction_periodic(1, 2, 5)
[[1]]
```

If the discriminator is zero or a perfect square then the number will be a rational number:

```
>>> continued_fraction_periodic(4, 3, 0)
[1, 3]
>>> continued_fraction_periodic(4, 3, 49)
[3, 1, 2]
```

diofant.ntheory.continued_fraction.**continued_fraction_reduce**(*cf*)

Reduce a continued fraction to a rational or quadratic irrational.

Compute the rational or quadratic irrational number from its terminating or periodic continued fraction expansion. The continued fraction expansion (cf) should be supplied as a terminating iterator supplying the terms of the expansion. For terminating continued fractions, this is equivalent to `list(continued_fraction_convergents(cf))[-1]`, only a little more efficient. If the expansion has a repeating part, a list of the repeating terms should be returned as the last element from the iterator. This is the format returned by continued_fraction_periodic.

For quadratic irrationals, returns the largest solution found, which is generally the one sought, if the fraction is in canonical form (all terms positive except possibly the first).

See also:

continued_fraction_periodic (page 268)

Examples

```
>>> continued_fraction_reduce([1, 2, 3, 4, 5])
225/157
>>> continued_fraction_reduce([-2, 1, 9, 7, 1, 2])
-256/233
>>> continued_fraction_reduce([2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8]).n(10)
2.718281835
>>> continued_fraction_reduce([1, 4, 2, [3, 1]])
(sqrt(21) + 287)/238
>>> continued_fraction_reduce([[1]])
1/2 + sqrt(5)/2
>>> continued_fraction_reduce(continued_fraction_periodic(8, 5, 13))
(sqrt(13) + 8)/5
```

class diofant.ntheory.**mobius**

Möbius function maps natural number to {-1, 0, 1}

It is defined as follows:

1. 1 if $n = 1$.
2. 0 if n has a squared prime factor.
3. $(-1)^k$ if n is a square-free positive integer with k number of prime factors.

It is an important multiplicative function in number theory and combinatorics. It has applications in mathematical series, algebraic number theory and also physics (Fermion operator has very concrete realization with Möbius Function model).

Parameters n : positive integer

References

[R440] (page 1252), [R441] (page 1252)

Examples

```
>>> mobius(13*7)
1
>>> mobius(1)
1
>>> mobius(13*7*5)
-1
>>> mobius(13**2)
0
```

classmethod eval(*n*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

diofant.ntheory.egyptian_fraction.**egyptian_fraction**(*r*, *algorithm='Greedy'*)

Compute an Egyptian fraction of the rational r.

Parameters r : Rational

a positive rational number.

algorithm : { "Greedy", "Graham Jewett", "Takenouchi", "Golomb" }, optional

Denotes the algorithm to be used (the default is "Greedy").

Returns list

The list of denominators of an Egyptian fraction expansion [R442] (page 1252).

See also:

diofant.core.numbers.Rational (page 88)

Notes

Currently the following algorithms are supported:

1. Greedy Algorithm

   Also called the Fibonacci-Sylvester algorithm [R443] (page 1252). At each step, extract the largest unit fraction less than the target and replace the target with the remainder.

   It has some distinct properties:

   (a) Given p/q in lowest terms, generates an expansion of maximum length p. Even as the numerators get large, the number of terms is seldom more than a handful.

   (b) Uses minimal memory.

   (c) The terms can blow up (standard examples of this are 5/121 and 31/311). The denominator is at most squared at each step (doubly-exponential growth) and typically exhibits singly-exponential growth.

2. Graham Jewett Algorithm

   The algorithm suggested by the result of Graham and Jewett. Note that this has a tendency to blow up: the length of the resulting expansion is always `2**(x/gcd(x, y)) - 1`. See [R444] (page 1252).

3. Takenouchi Algorithm

   The algorithm suggested by Takenouchi (1921). Differs from the Graham-Jewett algorithm only in the handling of duplicates. See [R444] (page 1252).

4. Golomb's Algorithm

   A method given by Golumb (1962), using modular arithmetic and inverses. It yields the same results as a method using continued fractions proposed by Bleicher (1972). See [R445] (page 1252).

If the given rational is greater than or equal to 1, a greedy algorithm of summing the harmonic sequence 1/1 + 1/2 + 1/3 + … is used, taking all the unit fractions of this sequence until adding one more would be greater than the given number. This list of denominators is prefixed to the result from the requested algorithm used on the remainder. For example, if r is 8/3, using the Greedy algorithm, we get [1, 2, 3, 4, 5, 6, 7, 14, 420], where the beginning of the sequence, [1, 2, 3, 4, 5, 6, 7] is part of the harmonic sequence summing to 363/140, leaving a remainder of 31/420, which yields [14, 420] by the Greedy algorithm. The result of egyptian_fraction(Rational(8, 3), "Golomb") is [1, 2, 3, 4, 5, 6, 7, 14, 574, 2788, 6460, 11590, 33062, 113820], and so on.

References

[R442] (page 1252), [R443] (page 1252), [R444] (page 1252), [R445] (page 1252)

Examples

```
>>> egyptian_fraction(Rational(3, 7))
[3, 11, 231]
>>> egyptian_fraction(Rational(3, 7), "Graham Jewett")
[7, 8, 9, 56, 57, 72, 3192]
>>> egyptian_fraction(Rational(3, 7), "Takenouchi")
[4, 7, 28]
>>> egyptian_fraction(Rational(3, 7), "Golomb")
[3, 15, 35]
>>> egyptian_fraction(Rational(11, 5), "Golomb")
[1, 2, 3, 4, 9, 234, 1118, 2580]
```

3.4 Concrete Mathematics

3.4.1 Hypergeometric terms

The center stage, in recurrence solving and summations, play hypergeometric terms. Formally these are sequences annihilated by first order linear recurrence operators. In simple words if we are given term $a(n)$ then it is hypergeometric if its consecutive term ratio is a rational function in n.

To check if a sequence is of this type you can use the `is_hypergeometric` method which is available in Basic class. Here is simple example involving a polynomial:

```
>>> (n**2 + 1).is_hypergeometric(n)
True
```

Of course polynomials are hypergeometric but are there any more complicated sequences of this type? Here are some trivial examples:

```
>>> factorial(n).is_hypergeometric(n)
True
>>> binomial(n, k).is_hypergeometric(n)
True
>>> rf(n, k).is_hypergeometric(n)
True
>>> ff(n, k).is_hypergeometric(n)
True
>>> gamma(n).is_hypergeometric(n)
True
>>> (2**n).is_hypergeometric(n)
True
```

We see that all species used in summations and other parts of concrete mathematics are hypergeometric. Note also that binomial coefficients and both rising and falling factorials are hypergeometric in both their arguments:

```
>>> binomial(n, k).is_hypergeometric(k)
True
>>> rf(n, k).is_hypergeometric(k)
True
>>> ff(n, k).is_hypergeometric(k)
True
```

To say more, all previously shown examples are valid for integer linear arguments:

```
>>> factorial(2*n).is_hypergeometric(n)
True
>>> binomial(3*n+1, k).is_hypergeometric(n)
True
>>> rf(n+1, k-1).is_hypergeometric(n)
True
>>> ff(n-1, k+1).is_hypergeometric(n)
True
>>> gamma(5*n).is_hypergeometric(n)
True
>>> (2**(n-7)).is_hypergeometric(n)
True
```

However nonlinear arguments make those sequences fail to be hypergeometric:

```
>>> factorial(n**2).is_hypergeometric(n)
False
>>> (2**(n**3 + 1)).is_hypergeometric(n)
False
```

If not only the knowledge of being hypergeometric or not is needed, you can use `hypersimp()` function. It will try to simplify combinatorial expression and if the term given is hypergeometric it will return a quotient of polynomials of minimal degree. Otherwise is will return $None$ to say that sequence is not hypergeometric:

```
>>> hypersimp(factorial(2*n), n)
2*(n + 1)*(2*n + 1)
>>> hypersimp(factorial(n**2), n)
```

3.4.2 Concrete Class Reference

class `diofant.concrete.summations.`**`Sum`**

Represents unevaluated summation.

`Sum` represents a finite or infinite series, with the first argument being the general form of terms in the series, and the second argument being `(dummy_variable, start, end)`, with `dummy_variable` taking all integer values from `start` through `end`. In accordance with long-standing mathematical convention, the end term is included in the summation.

For finite sums (and sums with symbolic limits assumed to be finite) we follow the summation convention described by Karr [1], especially definition 3 of section 1.4. The sum:

$$\sum_{m \leq i < n} f(i)$$

has *the obvious meaning* for $m < n$, namely:

$$\sum_{m \leq i < n} f(i) = f(m) + f(m+1) + \ldots + f(n-2) + f(n-1)$$

with the upper limit value $f(n)$ excluded. The sum over an empty set is zero if and only if $m = n$:

$$\sum_{m \leq i < n} f(i) = 0 \quad \text{for} \quad m = n$$

Finally, for all other sums over empty sets we assume the following definition:

$$\sum_{m \leq i < n} f(i) = - \sum_{n \leq i < m} f(i) \quad \text{for} \quad m > n$$

It is important to note that Karr defines all sums with the upper limit being exclusive. This is in contrast to the usual mathematical notation, but does not affect the summation convention. Indeed we have:

$$\sum_{m \leq i < n} f(i) = \sum_{i=m}^{n-1} f(i)$$

where the difference in notation is intentional to emphasize the meaning, with limits typeset on the top being inclusive.

See also:

diofant.concrete.summations.summation (page 287), *diofant.concrete.products.Product* (page 277), *diofant.concrete.products.product* (page 287)

References

[R65] (page 1252), [R66] (page 1252), [R67] (page 1252)

Examples

```
>>> from diofant.abc import i
```

```
>>> Sum(k, (k, 1, m))
Sum(k, (k, 1, m))
>>> Sum(k, (k, 1, m)).doit()
m**2/2 + m/2
>>> Sum(k**2, (k, 1, m))
Sum(k**2, (k, 1, m))
>>> Sum(k**2, (k, 1, m)).doit()
m**3/3 + m**2/2 + m/6
>>> Sum(x**k, (k, 0, oo))
Sum(x**k, (k, 0, oo))
>>> Sum(x**k, (k, 0, oo)).doit()
Piecewise((1/(-x + 1), Abs(x) < 1), (Sum(x**k, (k, 0, oo)), true))
>>> Sum(x**k/factorial(k), (k, 0, oo)).doit()
E**x
```

Here are examples to do summation with symbolic indices. You can use either Function of IndexedBase classes:

```
>>> f = Function('f')
```

```
>>> Sum(f(n), (n, 0, 3)).doit()
f(0) + f(1) + f(2) + f(3)
>>> Sum(f(n), (n, 0, oo)).doit()
Sum(f(n), (n, 0, oo))
>>> f = IndexedBase('f')
>>> Sum(f[n]**2, (n, 0, 3)).doit()
f[0]**2 + f[1]**2 + f[2]**2 + f[3]**2
```

An example showing that the symbolic result of a summation is still valid for seemingly nonsensical values of the limits. Then the Karr convention allows us to give a perfectly valid interpretation to those sums by interchanging the limits according to the above rules:

```
>>> S = Sum(i, (i, 1, n)).doit()
>>> S
n**2/2 + n/2
>>> S.subs(n, -4)
6
>>> Sum(i, (i, 1, -4)).doit()
6
```

(continues on next page)

(continued from previous page)

```
>>> Sum(-i, (i, -3, 0)).doit()
6
```

An explicit example of the Karr summation convention:

```
>>> S1 = Sum(i**2, (i, m, m+n-1)).doit()
>>> S1
m**2*n + m*n**2 - m*n + n**3/3 - n**2/2 + n/6
>>> S2 = Sum(i**2, (i, m+n, m-1)).doit()
>>> S2
-m**2*n - m*n**2 + m*n - n**3/3 + n**2/2 - n/6
>>> S1 + S2
0
>>> S3 = Sum(i, (i, m, m-1)).doit()
>>> S3
0
```

doit(***hints*)

Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

euler_maclaurin(*m=0, n=0, eps=0, eval_integral=True*)

Return an Euler-Maclaurin approximation of self, where m is the number of leading terms to sum directly and n is the number of terms in the tail.

With m = n = 0, this is simply the corresponding integral plus a first-order endpoint correction.

Returns (s, e) where s is the Euler-Maclaurin approximation and e is the estimated error (taken to be the magnitude of the first omitted term in the tail):

```
>>> from diofant.abc import a, b
```

```
>>> Sum(1/k, (k, 2, 5)).doit().evalf()
1.28333333333333
>>> s, e = Sum(1/k, (k, 2, 5)).euler_maclaurin()
>>> s
-log(2) + 7/20 + log(5)
>>> print(sstr((s.evalf(), e.evalf()), full_prec=True))
(1.26629073187415, 0.0175000000000000)
```

The endpoints may be symbolic:

```
>>> s, e = Sum(1/k, (k, a, b)).euler_maclaurin()
>>> s
-log(a) + log(b) + 1/(2*b) + 1/(2*a)
>>> e
Abs(1/(12*b**2) - 1/(12*a**2))
```

If the function is a polynomial of degree at most 2n+1, the Euler-Maclaurin formula becomes exact (and e = 0 is returned):

```
>>> Sum(k, (k, 2, b)).euler_maclaurin()
(b**2/2 + b/2 - 1, 0)
>>> Sum(k, (k, 2, b)).doit()
b**2/2 + b/2 - 1
```

With a nonzero eps specified, the summation is ended as soon as the remainder term is less than the epsilon.

findrecur(*F=Function('F'), n=None*)

Find a recurrence formula for the summand of the sum.

Given a sum $f(n) = \sum_k F(n,k)$, where $F(n,k)$ is doubly hypergeometric (that's, both $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are rational functions of n and k), we find a recurrence for the summand $F(n,k)$ of the form

$$\sum_{i=0}^{I}\sum_{j=0}^{J} a_{i,j} F(n-j, k-i) = 0$$

Notes

We use Sister Celine's algorithm, see [R68] (page 1252).

References

[R68] (page 1252)

Examples

```
>>> s = Sum(factorial(n)/(factorial(k)*factorial(n - k)), (k, 0, oo))
>>> s.findrecur()
-F(n, k) + F(n - 1, k) + F(n - 1, k - 1)
```

reverse_order(**indices*)

Reverse the order of a limit in a Sum.

Parameters *indices : list

The selectors in the argument `indices` specify some indices whose limits get reversed. These selectors are either variable names or numerical indices counted starting from the inner-most limit tuple.

See also:

diofant.concrete.expr_with_intlimits.ExprWithIntLimits.index (page 285), *diofant.concrete.expr_with_intlimits.ExprWithIntLimits.reorder_limit* (page 286), *diofant.concrete.expr_with_intlimits.ExprWithIntLimits.reorder* (page 285)

References

[R69] (page 1252)

Examples

```
>>> from diofant.abc import a, b, c, d
```

```
>>> Sum(x, (x, 0, 3)).reverse_order(x)
Sum(-x, (x, 4, -1))
>>> Sum(x*y, (x, 1, 5), (y, 0, 6)).reverse_order(x, y)
Sum(x*y, (x, 6, 0), (y, 7, -1))
>>> Sum(x, (x, a, b)).reverse_order(x)
Sum(-x, (x, b + 1, a - 1))
>>> Sum(x, (x, a, b)).reverse_order(0)
Sum(-x, (x, b + 1, a - 1))
```

While one should prefer variable names when specifying which limits to reverse, the index counting notation comes in handy in case there are several symbols with the same name.

```
>>> S = Sum(x**2, (x, a, b), (x, c, d))
>>> S
Sum(x**2, (x, a, b), (x, c, d))
>>> S0 = S.reverse_order(0)
>>> S0
Sum(-x**2, (x, b + 1, a - 1), (x, c, d))
>>> S1 = S0.reverse_order(1)
>>> S1
Sum(x**2, (x, b + 1, a - 1), (x, d + 1, c - 1))
```

Of course we can mix both notations:

```
>>> Sum(x*y, (x, a, b), (y, 2, 5)).reverse_order(x, 1)
Sum(x*y, (x, b + 1, a - 1), (y, 6, 1))
>>> Sum(x*y, (x, a, b), (y, 2, 5)).reverse_order(y, x)
Sum(x*y, (x, b + 1, a - 1), (y, 6, 1))
```

class diofant.concrete.products.**Product**

Represents unevaluated products.

`Product` represents a finite or infinite product, with the first argument being the general form of terms in the series, and the second argument being `(dummy_variable, start, end)`, with `dummy_variable` taking all integer values from `start` through `end`. In accordance with long-standing mathematical convention, the end term is included in the product.

For finite products (and products with symbolic limits assumed to be finite) we follow the analogue of the summation convention described by Karr [1], especially definition 3 of section 1.4. The product:

$$\prod_{m \leq i < n} f(i)$$

has *the obvious meaning* for $m < n$, namely:

$$\prod_{m \leq i < n} f(i) = f(m) f(m+1) \cdot \ldots \cdot f(n-2) f(n-1)$$

with the upper limit value $f(n)$ excluded. The product over an empty set is one if and only if $m = n$:

$$\prod_{m \leq i < n} f(i) = 1 \quad \mathrm{for} \quad m = n$$

Finally, for all other products over empty sets we assume the following definition:

$$\prod_{m \leq i < n} f(i) = \frac{1}{\prod_{n \leq i < m} f(i)} \quad \mathrm{for} \quad m > n$$

It is important to note that above we define all products with the upper limit being exclusive. This is in contrast to the usual mathematical notation, but does not affect the product convention. Indeed we have:

$$\prod_{m \leq i < n} f(i) = \prod_{i = m}^{n - 1} f(i)$$

where the difference in notation is intentional to emphasize the meaning, with limits typeset on the top being inclusive.

See also:

diofant.concrete.summations.Sum (page 273), *diofant.concrete.summations.summation* (page 287), *diofant.concrete.products.product* (page 287)

References

[R70] (page 1252), *[R71]* (page 1252), *[R72]* (page 1252)

Examples

```
>>> from diofant.abc import a, b, i
```

```
>>> Product(k, (k, 1, m))
Product(k, (k, 1, m))
>>> Product(k, (k, 1, m)).doit()
factorial(m)
>>> Product(k**2, (k, 1, m))
Product(k**2, (k, 1, m))
>>> Product(k**2, (k, 1, m)).doit()
factorial(m)**2
```

Wallis' product for pi:

```
>>> W = Product(2*i/(2*i-1) * 2*i/(2*i+1), (i, 1, oo))
>>> W
Product(4*i**2/((2*i - 1)*(2*i + 1)), (i, 1, oo))
```

Direct computation currently fails:

```
>>> W.doit()
Product(4*i**2/((2*i - 1)*(2*i + 1)), (i, 1, oo))
```

But we can approach the infinite product by a limit of finite products:

```
>>> W2 = Product(2*i/(2*i-1)*2*i/(2*i+1), (i, 1, n))
>>> W2
Product(4*i**2/((2*i - 1)*(2*i + 1)), (i, 1, n))
>>> W2e = W2.doit()
>>> W2e
2**(-2*n)*4**n*factorial(n)**2/(RisingFactorial(1/2, n)*RisingFactorial(3/2, n))
>>> limit(W2e, n, oo)
pi/2
```

By the same formula we can compute sin(pi/2):

```
>>> P = pi * x * Product(1 - x**2/k**2, (k, 1, n))
>>> P = P.subs(x, pi/2)
>>> P
pi**2*Product(1 - pi**2/(4*k**2), (k, 1, n))/2
>>> Pe = P.doit()
>>> Pe
pi**2*RisingFactorial(1 + pi/2, n)*RisingFactorial(-pi/2 + 1, n)/
↪(2*factorial(n)**2)
>>> Pe = Pe.rewrite(gamma)
>>> Pe
pi**2*gamma(n + 1 + pi/2)*gamma(n - pi/2 + 1)/(2*gamma(1 + pi/2)*gamma(-pi/2 +␣
↪1)*gamma(n + 1)**2)
>>> Pe = simplify(Pe)
>>> Pe
sin(pi**2/2)*gamma(n + 1 + pi/2)*gamma(n - pi/2 + 1)/gamma(n + 1)**2
>>> limit(Pe, n, oo)
sin(pi**2/2)
```

Products with the lower limit being larger than the upper one:

```
>>> Product(1/i, (i, 6, 1)).doit()
120
>>> Product(i, (i, 2, 5)).doit()
120
```

The empty product:

```
>>> Product(i, (i, n, n-1)).doit()
1
```

An example showing that the symbolic result of a product is still valid for seemingly nonsensical values of the limits. Then the Karr convention allows us to give a perfectly valid interpretation to those products by interchanging the limits according to the above rules:

```
>>> P = Product(2, (i, 10, n)).doit()
>>> P
2**(n - 9)
>>> P.subs(n, 5)
1/16
>>> Product(2, (i, 10, 5)).doit()
1/16
>>> 1/Product(2, (i, 6, 9)).doit()
1/16
```

An explicit example of the Karr summation convention applied to products:

```
>>> P1 = Product(x, (i, a, b)).doit()
>>> P1
x**(-a + b + 1)
>>> P2 = Product(x, (i, b+1, a-1)).doit()
>>> P2
x**(a - b - 1)
>>> simplify(P1 * P2)
1
```

And another one:

```
>>> P1 = Product(i, (i, b, a)).doit()
>>> P1
RisingFactorial(b, a - b + 1)
>>> P2 = Product(i, (i, a+1, b-1)).doit()
>>> P2
RisingFactorial(a + 1, -a + b - 1)
>>> P1 * P2
RisingFactorial(b, a - b + 1)*RisingFactorial(a + 1, -a + b - 1)
>>> simplify(P1 * P2)
1
```

doit(***hints*)

Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

reverse_order(**indices*)

Reverse the order of a limit in a Product.

Parameters ***indices** : list

The selectors in the argument `indices` specify some indices whose limits get reversed. These selectors are either variable names or numerical indices counted starting from the inner-most limit tuple.

See also:

diofant.concrete.expr_with_intlimits.ExprWithIntLimits.index (page 285), *diofant.concrete.expr_with_intlimits.ExprWithIntLimits.reorder_limit* (page 286), *diofant.concrete.expr_with_intlimits.ExprWithIntLimits.reorder* (page 285)

References

[R73] (page 1252)

Examples

```
>>> from diofant.abc import a, b, c, d
```

```
>>> P = Product(x, (x, a, b))
>>> Pr = P.reverse_order(x)
>>> Pr
Product(1/x, (x, b + 1, a - 1))
>>> Pr = Pr.doit()
>>> Pr
1/RisingFactorial(b + 1, a - b - 1)
>>> simplify(Pr)
gamma(b + 1)/gamma(a)
>>> P = P.doit()
>>> P
RisingFactorial(a, -a + b + 1)
>>> simplify(P)
gamma(b + 1)/gamma(a)
```

While one should prefer variable names when specifying which limits to reverse, the index counting notation comes in handy in case there are several symbols with the same name.

```
>>> S = Sum(x*y, (x, a, b), (y, c, d))
>>> S
Sum(x*y, (x, a, b), (y, c, d))
>>> S0 = S.reverse_order(0)
>>> S0
Sum(-x*y, (x, b + 1, a - 1), (y, c, d))
>>> S1 = S0.reverse_order(1)
>>> S1
Sum(x*y, (x, b + 1, a - 1), (y, d + 1, c - 1))
```

Of course we can mix both notations:

```
>>> Sum(x*y, (x, a, b), (y, 2, 5)).reverse_order(x, 1)
Sum(x*y, (x, b + 1, a - 1), (y, 6, 1))
>>> Sum(x*y, (x, a, b), (y, 2, 5)).reverse_order(y, x)
Sum(x*y, (x, b + 1, a - 1), (y, 6, 1))
```

class diofant.concrete.expr_with_limits.**ExprWithLimits**

as_dummy()

Replace instances of the given dummy variables with explicit dummy counterparts to make clear what are dummy variables and what are real-world symbols in an object.

See also:

diofant.concrete.expr_with_limits.ExprWithLimits.variables **(page 283)**
Lists the integration variables

Examples

```
>>> Integral(x, (x, x, y), (y, x, y)).as_dummy()
Integral(_x, (_x, x, _y), (_y, x, y))
```

If the object supports the "integral at" limit `(x,)` it is not treated as a dummy, but the explicit form, `(x, x)` of length 2 does treat the variable as a dummy.

```
>>> Integral(x, x).as_dummy()
Integral(x, x)
>>> Integral(x, (x, x)).as_dummy()
Integral(_x, (_x, x))
```

If there were no dummies in the original expression, then the the symbols which cannot be changed by subs() are clearly seen as those with an underscore prefix.

free_symbols

This method returns the symbols in the object, excluding those that take on a specific value (i.e. the dummy symbols).

Examples

```
>>> Sum(x, (x, y, 1)).free_symbols
{y}
```

function

Return the function applied across limits.

See also:

diofant.concrete.expr_with_limits.ExprWithLimits.limits (page 283), *diofant.concrete.expr_with_limits.ExprWithLimits.variables* (page 283), *diofant.concrete.expr_with_limits.ExprWithLimits.free_symbols* (page 282)

Examples

```
>>> Integral(x**2, x).function
x**2
```

is_number
Return True if the Sum has no free symbols, else False.

limits
Return the limits of expression.

See also:

diofant.concrete.expr_with_limits.ExprWithLimits.function (page 282), *diofant.concrete.expr_with_limits.ExprWithLimits.variables* (page 283), *diofant.concrete.expr_with_limits.ExprWithLimits.free_symbols* (page 282)

Examples

```
>>> from diofant.abc import i
>>> Integral(x**i, (i, 1, 3)).limits
((i, 1, 3),)
```

variables
Return a list of the dummy variables

```
>>> from diofant.abc import i
>>> Sum(x**i, (i, 1, 3)).variables
[i]
```

See also:

diofant.concrete.expr_with_limits.ExprWithLimits.function (page 282), *diofant.concrete.expr_with_limits.ExprWithLimits.limits* (page 283), *diofant.concrete.expr_with_limits.ExprWithLimits.free_symbols* (page 282)

diofant.concrete.expr_with_limits.ExprWithLimits.as_dummy **(page 282)**
Rename dummy variables

class diofant.concrete.expr_with_intlimits.**ExprWithIntLimits**

change_index(*var, trafo, newvar=None*)
Change index of a Sum or Product.

Perform a linear transformation $x \mapsto ax + b$ on the index variable x. For a the only values allowed are ± 1. A new variable to be used after the change of index can also be specified.

Parameters var : Symbol

specifies the index variable x to transform.

trafo : Expr

The linear transformation in terms of `var`.

newvar : Symbol, optional

Replacement symbol to be used instead of `var` in the final expression.

See also:

diofant.concrete.expr_with_intlimits.ExprWithIntLimits.index (page 285), *diofant.concrete.expr_with_intlimits.ExprWithIntLimits.reorder_limit* (page 286), *diofant.concrete.expr_with_intlimits.ExprWithIntLimits.reorder* (page 285), *diofant.concrete.summations.Sum.reverse_order* (page 276), *diofant.concrete.products.Product.reverse_order* (page 280)

Examples

```
>>> from diofant.abc import a, b, c, d, u, v, i, j, l
```

```
>>> S = Sum(x, (x, a, b))
>>> S.doit()
-a**2/2 + a/2 + b**2/2 + b/2
```

```
>>> Sn = S.change_index(x, x + 1, y)
>>> Sn
Sum(y - 1, (y, a + 1, b + 1))
>>> Sn.doit()
-a**2/2 + a/2 + b**2/2 + b/2
```

```
>>> Sn = S.change_index(x, -x, y)
>>> Sn
Sum(-y, (y, -b, -a))
>>> Sn.doit()
-a**2/2 + a/2 + b**2/2 + b/2
```

```
>>> Sn = S.change_index(x, x+u)
>>> Sn
Sum(-u + x, (x, a + u, b + u))
>>> Sn.doit()
-a**2/2 - a*u + a/2 + b**2/2 + b*u + b/2 - u*(-a + b + 1) + u
>>> simplify(Sn.doit())
-a**2/2 + a/2 + b**2/2 + b/2
```

```
>>> Sn = S.change_index(x, -x - u, y)
>>> Sn
Sum(-u - y, (y, -b - u, -a - u))
>>> Sn.doit()
-a**2/2 - a*u + a/2 + b**2/2 + b*u + b/2 - u*(-a + b + 1) + u
>>> simplify(Sn.doit())
-a**2/2 + a/2 + b**2/2 + b/2
```

```
>>> P = Product(i*j**2, (i, a, b), (j, c, d))
>>> P
Product(i*j**2, (i, a, b), (j, c, d))
>>> P2 = P.change_index(i, i+3, k)
>>> P2
Product(j**2*(k - 3), (k, a + 3, b + 3), (j, c, d))
>>> P3 = P2.change_index(j, -j, l)
>>> P3
Product(l**2*(k - 3), (k, a + 3, b + 3), (l, -d, -c))
```

When dealing with symbols only, we can make a general linear transformation:

```
>>> Sn = S.change_index(x, u*x+v, y)
>>> Sn
Sum((-v + y)/u, (y, b*u + v, a*u + v))
>>> Sn.doit()
-v*(a*u - b*u + 1)/u + (a**2*u**2/2 + a*u*v + a*u/2 - b**2*u**2/2 - b*u*v +
↪b*u/2 + v)/u
>>> simplify(Sn.doit())
a**2*u/2 + a/2 - b**2*u/2 + b/2
```

However, the last result can be inconsistent with usual summation where the index increment is always 1. This is obvious as we get back the original value only for u equal +1 or -1.

index(*x*)

Return the index of a dummy variable in the list of limits.

Note that we start counting with 0 at the inner-most limits tuple.

Parameters x : Symbol

a dummy variable

See also:

diofant.concrete.expr_with_intlimits.ExprWithIntLimits.reorder_limit (page 286), *diofant.concrete.expr_with_intlimits.ExprWithIntLimits.reorder* (page 285), *diofant.concrete.summations.Sum.reverse_order* (page 276), *diofant.concrete.products.Product.reverse_order* (page 280)

Examples

```
>>> from diofant.abc import a, b, c, d
>>> Sum(x*y, (x, a, b), (y, c, d)).index(x)
0
>>> Sum(x*y, (x, a, b), (y, c, d)).index(y)
1
>>> Product(x*y, (x, a, b), (y, c, d)).index(x)
0
>>> Product(x*y, (x, a, b), (y, c, d)).index(y)
1
```

reorder(**arg*)

Reorder limits in a expression containing a Sum or a Product.

Parameters *arg : list of tuples

These tuples can contain numerical indices or index variable names or involve both.

See also:

diofant.concrete.expr_with_intlimits.ExprWithIntLimits.index (page 285), *diofant.concrete.expr_with_intlimits.ExprWithIntLimits.reorder_limit* (page 286), *diofant.concrete.summations.Sum.reverse_order* (page 276), *diofant.concrete.products.Product.reverse_order* (page 280)

Examples

```
>>> from diofant.abc import a, b, c, d, e, f
```

```
>>> Sum(x*y, (x, a, b), (y, c, d)).reorder((x, y))
Sum(x*y, (y, c, d), (x, a, b))
```

```
>>> Sum(x*y*z, (x, a, b), (y, c, d), (z, e, f)).reorder((x, y), (x, z), (y,
↪z))
Sum(x*y*z, (z, e, f), (y, c, d), (x, a, b))
```

```
>>> P = Product(x*y*z, (x, a, b), (y, c, d), (z, e, f))
>>> P.reorder((x, y), (x, z), (y, z))
Product(x*y*z, (z, e, f), (y, c, d), (x, a, b))
```

We can also select the index variables by counting them, starting with the inner-most one:

```
>>> Sum(x**2, (x, a, b), (x, c, d)).reorder((0, 1))
Sum(x**2, (x, c, d), (x, a, b))
```

And of course we can mix both schemes:

```
>>> Sum(x*y, (x, a, b), (y, c, d)).reorder((y, x))
Sum(x*y, (y, c, d), (x, a, b))
>>> Sum(x*y, (x, a, b), (y, c, d)).reorder((y, 0))
Sum(x*y, (y, c, d), (x, a, b))
```

reorder_limit(*x*, *y*)
Interchange two limit tuples of a Sum or Product expression.

Parameters x, y: int

are integers corresponding to the index variables of the two limits which are to be interchanged.

See also:

diofant.concrete.expr_with_intlimits.ExprWithIntLimits.index (page 285), *diofant.concrete.expr_with_intlimits.ExprWithIntLimits.reorder* (page 285), *diofant.concrete.summations.Sum.reverse_order* (page 276), *diofant.concrete.products.Product.reverse_order* (page 280)

Examples

```
>>> from diofant.abc import a, b, c, d, e, f
```

```
>>> Sum(x*y*z, (x, a, b), (y, c, d), (z, e, f)).reorder_limit(0, 2)
Sum(x*y*z, (z, e, f), (y, c, d), (x, a, b))
>>> Sum(x**2, (x, a, b), (x, c, d)).reorder_limit(1, 0)
Sum(x**2, (x, c, d), (x, a, b))
```

```
>>> Product(x*y*z, (x, a, b), (y, c, d), (z, e, f)).reorder_limit(0, 2)
Product(x*y*z, (z, e, f), (y, c, d), (x, a, b))
```

3.4.3 Concrete Functions Reference

diofant.concrete.summations.**summation**(*f*, **symbols*, ***kwargs*)
Compute the summation of f with respect to symbols.

The notation for symbols is similar to the notation used in Integral. summation(f, (i, a, b)) computes the sum of f with respect to i from a to b, i.e.,

```
                              b
                            ____
                            \   `
summation(f, (i, a, b)) =    )    f
                            /___,
                            i = a
```

If it cannot compute the sum, it returns an unevaluated Sum object. Repeated sums can be computed by introducing additional symbols tuples:

```
>>> i = symbols('i', integer=True)
```

```
>>> summation(2*i - 1, (i, 1, n))
n**2
>>> summation(1/2**i, (i, 0, oo))
2
>>> summation(1/log(n)**n, (n, 2, oo))
Sum(log(n)**(-n), (n, 2, oo))
>>> summation(i, (i, 0, n), (n, 0, m))
m**3/6 + m**2/2 + m/3
```

```
>>> summation(x**n/factorial(n), (n, 0, oo))
E**x
```

See also:

diofant.concrete.summations.Sum (page 273), *diofant.concrete.products.Product* (page 277), *diofant.concrete.products.product* (page 287)

diofant.concrete.products.**product**(**args*, ***kwargs*)
Compute the product.

The notation for symbols is similar to the notation used in Sum or Integral. product(f, (i, a, b)) computes the product of f with respect to i from a to b, i.e.,

```
                               b
                             _____
product(f(n), (i, a, b)) =   |   | f(n)
                             |   |
                             i = a
```

If it cannot compute the product, it returns an unevaluated Product object. Repeated products can be computed by introducing additional symbols tuples:

```
>>> i = symbols('i', integer=True)
```

```
>>> product(i, (i, 1, k))
factorial(k)
>>> product(m, (i, 1, k))
```

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```
m**k
>>> product(i, (i, 1, k), (k, 1, n))
Product(factorial(k), (k, 1, n))
```

diofant.concrete.gosper.**gosper_normal**(*f*, *g*, *n*, *polys=True*)

Compute the Gosper's normal form of `f` and `g`.

Given relatively prime univariate polynomials `f` and `g`, rewrite their quotient to a normal form defined as follows:

$$\frac{f(n)}{g(n)} = Z \cdot \frac{A(n)C(n+1)}{B(n)C(n)}$$

where `Z` is an arbitrary constant and `A`, `B`, `C` are monic polynomials in `n` with the following properties:

1. gcd$(A(n), B(n+h)) = 1 \forall h \in \mathbb{N}$
2. gcd$(B(n), C(n+1)) = 1$
3. gcd$(A(n), C(n)) = 1$

This normal form, or rational factorization in other words, is a crucial step in Gosper's algorithm and in solving of difference equations. It can be also used to decide if two hypergeometric terms are similar or not.

This procedure will return a tuple containing elements of this factorization in the form `(Z*A, B, C)`.

Examples

```
>>> gosper_normal(4*n + 5, 2*(4*n + 1)*(2*n + 3), n, polys=False)
(1/4, n + 3/2, n + 1/4)
```

diofant.concrete.gosper.**gosper_term**(*f*, *n*)

Compute Gosper's hypergeometric term for `f`.

Suppose `f` is a hypergeometric term such that:

$$s_n = \sum_{k=0}^{n-1} f_k$$

and f_k doesn't depend on n. Returns a hypergeometric term g_n such that $g_{n+1} - g_n = f_n$.

Examples

```
>>> gosper_term((4*n + 1)*factorial(n)/factorial(2*n + 1), n)
(-n - 1/2)/(n + 1/4)
```

diofant.concrete.gosper.**gosper_sum**(*f*, *k*)

Gosper's hypergeometric summation algorithm.

Given a hypergeometric term `f` such that:

$$s_n = \sum_{k=0}^{n-1} f_k$$

and $f(n)$ doesn't depend on n, returns $g_n - g(0)$ where $g_{n+1} - g_n = f_n$, or None if s_n can not be expressed in closed form as a sum of hypergeometric terms.

References

[R74] (page 1252)

Examples

```
>>> from diofant.abc import i
```

```
>>> f = (4*k + 1)*factorial(k)/factorial(2*k + 1)
>>> gosper_sum(f, (k, 0, n))
(-factorial(n) + 2*factorial(2*n + 1))/factorial(2*n + 1)
>>> _.subs(n, 2) == sum(f.subs(k, i) for i in [0, 1, 2])
True
>>> gosper_sum(f, (k, 3, n))
(-60*factorial(n) + factorial(2*n + 1))/(60*factorial(2*n + 1))
>>> _.subs(n, 5) == sum(f.subs(k, i) for i in [3, 4, 5])
True
```

3.5 Mathematical Functions

All functions support the methods documented below, inherited from `diofant.core.function.Function` (page 132).

class diofant.core.function.**Function**

Base class for applied mathematical functions.

It also serves as a constructor for undefined function classes.

Examples

First example shows how to use Function as a constructor for undefined function classes:

```
>>> g = g(x)
>>> f
f
>>> f(x)
f(x)
>>> g
g(x)
>>> f(x).diff(x)
Derivative(f(x), x)
>>> g.diff(x)
Derivative(g(x), x)
```

In the following example Function is used as a base class for `my_func` that represents a mathematical function *my_func*. Suppose that it is well known, that *my_func(0)* is *1* and *my_func* at infinity goes to *0*, so we want those two simplifications to occur automatically.

Suppose also that *my_func(x)* is real exactly when *x* is real. Here is an implementation that honours those requirements:

```
>>> class my_func(Function):
...
...     @classmethod
...     def eval(cls, x):
...         if x.is_Number:
...             if x is S.Zero:
...                 return S.One
...             elif x is oo:
...                 return S.Zero
...
...     def _eval_is_real(self):
...         return self.args[0].is_real
...
>>> x = Symbol('x')
>>> my_func(0) + sin(0)
1
>>> my_func(oo)
0
>>> my_func(3.54).n() # Not yet implemented for my_func.
my_func(3.54)
>>> my_func(I).is_real
False
```

In order for `my_func` to become useful, several other methods would need to be implemented. See source code of some of the already implemented functions for more complete examples.

Also, if the function can take more than one argument, then `nargs` must be defined, e.g. if `my_func` can take one or two arguments then,

```
>>> class my_func(Function):
...     nargs = (1, 2)
...
>>>
```

as_base_exp()
Returns the method as the 2-tuple (base, exponent).

classmethod class_key()
Nice order of classes.

fdiff(*argindex=1*)
Returns the first derivative of the function.

3.5.1 Elementary

This module implements elementary functions such as trigonometric, hyperbolic as well as functions like `Abs`, `Max`, `sqrt` etc.

3.5.2 diofant.functions.elementary.complexes

re

class diofant.functions.elementary.complexes.**re**

Returns real part of expression.

This function performs only elementary analysis and so it will fail to decompose properly more complicated expressions. If completely simplified result is needed then use Basic.as_real_imag() or perform complex expansion on instance of this function.

See also:

diofant.functions.elementary.complexes.im (page 291)

Examples

```
>>> re(2*E)
2*E
>>> re(2*I + 17)
17
>>> re(2*I)
0
>>> re(im(x) + x*I + 2)
2
```

as_real_imag(*deep=True*, ***hints*)

Returns the real number with a zero imaginary part.

classmethod eval(*arg*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

im

class diofant.functions.elementary.complexes.**im**

Returns imaginary part of expression.

This function performs only elementary analysis and so it will fail to decompose properly more complicated expressions. If completely simplified result is needed then use Basic.as_real_imag() or perform complex expansion on instance of this function.

See also:

diofant.functions.elementary.complexes.re (page 291)

Examples

```
>>> im(2*E)
0
>>> re(2*I + 17)
17
>>> im(x*I)
re(x)
```

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```
>>> im(re(x) + y)
im(y)
```

as_real_imag(*deep=True*, ***hints*)

Return the imaginary part with a zero real part.

Examples

```
>>> im(2 + 3*I).as_real_imag()
(3, 0)
```

classmethod eval(*arg*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

sign

class diofant.functions.elementary.complexes.**sign**

Returns the complex sign of an expression.

If the expression is real the sign will be:

- 1 if expression is positive
- 0 if expression is equal to zero
- -1 if expression is negative

If the expression is imaginary the sign will be:

- I if im(expression) is positive
- -I if im(expression) is negative

Otherwise an unevaluated expression will be returned. When evaluated, the result (in general) will be `cos(arg(expr)) + I*sin(arg(expr))`.

See also:

diofant.functions.elementary.complexes.Abs (page 293), *diofant.functions.elementary.complexes.conjugate* (page 295)

Examples

```
>>> sign(-1)
-1
>>> sign(0)
0
>>> sign(-3*I)
-I
>>> sign(1 + I)
sign(1 + I)
```

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```
>>> _.evalf()
0.707106781186548 + 0.707106781186548*I
```

doit(***hints*)

Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

classmethod eval(*arg*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

Abs

class diofant.functions.elementary.complexes.**Abs**

Return the absolute value of the argument.

This is an extension of the built-in function abs() to accept symbolic values. If you pass a Diofant expression to the built-in abs(), it will pass it automatically to Abs().

See also:

diofant.functions.elementary.complexes.sign (page 292), *diofant.functions.elementary.complexes.conjugate* (page 295)

Examples

```
>>> Abs(-1)
1
>>> x = Symbol('x', extended_real=True)
>>> Abs(-x)
Abs(x)
>>> Abs(x**2)
x**2
>>> abs(-x) # The Python built-in
Abs(x)
```

Note that the Python built-in will return either an Expr or int depending on the argument:

```
>>> type(abs(-1))
<... 'int'>
>>> type(abs(S.NegativeOne))
<class 'diofant.core.numbers.One'>
```

Abs will always return a diofant object.

classmethod eval(*arg*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)

Get the first derivative of the argument to Abs().

Examples

```
>>> Abs(-x).fdiff()
sign(x)
```

adjoint

class diofant.functions.elementary.complexes.**adjoint**

Conjugate transpose or Hermite conjugation.

classmethod eval(*arg*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

arg

class diofant.functions.elementary.complexes.**arg**

Returns the argument (in radians) of a complex number.

For a real number, the argument is always 0.

Examples

```
>>> arg(2.0)
0
>>> arg(I)
pi/2
>>> arg(sqrt(2) + I*sqrt(2))
pi/4
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

conjugate

class diofant.functions.elementary.complexes.**conjugate**
Returns the complex conjugate of an argument.

In mathematics, the complex conjugate [R157] (page 1252) of a complex number is given by changing the sign of the imaginary part.

Thus, the conjugate of the complex number $a + ib$ (where a and b are real numbers) is $a - ib$

See also:

diofant.functions.elementary.complexes.sign (page 292), *diofant.functions.elementary.complexes.Abs* (page 293)

References

[R157] (page 1252)

Examples

```
>>> conjugate(2)
2
>>> conjugate(I)
-I
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

polar_lift

class diofant.functions.elementary.complexes.**polar_lift**
Lift argument to the Riemann surface of the logarithm, using the standard branch.

```
>>> p = Symbol('p', polar=True)
>>> polar_lift(4)
4*exp_polar(0)
>>> polar_lift(-4)
4*exp_polar(I*pi)
>>> polar_lift(-I)
exp_polar(-I*pi/2)
```

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```
>>> polar_lift(I + 2)
polar_lift(2 + I)
```

```
>>> polar_lift(4*x)
4*polar_lift(x)
>>> polar_lift(4*p)
4*p
```

See also:

diofant.functions.elementary.exponential.exp_polar (page 321), *diofant.functions.elementary.complexes.periodic_argument* (page 296)

classmethod eval(*arg*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

periodic_argument

class diofant.functions.elementary.complexes.**periodic_argument**

Represent the argument on a quotient of the Riemann surface of the logarithm. That is, given a period P, always return a value in (-P/2, P/2], by using exp(P*I) == 1.

```
>>> unbranched_argument(exp(5*I*pi))
pi
>>> unbranched_argument(exp_polar(5*I*pi))
5*pi
>>> periodic_argument(exp_polar(5*I*pi), 2*pi)
pi
>>> periodic_argument(exp_polar(5*I*pi), 3*pi)
-pi
>>> periodic_argument(exp_polar(5*I*pi), pi)
0
```

See also:

diofant.functions.elementary.exponential.exp_polar (page 321)

***diofant.functions.elementary.complexes.polar_lift* (page 295)** Lift argument to the Riemann surface of the logarithm

diofant.functions.elementary.complexes.principal_branch (page 296)

classmethod eval(*ar, period*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

principal_branch

class diofant.functions.elementary.complexes.**principal_branch**

Represent a polar number reduced to its principal branch on a quotient of the Riemann

surface of the logarithm.

This is a function of two arguments. The first argument is a polar number z, and the second one a positive real number of infinity, p. The result is "z mod exp_polar(I*p)".

```
>>> principal_branch(z, oo)
z
>>> principal_branch(exp_polar(2*pi*I)*3, 2*pi)
3*exp_polar(0)
>>> principal_branch(exp_polar(2*pi*I)*3*z, 2*pi)
3*principal_branch(z, 2*pi)
```

See also:

diofant.functions.elementary.exponential.exp_polar (page 321)

diofant.functions.elementary.complexes.polar_lift **(page 295)** Lift argument to the Riemann surface of the logarithm

diofant.functions.elementary.complexes.periodic_argument (page 296)

classmethod eval(*x*, *period*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

transpose

class diofant.functions.elementary.complexes.**transpose**
Linear map transposition.

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

3.5.3 diofant.functions.elementary.trigonometric

3.5.4 Trigonometric Functions

sin

class diofant.functions.elementary.trigonometric.**sin**
The sine function.

Returns the sine of x (measured in radians).

See also:

diofant.functions.elementary.trigonometric.csc (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301), *diofant.functions.elementary.trigonometric.asin* (page 304),

diofant.functions.elementary.trigonometric.acsc (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

This function will evaluate automatically in the case x/pi is some rational number *[R161]* (page 1252). For example, if x is a multiple of pi, pi/2, pi/3, pi/4 and pi/6.

References

[R158] (page 1252), *[R159]* (page 1252), *[R160]* (page 1252), *[R161]* (page 1252)

Examples

```
>>> sin(x**2).diff(x)
2*x*cos(x**2)
>>> sin(pi)
0
>>> sin(pi/2)
1
>>> sin(pi/6)
1/2
>>> sin(pi/12)
-sqrt(2)/4 + sqrt(6)/4
```

as_real_imag(*deep=True*, ***hints*)

Performs complex expansion on 'self' and returns a tuple containing collected both real and imaginary parts. This method can't be confused with re() and im() functions, which does not perform complex expansion at evaluation.

However it is possible to expand both re() and im() functions and get exactly the same results as with a single call to this function.

```
>>> x, y = symbols('x y', extended_real=True)
```

```
>>> (x + y*I).as_real_imag()
(x, y)
```

```
>>> from diofant.abc import w
```

```
>>> (z + w*I).as_real_imag()
(re(z) - im(w), re(w) + im(z))
```

classmethod eval(*arg*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

cos

class diofant.functions.elementary.trigonometric.**cos**
The cosine function.

Returns the cosine of x (measured in radians).

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301), *diofant.functions.elementary.trigonometric.asin* (page 304), *diofant.functions.elementary.trigonometric.acsc* (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

See *sin()* (page 297) for notes about automatic evaluation.

References

[R162] (page 1252), [R163] (page 1252), [R164] (page 1252)

Examples

```
>>> cos(x**2).diff(x)
-2*x*sin(x**2)
>>> cos(pi)
-1
>>> cos(pi/2)
0
>>> cos(2*pi/3)
-1/2
>>> cos(pi/12)
sqrt(2)/4 + sqrt(6)/4
```

as_real_imag(*deep=True*, ***hints*)
Performs complex expansion on 'self' and returns a tuple containing collected both

real and imaginary parts. This method can't be confused with re() and im() functions, which does not perform complex expansion at evaluation.

However it is possible to expand both re() and im() functions and get exactly the same results as with a single call to this function.

```
>>> x, y = symbols('x y', extended_real=True)
```

```
>>> (x + y*I).as_real_imag()
(x, y)
```

```
>>> from diofant.abc import w
```

```
>>> (z + w*I).as_real_imag()
(re(z) - im(w), re(w) + im(z))
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

static taylor_term(*x, *previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

tan

class diofant.functions.elementary.trigonometric.**tan**
The tangent function.

Returns the tangent of x (measured in radians).

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.cot* (page 301), *diofant.functions.elementary.trigonometric.asin* (page 304), *diofant.functions.elementary.trigonometric.acsc* (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

See *sin()* (page 297) for notes about automatic evaluation.

References

[R165] (page 1252), *[R166]* (page 1252), *[R167]* (page 1252)

Examples

```
>>> tan(x**2).diff(x)
2*x*(tan(x**2)**2 + 1)
>>> tan(pi/8).expand()
-1 + sqrt(2)
```

as_real_imag(*deep=True*, ***hints*)
Performs complex expansion on 'self' and returns a tuple containing collected both real and imaginary parts. This method can't be confused with re() and im() functions, which does not perform complex expansion at evaluation.

However it is possible to expand both re() and im() functions and get exactly the same results as with a single call to this function.

```
>>> x, y = symbols('x y', extended_real=True)
```

```
>>> (x + y*I).as_real_imag()
(x, y)
```

```
>>> from diofant.abc import w
```

```
>>> (z + w*I).as_real_imag()
(re(z) - im(w), re(w) + im(z))
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

cot

class diofant.functions.elementary.trigonometric.**cot**
The cotangent function.

Returns the cotangent of x (measured in radians).

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.asin* (page 304), *diofant.functions.elementary.trigonometric.acsc* (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

See *sin()* (page 297) for notes about automatic evaluation.

References

[R168] (page 1252), *[R169]* (page 1252), *[R170]* (page 1252)

Examples

```
>>> cot(x**2).diff(x)
2*x*(-cot(x**2)**2 - 1)
```

as_real_imag(*deep=True*, ***hints*)

Performs complex expansion on ‘self’ and returns a tuple containing collected both real and imaginary parts. This method can’t be confused with re() and im() functions, which does not perform complex expansion at evaluation.

However it is possible to expand both re() and im() functions and get exactly the same results as with a single call to this function.

```
>>> x, y = symbols('x y', extended_real=True)
```

```
>>> (x + y*I).as_real_imag()
(x, y)
```

```
>>> from diofant.abc import w
```

```
>>> (z + w*I).as_real_imag()
(re(z) - im(w), re(w) + im(z))
```

fdiff(*argindex=1*)

Returns the first derivative of the function.

inverse(*argindex=1*)

Returns the inverse of this function.

sec

class diofant.functions.elementary.trigonometric.**sec**

The secant function.

Returns the secant of x (measured in radians).

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301), *diofant.functions.elementary.trigonometric.asin* (page 304), *diofant.functions.elementary.trigonometric.acsc* (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

See *sin()* (page 297) for notes about automatic evaluation.

References

[R171] (page 1252), *[R172]* (page 1252), *[R173]* (page 1252)

Examples

```
>>> sec(x**2).diff(x)
2*x*tan(x**2)*sec(x**2)
```

fdiff(*argindex=1*)

Returns the first derivative of the function.

static taylor_term(*x*, **previous_terms*)

General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

csc

class diofant.functions.elementary.trigonometric.**csc**

The cosecant function.

Returns the cosecant of x (measured in radians).

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301), *diofant.functions.elementary.trigonometric.asin* (page 304), *diofant.functions.elementary.trigonometric.acsc* (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

See *sin()* (page 297) for notes about automatic evaluation.

References

[R174] (page 1253), *[R175]* (page 1253), *[R176]* (page 1253)

Examples

```
>>> csc(x**2).diff(x)
-2*x*cot(x**2)*csc(x**2)
```

fdiff(*argindex=1*)
Returns the first derivative of the function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

3.5.5 Trigonometric Inverses

asin

class diofant.functions.elementary.trigonometric.**asin**
The inverse sine function.

Returns the arcsine of x in radians.

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301),

diofant.functions.elementary.trigonometric.acsc (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

asin(x) will evaluate automatically in the cases oo, -oo, 0, 1, -1 and for some instances when the result is a rational multiple of pi (see the eval class method).

References

[R177] (page 1253), *[R178]* (page 1253), *[R179]* (page 1253)

Examples

```
>>> asin(1)
pi/2
>>> asin(-1)
-pi/2
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

acos

class diofant.functions.elementary.trigonometric.**acos**
The inverse cosine function.

Returns the arc cosine of x (measured in radians).

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301),

diofant.functions.elementary.trigonometric.asin (page 304), *diofant.functions.elementary.trigonometric.acsc* (page 310), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

`acos(x)` will evaluate automatically in the cases `oo`, `-oo`, `0`, `1`, `-1`.

`acos(zoo)` evaluates to `zoo` (see note in :py:class'diofant.functions.elementary.trigonometric.asec')

References

[R180] (page 1253), *[R181]* (page 1253), *[R182]* (page 1253)

Examples

```
>>> acos(1)
0
>>> acos(0)
pi/2
>>> acos(oo)
oo*I
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

asec

class diofant.functions.elementary.trigonometric.**asec**
The inverse secant function.

Returns the arc secant of x (measured in radians).

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301), *diofant.functions.elementary.trigonometric.asin* (page 304), *diofant.functions.elementary.trigonometric.acsc* (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

`asec(x)` will evaluate automatically in the cases `oo`, `-oo`, `0`, `1`, `-1`.

`asec(x)` has branch cut in the interval [-1, 1]. For complex arguments, it can be defined *[R186]* (page 1253) as

$$sec^{-1}(z) = -i * (log(\sqrt{1 - z^2} + 1)/z)$$

At `x = 0`, for positive branch cut, the limit evaluates to `zoo`. For negative branch cut, the limit

$$\lim_{z \to 0} -i * (log(-\sqrt{1 - z^2} + 1)/z)$$

simplifies to $-i * log(z/2 + O(z^3))$ which ultimately evaluates to `zoo`.

As `asex(x) = asec(1/x)`, a similar argument can be given for `acos(x)`.

References

[R183] (page 1253), *[R184]* (page 1253), *[R185]* (page 1253), *[R186]* (page 1253)

Examples

```
>>> asec(1)
0
>>> asec(-1)
pi
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

atan

class diofant.functions.elementary.trigonometric.**atan**

The inverse tangent function.

Returns the arc tangent of x (measured in radians).

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301), *diofant.functions.elementary.trigonometric.asin* (page 304), *diofant.functions.elementary.trigonometric.acsc* (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

atan(x) will evaluate automatically in the cases oo, -oo, 0, 1, -1.

References

[R187] (page 1253), [R188] (page 1253), [R189] (page 1253)

Examples

```
>>> atan(0)
0
>>> atan(1)
pi/4
>>> atan(oo)
pi/2
```

classmethod eval(*arg*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)

Returns the first derivative of the function.

inverse(*argindex=1*)

Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)

General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

acot

class diofant.functions.elementary.trigonometric.**acot**
The inverse cotangent function.

Returns the arc cotangent of x (measured in radians). This function has a branch cut discontinuity in the complex plane running from $-i$ to i.

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301), *diofant.functions.elementary.trigonometric.asin* (page 304), *diofant.functions.elementary.trigonometric.acsc* (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.atan2* (page 311)

References

[R190] (page 1253), *[R191]* (page 1253), *[R192]* (page 1253)

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

asec

class diofant.functions.elementary.trigonometric.**asec**
The inverse secant function.

Returns the arc secant of x (measured in radians).

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301),

diofant.functions.elementary.trigonometric.asin (page 304), *diofant.functions.elementary.trigonometric.acsc* (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

`asec(x)` will evaluate automatically in the cases `oo`, `-oo`, `0`, `1`, `-1`.

`asec(x)` has branch cut in the interval [-1, 1]. For complex arguments, it can be defined *[R196]* (page 1253) as

$$sec^{-1}(z) = -i * (log(\sqrt{1 - z^2} + 1)/z)$$

At `x = 0`, for positive branch cut, the limit evaluates to `zoo`. For negative branch cut, the limit

$$\lim_{z \to 0} -i * (log(-\sqrt{1 - z^2} + 1)/z)$$

simplifies to $-i * log(z/2 + O(z^3))$ which ultimately evaluates to `zoo`.

As `asex(x) = asec(1/x)`, a similar argument can be given for `acos(x)`.

References

[R193] (page 1253), *[R194]* (page 1253), *[R195]* (page 1253), *[R196]* (page 1253)

Examples

```
>>> asec(1)
0
>>> asec(-1)
pi
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

acsc

class diofant.functions.elementary.trigonometric.**acsc**
The inverse cosecant function.

Returns the arc cosecant of x (measured in radians).

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301), *diofant.functions.elementary.trigonometric.asin* (page 304), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309), *diofant.functions.elementary.trigonometric.atan2* (page 311)

Notes

acsc(x) will evaluate automatically in the cases oo, -oo, 0, 1, -1.

References

[R197] (page 1253), [R198] (page 1253), [R199] (page 1253)

Examples

```
>>> acsc(1)
pi/2
>>> acsc(-1)
-pi/2
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

atan2

class diofant.functions.elementary.trigonometric.**atan2**
The function `atan2(y, x)` computes $\operatorname{atan}(y/x)$ taking two arguments y and x. Signs of both y and x are considered to determine the appropriate quadrant of $\operatorname{atan}(y/x)$. The

range is $(-\pi, \pi]$. The complete definition reads as follows:

$$\operatorname{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & y \ge 0, x < 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & y < 0, x < 0 \\ +\frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

Attention: Note the role reversal of both arguments. The y-coordinate is the first argument and the x-coordinate the second.

See also:

diofant.functions.elementary.trigonometric.sin (page 297), *diofant.functions.elementary.trigonometric.csc* (page 303), *diofant.functions.elementary.trigonometric.cos* (page 299), *diofant.functions.elementary.trigonometric.sec* (page 303), *diofant.functions.elementary.trigonometric.tan* (page 300), *diofant.functions.elementary.trigonometric.cot* (page 301), *diofant.functions.elementary.trigonometric.asin* (page 304), *diofant.functions.elementary.trigonometric.acsc* (page 310), *diofant.functions.elementary.trigonometric.acos* (page 305), *diofant.functions.elementary.trigonometric.asec* (page 306), *diofant.functions.elementary.trigonometric.atan* (page 308), *diofant.functions.elementary.trigonometric.acot* (page 309)

References

[R200] (page 1253), *[R201]* (page 1253), *[R202]* (page 1253)

Examples

Going counter-clock wise around the origin we find the following angles:

```
>>> atan2(0, 1)
0
>>> atan2(1, 1)
pi/4
>>> atan2(1, 0)
pi/2
>>> atan2(1, -1)
3*pi/4
>>> atan2(0, -1)
pi
>>> atan2(-1, -1)
-3*pi/4
>>> atan2(-1, 0)
-pi/2
>>> atan2(-1, 1)
-pi/4
```

which are all correct. Compare this to the results of the ordinary atan function for the point $(x, y) = (-1, 1)$

```
>>> atan(Integer(1) / -1)
-pi/4
>>> atan2(1, -1)
3*pi/4
```

where only the atan2 function returns what we expect. We can differentiate the function with respect to both arguments:

```
>>> diff(atan2(y, x), x)
-y/(x**2 + y**2)
```

```
>>> diff(atan2(y, x), y)
x/(x**2 + y**2)
```

We can express the atan2 function in terms of complex logarithms:

```
>>> atan2(y, x).rewrite(log)
-I*log((x + I*y)/sqrt(x**2 + y**2))
```

and in terms of $(atan)$:

```
>>> atan2(y, x).rewrite(atan)
2*atan(y/(x + sqrt(x**2 + y**2)))
```

but note that this form is undefined on the negative real axis.

classmethod eval(*y*, *x*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex*)
Returns the first derivative of the function.

3.5.6 diofant.functions.elementary.hyperbolic

3.5.7 Hyperbolic Functions

HyperbolicFunction

class diofant.functions.elementary.hyperbolic.**HyperbolicFunction**
Base class for hyperbolic functions.

See also:

diofant.functions.elementary.hyperbolic.sinh (page 314), *diofant.functions.elementary.hyperbolic.cosh* (page 314), *diofant.functions.elementary.hyperbolic.tanh* (page 315), *diofant.functions.elementary.hyperbolic.coth* (page 316)

sinh

class diofant.functions.elementary.hyperbolic.**sinh**

The hyperbolic sine function, $\frac{e^x - e^{-x}}{2}$.

- sinh(x) -> Returns the hyperbolic sine of x

See also:

diofant.functions.elementary.hyperbolic.cosh (page 314), *diofant.functions.elementary.hyperbolic.tanh* (page 315), *diofant.functions.elementary.hyperbolic.asinh* (page 317)

as_real_imag(*deep=True*, ***hints*)

Returns this function as a complex coordinate.

classmethod eval(*arg*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)

Returns the first derivative of this function.

inverse(*argindex=1*)

Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)

Returns the next term in the Taylor series expansion.

cosh

class diofant.functions.elementary.hyperbolic.**cosh**

The hyperbolic cosine function, $\frac{e^x + e^{-x}}{2}$.

- cosh(x) -> Returns the hyperbolic cosine of x

See also:

diofant.functions.elementary.hyperbolic.sinh (page 314), *diofant.functions.elementary.hyperbolic.tanh* (page 315), *diofant.functions.elementary.hyperbolic.acosh* (page 318)

as_real_imag(*deep=True*, ***hints*)

Performs complex expansion on 'self' and returns a tuple containing collected both real and imaginary parts. This method can't be confused with re() and im() functions, which does not perform complex expansion at evaluation.

However it is possible to expand both re() and im() functions and get exactly the same results as with a single call to this function.

```
>>> x, y = symbols('x y', extended_real=True)
```

```
>>> (x + y*I).as_real_imag()
(x, y)
```

```
>>> from diofant.abc import w
```

```
>>> (z + w*I).as_real_imag()
(re(z) - im(w), re(w) + im(z))
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

tanh

class diofant.functions.elementary.hyperbolic.**tanh**
The hyperbolic tangent function, $\frac{\sinh(x)}{\cosh(x)}$.

- tanh(x) -> Returns the hyperbolic tangent of x

See also:

diofant.functions.elementary.hyperbolic.sinh (page 314), *diofant.functions.elementary.hyperbolic.cosh* (page 314), *diofant.functions.elementary.hyperbolic.atanh* (page 318)

as_real_imag(*deep=True*, ***hints*)
Performs complex expansion on 'self' and returns a tuple containing collected both real and imaginary parts. This method can't be confused with re() and im() functions, which does not perform complex expansion at evaluation.

However it is possible to expand both re() and im() functions and get exactly the same results as with a single call to this function.

```
>>> x, y = symbols('x y', extended_real=True)
```

```
>>> (x + y*I).as_real_imag()
(x, y)
```

```
>>> from diofant.abc import w
```

```
>>> (z + w*I).as_real_imag()
(re(z) - im(w), re(w) + im(z))
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

coth

class diofant.functions.elementary.hyperbolic.**coth**
The hyperbolic cotangent function, $\frac{\cosh(x)}{\sinh(x)}$.

- coth(x) -> Returns the hyperbolic cotangent of x

as_real_imag(*deep=True*, ***hints*)
Performs complex expansion on 'self' and returns a tuple containing collected both real and imaginary parts. This method can't be confused with re() and im() functions, which does not perform complex expansion at evaluation.

However it is possible to expand both re() and im() functions and get exactly the same results as with a single call to this function.

```
>>> x, y = symbols('x y', extended_real=True)
```

```
>>> (x + y*I).as_real_imag()
(x, y)
```

```
>>> from diofant.abc import w
```

```
>>> (z + w*I).as_real_imag()
(re(z) - im(w), re(w) + im(z))
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

sech

class diofant.functions.elementary.hyperbolic.**sech**

The hyperbolic secant function, $\frac{2}{e^x+e^{-x}}$

- sech(x) -> Returns the hyperbolic secant of x

See also:

diofant.functions.elementary.hyperbolic.sinh (page 314), *diofant.functions.elementary.hyperbolic.cosh* (page 314), *diofant.functions.elementary.hyperbolic.tanh* (page 315), *diofant.functions.elementary.hyperbolic.coth* (page 316), *diofant.functions.elementary.hyperbolic.csch* (page 317), *diofant.functions.elementary.hyperbolic.asinh* (page 317), *diofant.functions.elementary.hyperbolic.acosh* (page 318)

fdiff(*argindex=1*)

Returns the first derivative of the function.

static taylor_term(*x*, **previous_terms*)

General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

csch

class diofant.functions.elementary.hyperbolic.**csch**

The hyperbolic cosecant function, $\frac{2}{e^x-e^{-x}}$

- csch(x) -> Returns the hyperbolic cosecant of x

See also:

diofant.functions.elementary.hyperbolic.sinh (page 314), *diofant.functions.elementary.hyperbolic.cosh* (page 314), *diofant.functions.elementary.hyperbolic.tanh* (page 315), *diofant.functions.elementary.hyperbolic.sech* (page 317), *diofant.functions.elementary.hyperbolic.asinh* (page 317), *diofant.functions.elementary.hyperbolic.acosh* (page 318)

fdiff(*argindex=1*)

Returns the first derivative of this function

static taylor_term(*x*, **previous_terms*)

Returns the next term in the Taylor series expansion

3.5.8 Hyperbolic Inverses

asinh

class diofant.functions.elementary.hyperbolic.**asinh**

The inverse hyperbolic sine function.

- asinh(x) -> Returns the inverse hyperbolic sine of x

See also:

diofant.functions.elementary.hyperbolic.cosh (page 314), *diofant.functions.elementary.hyperbolic.tanh* (page 315), *diofant.functions.elementary.hyperbolic.sinh* (page 314)

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

acosh

class diofant.functions.elementary.hyperbolic.**acosh**
The inverse hyperbolic cosine function.

- acosh(x) -> Returns the inverse hyperbolic cosine of x

See also:

diofant.functions.elementary.hyperbolic.asinh (page 317), *diofant.functions.elementary.hyperbolic.atanh* (page 318), *diofant.functions.elementary.hyperbolic.cosh* (page 314)

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

atanh

class diofant.functions.elementary.hyperbolic.**atanh**
The inverse hyperbolic tangent function.

- atanh(x) -> Returns the inverse hyperbolic tangent of x

See also:

diofant.functions.elementary.hyperbolic.asinh (page 317), *diofant.functions.elementary.hyperbolic.acosh* (page 318), *diofant.functions.elementary.hyperbolic.tanh* (page 315)

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

acoth

class diofant.functions.elementary.hyperbolic.**acoth**
The inverse hyperbolic cotangent function.

- acoth(x) -> Returns the inverse hyperbolic cotangent of x

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns the inverse of this function.

static taylor_term(*x*, **previous_terms*)
General method for the taylor term.

This method is slow, because it differentiates n-times. Subclasses can redefine it to make it faster by using the "previous_terms".

3.5.9 diofant.functions.elementary.integers

ceiling

class diofant.functions.elementary.integers.**ceiling**
Ceiling is a univariate function which returns the smallest integer value not less than its argument. Ceiling function is generalized in this implementation to complex numbers.

See also:

diofant.functions.elementary.integers.floor (page 320)

References

[R203] (page 1253), *[R204]* (page 1253)

Examples

```
>>> ceiling(17)
17
>>> ceiling(Rational(23, 10))
3
>>> ceiling(2*E)
6
>>> ceiling(-Float(0.567))
0
>>> ceiling(I/2)
I
```

floor

class diofant.functions.elementary.integers.**floor**

Floor is a univariate function which returns the largest integer value not greater than its argument. However this implementation generalizes floor to complex numbers.

See also:

diofant.functions.elementary.integers.ceiling (page 319)

References

[R205] (page 1253), *[R206]* (page 1253)

Examples

```
>>> floor(17)
17
>>> floor(Rational(23, 10))
2
>>> floor(2*E)
5
>>> floor(-Float(0.567))
-1
>>> floor(-I/2)
-I
```

RoundFunction

class diofant.functions.elementary.integers.**RoundFunction**

The base class for rounding functions.

3.5.10 diofant.functions.elementary.exponential

exp

`diofant.functions.elementary.exponential.exp(arg, **kwargs)`

The exponential function, e^x.

See also:

diofant.functions.elementary.exponential.log (page 322)

exp_polar

`class diofant.functions.elementary.exponential.exp_polar`

Represent a 'polar number' (see g-function Sphinx documentation).

`exp_polar` represents the function $Exp : \mathbb{C} \to \mathcal{S}$, sending the complex number $z = a + bi$ to the polar number $r = exp(a), \theta = b$. It is one of the main functions to construct polar numbers.

The main difference is that polar numbers don't "wrap around" at 2π:

```
>>> exp(2*pi*I)
1
>>> exp_polar(2*pi*I)
exp_polar(2*I*pi)
```

apart from that they behave mostly like classical complex numbers:

```
>>> exp_polar(2)*exp_polar(3)
exp_polar(5)
```

See also:

diofant.simplify.powsimp.powsimp (page 792), *diofant.functions.elementary.complexes.polar_lift* (page 295), *diofant.functions.elementary.complexes.periodic_argument* (page 296), *diofant.functions.elementary.complexes.principal_branch* (page 296)

`as_base_exp()`

Returns the method as the 2-tuple (base, exponent).

`exp`

Returns the exponent of the function.

LambertW

`class diofant.functions.elementary.exponential.LambertW`

The Lambert W function $W(z)$ is defined as the inverse function of $w \exp(w)$ *[R207]* (page 1253).

In other words, the value of $W(z)$ is such that $z = W(z) \exp(W(z))$ for any complex number z. The Lambert W function is a multivalued function with infinitely many branches $W_k(z)$, indexed by $k \in \mathbb{Z}$. Each branch gives a different solution w of the equation $z = w \exp(w)$.

The Lambert W function has two partially real branches: the principal branch ($k = 0$) is real for real $z > -1/e$, and the $k = -1$ branch is real for $-1/e < z < 0$. All branches except $k = 0$ have a logarithmic singularity at $z = 0$.

References

[R207] (page 1253)

Examples

```
>>> LambertW(1.2)
0.635564016364870
>>> LambertW(1.2, -1).n()
-1.34747534407696 - 4.41624341514535*I
>>> LambertW(-1).is_extended_real
False
```

classmethod eval(*x*, *k=None*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Return the first derivative of this function.

log

class diofant.functions.elementary.exponential.**log**
The natural logarithm function $\ln(x)$ or $\log(x)$. Logarithms are taken with the natural base, e. To get a logarithm of a different base b, use `log(x, b)`, which is essentially short-hand for `log(x)/log(b)`.

See also:

diofant.functions.elementary.exponential.exp (page 321)

as_base_exp()
Returns this function in the form (base, exponent).

as_real_imag(*deep=True*, ***hints*)
Returns this function as a complex coordinate.

Examples

```
>>> log(x).as_real_imag()
(log(Abs(x)), arg(x))
>>> log(I).as_real_imag()
(0, pi/2)
>>> log(1 + I).as_real_imag()
(log(sqrt(2)), pi/4)
>>> log(I*x).as_real_imag()
(log(Abs(x)), arg(I*x))
```

classmethod eval(*arg*, *base=None*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

inverse(*argindex=1*)
Returns e^x, the inverse function of $\log(x)$.

3.5.11 diofant.functions.elementary.piecewise

ExprCondPair

class diofant.functions.elementary.piecewise.**ExprCondPair**
Represents an expression, condition pair.

cond
Returns the condition of this pair.

expr
Returns the expression of this pair.

Piecewise

class diofant.functions.elementary.piecewise.**Piecewise**
Represents a piecewise function.

Usage:

Piecewise((expr,cond), (expr,cond), …)

- Each argument is a 2-tuple defining an expression and condition
- The conds are evaluated in turn returning the first that is True. If any of the evaluated conds are not determined explicitly False, e.g. x < 1, the function is returned in symbolic form.
- If the function is evaluated at a place where all conditions are False, a ValueError exception will be raised.
- Pairs where the cond is explicitly False, will be removed.

See also:

diofant.functions.elementary.piecewise.piecewise_fold (page 324)

Examples

```
>>> f = x**2
>>> g = log(x)
>>> p = Piecewise((0, x<-1), (f, x<=1), (g, True))
>>> p.subs(x, 1)
1
>>> p.subs(x, 5)
log(5)
```

doit(***hints*)
Evaluate this piecewise function.

classmethod eval(**args*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

diofant.functions.elementary.piecewise.**piecewise_fold**(*expr*)
Takes an expression containing a piecewise function and returns the expression in piecewise form.

See also:

diofant.functions.elementary.piecewise.Piecewise (page 323)

Examples

```
>>> p = Piecewise((x, x < 1), (1, x >= 1))
>>> piecewise_fold(x*p)
Piecewise((x**2, x < 1), (x, x >= 1))
```

3.5.12 diofant.functions.elementary.miscellaneous

IdentityFunction

class diofant.functions.elementary.miscellaneous.**IdentityFunction**
The identity function

Examples

```
>>> x = Symbol('x')
>>> Id(x)
x
```

Min

class diofant.functions.elementary.miscellaneous.**Min**
Return, if possible, the minimum value of the list.

It is named `Min` and not `min` to avoid conflicts with the built-in function `min`.

See also:

diofant.functions.elementary.miscellaneous.Max **(page 325)** find maximum values

Examples

```
>>> p = Symbol('p', positive=True)
>>> n = Symbol('n', negative=True)
```

```
>>> Min(x, -2)
Min(-2, x)
>>> Min(x, -2).subs(x, 3)
-2
>>> Min(p, -3)
-3
>>> Min(x, y)
Min(x, y)
>>> Min(n, 8, p, -7, p, oo)
Min(-7, n)
```

Max

class diofant.functions.elementary.miscellaneous.**Max**

Return, if possible, the maximum value of the list.

When number of arguments is equal one, then return this argument.

When number of arguments is equal two, then return, if possible, the value from (a, b) that is >= the other.

In common case, when the length of list greater than 2, the task is more complicated. Return only the arguments, which are greater than others, if it is possible to determine directional relation.

If is not possible to determine such a relation, return a partially evaluated result.

Assumptions are used to make the decision too.

Also, only comparable arguments are permitted.

It is named `Max` and not `max` to avoid conflicts with the built-in function `max`.

See also:

diofant.functions.elementary.miscellaneous.Min **(page 324)** find minimum values

Notes

The task can be considered as searching of supremums in the directed complete partial orders *[R208]* (page 1253).

The source values are sequentially allocated by the isolated subsets in which supremums are searched and result as Max arguments.

If the resulted supremum is single, then it is returned.

The isolated subsets are the sets of values which are only the comparable with each other in the current set. E.g. natural numbers are comparable with each other, but not comparable with the x symbol. Another example: the symbol x with negative assumption is comparable with a natural number.

Also there are "least" elements, which are comparable with all others, and have a zero property (maximum or minimum for all elements). E.g. oo. In case of it the allocation operation is terminated and only this value is returned.

Assumption:

- if A > B > C then A > C
- if A == B then B can be removed

References

[R208] (page 1253), [R209] (page 1253)

Examples

```
>>> p = Symbol('p', positive=True)
>>> n = Symbol('n', negative=True)
```

```
>>> Max(x, -2)
Max(-2, x)
>>> Max(x, -2).subs(x, 3)
3
>>> Max(p, -2)
p
>>> Max(x, y)
Max(x, y)
>>> Max(x, y) == Max(y, x)
True
>>> Max(x, Max(y, z))
Max(x, y, z)
>>> Max(n, 8, p, 7, -oo)
Max(8, p)
>>> Max (1, x, oo)
oo
```

root

`diofant.functions.elementary.miscellaneous.`**`root`**(x, n, k) → Returns the k-th n-th root of x, defaulting to the principle root (k=0).

See also:

diofant.polys.rootoftools.RootOf (page 711), *diofant.core.power.integer_nthroot* (page 103), *diofant.functions.elementary.miscellaneous.sqrt* (page 328), *diofant.functions.elementary.miscellaneous.real_root* (page 328)

References

- https//en.wikipedia.org/wiki/Square_root
- https//en.wikipedia.org/wiki/Real_root

- https//en.wikipedia.org/wiki/Root_of_unity
- https//en.wikipedia.org/wiki/Principal_value
- http://mathworld.wolfram.com/CubeRoot.html

Examples

```
>>> root(x, 2)
sqrt(x)
```

```
>>> root(x, 3)
x**(1/3)
```

```
>>> root(x, n)
x**(1/n)
```

```
>>> root(x, -Rational(2, 3))
x**(-3/2)
```

To get the k-th n-th root, specify k:

```
>>> root(-2, 3, 2)
-(-1)**(2/3)*2**(1/3)
```

To get all n n-th roots you can use the RootOf function. The following examples show the roots of unity for n equal 2, 3 and 4:

```
>>> [RootOf(x**2 - 1, i) for i in range(2)]
[-1, 1]
```

```
>>> [RootOf(x**3 - 1, i) for i in range(3)]
[1, -1/2 - sqrt(3)*I/2, -1/2 + sqrt(3)*I/2]
```

```
>>> [RootOf(x**4 - 1, i) for i in range(4)]
[-1, 1, -I, I]
```

Diofant, like other symbolic algebra systems, returns the complex root of negative numbers. This is the principal root and differs from the text-book result that one might be expecting. For example, the cube root of -8 does not come back as -2:

```
>>> root(-8, 3)
2*(-1)**(1/3)
```

The real_root function can be used to either make the principle result real (or simply to return the real root directly):

```
>>> real_root(_)
-2
>>> real_root(-32, 5)
-2
```

Alternatively, the n//2-th n-th root of a negative number can be computed with root:

```
>>> root(-32, 5, 5//2)
-2
```

real_root

diofant.functions.elementary.miscellaneous.**real_root**(*arg*, *n=None*)

Return the real nth-root of arg if possible. If n is omitted then all instances of (-n)**(1/odd) will be changed to -n**(1/odd); this will only create a real root of a principle root – the presence of other factors may cause the result to not be real.

See also:

diofant.polys.rootoftools.RootOf (page 711), *diofant.core.power.integer_nthroot* (page 103), *diofant.functions.elementary.miscellaneous.root* (page 326), *diofant.functions.elementary.miscellaneous.sqrt* (page 328)

Examples

```
>>> real_root(-8, 3)
-2
>>> root(-8, 3)
2*(-1)**(1/3)
>>> real_root(_)
-2
```

If one creates a non-principle root and applies real_root, the result will not be real (so use with caution):

```
>>> root(-8, 3, 2)
-2*(-1)**(2/3)
>>> real_root(_)
-2*(-1)**(2/3)
```

sqrt

diofant.functions.elementary.miscellaneous.**sqrt**(*arg*, ***kwargs*)

The square root function

sqrt(x) -> Returns the principal square root of x.

See also:

diofant.polys.rootoftools.RootOf (page 711), *diofant.functions.elementary.miscellaneous.root* (page 326), *diofant.functions.elementary.miscellaneous.real_root* (page 328)

References

[R210] (page 1254), *[R211]* (page 1254)

Examples

```
>>> sqrt(x)
sqrt(x)
```

```
>>> sqrt(x)**2
x
```

Note that sqrt(x**2) does not simplify to x.

```
>>> sqrt(x**2)
sqrt(x**2)
```

This is because the two are not equal to each other in general. For example, consider x == -1:

```
>>> Eq(sqrt(x**2), x).subs(x, -1)
false
```

This is because sqrt computes the principal square root, so the square may put the argument in a different branch. This identity does hold if x is positive:

```
>>> y = Symbol('y', positive=True)
>>> sqrt(y**2)
y
```

You can force this simplification by using the powdenest() function with the force option set to True:

```
>>> sqrt(x**2)
sqrt(x**2)
>>> powdenest(sqrt(x**2), force=True)
x
```

To get both branches of the square root you can use the RootOf function:

```
>>> [RootOf(x**2 - 3, i) for i in (0, 1)]
[-sqrt(3), sqrt(3)]
```

3.5.13 Combinatorial

This module implements various combinatorial functions.

bell

class `diofant.functions.combinatorial.numbers.`**`bell`**

Bell numbers / Bell polynomials

The Bell numbers satisfy $B_0 = 1$ and

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k.$$

They are also given by:

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

The Bell polynomials are given by $B_0(x) = 1$ and

$$B_n(x) = x \sum_{k=1}^{n-1} \binom{n-1}{k-1} B_{k-1}(x).$$

The second kind of Bell polynomials (are sometimes called "partial" Bell polynomials or incomplete Bell polynomials) are defined as

$$B_{n,k}(x_1, x_2, \dots x_{n-k+1}) = \sum_{\substack{j_1+j_2+j_2+\cdots=k \\ j_1+2j_2+3j_2+\cdots=n}} \frac{n!}{j_1!j_2!\cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}.$$

- bell(n) gives the nth Bell number, B_n.
- bell(n, x) gives the nth Bell polynomial, $B_n(x)$.
- bell(n, k, (x1, x2, …)) gives Bell polynomials of the second kind, $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$.

See also:

diofant.functions.combinatorial.numbers.bernoulli (page 331), *diofant.functions.combinatorial.numbers.catalan* (page 333), *diofant.functions.combinatorial.numbers.euler* (page 335), *diofant.functions.combinatorial.numbers.fibonacci* (page 339), *diofant.functions.combinatorial.numbers.harmonic* (page 340), *diofant.functions.combinatorial.numbers.lucas* (page 342)

Notes

Not to be confused with Bernoulli numbers and Bernoulli polynomials, which use the same notation.

References

[R126] (page 1254), *[R127]* (page 1254), *[R128]* (page 1254)

Examples

```
>>> [bell(n) for n in range(11)]
[1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975]
>>> bell(30)
846749014511809332450147
>>> bell(4, Symbol('t'))
t**4 + 6*t**3 + 7*t**2 + t
>>> bell(6, 2, symbols('x:6')[1:])
6*x1*x5 + 15*x2*x4 + 10*x3**2
```

classmethod eval(*n*, *k_sym=None*, *symbols=None*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

bernoulli

class diofant.functions.combinatorial.numbers.**bernoulli**
Bernoulli numbers / Bernoulli polynomials

The Bernoulli numbers are a sequence of rational numbers defined by B_0 = 1 and the recursive relation (n > 0):

```
      n
     ___
    \      / n + 1 \
0 =  )     |       | * B .
    /___   \   k   /    k
    k = 0
```

They are also commonly defined by their exponential generating function, which is x/(exp(x) - 1). For odd indices > 1, the Bernoulli numbers are zero.

The Bernoulli polynomials satisfy the analogous formula:

```
          n
         ___
        \      / n \         n-k
B (x) =  )     |   | * B  * x    .
 n      /___   \ k /    k
        k = 0
```

Bernoulli numbers and Bernoulli polynomials are related as B_n(0) = B_n.

We compute Bernoulli numbers using Ramanujan's formula:

```
                         / n + 3 \
B   =  (A(n) - S(n))  /  |       |
 n                       \   n   /
```

where A(n) = (n+3)/3 when n = 0 or 2 (mod 6), A(n) = -(n+3)/6 when n = 4 (mod 6), and:

```
       [n/6]
        ___
       \      /  n + 3  \
S(n) =  )     |         | * B
       /___   \ n - 6*k /    n-6*k
       k = 1
```

This formula is similar to the sum given in the definition, but cuts 2/3 of the terms. For Bernoulli polynomials, we use the formula in the definition.

- bernoulli(n) gives the nth Bernoulli number, B_n
- bernoulli(n, x) gives the nth Bernoulli polynomial in x, B_n(x)

See also:

diofant.functions.combinatorial.numbers.bell (page 329), *diofant.functions.combinatorial.numbers.catalan* (page 333), *diofant.functions.combinatorial.numbers.euler* (page 335), *diofant.functions.combinatorial.numbers.fibonacci* (page 339), *diofant.functions.combinatorial.numbers.harmonic* (page 340), *diofant.functions.combinatorial.numbers.lucas* (page 342)

References

[R129] (page 1254), *[R130]* (page 1254), *[R131]* (page 1254), *[R132]* (page 1254)

Examples

```
>>> [bernoulli(n) for n in range(11)]
[1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, 0, 5/66]
>>> bernoulli(1000001)
0
```

classmethod eval(*n*, *sym=None*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

binomial

class diofant.functions.combinatorial.factorials.**binomial**

Implementation of the binomial coefficient. It can be defined in two ways depending on its desired interpretation:

C(n,k) = n!/(k!(n-k)!) or C(n, k) = ff(n, k)/k!

First, in a strict combinatorial sense it defines the number of ways we can choose 'k' elements from a set of 'n' elements. In this case both arguments are nonnegative integers and binomial is computed using an efficient algorithm based on prime factorization.

The other definition is generalization for arbitrary 'n', however 'k' must also be nonnegative. This case is very useful when evaluating summations.

For the sake of convenience for negative 'k' this function will return zero no matter what valued is the other argument.

To expand the binomial when n is a symbol, use either expand_func() or expand(func=True). The former will keep the polynomial in factored form while the latter will expand the polynomial itself. See examples for details.

Examples

```
>>> n = Symbol('n', integer=True, positive=True)
```

```
>>> binomial(15, 8)
6435
```

```
>>> binomial(n, -1)
0
```

Rows of Pascal's triangle can be generated with the binomial function:

```
>>> for N in range(8):
...     [ binomial(N, i) for i in range(N + 1)]
...
[1]
[1, 1]
[1, 2, 1]
[1, 3, 3, 1]
[1, 4, 6, 4, 1]
[1, 5, 10, 10, 5, 1]
[1, 6, 15, 20, 15, 6, 1]
[1, 7, 21, 35, 35, 21, 7, 1]
```

As can a given diagonal, e.g. the 4th diagonal:

```
>>> N = -4
>>> [binomial(N, i) for i in range(1 - N)]
[1, -4, 10, -20, 35]
```

```
>>> binomial(Rational(5, 4), 3)
-5/128
>>> binomial(Rational(-5, 4), 3)
-195/128
```

```
>>> binomial(n, 3)
binomial(n, 3)
```

```
>>> binomial(n, 3).expand(func=True)
n**3/6 - n**2/2 + n/3
```

```
>>> expand_func(binomial(n, 3))
n*(n - 2)*(n - 1)/6
```

classmethod eval(*n, k*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)

Returns the first derivative of the function.

catalan

class diofant.functions.combinatorial.numbers.**catalan**

Catalan numbers

The n-th catalan number is given by:

```
       1   / 2*n \
C  = ----- |     |
 n   n + 1 \  n  /
```

- catalan(n) gives the n-th Catalan number, C_n

See also:

diofant.functions.combinatorial.numbers.bell (page 329), *diofant.functions.combinatorial.numbers.bernoulli* (page 331), *diofant.functions.combinatorial.numbers.euler* (page 335), *diofant.functions.combinatorial.numbers.fibonacci* (page 339), *diofant.functions.combinatorial.numbers.harmonic* (page 340), *diofant.functions.combinatorial.numbers.lucas* (page 342), *diofant.functions.combinatorial.factorials.binomial* (page 332)

References

[R133] (page 1254), *[R134]* (page 1254), *[R135]* (page 1254), *[R136]* (page 1254)

Examples

```
>>> [catalan(i) for i in range(1, 10)]
[1, 2, 5, 14, 42, 132, 429, 1430, 4862]
```

```
>>> catalan(n)
catalan(n)
```

Catalan numbers can be transformed into several other, identical expressions involving other mathematical functions

```
>>> catalan(n).rewrite(binomial)
binomial(2*n, n)/(n + 1)
```

```
>>> catalan(n).rewrite(gamma)
4**n*gamma(n + 1/2)/(sqrt(pi)*gamma(n + 2))
```

```
>>> catalan(n).rewrite(hyper)
hyper((-n + 1, -n), (2,), 1)
```

For some non-integer values of n we can get closed form expressions by rewriting in terms of gamma functions:

```
>>> catalan(Rational(1, 2)).rewrite(gamma)
8/(3*pi)
```

We can differentiate the Catalan numbers C(n) interpreted as a continuous real function in n:

```
>>> diff(catalan(n), n)
(polygamma(0, n + 1/2) - polygamma(0, n + 2) + log(4))*catalan(n)
```

As a more advanced example consider the following ratio between consecutive numbers:

```
>>> combsimp((catalan(n + 1)/catalan(n)).rewrite(binomial))
2*(2*n + 1)/(n + 2)
```

The Catalan numbers can be generalized to complex numbers:

```
>>> catalan(I).rewrite(gamma)
4**I*gamma(1/2 + I)/(sqrt(pi)*gamma(2 + I))
```

and evaluated with arbitrary precision:

```
>>> catalan(I).evalf(20)
0.39764993382373624267 - 0.020884341620842555705*I
```

classmethod eval(*n*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

euler

class diofant.functions.combinatorial.numbers.**euler**
Euler numbers

The euler numbers are given by:

```
        2*n+1   k
         ___   ___            j          2*n+1
        \     \     / k \ (-1)  * (k-2*j)
E   = I  )     )    |   | --------------------
 2n     /___  /___  \ j /      k    k
        k = 1 j = 0           2  * I  * k

E     = 0
 2n+1
```

- euler(n) gives the n-th Euler number, E_n

See also:

diofant.functions.combinatorial.numbers.bell (page 329), *diofant.functions.combinatorial.numbers.bernoulli* (page 331), *diofant.functions.combinatorial.numbers.fibonacci* (page 339), *diofant.functions.combinatorial.numbers.harmonic* (page 340), *diofant.functions.combinatorial.numbers.lucas* (page 342)

References

[R137] (page 1254), *[R138]* (page 1254), *[R139]* (page 1254), *[R140]* (page 1254)

Examples

```
>>> from diofant.functions import euler
```

```
>>> [euler(n) for n in range(10)]
[1, 0, -1, 0, 5, 0, -61, 0, 1385, 0]
>>> euler(n+2*n)
euler(3*n)
```

classmethod eval(*m*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

factorial

class diofant.functions.combinatorial.factorials.**factorial**
Implementation of factorial function over nonnegative integers.

By convention (consistent with the gamma function and the binomial coefficients), factorial of a negative integer is complex infinity.

The factorial is very important in combinatorics where it gives the number of ways in which n objects can be permuted. It also arises in calculus, probability, number theory, etc.

There is strict relation of factorial with gamma function. In fact n! = gamma(n+1) for nonnegative integers. Rewrite of this kind is very useful in case of combinatorial simplification.

Computation of the factorial is done using two algorithms. For small arguments naive product is evaluated. However for bigger input algorithm Prime-Swing is used. It is the fastest algorithm known and computes n! via prime factorization of special class of numbers, called here the 'Swing Numbers'.

See also:

diofant.functions.combinatorial.factorials.factorial2 (page 338), *diofant.functions.combinatorial.factorials.RisingFactorial* (page 343), *diofant.functions.combinatorial.factorials.FallingFactorial* (page 339)

Examples

```
>>> factorial(0)
1
```

```
>>> factorial(7)
5040
```

```
>>> factorial(-2)
zoo
```

```
>>> factorial(n)
factorial(n)
```

```
>>> factorial(2*n)
factorial(2*n)
```

```
>>> factorial(Rational(1, 2))
factorial(1/2)
```

classmethod eval(*n*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

subfactorial

class diofant.functions.combinatorial.factorials.**subfactorial**
The subfactorial counts the derangements of n items and is defined for non-negative integers as:

```
      ,
     |  1                             for n = 0
!n = {  0                             for n = 1
     |  (n - 1)*(!(n - 1) + !(n - 2)) for n > 1
      `
```

It can also be written as int(round(n!/exp(1))) but the recursive definition with caching is implemented for this function.

An interesting analytic expression is the following *[R142]* (page 1254)

$$!x = \Gamma(x + 1, -1)/e$$

which is valid for non-negative integers x. The above formula is not very useful in case of non-integers. $\Gamma(x + 1, -1)$ is single-valued only for integral arguments x, elsewhere on the positive real axis it has an infinite number of branches none of which are real.

See also:

diofant.functions.combinatorial.factorials.factorial (page 336), *diofant.utilities.iterables.generate_derangements* (page 985), *diofant.functions.special.gamma_functions.uppergamma* (page 356)

References

[R141] (page 1254), *[R142]* (page 1254)

Examples

```
>>> subfactorial(n + 1)
subfactorial(n + 1)
>>> subfactorial(5)
44
```

classmethod eval(*arg*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

factorial2 / double factorial

class diofant.functions.combinatorial.factorials.**factorial2**
The double factorial n!!, not to be confused with (n!)!

The double factorial is defined for nonnegative integers and for odd negative integers as:

```
        ,
       |  n*(n - 2)*(n - 4)* ... * 1    for n positive odd
n!! = {  n*(n - 2)*(n - 4)* ... * 2    for n positive even
       |  1                             for n = 0
       |  (n+2)!! / (n+2)               for n negative odd
        `
```

See also:

diofant.functions.combinatorial.factorials.factorial (page 336), *diofant.functions.combinatorial.factorials.RisingFactorial* (page 343), *diofant.functions.combinatorial.factorials.FallingFactorial* (page 339)

References

[R143] (page 1254)

Examples

```
>>> factorial2(n + 1)
factorial2(n + 1)
>>> factorial2(5)
15
>>> factorial2(-1)
1
>>> factorial2(-5)
1/3
```

classmethod eval(*n*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

FallingFactorial

class diofant.functions.combinatorial.factorials.**FallingFactorial**

Falling factorial (related to rising factorial) is a double valued function arising in concrete mathematics, hypergeometric functions and series expansions.

It is defined by:

```
ff(x, k) = x * (x-1) * ... * (x - k+1)
```

where 'x' can be arbitrary expression and 'k' is an integer. For more information check "Concrete mathematics" by Graham, pp. 66 or visit http://mathworld.wolfram.com/FallingFactorial.html page.

```
>>> ff(x, 0)
1
```

```
>>> ff(5, 5)
120
```

```
>>> ff(x, 5) == x*(x-1)*(x-2)*(x-3)*(x-4)
True
```

See also:

diofant.functions.combinatorial.factorials.factorial (page 336), *diofant.functions.combinatorial.factorials.factorial2* (page 338), *diofant.functions.combinatorial.factorials.RisingFactorial* (page 343)

classmethod eval(*x*, *k*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fibonacci

class diofant.functions.combinatorial.numbers.**fibonacci**

Fibonacci numbers / Fibonacci polynomials

The Fibonacci numbers are the integer sequence defined by the initial terms F_0 = 0, F_1 = 1 and the two-term recurrence relation F_n = F_{n-1} + F_{n-2}. This definition extended to arbitrary real and complex arguments using the formula

$$F_z = \frac{\phi^z - \cos(\pi z)\phi^{-z}}{\sqrt{5}}$$

The Fibonacci polynomials are defined by F_1(x) = 1, F_2(x) = x, and F_n(x) = x*F_{n-1}(x) + F_{n-2}(x) for n > 2. For all positive integers n, F_n(1) = F_n.

- fibonacci(n) gives the nth Fibonacci number, F_n

- fibonacci(n, x) gives the nth Fibonacci polynomial in x, F_n(x)

See also:

diofant.functions.combinatorial.numbers.bell (page 329), *diofant.functions.combinatorial.numbers.bernoulli* (page 331), *diofant.functions.combinatorial.numbers.catalan* (page 333), *diofant.functions.combinatorial.numbers.euler* (page 335), *diofant.functions.combinatorial.numbers.harmonic* (page 340), *diofant.functions.combinatorial.numbers.lucas* (page 342)

References

[R144] (page 1254), *[R145]* (page 1254)

Examples

```
>>> [fibonacci(x) for x in range(11)]
[0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
>>> fibonacci(5, Symbol('t'))
t**4 + 3*t**2 + 1
```

classmethod eval(*n*, *sym=None*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

harmonic

class diofant.functions.combinatorial.numbers.**harmonic**

Harmonic numbers

The nth harmonic number is given by $\operatorname{H}_{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$.

More generally:

$$\operatorname{H}_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}$$

As $n \rightarrow \infty$, $\operatorname{H}_{n,m} \rightarrow \zeta(m)$, the Riemann zeta function.

- `harmonic(n)` gives the nth harmonic number, H_n
- `harmonic(n, m)` gives the nth generalized harmonic number of order m, $\operatorname{H}_{n,m}$, where `harmonic(n) == harmonic(n, 1)`

See also:

diofant.functions.combinatorial.numbers.bell (page 329), *diofant.functions.combinatorial.numbers.bernoulli* (page 331), *diofant.functions.combinatorial.numbers.catalan* (page 333), *diofant.functions.combinatorial.numbers.euler* (page 335), *diofant.functions.combinatorial.numbers.fibonacci* (page 339), *diofant.functions.combinatorial.numbers.lucas* (page 342)

References

[R146] (page 1254), [R147] (page 1254), [R148] (page 1254)

Examples

```
>>> [harmonic(n) for n in range(6)]
[0, 1, 3/2, 11/6, 25/12, 137/60]
>>> [harmonic(n, 2) for n in range(6)]
[0, 1, 5/4, 49/36, 205/144, 5269/3600]
>>> harmonic(oo, 2)
pi**2/6
```

```
>>> harmonic(n).rewrite(Sum)
Sum(1/_k, (_k, 1, n))
```

We can evaluate harmonic numbers for all integral and positive rational arguments:

```
>>> harmonic(8)
761/280
>>> harmonic(11)
83711/27720
```

```
>>> H = harmonic(1/Integer(3))
>>> H
harmonic(1/3)
>>> He = expand_func(H)
>>> He
-log(6) - sqrt(3)*pi/6 + 2*Sum(log(sin(pi*_k/3))*cos(2*pi*_k/3), (_k, 1, 1))
                       + 3*Sum(1/(3*_k + 1), (_k, 0, 0))
>>> He.doit()
-log(6) - sqrt(3)*pi/6 - log(sqrt(3)/2) + 3
>>> H = harmonic(25/Integer(7))
>>> He = simplify(expand_func(H).doit())
>>> He
log(sin(pi/7)**(-2*cos(pi/7))*sin(2*pi/7)**(2*cos(16*pi/7))*cos(pi/14)**(-
↪2*sin(pi/14))/14)
+ pi*tan(pi/14)/2 + 30247/9900
>>> He.n(40)
1.983697455232980674869851942390639915940
>>> harmonic(25/Integer(7)).n(40)
1.983697455232980674869851942390639915940
```

We can rewrite harmonic numbers in terms of polygamma functions:

```
>>> harmonic(n).rewrite(digamma)
polygamma(0, n + 1) + EulerGamma
```

```
>>> harmonic(n).rewrite(polygamma)
polygamma(0, n + 1) + EulerGamma
```

```
>>> harmonic(n, 3).rewrite(polygamma)
polygamma(2, n + 1)/2 - polygamma(2, 1)/2
```

```
>>> harmonic(n, m).rewrite(polygamma)
(-1)**m*(polygamma(m - 1, 1) - polygamma(m - 1, n + 1))/factorial(m - 1)
```

Integer offsets in the argument can be pulled out:

```
>>> expand_func(harmonic(n+4))
harmonic(n) + 1/(n + 4) + 1/(n + 3) + 1/(n + 2) + 1/(n + 1)
```

```
>>> expand_func(harmonic(n-4))
harmonic(n) - 1/(n - 1) - 1/(n - 2) - 1/(n - 3) - 1/n
```

Some limits can be computed as well:

```
>>> limit(harmonic(n), n, oo)
oo
```

```
>>> limit(harmonic(n, 2), n, oo)
pi**2/6
```

```
>>> limit(harmonic(n, 3), n, oo)
-polygamma(2, 1)/2
```

However we can not compute the general relation yet:

```
>>> limit(harmonic(n, m), n, oo)
harmonic(oo, m)
```

which equals `zeta(m)` for `m > 1`.

classmethod eval(*n*, *m=None*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

lucas

class `diofant.functions.combinatorial.numbers.`**lucas**

Lucas numbers

Lucas numbers satisfy a recurrence relation similar to that of the Fibonacci sequence, in which each term is the sum of the preceding two. They are generated by choosing the initial values L_0 = 2 and L_1 = 1.

- lucas(n) gives the nth Lucas number

See also:

diofant.functions.combinatorial.numbers.bell (page 329), *diofant.functions.combinatorial.numbers.bernoulli* (page 331), *diofant.functions.combinatorial.numbers.catalan* (page 333), *diofant.functions.combinatorial.numbers.euler* (page 335), *diofant.functions.combinatorial.numbers.fibonacci* (page 339), *diofant.functions.combinatorial.numbers.harmonic* (page 340)

References

[R149] (page 1254), [R150] (page 1254)

Examples

```
>>> [lucas(x) for x in range(11)]
[2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123]
```

classmethod eval(*n*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

MultiFactorial

class diofant.functions.combinatorial.factorials.**MultiFactorial**

RisingFactorial

class diofant.functions.combinatorial.factorials.**RisingFactorial**

Rising factorial (also called Pochhammer symbol) is a double valued function arising in concrete mathematics, hypergeometric functions and series expansions.

It is defined by:

```
rf(x, k) = x * (x+1) * ... * (x + k-1)
```

where 'x' can be arbitrary expression and 'k' is an integer. For more information check "Concrete mathematics" by Graham, pp. 66 or visit http://mathworld.wolfram.com/RisingFactorial.html page.

See also:

diofant.functions.combinatorial.factorials.factorial (page 336), *diofant.functions.combinatorial.factorials.factorial2* (page 338), *diofant.functions.combinatorial.factorials.FallingFactorial* (page 339)

Examples

```
>>> rf(x, 0)
1
```

```
>>> rf(1, 5)
120
```

```
>>> rf(x, 5) == x*(1 + x)*(2 + x)*(3 + x)*(4 + x)
True
```

classmethod eval(*x*, *k*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

stirling

diofant.functions.combinatorial.numbers.**stirling**(*n*, *k*, *d=None*, *kind=2*, *signed=False*)

Return Stirling number S(n, k) of the first or second (default) kind.

The sum of all Stirling numbers of the second kind for k = 1 through n is bell(n). The recurrence relationship for these numbers is:

```
{0}       {n}   {0}      {n + 1}     {n}   {   n   }
{ } = 1;  { } = { } = 0; {       } = j*{ } + {       }
{0}       {0}   {k}      {   k   }     {k}   {k - 1}
```

where j is:: n for Stirling numbers of the first kind -n for signed Stirling numbers of the first kind k for Stirling numbers of the second kind

The first kind of Stirling number counts the number of permutations of n distinct items that have k cycles; the second kind counts the ways in which n distinct items can be partitioned into k parts. If d is given, the "reduced Stirling number of the second kind" is returned: S^{d}(n, k) = S(n - d + 1, k - d + 1) with n >= k >= d. (This counts the ways to partition n consecutive integers into k groups with no pairwise difference less than d. See example below.)

To obtain the signed Stirling numbers of the first kind, use keyword signed=True. Using this keyword automatically sets kind to 1.

See also:

diofant.utilities.iterables.multiset_partitions (page 990)

References

[R151] (page 1254), [R152] (page 1254)

Examples

```
>>> from diofant.utilities.iterables import (multiset_partitions,
...                                          permutations, subsets)
```

First kind (unsigned by default):

```
>>> [stirling(6, i, kind=1) for i in range(7)]
[0, 120, 274, 225, 85, 15, 1]
>>> perms = list(permutations(range(4)))
>>> [sum(Permutation(p).cycles == i for p in perms) for i in range(5)]
[0, 6, 11, 6, 1]
```

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```
>>> [stirling(4, i, kind=1) for i in range(5)]
[0, 6, 11, 6, 1]
```

First kind (signed):

```
>>> [stirling(4, i, signed=True) for i in range(5)]
[0, -6, 11, -6, 1]
```

Second kind:

```
>>> [stirling(10, i) for i in range(12)]
[0, 1, 511, 9330, 34105, 42525, 22827, 5880, 750, 45, 1, 0]
>>> sum(_) == bell(10)
True
>>> len(list(multiset_partitions(range(4), 2))) == stirling(4, 2)
True
```

Reduced second kind:

```
>>> def delta(p):
...     if len(p) == 1:
...         return oo
...     return min(abs(i[0] - i[1]) for i in subsets(p, 2))
>>> parts = multiset_partitions(range(5), 3)
>>> d = 2
>>> sum(1 for p in parts if all(delta(i) >= d for i in p))
7
>>> stirling(5, 3, 2)
7
```

3.5.14 Enumeration

Three functions are available. Each of them attempts to efficiently compute a given combinatorial quantity for a given set or multiset which can be entered as an integer, sequence or multiset (dictionary with elements as keys and multiplicities as values). The `k` parameter indicates the number of elements to pick (or the number of partitions to make). When `k` is None, the sum of the enumeration for all `k` (from 0 through the number of items represented by `n`) is returned. A `replacement` parameter is recognized for combinations and permutations; this indicates that any item may appear with multiplicity as high as the number of items in the original set.

```
>>> from diofant.functions.combinatorial.numbers import nC, nP, nT
>>> items = 'baby'
```

`diofant.functions.combinatorial.numbers.nC`(*n*, *k=None*, *replacement=False*)

Return the number of combinations of `n` items taken `k` at a time.

Possible values for `n`:: integer - set of length `n` sequence - converted to a multiset internally multiset - {element: multiplicity}

If `k` is None then the total of all combinations of length 0 through the number of items represented in `n` will be returned.

If `replacement` is True then a given item can appear more than once in the `k` items. (For example, for 'ab' sets of 2 would include 'aa', 'ab', and 'bb'.) The multiplicity of elements

in n is ignored when replacement is True but the total number of elements is considered since no element can appear more times than the number of elements in n.

See also:

diofant.utilities.iterables.multiset_combinations (page 990)

References

[R153] (page 1254), [R154] (page 1254)

Examples

```
>>> from diofant.utilities.iterables import multiset_combinations
>>> nC(3, 2)
3
>>> nC('abc', 2)
3
>>> nC('aab', 2)
2
```

When replacement is True, each item can have multiplicity equal to the length represented by n:

```
>>> nC('aabc', replacement=True)
35
>>> [len(list(multiset_combinations('aaaabbbbcccc', i))) for i in range(5)]
[1, 3, 6, 10, 15]
>>> sum(_)
35
```

If there are k items with multiplicities m_1, m_2, ..., m_k then the total of all combinations of length 0 through k is the product, (m_1 + 1)*(m_2 + 1)*...*(m_k + 1). When the multiplicity of each item is 1 (i.e., k unique items) then there are 2**k combinations. For example, if there are 4 unique items, the total number of combinations is 16:

```
>>> sum(nC(4, i) for i in range(5))
16
```

diofant.functions.combinatorial.numbers.**nP**(*n*, *k=None*, *replacement=False*)

Return the number of permutations of n items taken k at a time.

Possible values for n:: integer - set of length n sequence - converted to a multiset internally multiset - {element: multiplicity}

If k is None then the total of all permutations of length 0 through the number of items represented by n will be returned.

If replacement is True then a given item can appear more than once in the k items. (For example, for 'ab' permutations of 2 would include 'aa', 'ab', 'ba' and 'bb'.) The multiplicity of elements in n is ignored when replacement is True but the total number of elements is considered since no element can appear more times than the number of elements in n.

See also:

diofant.utilities.iterables.multiset_permutations (page 992)

References

[R155] (page 1254)

Examples

```
>>> from diofant.utilities.iterables import multiset_permutations, multiset
```

```
>>> nP(3, 2)
6
>>> nP('abc', 2) == nP(multiset('abc'), 2) == 6
True
>>> nP('aab', 2)
3
>>> nP([1, 2, 2], 2)
3
>>> [nP(3, i) for i in range(4)]
[1, 3, 6, 6]
>>> nP(3) == sum(_)
True
```

When `replacement` is True, each item can have multiplicity equal to the length represented by `n`:

```
>>> nP('aabc', replacement=True)
121
>>> [len(list(multiset_permutations('aaaabbbbcccc', i))) for i in range(5)]
[1, 3, 9, 27, 81]
>>> sum(_)
121
```

`diofant.functions.combinatorial.numbers.`**`nT`**(*n*, *k=None*)

Return the number of `k`-sized partitions of `n` items.

Possible values for `n`:: integer - `n` identical items sequence - converted to a multiset internally multiset - {element: multiplicity}

Note: the convention for `nT` is different than that of `nC` and `nP` in that here an integer indicates `n` *identical* items instead of a set of length `n`; this is in keeping with the `partitions` function which treats its integer-`n` input like a list of `n` 1s. One can use `range(n)` for `n` to indicate `n` distinct items.

If `k` is None then the total number of ways to partition the elements represented in `n` will be returned.

See also:

diofant.utilities.iterables.partitions (page 994), *diofant.utilities.iterables.multiset_partitions* (page 990)

References

[R156] (page 1254)

Examples

Partitions of the given multiset:

```
>>> [nT('aabbc', i) for i in range(1, 7)]
[1, 8, 11, 5, 1, 0]
>>> nT('aabbc') == sum(_)
True
```

```
>>> [nT("mississippi", i) for i in range(1, 12)]
[1, 74, 609, 1521, 1768, 1224, 579, 197, 50, 9, 1]
```

Partitions when all items are identical:

```
>>> [nT(5, i) for i in range(1, 6)]
[1, 2, 2, 1, 1]
>>> nT('1'*5) == sum(_)
True
```

When all items are different:

```
>>> [nT(range(5), i) for i in range(1, 6)]
[1, 15, 25, 10, 1]
>>> nT(range(5)) == sum(_)
True
```

Note that the integer for `n` indicates *identical* items for `nT` but indicates `n` *different* items for `nC` and `nP`.

3.5.15 Special

DiracDelta

class `diofant.functions.special.delta_functions.`**`DiracDelta`**

The DiracDelta function and its derivatives.

DiracDelta function has the following properties:

1. `diff(Heaviside(x),x) = DiracDelta(x)`
2. `integrate(DiracDelta(x-a)*f(x),(x,-oo,oo)) = f(a)` and `integrate(DiracDelta(x-a)*f(x),(x,a-e,a+e)) = f(a)`
3. `DiracDelta(x) = 0` for all `x != 0`
4. `DiracDelta(g(x)) = Sum_i(DiracDelta(x-x_i)/abs(g'(x_i)))` Where `x_i`-s are the roots of `g`

Derivatives of `k`-th order of DiracDelta have the following property:

5. `DiracDelta(x,k) = 0`, for all `x != 0`

See also:

`diofant.functions.special.delta_functions.Heaviside` (page 350), *`diofant.simplify.simplify.simplify`* (page 776), *`diofant.functions.special.delta_functions.DiracDelta.is_simple`* (page 349), *`diofant.functions.special.tensor_functions.KroneckerDelta`* (page 418)

References

[R212] (page 1254)

classmethod eval(*arg*, *k=0*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)

Returns the first derivative of the function.

is_simple(*self*, *x*)

Tells whether the argument(args[0]) of DiracDelta is a linear expression in x.

x can be:

- a symbol

See also:

diofant.simplify.simplify.simplify (page 776), *diofant.functions.special.delta_functions.DiracDelta* (page 348)

Examples

```
>>> DiracDelta(x*y).is_simple(x)
True
>>> DiracDelta(x*y).is_simple(y)
True
```

```
>>> DiracDelta(x**2+x-2).is_simple(x)
False
```

```
>>> DiracDelta(cos(x)).is_simple(x)
False
```

simplify(*self*, *x*)

Compute a simplified representation of the function using property number 4.

x can be:

- a symbol

See also:

diofant.functions.special.delta_functions.DiracDelta.is_simple (page 349), *diofant.functions.special.delta_functions.DiracDelta* (page 348)

Examples

```
>>> DiracDelta(x*y).simplify(x)
DiracDelta(x)/Abs(y)
>>> DiracDelta(x*y).simplify(y)
DiracDelta(y)/Abs(x)
```

```
>>> DiracDelta(x**2 + x - 2).simplify(x)
DiracDelta(x - 1)/3 + DiracDelta(x + 2)/3
```

Heaviside

class diofant.functions.special.delta_functions.**Heaviside**

Heaviside step function *[R213]* (page 1254)

$$H(x) = \begin{cases} 0, x < 0 \\ 1/2, x = 0 \\ 1, x > 0 \end{cases}$$

See also:

diofant.functions.special.delta_functions.DiracDelta (page 348)

References

[R213] (page 1254)

classmethod eval(*arg*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)

Returns the first derivative of the function.

Gamma, Beta and related Functions

class diofant.functions.special.gamma_functions.**gamma**

The gamma function

$$\Gamma(x) := \int_0^\infty t^{x-1} e^t \mathrm{d}t.$$

The gamma function implements the function which passes through the values of the factorial function, i.e. $\Gamma(n) = (n-1)!$ when n is an integer. More general, $\Gamma(z)$ is defined in the whole complex plane except at the negative integers where there are simple poles.

See also:

***lowergamma* (page 357)** Lower incomplete gamma function.

***uppergamma* (page 356)** Upper incomplete gamma function.

***polygamma* (page 353)** Polygamma function.

loggamma **(page 351)** Log Gamma function.

digamma **(page 355)** Digamma function.

trigamma **(page 355)** Trigamma function.

diofant.functions.special.beta_functions.beta **(page 358)** Euler Beta function.

References

[R214] (page 1255), [R215] (page 1255), [R216] (page 1255), [R217] (page 1255)

Examples

Several special values are known:

```
>>> gamma(1)
1
>>> gamma(4)
6
>>> gamma(Rational(3, 2))
sqrt(pi)/2
```

The Gamma function obeys the mirror symmetry:

```
>>> conjugate(gamma(x))
gamma(conjugate(x))
```

Differentiation with respect to x is supported:

```
>>> diff(gamma(x), x)
gamma(x)*polygamma(0, x)
```

Series expansion is also supported:

```
>>> series(gamma(x), x, 0, 3)
1/x - EulerGamma + x*(EulerGamma**2/2 + pi**2/12) + x**2*(-EulerGamma*pi**2/12 +
↪polygamma(2, 1)/6 - EulerGamma**3/6) + O(x**3)
```

We can numerically evaluate the gamma function to arbitrary precision on the whole complex plane:

```
>>> gamma(pi).evalf(40)
2.288037795340032417959588909060233922890
>>> gamma(1+I).evalf(20)
0.49801566811835604271 - 0.15494982830181068512*I
```

classmethod eval(*arg*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)

Returns the first derivative of the function.

class diofant.functions.special.gamma_functions.**loggamma**

The loggamma function implements the logarithm of the gamma function i.e, $\log\Gamma(x)$.

See also:

gamma **(page 350)** Gamma function.

lowergamma **(page 357)** Lower incomplete gamma function.

uppergamma **(page 356)** Upper incomplete gamma function.

polygamma **(page 353)** Polygamma function.

digamma **(page 355)** Digamma function.

trigamma **(page 355)** Trigamma function.

diofant.functions.special.beta_functions.beta **(page 358)** Euler Beta function.

References

[R218] (page 1255), [R219] (page 1255), [R220] (page 1255), [R221] (page 1255)

Examples

Several special values are known. For numerical integral arguments we have:

```
>>> loggamma(-2)
oo
>>> loggamma(0)
oo
>>> loggamma(1)
0
>>> loggamma(2)
0
>>> loggamma(3)
log(2)
```

and for symbolic values:

```
>>> n = Symbol("n", integer=True, positive=True)
>>> loggamma(n)
log(gamma(n))
>>> loggamma(-n)
oo
```

for half-integral values:

```
>>> loggamma(Rational(5, 2))
log(3*sqrt(pi)/4)
>>> loggamma(n/2)
log(2**(-n + 1)*sqrt(pi)*gamma(n)/gamma(n/2 + 1/2))
```

and general rational arguments:

```
>>> L = loggamma(Rational(16, 3))
>>> expand_func(L).doit()
-5*log(3) + loggamma(1/3) + log(4) + log(7) + log(10) + log(13)
>>> L = loggamma(Rational(19, 4))
>>> expand_func(L).doit()
-4*log(4) + loggamma(3/4) + log(3) + log(7) + log(11) + log(15)
>>> L = loggamma(Rational(23, 7))
>>> expand_func(L).doit()
-3*log(7) + log(2) + loggamma(2/7) + log(9) + log(16)
```

The loggamma function has the following limits towards infinity:

```
>>> loggamma(oo)
oo
>>> loggamma(-oo)
zoo
```

The loggamma function obeys the mirror symmetry if $x \in \mathbb{C} \setminus \{-\infty, 0\}$:

```
>>> c = Symbol('c', complex=True, extended_real=False)
>>> conjugate(loggamma(c))
loggamma(conjugate(c))
```

Differentiation with respect to x is supported:

```
>>> diff(loggamma(x), x)
polygamma(0, x)
```

Series expansion is also supported:

```
>>> series(loggamma(x), x, 0, 4)
-log(x) - EulerGamma*x + pi**2*x**2/12 + x**3*polygamma(2, 1)/6 + O(x**4)
```

We can numerically evaluate the gamma function to arbitrary precision on the whole complex plane:

```
>>> loggamma(5).evalf(30)
3.17805383034794561964694160130
>>> loggamma(I).evalf(20)
-0.65092319930185633889 - 1.8724366472624298171*I
```

classmethod eval(*z*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)

Returns the first derivative of the function.

class diofant.functions.special.gamma_functions.**polygamma**

The function `polygamma(n, z)` returns `log(gamma(z)).diff(n + 1)`.

It is a meromorphic function on $\mathbb{C}$ and defined as the (n+1)-th derivative of the logarithm of the gamma function:

$$\psi^{(n)}(z) := \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} \log\Gamma(z).$$

See also:

gamma **(page 350)** Gamma function.

lowergamma **(page 357)** Lower incomplete gamma function.

uppergamma **(page 356)** Upper incomplete gamma function.

loggamma **(page 351)** Log Gamma function.

digamma **(page 355)** Digamma function.

trigamma **(page 355)** Trigamma function.

diofant.functions.special.beta_functions.beta **(page 358)** Euler Beta function.

References

[R222] (page 1255), [R223] (page 1255), [R224] (page 1255), [R225] (page 1255)

Examples

Several special values are known:

```
>>> polygamma(0, 1)
-EulerGamma
>>> polygamma(0, 1/Integer(2))
-2*log(2) - EulerGamma
>>> polygamma(0, 1/Integer(3))
-3*log(3)/2 - sqrt(3)*pi/6 - EulerGamma
>>> polygamma(0, 1/Integer(4))
-3*log(2) - pi/2 - EulerGamma
>>> polygamma(0, 2)
-EulerGamma + 1
>>> polygamma(0, 23)
-EulerGamma + 19093197/5173168
```

```
>>> polygamma(0, oo)
oo
>>> polygamma(0, -oo)
oo
>>> polygamma(0, I*oo)
oo
>>> polygamma(0, -I*oo)
oo
```

Differentiation with respect to x is supported:

```
>>> diff(polygamma(0, x), x)
polygamma(1, x)
>>> diff(polygamma(0, x), x, 2)
polygamma(2, x)
>>> diff(polygamma(0, x), x, 3)
polygamma(3, x)
>>> diff(polygamma(1, x), x)
polygamma(2, x)
>>> diff(polygamma(1, x), x, 2)
```

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```
polygamma(3, x)
>>> diff(polygamma(2, x), x)
polygamma(3, x)
>>> diff(polygamma(2, x), x, 2)
polygamma(4, x)
```

```
>>> diff(polygamma(n, x), x)
polygamma(n + 1, x)
>>> diff(polygamma(n, x), x, 2)
polygamma(n + 2, x)
```

We can rewrite polygamma functions in terms of harmonic numbers:

```
>>> polygamma(0, x).rewrite(harmonic)
harmonic(x - 1) - EulerGamma
>>> polygamma(2, x).rewrite(harmonic)
2*harmonic(x - 1, 3) - 2*zeta(3)
>>> ni = Symbol("n", integer=True)
>>> polygamma(ni, x).rewrite(harmonic)
(-1)**(n + 1)*(-harmonic(x - 1, n + 1) + zeta(n + 1))*factorial(n)
```

classmethod eval(*n*, *z*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=2*)
Returns the first derivative of the function.

diofant.functions.special.gamma_functions.**digamma**(*x*)
The digamma function is the first derivative of the loggamma function i.e,

$$\psi(x) := \frac{\mathrm{d}}{\mathrm{d}z} \log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

In this case, `digamma(z) = polygamma(0, z)`.

See also:

gamma **(page 350)** Gamma function.

lowergamma **(page 357)** Lower incomplete gamma function.

uppergamma **(page 356)** Upper incomplete gamma function.

polygamma **(page 353)** Polygamma function.

loggamma **(page 351)** Log Gamma function.

trigamma **(page 355)** Trigamma function.

diofant.functions.special.beta_functions.beta **(page 358)** Euler Beta function.

References

[R226] (page 1255), [R227] (page 1255), [R228] (page 1255)

`diofant.functions.special.gamma_functions.`**`trigamma`**`(x)`

The trigamma function is the second derivative of the loggamma function i.e,

$$\psi^{(1)}(z) := \frac{\mathrm{d}^2}{\mathrm{d}z^2} \log\Gamma(z).$$

In this case, `trigamma(z) = polygamma(1, z)`.

See also:

gamma **(page 350)** Gamma function.

lowergamma **(page 357)** Lower incomplete gamma function.

uppergamma **(page 356)** Upper incomplete gamma function.

polygamma **(page 353)** Polygamma function.

loggamma **(page 351)** Log Gamma function.

digamma **(page 355)** Digamma function.

diofant.functions.special.beta_functions.beta **(page 358)** Euler Beta function.

References

[R229] (page 1255), *[R230]* (page 1255), *[R231]* (page 1255)

class `diofant.functions.special.gamma_functions.`**`uppergamma`**

The upper incomplete gamma function.

It can be defined as the meromorphic continuation of

$$\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} \mathrm{d}t = \Gamma(s) - \gamma(s, x).$$

where $\gamma(s, x)$ is the lower incomplete gamma function, *lowergamma* (page 357). This can be shown to be the same as

$$\Gamma(s, x) = \Gamma(s) - \frac{x^s}{s} {}_1F_1\left(\begin{matrix} s \\ s+1 \end{matrix}\middle| -x\right),$$

where ${}_1F_1$ is the (confluent) hypergeometric function.

The upper incomplete gamma function is also essentially equivalent to the generalized exponential integral:

$$\mathrm{E}_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} \, dt = x^{n-1}\Gamma(1-n, x).$$

See also:

gamma **(page 350)** Gamma function.

lowergamma **(page 357)** Lower incomplete gamma function.

polygamma **(page 353)** Polygamma function.

loggamma **(page 351)** Log Gamma function.

digamma **(page 355)** Digamma function.

trigamma **(page 355)** Trigamma function.

diofant.functions.special.beta_functions.beta **(page 358)** Euler Beta function.

References

[R232] (page 1255), [R233] (page 1255), [R234] (page 1255), [R235] (page 1255), [R236] (page 1255), [R237] (page 1255)

Examples

```
>>> from diofant.abc import s
>>> uppergamma(s, x)
uppergamma(s, x)
>>> uppergamma(3, x)
E**(-x)*x**2 + 2*E**(-x)*x + 2*E**(-x)
>>> uppergamma(-Rational(1, 2), x)
-2*sqrt(pi)*(-erf(sqrt(x)) + 1) + 2*E**(-x)/sqrt(x)
>>> uppergamma(-2, x)
expint(3, x)/x**2
```

classmethod eval(*a*, *z*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=2*)
Returns the first derivative of the function.

class diofant.functions.special.gamma_functions.**lowergamma**
The lower incomplete gamma function.

It can be defined as the meromorphic continuation of

$$\gamma(s, x) := \int_0^x t^{s-1} e^{-t} \mathrm{d}t = \Gamma(s) - \Gamma(s, x).$$

This can be shown to be the same as

$$\gamma(s, x) = \frac{x^s}{s} {}_1F_1\left(\begin{matrix} s \\ s+1 \end{matrix} \middle| -x\right),$$

where ${}_1F_1$ is the (confluent) hypergeometric function.

See also:

gamma **(page 350)** Gamma function.

uppergamma **(page 356)** Upper incomplete gamma function.

polygamma **(page 353)** Polygamma function.

loggamma **(page 351)** Log Gamma function.

digamma **(page 355)** Digamma function.

trigamma **(page 355)** Trigamma function.

diofant.functions.special.beta_functions.beta **(page 358)** Euler Beta function.

References

[R238] (page 1255), [R239] (page 1255), [R240] (page 1255), [R241] (page 1255), [R242] (page 1255)

Examples

```
>>> from diofant.abc import s
>>> lowergamma(s, x)
lowergamma(s, x)
>>> lowergamma(3, x)
2 - E**(-x)*x**2 - 2*E**(-x)*x - 2*E**(-x)
>>> lowergamma(-Rational(1, 2), x)
-2*sqrt(pi)*erf(sqrt(x)) - 2*E**(-x)/sqrt(x)
```

classmethod eval(*a*, *x*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=2*)
Returns the first derivative of the function.

class diofant.functions.special.beta_functions.**beta**
The beta integral is called the Eulerian integral of the first kind by Legendre:

$$\mathrm{B}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t.$$

Beta function or Euler's first integral is closely associated with gamma function. The Beta function often used in probability theory and mathematical statistics. It satisfies properties like:

$$\mathrm{B}(a,1) = \frac{1}{a}$$
$$\mathrm{B}(a,b) = \mathrm{B}(b,a)$$
$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Therefore for integral values of a and b:

$$\mathrm{B} = \frac{(a-1)!(b-1)!}{(a+b-1)!}$$

See also:

diofant.functions.special.gamma_functions.gamma **(page 350)** Gamma function.

diofant.functions.special.gamma_functions.uppergamma **(page 356)** Upper incomplete gamma function.

diofant.functions.special.gamma_functions.lowergamma **(page 357)** Lower incomplete gamma function.

diofant.functions.special.gamma_functions.polygamma **(page 353)** Polygamma function.

diofant.functions.special.gamma_functions.loggamma **(page 351)** Log Gamma function.

diofant.functions.special.gamma_functions.digamma **(page 355)** Digamma function.

diofant.functions.special.gamma_functions.trigamma **(page 355)** Trigamma function.

References

[R243] (page 1255), *[R244]* (page 1255), *[R245]* (page 1255)

Examples

The Beta function obeys the mirror symmetry:

```
>>> conjugate(beta(x, y))
beta(conjugate(x), conjugate(y))
```

Differentiation with respect to both x and y is supported:

```
>>> diff(beta(x, y), x)
(polygamma(0, x) - polygamma(0, x + y))*beta(x, y)
```

```
>>> diff(beta(x, y), y)
(polygamma(0, y) - polygamma(0, x + y))*beta(x, y)
```

We can numerically evaluate the gamma function to arbitrary precision on the whole complex plane:

```
>>> beta(pi, pi).evalf(40)
0.02671848900111377452242355235388489324562
```

```
>>> beta(1 + I, 1 + I).evalf(20)
-0.2112723729365330143 - 0.7655283165378005676*I
```

classmethod eval(*x*, *y*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex*)

Returns the first derivative of the function.

Error Functions and Fresnel Integrals

class diofant.functions.special.error_functions.**erf**

The Gauss error function. This function is defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t.$$

See also:

erfc **(page 361)** Complementary error function.

erfi **(page 362)** Imaginary error function.

erf2 **(page 363)** Two-argument error function.

erfinv **(page 364)** Inverse error function.

erfcinv **(page 365)** Inverse Complementary error function.

erf2inv **(page 365)** Inverse two-argument error function.

References

[R246] (page 1255), [R247] (page 1255), [R248] (page 1255), [R249] (page 1256)

Examples

Several special values are known:

```
>>> erf(0)
0
>>> erf(oo)
1
>>> erf(-oo)
-1
>>> erf(I*oo)
oo*I
>>> erf(-I*oo)
-oo*I
```

In general one can pull out factors of -1 and I from the argument:

```
>>> erf(-z)
-erf(z)
```

The error function obeys the mirror symmetry:

```
>>> conjugate(erf(z))
erf(conjugate(z))
```

Differentiation with respect to z is supported:

```
>>> diff(erf(z), z)
2*E**(-z**2)/sqrt(pi)
```

We can numerically evaluate the error function to arbitrary precision on the whole complex plane:

```
>>> erf(4).evalf(30)
0.999999984582742099719981147840
```

```
>>> erf(-4*I).evalf(30)
-1296959.73071763923152794095062*I
```

class diofant.functions.special.error_functions.**erfc**

Complementary Error Function. The function is defined as:

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \mathrm{d}t$$

See also:

erf **(page 360)** Gaussian error function.

erfi **(page 362)** Imaginary error function.

erf2 **(page 363)** Two-argument error function.

erfinv **(page 364)** Inverse error function.

erfcinv **(page 365)** Inverse Complementary error function.

erf2inv **(page 365)** Inverse two-argument error function.

References

[R250] (page 1256), *[R251]* (page 1256), *[R252]* (page 1256), *[R253]* (page 1256)

Examples

Several special values are known:

```
>>> erfc(0)
1
>>> erfc(oo)
0
>>> erfc(-oo)
2
>>> erfc(I*oo)
-oo*I
>>> erfc(-I*oo)
oo*I
```

The error function obeys the mirror symmetry:

```
>>> conjugate(erfc(z))
erfc(conjugate(z))
```

Differentiation with respect to z is supported:

```
>>> diff(erfc(z), z)
-2*E**(-z**2)/sqrt(pi)
```

It also follows

```
>>> erfc(-z)
-erfc(z) + 2
```

We can numerically evaluate the complementary error function to arbitrary precision on the whole complex plane:

```
>>> erfc(4).evalf(30)
0.0000000154172579002800188521596734869
```

```
>>> erfc(4*I).evalf(30)
1.0 - 1296959.73071763923152794095062*I
```

class diofant.functions.special.error_functions.**erfi**

Imaginary error function. The function erfi is defined as:

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} \mathrm{d}t$$

See also:

***erf* (page 360)** Gaussian error function.

***erfc* (page 361)** Complementary error function.

***erf2* (page 363)** Two-argument error function.

***erfinv* (page 364)** Inverse error function.

***erfcinv* (page 365)** Inverse Complementary error function.

***erf2inv* (page 365)** Inverse two-argument error function.

References

[R254] (page 1256), *[R255]* (page 1256), *[R256]* (page 1256)

Examples

Several special values are known:

```
>>> erfi(0)
0
>>> erfi(oo)
oo
>>> erfi(-oo)
-oo
>>> erfi(I*oo)
I
>>> erfi(-I*oo)
-I
```

In general one can pull out factors of -1 and I from the argument:

```
>>> erfi(-z)
-erfi(z)
```

```
>>> conjugate(erfi(z))
erfi(conjugate(z))
```

Differentiation with respect to z is supported:

```
>>> diff(erfi(z), z)
2*E**(z**2)/sqrt(pi)
```

We can numerically evaluate the imaginary error function to arbitrary precision on the whole complex plane:

```
>>> erfi(2).evalf(30)
18.5648024145755525987042919132
```

```
>>> erfi(-2*I).evalf(30)
-0.995322265018952734162069256367*I
```

class `diofant.functions.special.error_functions.`**`erf2`**

Two-argument error function. This function is defined as:

$$\operatorname{erf2}(x, y) = \frac{2}{\sqrt{\pi}} \int_x^y e^{-t^2} \mathrm{d}t$$

See also:

erf **(page 360)** Gaussian error function.

erfc **(page 361)** Complementary error function.

erfi **(page 362)** Imaginary error function.

erfinv **(page 364)** Inverse error function.

erfcinv **(page 365)** Inverse Complementary error function.

erf2inv **(page 365)** Inverse two-argument error function.

References

[R257] (page 1256)

Examples

Several special values are known:

```
>>> erf2(0, 0)
0
>>> erf2(x, x)
0
>>> erf2(x, oo)
```

(continues on next page)

(continued from previous page)

```
-erf(x) + 1
>>> erf2(x, -oo)
-erf(x) - 1
>>> erf2(oo, y)
erf(y) - 1
>>> erf2(-oo, y)
erf(y) + 1
```

In general one can pull out factors of -1:

```
>>> erf2(-x, -y)
-erf2(x, y)
```

The error function obeys the mirror symmetry:

```
>>> conjugate(erf2(x, y))
erf2(conjugate(x), conjugate(y))
```

Differentiation with respect to x, y is supported:

```
>>> diff(erf2(x, y), x)
-2*E**(-x**2)/sqrt(pi)
>>> diff(erf2(x, y), y)
2*E**(-y**2)/sqrt(pi)
```

class diofant.functions.special.error_functions.**erfinv**

Inverse Error Function. The erfinv function is defined as:

$$\operatorname{erf}(x) = y \quad \Rightarrow \quad \operatorname{erfinv}(y) = x$$

See also:

erf **(page 360)** Gaussian error function.

erfc **(page 361)** Complementary error function.

erfi **(page 362)** Imaginary error function.

erf2 **(page 363)** Two-argument error function.

erfcinv **(page 365)** Inverse Complementary error function.

erf2inv **(page 365)** Inverse two-argument error function.

References

[R258] (page 1256), *[R259]* (page 1256)

Examples

Several special values are known:

```
>>> erfinv(0)
0
>>> erfinv(1)
oo
```

Differentiation with respect to x is supported:

```
>>> diff(erfinv(x), x)
E**(erfinv(x)**2)*sqrt(pi)/2
```

We can numerically evaluate the inverse error function to arbitrary precision on [-1, 1]:

```
>>> erfinv(0.2)
0.179143454621292
```

class diofant.functions.special.error_functions.**erfcinv**

Inverse Complementary Error Function. The erfcinv function is defined as:

$$\operatorname{erfc}(x) = y \quad \Rightarrow \quad \operatorname{erfcinv}(y) = x$$

See also:

***erf* (page 360)** Gaussian error function.

***erfc* (page 361)** Complementary error function.

***erfi* (page 362)** Imaginary error function.

***erf2* (page 363)** Two-argument error function.

***erfinv* (page 364)** Inverse error function.

***erf2inv* (page 365)** Inverse two-argument error function.

References

[R260] (page 1256), *[R261]* (page 1256)

Examples

Several special values are known:

```
>>> erfcinv(1)
0
>>> erfcinv(0)
oo
```

Differentiation with respect to x is supported:

```
>>> diff(erfcinv(x), x)
-E**(erfcinv(x)**2)*sqrt(pi)/2
```

class diofant.functions.special.error_functions.**erf2inv**

Two-argument Inverse error function. The erf2inv function is defined as:

$$\operatorname{erf2}(x, w) = y \quad \Rightarrow \quad \operatorname{erf2inv}(x, y) = w$$

See also:

***erf* (page 360)** Gaussian error function.

***erfc* (page 361)** Complementary error function.

***erfi* (page 362)** Imaginary error function.

***erf2* (page 363)** Two-argument error function.

***erfinv* (page 364)** Inverse error function.

***erfcinv* (page 365)** Inverse complementary error function.

References

[R262] (page 1256)

Examples

Several special values are known:

```
>>> erf2inv(0, 0)
0
>>> erf2inv(1, 0)
1
>>> erf2inv(0, 1)
oo
>>> erf2inv(0, y)
erfinv(y)
>>> erf2inv(oo, y)
erfcinv(-y)
```

Differentiation with respect to x and y is supported:

```
>>> diff(erf2inv(x, y), x)
E**(-x**2 + erf2inv(x, y)**2)
>>> diff(erf2inv(x, y), y)
E**(erf2inv(x, y)**2)*sqrt(pi)/2
```

class diofant.functions.special.error_functions.**FresnelIntegral**

Base class for the Fresnel integrals.

as_real_imag(*deep=True*, ***hints*)

Performs complex expansion on 'self' and returns a tuple containing collected both real and imaginary parts. This method can't be confused with re() and im() functions, which does not perform complex expansion at evaluation.

However it is possible to expand both re() and im() functions and get exactly the same results as with a single call to this function.

```
>>> x, y = symbols('x y', extended_real=True)
```

```
>>> (x + y*I).as_real_imag()
(x, y)
```

```
>>> from diofant.abc import w
```

```
>>> (z + w*I).as_real_imag()
(re(z) - im(w), re(w) + im(z))
```

classmethod eval(*z*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=1*)
Returns the first derivative of the function.

class diofant.functions.special.error_functions.**fresnels**
Fresnel integral S.

This function is defined by

$$\operatorname{S}(z) = \int_0^z \sin{\frac{\pi}{2} t^2} \mathrm{d}t.$$

It is an entire function.

See also:

fresnelc **(page 368)** Fresnel cosine integral.

References

[R263] (page 1256), [R264] (page 1256), [R265] (page 1256), [R266] (page 1256), [R267] (page 1256)

Examples

Several special values are known:

```
>>> fresnels(0)
0
>>> fresnels(oo)
1/2
>>> fresnels(-oo)
-1/2
>>> fresnels(I*oo)
-I/2
>>> fresnels(-I*oo)
I/2
```

In general one can pull out factors of -1 and i from the argument:

```
>>> fresnels(-z)
-fresnels(z)
>>> fresnels(I*z)
-I*fresnels(z)
```

The Fresnel S integral obeys the mirror symmetry $\overline{S(z)} = S(\bar{z})$:

```
>>> conjugate(fresnels(z))
fresnels(conjugate(z))
```

Differentiation with respect to z is supported:

```
>>> diff(fresnels(z), z)
sin(pi*z**2/2)
```

Defining the Fresnel functions via an integral

```
>>> integrate(sin(pi*z**2/2), z)
3*fresnels(z)*gamma(3/4)/(4*gamma(7/4))
>>> expand_func(integrate(sin(pi*z**2/2), z))
fresnels(z)
```

We can numerically evaluate the Fresnel integral to arbitrary precision on the whole complex plane:

```
>>> fresnels(2).evalf(30)
0.343415678363698242195300815958
```

```
>>> fresnels(-2*I).evalf(30)
0.343415678363698242195300815958*I
```

class diofant.functions.special.error_functions.**fresnelc**

Fresnel integral C.

This function is defined by

$$\mathrm{C}(z) = \int_0^z \cos{\frac{\pi}{2} t^2} \mathrm{d}t.$$

It is an entire function.

See also:

fresnels **(page 367)** Fresnel sine integral.

References

[R268] (page 1256), [R269] (page 1256), [R270] (page 1256), [R271] (page 1256), [R272] (page 1256)

Examples

Several special values are known:

```
>>> fresnelc(0)
0
>>> fresnelc(oo)
1/2
>>> fresnelc(-oo)
-1/2
>>> fresnelc(I*oo)
I/2
>>> fresnelc(-I*oo)
-I/2
```

In general one can pull out factors of -1 and i from the argument:

```
>>> fresnelc(-z)
-fresnelc(z)
>>> fresnelc(I*z)
I*fresnelc(z)
```

The Fresnel C integral obeys the mirror symmetry $\overline{C(z)} = C(\bar{z})$:

```
>>> conjugate(fresnelc(z))
fresnelc(conjugate(z))
```

Differentiation with respect to z is supported:

```
>>> diff(fresnelc(z), z)
cos(pi*z**2/2)
```

Defining the Fresnel functions via an integral

```
>>> integrate(cos(pi*z**2/2), z)
fresnelc(z)*gamma(1/4)/(4*gamma(5/4))
>>> expand_func(integrate(cos(pi*z**2/2), z))
fresnelc(z)
```

We can numerically evaluate the Fresnel integral to arbitrary precision on the whole complex plane:

```
>>> fresnelc(2).evalf(30)
0.488253406075340754500223503357
```

```
>>> fresnelc(-2*I).evalf(30)
-0.488253406075340754500223503357*I
```

Exponential, Logarithmic and Trigonometric Integrals

class diofant.functions.special.error_functions.**Ei**

The classical exponential integral.

For use in Diofant, this function is defined as

$$\operatorname{Ei}(x) = \sum_{n=1}^\infty \frac{x^n}{n\, n!} + \log(x) + \gamma,$$

where γ is the Euler-Mascheroni constant.

If x is a polar number, this defines an analytic function on the Riemann surface of the logarithm. Otherwise this defines an analytic function in the cut plane $\mathbb{C} \setminus (-\infty, 0]$.

Background

The name *exponential integral* comes from the following statement:

$$\operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} \mathrm{d}t$$

If the integral is interpreted as a Cauchy principal value, this statement holds for $x > 0$ and $\operatorname{Ei}(x)$ as defined above.

Note that we carefully avoided defining $\mathrm{Ei}(x)$ for negative real x. This is because above integral formula does not hold for any polar lift of such x, indeed all branches of $\mathrm{Ei}(x)$ above the negative reals are imaginary.

However, the following statement holds for all $x \in \mathbb{R}^*$:

$$\int_{-\infty}^{x} \frac{e^t}{t}\mathrm{d}t = \frac{\mathrm{Ei}\left(|x|e^{i \arg(x)}\right) + \mathrm{Ei}\left(|x|e^{- i \arg(x)}\right)}{2},$$

where the integral is again understood to be a principal value if $x > 0$, and $|x|e^{i \arg(x)}$, $|x|e^{- i \arg(x)}$ denote two conjugate polar lifts of x.

See also:

expint **(page 371)** Generalized exponential integral.

E1 **(page 372)** Special case of the generalized exponential integral.

li **(page 373)** Logarithmic integral.

Li **(page 374)** Offset logarithmic integral.

Si **(page 375)** Sine integral.

Ci **(page 376)** Cosine integral.

Shi **(page 377)** Hyperbolic sine integral.

Chi **(page 378)** Hyperbolic cosine integral.

diofant.functions.special.gamma_functions.uppergamma **(page 356)** Upper incomplete gamma function.

References

[R273] (page 1256), *[R274]* (page 1256), *[R275]* (page 1256)

Examples

The exponential integral in Diofant is strictly undefined for negative values of the argument. For convenience, exponential integrals with negative arguments are immediately converted into an expression that agrees with the classical integral definition:

```
>>> Ei(-1)
-I*pi + Ei(exp_polar(I*pi))
```

This yields a real value:

```
>>> Ei(-1).n(chop=True)
-0.219383934395520
```

On the other hand the analytic continuation is not real:

```
>>> Ei(polar_lift(-1)).n(chop=True)
-0.21938393439552 + 3.14159265358979*I
```

The exponential integral has a logarithmic branch point at the origin:

```
>>> Ei(x*exp_polar(2*I*pi))
Ei(x) + 2*I*pi
```

Differentiation is supported:

```
>>> Ei(x).diff(x)
E**x/x
```

The exponential integral is related to many other special functions. For example:

```
>>> Ei(x).rewrite(expint)
-expint(1, x*exp_polar(I*pi)) - I*pi
>>> Ei(x).rewrite(Shi)
Chi(x) + Shi(x)
```

class diofant.functions.special.error_functions.**expint**

Generalized exponential integral.

This function is defined as

$$\operatorname{E}_\nu(z) = z^{\nu - 1} \Gamma(1 - \nu, z),$$

where $\Gamma(1 - \nu, z)$ is the upper incomplete gamma function (uppergamma).

Hence for z with positive real part we have

$$\operatorname{E}_\nu(z) = \int_1^\infty \frac{e^{-zt}}{t^\nu} \mathrm{d}t,$$

which explains the name.

The representation as an incomplete gamma function provides an analytic continuation for $\operatorname{E}_\nu(z)$. If ν is a non-positive integer the exponential integral is thus an unbranched function of z, otherwise there is a branch point at the origin. Refer to the incomplete gamma function documentation for details of the branching behavior.

See also:

Ei **(page 369)** Another related function called exponential integral.

E1 **(page 372)** The classical case, returns expint(1, z).

li **(page 373)** Logarithmic integral.

Li **(page 374)** Offset logarithmic integral.

Si **(page 375)** Sine integral.

Ci **(page 376)** Cosine integral.

Shi **(page 377)** Hyperbolic sine integral.

Chi **(page 378)** Hyperbolic cosine integral.

diofant.functions.special.gamma_functions.uppergamma (page 356)

References

[R276] (page 1256), [R277] (page 1256), [R278] (page 1256)

Examples

```
>>> from diofant.abc import nu
```

Differentiation is supported. Differentiation with respect to z explains further the name: for integral orders, the exponential integral is an iterated integral of the exponential function.

```
>>> expint(nu, z).diff(z)
-expint(nu - 1, z)
```

Differentiation with respect to nu has no classical expression:

```
>>> expint(nu, z).diff(nu)
-z**(nu - 1)*meijerg(((), (1, 1)), ((0, 0, -nu + 1), ()), z)
```

At non-postive integer orders, the exponential integral reduces to the exponential function:

```
>>> expint(0, z)
E**(-z)/z
>>> expint(-1, z)
E**(-z)/z + E**(-z)/z**2
```

At half-integers it reduces to error functions:

```
>>> expint(Rational(1, 2), z)
-sqrt(pi)*erf(sqrt(z))/sqrt(z) + sqrt(pi)/sqrt(z)
```

At positive integer orders it can be rewritten in terms of exponentials and expint(1, z). Use expand_func() to do this:

```
>>> expand_func(expint(5, z))
z**4*expint(1, z)/24 + E**(-z)*(-z**3 + z**2 - 2*z + 6)/24
```

The generalized exponential integral is essentially equivalent to the incomplete gamma function:

```
>>> expint(nu, z).rewrite(uppergamma)
z**(nu - 1)*uppergamma(-nu + 1, z)
```

As such it is branched at the origin:

```
>>> expint(4, z*exp_polar(2*pi*I))
I*pi*z**3/3 + expint(4, z)
>>> expint(nu, z*exp_polar(2*pi*I))
z**(nu - 1)*(E**(2*I*pi*nu) - 1)*gamma(-nu + 1) + expint(nu, z)
```

`diofant.functions.special.error_functions.`**`E1`**(*z*)

Classical case of the generalized exponential integral.

This is equivalent to `expint(1, z)`.

See also:

Ei **(page 369)** Exponential integral.

expint **(page 371)** Generalized exponential integral.

li **(page 373)** Logarithmic integral.

Li **(page 374)** Offset logarithmic integral.

Si **(page 375)** Sine integral.

Ci **(page 376)** Cosine integral.

Shi **(page 377)** Hyperbolic sine integral.

Chi **(page 378)** Hyperbolic cosine integral.

class `diofant.functions.special.error_functions.li`

The classical logarithmic integral.

For the use in Diofant, this function is defined as

$$\mathrm{li}(x) = \int_0^x \frac{1}{\log(t)} \mathrm{d}t \,.$$

See also:

Li **(page 374)** Offset logarithmic integral.

Ei **(page 369)** Exponential integral.

expint **(page 371)** Generalized exponential integral.

E1 **(page 372)** Special case of the generalized exponential integral.

Si **(page 375)** Sine integral.

Ci **(page 376)** Cosine integral.

Shi **(page 377)** Hyperbolic sine integral.

Chi **(page 378)** Hyperbolic cosine integral.

References

[R279] (page 1256), *[R280]* (page 1256), *[R281]* (page 1256), *[R282]* (page 1256)

Examples

Several special values are known:

```
>>> li(0)
0
>>> li(1)
-oo
>>> li(oo)
oo
```

Differentiation with respect to z is supported:

```
>>> diff(li(z), z)
1/log(z)
```

Defining the li function via an integral:

The logarithmic integral can also be defined in terms of Ei:

```
>>> li(z).rewrite(Ei)
Ei(log(z))
>>> diff(li(z).rewrite(Ei), z)
1/log(z)
```

We can numerically evaluate the logarithmic integral to arbitrary precision on the whole complex plane (except the singular points):

```
>>> li(2).evalf(30)
1.04516378011749278484458888919
```

```
>>> li(2*I).evalf(30)
1.0652795784357498247001125598 + 3.08346052231061726610939702133*I
```

We can even compute Soldner's constant by the help of mpmath:

```
>>> from mpmath import findroot
>>> print(findroot(li, 2))
1.45136923488338
```

Further transformations include rewriting li in terms of the trigonometric integrals Si, Ci, Shi and Chi:

```
>>> li(z).rewrite(Si)
-log(I*log(z)) - log(1/log(z))/2 + log(log(z))/2 + Ci(I*log(z)) + Shi(log(z))
>>> li(z).rewrite(Ci)
-log(I*log(z)) - log(1/log(z))/2 + log(log(z))/2 + Ci(I*log(z)) + Shi(log(z))
>>> li(z).rewrite(Shi)
-log(1/log(z))/2 + log(log(z))/2 + Chi(log(z)) - Shi(log(z))
>>> li(z).rewrite(Chi)
-log(1/log(z))/2 + log(log(z))/2 + Chi(log(z)) - Shi(log(z))
```

class `diofant.functions.special.error_functions.`**`Li`**

The offset logarithmic integral.

For the use in Diofant, this function is defined as

$$\operatorname{Li}(x) = \operatorname{li}(x) - \operatorname{li}(2)$$

See also:

`li` **(page 373)** Logarithmic integral.

`Ei` **(page 369)** Exponential integral.

`expint` **(page 371)** Generalized exponential integral.

`E1` **(page 372)** Special case of the generalized exponential integral.

`Si` **(page 375)** Sine integral.

`Ci` **(page 376)** Cosine integral.

`Shi` **(page 377)** Hyperbolic sine integral.

`Chi` **(page 378)** Hyperbolic cosine integral.

References

[R283] (page 1256), [R284] (page 1257), [R285] (page 1257)

Examples

The following special value is known:

```
>>> Li(2)
0
```

Differentiation with respect to z is supported:

```
>>> diff(Li(z), z)
1/log(z)
```

The shifted logarithmic integral can be written in terms of $li(z)$:

```
>>> Li(z).rewrite(li)
li(z) - li(2)
```

We can numerically evaluate the logarithmic integral to arbitrary precision on the whole complex plane (except the singular points):

```
>>> Li(2).evalf(30)
0
```

```
>>> Li(4).evalf(30)
1.92242131492155809316615998938
```

class diofant.functions.special.error_functions.**Si**

Sine integral.

This function is defined by

$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t} \mathrm{d}t.$$

It is an entire function.

See also:

***Ci* (page 376)** Cosine integral.

***Shi* (page 377)** Hyperbolic sine integral.

***Chi* (page 378)** Hyperbolic cosine integral.

***Ei* (page 369)** Exponential integral.

***expint* (page 371)** Generalized exponential integral.

***E1* (page 372)** Special case of the generalized exponential integral.

***li* (page 373)** Logarithmic integral.

***Li* (page 374)** Offset logarithmic integral.

References

[R286] (page 1257)

Examples

The sine integral is an antiderivative of sin(z)/z:

```
>>> Si(z).diff(z)
sin(z)/z
```

It is unbranched:

```
>>> Si(z*exp_polar(2*I*pi))
Si(z)
```

Sine integral behaves much like ordinary sine under multiplication by `I`:

```
>>> Si(I*z)
I*Shi(z)
>>> Si(-z)
-Si(z)
```

It can also be expressed in terms of exponential integrals, but beware that the latter is branched:

```
>>> Si(z).rewrite(expint)
-I*(-expint(1, z*exp_polar(-I*pi/2))/2 +
     expint(1, z*exp_polar(I*pi/2))/2) + pi/2
```

class `diofant.functions.special.error_functions.`**`Ci`**

Cosine integral.

This function is defined for positive x by

$$\operatorname{Ci}(x) = \gamma + \log x + \int_0^x \frac{\cos t - 1}{t} \mathrm{d}t = -\int_x^\infty \frac{\cos t}{t} \mathrm{d}t,$$

where γ is the Euler-Mascheroni constant.

We have

$$\operatorname{Ci}(z) = -\frac{\operatorname{E}_1\left(e^{i\pi/2} z\right) + \operatorname{E}_1\left(e^{-i\pi/2} z\right)}{2}$$

which holds for all polar z and thus provides an analytic continuation to the Riemann surface of the logarithm.

The formula also holds as stated for $z \in \mathbb{C}$ with $\Re(z) > 0$. By lifting to the principal branch we obtain an analytic function on the cut complex plane.

See also:

Si **(page 375)** Sine integral.

Shi **(page 377)** Hyperbolic sine integral.

Chi **(page 378)** Hyperbolic cosine integral.

Ei **(page 369)** Exponential integral.

expint **(page 371)** Generalized exponential integral.

E1 **(page 372)** Special case of the generalized exponential integral.

li **(page 373)** Logarithmic integral.

Li **(page 374)** Offset logarithmic integral.

References

[R287] (page 1257)

Examples

The cosine integral is a primitive of $\cos(z)/z$:

```
>>> Ci(z).diff(z)
cos(z)/z
```

It has a logarithmic branch point at the origin:

```
>>> Ci(z*exp_polar(2*I*pi))
Ci(z) + 2*I*pi
```

The cosine integral behaves somewhat like ordinary cos under multiplication by i:

```
>>> Ci(polar_lift(I)*z)
Chi(z) + I*pi/2
>>> Ci(polar_lift(-1)*z)
Ci(z) + I*pi
```

It can also be expressed in terms of exponential integrals:

```
>>> Ci(z).rewrite(expint)
-expint(1, z*exp_polar(-I*pi/2))/2 - expint(1, z*exp_polar(I*pi/2))/2
```

class diofant.functions.special.error_functions.**Shi**

Sinh integral.

This function is defined by

$$\operatorname{Shi}(z) = \int_0^z \frac{\sinh t}{t} \mathrm{d}t.$$

It is an entire function.

See also:

Si **(page 375)** Sine integral.

Ci **(page 376)** Cosine integral.

Chi **(page 378)** Hyperbolic cosine integral.

Ei **(page 369)** Exponential integral.

expint **(page 371)** Generalized exponential integral.

E1 **(page 372)** Special case of the generalized exponential integral.

li **(page 373)** Logarithmic integral.

Li **(page 374)** Offset logarithmic integral.

References

[R288] (page 1257)

Examples

The Sinh integral is a primitive of $\sinh(z)/z$:

```
>>> Shi(z).diff(z)
sinh(z)/z
```

It is unbranched:

```
>>> Shi(z*exp_polar(2*I*pi))
Shi(z)
```

The sinh integral behaves much like ordinary sinh under multiplication by i:

```
>>> Shi(I*z)
I*Si(z)
>>> Shi(-z)
-Shi(z)
```

It can also be expressed in terms of exponential integrals, but beware that the latter is branched:

```
>>> Shi(z).rewrite(expint)
expint(1, z)/2 - expint(1, z*exp_polar(I*pi))/2 - I*pi/2
```

class diofant.functions.special.error_functions.**Chi**

Cosh integral.

This function is defined for positive x by

$$\operatorname{Chi}(x) = \gamma + \log x + \int_0^x \frac{\cosh t - 1}{t} \mathrm{d}t,$$

where γ is the Euler-Mascheroni constant.

We have

$$\operatorname{Chi}(z) = \operatorname{Ci}\left(e^{i\pi/2} z\right) - i\frac{\pi}{2},$$

which holds for all polar z and thus provides an analytic continuation to the Riemann surface of the logarithm. By lifting to the principal branch we obtain an analytic function on the cut complex plane.

See also:

Si **(page 375)** Sine integral.

Ci **(page 376)** Cosine integral.

Shi **(page 377)** Hyperbolic sine integral.

Ei **(page 369)** Exponential integral.

expint **(page 371)** Generalized exponential integral.

E1 **(page 372)** Special case of the generalized exponential integral.

li **(page 373)** Logarithmic integral.

Li **(page 374)** Offset logarithmic integral.

References

[R289] (page 1257)

Examples

The cosh integral is a primitive of $\cosh(z)/z$:

```
>>> Chi(z).diff(z)
cosh(z)/z
```

It has a logarithmic branch point at the origin:

```
>>> Chi(z*exp_polar(2*I*pi))
Chi(z) + 2*I*pi
```

The cosh integral behaves somewhat like ordinary cosh under multiplication by i:

```
>>> Chi(polar_lift(I)*z)
Ci(z) + I*pi/2
>>> Chi(polar_lift(-1)*z)
Chi(z) + I*pi
```

It can also be expressed in terms of exponential integrals:

```
>>> Chi(z).rewrite(expint)
-expint(1, z)/2 - expint(1, z*exp_polar(I*pi))/2 - I*pi/2
```

Bessel Type Functions

class diofant.functions.special.bessel.**BesselBase**

Abstract base class for bessel-type functions.

This class is meant to reduce code duplication. All Bessel type functions can 1) be differentiated, and the derivatives expressed in terms of similar functions and 2) be rewritten in terms of other bessel-type functions.

Here "bessel-type functions" are assumed to have one complex parameter.

To use this base class, define class attributes _a and _b such that `2*F_n' = -_a*F_{n+1} + b*F_{n-1}`.

argument

The argument of the bessel-type function.

classmethod eval(*nu*, *z*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=2*)
Returns the first derivative of the function.

order
The order of the bessel-type function.

class diofant.functions.special.bessel.**besselj**
Bessel function of the first kind.

The Bessel J function of order ν is defined to be the function satisfying Bessel's differential equation

$$z^2 \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + z \frac{\mathrm{d}w}{\mathrm{d}z} + (z^2 - \nu^2) w = 0,$$

with Laurent expansion

$$J_\nu(z) = z^\nu \left(\frac{1}{\Gamma(\nu + 1) 2^\nu} + O(z^2) \right),$$

if ν is not a negative integer. If $\nu = -n \in \mathbb{Z}_{<0}$ *is* a negative integer, then the definition is

$$J_{-n}(z) = (-1)^n J_n(z).$$

See also:

bessely (page 381), *besseli* (page 381), *besselk* (page 382)

References

[R290] (page 1257), *[R291]* (page 1257), *[R292]* (page 1257), *[R293]* (page 1257)

Examples

Create a Bessel function object:

```
>>> b = besselj(n, z)
```

Differentiate it:

```
>>> b.diff(z)
besselj(n - 1, z)/2 - besselj(n + 1, z)/2
```

Rewrite in terms of spherical Bessel functions:

```
>>> b.rewrite(jn)
sqrt(2)*sqrt(z)*jn(n - 1/2, z)/sqrt(pi)
```

Access the parameter and argument:

```
>>> b.order
n
>>> b.argument
z
```

class diofant.functions.special.bessel.**bessely**

Bessel function of the second kind.

The Bessel Y function of order ν is defined as

$$Y_\nu(z) = \lim_{\mu \to \nu} \frac{J_\mu(z) \cos(\pi \mu) - J_{-\mu}(z)}{\sin(\pi \mu)},$$

where $J_\mu(z)$ is the Bessel function of the first kind.

It is a solution to Bessel's equation, and linearly independent from J_ν.

See also:

besselj (page 380), *besseli* (page 381), *besselk* (page 382)

References

[R294] (page 1257)

Examples

```
>>> b = bessely(n, z)
>>> b.diff(z)
bessely(n - 1, z)/2 - bessely(n + 1, z)/2
>>> b.rewrite(yn)
sqrt(2)*sqrt(z)*yn(n - 1/2, z)/sqrt(pi)
```

class diofant.functions.special.bessel.**besseli**

Modified Bessel function of the first kind.

The Bessel I function is a solution to the modified Bessel equation

$$z^2 \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + z \frac{\mathrm{d}w}{\mathrm{d}z} + (z^2 + \nu^2)^2 w = 0.$$

It can be defined as

$$I_\nu(z) = i^{-\nu} J_\nu(iz),$$

where $J_\nu(z)$ is the Bessel function of the first kind.

See also:

besselj (page 380), *bessely* (page 381), *besselk* (page 382)

References

[R295] (page 1257)

Examples

```
>>> besseli(n, z).diff(z)
besseli(n - 1, z)/2 + besseli(n + 1, z)/2
```

class diofant.functions.special.bessel.**besselk**

Modified Bessel function of the second kind.

The Bessel K function of order ν is defined as

$$K_\nu(z) = \lim_{\mu \to \nu} \frac{\pi}{2} \frac{I_{-\mu}(z) - I_\mu(z)}{\sin(\pi\mu)},$$

where $I_\mu(z)$ is the modified Bessel function of the first kind.

It is a solution of the modified Bessel equation, and linearly independent from Y_ν.

See also:

besselj (page 380), *besseli* (page 381), *bessely* (page 381)

References

[R296] (page 1257)

Examples

```
>>> besselk(n, z).diff(z)
-besselk(n - 1, z)/2 - besselk(n + 1, z)/2
```

class diofant.functions.special.bessel.**hankel1**

Hankel function of the first kind.

This function is defined as

$$H_\nu^{(1)} = J_\nu(z) + iY_\nu(z),$$

where $J_\nu(z)$ is the Bessel function of the first kind, and $Y_\nu(z)$ is the Bessel function of the second kind.

It is a solution to Bessel's equation.

See also:

hankel2 (page 382), *besselj* (page 380), *bessely* (page 381)

References

[R297] (page 1257)

Examples

```
>>> hankel1(n, z).diff(z)
hankel1(n - 1, z)/2 - hankel1(n + 1, z)/2
```

class diofant.functions.special.bessel.**hankel2**

Hankel function of the second kind.

This function is defined as

$$H_\nu^{(2)} = J_\nu(z) - iY_\nu(z),$$

where $J_\nu(z)$ is the Bessel function of the first kind, and $Y_\nu(z)$ is the Bessel function of the second kind.

It is a solution to Bessel's equation, and linearly independent from $H_\nu^{(1)}$.

See also:

hankel1 (page 382), *besselj* (page 380), *bessely* (page 381)

References

[R298] (page 1257)

Examples

```
>>> hankel2(n, z).diff(z)
hankel2(n - 1, z)/2 - hankel2(n + 1, z)/2
```

class diofant.functions.special.bessel.**jn**

Spherical Bessel function of the first kind.

This function is a solution to the spherical Bessel equation

$$z^2 \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + 2z \frac{\mathrm{d}w}{\mathrm{d}z} + (z^2 - \nu(\nu + 1)) w = 0.$$

It can be defined as

$$j_\nu(z) = \sqrt{\frac{\pi}{2z}} J_{\nu + \frac{1}{2}}(z),$$

where $J_\nu(z)$ is the Bessel function of the first kind.

See also:

besselj (page 380), *bessely* (page 381), *besselk* (page 382), *yn* (page 384)

Examples

```
>>> print(jn(0, z).expand(func=True))
sin(z)/z
>>> jn(1, z).expand(func=True) == sin(z)/z**2 - cos(z)/z
True
>>> expand_func(jn(3, z))
(-6/z**2 + 15/z**4)*sin(z) + (1/z - 15/z**3)*cos(z)
```

The spherical Bessel functions of integral order are calculated using the formula:

$$j_n(z) = f_n(z) \sin z + (-1)^{n+1} f_{-n-1}(z) \cos z,$$

where the coefficients $f_n(z)$ are available as *diofant.polys.orthopolys.spherical_bessel_fn()* (page 715).

class diofant.functions.special.bessel.**yn**

Spherical Bessel function of the second kind.

This function is another solution to the spherical Bessel equation, and linearly independent from j_n. It can be defined as

$$j_\nu(z) = \sqrt{\frac{\pi}{2z}} Y_{\nu+\frac{1}{2}}(z),$$

where $Y_\nu(z)$ is the Bessel function of the second kind.

See also:

besselj (page 380), *bessely* (page 381), *besselk* (page 382), *jn* (page 383)

Examples

```
>>> expand_func(yn(0, z))
-cos(z)/z
>>> expand_func(yn(1, z)) == -cos(z)/z**2-sin(z)/z
True
```

For integral orders n, y_n is calculated using the formula:

$$y_n(z) = (-1)^{n+1} j_{-n-1}(z)$$

diofant.functions.special.bessel.**jn_zeros**(*n*, *k*, *method='diofant'*, *dps=15*)

Zeros of the spherical Bessel function of the first kind.

This returns an array of zeros of jn up to the k-th zero.

- method = "diofant": uses mpmath's function `besseljzero`
- method = "scipy": uses `scipy.special.jn_zeros()`. and `scipy.optimize.newton()` to find all roots, which is faster than computing the zeros using a general numerical solver, but it requires SciPy and only works with low precision floating point numbers. [The function used with method="diofant" is a recent addition to mpmath, before that a general solver was used.]

See also:

jn (page 383), *yn* (page 384), *besselj* (page 380), *besselk* (page 382), *bessely* (page 381)

Examples

```
>>> jn_zeros(2, 4, dps=5)
[5.7635, 9.095, 12.323, 15.515]
```

Airy Functions

class diofant.functions.special.bessel.**AiryBase**
Abstract base class for Airy functions.

This class is meant to reduce code duplication.

as_real_imag(*deep=True*, ***hints*)
Performs complex expansion on 'self' and returns a tuple containing collected both real and imaginary parts. This method can't be confused with re() and im() functions, which does not perform complex expansion at evaluation.

However it is possible to expand both re() and im() functions and get exactly the same results as with a single call to this function.

```
>>> x, y = symbols('x y', extended_real=True)
```

```
>>> (x + y*I).as_real_imag()
(x, y)
```

```
>>> from diofant.abc import w
```

```
>>> (z + w*I).as_real_imag()
(re(z) - im(w), re(w) + im(z))
```

class diofant.functions.special.bessel.**airyai**
The Airy function Ai of the first kind.

The Airy function $\operatorname{Ai}(z)$ is defined to be the function satisfying Airy's differential equation

$$\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2} - zw(z) = 0.$$

Equivalently, for real z

$$\operatorname{Ai}(z) := \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + zt\right) \mathrm{d}t.$$

See also:

airybi **(page 386)** Airy function of the second kind.

airyaiprime **(page 387)** Derivative of the Airy function of the first kind.

airybiprime **(page 389)** Derivative of the Airy function of the second kind.

References

[R299] (page 1257), *[R300]* (page 1257), *[R301]* (page 1257), *[R302]* (page 1257)

Examples

Create an Airy function object:

```
>>> airyai(z)
airyai(z)
```

Several special values are known:

```
>>> airyai(0)
3**(1/3)/(3*gamma(2/3))
>>> airyai(oo)
0
>>> airyai(-oo)
0
```

The Airy function obeys the mirror symmetry:

```
>>> conjugate(airyai(z))
airyai(conjugate(z))
```

Differentiation with respect to z is supported:

```
>>> diff(airyai(z), z)
airyaiprime(z)
>>> diff(airyai(z), z, 2)
z*airyai(z)
```

Series expansion is also supported:

```
>>> series(airyai(z), z, 0, 3)
3**(5/6)*gamma(1/3)/(6*pi) - 3**(1/6)*z*gamma(2/3)/(2*pi) + O(z**3)
```

We can numerically evaluate the Airy function to arbitrary precision on the whole complex plane:

```
>>> airyai(-2).evalf(50)
0.22740742820168557599192443603787379946077222541710
```

Rewrite Ai(z) in terms of hypergeometric functions:

```
>>> airyai(z).rewrite(hyper)
-3**(2/3)*z*hyper((), (4/3,), z**3/9)/(3*gamma(1/3)) + 3**(1/3)*hyper((), (2/3,),␣
↪z**3/9)/(3*gamma(2/3))
```

class `diofant.functions.special.bessel.`**`airybi`**

The Airy function Bi of the second kind.

The Airy function $\operatorname{Bi}(z)$ is defined to be the function satisfying Airy's differential equation

$$\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2} - zw(z) = 0.$$

Equivalently, for real z

$$\operatorname{Bi}(z) := \frac{1}{\pi} \int_0^\infty \exp\left(-\frac{t^3}{3} + zt\right) + \sin\left(\frac{t^3}{3} + zt\right) \mathrm{d}t.$$

See also:

airyai **(page 385)** Airy function of the first kind.

airyaiprime **(page 387)** Derivative of the Airy function of the first kind.

airybiprime **(page 389)** Derivative of the Airy function of the second kind.

References

[R303] (page 1257), [R304] (page 1257), [R305] (page 1257), [R306] (page 1257)

Examples

Create an Airy function object:

```
>>> airybi(z)
airybi(z)
```

Several special values are known:

```
>>> airybi(0)
3**(5/6)/(3*gamma(2/3))
>>> airybi(oo)
oo
>>> airybi(-oo)
0
```

The Airy function obeys the mirror symmetry:

```
>>> conjugate(airybi(z))
airybi(conjugate(z))
```

Differentiation with respect to z is supported:

```
>>> diff(airybi(z), z)
airybiprime(z)
>>> diff(airybi(z), z, 2)
z*airybi(z)
```

Series expansion is also supported:

```
>>> series(airybi(z), z, 0, 3)
3**(1/3)*gamma(1/3)/(2*pi) + 3**(2/3)*z*gamma(2/3)/(2*pi) + O(z**3)
```

We can numerically evaluate the Airy function to arbitrary precision on the whole complex plane:

```
>>> airybi(-2).evalf(50)
-0.41230258795639848808323405461146104203453483447240
```

Rewrite Bi(z) in terms of hypergeometric functions:

```
>>> airybi(z).rewrite(hyper)
3**(1/6)*z*hyper((), (4/3,), z**3/9)/gamma(1/3) + 3**(5/6)*hyper((), (2/3,), z**3/
↪9)/(3*gamma(2/3))
```

class diofant.functions.special.bessel.**airyaiprime**

The derivative $\operatorname{Ai}^\prime$ of the Airy function of the first kind.

The Airy function $\operatorname{Ai}^{\prime}(z)$ is defined to be the function

$$\operatorname{Ai}^{\prime}(z) := \frac{\mathrm{d} \operatorname{Ai}(z)}{\mathrm{d} z}.$$

See also:

airyai **(page 385)** Airy function of the first kind.

airybi **(page 386)** Airy function of the second kind.

airybiprime **(page 389)** Derivative of the Airy function of the second kind.

References

[R307] (page 1257), *[R308]* (page 1257), *[R309]* (page 1257), *[R310]* (page 1257)

Examples

Create an Airy function object:

```
>>> airyaiprime(z)
airyaiprime(z)
```

Several special values are known:

```
>>> airyaiprime(0)
-3**(2/3)/(3*gamma(1/3))
>>> airyaiprime(oo)
0
```

The Airy function obeys the mirror symmetry:

```
>>> conjugate(airyaiprime(z))
airyaiprime(conjugate(z))
```

Differentiation with respect to z is supported:

```
>>> diff(airyaiprime(z), z)
z*airyai(z)
>>> diff(airyaiprime(z), z, 2)
z*airyaiprime(z) + airyai(z)
```

Series expansion is also supported:

```
>>> series(airyaiprime(z), z, 0, 3)
-3**(2/3)/(3*gamma(1/3)) + 3**(1/3)*z**2/(6*gamma(2/3)) + O(z**3)
```

We can numerically evaluate the Airy function to arbitrary precision on the whole complex plane:

```
>>> airyaiprime(-2).evalf(50)
0.61825902074169104140626429133247528291577794512415
```

Rewrite Ai’(z) in terms of hypergeometric functions:

```
>>> airyaiprime(z).rewrite(hyper)
3**(1/3)*z**2*hyper((), (5/3,), z**3/9)/(6*gamma(2/3)) - 3**(2/3)*hyper((), (1/3,
↪), z**3/9)/(3*gamma(1/3))
```

class diofant.functions.special.bessel.**airybiprime**

The derivative $\operatorname{Bi}^\prime$ of the Airy function of the first kind.

The Airy function $\operatorname{Bi}^\prime(z)$ is defined to be the function

$$\operatorname{Bi}^\prime(z) := \frac{\mathrm{d} \operatorname{Bi}(z)}{\mathrm{d} z}.$$

See also:

***airyai* (page 385)** Airy function of the first kind.

***airybi* (page 386)** Airy function of the second kind.

***airyaiprime* (page 387)** Derivative of the Airy function of the first kind.

References

[R311] (page 1257), *[R312]* (page 1257), *[R313]* (page 1257), *[R314]* (page 1257)

Examples

Create an Airy function object:

```
>>> airybiprime(z)
airybiprime(z)
```

Several special values are known:

```
>>> airybiprime(0)
3**(1/6)/gamma(1/3)
>>> airybiprime(oo)
oo
>>> airybiprime(-oo)
0
```

The Airy function obeys the mirror symmetry:

```
>>> conjugate(airybiprime(z))
airybiprime(conjugate(z))
```

Differentiation with respect to z is supported:

```
>>> diff(airybiprime(z), z)
z*airybi(z)
>>> diff(airybiprime(z), z, 2)
z*airybiprime(z) + airybi(z)
```

Series expansion is also supported:

```
>>> series(airybiprime(z), z, 0, 3)
3**(1/6)/gamma(1/3) + 3**(5/6)*z**2/(6*gamma(2/3)) + O(z**3)
```

We can numerically evaluate the Airy function to arbitrary precision on the whole complex plane:

```
>>> airybiprime(-2).evalf(50)
0.27879516692116952268509756941098324140300059345163
```

Rewrite Bi'(z) in terms of hypergeometric functions:

```
>>> airybiprime(z).rewrite(hyper)
3**(5/6)*z**2*hyper((), (5/3,), z**3/9)/(6*gamma(2/3)) + 3**(1/6)*hyper((), (1/3,
↪), z**3/9)/gamma(1/3)
```

B-Splines

diofant.functions.special.bsplines.**bspline_basis**(*d, knots, n, x, close=True*)

The n-th B-spline at x of degree d with knots.

B-Splines are piecewise polynomials of degree d *[R315]* (page 1257). They are defined on a set of knots, which is a sequence of integers or floats.

The 0th degree splines have a value of one on a single interval:

```
>>> d = 0
>>> knots = range(5)
>>> bspline_basis(d, knots, 0, x)
Piecewise((1, And(x <= 1, x >= 0)), (0, true))
```

For a given `(d, knots)` there are `len(knots)-d-1` B-splines defined, that are indexed by `n` (starting at 0).

Here is an example of a cubic B-spline:

```
>>> bspline_basis(3, range(5), 0, x)
Piecewise((x**3/6, And(x < 1, x >= 0)),
          (-x**3/2 + 2*x**2 - 2*x + 2/3, And(x < 2, x >= 1)),
          (x**3/2 - 4*x**2 + 10*x - 22/3, And(x < 3, x >= 2)),
          (-x**3/6 + 2*x**2 - 8*x + 32/3, And(x <= 4, x >= 3)),
          (0, true))
```

By repeating knot points, you can introduce discontinuities in the B-splines and their derivatives:

```
>>> d = 1
>>> knots = [0, 0, 2, 3, 4]
>>> bspline_basis(d, knots, 0, x)
Piecewise((-x/2 + 1, And(x <= 2, x >= 0)), (0, true))
```

It is quite time consuming to construct and evaluate B-splines. If you need to evaluate a B-splines many times, it is best to lambdify them first:

```
>>> d = 3
>>> knots = range(10)
>>> b0 = bspline_basis(d, knots, 0, x)
```

(continues on next page)

(continued from previous page)

```
>>> f = lambdify(x, b0)
>>> y = f(0.5)
```

See also:

diofant.functions.special.bsplines.bspline_basis_set (page 391)

References

[R315] (page 1257)

diofant.functions.special.bsplines.**bspline_basis_set**(*d*, *knots*, *x*)

Return the `len(knots)-d-1` B-splines at `x` of degree `d` with `knots`.

This function returns a list of Piecewise polynomials that are the `len(knots)-d-1` B-splines of degree `d` for the given knots. This function calls `bspline_basis(d, knots, n, x)` for different values of `n`.

See also:

diofant.functions.special.bsplines.bspline_basis (page 390)

Examples

```
>>> d = 2
>>> knots = range(5)
>>> splines = bspline_basis_set(d, knots, x)
>>> splines
[Piecewise((x**2/2, And(x < 1, x >= 0)),
           (-x**2 + 3*x - 3/2, And(x < 2, x >= 1)),
           (x**2/2 - 3*x + 9/2, And(x <= 3, x >= 2)),
           (0, true)),
 Piecewise((x**2/2 - x + 1/2, And(x < 2, x >= 1)),
           (-x**2 + 5*x - 11/2, And(x < 3, x >= 2)),
           (x**2/2 - 4*x + 8, And(x <= 4, x >= 3)),
           (0, true))]
```

Riemann Zeta and Related Functions

class diofant.functions.special.zeta_functions.**zeta**

Hurwitz zeta function (or Riemann zeta function).

For $\operatorname{Re}(a) > 0$ and $\operatorname{Re}(s) > 1$, this function is defined as

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},$$

where the standard choice of argument for $n + a$ is used. For fixed a with $\operatorname{Re}(a) > 0$ the Hurwitz zeta function admits a meromorphic continuation to all of $\mathbb{C}$, it is an unbranched function with a simple pole at $s = 1$.

Analytic continuation to other a is possible under some circumstances, but this is not typically done.

The Hurwitz zeta function is a special case of the Lerch transcendent:

$$\zeta(s, a) = \Phi(1, s, a).$$

This formula defines an analytic continuation for all possible values of s and a (also $\operatorname{Re}(a) < 0$), see the documentation of *lerchphi* (page 395) for a description of the branching behavior.

If no value is passed for a, by this function assumes a default value of $a = 1$, yielding the Riemann zeta function.

See also:

dirichlet_eta (page 393), *lerchphi* (page 395), *polylog* (page 393)

References

[R316] (page 1257), [R317] (page 1257)

Examples

For $a = 1$ the Hurwitz zeta function reduces to the famous Riemann zeta function:

$$\zeta(s, 1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

```
>>> from diofant.abc import s
>>> zeta(s, 1)
zeta(s)
>>> zeta(s)
zeta(s)
```

The Riemann zeta function can also be expressed using the Dirichlet eta function:

```
>>> zeta(s).rewrite(dirichlet_eta)
dirichlet_eta(s)/(-2**(-s + 1) + 1)
```

The Riemann zeta function at positive even integer and negative odd integer values is related to the Bernoulli numbers:

```
>>> zeta(2)
pi**2/6
>>> zeta(4)
pi**4/90
>>> zeta(-1)
-1/12
```

The specific formulae are:

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n} (2\pi)^{2n}}{2(2n)!}$$

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

At negative even integers the Riemann zeta function is zero:

```
>>> zeta(-4)
0
```

No closed-form expressions are known at positive odd integers, but numerical evaluation is possible:

```
>>> zeta(3).n()
1.20205690315959
```

The derivative of $\zeta(s, a)$ with respect to a is easily computed:

```
>>> from diofant.abc import a
>>> zeta(s, a).diff(a)
-s*zeta(s + 1, a)
```

However the derivative with respect to s has no useful closed form expression:

```
>>> zeta(s, a).diff(s)
Derivative(zeta(s, a), s)
```

The Hurwitz zeta function can be expressed in terms of the Lerch transcendent, *diofant.functions.special.zeta_functions.lerchphi* (page 395):

```
>>> zeta(s, a).rewrite(lerchphi)
lerchphi(1, s, a)
```

class diofant.functions.special.zeta_functions.**dirichlet_eta**

Dirichlet eta function.

For $\operatorname{Re}(s) > 0$, this function is defined as

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

It admits a unique analytic continuation to all of $\mathbb{C}$. It is an entire, unbranched function.

See also:

zeta (page 391)

References

[R318] (page 1257), [R319] (page 1258)

Examples

The Dirichlet eta function is closely related to the Riemann zeta function:

```
>>> from diofant.abc import s
>>> dirichlet_eta(s).rewrite(zeta)
(-2**(-s + 1) + 1)*zeta(s)
```

class diofant.functions.special.zeta_functions.**polylog**

Polylogarithm function.

For $|z| < 1$ and $s \in \mathbb{C}$, the polylogarithm is defined by

$$\operatorname{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s},$$

where the standard branch of the argument is used for n. It admits an analytic continuation which is branched at $z = 1$ (notably not on the sheet of initial definition), $z = 0$ and $z = \infty$.

The name polylogarithm comes from the fact that for $s = 1$, the polylogarithm is related to the ordinary logarithm (see examples), and that

$$\operatorname{Li}_{s+1}(z) = \int_0^z \frac{\operatorname{Li}_s(t)}{t} \mathrm{d}t.$$

The polylogarithm is a special case of the Lerch transcendent:

$$\operatorname{Li}_s(z) = z\Phi(z, s, 1)$$

See also:

zeta (page 391), *lerchphi* (page 395)

References

[R320] (page 1258), [R321] (page 1258)

Examples

For $z \in \{0, 1, -1\}$, the polylogarithm is automatically expressed using other functions:

```
>>> from diofant.abc import s
>>> polylog(s, 0)
0
>>> polylog(s, 1)
zeta(s)
>>> polylog(s, -1)
-dirichlet_eta(s)
```

If s is a negative integer, 0 or 1, the polylogarithm can be expressed using elementary functions. This can be done using expand_func():

```
>>> expand_func(polylog(1, z))
-log(-z + 1)
>>> expand_func(polylog(0, z))
z/(-z + 1)
```

The derivative with respect to z can be computed in closed form:

```
>>> polylog(s, z).diff(z)
polylog(s - 1, z)/z
```

The polylogarithm can be expressed in terms of the lerch transcendent:

```
>>> polylog(s, z).rewrite(lerchphi)
z*lerchphi(z, s, 1)
```

class diofant.functions.special.zeta_functions.**lerchphi**

Lerch transcendent (Lerch phi function).

For Re$(a) > 0$, $|z| < 1$ and $s \in \mathbb{C}$, the Lerch transcendent is defined as

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s},$$

where the standard branch of the argument is used for $n+a$, and by analytic continuation for other values of the parameters.

A commonly used related function is the Lerch zeta function, defined by

$$L(q, s, a) = \Phi(e^{2\pi i q}, s, a).$$

Analytic Continuation and Branching Behavior

It can be shown that

$$\Phi(z, s, a) = z\Phi(z, s, a+1) + a^{-s}.$$

This provides the analytic continuation to Re$(a) \le 0$.

Assume now Re$(a) > 0$. The integral representation

$$\Phi_0(z, s, a) = \int_0^\infty \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} \frac{\mathrm{d}t}{\Gamma(s)}$$

provides an analytic continuation to $\mathbb{C} - [1, \infty)$. Finally, for $x \in (1, \infty)$ we find

$$\lim_{\epsilon \to 0^+} \Phi_0(x + i\epsilon, s, a) - \lim_{\epsilon \to 0^+} \Phi_0(x - i\epsilon, s, a) = \frac{2\pi i \log^{s-1} x}{x^a \Gamma(s)},$$

using the standard branch for both $\log x$ and $\log\log x$ (a branch of $\log\log x$ is needed to evaluate $\log x^{s-1}$). This concludes the analytic continuation. The Lerch transcendent is thus branched at $z \in \{0, 1, \infty\}$ and $a \in \mathbb{Z}_{\le 0}$. For fixed z, a outside these branch points, it is an entire function of s.

See also:

polylog (page 393), *zeta* (page 391)

References

[R322] (page 1258), [R323] (page 1258), [R324] (page 1258)

Examples

The Lerch transcendent is a fairly general function, for this reason it does not automatically evaluate to simpler functions. Use expand_func() to achieve this.

If $z = 1$, the Lerch transcendent reduces to the Hurwitz zeta function:

```
>>> from diofant.abc import s, a
>>> expand_func(lerchphi(1, s, a))
zeta(s, a)
```

More generally, if z is a root of unity, the Lerch transcendent reduces to a sum of Hurwitz zeta functions:

```
>>> expand_func(lerchphi(-1, s, a))
2**(-s)*zeta(s, a/2) - 2**(-s)*zeta(s, a/2 + 1/2)
```

If $a = 1$, the Lerch transcendent reduces to the polylogarithm:

```
>>> expand_func(lerchphi(z, s, 1))
polylog(s, z)/z
```

More generally, if a is rational, the Lerch transcendent reduces to a sum of polylogarithms:

```
>>> expand_func(lerchphi(z, s, Rational(1, 2)))
2**(s - 1)*(polylog(s, sqrt(z))/sqrt(z) -
            polylog(s, sqrt(z)*exp_polar(I*pi))/sqrt(z))
>>> expand_func(lerchphi(z, s, Rational(3, 2)))
-2**s/z + 2**(s - 1)*(polylog(s, sqrt(z))/sqrt(z) -
                      polylog(s, sqrt(z)*exp_polar(I*pi))/sqrt(z))/z
```

The derivatives with respect to z and a can be computed in closed form:

```
>>> lerchphi(z, s, a).diff(z)
(-a*lerchphi(z, s, a) + lerchphi(z, s - 1, a))/z
>>> lerchphi(z, s, a).diff(a)
-s*lerchphi(z, s + 1, a)
```

Hypergeometric Functions

class diofant.functions.special.hyper.**hyper**

The (generalized) hypergeometric function is defined by a series where the ratios of successive terms are a rational function of the summation index. When convergent, it is continued analytically to the largest possible domain.

The hypergeometric function depends on two vectors of parameters, called the numerator parameters a_p, and the denominator parameters b_q. It also has an argument z. The series definition is

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| z\right) = \sum_{n=0}^\infty \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!},$$

where $(a)_n = (a)(a+1)\ldots(a+n-1)$ denotes the rising factorial.

If one of the b_q is a non-positive integer then the series is undefined unless one of the a_p is a larger (i.e. smaller in magnitude) non-positive integer. If none of the b_q is a non-positive integer and one of the a_p is a non-positive integer, then the series reduces to a polynomial. To simplify the following discussion, we assume that none of the a_p or b_q is a non-positive integer. For more details, see the references.

The series converges for all z if $p \le q$, and thus defines an entire single-valued function in this case. If $p = q+1$ the series converges for $|z| < 1$, and can be continued analytically into a half-plane. If $p > q+1$ the series is divergent for all z.

Note: The hypergeometric function constructor currently does *not* check if the parameters actually yield a well-defined function.

See also:

diofant.simplify.hyperexpand (page 799), *diofant.functions.special.gamma_functions.gamma* (page 350), *diofant.functions.special.hyper.meijerg* (page 398)

References

[R325] (page 1258), *[R326]* (page 1258)

Examples

The parameters a_p and b_q can be passed as arbitrary iterables, for example:

```
>>> from diofant.abc import a
>>> hyper((1, 2, 3), [3, 4], x)
hyper((1, 2, 3), (3, 4), x)
```

There is also pretty printing (it looks better using unicode):

```
>>> pprint(hyper((1, 2, 3), [3, 4], x), use_unicode=False)
  _
 |_  /1, 2, 3 |  \
 |   |        | x|
3  2 \  3, 4  |  /
```

The parameters must always be iterables, even if they are vectors of length one or zero:

```
>>> hyper([1], [], x)
hyper((1,), (), x)
```

But of course they may be variables (but if they depend on x then you should not expect much implemented functionality):

```
>>> hyper([n, a], [n**2], x)
hyper((n, a), (n**2,), x)
```

The hypergeometric function generalizes many named special functions. The function hyperexpand() tries to express a hypergeometric function using named special functions. For example:

```
>>> hyperexpand(hyper([], [], x))
E**x
```

You can also use expand_func:

```
>>> expand_func(x*hyper([1, 1], [2], -x))
log(x + 1)
```

More examples:

```
>>> hyperexpand(hyper([], [Rational(1, 2)], -x**2/4))
cos(x)
>>> hyperexpand(x*hyper([Rational(1, 2), Rational(1, 2)], [Rational(3, 2)], x**2))
asin(x)
```

We can also sometimes hyperexpand parametric functions:

```
>>> from diofant.abc import a
>>> hyperexpand(hyper([-a], [], x))
(-x + 1)**a
```

ap
Numerator parameters of the hypergeometric function.

argument
Argument of the hypergeometric function.

bq
Denominator parameters of the hypergeometric function.

convergence_statement
Return a condition on z under which the series converges.

eta
A quantity related to the convergence of the series.

classmethod eval(*ap*, *bq*, *z*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=3*)
Returns the first derivative of the function.

radius_of_convergence
Compute the radius of convergence of the defining series.

Note that even if this is not oo, the function may still be evaluated outside of the radius of convergence by analytic continuation. But if this is zero, then the function is not actually defined anywhere else.

```
>>> hyper((1, 2), [3], z).radius_of_convergence
1
>>> hyper((1, 2, 3), [4], z).radius_of_convergence
0
>>> hyper((1, 2), (3, 4), z).radius_of_convergence
oo
```

class diofant.functions.special.hyper.**meijerg**
The Meijer G-function is defined by a Mellin-Barnes type integral that resembles an inverse Mellin transform. It generalizes the hypergeometric functions.

The Meijer G-function depends on four sets of parameters. There are "*numerator parameters*" $a_1, \ldots, a_n$ and $a_{n+1}, \ldots, a_p$, and there are "*denominator parameters*" $b_1, \ldots, b_m$ and $b_{m+1}, \ldots, b_q$. Confusingly, it is traditionally denoted as follows (note the position of m, n, p, q, and how they relate to the lengths of the four parameter vectors):

$$G_{p,q}^{m,n} \left(\begin{matrix} a_1, \ldots, a_n & a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m & b_{m+1}, \ldots, b_q \end{matrix} \middle| z \right).$$

However, in diofant the four parameter vectors are always available separately (see examples), so that there is no need to keep track of the decorating sub- and super-scripts on the G symbol.

The G function is defined as the following integral:

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - s)\prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s)\prod_{j=n+1}^p \Gamma(a_j - s)} z^s \mathrm{d}s,$$

where $\Gamma(z)$ is the gamma function. There are three possible contours which we will not describe in detail here (see the references). If the integral converges along more than one of them the definitions agree. The contours all separate the poles of $\Gamma(1-a_j+s)$ from the poles of $\Gamma(b_k - s)$, so in particular the G function is undefined if $a_j - b_k \in \mathbb{Z}_{>0}$ for some $j \leq n$ and $k \leq m$.

The conditions under which one of the contours yields a convergent integral are complicated and we do not state them here, see the references.

Note: Currently the Meijer G-function constructor does *not* check any convergence conditions.

See also:

diofant.functions.special.hyper.hyper (page 396), *diofant.simplify.hyperexpand* (page 799)

References

[R327] (page 1258), [R328] (page 1258)

Examples

You can pass the parameters either as four separate vectors:

```
>>> from diofant.abc import a
>>> pprint(meijerg([1, 2], [a, 4], [5], [], x), use_unicode=False)
 __1, 2 /1, 2  a, 4 |  \
/__     |           | x|
\_|4, 1 \ 5         |  /
```

or as two nested vectors:

```
>>> pprint(meijerg(([1, 2], [3, 4]), ([5], []), x), use_unicode=False)
 __1, 2 /1, 2  3, 4 |  \
/__     |           | x|
\_|4, 1 \ 5         |  /
```

As with the hypergeometric function, the parameters may be passed as arbitrary iterables. Vectors of length zero and one also have to be passed as iterables. The parameters need not be constants, but if they depend on the argument then not much implemented functionality should be expected.

All the subvectors of parameters are available:

```
>>> g = meijerg([1], [2], [3], [4], x)
>>> pprint(g, use_unicode=False)
 __1, 1 /1  2 |  \
/__     |     | x|
\_|2, 2 \3  4 |  /
>>> g.an
(1,)
>>> g.ap
(1, 2)
>>> g.aother
(2,)
>>> g.bm
(3,)
>>> g.bq
(3, 4)
>>> g.bother
(4,)
```

The Meijer G-function generalizes the hypergeometric functions. In some cases it can be expressed in terms of hypergeometric functions, using Slater's theorem. For example:

```
>>> from diofant.abc import a, b, c
>>> hyperexpand(meijerg([a], [], [c], [b], x), allow_hyper=True)
x**c*gamma(-a + c + 1)*hyper((-a + c + 1,),
                             (-b + c + 1,), -x)/gamma(-b + c + 1)
```

Thus the Meijer G-function also subsumes many named functions as special cases. You can use expand_func or hyperexpand to (try to) rewrite a Meijer G-function in terms of named special functions. For example:

```
>>> expand_func(meijerg([[], []], [[0], []], -x))
E**x
>>> hyperexpand(meijerg([[], []], [[Rational(1, 2)], [0]], (x/2)**2))
sin(x)/sqrt(pi)
```

an
First set of numerator parameters.

aother
Second set of numerator parameters.

ap
Combined numerator parameters.

argument
Argument of the Meijer G-function.

bm
First set of denominator parameters.

bother
Second set of denominator parameters.

bq
Combined denominator parameters.

delta
A quantity related to the convergence region of the integral, c.f. references.

fdiff(*argindex=3*)
Returns the first derivative of the function.

get_period()
Return a number P such that G(x*exp(I*P)) == G(x).

```
>>> meijerg([1], [], [], [], z).get_period()
2*pi
>>> meijerg([pi], [], [], [], z).get_period()
oo
>>> meijerg([1, 2], [], [], [], z).get_period()
oo
>>> meijerg([1, 1], [2], [1, Rational(1, 2), Rational(1, 3)], [1], z).get_
↪period()
12*pi
```

integrand(*s*)
Get the defining integrand D(s).

nu
A quantity related to the convergence region of the integral, c.f. references.

Elliptic integrals

class diofant.functions.special.elliptic_integrals.**elliptic_k**
The complete elliptic integral of the first kind, defined by

$$K(z) = F\left(\frac{\pi}{2}\middle| z\right)$$

where $F\left(z\middle| m\right)$ is the Legendre incomplete elliptic integral of the first kind.

The function $K(z)$ is a single-valued function on the complex plane with branch cut along the interval $(1, \infty)$.

See also:

elliptic_f (page 401)

References

[R329] (page 1258), [R330] (page 1258)

Examples

```
>>> elliptic_k(0)
pi/2
>>> elliptic_k(1.0 + I)
1.50923695405127 + 0.625146415202697*I
>>> elliptic_k(z).series(z, n=3)
pi/2 + pi*z/8 + 9*pi*z**2/128 + O(z**3)
```

class diofant.functions.special.elliptic_integrals.**elliptic_f**
The Legendre incomplete elliptic integral of the first kind, defined by

$$F\left(z\middle| m\right) = \int_0^z \frac{dt}{\sqrt{1 - m \sin^2 t}}$$

This function reduces to a complete elliptic integral of the first kind, $K(m)$, when $z = \pi/2$.

See also:

elliptic_k (page 401)

References

[R331] (page 1258), [R332] (page 1258)

Examples

```
>>> elliptic_f(z, m).series(z)
z + z**5*(3*m**2/40 - m/30) + m*z**3/6 + O(z**6)
>>> elliptic_f(3.0 + I/2, 1.0 + I)
2.909449841483 + 1.74720545502474*I
```

class diofant.functions.special.elliptic_integrals.**elliptic_e**

Called with two arguments z and m, evaluates the incomplete elliptic integral of the second kind, defined by

$$E\left(z\middle| m\right) = \int_0^z \sqrt{1 - m \sin^2 t} dt$$

Called with a single argument z, evaluates the Legendre complete elliptic integral of the second kind

$$E(z) = E\left(\tfrac{\pi}{2}\middle| z\right)$$

The function $E(z)$ is a single-valued function on the complex plane with branch cut along the interval $(1, \infty)$.

References

[R333] (page 1258), [R334] (page 1258), [R335] (page 1258)

Examples

```
>>> elliptic_e(z, m).series(z)
z + z**5*(-m**2/40 + m/30) - m*z**3/6 + O(z**6)
>>> elliptic_e(z).series(z, n=4)
pi/2 - pi*z/8 - 3*pi*z**2/128 - 5*pi*z**3/512 + O(z**4)
>>> elliptic_e(1 + I, 2 - I/2).n()
1.55203744279187 + 0.290764986058437*I
>>> elliptic_e(0)
pi/2
>>> elliptic_e(2.0 - I)
0.991052601328069 + 0.81879421395609*I
```

class diofant.functions.special.elliptic_integrals.**elliptic_pi**

Called with three arguments n, z and m, evaluates the Legendre incomplete elliptic integral of the third kind, defined by

$$\Pi\left(n; z\middle| m\right) = \int_0^z \frac{dt}{\left(1 - n\sin^2 t\right)\sqrt{1 - m\sin^2 t}}$$

Called with two arguments n and m, evaluates the complete elliptic integral of the third kind:

$$\Pi\left(n\middle| m\right) = \Pi\left(n; \tfrac{\pi}{2}\middle| m\right)$$

References

[R336] (page 1258), *[R337]* (page 1258), *[R338]* (page 1258)

Examples

```
>>> elliptic_pi(n, z, m).series(z, n=4)
z + z**3*(m/6 + n/3) + O(z**4)
>>> elliptic_pi(0.5 + I, 1.0 - I, 1.2)
2.50232379629182 - 0.760939574180767*I
>>> elliptic_pi(0, 0)
pi/2
>>> elliptic_pi(1.0 - I/3, 2.0 + I)
3.29136443417283 + 0.32555634906645*I
```

Orthogonal Polynomials

This module mainly implements special orthogonal polynomials.

See also functions.combinatorial.numbers which contains some combinatorial polynomials.

Jacobi Polynomials

class diofant.functions.special.polynomials.**jacobi**

Jacobi polynomial $P_n^{\left(\alpha,\beta\right)}(x)$

jacobi(n, alpha, beta, x) gives the nth Jacobi polynomial in x, $P_n^{\left(\alpha,\beta\right)}(x)$.

The Jacobi polynomials are orthogonal on $[-1, 1]$ with respect to the weight $\left(1-x\right)^\alpha \left(1+x\right)^\beta$.

See also:

gegenbauer (page 405), *chebyshevt_root* (page 408), *chebyshevu* (page 407), *chebyshevu_root* (page 409), *legendre* (page 410), *assoc_legendre* (page 410), *hermite* (page 411), *laguerre* (page 412), *assoc_laguerre* (page 413), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly* (page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

References

[R339] (page 1258), [R340] (page 1258), [R341] (page 1258)

Examples

```
>>> from diofant.abc import a, b
```

```
>>> jacobi(0, a, b, x)
1
>>> jacobi(1, a, b, x)
a/2 - b/2 + x*(a/2 + b/2 + 1)
```

```
>>> jacobi(n, a, b, x)
jacobi(n, a, b, x)
```

```
>>> jacobi(n, a, a, x)
RisingFactorial(a + 1, n)*gegenbauer(n,
    a + 1/2, x)/RisingFactorial(2*a + 1, n)
```

```
>>> jacobi(n, 0, 0, x)
legendre(n, x)
```

```
>>> jacobi(n, Rational(1, 2), Rational(1, 2), x)
RisingFactorial(3/2, n)*chebyshevu(n, x)/factorial(n + 1)
```

```
>>> jacobi(n, -Rational(1, 2), -Rational(1, 2), x)
RisingFactorial(1/2, n)*chebyshevt(n, x)/factorial(n)
```

```
>>> jacobi(n, a, b, -x)
(-1)**n*jacobi(n, b, a, x)
```

```
>>> jacobi(n, a, b, 0)
2**(-n)*gamma(a + n + 1)*hyper((-b - n, -n), (a + 1,), -1)/(factorial(n)*gamma(a␣
↪+ 1))
>>> jacobi(n, a, b, 1)
RisingFactorial(a + 1, n)/factorial(n)
```

```
>>> conjugate(jacobi(n, a, b, x))
jacobi(n, conjugate(a), conjugate(b), conjugate(x))
```

```
>>> diff(jacobi(n, a, b, x), x)
(a/2 + b/2 + n/2 + 1/2)*jacobi(n - 1, a + 1, b + 1, x)
```

classmethod eval(*n, a, b, x*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=4*)

Returns the first derivative of the function.

diofant.functions.special.polynomials.**jacobi_normalized**(*n*, *a*, *b*, *x*)

Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$

jacobi_normalized(n, alpha, beta, x) gives the nth Jacobi polynomial in x, $P_n^{(\alpha,\beta)}(x)$.

The Jacobi polynomials are orthogonal on $[-1,1]$ with respect to the weight $(1-x)^\alpha (1+x)^\beta$.

This functions returns the polynomials normilzed:

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta \mathrm{d}x = \delta_{m,n}$$

See also:

gegenbauer (page 405), *chebyshevt_root* (page 408), *chebyshevu* (page 407), *chebyshevu_root* (page 409), *legendre* (page 410), *assoc_legendre* (page 410), *hermite* (page 411), *laguerre* (page 412), *assoc_laguerre* (page 413), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly* (page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

References

[R342] (page 1258), [R343] (page 1258), [R344] (page 1258)

Examples

```
>>> from diofant.abc import a, b
```

```
>>> jacobi_normalized(n, a, b, x)
jacobi(n, a, b, x)/sqrt(2**(a + b + 1)*gamma(a + n + 1)*gamma(b + n + 1)/((a + b␣
↪+ 2*n + 1)*factorial(n)*gamma(a + b + n + 1)))
```

Gegenbauer Polynomials

class diofant.functions.special.polynomials.**gegenbauer**

Gegenbauer polynomial $C_n^{(\alpha)}(x)$

gegenbauer(n, alpha, x) gives the nth Gegenbauer polynomial in x, $C_n^{(\alpha)}(x)$.

The Gegenbauer polynomials are orthogonal on $[-1,1]$ with respect to the weight $\left(1-x^2\right)^{\alpha-\frac{1}{2}}$.

See also:

jacobi (page 403), *chebyshevt_root* (page 408), *chebyshevu* (page 407), *chebyshevu_root* (page 409), *legendre* (page 410), *assoc_legendre* (page 410), *hermite* (page 411), *laguerre* (page 412), *assoc_laguerre* (page 413), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly*

(page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

References

[R345] (page 1258), [R346] (page 1258), [R347] (page 1258)

Examples

```
>>> from diofant.abc import a
>>> gegenbauer(0, a, x)
1
>>> gegenbauer(1, a, x)
2*a*x
>>> gegenbauer(2, a, x)
-a + x**2*(2*a**2 + 2*a)
>>> gegenbauer(3, a, x)
x**3*(4*a**3/3 + 4*a**2 + 8*a/3) + x*(-2*a**2 - 2*a)
```

```
>>> gegenbauer(n, a, x)
gegenbauer(n, a, x)
>>> gegenbauer(n, a, -x)
(-1)**n*gegenbauer(n, a, x)
```

```
>>> gegenbauer(n, a, 0)
2**n*sqrt(pi)*gamma(a + n/2)/(gamma(a)*gamma(-n/2 + 1/2)*gamma(n + 1))
>>> gegenbauer(n, a, 1)
gamma(2*a + n)/(gamma(2*a)*gamma(n + 1))
```

```
>>> conjugate(gegenbauer(n, a, x))
gegenbauer(n, conjugate(a), conjugate(x))
```

```
>>> diff(gegenbauer(n, a, x), x)
2*a*gegenbauer(n - 1, a + 1, x)
```

classmethod eval(*n, a, x*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=3*)
Returns the first derivative of the function.

Chebyshev Polynomials

class diofant.functions.special.polynomials.**chebyshevt**
Chebyshev polynomial of the first kind, $T_n(x)$

chebyshevt(n, x) gives the nth Chebyshev polynomial (of the first kind) in x, $T_n(x)$.

The Chebyshev polynomials of the first kind are orthogonal on $[-1, 1]$ with respect to the weight $\frac{1}{\sqrt{1-x^2}}$.

See also:

jacobi (page 403), *gegenbauer* (page 405), *chebyshevt_root* (page 408), *chebyshevu* (page 407), *chebyshevu_root* (page 409), *legendre* (page 410), *assoc_legendre* (page 410), *hermite* (page 411), *laguerre* (page 412), *assoc_laguerre* (page 413), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly* (page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

References

[R348] (page 1258), *[R349]* (page 1258), *[R350]* (page 1258), *[R351]* (page 1258), *[R352]* (page 1258)

Examples

```
>>> chebyshevt(0, x)
1
>>> chebyshevt(1, x)
x
>>> chebyshevt(2, x)
2*x**2 - 1
```

```
>>> chebyshevt(n, x)
chebyshevt(n, x)
>>> chebyshevt(n, -x)
(-1)**n*chebyshevt(n, x)
>>> chebyshevt(-n, x)
chebyshevt(n, x)
```

```
>>> chebyshevt(n, 0)
cos(pi*n/2)
>>> chebyshevt(n, -1)
(-1)**n
```

```
>>> diff(chebyshevt(n, x), x)
n*chebyshevu(n - 1, x)
```

classmethod eval(*n*, *x*)
Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=2*)
Returns the first derivative of the function.

class diofant.functions.special.polynomials.**chebyshevu**
Chebyshev polynomial of the second kind, $U_n(x)$

chebyshevu(n, x) gives the nth Chebyshev polynomial of the second kind in x, $U_n(x)$.

The Chebyshev polynomials of the second kind are orthogonal on $[-1, 1]$ with respect to the weight $\sqrt{1-x^2}$.

See also:

jacobi (page 403), *gegenbauer* (page 405), *chebyshevt* (page 406), *chebyshevt_root* (page 408), *chebyshevu_root* (page 409), *legendre* (page 410), *assoc_legendre* (page 410), *hermite* (page 411), *laguerre* (page 412), *assoc_laguerre* (page 413), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly* (page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

References

[R353] (page 1258), [R354] (page 1259), [R355] (page 1259), [R356] (page 1259), [R357] (page 1259)

Examples

```
>>> chebyshevu(0, x)
1
>>> chebyshevu(1, x)
2*x
>>> chebyshevu(2, x)
4*x**2 - 1
```

```
>>> chebyshevu(n, x)
chebyshevu(n, x)
>>> chebyshevu(n, -x)
(-1)**n*chebyshevu(n, x)
>>> chebyshevu(-n, x)
-chebyshevu(n - 2, x)
```

```
>>> chebyshevu(n, 0)
cos(pi*n/2)
>>> chebyshevu(n, 1)
n + 1
```

```
>>> diff(chebyshevu(n, x), x)
(-x*chebyshevu(n, x) + (n + 1)*chebyshevt(n + 1, x))/(x**2 - 1)
```

classmethod eval(*n*, *x*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=2*)

Returns the first derivative of the function.

class diofant.functions.special.polynomials.**chebyshevt_root**

chebyshev_root(n, k) returns the kth root (indexed from zero) of the nth Chebyshev polynomial of the first kind; that is, if 0 <= k < n, chebyshevt(n, chebyshevt_root(n, k)) == 0.

See also:

jacobi (page 403), *gegenbauer* (page 405), *chebyshevt* (page 406), *chebyshevu* (page 407), *chebyshevu_root* (page 409), *legendre* (page 410), *assoc_legendre* (page 410), *hermite* (page 411), *laguerre* (page 412), *assoc_laguerre* (page 413), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly* (page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

Examples

```
>>> chebyshevt_root(3, 2)
-sqrt(3)/2
>>> chebyshevt(3, chebyshevt_root(3, 2))
0
```

classmethod eval(n, k)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

class diofant.functions.special.polynomials.**chebyshevu_root**

chebyshevu_root(n, k) returns the kth root (indexed from zero) of the nth Chebyshev polynomial of the second kind; that is, if 0 <= k < n, chebyshevu(n, chebyshevu_root(n, k)) == 0.

See also:

chebyshevt (page 406), *chebyshevt_root* (page 408), *chebyshevu* (page 407), *legendre* (page 410), *assoc_legendre* (page 410), *hermite* (page 411), *laguerre* (page 412), *assoc_laguerre* (page 413), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly* (page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

Examples

```
>>> chebyshevu_root(3, 2)
-sqrt(2)/2
>>> chebyshevu(3, chebyshevu_root(3, 2))
0
```

classmethod eval(n, k)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

Legendre Polynomials

class `diofant.functions.special.polynomials.`**`legendre`**

legendre(n, x) gives the nth Legendre polynomial of x, $P_n(x)$

The Legendre polynomials are orthogonal on [-1, 1] with respect to the constant weight 1. They satisfy $P_n(1) = 1$ for all n; further, P_n is odd for odd n and even for even n.

See also:

jacobi (page 403), *gegenbauer* (page 405), *chebyshevt* (page 406), *chebyshevt_root* (page 408), *chebyshevu* (page 407), *chebyshevu_root* (page 409), *assoc_legendre* (page 410), *hermite* (page 411), *laguerre* (page 412), *assoc_laguerre* (page 413), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly* (page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

References

[R358] (page 1259), [R359] (page 1259), [R360] (page 1259), [R361] (page 1259)

Examples

```
>>> legendre(0, x)
1
>>> legendre(1, x)
x
>>> legendre(2, x)
3*x**2/2 - 1/2
>>> legendre(n, x)
legendre(n, x)
>>> diff(legendre(n, x), x)
n*(x*legendre(n, x) - legendre(n - 1, x))/(x**2 - 1)
```

classmethod eval(*n, x*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=2*)

Returns the first derivative of the function.

class `diofant.functions.special.polynomials.`**`assoc_legendre`**

assoc_legendre(n,m, x) gives $P_n^m(x)$, where n and m are the degree and order or an expression which is related to the nth order Legendre polynomial, $P_n(x)$ in the following

manner:

$$P_n^m(x) = (-1)^m (1 - x^2)^{\frac{m}{2}} \frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m}$$

Associated Legendre polynomial are orthogonal on [-1, 1] with:

- weight = 1 for the same m, and different n.
- weight = 1/(1-x**2) for the same n, and different m.

See also:

jacobi (page 403), *gegenbauer* (page 405), *chebyshevt* (page 406), *chebyshevt_root* (page 408), *chebyshevu* (page 407), *chebyshevu_root* (page 409), *legendre* (page 410), *hermite* (page 411), *laguerre* (page 412), *assoc_laguerre* (page 413), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly* (page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

References

[R362] (page 1259), *[R363]* (page 1259), *[R364]* (page 1259), *[R365]* (page 1259)

Examples

```
>>> assoc_legendre(0, 0, x)
1
>>> assoc_legendre(1, 0, x)
x
>>> assoc_legendre(1, 1, x)
-sqrt(-x**2 + 1)
>>> assoc_legendre(n, m, x)
assoc_legendre(n, m, x)
```

classmethod eval(*n*, *m*, *x*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=3*)

Returns the first derivative of the function.

Hermite Polynomials

class diofant.functions.special.polynomials.**hermite**

hermite(n, x) gives the nth Hermite polynomial in x, $H_n(x)$

The Hermite polynomials are orthogonal on $(-\infty, \infty)$ with respect to the weight $\exp\left(-\frac{x^2}{2}\right)$.

See also:

jacobi (page 403), *gegenbauer* (page 405), *chebyshevt* (page 406), *chebyshevt_root* (page 408), *chebyshevu* (page 407), *chebyshevu_root* (page 409), *legendre* (page 410), *assoc_legendre* (page 410), *laguerre* (page 412), *assoc_laguerre* (page 413), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly* (page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

References

[R366] (page 1259), *[R367]* (page 1259), *[R368]* (page 1259)

Examples

```
>>> hermite(0, x)
1
>>> hermite(1, x)
2*x
>>> hermite(2, x)
4*x**2 - 2
>>> hermite(n, x)
hermite(n, x)
>>> diff(hermite(n, x), x)
2*n*hermite(n - 1, x)
>>> hermite(n, -x)
(-1)**n*hermite(n, x)
```

classmethod eval(*n*, *x*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=2*)

Returns the first derivative of the function.

Laguerre Polynomials

class diofant.functions.special.polynomials.**laguerre**

Returns the nth Laguerre polynomial in x, $L_n(x)$.

Parameters n : int

Degree of Laguerre polynomial. Must be `n >= 0`.

See also:

jacobi (page 403), *gegenbauer* (page 405), *chebyshevt* (page 406), *chebyshevt_root* (page 408), *chebyshevu* (page 407), *chebyshevu_root* (page 409), *legendre* (page 410), *assoc_legendre* (page 410), *hermite* (page 411), *assoc_laguerre*

(page 413), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly* (page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

References

[R369] (page 1259), *[R370]* (page 1259), *[R371]* (page 1259), *[R372]* (page 1259)

Examples

```
>>> laguerre(0, x)
1
>>> laguerre(1, x)
-x + 1
>>> laguerre(2, x)
x**2/2 - 2*x + 1
>>> laguerre(3, x)
-x**3/6 + 3*x**2/2 - 3*x + 1
```

```
>>> laguerre(n, x)
laguerre(n, x)
```

```
>>> diff(laguerre(n, x), x)
-assoc_laguerre(n - 1, 1, x)
```

classmethod eval(*n*, *x*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=2*)

Returns the first derivative of the function.

class diofant.functions.special.polynomials.**assoc_laguerre**

Returns the nth generalized Laguerre polynomial in x, $L_n(x)$.

Parameters n : int

Degree of Laguerre polynomial. Must be `n >= 0`.

alpha : Expr

Arbitrary expression. For `alpha=0` regular Laguerre polynomials will be generated.

See also:

jacobi (page 403), *gegenbauer* (page 405), *chebyshevt* (page 406), *chebyshevt_root* (page 408), *chebyshevu* (page 407), *chebyshevu_root* (page 409), *legendre* (page 410), *assoc_legendre* (page 410), *hermite* (page 411), *laguerre* (page 412), *diofant.polys.orthopolys.jacobi_poly* (page 715), *diofant.polys.orthopolys.gegenbauer_poly*

(page 715), *diofant.polys.orthopolys.chebyshevt_poly* (page 715), *diofant.polys.orthopolys.chebyshevu_poly* (page 715), *diofant.polys.orthopolys.hermite_poly* (page 715), *diofant.polys.orthopolys.legendre_poly* (page 715), *diofant.polys.orthopolys.laguerre_poly* (page 715)

References

[R373] (page 1259), *[R374]* (page 1259), *[R375]* (page 1259), *[R376]* (page 1259)

Examples

```
>>> from diofant.abc import a
>>> assoc_laguerre(0, a, x)
1
>>> assoc_laguerre(1, a, x)
a - x + 1
>>> assoc_laguerre(2, a, x)
a**2/2 + 3*a/2 + x**2/2 + x*(-a - 2) + 1
>>> assoc_laguerre(3, a, x)
a**3/6 + a**2 + 11*a/6 - x**3/6 + x**2*(a/2 + 3/2) +
    x*(-a**2/2 - 5*a/2 - 3) + 1
```

```
>>> assoc_laguerre(n, a, 0)
binomial(a + n, a)
```

```
>>> assoc_laguerre(n, a, x)
assoc_laguerre(n, a, x)
```

```
>>> assoc_laguerre(n, 0, x)
laguerre(n, x)
```

```
>>> diff(assoc_laguerre(n, a, x), x)
-assoc_laguerre(n - 1, a + 1, x)
```

```
>>> diff(assoc_laguerre(n, a, x), a)
Sum(assoc_laguerre(_k, a, x)/(-a + n), (_k, 0, n - 1))
```

classmethod eval(*n, alpha, x*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

fdiff(*argindex=3*)

Returns the first derivative of the function.

Spherical Harmonics

class diofant.functions.special.spherical_harmonics.**Ynm**

Spherical harmonics defined as

$$Y_n^m(\theta, \varphi) := \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \exp(im\varphi) \mathrm{P}_n^m(\cos(\theta))$$

Ynm() gives the spherical harmonic function of order n and m in θ and φ, $Y_n^m(\theta, \varphi)$. The four parameters are as follows: $n \geq 0$ an integer and m an integer such that $-n \leq m \leq n$ holds. The two angles are real-valued with $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$.

See also:

diofant.functions.special.spherical_harmonics.Ynm_c (page 416), *diofant.functions.special.spherical_harmonics.Znm* (page 417)

References

[R377] (page 1259), *[R378]* (page 1259), *[R379]* (page 1259), *[R380]* (page 1259)

Examples

```
>>> theta = Symbol("theta")
>>> phi = Symbol("phi")
```

```
>>> Ynm(n, m, theta, phi)
Ynm(n, m, theta, phi)
```

Several symmetries are known, for the order

```
>>> theta = Symbol("theta")
>>> phi = Symbol("phi")
```

```
>>> Ynm(n, -m, theta, phi)
(-1)**m*E**(-2*I*m*phi)*Ynm(n, m, theta, phi)
```

as well as for the angles

```
>>> theta = Symbol("theta")
>>> phi = Symbol("phi")
```

```
>>> Ynm(n, m, -theta, phi)
Ynm(n, m, theta, phi)
```

```
>>> Ynm(n, m, theta, -phi)
E**(-2*I*m*phi)*Ynm(n, m, theta, phi)
```

For specific integers n and m we can evaluate the harmonics to more useful expressions

```
>>> simplify(Ynm(0, 0, theta, phi).expand(func=True))
1/(2*sqrt(pi))
```

```
>>> simplify(Ynm(1, -1, theta, phi).expand(func=True))
sqrt(6)*E**(-I*phi)*sin(theta)/(4*sqrt(pi))
```

```
>>> simplify(Ynm(1, 0, theta, phi).expand(func=True))
sqrt(3)*cos(theta)/(2*sqrt(pi))
```

```
>>> simplify(Ynm(1, 1, theta, phi).expand(func=True))
-sqrt(6)*E**(I*phi)*sin(theta)/(4*sqrt(pi))
```

```
>>> simplify(Ynm(2, -2, theta, phi).expand(func=True))
sqrt(30)*E**(-2*I*phi)*sin(theta)**2/(8*sqrt(pi))
```

```
>>> simplify(Ynm(2, -1, theta, phi).expand(func=True))
sqrt(30)*E**(-I*phi)*sin(2*theta)/(8*sqrt(pi))
```

```
>>> simplify(Ynm(2, 0, theta, phi).expand(func=True))
sqrt(5)*(3*cos(theta)**2 - 1)/(4*sqrt(pi))
```

```
>>> simplify(Ynm(2, 1, theta, phi).expand(func=True))
-sqrt(30)*E**(I*phi)*sin(2*theta)/(8*sqrt(pi))
```

```
>>> simplify(Ynm(2, 2, theta, phi).expand(func=True))
sqrt(30)*E**(2*I*phi)*sin(theta)**2/(8*sqrt(pi))
```

We can differentiate the functions with respect to both angles

```
>>> theta = Symbol("theta")
>>> phi = Symbol("phi")
```

```
>>> diff(Ynm(n, m, theta, phi), theta)
m*cot(theta)*Ynm(n, m, theta, phi) + E**(-I*phi)*sqrt((-m + n)*(m + n + 1))*Ynm(n,
↪ m + 1, theta, phi)
```

```
>>> diff(Ynm(n, m, theta, phi), phi)
I*m*Ynm(n, m, theta, phi)
```

Further we can compute the complex conjugation

```
>>> theta = Symbol("theta")
>>> phi = Symbol("phi")
>>> m = Symbol('m')
```

```
>>> conjugate(Ynm(n, m, theta, phi))
(-1)**(2*m)*E**(-2*I*m*phi)*Ynm(n, m, theta, phi)
```

To get back the well known expressions in spherical coordinates we use full expansion

```
>>> theta = Symbol("theta")
>>> phi = Symbol("phi")
```

```
>>> expand_func(Ynm(n, m, theta, phi))
E**(I*m*phi)*sqrt((2*n + 1)*factorial(-m + n)/factorial(m + n))*assoc_legendre(n,␣
↪m, cos(theta))/(2*sqrt(pi))
```

diofant.functions.special.spherical_harmonics.**Ynm_c**(*n*, *m*, *theta*, *phi*)
Conjugate spherical harmonics defined as

$$\overline{Y_n^m(\theta, \varphi)} := (-1)^m Y_n^{-m}(\theta, \varphi)$$

See also:

diofant.functions.special.spherical_harmonics.Ynm (page 415), *diofant.functions.special.spherical_harmonics.Znm* (page 417)

References

[R381] (page 1259), [R382] (page 1259), [R383] (page 1259)

class diofant.functions.special.spherical_harmonics.**Znm**
Real spherical harmonics defined as

$$Z_n^m(\theta, \varphi) := \begin{cases} \frac{Y_n^m(\theta, \varphi) + \overline{Y_n^m(\theta, \varphi)}}{\sqrt{2}} & m > 0 \\ Y_n^m(\theta, \varphi) & m = 0 \\ \frac{Y_n^m(\theta, \varphi) - \overline{Y_n^m(\theta, \varphi)}}{i\sqrt{2}} & m < 0 \end{cases}$$

which gives in simplified form

$$Z_n^m(\theta, \varphi) = \begin{cases} \frac{Y_n^m(\theta, \varphi) + (-1)^m Y_n^{-m}(\theta, \varphi)}{\sqrt{2}} & m > 0 \\ Y_n^m(\theta, \varphi) & m = 0 \\ \frac{Y_n^m(\theta, \varphi) - (-1)^m Y_n^{-m}(\theta, \varphi)}{i\sqrt{2}} & m < 0 \end{cases}$$

See also:

diofant.functions.special.spherical_harmonics.Ynm (page 415), *diofant.functions.special.spherical_harmonics.Ynm_c* (page 416)

References

[R384] (page 1259), [R385] (page 1259), [R386] (page 1259)

Tensor Functions

diofant.functions.special.tensor_functions.**Eijk**(**args*, ***kwargs*)
Represent the Levi-Civita symbol.

This is just compatibility wrapper to `LeviCivita()`.

See also:

diofant.functions.special.tensor_functions.LeviCivita (page 417)

diofant.functions.special.tensor_functions.**eval_levicivita**(**args*)
Evaluate Levi-Civita symbol.

class diofant.functions.special.tensor_functions.**LeviCivita**

Represent the Levi-Civita symbol.

For even permutations of indices it returns 1, for odd permutations -1, and for everything else (a repeated index) it returns 0.

Thus it represents an alternating pseudotensor.

See also:

diofant.functions.special.tensor_functions.Eijk (page 417)

Examples

```
>>> from diofant.abc import i, j
>>> LeviCivita(1, 2, 3)
1
>>> LeviCivita(1, 3, 2)
-1
>>> LeviCivita(1, 2, 2)
0
>>> LeviCivita(i, j, k)
LeviCivita(i, j, k)
>>> LeviCivita(i, j, i)
0
```

doit(***hints*)

Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

classmethod eval(**args*)

Returns a canonical form of cls applied to arguments args.

The eval() method is called when the class cls is about to be instantiated and it should return either some simplified instance (possible of some other class), or if the class cls should be unmodified, return None.

class diofant.functions.special.tensor_functions.**KroneckerDelta**

The discrete, or Kronecker, delta function.

A function that takes in two integers i and j. It returns 0 if i and j are not equal or it returns 1 if i and j are equal.

Parameters **i** : Number, Symbol

The first index of the delta function.

j : Number, Symbol

The second index of the delta function.

See also:

diofant.functions.special.tensor_functions.KroneckerDelta.eval (page 419), *diofant.functions.special.delta_functions.DiracDelta* (page 348)

References

[R387] (page 1259)

Examples

A simple example with integer indices:

```
>>> KroneckerDelta(1, 2)
0
>>> KroneckerDelta(3, 3)
1
```

Symbolic indices:

```
>>> from diofant.abc import i, j
>>> KroneckerDelta(i, j)
KroneckerDelta(i, j)
>>> KroneckerDelta(i, i)
1
>>> KroneckerDelta(i, i + 1)
0
>>> KroneckerDelta(i, i + 1 + k)
KroneckerDelta(i, i + k + 1)
```

classmethod eval(*i*, *j*)

Evaluates the discrete delta function.

Examples

```
>>> from diofant.abc import i, j
```

```
>>> KroneckerDelta(i, j)
KroneckerDelta(i, j)
>>> KroneckerDelta(i, i)
1
>>> KroneckerDelta(i, i + 1)
0
>>> KroneckerDelta(i, i + 1 + k)
KroneckerDelta(i, i + k + 1)
```

indices_contain_equal_information

Returns True if indices are either both above or below fermi.

Examples

```
>>> a = Symbol('a', above_fermi=True)
>>> i = Symbol('i', below_fermi=True)
>>> p = Symbol('p')
>>> q = Symbol('q')
>>> KroneckerDelta(p, q).indices_contain_equal_information
True
>>> KroneckerDelta(p, q+1).indices_contain_equal_information
True
>>> KroneckerDelta(i, p).indices_contain_equal_information
False
```

is_above_fermi

True if Delta can be non-zero above fermi

See also:

diofant.functions.special.tensor_functions.KroneckerDelta.is_below_fermi (page 420), *diofant.functions.special.tensor_functions.KroneckerDelta.is_only_below_fermi* (page 421), *diofant.functions.special.tensor_functions.KroneckerDelta.is_only_above_fermi* (page 421)

Examples

```
>>> a = Symbol('a', above_fermi=True)
>>> i = Symbol('i', below_fermi=True)
>>> p = Symbol('p')
>>> q = Symbol('q')
>>> KroneckerDelta(p, a).is_above_fermi
True
>>> KroneckerDelta(p, i).is_above_fermi
False
>>> KroneckerDelta(p, q).is_above_fermi
True
```

is_below_fermi

True if Delta can be non-zero below fermi

See also:

diofant.functions.special.tensor_functions.KroneckerDelta.is_above_fermi (page 420), *diofant.functions.special.tensor_functions.KroneckerDelta.is_only_above_fermi* (page 421), *diofant.functions.special.tensor_functions.KroneckerDelta.is_only_below_fermi* (page 421)

Examples

```
>>> a = Symbol('a', above_fermi=True)
>>> i = Symbol('i', below_fermi=True)
```

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```
>>> p = Symbol('p')
>>> q = Symbol('q')
>>> KroneckerDelta(p, a).is_below_fermi
False
>>> KroneckerDelta(p, i).is_below_fermi
True
>>> KroneckerDelta(p, q).is_below_fermi
True
```

is_only_above_fermi

True if Delta is restricted to above fermi

See also:

diofant.functions.special.tensor_functions.KroneckerDelta.is_above_fermi (page 420), *diofant.functions.special.tensor_functions.KroneckerDelta.is_below_fermi* (page 420), *diofant.functions.special.tensor_functions.KroneckerDelta.is_only_below_fermi* (page 421)

Examples

```
>>> a = Symbol('a', above_fermi=True)
>>> i = Symbol('i', below_fermi=True)
>>> p = Symbol('p')
>>> q = Symbol('q')
>>> KroneckerDelta(p, a).is_only_above_fermi
True
>>> KroneckerDelta(p, q).is_only_above_fermi
False
>>> KroneckerDelta(p, i).is_only_above_fermi
False
```

is_only_below_fermi

True if Delta is restricted to below fermi

See also:

diofant.functions.special.tensor_functions.KroneckerDelta.is_above_fermi (page 420), *diofant.functions.special.tensor_functions.KroneckerDelta.is_below_fermi* (page 420), *diofant.functions.special.tensor_functions.KroneckerDelta.is_only_above_fermi* (page 421)

Examples

```
>>> a = Symbol('a', above_fermi=True)
>>> i = Symbol('i', below_fermi=True)
>>> p = Symbol('p')
>>> q = Symbol('q')
>>> KroneckerDelta(p, i).is_only_below_fermi
True
>>> KroneckerDelta(p, q).is_only_below_fermi
False
>>> KroneckerDelta(p, a).is_only_below_fermi
False
```

killable_index

Returns the index which is preferred to substitute in the final expression.

The index to substitute is the index with less information regarding fermi level. If indices contain same information, 'a' is preferred before 'b'.

See also:

diofant.functions.special.tensor_functions.KroneckerDelta.preferred_index (page 422)

Examples

```
>>> a = Symbol('a', above_fermi=True)
>>> i = Symbol('i', below_fermi=True)
>>> j = Symbol('j', below_fermi=True)
>>> p = Symbol('p')
>>> KroneckerDelta(p, i).killable_index
p
>>> KroneckerDelta(p, a).killable_index
p
>>> KroneckerDelta(i, j).killable_index
j
```

preferred_index

Returns the index which is preferred to keep in the final expression.

The preferred index is the index with more information regarding fermi level. If indices contain same information, 'a' is preferred before 'b'.

See also:

diofant.functions.special.tensor_functions.KroneckerDelta.killable_index (page 421)

Examples

```
>>> a = Symbol('a', above_fermi=True)
>>> i = Symbol('i', below_fermi=True)
>>> j = Symbol('j', below_fermi=True)
>>> p = Symbol('p')
>>> KroneckerDelta(p, i).preferred_index
i
>>> KroneckerDelta(p, a).preferred_index
a
>>> KroneckerDelta(i, j).preferred_index
i
```

3.6 Geometry

3.6.1 Introduction

The geometry module for Diofant allows one to create two-dimensional geometrical entities, such as lines and circles, and query for information about these entities. This could include

asking the area of an ellipse, checking for collinearity of a set of points, or finding the intersection between two lines. The primary use case of the module involves entities with numerical values, but it is possible to also use symbolic representations.

3.6.2 Available Entities

The following entities are currently available in the geometry module:

- `Point`
- `Line`, `Ray`, `Segment`
- `Ellipse`, `Circle`
- `Polygon`, `RegularPolygon`, `Triangle`

Most of the work one will do will be through the properties and methods of these entities, but several global methods exist:

- `intersection(entity1, entity2)`
- `are_similar(entity1, entity2)`
- `convex_hull(points)`

For a full API listing and an explanation of the methods and their return values please see the list of classes at the end of this document.

3.6.3 Example Usage

The following Python session gives one an idea of how to work with some of the geometry module.

```
>>> x = Point(0, 0)
>>> y = Point(1, 1)
>>> z = Point(2, 2)
>>> zp = Point(1, 0)
>>> Point.is_collinear(x, y, z)
True
>>> Point.is_collinear(x, y, zp)
False
>>> t = Triangle(zp, y, x)
>>> t.area
1/2
>>> t.medians[x]
Segment(Point2D(0, 0), Point2D(1, 1/2))
>>> Segment(Point(1, Rational(1, 2)), Point(0, 0))
Segment(Point2D(0, 0), Point2D(1, 1/2))
>>> m = t.medians
>>> intersection(m[x], m[y], m[zp])
[Point2D(2/3, 1/3)]
>>> c = Circle(x, 5)
>>> l = Line(Point(5, -5), Point(5, 5))
>>> c.is_tangent(l) # is l tangent to c?
True
>>> l = Line(x, y)
>>> c.is_tangent(l) # is l tangent to c?
False
```

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```
>>> intersection(c, l)
[Point2D(-5*sqrt(2)/2, -5*sqrt(2)/2), Point2D(5*sqrt(2)/2, 5*sqrt(2)/2)]
```

3.6.4 Intersection of medians

```
>>> a, b = symbols("a b", positive=True)

>>> x = Point(0, 0)
>>> y = Point(a, 0)
>>> z = Point(2*a, b)
>>> t = Triangle(x, y, z)

>>> t.area
a*b/2

>>> t.medians[x]
Segment(Point2D(0, 0), Point2D(3*a/2, b/2))

>>> intersection(t.medians[x], t.medians[y], t.medians[z])
[Point2D(a, b/3)]
```

3.6.5 An in-depth example: Pappus' Hexagon Theorem

From Wikipedia (*[WikiPappus]* (page 1259)):

> Given one set of collinear points A, B, C, and another set of collinear points a, b, c, then the intersection points X, Y, Z of line pairs Ab and aB, Ac and aC, Bc and bC are collinear.

```
>>> l1 = Line(Point(0, 0), Point(5, 6))
>>> l2 = Line(Point(0, 0), Point(2, -2))
>>>
>>> def subs_point(l, val):
...     """Take an arbitrary point and make it a fixed point."""
...     t = Symbol('t', extended_real=True)
...     ap = l.arbitrary_point()
...     return Point(ap.x.subs(t, val), ap.y.subs(t, val))
...
>>> p11 = subs_point(l1, 5)
>>> p12 = subs_point(l1, 6)
>>> p13 = subs_point(l1, 11)
>>>
>>> p21 = subs_point(l2, -1)
>>> p22 = subs_point(l2, 2)
>>> p23 = subs_point(l2, 13)
>>>
>>> ll1 = Line(p11, p22)
>>> ll2 = Line(p11, p23)
>>> ll3 = Line(p12, p21)
>>> ll4 = Line(p12, p23)
>>> ll5 = Line(p13, p21)
>>> ll6 = Line(p13, p22)
>>>
```

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```
>>> pp1 = intersection(ll1, ll3)[0]
>>> pp2 = intersection(ll2, ll5)[0]
>>> pp3 = intersection(ll4, ll6)[0]
>>>
>>> Point.is_collinear(pp1, pp2, pp3)
True
```

References

3.6.6 Miscellaneous Notes

- The area property of `Polygon` and `Triangle` may return a positive or negative value, depending on whether or not the points are oriented counter-clockwise or clockwise, respectively. If you always want a positive value be sure to use the `abs` function.
- Although `Polygon` can refer to any type of polygon, the code has been written for simple polygons. Hence, expect potential problems if dealing with complex polygons (overlapping sides).
- Since Diofant is still in its infancy some things may not simplify properly and hence some things that should return `True` (e.g., `Point.is_collinear`) may not actually do so. Similarly, attempting to find the intersection of entities that do intersect may result in an empty result.

3.6.7 Future Work

Truth Setting Expressions

When one deals with symbolic entities, it often happens that an assertion cannot be guaranteed. For example, consider the following code:

```
>>> x, y, z = symbols('x y z')
>>> p1, p2, p3 = Point(x, y), Point(y, z), Point(2*x*y, y)
>>> Point.is_collinear(p1, p2, p3)
False
```

Even though the result is currently `False`, this is not *always* true. If the quantity $z - y - 2 * y * z + 2 * y * *2 == 0$ then the points will be collinear. It would be really nice to inform the user of this because such a quantity may be useful to a user for further calculation and, at the very least, being nice to know. This could be potentially done by returning an object (e.g., GeometryResult) that the user could use. This actually would not involve an extensive amount of work.

Three Dimensions and Beyond

Currently there are no plans for extending the module to three dimensions, but it certainly would be a good addition. This would probably involve a fair amount of work since many of the algorithms used are specific to two dimensions.

Submodules

Entities

class diofant.geometry.entity.**GeometryEntity**

The base class for all geometrical entities.

This class doesn't represent any particular geometric entity, it only provides the implementation of some methods common to all subclasses.

ambient_dimension

What is the dimension of the space that the object is contained in?

encloses(*o*)

Return True if o is inside (not on or outside) the boundaries of self.

The object will be decomposed into Points and individual Entities need only define an encloses_point method for their class.

See also:

diofant.geometry.ellipse.Ellipse.encloses_point (page 478), *diofant.geometry.polygon.Polygon.encloses_point* (page 492)

Examples

```
>>> t  = Polygon(*RegularPolygon(Point(0, 0), 1, 3).vertices)
>>> t2 = Polygon(*RegularPolygon(Point(0, 0), 2, 3).vertices)
>>> t2.encloses(t)
True
>>> t.encloses(t2)
False
```

intersection(*o*)

Returns a list of all of the intersections of self with o.

See also:

diofant.geometry.util.intersection (page 428)

Notes

An entity is not required to implement this method.

is_similar(*other*)

Is this geometrical entity similar to another geometrical entity?

Two entities are similar if a uniform scaling (enlarging or shrinking) of one of the entities will allow one to obtain the other.

See also:

scale (page 427)

Notes

This method is not intended to be used directly but rather through the $are_similar$ function found in util.py. An entity is not required to implement this method. If two different types of entities can be similar, it is only required that one of them be able to determine this.

rotate(*angle*, *pt=None*)

Rotate `angle` radians counterclockwise about Point `pt`.

The default pt is the origin, Point(0, 0)

See also:

scale (page 427), *translate* (page 427)

Examples

```
>>> t = Polygon(*RegularPolygon(Point(0, 0), 1, 3).vertices)
>>> t # vertex on x axis
Triangle(Point2D(1, 0), Point2D(-1/2, sqrt(3)/2), Point2D(-1/2, -sqrt(3)/2))
>>> t.rotate(pi/2) # vertex on y axis now
Triangle(Point2D(0, 1), Point2D(-sqrt(3)/2, -1/2), Point2D(sqrt(3)/2, -1/2))
```

scale(*x=1*, *y=1*, *pt=None*)

Scale the object by multiplying the x,y-coordinates by x and y.

If pt is given, the scaling is done relative to that point; the object is shifted by -pt, scaled, and shifted by pt.

See also:

rotate (page 427), *translate* (page 427)

Examples

```
>>> t = Polygon(*RegularPolygon(Point(0, 0), 1, 3).vertices)
>>> t
Triangle(Point2D(1, 0), Point2D(-1/2, sqrt(3)/2), Point2D(-1/2, -sqrt(3)/2))
>>> t.scale(2)
Triangle(Point2D(2, 0), Point2D(-1, sqrt(3)/2), Point2D(-1, -sqrt(3)/2))
>>> t.scale(2, 2)
Triangle(Point2D(2, 0), Point2D(-1, sqrt(3)), Point2D(-1, -sqrt(3)))
```

translate(*x=0*, *y=0*)

Shift the object by adding to the x,y-coordinates the values x and y.

See also:

rotate (page 427), *scale* (page 427)

Examples

```
>>> t = Polygon(*RegularPolygon(Point(0, 0), 1, 3).vertices)
>>> t
Triangle(Point2D(1, 0), Point2D(-1/2, sqrt(3)/2), Point2D(-1/2, -sqrt(3)/2))
>>> t.translate(2)
Triangle(Point2D(3, 0), Point2D(3/2, sqrt(3)/2), Point2D(3/2, -sqrt(3)/2))
>>> t.translate(2, 2)
Triangle(Point2D(3, 2), Point2D(3/2, sqrt(3)/2 + 2),
    Point2D(3/2, -sqrt(3)/2 + 2))
```

Utils

diofant.geometry.util.**idiff**(*eq, y, x, n=1*)

Return `dy/dx` assuming that `eq == 0`.

Parameters y : the dependent variable or a list of dependent variables (with y first)

x : the variable that the derivative is being taken with respect to

n : the order of the derivative (default is 1)

See also:

diofant.core.function.Derivative **(page 128)** represents unevaluated derivatives

diofant.core.function.diff **(page 131)** explicitly differentiates wrt symbols

Examples

```
>>> from diofant.abc import a
```

```
>>> circ = x**2 + y**2 - 4
>>> idiff(circ, y, x)
-x/y
>>> idiff(circ, y, x, 2).simplify()
-(x**2 + y**2)/y**3
```

Here, `a` is assumed to be independent of `x`:

```
>>> idiff(x + a + y, y, x)
-1
```

Now the x-dependence of `a` is made explicit by listing `a` after `y` in a list.

```
>>> idiff(x + a + y, [y, a], x)
-Derivative(a, x) - 1
```

diofant.geometry.util.**intersection**(**entities*)

The intersection of a collection of GeometryEntity instances.

Parameters entities : sequence of GeometryEntity

Returns intersection : list of GeometryEntity

Raises NotImplementedError

When unable to calculate intersection.

See also:

diofant.geometry.entity.GeometryEntity.intersection (page 426)

Notes

The intersection of any geometrical entity with itself should return a list with one item: the entity in question. An intersection requires two or more entities. If only a single entity is given then the function will return an empty list. It is possible for $intersection$ to miss intersections that one knows exists because the required quantities were not fully simplified internally. Reals should be converted to Rationals, e.g. Rational(str(real_num)) or else failures due to floating point issues may result.

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 1), Point(-1, 5)
>>> l1, l2 = Line(p1, p2), Line(p3, p2)
>>> c = Circle(p2, 1)
>>> intersection(l1, p2)
[Point2D(1, 1)]
>>> intersection(l1, l2)
[Point2D(1, 1)]
>>> intersection(c, p2)
[]
>>> intersection(c, Point(1, 0))
[Point2D(1, 0)]
>>> intersection(c, l2)
[Point2D(-sqrt(5)/5 + 1, 2*sqrt(5)/5 + 1),
 Point2D(sqrt(5)/5 + 1, -2*sqrt(5)/5 + 1)]
```

diofant.geometry.util.**convex_hull**(**args*)

The convex hull surrounding the Points contained in the list of entities.

Parameters args : a collection of Points, Segments and/or Polygons

Returns convex_hull : Polygon

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.polygon.Polygon* (page 489)

Notes

This can only be performed on a set of non-symbolic points.

References

[1] https//en.wikipedia.org/wiki/Graham_scan

[2] Andrew's Monotone Chain Algorithm (A.M. Andrew, "Another Efficient Algorithm for Convex Hulls in Two Dimensions", 1979) http://geomalgorithms.com/a10-_hull-1.html

Examples

```
>>> points = [(1, 1), (1, 2), (3, 1), (-5, 2), (15, 4)]
>>> convex_hull(*points)
Polygon(Point2D(-5, 2), Point2D(1, 1), Point2D(3, 1), Point2D(15, 4))
```

diofant.geometry.util.**are_similar**(*e1*, *e2*)

Are two geometrical entities similar.

Can one geometrical entity be uniformly scaled to the other?

Parameters **e1** : GeometryEntity

e2 : GeometryEntity

Returns **are_similar** : boolean

Raises **GeometryError**

When $e1$ and $e2$ cannot be compared.

See also:

diofant.geometry.entity.GeometryEntity.is_similar (page 426)

Notes

If the two objects are equal then they are similar.

Examples

```
>>> c1, c2 = Circle(Point(0, 0), 4), Circle(Point(1, 4), 3)
>>> t1 = Triangle(Point(0, 0), Point(1, 0), Point(0, 1))
>>> t2 = Triangle(Point(0, 0), Point(2, 0), Point(0, 2))
>>> t3 = Triangle(Point(0, 0), Point(3, 0), Point(0, 1))
>>> are_similar(t1, t2)
True
>>> are_similar(t1, t3)
False
```

diofant.geometry.util.**centroid**(**args*)

Find the centroid (center of mass) of the collection containing only Points, Segments or Polygons. The centroid is the weighted average of the individual centroid where the weights are the lengths (of segments) or areas (of polygons). Overlapping regions will add to the weight of that region.

If there are no objects (or a mixture of objects) then None is returned.

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.line.Segment* (page 455), *diofant.geometry.polygon.Polygon* (page 489)

Examples

```
>>> p = Polygon((0, 0), (10, 0), (10, 10))
>>> q = p.translate(0, 20)
>>> p.centroid, q.centroid
(Point2D(20/3, 10/3), Point2D(20/3, 70/3))
>>> centroid(p, q)
Point2D(20/3, 40/3)
>>> p, q = Segment((0, 0), (2, 0)), Segment((0, 0), (2, 2))
>>> centroid(p, q)
Point2D(1, -sqrt(2) + 2)
>>> centroid(Point(0, 0), Point(2, 0))
Point2D(1, 0)
```

Stacking 3 polygons on top of each other effectively triples the weight of that polygon:

```
>>> p = Polygon((0, 0), (1, 0), (1, 1), (0, 1))
>>> q = Polygon((1, 0), (3, 0), (3, 1), (1, 1))
>>> centroid(p, q)
Point2D(3/2, 1/2)
>>> centroid(p, p, p, q) # centroid x-coord shifts left
Point2D(11/10, 1/2)
```

Stacking the squares vertically above and below p has the same effect:

```
>>> centroid(p, p.translate(0, 1), p.translate(0, -1), q)
Point2D(11/10, 1/2)
```

Points

class diofant.geometry.point.**Point**

A point in a n-dimensional Euclidean space.

Parameters coords : sequence of n-coordinate values. In the special

case where n=2 or 3, a Point2D or Point3D will be created

as appropriate.

Raises TypeError

When trying to add or subtract points with different dimensions. When *intersection* is called with object other than a Point.

See also:

diofant.geometry.line.Segment **(page 455)** Connects two Points

Examples

```
>>> Point(1, 2, 3)
Point3D(1, 2, 3)
>>> Point([1, 2])
Point2D(1, 2)
>>> Point(0, x)
Point2D(0, x)
```

Floats are automatically converted to Rational unless the evaluate flag is False:

```
>>> Point(0.5, 0.25)
Point2D(1/2, 1/4)
>>> print(Point(0.5, 0.25, evaluate=False))
Point2D(0.5, 0.25)
```

Attributes

length (page 434)	Treating a Point as a Line, this returns 0 for the length of a Point.

origin: A 'Point' representing the origin of the	appropriately-dimensioned space.

ambient_dimension
The dimension of the ambient space the point is in. I.e., if the point is in R^n, the ambient dimension will be n

distance(*p*)
The Euclidean distance from self to point p.

Parameters p : Point

Returns distance : number or symbolic expression.

See also:

diofant.geometry.line.Segment.length (page 456)

Examples

```
>>> p1, p2 = Point(1, 1), Point(4, 5)
>>> p1.distance(p2)
5
```

```
>>> p3 = Point(x, y)
>>> p3.distance(Point(0, 0))
sqrt(x**2 + y**2)
```

dot(*p2*)
Return dot product of self with another Point.

equals(*other*)
Returns whether the coordinates of self and other agree.

evalf(*dps=15*, ***options*)
Evaluate the coordinates of the point.

This method will, where possible, create and return a new Point where the coordinates are evaluated as floating point numbers to the decimal precision dps.

Returns point : Point

Examples

```
>>> p1 = Point(Rational(1, 2), Rational(3, 2))
>>> p1
Point2D(1/2, 3/2)
>>> print(p1.evalf())
Point2D(0.5, 1.5)
```

intersection(*o*)

The intersection between this point and another point.

Parameters other : Point

Returns intersection : list of Points

Notes

The return value will either be an empty list if there is no intersection, otherwise it will contain this point.

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 1), Point(0, 0)
>>> p1.intersection(p2)
[]
>>> p1.intersection(p3)
[Point2D(0, 0)]
```

is_collinear()

Is a sequence of points collinear?

Test whether or not a set of points are collinear. Returns True if the set of points are collinear, or False otherwise.

Parameters points : sequence of Point

Returns is_collinear : boolean

See also:

diofant.geometry.line.Line (page 449)

Notes

Slope is preserved everywhere on a line, so the slope between any two points on the line should be the same. Take the first two points, p1 and p2, and create a translated point v1 with p1 as the origin. Now for every other point we create a translated point, vi with p1 also as the origin. Note that these translations preserve slope since everything is consistently translated to a new origin of p1. Since slope is preserved then we have the following equality:

- v1_slope = vi_slope
- v1.y/v1.x = vi.y/vi.x (due to translation)
- v1.y*vi.x = vi.y*v1.x

- v1.y*vi.x - vi.y*v1.x = 0 (*)

Hence, if we have a vi such that the equality in (*) is False then the points are not collinear. We do this test for every point in the list, and if all pass then they are collinear.

Examples

```
>>> p1, p2 = Point(0, 0), Point(1, 1)
>>> p3, p4, p5 = Point(2, 2), Point(x, x), Point(1, 2)
>>> Point.is_collinear(p1, p2, p3, p4)
True
>>> Point.is_collinear(p1, p2, p3, p5)
False
```

is_scalar_multiple(*other*)
Returns whether $self$ and $other$ are scalar multiples of each other.

is_zero
True if every coordinate is zero, otherwise False.

length
Treating a Point as a Line, this returns 0 for the length of a Point.

Examples

```
>>> p = Point(0, 1)
>>> p.length
0
```

midpoint(*p*)
The midpoint between self and point p.

Parameters p : Point

Returns midpoint : Point

See also:

diofant.geometry.line.Segment.midpoint (page 457)

Examples

```
>>> p1, p2 = Point(1, 1), Point(13, 5)
>>> p1.midpoint(p2)
Point2D(7, 3)
```

n(*dps=15, **options*)
Evaluate the coordinates of the point.

This method will, where possible, create and return a new Point where the coordinates are evaluated as floating point numbers to the decimal precision dps.

Returns point : Point

Examples

```
>>> p1 = Point(Rational(1, 2), Rational(3, 2))
>>> p1
Point2D(1/2, 3/2)
>>> print(p1.evalf())
Point2D(0.5, 1.5)
```

origin
A point of all zeros of the same ambient dimension as the current point

class diofant.geometry.point.**Point2D**
A point in a 2-dimensional Euclidean space.

Parameters coords : sequence of 2 coordinate values.

Raises TypeError

When trying to add or subtract points with different dimensions. When trying to create a point with more than two dimensions. When *intersection* is called with object other than a Point.

See also:

diofant.geometry.line.Segment **(page 455)** Connects two Points

Examples

```
>>> Point2D(1, 2)
Point2D(1, 2)
>>> Point2D([1, 2])
Point2D(1, 2)
>>> Point2D(0, x)
Point2D(0, x)
```

Floats are automatically converted to Rational unless the evaluate flag is False:

```
>>> Point2D(0.5, 0.25)
Point2D(1/2, 1/4)
>>> print(Point2D(0.5, 0.25, evaluate=False))
Point2D(0.5, 0.25)
```

Attributes

x (page 437)	Returns the X coordinate of the Point.
y (page 437)	Returns the Y coordinate of the Point.

diofant.geometry.Point.length	

bounds
Return a tuple (xmin, ymin, xmax, ymax) representing the bounding rectangle for the geometric figure.

is_concyclic()

Is a sequence of points concyclic?

Test whether or not a sequence of points are concyclic (i.e., they lie on a circle).

Parameters points : sequence of Points

Returns is_concyclic : boolean

True if points are concyclic, False otherwise.

See also:

diofant.geometry.ellipse.Circle (page 486)

Notes

No points are not considered to be concyclic. One or two points are definitely concyclic and three points are conyclic iff they are not collinear.

For more than three points, create a circle from the first three points. If the circle cannot be created (i.e., they are collinear) then all of the points cannot be concyclic. If the circle is created successfully then simply check the remaining points for containment in the circle.

Examples

```
>>> p1, p2 = Point(-1, 0), Point(1, 0)
>>> p3, p4 = Point(0, 1), Point(-1, 2)
>>> Point.is_concyclic(p1, p2, p3)
True
>>> Point.is_concyclic(p1, p2, p3, p4)
False
```

rotate(*angle*, *pt=None*)

Rotate `angle` radians counterclockwise about Point `pt`.

See also:

rotate (page 436), *scale* (page 436)

Examples

```
>>> t = Point2D(1, 0)
>>> t.rotate(pi/2)
Point2D(0, 1)
>>> t.rotate(pi/2, (2, 0))
Point2D(2, -1)
```

scale(*x=1*, *y=1*, *pt=None*)

Scale the coordinates of the Point by multiplying by `x` and `y` after subtracting `pt` – default is (0, 0) – and then adding `pt` back again (i.e. `pt` is the point of reference for the scaling).

See also:

rotate (page 436), *translate* (page 437)

Examples

```
>>> t = Point2D(1, 1)
>>> t.scale(2)
Point2D(2, 1)
>>> t.scale(2, 2)
Point2D(2, 2)
```

transform(*matrix*)
Return the point after applying the transformation described by the 3x3 Matrix, `matrix`.

See also:

diofant.geometry.entity.GeometryEntity.rotate (page 427), *diofant.geometry.entity.GeometryEntity.scale* (page 427), *diofant.geometry.entity.GeometryEntity.translate* (page 427)

translate(*x=0, y=0*)
Shift the Point by adding x and y to the coordinates of the Point.

See also:

rotate (page 436), *scale* (page 436)

Examples

```
>>> t = Point2D(0, 1)
>>> t.translate(2)
Point2D(2, 1)
>>> t.translate(2, 2)
Point2D(2, 3)
>>> t + Point2D(2, 2)
Point2D(2, 3)
```

x
Returns the X coordinate of the Point.

Examples

```
>>> p = Point2D(0, 1)
>>> p.x
0
```

y
Returns the Y coordinate of the Point.

Examples

```
>>> p = Point2D(0, 1)
>>> p.y
1
```

class diofant.geometry.point.**Point3D**

A point in a 3-dimensional Euclidean space.

Parameters coords : sequence of 3 coordinate values.

Raises TypeError

When trying to add or subtract points with different dimensions. When *intersection* is called with object other than a Point.

Notes

Currently only 2-dimensional and 3-dimensional points are supported.

Examples

```
>>> Point3D(1, 2, 3)
Point3D(1, 2, 3)
>>> Point3D([1, 2, 3])
Point3D(1, 2, 3)
>>> Point3D(0, x, 3)
Point3D(0, x, 3)
```

Floats are automatically converted to Rational unless the evaluate flag is False:

```
>>> Point3D(0.5, 0.25, 2)
Point3D(1/2, 1/4, 2)
>>> print(Point3D(0.5, 0.25, 3, evaluate=False))
Point3D(0.5, 0.25, 3)
```

Attributes

x (page 441)	Returns the X coordinate of the Point.
y (page 441)	Returns the Y coordinate of the Point.
z (page 441)	Returns the Z coordinate of the Point.

diofant.geometry.Point.length	

static are_collinear()

Is a sequence of points collinear?

Test whether or not a set of points are collinear. Returns True if the set of points are collinear, or False otherwise.

Parameters points : sequence of Point

Returns are_collinear : boolean

See also:

diofant.geometry.line3d.Line3D (page 458)

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(1, 1, 1)
>>> p3, p4, p5 = Point3D(2, 2, 2), Point3D(x, x, x), Point3D(1, 2, 6)
>>> Point3D.are_collinear(p1, p2, p3, p4)
True
>>> Point3D.are_collinear(p1, p2, p3, p5)
False
```

static are_coplanar()

This function tests whether passed points are coplanar or not. It uses the fact that the triple scalar product of three vectors vanishes if the vectors are coplanar. Which means that the volume of the solid described by them will have to be zero for coplanarity.

Parameters A set of points 3D points

Returns boolean

Examples

```
>>> p1 = Point3D(1, 2, 2)
>>> p2 = Point3D(2, 7, 2)
>>> p3 = Point3D(0, 0, 2)
>>> p4 = Point3D(1, 1, 2)
>>> Point3D.are_coplanar(p1, p2, p3, p4)
True
>>> p5 = Point3D(0, 1, 3)
>>> Point3D.are_coplanar(p1, p2, p3, p5)
False
```

direction_cosine(*point*)

Gives the direction cosine between 2 points

Parameters p : Point3D

Returns list

Examples

```
>>> p1 = Point3D(1, 2, 3)
>>> p1.direction_cosine(Point3D(2, 3, 5))
[sqrt(6)/6, sqrt(6)/6, sqrt(6)/3]
```

direction_ratio(*point*)

Gives the direction ratio between 2 points

Parameters p : Point3D

Returns list

Examples

```
>>> p1 = Point3D(1, 2, 3)
>>> p1.direction_ratio(Point3D(2, 3, 5))
[1, 1, 2]
```

intersection(*o*)

The intersection between this point and another point.

Parameters other : Point

Returns intersection : list of Points

Notes

The return value will either be an empty list if there is no intersection, otherwise it will contain this point.

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 0), Point3D(1, 1, 1), Point3D(0, 0, 0)
>>> p1.intersection(p2)
[]
>>> p1.intersection(p3)
[Point3D(0, 0, 0)]
```

scale(*x=1, y=1, z=1, pt=None*)

Scale the coordinates of the Point by multiplying by `x` and `y` after subtracting `pt` – default is (0, 0) – and then adding `pt` back again (i.e. `pt` is the point of reference for the scaling).

See also:

translate (page 440)

Examples

```
>>> t = Point3D(1, 1, 1)
>>> t.scale(2)
Point3D(2, 1, 1)
>>> t.scale(2, 2)
Point3D(2, 2, 1)
```

transform(*matrix*)

Return the point after applying the transformation described by the 4x4 Matrix, `matrix`.

See also:

diofant.geometry.entity.GeometryEntity.rotate (page 427), *diofant.geometry.entity.GeometryEntity.scale* (page 427), *diofant.geometry.entity.GeometryEntity.translate* (page 427)

translate(*x=0, y=0, z=0*)

Shift the Point by adding x and y to the coordinates of the Point.

See also:

diofant.geometry.entity.GeometryEntity.rotate (page 427), *diofant.geometry.entity.GeometryEntity.scale* (page 427)

Examples

```
>>> t = Point3D(0, 1, 1)
>>> t.translate(2)
Point3D(2, 1, 1)
>>> t.translate(2, 2)
Point3D(2, 3, 1)
>>> t + Point3D(2, 2, 2)
Point3D(2, 3, 3)
```

x

Returns the X coordinate of the Point.

Examples

```
>>> p = Point3D(0, 1, 3)
>>> p.x
0
```

y

Returns the Y coordinate of the Point.

Examples

```
>>> p = Point3D(0, 1, 2)
>>> p.y
1
```

z

Returns the Z coordinate of the Point.

Examples

```
>>> p = Point3D(0, 1, 1)
>>> p.z
1
```

Lines

class diofant.geometry.line.**LinearEntity**

A base class for all linear entities (line, ray and segment) in a 2-dimensional Euclidean space.

See also:

diofant.geometry.entity.GeometryEntity (page 426)

Notes

This is an abstract class and is not meant to be instantiated.

Attributes

p1 (page 446)	The first defining point of a linear entity.
p2 (page 446)	The second defining point of a linear entity.
coefficients (page 443)	The coefficients (a, b, c) for $ax + by + c = 0$.
slope (page 449)	The slope of this linear entity, or infinity if vertical.
points (page 448)	The two points used to define this linear entity.

ambient_dimension
What is the dimension of the space that the object is contained in?

angle_between(*other*)
The angle formed between the two linear entities.

Parameters self : LinearEntity

other : LinearEntity

Returns angle : angle in radians

See also:

is_perpendicular (page 445)

Notes

From the dot product of vectors v1 and v2 it is known that:

```
dot(v1, v2) = |v1|*|v2|*cos(A)
```

where A is the angle formed between the two vectors. We can get the directional vectors of the two lines and readily find the angle between the two using the above formula.

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(0, 4), Point(2, 0)
>>> l1, l2 = Line(p1, p2), Line(p1, p3)
>>> l1.angle_between(l2)
pi/2
```

arbitrary_point(*parameter='t'*)
A parameterized point on the Line.

Parameters parameter : str, optional

The name of the parameter which will be used for the parametric point. The default value is 't'. When this parameter is 0, the first point used to define the line will be returned, and when it is 1 the second point will be returned.

Returns point : Point

Raises ValueError

When `parameter` already appears in the Line's definition.

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> p1, p2 = Point(1, 0), Point(5, 3)
>>> l1 = Line(p1, p2)
>>> l1.arbitrary_point()
Point2D(4*t + 1, 3*t)
```

static are_concurrent()

Is a sequence of linear entities concurrent?

Two or more linear entities are concurrent if they all intersect at a single point.

Parameters lines : a sequence of linear entities.

Returns True : if the set of linear entities are concurrent,

False : otherwise.

See also:

diofant.geometry.util.intersection (page 428)

Notes

Simply take the first two lines and find their intersection. If there is no intersection, then the first two lines were parallel and had no intersection so concurrency is impossible amongst the whole set. Otherwise, check to see if the intersection point of the first two lines is a member on the rest of the lines. If so, the lines are concurrent.

Examples

```
>>> p1, p2 = Point(0, 0), Point(3, 5)
>>> p3, p4 = Point(-2, -2), Point(0, 2)
>>> l1, l2, l3 = Line(p1, p2), Line(p1, p3), Line(p1, p4)
>>> Line.are_concurrent(l1, l2, l3)
True
```

```
>>> l4 = Line(p2, p3)
>>> Line.are_concurrent(l2, l3, l4)
False
```

coefficients

The coefficients (a, b, c) for $ax + by + c = 0$.

See also:

diofant.geometry.line.Line.equation (page 451)

Examples

```
>>> p1, p2 = Point(0, 0), Point(5, 3)
>>> l = Line(p1, p2)
>>> l.coefficients
(-3, 5, 0)
```

```
>>> p3 = Point(x, y)
>>> l2 = Line(p1, p3)
>>> l2.coefficients
(-y, x, 0)
```

contains(*other*)

Subclasses should implement this method and should return True if other is on the boundaries of self; False if not on the boundaries of self; None if a determination cannot be made.

intersection(*o*)

The intersection with another geometrical entity.

Parameters o : Point or LinearEntity

Returns intersection : list of geometrical entities

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 1), Point(7, 7)
>>> l1 = Line(p1, p2)
>>> l1.intersection(p3)
[Point2D(7, 7)]
```

```
>>> p4, p5 = Point(5, 0), Point(0, 3)
>>> l2 = Line(p4, p5)
>>> l1.intersection(l2)
[Point2D(15/8, 15/8)]
```

```
>>> p6, p7 = Point(0, 5), Point(2, 6)
>>> s1 = Segment(p6, p7)
>>> l1.intersection(s1)
[]
```

is_parallel(*other*)

Are two linear entities parallel?

Parameters self : LinearEntity

other : LinearEntity

Returns True : if self and other are parallel,

False : otherwise.

See also:

coefficients (page 443)

Examples

```
>>> p1, p2 = Point(0, 0), Point(1, 1)
>>> p3, p4 = Point(3, 4), Point(6, 7)
>>> l1, l2 = Line(p1, p2), Line(p3, p4)
>>> Line.is_parallel(l1, l2)
True
```

```
>>> p5 = Point(6, 6)
>>> l3 = Line(p3, p5)
>>> Line.is_parallel(l1, l3)
False
```

is_perpendicular(*other*)

Are two linear entities perpendicular?

Parameters self : LinearEntity

other : LinearEntity

Returns True : if self and other are perpendicular,

False : otherwise.

See also:

coefficients (page 443)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 1), Point(-1, 1)
>>> l1, l2 = Line(p1, p2), Line(p1, p3)
>>> l1.is_perpendicular(l2)
True
```

```
>>> p4 = Point(5, 3)
>>> l3 = Line(p1, p4)
>>> l1.is_perpendicular(l3)
False
```

is_similar(*other*)

Return True if self and other are contained in the same line.

Examples

```
>>> p1, p2, p3 = Point(0, 1), Point(3, 4), Point(2, 3)
>>> l1 = Line(p1, p2)
>>> l2 = Line(p1, p3)
>>> l1.is_similar(l2)
True
```

length

The length of the line.

Examples

```
>>> p1, p2 = Point(0, 0), Point(3, 5)
>>> l1 = Line(p1, p2)
>>> l1.length
oo
```

p1

The first defining point of a linear entity.

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> p1, p2 = Point(0, 0), Point(5, 3)
>>> l = Line(p1, p2)
>>> l.p1
Point2D(0, 0)
```

p2

The second defining point of a linear entity.

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> p1, p2 = Point(0, 0), Point(5, 3)
>>> l = Line(p1, p2)
>>> l.p2
Point2D(5, 3)
```

parallel_line(p)

Create a new Line parallel to this linear entity which passes through the point p.

Parameters p : Point

Returns line : Line

See also:

is_parallel (page 444)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(2, 3), Point(-2, 2)
>>> l1 = Line(p1, p2)
>>> l2 = l1.parallel_line(p3)
>>> p3 in l2
True
>>> l1.is_parallel(l2)
True
```

perpendicular_line(*p*)

Create a new Line perpendicular to this linear entity which passes through the point p.

Parameters p : Point

Returns line : Line

See also:

is_perpendicular (page 445), *perpendicular_segment* (page 447)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(2, 3), Point(-2, 2)
>>> l1 = Line(p1, p2)
>>> l2 = l1.perpendicular_line(p3)
>>> p3 in l2
True
>>> l1.is_perpendicular(l2)
True
```

perpendicular_segment(*p*)

Create a perpendicular line segment from p to this line.

The enpoints of the segment are `p` and the closest point in the line containing self. (If self is not a line, the point might not be in self.)

Parameters p : Point

Returns segment : Segment

See also:

perpendicular_line (page 447)

Notes

Returns p itself if p is on this linear entity.

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 1), Point(0, 2)
>>> l1 = Line(p1, p2)
>>> s1 = l1.perpendicular_segment(p3)
>>> l1.is_perpendicular(s1)
True
>>> p3 in s1
True
>>> l1.perpendicular_segment(Point(4, 0))
Segment(Point2D(2, 2), Point2D(4, 0))
```

points

The two points used to define this linear entity.

Returns points : tuple of Points

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> p1, p2 = Point(0, 0), Point(5, 11)
>>> l1 = Line(p1, p2)
>>> l1.points
(Point2D(0, 0), Point2D(5, 11))
```

projection(o)

Project a point, line, ray, or segment onto this linear entity.

Parameters other : Point or LinearEntity (Line, Ray, Segment)

Returns projection : Point or LinearEntity (Line, Ray, Segment)

The return type matches the type of the parameter `other`.

Raises GeometryError

When method is unable to perform projection.

See also:

diofant.geometry.point.Point (page 431), *perpendicular_line* (page 447)

Notes

A projection involves taking the two points that define the linear entity and projecting those points onto a Line and then reforming the linear entity using these projections. A point P is projected onto a line L by finding the point on L that is closest to P. This point is the intersection of L and the line perpendicular to L that passes through P.

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 1), Point(Rational(1, 2), 0)
>>> l1 = Line(p1, p2)
>>> l1.projection(p3)
Point2D(1/4, 1/4)
```

```
>>> p4, p5 = Point(10, 0), Point(12, 1)
>>> s1 = Segment(p4, p5)
>>> l1.projection(s1)
Segment(Point2D(5, 5), Point2D(13/2, 13/2))
```

random_point()

A random point on a LinearEntity.

Returns point : Point

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> p1, p2 = Point(0, 0), Point(5, 3)
>>> l1 = Line(p1, p2)
>>> p3 = l1.random_point()
>>> # random point - don't know its coords in advance
>>> p3
Point2D(...)
>>> # point should belong to the line
>>> p3 in l1
True
```

slope

The slope of this linear entity, or infinity if vertical.

Returns slope : number or diofant expression

See also:

coefficients (page 443)

Examples

```
>>> p1, p2 = Point(0, 0), Point(3, 5)
>>> l1 = Line(p1, p2)
>>> l1.slope
5/3
```

```
>>> p3 = Point(0, 4)
>>> l2 = Line(p1, p3)
>>> l2.slope
oo
```

class diofant.geometry.line.**Line**

An infinite line in space.

A line is declared with two distinct points or a point and slope as defined using keyword *slope*.

Parameters p1 : Point

pt : Point

slope : diofant expression

See also:

diofant.geometry.point.Point (page 431)

Notes

At the moment only lines in a 2D space can be declared, because Points can be defined only for 2D spaces.

Examples

```
>>> from diofant.abc import L
>>> L = Line(Point(2, 3), Point(3, 5))
>>> L
Line(Point2D(2, 3), Point2D(3, 5))
>>> L.points
(Point2D(2, 3), Point2D(3, 5))
>>> L.equation()
-2*x + y + 1
>>> L.coefficients
(-2, 1, 1)
```

Instantiate with keyword slope:

```
>>> Line(Point(0, 0), slope=0)
Line(Point2D(0, 0), Point2D(1, 0))
```

Instantiate with another linear object

```
>>> s = Segment((0, 0), (0, 1))
>>> Line(s).equation()
x
```

contains(*o*)

Return True if o is on this Line, or False otherwise.

Examples

```
>>> p1, p2 = Point(0, 1), Point(3, 4)
>>> l = Line(p1, p2)
>>> l.contains(p1)
True
```

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```
>>> l.contains((0, 1))
True
>>> l.contains((0, 0))
False
```

distance(*o*)

Finds the shortest distance between a line and a point.

Raises NotImplementedError is raised if o is not a Point

Examples

```
>>> p1, p2 = Point(0, 0), Point(1, 1)
>>> s = Line(p1, p2)
>>> s.distance(Point(-1, 1))
sqrt(2)
>>> s.distance((-1, 2))
3*sqrt(2)/2
```

equal(*other*)

Returns True if self and other are the same mathematical entities

equation(*x='x'*, *y='y'*)

The equation of the line: ax + by + c.

Parameters x : str, optional

The name to use for the x-axis, default value is 'x'.

y : str, optional

The name to use for the y-axis, default value is 'y'.

Returns equation : diofant expression

See also:

LinearEntity.coefficients (page 443)

Examples

```
>>> p1, p2 = Point(1, 0), Point(5, 3)
>>> l1 = Line(p1, p2)
>>> l1.equation()
-3*x + 4*y + 3
```

plot_interval(*parameter='t'*)

The plot interval for the default geometric plot of line. Gives values that will produce a line that is +/- 5 units long (where a unit is the distance between the two points that define the line).

Parameters parameter : str, optional

Default value is 't'.

Returns plot_interval : list (plot interval)

[parameter, lower_bound, upper_bound]

Examples

```
>>> p1, p2 = Point(0, 0), Point(5, 3)
>>> l1 = Line(p1, p2)
>>> l1.plot_interval()
[t, -5, 5]
```

class diofant.geometry.line.**Ray**

A Ray is a semi-line in the space with a source point and a direction.

Parameters p1 : Point

The source of the Ray

p2 : Point or radian value

This point determines the direction in which the Ray propagates. If given as an angle it is interpreted in radians with the positive direction being ccw.

See also:

diofant.geometry.point.Point (page 431), *Line* (page 449)

Notes

At the moment only rays in a 2D space can be declared, because Points can be defined only for 2D spaces.

Examples

```
>>> from diofant.abc import r
>>> r = Ray(Point(2, 3), Point(3, 5))
>>> r = Ray(Point(2, 3), Point(3, 5))
>>> r
Ray(Point2D(2, 3), Point2D(3, 5))
>>> r.points
(Point2D(2, 3), Point2D(3, 5))
>>> r.source
Point2D(2, 3)
>>> r.xdirection
oo
>>> r.ydirection
oo
>>> r.slope
2
>>> Ray(Point(0, 0), angle=pi/4).slope
1
```

Attributes

source (page 454)	The point from which the ray emanates.

Continued on next page

Table 5 – continued from previous page

xdirection (page 454)	The x direction of the ray.
ydirection (page 455)	The y direction of the ray.

contains(*o*)
Is other GeometryEntity contained in this Ray?

Examples

```
>>> p1, p2 = Point(0, 0), Point(4, 4)
>>> r = Ray(p1, p2)
>>> r.contains(p1)
True
>>> r.contains((1, 1))
True
>>> r.contains((1, 3))
False
>>> s = Segment((1, 1), (2, 2))
>>> r.contains(s)
True
>>> s = Segment((1, 2), (2, 5))
>>> r.contains(s)
False
>>> r1 = Ray((2, 2), (3, 3))
>>> r.contains(r1)
True
>>> r1 = Ray((2, 2), (3, 5))
>>> r.contains(r1)
False
```

direction
The direction in which the ray emanates.

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> p1, p2 = Point(0, 0), Point(4, 1)
>>> r1 = Ray(p1, p2)
>>> r1.direction
Point2D(4, 1)
```

distance(*o*)
Finds the shortest distance between the ray and a point.

Raises NotImplementedError is raised if o is not a Point

Examples

```
>>> p1, p2 = Point(0, 0), Point(1, 1)
>>> s = Ray(p1, p2)
>>> s.distance(Point(-1, -1))
sqrt(2)
>>> s.distance((-1, 2))
3*sqrt(2)/2
```

equals(*other*)
Returns True if self and other are the same mathematical entities

plot_interval(*parameter='t'*)
The plot interval for the default geometric plot of the Ray. Gives values that will produce a ray that is 10 units long (where a unit is the distance between the two points that define the ray).

Parameters parameter : str, optional

Default value is 't'.

Returns plot_interval : list

[parameter, lower_bound, upper_bound]

Examples

```
>>> r = Ray((0, 0), angle=pi/4)
>>> r.plot_interval()
[t, 0, 10]
```

source
The point from which the ray emanates.

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> p1, p2 = Point(0, 0), Point(4, 1)
>>> r1 = Ray(p1, p2)
>>> r1.source
Point2D(0, 0)
```

xdirection
The x direction of the ray.

Positive infinity if the ray points in the positive x direction, negative infinity if the ray points in the negative x direction, or 0 if the ray is vertical.

See also:

ydirection (page 455)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 1), Point(0, -1)
>>> r1, r2 = Ray(p1, p2), Ray(p1, p3)
>>> r1.xdirection
oo
>>> r2.xdirection
0
```

ydirection

The y direction of the ray.

Positive infinity if the ray points in the positive y direction, negative infinity if the ray points in the negative y direction, or 0 if the ray is horizontal.

See also:

xdirection (page 454)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(-1, -1), Point(-1, 0)
>>> r1, r2 = Ray(p1, p2), Ray(p1, p3)
>>> r1.ydirection
-oo
>>> r2.ydirection
0
```

class diofant.geometry.line.**Segment**

A undirected line segment in space.

Parameters p1 : Point

p2 : Point

See also:

diofant.geometry.point.Point (page 431), *Line* (page 449)

Notes

At the moment only segments in a 2D space can be declared, because Points can be defined only for 2D spaces.

Examples

```
>>> from diofant.abc import s
>>> Segment((1, 0), (1, 1)) # tuples are interpreted as pts
Segment(Point2D(1, 0), Point2D(1, 1))
>>> s = Segment(Point(4, 3), Point(1, 1))
>>> s
Segment(Point2D(1, 1), Point2D(4, 3))
>>> s.points
(Point2D(1, 1), Point2D(4, 3))
```

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```
>>> s.slope
2/3
>>> s.length
sqrt(13)
>>> s.midpoint
Point2D(5/2, 2)
```

Attributes

length (page 456)	The length of the line segment.
midpoint (page 457)	The midpoint of the line segment.

contains(*other*)
Is the other GeometryEntity contained within this Segment?

Examples

```
>>> p1, p2 = Point(0, 1), Point(3, 4)
>>> s = Segment(p1, p2)
>>> s2 = Segment(p2, p1)
>>> s.contains(s2)
True
```

distance(*o*)
Finds the shortest distance between a line segment and a point.

Raises NotImplementedError is raised if o is not a Point

Examples

```
>>> p1, p2 = Point(0, 1), Point(3, 4)
>>> s = Segment(p1, p2)
>>> s.distance(Point(10, 15))
sqrt(170)
>>> s.distance((0, 12))
sqrt(73)
```

length
The length of the line segment.

See also:

diofant.geometry.point.Point.distance (page 432)

Examples

```
>>> p1, p2 = Point(0, 0), Point(4, 3)
>>> s1 = Segment(p1, p2)
```

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```
>>> s1.length
5
```

midpoint

The midpoint of the line segment.

See also:

diofant.geometry.point.Point.midpoint (page 434)

Examples

```
>>> p1, p2 = Point(0, 0), Point(4, 3)
>>> s1 = Segment(p1, p2)
>>> s1.midpoint
Point2D(2, 3/2)
```

perpendicular_bisector(*p=None*)

The perpendicular bisector of this segment.

If no point is specified or the point specified is not on the bisector then the bisector is returned as a Line. Otherwise a Segment is returned that joins the point specified and the intersection of the bisector and the segment.

Parameters p : Point

Returns bisector : Line or Segment

See also:

LinearEntity.perpendicular_segment (page 447)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(6, 6), Point(5, 1)
>>> s1 = Segment(p1, p2)
>>> s1.perpendicular_bisector()
Line(Point2D(3, 3), Point2D(9, -3))
```

```
>>> s1.perpendicular_bisector(p3)
Segment(Point2D(3, 3), Point2D(5, 1))
```

plot_interval(*parameter='t'*)

The plot interval for the default geometric plot of the Segment gives values that will produce the full segment in a plot.

Parameters parameter : str, optional

Default value is 't'.

Returns plot_interval : list

[parameter, lower_bound, upper_bound]

Examples

```
>>> p1, p2 = Point(0, 0), Point(5, 3)
>>> s1 = Segment(p1, p2)
>>> s1.plot_interval()
[t, 0, 1]
```

3D Line

Line-like geometrical entities.

Contains

LinearEntity3D Line3D Ray3D Segment3D

class `diofant.geometry.line3d.`**`Line3D`**

An infinite 3D line in space.

A line is declared with two distinct points or a point and direction_ratio as defined using keyword $direction_ratio$.

Parameters p1 : Point3D

pt : Point3D

direction_ratio : list

See also:

diofant.geometry.point.Point3D (page 437)

Examples

```
>>> from diofant.abc import L
>>> L = Line3D(Point3D(2, 3, 4), Point3D(3, 5, 1))
>>> L
Line3D(Point3D(2, 3, 4), Point3D(3, 5, 1))
>>> L.points
(Point3D(2, 3, 4), Point3D(3, 5, 1))
```

contains(o)

Return True if o is on this Line, or False otherwise.

Examples

```
>>> a = (0, 0, 0)
>>> b = (1, 1, 1)
>>> c = (2, 2, 2)
>>> l1 = Line3D(a, b)
>>> l2 = Line3D(b, a)
>>> l1 == l2
False
```

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```
>>> l1 in l2
True
```

distance(*o*)

Finds the shortest distance between a line and a point.

Raises NotImplementedError is raised if o is not an instance of Point3D

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(1, 1, 1)
>>> s = Line3D(p1, p2)
>>> s.distance(Point3D(-1, 1, 1))
2*sqrt(6)/3
>>> s.distance((-1, 1, 1))
2*sqrt(6)/3
```

equals(*other*)

Returns True if self and other are the same mathematical entities

equation(*x='x', y='y', z='z', k='k'*)

The equation of the line in 3D

Parameters x : str, optional

The name to use for the x-axis, default value is 'x'.

y : str, optional

The name to use for the y-axis, default value is 'y'.

z : str, optional

The name to use for the x-axis, default value is 'z'.

Returns equation : tuple

Examples

```
>>> p1, p2 = Point3D(1, 0, 0), Point3D(5, 3, 0)
>>> l1 = Line3D(p1, p2)
>>> l1.equation()
(x/4 - 1/4, y/3, zoo*z, k)
```

plot_interval(*parameter='t'*)

The plot interval for the default geometric plot of line. Gives values that will produce a line that is +/- 5 units long (where a unit is the distance between the two points that define the line).

Parameters parameter : str, optional

Default value is 't'.

Returns plot_interval : list (plot interval)

[parameter, lower_bound, upper_bound]

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(5, 3, 1)
>>> l1 = Line3D(p1, p2)
>>> l1.plot_interval()
[t, -5, 5]
```

class diofant.geometry.line3d.**LinearEntity3D**

An base class for all linear entities (line, ray and segment) in a 3-dimensional Euclidean space.

Notes

This is a base class and is not meant to be instantiated.

Attributes

p1 (page 464)	The first defining point of a linear entity.
p2 (page 464)	The second defining point of a linear entity.
direction_ratio (page 462)	The direction ratio of a given line in 3D.
direction_cosine (page 462)	The normalized direction ratio of a given line in 3D.
points (page 466)	The two points used to define this linear entity.

angle_between(*other*)

The angle formed between the two linear entities.

Parameters self : LinearEntity

other : LinearEntity

Returns angle : angle in radians

See also:

is_perpendicular (page 463)

Notes

From the dot product of vectors v1 and v2 it is known that:

```
dot(v1, v2) = |v1|*|v2|*cos(A)
```

where A is the angle formed between the two vectors. We can get the directional vectors of the two lines and readily find the angle between the two using the above formula.

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 0), Point3D(1, 1, 1), Point3D(-1, 2, 0)
>>> l1, l2 = Line3D(p1, p2), Line3D(p2, p3)
>>> l1.angle_between(l2)
acos(-sqrt(2)/3)
```

arbitrary_point(*parameter='t'*)

A parameterized point on the Line.

Parameters parameter : str, optional

The name of the parameter which will be used for the parametric point. The default value is 't'. When this parameter is 0, the first point used to define the line will be returned, and when it is 1 the second point will be returned.

Returns point : Point3D

Raises ValueError

When `parameter` already appears in the Line's definition.

See also:

diofant.geometry.point.Point3D (page 437)

Examples

```
>>> p1, p2 = Point3D(1, 0, 0), Point3D(5, 3, 1)
>>> l1 = Line3D(p1, p2)
>>> l1.arbitrary_point()
Point3D(4*t + 1, 3*t, t)
```

static are_concurrent()

Is a sequence of linear entities concurrent?

Two or more linear entities are concurrent if they all intersect at a single point.

Parameters lines : a sequence of linear entities.

Returns True : if the set of linear entities are concurrent,

False : otherwise.

See also:

diofant.geometry.util.intersection (page 428)

Notes

Simply take the first two lines and find their intersection. If there is no intersection, then the first two lines were parallel and had no intersection so concurrency is impossible amongst the whole set. Otherwise, check to see if the intersection point of the first two lines is a member on the rest of the lines. If so, the lines are concurrent.

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(3, 5, 2)
>>> p3, p4 = Point3D(-2, -2, -2), Point3D(0, 2, 1)
>>> l1, l2, l3 = Line3D(p1, p2), Line3D(p1, p3), Line3D(p1, p4)
>>> Line3D.are_concurrent(l1, l2, l3)
True
```

```
>>> l4 = Line3D(p2, p3)
>>> Line3D.are_concurrent(l2, l3, l4)
False
```

contains(*other*)

Subclasses should implement this method and should return True if other is on the boundaries of self; False if not on the boundaries of self; None if a determination cannot be made.

direction_cosine

The normalized direction ratio of a given line in 3D.

See also:

diofant.geometry.line.Line.equation (page 451)

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(5, 3, 1)
>>> l = Line3D(p1, p2)
>>> l.direction_cosine
[sqrt(35)/7, 3*sqrt(35)/35, sqrt(35)/35]
>>> sum(i**2 for i in _)
1
```

direction_ratio

The direction ratio of a given line in 3D.

See also:

diofant.geometry.line.Line.equation (page 451)

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(5, 3, 1)
>>> l = Line3D(p1, p2)
>>> l.direction_ratio
[5, 3, 1]
```

intersection(*o*)

The intersection with another geometrical entity.

Parameters o : Point or LinearEntity3D

Returns intersection : list of geometrical entities

See also:

diofant.geometry.point.Point3D (page 437)

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 0), Point3D(1, 1, 1), Point3D(7, 7, 7)
>>> l1 = Line3D(p1, p2)
>>> l1.intersection(p3)
[Point3D(7, 7, 7)]
```

```
>>> l1 = Line3D(Point3D(4, 19, 12), Point3D(5, 25, 17))
>>> l2 = Line3D(Point3D(-3, -15, -19), direction_ratio=[2, 8, 8])
>>> l1.intersection(l2)
[Point3D(1, 1, -3)]
```

```
>>> p6, p7 = Point3D(0, 5, 2), Point3D(2, 6, 3)
>>> s1 = Segment3D(p6, p7)
>>> l1.intersection(s1)
[]
```

is_parallel(*other*)

Are two linear entities parallel?

Parameters **self** : LinearEntity

other : LinearEntity

Returns **True** : if self and other are parallel,

False : otherwise.

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(3, 4, 5)
>>> p3, p4 = Point3D(2, 1, 1), Point3D(8, 9, 11)
>>> l1, l2 = Line3D(p1, p2), Line3D(p3, p4)
>>> Line3D.is_parallel(l1, l2)
True
```

```
>>> p5 = Point3D(6, 6, 6)
>>> l3 = Line3D(p3, p5)
>>> Line3D.is_parallel(l1, l3)
False
```

is_perpendicular(*other*)

Are two linear entities perpendicular?

Parameters **self** : LinearEntity

other : LinearEntity

Returns **True** : if self and other are perpendicular,

False : otherwise.

See also:

direction_ratio (page 462)

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 0), Point3D(1, 1, 1), Point3D(-1, 2, 0)
>>> l1, l2 = Line3D(p1, p2), Line3D(p2, p3)
>>> l1.is_perpendicular(l2)
False
```

```
>>> p4 = Point3D(5, 3, 7)
>>> l3 = Line3D(p1, p4)
>>> l1.is_perpendicular(l3)
False
```

is_similar(*other*)
Return True if self and other are contained in the same line.

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 0), Point3D(1, 1, 1), Point3D(2, 2, 2)
>>> l1 = Line3D(p1, p2)
>>> l2 = Line3D(p1, p3)
>>> l1.is_similar(l2)
True
```

length
The length of the line.

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(3, 5, 1)
>>> l1 = Line3D(p1, p2)
>>> l1.length
oo
```

p1
The first defining point of a linear entity.

See also:

diofant.geometry.point.Point3D (page 437)

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(5, 3, 1)
>>> l = Line3D(p1, p2)
>>> l.p1
Point3D(0, 0, 0)
```

p2
The second defining point of a linear entity.

See also:

diofant.geometry.point.Point3D (page 437)

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(5, 3, 1)
>>> l = Line3D(p1, p2)
>>> l.p2
Point3D(5, 3, 1)
```

parallel_line(*p*)

Create a new Line parallel to this linear entity which passes through the point p.

Parameters p : Point3D

Returns line : Line3D

See also:

is_parallel (page 463)

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 0), Point3D(2, 3, 4), Point3D(-2, 2, 0)
>>> l1 = Line3D(p1, p2)
>>> l2 = l1.parallel_line(p3)
>>> p3 in l2
True
>>> l1.is_parallel(l2)
True
```

perpendicular_line(*p*)

Create a new Line perpendicular to this linear entity which passes through the point p.

Parameters p : Point3D

Returns line : Line3D

See also:

is_perpendicular (page 463), *perpendicular_segment* (page 465)

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 0), Point3D(2, 3, 4), Point3D(-2, 2, 0)
>>> l1 = Line3D(p1, p2)
>>> l2 = l1.perpendicular_line(p3)
>>> p3 in l2
True
>>> l1.is_perpendicular(l2)
True
```

perpendicular_segment(*p*)

Create a perpendicular line segment from p to this line.

The enpoints of the segment are p and the closest point in the line containing self. (If self is not a line, the point might not be in self.)

Parameters p : Point3D

Returns segment : Segment3D

See also:

perpendicular_line (page 465)

Notes

Returns p itself if p is on this linear entity.

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 0), Point3D(1, 1, 1), Point3D(0, 2, 0)
>>> l1 = Line3D(p1, p2)
>>> s1 = l1.perpendicular_segment(p3)
>>> l1.is_perpendicular(s1)
True
>>> p3 in s1
True
>>> l1.perpendicular_segment(Point3D(4, 0, 0))
Segment3D(Point3D(4/3, 4/3, 4/3), Point3D(4, 0, 0))
```

points

The two points used to define this linear entity.

Returns points : tuple of Points

See also:

diofant.geometry.point.Point3D (page 437)

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(5, 11, 1)
>>> l1 = Line3D(p1, p2)
>>> l1.points
(Point3D(0, 0, 0), Point3D(5, 11, 1))
```

projection(o)

Project a point, line, ray, or segment onto this linear entity.

Parameters other : Point or LinearEntity (Line, Ray, Segment)

Returns projection : Point or LinearEntity (Line, Ray, Segment)

The return type matches the type of the parameter `other`.

Raises GeometryError

When method is unable to perform projection.

See also:

diofant.geometry.point.Point3D (page 437), *perpendicular_line* (page 465)

Notes

A projection involves taking the two points that define the linear entity and projecting those points onto a Line and then reforming the linear entity using these projections. A point P is projected onto a line L by finding the point on L that is closest to P. This point is the intersection of L and the line perpendicular to L that passes through P.

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 1), Point3D(1, 1, 2), Point3D(2, 0, 1)
>>> l1 = Line3D(p1, p2)
>>> l1.projection(p3)
Point3D(2/3, 2/3, 5/3)
```

```
>>> p4, p5 = Point3D(10, 0, 1), Point3D(12, 1, 3)
>>> s1 = Segment3D(p4, p5)
>>> l1.projection(s1)
[Segment3D(Point3D(10/3, 10/3, 13/3), Point3D(5, 5, 6))]
```

class diofant.geometry.line3d.**Ray3D**

A Ray is a semi-line in the space with a source point and a direction.

Parameters p1 : Point3D

The source of the Ray

p2 : Point or a direction vector

direction_ratio: Determines the direction in which the Ray propagates.

See also:

diofant.geometry.point.Point3D (page 437), *Line3D* (page 458)

Examples

```
>>> from diofant.abc import r
>>> r = Ray3D(Point3D(2, 3, 4), Point3D(3, 5, 0))
>>> r
Ray3D(Point3D(2, 3, 4), Point3D(3, 5, 0))
>>> r.points
(Point3D(2, 3, 4), Point3D(3, 5, 0))
>>> r.source
Point3D(2, 3, 4)
>>> r.xdirection
oo
>>> r.ydirection
oo
>>> r.direction_ratio
[1, 2, -4]
```

Attributes

source (page 468)	The point from which the ray emanates.
xdirection (page 469)	The x direction of the ray.
ydirection (page 469)	The y direction of the ray.
zdirection (page 469)	The z direction of the ray.

contains(*o*)
Is other GeometryEntity contained in this Ray?

distance(*o*)
Finds the shortest distance between the ray and a point.

Raises NotImplementedError is raised if o is not a Point

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(1, 1, 2)
>>> s = Ray3D(p1, p2)
>>> s.distance(Point3D(-1, -1, 2))
sqrt(6)
>>> s.distance((-1, -1, 2))
sqrt(6)
```

equals(*other*)
Returns True if self and other are the same mathematical entities

plot_interval(*parameter='t'*)
The plot interval for the default geometric plot of the Ray. Gives values that will produce a ray that is 10 units long (where a unit is the distance between the two points that define the ray).

Parameters parameter : str, optional

Default value is 't'.

Returns plot_interval : list

[parameter, lower_bound, upper_bound]

Examples

```
>>> r = Ray3D(Point3D(0, 0, 0), Point3D(1, 1, 1))
>>> r.plot_interval()
[t, 0, 10]
```

source
The point from which the ray emanates.

See also:

diofant.geometry.point.Point3D (page 437)

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(4, 1, 5)
>>> r1 = Ray3D(p1, p2)
>>> r1.source
Point3D(0, 0, 0)
```

xdirection

The x direction of the ray.

Positive infinity if the ray points in the positive x direction, negative infinity if the ray points in the negative x direction, or 0 if the ray is vertical.

See also:

ydirection (page 469)

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 0), Point3D(1, 1, 1), Point3D(0, -1, 0)
>>> r1, r2 = Ray3D(p1, p2), Ray3D(p1, p3)
>>> r1.xdirection
oo
>>> r2.xdirection
0
```

ydirection

The y direction of the ray.

Positive infinity if the ray points in the positive y direction, negative infinity if the ray points in the negative y direction, or 0 if the ray is horizontal.

See also:

xdirection (page 469)

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 0), Point3D(-1, -1, -1), Point3D(-1, 0, 0)
>>> r1, r2 = Ray3D(p1, p2), Ray3D(p1, p3)
>>> r1.ydirection
-oo
>>> r2.ydirection
0
```

zdirection

The z direction of the ray.

Positive infinity if the ray points in the positive z direction, negative infinity if the ray points in the negative z direction, or 0 if the ray is horizontal.

See also:

xdirection (page 469)

Examples

```
>>> p1, p2, p3 = Point3D(0, 0, 0), Point3D(-1, -1, -1), Point3D(-1, 0, 0)
>>> r1, r2 = Ray3D(p1, p2), Ray3D(p1, p3)
>>> r1.ydirection
-oo
>>> r2.ydirection
0
>>> r2.zdirection
0
```

class diofant.geometry.line3d.**Segment3D**

A undirected line segment in a 3D space.

Parameters p1 : Point3D

p2 : Point3D

See also:

diofant.geometry.point.Point3D (page 437), *Line3D* (page 458)

Examples

```
>>> from diofant.abc import s
>>> Segment3D((1, 0, 0), (1, 1, 1)) # tuples are interpreted as pts
Segment3D(Point3D(1, 0, 0), Point3D(1, 1, 1))
>>> s = Segment3D(Point3D(4, 3, 9), Point3D(1, 1, 7))
>>> s
Segment3D(Point3D(1, 1, 7), Point3D(4, 3, 9))
>>> s.points
(Point3D(1, 1, 7), Point3D(4, 3, 9))
>>> s.length
sqrt(17)
>>> s.midpoint
Point3D(5/2, 2, 8)
```

Attributes

length (page 471)	The length of the line segment.
midpoint (page 471)	The midpoint of the line segment.

contains(*other*)

Is the other GeometryEntity contained within this Segment?

Examples

```
>>> p1, p2 = Point3D(0, 1, 1), Point3D(3, 4, 5)
>>> s = Segment3D(p1, p2)
>>> s2 = Segment3D(p2, p1)
>>> s.contains(s2)
True
```

distance(*o*)
Finds the shortest distance between a line segment and a point.

Raises NotImplementedError is raised if o is not a Point3D

Examples

```
>>> p1, p2 = Point3D(0, 0, 3), Point3D(1, 1, 4)
>>> s = Segment3D(p1, p2)
>>> s.distance(Point3D(10, 15, 12))
sqrt(341)
>>> s.distance((10, 15, 12))
sqrt(341)
```

length
The length of the line segment.

See also:

diofant.geometry.point.Point.distance (page 432)

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(4, 3, 3)
>>> s1 = Segment3D(p1, p2)
>>> s1.length
sqrt(34)
```

midpoint
The midpoint of the line segment.

See also:

diofant.geometry.point.Point.midpoint (page 434)

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(4, 3, 3)
>>> s1 = Segment3D(p1, p2)
>>> s1.midpoint
Point3D(2, 3/2, 3/2)
```

plot_interval(*parameter='t'*)
The plot interval for the default geometric plot of the Segment gives values that will produce the full segment in a plot.

Parameters parameter : str, optional

Default value is 't'.

Returns plot_interval : list

[parameter, lower_bound, upper_bound]

Examples

```
>>> p1, p2 = Point3D(0, 0, 0), Point3D(5, 3, 0)
>>> s1 = Segment3D(p1, p2)
>>> s1.plot_interval()
[t, 0, 1]
```

Curves

class diofant.geometry.curve.**Curve**

A curve in space.

A curve is defined by parametric functions for the coordinates, a parameter and the lower and upper bounds for the parameter value.

Parameters function : list of functions

limits : 3-tuple

Function parameter and lower and upper bounds.

Raises ValueError

When $functions$ are specified incorrectly. When $limits$ are specified incorrectly.

See also:

diofant.core.function.Function (page 132), *diofant.polys.polyfuncs.interpolate* (page 707)

Examples

```
>>> from diofant.abc import t, a
>>> C = Curve((sin(t), cos(t)), (t, 0, 2))
>>> C.functions
(sin(t), cos(t))
>>> C.limits
(t, 0, 2)
>>> C.parameter
t
>>> C = Curve((t, interpolate([1, 4, 9, 16], t)), (t, 0, 1)); C
Curve((t, t**2), (t, 0, 1))
>>> C.subs(t, 4)
Point2D(4, 16)
>>> C.arbitrary_point(a)
Point2D(a, a**2)
```

Attributes

functions (page 473)	The functions specifying the curve.
parameter (page 474)	The curve function variable.
limits (page 474)	The limits for the curve.

arbitrary_point(*parameter='t'*)

A parameterized point on the curve.

Parameters parameter : str or Symbol, optional

Default value is 't'; the Curve's parameter is selected with None or self.parameter otherwise the provided symbol is used.

Returns arbitrary_point : Point

Raises ValueError

When $parameter$ already appears in the functions.

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> from diofant.abc import s
>>> C = Curve([2*s, s**2], (s, 0, 2))
>>> C.arbitrary_point()
Point2D(2*t, t**2)
>>> C.arbitrary_point(C.parameter)
Point2D(2*s, s**2)
>>> C.arbitrary_point(None)
Point2D(2*s, s**2)
>>> C.arbitrary_point(Symbol('a'))
Point2D(2*a, a**2)
```

free_symbols

Return a set of symbols other than the bound symbols used to parametrically define the Curve.

Examples

```
>>> from diofant.abc import t, a
>>> Curve((t, t**2), (t, 0, 2)).free_symbols
set()
>>> Curve((t, t**2), (t, a, 2)).free_symbols
{a}
```

functions

The functions specifying the curve.

Returns functions : list of parameterized coordinate functions.

See also:

parameter (page 474)

Examples

```
>>> from diofant.abc import t
>>> C = Curve((t, t**2), (t, 0, 2))
>>> C.functions
(t, t**2)
```

limits

The limits for the curve.

Returns limits : tuple

Contains parameter and lower and upper limits.

See also:

plot_interval (page 474)

Examples

```
>>> from diofant.abc import t
>>> C = Curve([t, t**3], (t, -2, 2))
>>> C.limits
(t, -2, 2)
```

parameter

The curve function variable.

Returns parameter : Diofant symbol

See also:

functions (page 473)

Examples

```
>>> from diofant.abc import t
>>> C = Curve([t, t**2], (t, 0, 2))
>>> C.parameter
t
```

plot_interval(*parameter='t'*)

The plot interval for the default geometric plot of the curve.

Parameters parameter : str or Symbol, optional

Default value is 't'; otherwise the provided symbol is used.

Returns plot_interval : list (plot interval)

[parameter, lower_bound, upper_bound]

See also:

***limits* (page 474)** Returns limits of the parameter interval

Examples

```
>>> from diofant.abc import t, s
>>> Curve((x, sin(x)), (x, 1, 2)).plot_interval()
[t, 1, 2]
>>> Curve((x, sin(x)), (x, 1, 2)).plot_interval(s)
[s, 1, 2]
```

rotate(*angle=0*, *pt=None*)
Rotate `angle` radians counterclockwise about Point `pt`.

The default pt is the origin, Point(0, 0).

Examples

```
>>> Curve((x, x), (x, 0, 1)).rotate(pi/2)
Curve((-x, x), (x, 0, 1))
```

scale(*x=1*, *y=1*, *pt=None*)
Override GeometryEntity.scale since Curve is not made up of Points.

Examples

```
>>> Curve((x, x), (x, 0, 1)).scale(2)
Curve((2*x, x), (x, 0, 1))
```

translate(*x=0*, *y=0*)
Translate the Curve by (x, y).

Examples

```
>>> Curve((x, x), (x, 0, 1)).translate(1, 2)
Curve((x + 1, x + 2), (x, 0, 1))
```

Ellipses

class diofant.geometry.ellipse.**Ellipse**
An elliptical GeometryEntity.

Parameters center : Point, optional

Default value is Point(0, 0)

hradius : number or Diofant expression, optional

vradius : number or Diofant expression, optional

eccentricity : number or Diofant expression, optional

Two of $hradius$, $vradius$ and $eccentricity$ must be supplied to create an Ellipse. The third is derived from the two supplied.

Raises GeometryError

When *hradius*, *vradius* and *eccentricity* are incorrectly supplied as parameters.

TypeError

When *center* is not a Point.

See also:

Circle (page 486)

Notes

Constructed from a center and two radii, the first being the horizontal radius (along the x-axis) and the second being the vertical radius (along the y-axis).

When symbolic value for hradius and vradius are used, any calculation that refers to the foci or the major or minor axis will assume that the ellipse has its major radius on the x-axis. If this is not true then a manual rotation is necessary.

Examples

```
>>> e1 = Ellipse(Point(0, 0), 5, 1)
>>> e1.hradius, e1.vradius
(5, 1)
>>> e2 = Ellipse(Point(3, 1), hradius=3, eccentricity=Rational(4, 5))
>>> e2
Ellipse(Point2D(3, 1), 3, 9/5)
```

Attributes

center (page 477)	The center of the ellipse.
hradius (page 480)	The horizontal radius of the ellipse.
vradius (page 486)	The vertical radius of the ellipse.
area (page 477)	The area of the ellipse.
circumference (page 478)	The circumference of the ellipse.
eccentricity (page 478)	The eccentricity of the ellipse.
periapsis (page 483)	The periapsis of the ellipse.
apoapsis (page 476)	The apoapsis of the ellipse.
focus_distance (page 480)	The focale distance of the ellipse.
foci (page 479)	The foci of the ellipse.

ambient_dimension

What is the dimension of the space that the object is contained in?

apoapsis

The apoapsis of the ellipse.

The greatest distance between the focus and the contour.

Returns apoapsis : number

See also:

periapsis **(page 483)** Returns shortest distance between foci and contour

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, 1)
>>> e1.apoapsis
2*sqrt(2) + 3
```

arbitrary_point(*parameter='t'*)

A parameterized point on the ellipse.

Parameters parameter : str, optional

Default value is 't'.

Returns arbitrary_point : Point

Raises ValueError

When $parameter$ already appears in the functions.

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> e1 = Ellipse(Point(0, 0), 3, 2)
>>> e1.arbitrary_point()
Point2D(3*cos(t), 2*sin(t))
```

area

The area of the ellipse.

Returns area : number

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, 1)
>>> e1.area
3*pi
```

center

The center of the ellipse.

Returns center : number

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, 1)
>>> e1.center
Point2D(0, 0)
```

circumference
The circumference of the ellipse.

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, 1)
>>> e1.circumference
12*Integral(sqrt((-8*_x**2/9 + 1)/(-_x**2 + 1)), (_x, 0, 1))
```

eccentricity
The eccentricity of the ellipse.

Returns eccentricity : number

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, sqrt(2))
>>> e1.eccentricity
sqrt(7)/3
```

encloses_point(*p*)
Return True if p is enclosed by (is inside of) self.

Parameters p : Point

Returns encloses_point : True, False or None

See also:

diofant.geometry.point.Point (page 431)

Notes

Being on the border of self is considered False.

Examples

```
>>> from diofant.abc import t
>>> e = Ellipse((0, 0), 3, 2)
>>> e.encloses_point((0, 0))
True
>>> e.encloses_point(e.arbitrary_point(t).subs(t, S.Half))
False
```

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```
>>> e.encloses_point((4, 0))
False
```

equation(*x='x'*, *y='y'*)
The equation of the ellipse.

Parameters x : str, optional

Label for the x-axis. Default value is 'x'.

y : str, optional

Label for the y-axis. Default value is 'y'.

Returns equation : diofant expression

See also:

arbitrary_point **(page 477)** Returns parameterized point on ellipse

Examples

```
>>> e1 = Ellipse(Point(1, 0), 3, 2)
>>> e1.equation()
y**2/4 + (x/3 - 1/3)**2 - 1
```

evolute(*x='x'*, *y='y'*)
The equation of evolute of the ellipse.

Parameters x : str, optional

Label for the x-axis. Default value is 'x'.

y : str, optional

Label for the y-axis. Default value is 'y'.

Returns equation : diofant expression

Examples

```
>>> e1 = Ellipse(Point(1, 0), 3, 2)
>>> e1.evolute()
2**(2/3)*y**(2/3) + (3*x - 3)**(2/3) - 5**(2/3)
```

foci
The foci of the ellipse.

Raises ValueError

When the major and minor axis cannot be determined.

See also:

diofant.geometry.point.Point (page 431)

focus_distance **(page 480)** Returns the distance between focus and center

Notes

The foci can only be calculated if the major/minor axes are known.

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, 1)
>>> e1.foci
(Point2D(-2*sqrt(2), 0), Point2D(2*sqrt(2), 0))
```

focus_distance

The focale distance of the ellipse.

The distance between the center and one focus.

Returns focus_distance : number

See also:

foci (page 479)

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, 1)
>>> e1.focus_distance
2*sqrt(2)
```

hradius

The horizontal radius of the ellipse.

Returns hradius : number

See also:

vradius (page 486), *major* (page 482), *minor* (page 482)

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, 1)
>>> e1.hradius
3
```

intersection(o)

The intersection of this ellipse and another geometrical entity o.

Parameters o : GeometryEntity

Returns intersection : list of GeometryEntity objects

See also:

diofant.geometry.entity.GeometryEntity (page 426)

Notes

Currently supports intersections with Point, Line, Segment, Ray, Circle and Ellipse types.

Examples

```
>>> e = Ellipse(Point(0, 0), 5, 7)
>>> e.intersection(Point(0, 0))
[]
>>> e.intersection(Point(5, 0))
[Point2D(5, 0)]
>>> e.intersection(Line(Point(0, 0), Point(0, 1)))
[Point2D(0, -7), Point2D(0, 7)]
>>> e.intersection(Line(Point(5, 0), Point(5, 1)))
[Point2D(5, 0)]
>>> e.intersection(Line(Point(6, 0), Point(6, 1)))
[]
>>> e = Ellipse(Point(-1, 0), 4, 3)
>>> e.intersection(Ellipse(Point(1, 0), 4, 3))
[Point2D(0, -3*sqrt(15)/4), Point2D(0, 3*sqrt(15)/4)]
>>> e.intersection(Ellipse(Point(5, 0), 4, 3))
[Point2D(2, -3*sqrt(7)/4), Point2D(2, 3*sqrt(7)/4)]
>>> e.intersection(Ellipse(Point(100500, 0), 4, 3))
[]
>>> e.intersection(Ellipse(Point(0, 0), 3, 4))
[Point2D(-363/175, -48*sqrt(111)/175), Point2D(-363/175, 48*sqrt(111)/175),
↪Point2D(3, 0)]
```

```
>>> e.intersection(Ellipse(Point(-1, 0), 3, 4))
[Point2D(-17/5, -12/5), Point2D(-17/5, 12/5), Point2D(7/5, -12/5), Point2D(7/
↪5, 12/5)]
```

is_tangent(o)

Is o tangent to the ellipse?

Parameters o : GeometryEntity

An Ellipse, LinearEntity or Polygon

Returns is_tangent: boolean

True if o is tangent to the ellipse, False otherwise.

Raises NotImplementedError

When the wrong type of argument is supplied.

See also:

tangent_lines (page 485)

Examples

```
>>> p0, p1, p2 = Point(0, 0), Point(3, 0), Point(3, 3)
>>> e1 = Ellipse(p0, 3, 2)
>>> l1 = Line(p1, p2)
>>> e1.is_tangent(l1)
True
```

major

Longer axis of the ellipse (if it can be determined) else hradius.

Returns major : number or expression

See also:

hradius (page 480), *vradius* (page 486), *minor* (page 482)

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, 1)
>>> e1.major
3
```

```
>>> a = Symbol('a')
>>> b = Symbol('b')
>>> Ellipse(p1, a, b).major
a
>>> Ellipse(p1, b, a).major
b
```

```
>>> m = Symbol('m')
>>> M = m + 1
>>> Ellipse(p1, m, M).major
m + 1
```

minor

Shorter axis of the ellipse (if it can be determined) else vradius.

Returns minor : number or expression

See also:

hradius (page 480), *vradius* (page 486), *major* (page 482)

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, 1)
>>> e1.minor
1
```

```
>>> a = Symbol('a')
>>> b = Symbol('b')
>>> Ellipse(p1, a, b).minor
b
```

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```
>>> Ellipse(p1, b, a).minor
a
```

```
>>> m = Symbol('m')
>>> M = m + 1
>>> Ellipse(p1, m, M).minor
m
```

normal_lines(*p*, *prec=None*)

Normal lines between p and the ellipse.

Parameters p : Point

Returns normal_lines : list with 1, 2 or 4 Lines

Examples

```
>>> e = Ellipse((0, 0), 2, 3)
>>> c = e.center
>>> e.normal_lines(c + Point(1, 0))
[Line(Point2D(0, 0), Point2D(1, 0))]
>>> e.normal_lines(c)
[Line(Point2D(0, 0), Point2D(0, 1)), Line(Point2D(0, 0), Point2D(1, 0))]
```

Off-axis points require the solution of a quartic equation. This often leads to very large expressions that may be of little practical use. An approximate solution of $prec$ digits can be obtained by passing in the desired value:

```
>>> e.normal_lines((3, 3), prec=2)
[Line(Point2D(-38/47, -85/31), Point2D(9/47, -21/17)),
Line(Point2D(19/13, -43/21), Point2D(32/13, -8/3))]
```

Whereas the above solution has an operation count of 12, the exact solution has an operation count of 2020.

periapsis

The periapsis of the ellipse.

The shortest distance between the focus and the contour.

Returns periapsis : number

See also:

apoapsis **(page 476)** Returns greatest distance between focus and contour

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, 1)
>>> e1.periapsis
-2*sqrt(2) + 3
```

plot_interval(*parameter='t'*)

The plot interval for the default geometric plot of the Ellipse.

Parameters parameter : str, optional

Default value is 't'.

Returns plot_interval : list

[parameter, lower_bound, upper_bound]

Examples

```
>>> e1 = Ellipse(Point(0, 0), 3, 2)
>>> e1.plot_interval()
[t, -pi, pi]
```

random_point(*seed=None*)

A random point on the ellipse.

Returns point : Point

See also:

diofant.geometry.point.Point (page 431)

arbitrary_point **(page 477)** Returns parameterized point on ellipse

Notes

A random point may not appear to be on the ellipse, ie, $p in e$ may return False. This is because the coordinates of the point will be floating point values, and when these values are substituted into the equation for the ellipse the result may not be zero because of floating point rounding error.

An arbitrary_point with a random value of t substituted into it may not test as being on the ellipse because the expression tested that a point is on the ellipse doesn't simplify to zero and doesn't evaluate exactly to zero:

```
>>> from diofant.abc import t
>>> e1.arbitrary_point(t)
Point2D(3*cos(t), 2*sin(t))
>>> p2 = _.subs(t, 0.1)
>>> p2 in e1
False
```

Note that arbitrary_point routine does not take this approach. A value for cos(t) and sin(t) (not t) is substituted into the arbitrary point. There is a small chance that this will give a point that will not test as being in the ellipse, so the process is repeated (up to 10 times) until a valid point is obtained.

Examples

```
>>> e1 = Ellipse(Point(0, 0), 3, 2)
>>> e1.random_point() # gives some random point
Point2D(...)
>>> p1 = e1.random_point(seed=0); p1.n(2)
Point2D(2.1, 1.4)
```

The random_point method assures that the point will test as being in the ellipse:

```
>>> p1 in e1
True
```

reflect(*line*)

Override GeometryEntity.reflect since the radius is not a GeometryEntity.

Notes

Until the general ellipse (with no axis parallel to the x-axis) is supported a NotImplemented error is raised and the equation whose zeros define the rotated ellipse is given.

Examples

```
>>> Circle((0, 1), 1).reflect(Line((0, 0), (1, 1)))
Circle(Point2D(1, 0), -1)
>>> Ellipse(Point(3, 4), 1, 3).reflect(Line(Point(0, -4), Point(5, 0)))
Traceback (most recent call last):
...
NotImplementedError:
General Ellipse is not supported but the equation of the reflected
Ellipse is given by the zeros of: f(x, y) = (9*x/41 + 40*y/41 +
37/41)**2 + (40*x/123 - 3*y/41 - 364/123)**2 - 1
```

rotate(*angle=0, pt=None*)

Rotate `angle` radians counterclockwise about Point `pt`.

Note: since the general ellipse is not supported, only rotations that are integer multiples of pi/2 are allowed.

Examples

```
>>> Ellipse((1, 0), 2, 1).rotate(pi/2)
Ellipse(Point2D(0, 1), 1, 2)
>>> Ellipse((1, 0), 2, 1).rotate(pi)
Ellipse(Point2D(-1, 0), 2, 1)
```

scale(*x=1, y=1, pt=None*)

Override GeometryEntity.scale since it is the major and minor axes which must be scaled and they are not GeometryEntities.

Examples

```
>>> Ellipse((0, 0), 2, 1).scale(2, 4)
Circle(Point2D(0, 0), 4)
>>> Ellipse((0, 0), 2, 1).scale(2)
Ellipse(Point2D(0, 0), 4, 1)
```

tangent_lines(p)

Tangent lines between p and the ellipse.

If p is on the ellipse, returns the tangent line through point p. Otherwise, returns the tangent line(s) from p to the ellipse, or None if no tangent line is possible (e.g., p inside ellipse).

Parameters p : Point

Returns tangent_lines : list with 1 or 2 Lines

Raises NotImplementedError

Can only find tangent lines for a point, p, on the ellipse.

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.line.Line* (page 449)

Examples

```
>>> e1 = Ellipse(Point(0, 0), 3, 2)
>>> e1.tangent_lines(Point(3, 0))
[Line(Point2D(3, 0), Point2D(3, -12))]
```

vradius

The vertical radius of the ellipse.

Returns vradius : number

See also:

hradius (page 480), *major* (page 482), *minor* (page 482)

Examples

```
>>> p1 = Point(0, 0)
>>> e1 = Ellipse(p1, 3, 1)
>>> e1.vradius
1
```

class diofant.geometry.ellipse.**Circle**

A circle in space.

Constructed simply from a center and a radius, or from three non-collinear points.

Parameters center : Point

radius : number or diofant expression

points : sequence of three Points

Raises GeometryError

When trying to construct circle from three collinear points. When trying to construct circle from incorrect parameters.

See also:

Ellipse (page 475), *diofant.geometry.point.Point* (page 431)

Examples

```
>>> # a circle constructed from a center and radius
>>> c1 = Circle(Point(0, 0), 5)
>>> c1.hradius, c1.vradius, c1.radius
(5, 5, 5)
```

```
>>> # a circle costructed from three points
>>> c2 = Circle(Point(0, 0), Point(1, 1), Point(1, 0))
>>> c2.hradius, c2.vradius, c2.radius, c2.center
(sqrt(2)/2, sqrt(2)/2, sqrt(2)/2, Point2D(1/2, 1/2))
```

Attributes

circumference (page 487)	The circumference of the circle.
equation (page 487)([x, y])	The equation of the circle.

radius (synonymous with hradius, vradius, major and minor)	

circumference
The circumference of the circle.

Returns circumference : number or Diofant expression

Examples

```
>>> c1 = Circle(Point(3, 4), 6)
>>> c1.circumference
12*pi
```

equation(*x='x'*, *y='y'*)
The equation of the circle.

Parameters x : str or Symbol, optional

Default value is 'x'.

y : str or Symbol, optional

Default value is 'y'.

Returns equation : Diofant expression

Examples

```
>>> c1 = Circle(Point(0, 0), 5)
>>> c1.equation()
x**2 + y**2 - 25
```

intersection(*o*)
The intersection of this circle with another geometrical entity.

Parameters o : GeometryEntity

Returns intersection : list of GeometryEntities

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(5, 5), Point(6, 0)
>>> p4 = Point(5, 0)
>>> c1 = Circle(p1, 5)
>>> c1.intersection(p2)
[]
>>> c1.intersection(p4)
[Point2D(5, 0)]
>>> c1.intersection(Ray(p1, p2))
[Point2D(5*sqrt(2)/2, 5*sqrt(2)/2)]
>>> c1.intersection(Line(p2, p3))
[]
```

radius

The radius of the circle.

Returns radius : number or diofant expression

See also:

Ellipse.major (page 482), *Ellipse.minor* (page 482), *Ellipse.hradius* (page 480), *Ellipse.vradius* (page 486)

Examples

```
>>> c1 = Circle(Point(3, 4), 6)
>>> c1.radius
6
```

reflect(*line*)

Override GeometryEntity.reflect since the radius is not a GeometryEntity.

Examples

```
>>> Circle((0, 1), 1).reflect(Line((0, 0), (1, 1)))
Circle(Point2D(1, 0), -1)
```

scale(*x=1, y=1, pt=None*)

Override GeometryEntity.scale since the radius is not a GeometryEntity.

Examples

```
>>> Circle((0, 0), 1).scale(2, 2)
Circle(Point2D(0, 0), 2)
>>> Circle((0, 0), 1).scale(2, 4)
Ellipse(Point2D(0, 0), 2, 4)
```

vradius

This Ellipse property is an alias for the Circle's radius.

Whereas hradius, major and minor can use Ellipse's conventions, the vradius does not exist for a circle. It is always a positive value in order that the Circle, like Polygons, will have an area that can be positive or negative as determined by the sign of the hradius.

Examples

```
>>> c1 = Circle(Point(3, 4), 6)
>>> c1.vradius
6
```

Polygons

class diofant.geometry.polygon.**Polygon**

A two-dimensional polygon.

A simple polygon in space. Can be constructed from a sequence of points or from a center, radius, number of sides and rotation angle.

Parameters vertices : sequence of Points

Raises GeometryError

If all parameters are not Points.

If the Polygon has intersecting sides.

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.line.Segment* (page 455), *Triangle* (page 502)

Notes

Polygons are treated as closed paths rather than 2D areas so some calculations can be be negative or positive (e.g., area) based on the orientation of the points.

Any consecutive identical points are reduced to a single point and any points collinear and between two points will be removed unless they are needed to define an explicit intersection (see examples).

A Triangle, Segment or Point will be returned when there are 3 or fewer points provided.

Examples

```
>>> p1, p2, p3, p4, p5 = [(0, 0), (1, 0), (5, 1), (0, 1), (3, 0)]
>>> Polygon(p1, p2, p3, p4)
Polygon(Point2D(0, 0), Point2D(1, 0), Point2D(5, 1), Point2D(0, 1))
>>> Polygon(p1, p2)
Segment(Point2D(0, 0), Point2D(1, 0))
```

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```
>>> Polygon(p1, p2, p5)
Segment(Point2D(0, 0), Point2D(3, 0))
```

While the sides of a polygon are not allowed to cross implicitly, they can do so explicitly. For example, a polygon shaped like a Z with the top left connecting to the bottom right of the Z must have the point in the middle of the Z explicitly given:

```
>>> mid = Point(1, 1)
>>> Polygon((0, 2), (2, 2), mid, (0, 0), (2, 0), mid).area
0
>>> Polygon((0, 2), (2, 2), mid, (2, 0), (0, 0), mid).area
-2
```

When the the keyword n is used to define the number of sides of the Polygon then a RegularPolygon is created and the other arguments are interpreted as center, radius and rotation. The unrotated RegularPolygon will always have a vertex at Point(r, 0) where r is the radius of the circle that circumscribes the RegularPolygon. Its method $spin$ can be used to increment that angle.

```
>>> p = Polygon((0, 0), 1, n=3)
>>> p
RegularPolygon(Point2D(0, 0), 1, 3, 0)
>>> p.vertices[0]
Point2D(1, 0)
>>> p.args[0]
Point2D(0, 0)
>>> p.spin(pi/2)
>>> p.vertices[0]
Point2D(0, 1)
```

Attributes

area (page 491)	The area of the polygon.
angles (page 490)	The internal angle at each vertex.
perimeter (page 493)	The perimeter of the polygon.
vertices (page 495)	The vertices of the polygon.
centroid (page 492)	The centroid of the polygon.
sides (page 494)	The line segments that form the sides of the polygon.

angles

The internal angle at each vertex.

Returns angles : dict

A dictionary where each key is a vertex and each value is the internal angle at that vertex. The vertices are represented as Points.

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.line.LinearEntity.angle_between* (page 442)

Examples

```
>>> p1, p2, p3, p4 = map(Point, [(0, 0), (1, 0), (5, 1), (0, 1)])
>>> poly = Polygon(p1, p2, p3, p4)
>>> poly.angles[p1]
pi/2
>>> poly.angles[p2]
acos(-4*sqrt(17)/17)
```

arbitrary_point(*parameter='t'*)

A parameterized point on the polygon.

The parameter, varying from 0 to 1, assigns points to the position on the perimeter that is that fraction of the total perimeter. So the point evaluated at t=1/2 would return the point from the first vertex that is 1/2 way around the polygon.

Parameters parameter : str, optional

Default value is 't'.

Returns arbitrary_point : Point

Raises ValueError

When $parameter$ already appears in the Polygon's definition.

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> t = Symbol('t', extended_real=True)
>>> tri = Polygon((0, 0), (1, 0), (1, 1))
>>> p = tri.arbitrary_point('t')
>>> perimeter = tri.perimeter
>>> s1, s2 = [s.length for s in tri.sides[:2]]
>>> p.subs(t, (s1 + s2/2)/perimeter)
Point2D(1, 1/2)
```

area

The area of the polygon.

See also:

diofant.geometry.ellipse.Ellipse.area (page 477)

Notes

The area calculation can be positive or negative based on the orientation of the points.

Examples

```
>>> p1, p2, p3, p4 = map(Point, [(0, 0), (1, 0), (5, 1), (0, 1)])
>>> poly = Polygon(p1, p2, p3, p4)
>>> poly.area
3
```

centroid
The centroid of the polygon.

Returns centroid : Point

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.util.centroid* (page 430)

Examples

```
>>> p1, p2, p3, p4 = map(Point, [(0, 0), (1, 0), (5, 1), (0, 1)])
>>> poly = Polygon(p1, p2, p3, p4)
>>> poly.centroid
Point2D(31/18, 11/18)
```

distance(*o*)
Returns the shortest distance between self and o.

If o is a point, then self does not need to be convex. If o is another polygon self and o must be complex.

Examples

```
>>> p1, p2 = map(Point, [(0, 0), (7, 5)])
>>> poly = Polygon(*RegularPolygon(p1, 1, 3).vertices)
>>> poly.distance(p2)
sqrt(61)
```

encloses_point(*p*)
Return True if p is enclosed by (is inside of) self.

Parameters p : Point

Returns encloses_point : True, False or None

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.ellipse.Ellipse.encloses_point* (page 478)

Notes

Being on the border of self is considered False.

References

[R388] (page 1260)

Examples

```
>>> from diofant.abc import t
>>> p = Polygon((0, 0), (4, 0), (4, 4))
>>> p.encloses_point(Point(2, 1))
True
>>> p.encloses_point(Point(2, 2))
False
>>> p.encloses_point(Point(5, 5))
False
```

intersection(*o*)

The intersection of two polygons.

The intersection may be empty and can contain individual Points and complete Line Segments.

Parameters other: Polygon

Returns intersection : list

The list of Segments and Points

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.line.Segment* (page 455)

Examples

```
>>> p1, p2, p3, p4 = map(Point, [(0, 0), (1, 0), (5, 1), (0, 1)])
>>> poly1 = Polygon(p1, p2, p3, p4)
>>> p5, p6, p7 = map(Point, [(3, 2), (1, -1), (0, 2)])
>>> poly2 = Polygon(p5, p6, p7)
>>> poly1.intersection(poly2)
[Point2D(2/3, 0), Point2D(9/5, 1/5), Point2D(7/3, 1), Point2D(1/3, 1)]
```

is_convex()

Is the polygon convex?

A polygon is convex if all its interior angles are less than 180 degrees.

Returns is_convex : boolean

True if this polygon is convex, False otherwise.

See also:

diofant.geometry.util.convex_hull (page 429)

Examples

```
>>> p1, p2, p3, p4 = map(Point, [(0, 0), (1, 0), (5, 1), (0, 1)])
>>> poly = Polygon(p1, p2, p3, p4)
>>> poly.is_convex()
True
```

perimeter
The perimeter of the polygon.

Returns perimeter : number or Basic instance

See also:

diofant.geometry.line.Segment.length (page 456)

Examples

```
>>> p1, p2, p3, p4 = map(Point, [(0, 0), (1, 0), (5, 1), (0, 1)])
>>> poly = Polygon(p1, p2, p3, p4)
>>> poly.perimeter
sqrt(17) + 7
```

plot_interval(*parameter='t'*)
The plot interval for the default geometric plot of the polygon.

Parameters parameter : str, optional

Default value is 't'.

Returns plot_interval : list (plot interval)

[parameter, lower_bound, upper_bound]

Examples

```
>>> p = Polygon((0, 0), (1, 0), (1, 1))
>>> p.plot_interval()
[t, 0, 1]
```

sides
The line segments that form the sides of the polygon.

Returns sides : list of sides

Each side is a Segment.

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.line.Segment* (page 455)

Notes

The Segments that represent the sides are an undirected line segment so cannot be used to tell the orientation of the polygon.

Examples

```
>>> p1, p2, p3, p4 = map(Point, [(0, 0), (1, 0), (5, 1), (0, 1)])
>>> poly = Polygon(p1, p2, p3, p4)
>>> poly.sides
[Segment(Point2D(0, 0), Point2D(1, 0)),
Segment(Point2D(1, 0), Point2D(5, 1)),
Segment(Point2D(0, 1), Point2D(5, 1)), Segment(Point2D(0, 0), Point2D(0, 1))]
```

vertices

The vertices of the polygon.

Returns vertices : tuple of Points

See also:

diofant.geometry.point.Point (page 431)

Notes

When iterating over the vertices, it is more efficient to index self rather than to request the vertices and index them. Only use the vertices when you want to process all of them at once. This is even more important with RegularPolygons that calculate each vertex.

Examples

```
>>> p1, p2, p3, p4 = map(Point, [(0, 0), (1, 0), (5, 1), (0, 1)])
>>> poly = Polygon(p1, p2, p3, p4)
>>> poly.vertices
(Point2D(0, 0), Point2D(1, 0), Point2D(5, 1), Point2D(0, 1))
>>> poly.args[0]
Point2D(0, 0)
```

class diofant.geometry.polygon.**RegularPolygon**

A regular polygon.

Such a polygon has all internal angles equal and all sides the same length.

Parameters center : Point

radius : number or Basic instance

The distance from the center to a vertex

n : int

The number of sides

Raises GeometryError

If the $center$ is not a Point, or the $radius$ is not a number or Basic instance, or the number of sides, n, is less than three.

See also:

diofant.geometry.point.Point (page 431), *Polygon* (page 489)

Notes

A RegularPolygon can be instantiated with Polygon with the kwarg n.

Regular polygons are instantiated with a center, radius, number of sides and a rotation angle. Whereas the arguments of a Polygon are vertices, the vertices of the RegularPolygon must be obtained with the vertices method.

Examples

```
>>> r = RegularPolygon(Point(0, 0), 5, 3)
>>> r
RegularPolygon(Point2D(0, 0), 5, 3, 0)
>>> r.vertices[0]
Point2D(5, 0)
```

Attributes

vertices (page 502)	The vertices of the RegularPolygon.
center (page 497)	The center of the RegularPolygon
radius (page 500)	Radius of the RegularPolygon
rotation (page 501)	CCW angle by which the RegularPolygon is rotated
apothem (page 496)	The inradius of the RegularPolygon.
interior_angle (page 500)	Measure of the interior angles.
exterior_angle (page 499)	Measure of the exterior angles.
circumcircle (page 498)	The circumcircle of the RegularPolygon.
incircle (page 499)	The incircle of the RegularPolygon.
angles (page 496)	Returns a dictionary with keys, the vertices of the Polygon, and values, the interior angle at each vertex.

angles

Returns a dictionary with keys, the vertices of the Polygon, and values, the interior angle at each vertex.

Examples

```
>>> r = RegularPolygon(Point2D(0, 0), 5, 3)
>>> r.angles
{Point2D(-5/2, -5*sqrt(3)/2): pi/3,
 Point2D(-5/2, 5*sqrt(3)/2): pi/3,
 Point2D(5, 0): pi/3}
```

apothem

The inradius of the RegularPolygon.

The apothem/inradius is the radius of the inscribed circle.

Returns apothem : number or instance of Basic

See also:

diofant.geometry.line.Segment.length (page 456), *diofant.geometry.ellipse.Circle.radius* (page 488)

Examples

```
>>> radius = Symbol('r')
>>> rp = RegularPolygon(Point(0, 0), radius, 4)
>>> rp.apothem
sqrt(2)*r/2
```

area
Returns the area.

Examples

```
>>> square = RegularPolygon((0, 0), 1, 4)
>>> square.area
2
>>> _ == square.length**2
True
```

args
Returns the center point, the radius, the number of sides, and the orientation angle.

Examples

```
>>> r = RegularPolygon(Point(0, 0), 5, 3)
>>> r.args
(Point2D(0, 0), 5, 3, 0)
```

center
The center of the RegularPolygon

This is also the center of the circumscribing circle.

Returns center : Point

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.ellipse.Ellipse.center* (page 477)

Examples

```
>>> rp = RegularPolygon(Point(0, 0), 5, 4)
>>> rp.center
Point2D(0, 0)
```

centroid
The center of the RegularPolygon

This is also the center of the circumscribing circle.

Returns center : Point

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.ellipse.Ellipse.center* (page 477)

Examples

```
>>> rp = RegularPolygon(Point(0, 0), 5, 4)
>>> rp.center
Point2D(0, 0)
```

circumcenter

Alias for center.

Examples

```
>>> rp = RegularPolygon(Point(0, 0), 5, 4)
>>> rp.circumcenter
Point2D(0, 0)
```

circumcircle

The circumcircle of the RegularPolygon.

Returns circumcircle : Circle

See also:

circumcenter (page 498), *diofant.geometry.ellipse.Circle* (page 486)

Examples

```
>>> rp = RegularPolygon(Point(0, 0), 4, 8)
>>> rp.circumcircle
Circle(Point2D(0, 0), 4)
```

circumradius

Alias for radius.

Examples

```
>>> radius = Symbol('r')
>>> rp = RegularPolygon(Point(0, 0), radius, 4)
>>> rp.circumradius
r
```

encloses_point(*p*)

Return True if p is enclosed by (is inside of) self.

Parameters p : Point

Returns encloses_point : True, False or None

See also:

diofant.geometry.ellipse.Ellipse.encloses_point (page 478)

Notes

Being on the border of self is considered False.

The general Polygon.encloses_point method is called only if a point is not within or beyond the incircle or circumcircle, respectively.

Examples

```
>>> p = RegularPolygon((0, 0), 3, 4)
>>> p.encloses_point(Point(0, 0))
True
>>> r, R = p.inradius, p.circumradius
>>> p.encloses_point(Point((r + R)/2, 0))
True
>>> p.encloses_point(Point(R/2, R/2 + (R - r)/10))
False
>>> t = Symbol('t', extended_real=True)
>>> p.encloses_point(p.arbitrary_point().subs(t, S.Half))
False
>>> p.encloses_point(Point(5, 5))
False
```

exterior_angle

Measure of the exterior angles.

Returns exterior_angle : number

See also:

diofant.geometry.line.LinearEntity.angle_between (page 442)

Examples

```
>>> rp = RegularPolygon(Point(0, 0), 4, 8)
>>> rp.exterior_angle
pi/4
```

incircle

The incircle of the RegularPolygon.

Returns incircle : Circle

See also:

inradius (page 500), *diofant.geometry.ellipse.Circle* (page 486)

Examples

```
>>> rp = RegularPolygon(Point(0, 0), 4, 7)
>>> rp.incircle
Circle(Point2D(0, 0), 4*cos(pi/7))
```

inradius

Alias for apothem.

Examples

```
>>> radius = Symbol('r')
>>> rp = RegularPolygon(Point(0, 0), radius, 4)
>>> rp.inradius
sqrt(2)*r/2
```

interior_angle

Measure of the interior angles.

Returns interior_angle : number

See also:

diofant.geometry.line.LinearEntity.angle_between (page 442)

Examples

```
>>> rp = RegularPolygon(Point(0, 0), 4, 8)
>>> rp.interior_angle
3*pi/4
```

length

Returns the length of the sides.

The half-length of the side and the apothem form two legs of a right triangle whose hypotenuse is the radius of the regular polygon.

Examples

```
>>> s = square_in_unit_circle = RegularPolygon((0, 0), 1, 4)
>>> s.length
sqrt(2)
>>> sqrt((_/2)**2 + s.apothem**2) == s.radius
True
```

radius

Radius of the RegularPolygon

This is also the radius of the circumscribing circle.

Returns radius : number or instance of Basic

See also:

diofant.geometry.line.Segment.length (page 456), *diofant.geometry.ellipse.Circle.radius* (page 488)

Examples

```
>>> radius = Symbol('r')
>>> rp = RegularPolygon(Point(0, 0), radius, 4)
>>> rp.radius
r
```

reflect(*line*)

Override GeometryEntity.reflect since this is not made of only points.

```
>>> RegularPolygon((0, 0), 1, 4).reflect(Line((0, 1), slope=-2))
RegularPolygon(Point2D(4/5, 2/5), -1, 4, acos(3/5))
```

rotate(*angle*, *pt=None*)

Override GeometryEntity.rotate to first rotate the RegularPolygon about its center.

```
>>> t = RegularPolygon(Point(1, 0), 1, 3)
>>> t.vertices[0] # vertex on x-axis
Point2D(2, 0)
>>> t.rotate(pi/2).vertices[0] # vertex on y axis now
Point2D(0, 2)
```

See also:

rotation (page 501)

***spin* (page 501)** Rotates a RegularPolygon in place

rotation

CCW angle by which the RegularPolygon is rotated

Returns rotation : number or instance of Basic

Examples

```
>>> RegularPolygon(Point(0, 0), 3, 4, pi).rotation
pi
```

scale(*x=1*, *y=1*, *pt=None*)

Override GeometryEntity.scale since it is the radius that must be scaled (if x == y) or else a new Polygon must be returned.

Symmetric scaling returns a RegularPolygon:

```
>>> RegularPolygon((0, 0), 1, 4).scale(2, 2)
RegularPolygon(Point2D(0, 0), 2, 4, 0)
```

Asymmetric scaling returns a kite as a Polygon:

```
>>> RegularPolygon((0, 0), 1, 4).scale(2, 1)
Polygon(Point2D(2, 0), Point2D(0, 1), Point2D(-2, 0), Point2D(0, -1))
```

spin(*angle*)

Increment *in place* the virtual Polygon's rotation by ccw angle.

See also: rotate method which moves the center.

```
>>> r = Polygon(Point(0, 0), 1, n=3)
>>> r.vertices[0]
Point2D(1, 0)
>>> r.spin(pi/6)
>>> r.vertices[0]
Point2D(sqrt(3)/2, 1/2)
```

See also:

rotation (page 501)

***rotate* (page 501)** Creates a copy of the RegularPolygon rotated about a Point

vertices

The vertices of the RegularPolygon.

Returns vertices : list

Each vertex is a Point.

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> rp = RegularPolygon(Point(0, 0), 5, 4)
>>> rp.vertices
[Point2D(5, 0), Point2D(0, 5), Point2D(-5, 0), Point2D(0, -5)]
```

class diofant.geometry.polygon.**Triangle**

A polygon with three vertices and three sides.

Parameters points : sequence of Points

keyword: asa, sas, or sss to specify sides/angles of the triangle

Raises GeometryError

If the number of vertices is not equal to three, or one of the vertices is not a Point, or a valid keyword is not given.

See also:

diofant.geometry.point.Point (page 431), *Polygon* (page 489)

Examples

```
>>> Triangle(Point(0, 0), Point(4, 0), Point(4, 3))
Triangle(Point2D(0, 0), Point2D(4, 0), Point2D(4, 3))
```

Keywords sss, sas, or asa can be used to give the desired side lengths (in order) and interior angles (in degrees) that define the triangle:

```
>>> Triangle(sss=(3, 4, 5))
Triangle(Point2D(0, 0), Point2D(3, 0), Point2D(3, 4))
>>> Triangle(asa=(30, 1, 30))
Triangle(Point2D(0, 0), Point2D(1, 0), Point2D(1/2, sqrt(3)/6))
>>> Triangle(sas=(1, 45, 2))
Triangle(Point2D(0, 0), Point2D(2, 0), Point2D(sqrt(2)/2, sqrt(2)/2))
```

Attributes

vertices (page 508)	The triangle's vertices
altitudes (page 503)	The altitudes of the triangle.
orthocenter (page 508)	The orthocenter of the triangle.
circumcenter (page 504)	The circumcenter of the triangle
circumradius (page 504)	The radius of the circumcircle of the triangle.
circumcircle (page 504)	The circle which passes through the three vertices of the triangle.
inradius (page 505)	The radius of the incircle.
incircle (page 505)	The incircle of the triangle.
medians (page 508)	The medians of the triangle.
medial (page 507)	The medial triangle of the triangle.

altitudes

The altitudes of the triangle.

An altitude of a triangle is a segment through a vertex, perpendicular to the opposite side, with length being the height of the vertex measured from the line containing the side.

Returns altitudes : dict

The dictionary consists of keys which are vertices and values which are Segments.

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.line.Segment.length* (page 456)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 0), Point(0, 1)
>>> t = Triangle(p1, p2, p3)
>>> t.altitudes[p1]
Segment(Point2D(0, 0), Point2D(1/2, 1/2))
```

bisectors()

The angle bisectors of the triangle.

An angle bisector of a triangle is a straight line through a vertex which cuts the corresponding angle in half.

Returns bisectors : dict

Each key is a vertex (Point) and each value is the corresponding bisector (Segment).

See also:

diofant.geometry.point.Point (page 431), *diofant.geometry.line.Segment* (page 455)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 0), Point(0, 1)
>>> t = Triangle(p1, p2, p3)
>>> t.bisectors()[p2] == Segment(Point(0, sqrt(2) - 1), Point(1, 0))
True
```

circumcenter

The circumcenter of the triangle

The circumcenter is the center of the circumcircle.

Returns circumcenter : Point

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 0), Point(0, 1)
>>> t = Triangle(p1, p2, p3)
>>> t.circumcenter
Point2D(1/2, 1/2)
```

circumcircle

The circle which passes through the three vertices of the triangle.

Returns circumcircle : Circle

See also:

diofant.geometry.ellipse.Circle (page 486)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 0), Point(0, 1)
>>> t = Triangle(p1, p2, p3)
>>> t.circumcircle
Circle(Point2D(1/2, 1/2), sqrt(2)/2)
```

circumradius

The radius of the circumcircle of the triangle.

Returns circumradius : number of Basic instance

See also:

diofant.geometry.ellipse.Circle.radius (page 488)

Examples

```
>>> a = Symbol('a')
>>> p1, p2, p3 = Point(0, 0), Point(1, 0), Point(0, a)
>>> t = Triangle(p1, p2, p3)
>>> t.circumradius
sqrt(a**2/4 + 1/4)
```

incenter

The center of the incircle.

The incircle is the circle which lies inside the triangle and touches all three sides.

Returns incenter : Point

See also:

incircle (page 505), *diofant.geometry.point.Point* (page 431)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 0), Point(0, 1)
>>> t = Triangle(p1, p2, p3)
>>> t.incenter
Point2D(-sqrt(2)/2 + 1, -sqrt(2)/2 + 1)
```

incircle

The incircle of the triangle.

The incircle is the circle which lies inside the triangle and touches all three sides.

Returns incircle : Circle

See also:

diofant.geometry.ellipse.Circle (page 486)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(2, 0), Point(0, 2)
>>> t = Triangle(p1, p2, p3)
>>> t.incircle
Circle(Point2D(-sqrt(2) + 2, -sqrt(2) + 2), -sqrt(2) + 2)
```

inradius

The radius of the incircle.

Returns inradius : number of Basic instance

See also:

incircle (page 505), *diofant.geometry.ellipse.Circle.radius* (page 488)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(4, 0), Point(0, 3)
>>> t = Triangle(p1, p2, p3)
>>> t.inradius
1
```

is_equilateral()

Are all the sides the same length?

Returns is_equilateral : boolean

See also:

diofant.geometry.entity.GeometryEntity.is_similar (page 426), *RegularPolygon* (page 495), *is_isosceles* (page 506), *is_right* (page 506), *is_scalene* (page 507)

Examples

```
>>> t1 = Triangle(Point(0, 0), Point(4, 0), Point(4, 3))
>>> t1.is_equilateral()
False
```

```
>>> t2 = Triangle(Point(0, 0), Point(10, 0), Point(5, 5*sqrt(3)))
>>> t2.is_equilateral()
True
```

is_isosceles()

Are two or more of the sides the same length?

Returns is_isosceles : boolean

See also:

is_equilateral (page 506), *is_right* (page 506), *is_scalene* (page 507)

Examples

```
>>> t1 = Triangle(Point(0, 0), Point(4, 0), Point(2, 4))
>>> t1.is_isosceles()
True
```

is_right()

Is the triangle right-angled.

Returns is_right : boolean

See also:

diofant.geometry.line.LinearEntity.is_perpendicular (page 445), *is_equilateral* (page 506), *is_isosceles* (page 506), *is_scalene* (page 507)

Examples

```
>>> t1 = Triangle(Point(0, 0), Point(4, 0), Point(4, 3))
>>> t1.is_right()
True
```

is_scalene()
Are all the sides of the triangle of different lengths?

Returns is_scalene : boolean

See also:

is_equilateral (page 506), *is_isosceles* (page 506), *is_right* (page 506)

Examples

```
>>> t1 = Triangle(Point(0, 0), Point(4, 0), Point(1, 4))
>>> t1.is_scalene()
True
```

is_similar(*other*)
Is another triangle similar to this one.

Two triangles are similar if one can be uniformly scaled to the other.

Parameters other: Triangle

Returns is_similar : boolean

See also:

diofant.geometry.entity.GeometryEntity.is_similar (page 426)

Examples

```
>>> t1 = Triangle(Point(0, 0), Point(4, 0), Point(4, 3))
>>> t2 = Triangle(Point(0, 0), Point(-4, 0), Point(-4, -3))
>>> t1.is_similar(t2)
True
```

```
>>> t2 = Triangle(Point(0, 0), Point(-4, 0), Point(-4, -4))
>>> t1.is_similar(t2)
False
```

medial
The medial triangle of the triangle.

The triangle which is formed from the midpoints of the three sides.

Returns medial : Triangle

See also:

diofant.geometry.line.Segment.midpoint (page 457)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 0), Point(0, 1)
>>> t = Triangle(p1, p2, p3)
>>> t.medial
Triangle(Point2D(1/2, 0), Point2D(1/2, 1/2), Point2D(0, 1/2))
```

medians

The medians of the triangle.

A median of a triangle is a straight line through a vertex and the midpoint of the opposite side, and divides the triangle into two equal areas.

Returns medians : dict

Each key is a vertex (Point) and each value is the median (Segment) at that point.

See also:

diofant.geometry.point.Point.midpoint (page 434), *diofant.geometry.line.Segment.midpoint* (page 457)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 0), Point(0, 1)
>>> t = Triangle(p1, p2, p3)
>>> t.medians[p1]
Segment(Point2D(0, 0), Point2D(1/2, 1/2))
```

orthocenter

The orthocenter of the triangle.

The orthocenter is the intersection of the altitudes of a triangle. It may lie inside, outside or on the triangle.

Returns orthocenter : Point

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> p1, p2, p3 = Point(0, 0), Point(1, 0), Point(0, 1)
>>> t = Triangle(p1, p2, p3)
>>> t.orthocenter
Point2D(0, 0)
```

vertices

The triangle's vertices

Returns vertices : tuple

Each element in the tuple is a Point

See also:

diofant.geometry.point.Point (page 431)

Examples

```
>>> t = Triangle(Point(0, 0), Point(4, 0), Point(4, 3))
>>> t.vertices
(Point2D(0, 0), Point2D(4, 0), Point2D(4, 3))
```

Plane

Geometrical Planes.

Contains

Plane

class diofant.geometry.plane.**Plane**

A plane is a flat, two-dimensional surface. A plane is the two-dimensional analogue of a point (zero-dimensions), a line (one-dimension) and a solid (three-dimensions). A plane can generally be constructed by two types of inputs. They are three non-collinear points and a point and the plane's normal vector.

Examples

```
>>> Plane(Point3D(1, 1, 1), Point3D(2, 3, 4), Point3D(2, 2, 2))
Plane(Point3D(1, 1, 1), (-1, 2, -1))
>>> Plane((1, 1, 1), (2, 3, 4), (2, 2, 2))
Plane(Point3D(1, 1, 1), (-1, 2, -1))
>>> Plane(Point3D(1, 1, 1), normal_vector=(1, 4, 7))
Plane(Point3D(1, 1, 1), (1, 4, 7))
```

Attributes

p1 (page 512)	The only defining point of the plane.
normal_vector (page 512)	Normal vector of the given plane.

angle_between(*o*)

Angle between the plane and other geometric entity.

Parameters LinearEntity3D, Plane.

Returns angle : angle in radians

Notes

This method accepts only 3D entities as it's parameter, but if you want to calculate the angle between a 2D entity and a plane you should first convert to a 3D entity by projecting onto a desired plane and then proceed to calculate the angle.

Examples

```
>>> a = Plane(Point3D(1, 2, 2), normal_vector=(1, 2, 3))
>>> b = Line3D(Point3D(1, 3, 4), Point3D(2, 2, 2))
>>> a.angle_between(b)
-asin(sqrt(21)/6)
```

arbitrary_point(*t=None*)
Returns an arbitrary point on the Plane; varying t from 0 to 2*pi will move the point in a circle of radius 1 about p1 of the Plane.

Returns Point3D

Examples

```
>>> from diofant.abc import t
>>> p = Plane((0, 0, 0), (0, 0, 1), (0, 1, 0))
>>> p.arbitrary_point(t)
Point3D(0, cos(t), sin(t))
>>> _.distance(p.p1).simplify()
1
```

static are_concurrent()
Is a sequence of Planes concurrent?

Two or more Planes are concurrent if their intersections are a common line.

Parameters planes: list

Returns Boolean

Examples

```
>>> a = Plane(Point3D(5, 0, 0), normal_vector=(1, -1, 1))
>>> b = Plane(Point3D(0, -2, 0), normal_vector=(3, 1, 1))
>>> c = Plane(Point3D(0, -1, 0), normal_vector=(5, -1, 9))
>>> Plane.are_concurrent(a, b)
True
>>> Plane.are_concurrent(a, b, c)
False
```

distance(*o*)
Distance between the plane and another geometric entity.

Parameters Point3D, LinearEntity3D, Plane.

Returns distance

Notes

This method accepts only 3D entities as it's parameter, but if you want to calculate the distance between a 2D entity and a plane you should first convert to a 3D entity by projecting onto a desired plane and then proceed to calculate the distance.

Examples

```
>>> a = Plane(Point3D(1, 1, 1), normal_vector=(1, 1, 1))
>>> b = Point3D(1, 2, 3)
>>> a.distance(b)
sqrt(3)
>>> c = Line3D(Point3D(2, 3, 1), Point3D(1, 2, 2))
>>> a.distance(c)
0
```

equation(*x=None, y=None, z=None*)
The equation of the Plane.

Examples

```
>>> a = Plane(Point3D(1, 1, 2), Point3D(2, 4, 7), Point3D(3, 5, 1))
>>> a.equation()
-23*x + 11*y - 2*z + 16
>>> a = Plane(Point3D(1, 4, 2), normal_vector=(6, 6, 6))
>>> a.equation()
6*x + 6*y + 6*z - 42
```

intersection(*o*)
The intersection with other geometrical entity.

Parameters Point, Point3D, LinearEntity, LinearEntity3D, Plane

Returns List

Examples

```
>>> a = Plane(Point3D(1, 2, 3), normal_vector=(1, 1, 1))
>>> b = Point3D(1, 2, 3)
>>> a.intersection(b)
[Point3D(1, 2, 3)]
>>> c = Line3D(Point3D(1, 4, 7), Point3D(2, 2, 2))
>>> a.intersection(c)
[Point3D(2, 2, 2)]
>>> d = Plane(Point3D(6, 0, 0), normal_vector=(2, -5, 3))
>>> e = Plane(Point3D(2, 0, 0), normal_vector=(3, 4, -3))
>>> d.intersection(e)
[Line3D(Point3D(78/23, -24/23, 0), Point3D(147/23, 321/23, 23))]
```

is_coplanar(*o*)
Returns True if o is coplanar with self, else False.

Examples

```
>>> o = (0, 0, 0)
>>> p = Plane(o, (1, 1, 1))
>>> p2 = Plane(o, (2, 2, 2))
>>> p == p2
False
>>> p.is_coplanar(p2)
True
```

is_parallel(*l*)
Is the given geometric entity parallel to the plane?

Parameters **LinearEntity3D or Plane**

Returns Boolean

Examples

```
>>> a = Plane(Point3D(1, 4, 6), normal_vector=(2, 4, 6))
>>> b = Plane(Point3D(3, 1, 3), normal_vector=(4, 8, 12))
>>> a.is_parallel(b)
True
```

is_perpendicular(*l*)
is the given geometric entity perpendicular to the given plane?

Parameters **LinearEntity3D or Plane**

Returns Boolean

Examples

```
>>> a = Plane(Point3D(1, 4, 6), normal_vector=(2, 4, 6))
>>> b = Plane(Point3D(2, 2, 2), normal_vector=(-1, 2, -1))
>>> a.is_perpendicular(b)
True
```

normal_vector
Normal vector of the given plane.

Examples

```
>>> a = Plane(Point3D(1, 1, 1), Point3D(2, 3, 4), Point3D(2, 2, 2))
>>> a.normal_vector
(-1, 2, -1)
>>> a = Plane(Point3D(1, 1, 1), normal_vector=(1, 4, 7))
>>> a.normal_vector
(1, 4, 7)
```

p1
The only defining point of the plane. Others can be obtained from the arbitrary_point method.

See also:

diofant.geometry.point.Point3D (page 437)

Examples

```
>>> a = Plane(Point3D(1, 1, 1), Point3D(2, 3, 4), Point3D(2, 2, 2))
>>> a.p1
Point3D(1, 1, 1)
```

parallel_plane(*pt*)
Plane parallel to the given plane and passing through the point pt.

Parameters pt: Point3D

Returns Plane

Examples

```
>>> a = Plane(Point3D(1, 4, 6), normal_vector=(2, 4, 6))
>>> a.parallel_plane(Point3D(2, 3, 5))
Plane(Point3D(2, 3, 5), (2, 4, 6))
```

perpendicular_line(*pt*)
A line perpendicular to the given plane.

Parameters pt: Point3D

Returns Line3D

Examples

```
>>> a = Plane(Point3D(1, 4, 6), normal_vector=(2, 4, 6))
>>> a.perpendicular_line(Point3D(9, 8, 7))
Line3D(Point3D(9, 8, 7), Point3D(11, 12, 13))
```

perpendicular_plane(**pts*)
Return a perpendicular passing through the given points. If the direction ratio between the points is the same as the Plane's normal vector then, to select from the infinite number of possible planes, a third point will be chosen on the z-axis (or the y-axis if the normal vector is already parallel to the z-axis). If less than two points are given they will be supplied as follows: if no point is given then pt1 will be self.p1; if a second point is not given it will be a point through pt1 on a line parallel to the z-axis (if the normal is not already the z-axis, otherwise on the line parallel to the y-axis).

Parameters pts: 0, 1 or 2 Point3D

Returns Plane

Examples

```
>>> a, b = Point3D(0, 0, 0), Point3D(0, 1, 0)
>>> Z = (0, 0, 1)
>>> p = Plane(a, normal_vector=Z)
>>> p.perpendicular_plane(a, b)
Plane(Point3D(0, 0, 0), (1, 0, 0))
```

projection(*pt*)

Project the given point onto the plane along the plane normal.

Parameters **Point or Point3D**

Returns Point3D

Examples

```
>>> A = Plane(Point3D(1, 1, 2), normal_vector=(1, 1, 1))
```

The projection is along the normal vector direction, not the z axis, so (1, 1) does not project to (1, 1, 2) on the plane A:

```
>>> b = Point(1, 1)
>>> A.projection(b)
Point3D(5/3, 5/3, 2/3)
>>> _ in A
True
```

But the point (1, 1, 2) projects to (1, 1) on the XY-plane:

```
>>> XY = Plane((0, 0, 0), (0, 0, 1))
>>> XY.projection((1, 1, 2))
Point3D(1, 1, 0)
```

projection_line(*line*)

Project the given line onto the plane through the normal plane containing the line.

Parameters **LinearEntity or LinearEntity3D**

Returns Point3D, Line3D, Ray3D or Segment3D

Notes

For the interaction between 2D and 3D lines(segments, rays), you should convert the line to 3D by using this method. For example for finding the intersection between a 2D and a 3D line, convert the 2D line to a 3D line by projecting it on a required plane and then proceed to find the intersection between those lines.

Examples

```
>>> a = Plane(Point3D(1, 1, 1), normal_vector=(1, 1, 1))
>>> b = Line(Point(1, 1), Point(2, 2))
>>> a.projection_line(b)
```

(continues on next page)

(continued from previous page)

```
Line3D(Point3D(4/3, 4/3, 1/3), Point3D(5/3, 5/3, -1/3))
>>> c = Line3D(Point3D(1, 1, 1), Point3D(2, 2, 2))
>>> a.projection_line(c)
Point3D(1, 1, 1)
```

random_point(*seed=None*)
Returns a random point on the Plane.

Returns Point3D

3.7 Integrals

The `integrals` module in Diofant implements methods to calculate definite and indefinite integrals of expressions.

Principal method in this module is *integrate()* (page 522)

- `integrate(f, x)` returns the indefinite integral $\int f\,dx$
- `integrate(f, (x, a, b))` returns the definite integral $\int_a^b f\,dx$

3.7.1 Examples

Diofant can integrate a vast array of functions. It can integrate polynomial functions:

```
>>> init_printing(pretty_print=True, use_unicode=False, wrap_line=False, no_
↪global=True)
>>> x = Symbol('x')
>>> integrate(x**2 + x + 1, x)
 3    2
x    x
-- + -- + x
3    2
```

Rational functions:

```
>>> integrate(x/(x**2+2*x+1), x)
               1
log(x + 1) + -----
             x + 1
```

Exponential-polynomial functions. These multiplicative combinations of polynomials and the functions `exp`, `cos` and `sin` can be integrated by hand using repeated integration by parts, which is an extremely tedious process. Happily, Diofant will deal with these integrals.

```
>>> integrate(x**2 * exp(x) * cos(x), x)
 x  2           x  2                         x           x
E *x *sin(x)   E *x *cos(x)    x            E *sin(x)   E *cos(x)
------------ + ------------ - E *x*sin(x) + --------- - ---------
     2              2                           2           2
```

even a few nonelementary integrals (in particular, some integrals involving the error function) can be evaluated:

```
>>> integrate(exp(-x**2)*erf(x), x)
  ____    2
\/ pi *erf (x)
--------------
      4
```

3.7.2 Integral Transforms

Diofant has special support for definite integrals, and integral transforms.

diofant.integrals.transforms.**mellin_transform**(*f*, *x*, *s*, ***hints*)

Compute the Mellin transform $F(s)$ of $f(x)$,

$$F(s) = \int_0^\infty x^{s-1} f(x) \mathrm{d}x.$$

For all "sensible" functions, this converges absolutely in a strip $a < \operatorname{Re}(s) < b$.

The Mellin transform is related via change of variables to the Fourier transform, and also to the (bilateral) Laplace transform.

This function returns `(F, (a, b), cond)` where `F` is the Mellin transform of `f`, `(a, b)` is the fundamental strip (as above), and `cond` are auxiliary convergence conditions.

If the integral cannot be computed in closed form, this function returns an unevaluated *MellinTransform* (page 533) object.

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532). If `noconds=False`, then only F will be returned (i.e. not `cond`, and also not the strip `(a, b)`).

```
>>> from diofant.abc import s
>>> mellin_transform(exp(-x), x, s)
(gamma(s), (0, oo), True)
```

See also:

inverse_mellin_transform (page 516), *laplace_transform* (page 517), *fourier_transform* (page 518), *hankel_transform* (page 521), *inverse_hankel_transform* (page 521)

diofant.integrals.transforms.**inverse_mellin_transform**(*F*, *s*, *x*, *strip*, ***hints*)

Compute the inverse Mellin transform of $F(s)$ over the fundamental strip given by `strip=(a, b)`.

This can be defined as

$$f(x) = \int_{c - i\infty}^{c + i\infty} x^{-s} F(s) \mathrm{d}s,$$

for any c in the fundamental strip. Under certain regularity conditions on F and/or f, this recovers f from its Mellin transform F (and vice versa), for positive real x.

One of a or b may be passed as `None`; a suitable c will be inferred.

If the integral cannot be computed in closed form, this function returns an unevaluated *InverseMellinTransform* (page 533) object.

Note that this function will assume x to be positive and real, regardless of the diofant assumptions!

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532).

```
>>> from diofant.abc import s
>>> inverse_mellin_transform(gamma(s), s, x, (0, oo))
E**(-x)
```

The fundamental strip matters:

```
>>> f = 1/(s**2 - 1)
>>> inverse_mellin_transform(f, s, x, (-oo, -1))
(x/2 - 1/(2*x))*Heaviside(x - 1)
>>> inverse_mellin_transform(f, s, x, (-1, 1))
-x*Heaviside(-x + 1)/2 - Heaviside(x - 1)/(2*x)
>>> inverse_mellin_transform(f, s, x, (1, oo))
(-x/2 + 1/(2*x))*Heaviside(-x + 1)
```

See also:

mellin_transform (page 516), *hankel_transform* (page 521), *inverse_hankel_transform* (page 521)

diofant.integrals.transforms.**laplace_transform**(*f, t, s, **hints*)

Compute the Laplace Transform $F(s)$ of $f(t)$,

$$F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$

For all "sensible" functions, this converges absolutely in a half plane $a < \mathrm{Re}(s)$.

This function returns `(F, a, cond)` where `F` is the Laplace transform of `f`, $\mathrm{Re}(s) > a$ is the half-plane of convergence, and `cond` are auxiliary convergence conditions.

If the integral cannot be computed in closed form, this function returns an unevaluated *LaplaceTransform* (page 533) object.

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532). If `noconds=True`, only F will be returned (i.e. not `cond`, and also not the plane `a`).

```
>>> from diofant.abc import t, s, a
>>> laplace_transform(t**a, t, s)
(s**(-a)*gamma(a + 1)/s, 0, -re(a) < 1)
```

See also:

inverse_laplace_transform (page 517), *mellin_transform* (page 516), *fourier_transform* (page 518), *hankel_transform* (page 521), *inverse_hankel_transform* (page 521)

diofant.integrals.transforms.**inverse_laplace_transform**(*F, s, t, plane=None, **hints*)

Compute the inverse Laplace transform of $F(s)$, defined as

$$f(t) = \int_{c-i\infty}^{c+i\infty} e^{st} F(s) \mathrm{d}s,$$

for c so large that $F(s)$ has no singularites in the half-plane $\mathrm{Re}(s) > c - \epsilon$.

The plane can be specified by argument `plane`, but will be inferred if passed as None.

Under certain regularity conditions, this recovers $f(t)$ from its Laplace Transform $F(s)$, for non-negative t, and vice versa.

If the integral cannot be computed in closed form, this function returns an unevaluated *InverseLaplaceTransform* (page 533) object.

Note that this function will always assume t to be real, regardless of the diofant assumption on t.

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532).

```
>>> from diofant.abc import s, t
>>> a = Symbol('a', positive=True)
>>> inverse_laplace_transform(exp(-a*s)/s, s, t)
Heaviside(-a + t)
```

See also:

laplace_transform (page 517), *hankel_transform* (page 521), *inverse_hankel_transform* (page 521)

diofant.integrals.transforms.**fourier_transform**(*f*, *x*, *k*, ***hints*)

Compute the unitary, ordinary-frequency Fourier transform of f, defined as

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x k} \mathrm{d}x.$$

If the transform cannot be computed in closed form, this function returns an unevaluated *FourierTransform* (page 533) object.

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532). Note that for this transform, by default noconds=True.

```
>>> fourier_transform(exp(-x**2), x, k)
E**(-pi**2*k**2)*sqrt(pi)
>>> fourier_transform(exp(-x**2), x, k, noconds=False)
(E**(-pi**2*k**2)*sqrt(pi), True)
```

See also:

inverse_fourier_transform (page 518), *sine_transform* (page 519), *inverse_sine_transform* (page 519), *cosine_transform* (page 520), *inverse_cosine_transform* (page 520), *hankel_transform* (page 521), *inverse_hankel_transform* (page 521), *mellin_transform* (page 516), *laplace_transform* (page 517)

diofant.integrals.transforms.**inverse_fourier_transform**(*F*, *k*, *x*, ***hints*)

Compute the unitary, ordinary-frequency inverse Fourier transform of F, defined as

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi i x k} \mathrm{d}k.$$

If the transform cannot be computed in closed form, this function returns an unevaluated *InverseFourierTransform* (page 533) object.

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532). Note that for this transform, by default noconds=True.

```
>>> inverse_fourier_transform(sqrt(pi)*exp(-(pi*k)**2), k, x)
E**(-x**2)
>>> inverse_fourier_transform(sqrt(pi)*exp(-(pi*k)**2), k, x, noconds=False)
(E**(-x**2), True)
```

See also:

fourier_transform (page 518), *sine_transform* (page 519), *inverse_sine_transform* (page 519), *cosine_transform* (page 520), *inverse_cosine_transform* (page 520), *hankel_transform* (page 521), *inverse_hankel_transform* (page 521), *mellin_transform* (page 516), *laplace_transform* (page 517)

diofant.integrals.transforms.**sine_transform**(*f*, *x*, *k*, ***hints*)

Compute the unitary, ordinary-frequency sine transform of f, defined as

$$F(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(2\pi x k) \mathrm{d}x.$$

If the transform cannot be computed in closed form, this function returns an unevaluated *SineTransform* (page 534) object.

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532). Note that for this transform, by default noconds=True.

```
>>> from diofant.abc import a
>>> sine_transform(x*exp(-a*x**2), x, k)
sqrt(2)*E**(-k**2/(4*a))*k/(4*a**(3/2))
>>> sine_transform(x**(-a), x, k)
2**(-a + 1/2)*k**(a - 1)*gamma(-a/2 + 1)/gamma(a/2 + 1/2)
```

See also:

fourier_transform (page 518), *inverse_fourier_transform* (page 518), *inverse_sine_transform* (page 519), *cosine_transform* (page 520), *inverse_cosine_transform* (page 520), *hankel_transform* (page 521), *inverse_hankel_transform* (page 521), *mellin_transform* (page 516), *laplace_transform* (page 517)

diofant.integrals.transforms.**inverse_sine_transform**(*F*, *k*, *x*, ***hints*)

Compute the unitary, ordinary-frequency inverse sine transform of F, defined as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(k) \sin(2\pi x k) \mathrm{d}k.$$

If the transform cannot be computed in closed form, this function returns an unevaluated *InverseSineTransform* (page 534) object.

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532). Note that for this transform, by default noconds=True.

```
>>> from diofant.abc import a
>>> inverse_sine_transform(2**((1-2*a)/2)*k**(a - 1)*
...     gamma(-a/2 + 1)/gamma((a+1)/2), k, x)
x**(-a)
>>> inverse_sine_transform(sqrt(2)*k*exp(-k**2/(4*a))/(4*sqrt(a)**3), k, x)
E**(-a*x**2)*x
```

See also:

fourier_transform (page 518), *inverse_fourier_transform* (page 518), *sine_transform* (page 519), *cosine_transform* (page 520), *inverse_cosine_transform* (page 520), *hankel_transform* (page 521), *inverse_hankel_transform* (page 521), *mellin_transform* (page 516), *laplace_transform* (page 517)

`diofant.integrals.transforms.`**`cosine_transform`**(*f*, *x*, *k*, ***hints*)

Compute the unitary, ordinary-frequency cosine transform of f, defined as

$$F(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(2\pi xk) \mathrm{d}x.$$

If the transform cannot be computed in closed form, this function returns an unevaluated *CosineTransform* (page 534) object.

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532). Note that for this transform, by default noconds=True.

```
>>> from diofant.abc import a
>>> cosine_transform(exp(-a*x), x, k)
sqrt(2)*a/(sqrt(pi)*(a**2 + k**2))
>>> cosine_transform(exp(-a*sqrt(x))*cos(a*sqrt(x)), x, k)
E**(-a**2/(2*k))*a/(2*k**(3/2))
```

See also:

fourier_transform (page 518), *inverse_fourier_transform* (page 518), *sine_transform* (page 519), *inverse_sine_transform* (page 519), *inverse_cosine_transform* (page 520), *hankel_transform* (page 521), *inverse_hankel_transform* (page 521), *mellin_transform* (page 516), *laplace_transform* (page 517)

`diofant.integrals.transforms.`**`inverse_cosine_transform`**(*F*, *k*, *x*, ***hints*)

Compute the unitary, ordinary-frequency inverse cosine transform of F, defined as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(k) \cos(2\pi xk) \mathrm{d}k.$$

If the transform cannot be computed in closed form, this function returns an unevaluated *InverseCosineTransform* (page 534) object.

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532). Note that for this transform, by default noconds=True.

```
>>> from diofant.abc import a
>>> inverse_cosine_transform(sqrt(2)*a/(sqrt(pi)*(a**2 + k**2)), k, x)
E**(-a*x)
>>> inverse_cosine_transform(1/sqrt(k), k, x)
1/sqrt(x)
```

See also:

fourier_transform (page 518), *inverse_fourier_transform* (page 518), *sine_transform* (page 519), *inverse_sine_transform* (page 519), *cosine_transform* (page 520), *hankel_transform* (page 521), *inverse_hankel_transform* (page 521), *mellin_transform* (page 516), *laplace_transform* (page 517)

diofant.integrals.transforms.**hankel_transform**(*f*, *r*, *k*, *nu*, ***hints*)

Compute the Hankel transform of f, defined as

$$F_\nu(k) = \int_0^\infty f(r) J_\nu(k r) r \mathrm{d} r.$$

If the transform cannot be computed in closed form, this function returns an unevaluated *HankelTransform* (page 534) object.

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532). Note that for this transform, by default noconds=True.

```
>>> from diofant.abc import r, nu, a, k
```

```
>>> ht = hankel_transform(1/r**m, r, k, nu)
>>> ht
2*2**(-m)*k**(m - 2)*gamma(-m/2 + nu/2 + 1)/gamma(m/2 + nu/2)
```

```
>>> inverse_hankel_transform(ht, k, r, nu)
r**(-m)
```

```
>>> ht = hankel_transform(exp(-a*r), r, k, 0)
>>> ht
a/(k**3*(a**2/k**2 + 1)**(3/2))
```

```
>>> inverse_hankel_transform(ht, k, r, 0)
E**(-a*r)
```

See also:

fourier_transform (page 518), *inverse_fourier_transform* (page 518), *sine_transform* (page 519), *inverse_sine_transform* (page 519), *cosine_transform* (page 520), *inverse_cosine_transform* (page 520), *inverse_hankel_transform* (page 521), *mellin_transform* (page 516), *laplace_transform* (page 517)

diofant.integrals.transforms.**inverse_hankel_transform**(*F*, *k*, *r*, *nu*, ***hints*)

Compute the inverse Hankel transform of F defined as

$$f(r) = \int_0^\infty F_\nu(k) J_\nu(k r) k \mathrm{d} k.$$

If the transform cannot be computed in closed form, this function returns an unevaluated *InverseHankelTransform* (page 534) object.

For a description of possible hints, refer to the docstring of *diofant.integrals.transforms.IntegralTransform.doit()* (page 532). Note that for this transform, by default noconds=True.

```
>>> from diofant.abc import r, nu, a, k
```

```
>>> ht = hankel_transform(1/r**m, r, k, nu)
>>> ht
2*2**(-m)*k**(m - 2)*gamma(-m/2 + nu/2 + 1)/gamma(m/2 + nu/2)
```

```
>>> inverse_hankel_transform(ht, k, r, nu)
r**(-m)
```

```
>>> ht = hankel_transform(exp(-a*r), r, k, 0)
>>> ht
a/(k**3*(a**2/k**2 + 1)**(3/2))
```

```
>>> inverse_hankel_transform(ht, k, r, 0)
E**(-a*r)
```

See also:

fourier_transform (page 518), *inverse_fourier_transform* (page 518), *sine_transform* (page 519), *inverse_sine_transform* (page 519), *cosine_transform* (page 520), *inverse_cosine_transform* (page 520), *hankel_transform* (page 521), *mellin_transform* (page 516), *laplace_transform* (page 517)

3.7.3 Internals

There is a general method for calculating antiderivatives of elementary functions, called the *Risch algorithm*. The Risch algorithm is a decision procedure that can determine whether an elementary solution exists, and in that case calculate it. It can be extended to handle many nonelementary functions in addition to the elementary ones.

Diofant currently uses a simplified version of the Risch algorithm, called the *Risch-Norman algorithm*. This algorithm is much faster, but may fail to find an antiderivative, although it is still very powerful. Diofant also uses pattern matching and heuristics to speed up evaluation of some types of integrals, e.g. polynomials.

For non-elementary definite integrals, Diofant uses so-called Meijer G-functions. Details are described *here* (page 1165).

3.7.4 API reference

diofant.integrals.integrals.**integrate**(*f*, *var*, *...*)

Compute definite or indefinite integral of one or more variables using Risch-Norman algorithm and table lookup. This procedure is able to handle elementary algebraic and transcendental functions and also a huge class of special functions, including Airy, Bessel, Whittaker and Lambert.

var can be:

- a symbol – indefinite integration
- **a tuple (symbol, a) – indefinite integration with result** given with a replacing $symbol$
- a tuple (symbol, a, b) – definite integration

Several variables can be specified, in which case the result is multiple integration. (If var is omitted and the integrand is univariate, the indefinite integral in that variable will be performed.)

Indefinite integrals are returned without terms that are independent of the integration variables. (see examples)

Definite improper integrals often entail delicate convergence conditions. Pass conds='piecewise', 'separate' or 'none' to have these returned, respectively, as a Piecewise function, as a separate result (i.e. result will be a tuple), or not at all (default is 'piecewise').

Strategy

Diofant uses various approaches to definite integration. One method is to find an antiderivative for the integrand, and then use the fundamental theorem of calculus. Various functions are implemented to integrate polynomial, rational and trigonometric functions, and integrands containing DiracDelta terms.

Diofant also implements the part of the Risch algorithm, which is a decision procedure for integrating elementary functions, i.e., the algorithm can either find an elementary antiderivative, or prove that one does not exist. There is also a (very successful, albeit somewhat slow) general implementation of the heuristic Risch algorithm. This algorithm will eventually be phased out as more of the full Risch algorithm is implemented. See the docstring of Integral._eval_integral() for more details on computing the antiderivative using algebraic methods.

The option risch=True can be used to use only the (full) Risch algorithm. This is useful if you want to know if an elementary function has an elementary antiderivative. If the indefinite Integral returned by this function is an instance of NonElementaryIntegral, that means that the Risch algorithm has proven that integral to be non-elementary. Note that by default, additional methods (such as the Meijer G method outlined below) are tried on these integrals, as they may be expressible in terms of special functions, so if you only care about elementary answers, use risch=True. Also note that an unevaluated Integral returned by this function is not necessarily a NonElementaryIntegral, even with risch=True, as it may just be an indication that the particular part of the Risch algorithm needed to integrate that function is not yet implemented.

Another family of strategies comes from re-writing the integrand in terms of so-called Meijer G-functions. Indefinite integrals of a single G-function can always be computed, and the definite integral of a product of two G-functions can be computed from zero to infinity. Various strategies are implemented to rewrite integrands as G-functions, and use this information to compute integrals (see the `meijerint` module).

In general, the algebraic methods work best for computing antiderivatives of (possibly complicated) combinations of elementary functions. The G-function methods work best for computing definite integrals from zero to infinity of moderately complicated combinations of special functions, or indefinite integrals of very simple combinations of special functions.

The strategy employed by the integration code is as follows:

- If computing a definite integral, and both limits are real, and at least one limit is +-oo, try the G-function method of definite integration first.
- Try to find an antiderivative, using all available methods, ordered by performance (that is try fastest method first, slowest last; in particular polynomial integration is tried first, Meijer G-functions second to last, and heuristic Risch last).
- If still not successful, try G-functions irrespective of the limits.

The option meijerg=True, False, None can be used to, respectively: always use G-function methods and no others, never use G-function methods, or use all available methods (in order as described above). It defaults to None.

See also:

diofant.integrals.integrals.Integral (page 529), *diofant.integrals.integrals.Integral.doit* (page 530)

Examples

```
>>> from diofant.abc import a
```

```
>>> integrate(x*y, x)
x**2*y/2
```

```
>>> integrate(log(x), x)
x*log(x) - x
```

```
>>> integrate(log(x), (x, 1, a))
a*log(a) - a + 1
```

```
>>> integrate(x)
x**2/2
```

Terms that are independent of x are dropped by indefinite integration:

```
>>> integrate(sqrt(1 + x), (x, 0, x))
2*(x + 1)**(3/2)/3 - 2/3
>>> integrate(sqrt(1 + x), x)
2*(x + 1)**(3/2)/3
```

```
>>> integrate(x*y)
Traceback (most recent call last):
...
ValueError: specify integration variables to integrate x*y
```

Note that `integrate(x)` syntax is meant only for convenience in interactive sessions and should be avoided in library code.

```
>>> integrate(x**a*exp(-x), (x, 0, oo)) # same as conds='piecewise'
Piecewise((gamma(a + 1), -re(a) < 1),
    (Integral(E**(-x)*x**a, (x, 0, oo)), true))
```

```
>>> integrate(x**a*exp(-x), (x, 0, oo), conds='none')
gamma(a + 1)
```

```
>>> integrate(x**a*exp(-x), (x, 0, oo), conds='separate')
(gamma(a + 1), -re(a) < 1)
```

`diofant.integrals.integrals.`**`line_integrate`**(*field, Curve, variables*)

Compute the line integral.

See also:

diofant.integrals.integrals.integrate (page 522), *diofant.integrals.integrals.Integral* (page 529)

Examples

```
>>> from diofant.abc import t
>>> C = Curve([E**t + 1, E**t - 1], (t, 0, ln(2)))
>>> line_integrate(x + y, C, [x, y])
3*sqrt(2)
```

diofant.integrals.deltafunctions.**deltaintegrate**(*f*, *x*)

The idea for integration is the following:

- If we are dealing with a DiracDelta expression, i.e. DiracDelta(g(x)), we try to simplify it.

  If we could simplify it, then we integrate the resulting expression. We already know we can integrate a simplified expression, because only simple DiracDelta expressions are involved.

  If we couldn't simplify it, there are two cases:

  1. The expression is a simple expression: we return the integral, taking care if we are dealing with a Derivative or with a proper DiracDelta.
  2. The expression is not simple (i.e. DiracDelta(cos(x))): we can do nothing at all.

- If the node is a multiplication node having a DiracDelta term:

  First we expand it.

  If the expansion did work, then we try to integrate the expansion.

  If not, we try to extract a simple DiracDelta term, then we have two cases:

  1. We have a simple DiracDelta term, so we return the integral.
  2. We didn't have a simple term, but we do have an expression with simplified DiracDelta terms, so we integrate this expression.

See also:

diofant.functions.special.delta_functions.DiracDelta (page 348), *diofant.integrals.integrals.Integral* (page 529)

Examples

```
>>> deltaintegrate(x*sin(x)*cos(x)*DiracDelta(x - 1), x)
sin(1)*cos(1)*Heaviside(x - 1)
>>> deltaintegrate(y**2*DiracDelta(x - z)*DiracDelta(y - z), y)
z**2*DiracDelta(x - z)*Heaviside(y - z)
```

diofant.integrals.rationaltools.**ratint**(*f*, *x*, ***flags*)

Performs indefinite integration of rational functions.

Given a field K and a rational function $f = p/q$, where p and q are polynomials in $K[x]$, returns a function g such that $f = g'$.

```
>>> ratint(36/(x**5 - 2*x**4 - 2*x**3 + 4*x**2 + x - 2), x)
(12*x + 6)/(x**2 - 1) + 4*log(x - 2) - 4*log(x + 1)
```

See also:

diofant.integrals.integrals.Integral.doit (page 530), *diofant.integrals.rationaltools.ratint_logpart* (page 526), *diofant.integrals.rationaltools.ratint_ratpart* (page 526)

References

[Bro05388] (page 1260)

diofant.integrals.rationaltools.**ratint_logpart**(*f, g, x, t=None*)

Lazard-Rioboo-Trager algorithm.

Given a field K and polynomials f and g in K[x], such that f and g are coprime, deg(f) < deg(g) and g is square-free, returns a list of tuples (s_i, q_i) of polynomials, for i = 1..n, such that s_i in K[t, x] and q_i in K[t], and:

```
           ___    ___
d  f   d   \  `   \  `
-- - = --   )      )   a log(s_i(a, x))
dx g   dx  /__,   /__,
          i=1..n a | q_i(a) = 0
```

See also:

diofant.integrals.rationaltools.ratint (page 525), *diofant.integrals.rationaltools.ratint_ratpart* (page 526)

Examples

```
>>> ratint_logpart(Poly(1, x), Poly(x**2 + x + 1, x), x)
[(Poly(x + 3*_t/2 + 1/2, x, domain='QQ[_t]'),
  Poly(3*_t**2 + 1, _t, domain='ZZ'))]
>>> ratint_logpart(Poly(12, x), Poly(x**2 - x - 2, x), x)
[(Poly(x - 3*_t/8 - 1/2, x, domain='QQ[_t]'),
  Poly(-_t**2 + 16, _t, domain='ZZ'))]
```

diofant.integrals.rationaltools.**ratint_ratpart**(*f, g, x*)

Horowitz-Ostrogradsky algorithm.

Given a field K and polynomials f and g in K[x], such that f and g are coprime and deg(f) < deg(g), returns fractions A and B in K(x), such that f/g = A' + B and B has square-free denominator.

See also:

diofant.integrals.rationaltools.ratint (page 525), *diofant.integrals.rationaltools.ratint_logpart* (page 526)

Examples

```
>>> ratint_ratpart(Poly(1, x), Poly(x + 1, x), x)
(0, 1/(x + 1))
>>> ratint_ratpart(Poly(1, x, domain='EX'),
...                Poly(x**2 + y**2, x, domain='EX'), x)
(0, 1/(x**2 + y**2))
```

(continues on next page)

(continued from previous page)

```
>>> ratint_ratpart(Poly(36, x),
...                Poly(x**5 - 2*x**4 - 2*x**3 + 4*x**2 + x - 2, x), x)
((12*x + 6)/(x**2 - 1), 12/(x**2 - x - 2))
```

diofant.integrals.heurisch.**components**(*f*, *x*)

Returns a set of all functional components of the given expression which includes symbols, function applications and compositions and non-integer powers. Fractional powers are collected with with minimal, positive exponents.

```
>>> components(sin(x)*cos(x)**2, x)
{x, sin(x), cos(x)}
```

See also:

diofant.integrals.heurisch.heurisch (page 527)

diofant.integrals.heurisch.**heurisch**(*f*, *x*, *rewrite=False*, *hints=None*, *mappings=None*, *retries=3*, *degree_offset=0*, *unnecessary_permutations=None*)

Compute indefinite integral using heuristic Risch algorithm.

This is a heuristic approach to indefinite integration in finite terms using the extended heuristic (parallel) Risch algorithm, based on Manuel Bronstein's "Poor Man's Integrator" *[R390]* (page 1260).

The algorithm supports various classes of functions including transcendental elementary or special functions like Airy, Bessel, Whittaker and Lambert.

Note that this algorithm is not a decision procedure. If it isn't able to compute the antiderivative for a given function, then this is not a proof that such a functions does not exist. One should use recursive Risch algorithm in such case. It's an open question if this algorithm can be made a full decision procedure.

This is an internal integrator procedure. You should use toplevel 'integrate' function in most cases, as this procedure needs some preprocessing steps and otherwise may fail.

Parameters **f** : Expr

expression

x : Symbol

variable

rewrite : Boolean, optional

force rewrite 'f' in terms of 'tan' and 'tanh', default False.

hints : None or list

a list of functions that may appear in anti-derivate. If None (default) - no suggestions at all, if empty list - try to figure out.

See also:

diofant.integrals.integrals.Integral.doit (page 530), *diofant.integrals.integrals.Integral* (page 529), *diofant.integrals.heurisch.components* (page 527)

References

[R390] (page 1260), *[R391]* (page 1260), *[R392]* (page 1260), *[R393]* (page 1260), *[R394]* (page 1260)

Examples

```
>>> heurisch(y*tan(x), x)
y*log(tan(x)**2 + 1)/2
```

diofant.integrals.heurisch.**heurisch_wrapper**(*f*, *x*, *rewrite=False*, *hints=None*, *mappings=None*, *retries=3*, *degree_offset=0*, *unnecessary_permutations=None*)

A wrapper around the heurisch integration algorithm.

This method takes the result from heurisch and checks for poles in the denominator. For each of these poles, the integral is reevaluated, and the final integration result is given in terms of a Piecewise.

See also:

diofant.integrals.heurisch.heurisch (page 527)

Examples

```
>>> heurisch(cos(n*x), x)
sin(n*x)/n
>>> heurisch_wrapper(cos(n*x), x)
Piecewise((x, Eq(n, 0)), (sin(n*x)/n, true))
```

diofant.integrals.trigonometry.**trigintegrate**(*f*, *x*, *conds='piecewise'*)

Integrate f = Mul(trig) over x

```
>>> trigintegrate(sin(x)*cos(x), x)
sin(x)**2/2
```

```
>>> trigintegrate(sin(x)**2, x)
x/2 - sin(x)*cos(x)/2
```

```
>>> trigintegrate(tan(x)*sec(x), x)
1/cos(x)
```

```
>>> trigintegrate(sin(x)*tan(x), x)
-log(sin(x) - 1)/2 + log(sin(x) + 1)/2 - sin(x)
```

See also:

diofant.integrals.integrals.Integral.doit (page 530), *diofant.integrals.integrals.Integral* (page 529)

References

[R395] (page 1260)

The class $Integral$ represents an unevaluated integral and has some methods that help in the integration of an expression.

class diofant.integrals.integrals.**Integral**
Represents unevaluated integral.

is_commutative
Returns whether all the free symbols in the integral are commutative.

as_sum(*n*, *method='midpoint'*)
Approximates the definite integral by a sum.

method … one of: left, right, midpoint, trapezoid

These are all basically the rectangle method [1], the only difference is where the function value is taken in each interval to define the rectangle.

See also:

diofant.integrals.integrals.Integral.doit **(page 530)** Perform the integration using any hints

References

[R396] (page 1260)

Examples

```
>>> e = Integral(sin(x), (x, 3, 7))
>>> e
Integral(sin(x), (x, 3, 7))
```

For demonstration purposes, this interval will only be split into 2 regions, bounded by [3, 5] and [5, 7].

The left-hand rule uses function evaluations at the left of each interval:

```
>>> e.as_sum(2, 'left')
2*sin(5) + 2*sin(3)
```

The midpoint rule uses evaluations at the center of each interval:

```
>>> e.as_sum(2, 'midpoint')
2*sin(4) + 2*sin(6)
```

The right-hand rule uses function evaluations at the right of each interval:

```
>>> e.as_sum(2, 'right')
2*sin(5) + 2*sin(7)
```

The trapezoid rule uses function evaluations on both sides of the intervals. This is equivalent to taking the average of the left and right hand rule results:

```
>>> e.as_sum(2, 'trapezoid')
2*sin(5) + sin(3) + sin(7)
>>> (e.as_sum(2, 'left') + e.as_sum(2, 'right'))/2 == _
True
```

All but the trapexoid method may be used when dealing with a function with a discontinuity. Here, the discontinuity at x = 0 can be avoided by using the midpoint or right-hand method:

```
>>> e = Integral(1/sqrt(x), (x, 0, 1))
>>> e.as_sum(5).n(4)
1.730
>>> e.as_sum(10).n(4)
1.809
>>> e.doit().n(4)  # the actual value is 2
2.000
```

The left- or trapezoid method will encounter the discontinuity and return oo:

```
>>> e.as_sum(5, 'left')
oo
>>> e.as_sum(5, 'trapezoid')
oo
```

doit(***hints*)

Perform the integration using any hints given.

See also:

diofant.integrals.trigonometry.trigintegrate (page 528), *diofant.integrals.heurisch.heurisch* (page 527), *diofant.integrals.rationaltools.ratint* (page 525)

diofant.integrals.integrals.Integral.as_sum **(page 529)** Approximate the integral using a sum

Examples

```
>>> from diofant.abc import i
>>> Integral(x**i, (i, 1, 3)).doit()
Piecewise((2, Eq(log(x), 0)), (x**3/log(x) - x/log(x), true))
```

free_symbols

This method returns the symbols that will exist when the integral is evaluated. This is useful if one is trying to determine whether an integral depends on a certain symbol or not.

See also:

diofant.concrete.expr_with_limits.ExprWithLimits.function (page 282), *diofant.concrete.expr_with_limits.ExprWithLimits.limits* (page 283), *diofant.concrete.expr_with_limits.ExprWithLimits.variables* (page 283)

Examples

```
>>> Integral(x, (x, y, 1)).free_symbols
{y}
```

transform(*x*, *u*)

Performs a change of variables from x to u using the relationship given by x and u which will define the transformations f and F (which are inverses of each other) as follows:

1. If x is a Symbol (which is a variable of integration) then u will be interpreted as some function, f(u), with inverse F(u). This, in effect, just makes the substitution of x with f(x).
2. If u is a Symbol then x will be interpreted as some function, F(x), with inverse f(u). This is commonly referred to as u-substitution.

Once f and F have been identified, the transformation is made as follows:

$$\int_a^b x \mathrm{d}x \rightarrow \int_{F(a)}^{F(b)} f(x) \frac{\mathrm{d}}{\mathrm{d}x}$$

where $F(x)$ is the inverse of $f(x)$ and the limits and integrand have been corrected so as to retain the same value after integration.

See also:

diofant.concrete.expr_with_limits.ExprWithLimits.variables **(page 283)**
Lists the integration variables

diofant.concrete.expr_with_limits.ExprWithLimits.as_dummy **(page 282)**
Replace integration variables with dummy ones

Notes

The mappings, F(x) or f(u), must lead to a unique integral. Linear or rational linear expression, $2*x$, $1/x$ and $sqrt(x)$, will always work; quadratic expressions like $x**2-1$ are acceptable as long as the resulting integrand does not depend on the sign of the solutions (see examples).

The integral will be returned unchanged if x is not a variable of integration.

x must be (or contain) only one of of the integration variables. If u has more than one free symbol then it should be sent as a tuple (u, $uvar$) where $uvar$ identifies which variable is replacing the integration variable. XXX can it contain another integration variable?

Examples

```
>>> from diofant.abc import a, b, c, d, u
```

```
>>> i = Integral(x*cos(x**2 - 1), (x, 0, 1))
```

transform can change the variable of integration

```
>>> i.transform(x, u)
Integral(u*cos(u**2 - 1), (u, 0, 1))
```

transform can perform u-substitution as long as a unique integrand is obtained:

```
>>> i.transform(x**2 - 1, u)
Integral(cos(u)/2, (u, -1, 0))
```

This attempt fails because x = +/-sqrt(u + 1) and the sign does not cancel out of the integrand:

```
>>> Integral(cos(x**2 - 1), (x, 0, 1)).transform(x**2 - 1, u)
Traceback (most recent call last):
...
ValueError:
The mapping between F(x) and f(u) did not give a unique integrand.
```

transform can do a substitution. Here, the previous result is transformed back into the original expression using "u-substitution":

```
>>> ui = _
>>> _.transform(sqrt(u + 1), x) == i
True
```

We can accomplish the same with a regular substitution:

```
>>> ui.transform(u, x**2 - 1) == i
True
```

If the x does not contain a symbol of integration then the integral will be returned unchanged. Integral i does not have an integration variable a so no change is made:

```
>>> i.transform(a, x) == i
True
```

When u has more than one free symbol the symbol that is replacing x must be identified by passing u as a tuple:

```
>>> Integral(x, (x, 0, 1)).transform(x, (u + a, u))
Integral(a + u, (u, -a, -a + 1))
>>> Integral(x, (x, 0, 1)).transform(x, (u + a, a))
Integral(a + u, (a, -u, -u + 1))
```

class diofant.integrals.transforms.**IntegralTransform**

Base class for integral transforms.

This class represents unevaluated transforms.

To implement a concrete transform, derive from this class and implement the `_compute_transform(f, x, s, **hints)` and `_as_integral(f, x, s)` functions. If the transform cannot be computed, raise IntegralTransformError.

Also set `cls._name`.

Implement self._collapse_extra if your function returns more than just a number and possibly a convergence condition.

doit(***hints*)

Try to evaluate the transform in closed form.

This general function handles linearity, but apart from that leaves pretty much everything to _compute_transform.

Standard hints are the following:

- `simplify`: whether or not to simplify the result
- `noconds`: if True, don't return convergence conditions
- **`needeval`: if True, raise IntegralTransformError instead of** returning IntegralTransform objects

The default values of these hints depend on the concrete transform, usually the default is `(simplify, noconds, needeval) = (True, False, False)`.

`free_symbols`
This method returns the symbols that will exist when the transform is evaluated.

`function`
The function to be transformed.

`function_variable`
The dependent variable of the function to be transformed.

`transform_variable`
The independent transform variable.

class `diofant.integrals.transforms.`**`MellinTransform`**
Class representing unevaluated Mellin transforms.

See also:

`IntegralTransform` (page 532), *`mellin_transform`* (page 516)

class `diofant.integrals.transforms.`**`InverseMellinTransform`**
Class representing unevaluated inverse Mellin transforms.

See also:

`IntegralTransform` (page 532), *`inverse_mellin_transform`* (page 516)

class `diofant.integrals.transforms.`**`LaplaceTransform`**
Class representing unevaluated Laplace transforms.

See also:

`IntegralTransform` (page 532), *`laplace_transform`* (page 517)

class `diofant.integrals.transforms.`**`InverseLaplaceTransform`**
Class representing unevaluated inverse Laplace transforms.

See also:

`IntegralTransform` (page 532), *`inverse_laplace_transform`* (page 517)

class `diofant.integrals.transforms.`**`FourierTransform`**
Class representing unevaluated Fourier transforms.

See also:

`IntegralTransform` (page 532), *`fourier_transform`* (page 518)

class `diofant.integrals.transforms.`**`InverseFourierTransform`**
Class representing unevaluated inverse Fourier transforms.

See also:

IntegralTransform (page 532), *inverse_fourier_transform* (page 518)

class diofant.integrals.transforms.**SineTransform**
Class representing unevaluated sine transforms.

See also:

IntegralTransform (page 532), *sine_transform* (page 519)

class diofant.integrals.transforms.**InverseSineTransform**
Class representing unevaluated inverse sine transforms.

See also:

IntegralTransform (page 532), *inverse_sine_transform* (page 519)

class diofant.integrals.transforms.**CosineTransform**
Class representing unevaluated cosine transforms.

See also:

IntegralTransform (page 532), *cosine_transform* (page 520)

class diofant.integrals.transforms.**InverseCosineTransform**
Class representing unevaluated inverse cosine transforms.

See also:

IntegralTransform (page 532), *inverse_cosine_transform* (page 520)

class diofant.integrals.transforms.**HankelTransform**
Class representing unevaluated Hankel transforms.

See also:

IntegralTransform (page 532), *hankel_transform* (page 521)

class diofant.integrals.transforms.**InverseHankelTransform**
Class representing unevaluated inverse Hankel transforms.

See also:

IntegralTransform (page 532), *inverse_hankel_transform* (page 521)

3.7.5 Numeric Integrals

Diofant has functions to calculate points and weights for Gaussian quadrature of any order and any precision:

diofant.integrals.quadrature.**gauss_legendre**(*n*, *n_digits*)
Computes the Gauss-Legendre quadrature *[R397]* (page 1260) points and weights.

The Gauss-Legendre quadrature approximates the integral:

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

The nodes x_i of an order n quadrature rule are the roots of P_n and the weights w_i are given by:

$$w_i = \frac{2}{\left(1 - x_i^2\right)\left(P'_n(x_i)\right)^2}$$

Parameters **n** : the order of quadrature

n_digits : number of significant digits of the points and weights to return

Returns **(x, w)** : the x and w are lists of points and weights as Floats.

The points x_i and weights w_i are returned as (x, w) tuple of lists.

See also:

gauss_laguerre (page 535), *gauss_gen_laguerre* (page 537), *gauss_hermite* (page 536), *gauss_chebyshev_t* (page 538), *gauss_chebyshev_u* (page 538), *gauss_jacobi* (page 539)

References

[R397] (page 1260), [R398] (page 1260)

Examples

```
>>> x, w = gauss_legendre(3, 5)
>>> x
[-0.7746, 0, 0.7746]
>>> w
[0.55556, 0.88889, 0.55556]
```

```
>>> x, w = gauss_legendre(4, 5)
>>> x
[-0.86114, -0.33998, 0.33998, 0.86114]
>>> w
[0.34785, 0.65215, 0.65215, 0.34785]
```

diofant.integrals.quadrature.**gauss_laguerre**(*n*, *n_digits*)

Computes the Gauss-Laguerre quadrature [R399] (page 1260) points and weights.

The Gauss-Laguerre quadrature approximates the integral:

$$\int_0^{\infty} e^{-x} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

The nodes x_i of an order n quadrature rule are the roots of L_n and the weights w_i are given by:

$$w_i = \frac{x_i}{(n+1)^2 \left(L_{n+1}(x_i)\right)^2}$$

Parameters **n** : the order of quadrature

n_digits : number of significant digits of the points and weights to return

Returns **(x, w)** : the x and w are lists of points and weights as Floats.

The points x_i and weights w_i are returned as (x, w) tuple of lists.

See also:

gauss_legendre (page 534), *gauss_gen_laguerre* (page 537), *gauss_hermite* (page 536), *gauss_chebyshev_t* (page 538), *gauss_chebyshev_u* (page 538), *gauss_jacobi* (page 539)

References

[R399] (page 1260), [R400] (page 1260)

Examples

```
>>> x, w = gauss_laguerre(3, 5)
>>> x
[0.41577, 2.2943, 6.2899]
>>> w
[0.71109, 0.27852, 0.010389]
```

```
>>> x, w = gauss_laguerre(6, 5)
>>> x
[0.22285, 1.1889, 2.9927, 5.7751, 9.8375, 15.983]
>>> w
[0.45896, 0.417, 0.11337, 0.010399, 0.00026102, 8.9855e-7]
```

diofant.integrals.quadrature.**gauss_hermite**(*n*, *n_digits*)

Computes the Gauss-Hermite quadrature [R401] (page 1260) points and weights.

The Gauss-Hermite quadrature approximates the integral:

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

The nodes x_i of an order n quadrature rule are the roots of H_n and the weights w_i are given by:

$$w_i = \frac{2^{n-1} n! \sqrt{\pi}}{n^2 \left(H_{n-1}(x_i)\right)^2}$$

Parameters n : the order of quadrature

n_digits : number of significant digits of the points and weights to return

Returns (x, w) : the `x` and `w` are lists of points and weights as Floats.

The points x_i and weights w_i are returned as `(x, w)` tuple of lists.

See also:

gauss_legendre (page 534), *gauss_laguerre* (page 535), *gauss_gen_laguerre* (page 537), *gauss_chebyshev_t* (page 538), *gauss_chebyshev_u* (page 538), *gauss_jacobi* (page 539)

References

[R401] (page 1260), [R402] (page 1260), [R403] (page 1260)

Examples

```
>>> x, w = gauss_hermite(3, 5)
>>> x
[-1.2247, 0, 1.2247]
>>> w
[0.29541, 1.1816, 0.29541]
```

```
>>> x, w = gauss_hermite(6, 5)
>>> x
[-2.3506, -1.3358, -0.43608, 0.43608, 1.3358, 2.3506]
>>> w
[0.00453, 0.15707, 0.72463, 0.72463, 0.15707, 0.00453]
```

diofant.integrals.quadrature.**gauss_gen_laguerre**(*n*, *alpha*, *n_digits*)

Computes the generalized Gauss-Laguerre quadrature [R404] (page 1260) points and weights.

The generalized Gauss-Laguerre quadrature approximates the integral:

$$\int_0^{\infty} x^{\alpha} e^{-x} f(x)\,dx \approx \sum_{i=1}^n w_i f(x_i)$$

The nodes x_i of an order n quadrature rule are the roots of L_n^{α} and the weights w_i are given by:

$$w_i = \frac{\Gamma(\alpha+n)}{n\Gamma(n) L_{n-1}^{\alpha}(x_i) L_{n-1}^{\alpha+1}(x_i)}$$

Parameters n : the order of quadrature

alpha : the exponent of the singularity, $\alpha > -1$

n_digits : number of significant digits of the points and weights to return

Returns (x, w) : the x and w are lists of points and weights as Floats.

The points x_i and weights w_i are returned as `(x, w)` tuple of lists.

See also:

gauss_legendre (page 534), *gauss_laguerre* (page 535), *gauss_hermite* (page 536), *gauss_chebyshev_t* (page 538), *gauss_chebyshev_u* (page 538), *gauss_jacobi* (page 539)

References

[R404] (page 1260), [R405] (page 1260)

Examples

```
>>> x, w = gauss_gen_laguerre(3, -0.5, 5)
>>> x
[0.19016, 1.7845, 5.5253]
>>> w
[1.4493, 0.31413, 0.00906]
```

```
>>> x, w = gauss_gen_laguerre(4, 1.5, 5)
>>> x
[0.97851, 2.9904, 6.3193, 11.712]
>>> w
[0.53087, 0.67721, 0.11895, 0.0023152]
```

diofant.integrals.quadrature.**gauss_chebyshev_t**(*n*, *n_digits*)

Computes the Gauss-Chebyshev quadrature [R406] (page 1260) points and weights of the first kind.

The Gauss-Chebyshev quadrature of the first kind approximates the integral:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

The nodes x_i of an order n quadrature rule are the roots of T_n and the weights w_i are given by:

$$w_i = \frac{\pi}{n}$$

Parameters n : the order of quadrature

n_digits : number of significant digits of the points and weights to return

Returns (x, w) : the x and w are lists of points and weights as Floats.

The points x_i and weights w_i are returned as (x, w) tuple of lists.

See also:

gauss_legendre (page 534), *gauss_laguerre* (page 535), *gauss_hermite* (page 536), *gauss_gen_laguerre* (page 537), *gauss_chebyshev_u* (page 538), *gauss_jacobi* (page 539)

References

[R406] (page 1260), [R407] (page 1260)

Examples

```
>>> x, w = gauss_chebyshev_t(3, 5)
>>> x
[0.86602, 0, -0.86602]
>>> w
[1.0472, 1.0472, 1.0472]
```

```
>>> x, w = gauss_chebyshev_t(6, 5)
>>> x
[0.96593, 0.70711, 0.25882, -0.25882, -0.70711, -0.96593]
>>> w
[0.5236, 0.5236, 0.5236, 0.5236, 0.5236, 0.5236]
```

diofant.integrals.quadrature.**gauss_chebyshev_u**(*n*, *n_digits*)

Computes the Gauss-Chebyshev quadrature [R408] (page 1260) points and weights of the second kind.

The Gauss-Chebyshev quadrature of the second kind approximates the integral:

$$\int_{-1}^{1} \sqrt{1-x^2} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

The nodes x_i of an order n quadrature rule are the roots of U_n and the weights w_i are given by:

$$w_i = \frac{\pi}{n+1} \sin^2\left(\frac{i}{n+1}\pi\right)$$

Parameters n : the order of quadrature

n_digits : number of significant digits of the points and weights to return

Returns (x, w) : the x and w are lists of points and weights as Floats.

The points x_i and weights w_i are returned as (x, w) tuple of lists.

See also:

gauss_legendre (page 534), *gauss_laguerre* (page 535), *gauss_hermite* (page 536), *gauss_gen_laguerre* (page 537), *gauss_chebyshev_t* (page 538), *gauss_jacobi* (page 539)

References

[R408] (page 1260), *[R409]* (page 1260)

Examples

```
>>> x, w = gauss_chebyshev_u(3, 5)
>>> x
[0.70711, 0, -0.70711]
>>> w
[0.3927, 0.7854, 0.3927]
```

```
>>> x, w = gauss_chebyshev_u(6, 5)
>>> x
[0.90097, 0.62349, 0.22252, -0.22252, -0.62349, -0.90097]
>>> w
[0.084489, 0.27433, 0.42658, 0.42658, 0.27433, 0.084489]
```

diofant.integrals.quadrature.**gauss_jacobi**(*n, alpha, beta, n_digits*)

Computes the Gauss-Jacobi quadrature *[R410]* (page 1260) points and weights.

The Gauss-Jacobi quadrature of the first kind approximates the integral:

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

The nodes x_i of an order n quadrature rule are the roots of $P_n^{(\alpha,\beta)}$ and the weights w_i are given by:

$$w_i = -\frac{2n+\alpha+\beta+2}{n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)(n+1)!} \frac{2^{\alpha+\beta}}{P'_n(x_i) P^{(\alpha,\beta)}_{n+1}(x_i)}$$

Parameters **n** : the order of quadrature

alpha : the first parameter of the Jacobi Polynomial, $\alpha > -1$

beta : the second parameter of the Jacobi Polynomial, $\beta > -1$

n_digits : number of significant digits of the points and weights to return

Returns **(x, w)** : the `x` and `w` are lists of points and weights as Floats.

The points x_i and weights w_i are returned as `(x, w)` tuple of lists.

See also:

gauss_legendre (page 534), *gauss_laguerre* (page 535), *gauss_hermite* (page 536), *gauss_gen_laguerre* (page 537), *gauss_chebyshev_t* (page 538), *gauss_chebyshev_u* (page 538)

References

[R410] (page 1260), *[R411]* (page 1260), *[R412]* (page 1260)

Examples

```
>>> x, w = gauss_jacobi(3, 0.5, -0.5, 5)
>>> x
[-0.90097, -0.22252, 0.62349]
>>> w
[1.7063, 1.0973, 0.33795]
```

```
>>> x, w = gauss_jacobi(6, 1, 1, 5)
>>> x
[-0.87174, -0.5917, -0.2093, 0.2093, 0.5917, 0.87174]
>>> w
[0.050584, 0.22169, 0.39439, 0.39439, 0.22169, 0.050584]
```

3.8 Logic

3.8.1 Introduction

The logic module for Diofant allows to form and manipulate logic expressions using symbolic and Boolean values.

3.8.2 Forming logical expressions

You can build Boolean expressions with the standard python operators & (*And* (page 543)), | (*Or* (page 544)), ~ (*Not* (page 544)):

```
>>> y | (x & y)
Or(And(x, y), y)
>>> x | y
Or(x, y)
```

(continues on next page)

(continued from previous page)

```
>>> ~x
Not(x)
```

You can also form implications with >> and <<:

```
>>> x >> y
Implies(x, y)
>>> x << y
Implies(y, x)
```

Like most types in Diofant, Boolean expressions inherit from *Basic* (page 42):

```
>>> (y & x).subs({x: True, y: True})
true
>>> (x | y).atoms()
{x, y}
```

The logic module also includes the following functions to derive boolean expressions from their truth tables-

diofant.logic.boolalg.**SOPform**(*variables*, *minterms*, *dontcares=None*)

The SOPform function uses simplified_pairs and a redundant group- eliminating algorithm to convert the list of all input combos that generate '1' (the minterms) into the smallest Sum of Products form.

The variables must be given as the first argument.

Return a logical Or function (i.e., the "sum of products" or "SOP" form) that gives the desired outcome. If there are inputs that can be ignored, pass them as a list, too.

The result will be one of the (perhaps many) functions that satisfy the conditions.

References

[R413] (page 1260)

Examples

```
>>> w = Symbol('w')
>>> minterms = [[0, 0, 0, 1], [0, 0, 1, 1],
...             [0, 1, 1, 1], [1, 0, 1, 1], [1, 1, 1, 1]]
>>> dontcares = [[0, 0, 0, 0], [0, 0, 1, 0], [0, 1, 0, 1]]
>>> SOPform([w, x, y, z], minterms, dontcares)
Or(And(Not(w), z), And(y, z))
```

diofant.logic.boolalg.**POSform**(*variables*, *minterms*, *dontcares=None*)

The POSform function uses simplified_pairs and a redundant-group eliminating algorithm to convert the list of all input combinations that generate '1' (the minterms) into the smallest Product of Sums form.

The variables must be given as the first argument.

Return a logical And function (i.e., the "product of sums" or "POS" form) that gives the desired outcome. If there are inputs that can be ignored, pass them as a list, too.

The result will be one of the (perhaps many) functions that satisfy the conditions.

References

[R414] (page 1260)

Examples

```
>>> w = Symbol('w')
>>> minterms = [[0, 0, 0, 1], [0, 0, 1, 1], [0, 1, 1, 1],
...             [1, 0, 1, 1], [1, 1, 1, 1]]
>>> dontcares = [[0, 0, 0, 0], [0, 0, 1, 0], [0, 1, 0, 1]]
>>> POSform([w, x, y, z], minterms, dontcares)
And(Or(Not(w), y), z)
```

3.8.3 Boolean functions

class diofant.logic.boolalg.**BooleanTrue**

Diofant version of True, a singleton that can be accessed via S.true.

This is the Diofant version of True, for use in the logic module. The primary advantage of using true instead of True is that shorthand boolean operations like ~ and >> will work as expected on this class, whereas with True they act bitwise on 1. Functions in the logic module will return this class when they evaluate to true.

See also:

diofant.logic.boolalg.BooleanFalse (page 543)

Notes

There is liable to be some confusion as to when `True` should be used and when `S.true` should be used in various contexts throughout Diofant. An important thing to remember is that `sympify(True)` returns `S.true`. This means that for the most part, you can just use `True` and it will automatically be converted to `S.true` when necessary, similar to how you can generally use 1 instead of `S.One`.

The rule of thumb is:

"If the boolean in question can be replaced by an arbitrary symbolic `Boolean`, like `Or(x, y)` or `x > 1`, use `S.true`. Otherwise, use `True`".

In other words, use `S.true` only on those contexts where the boolean is being used as a symbolic representation of truth. For example, if the object ends up in the `.args` of any expression, then it must necessarily be `S.true` instead of `True`, as elements of `.args` must be `Basic`. On the other hand, == is not a symbolic operation in Diofant, since it always returns `True` or `False`, and does so in terms of structural equality rather than mathematical, so it should return `True`. The assumptions system should use `True` and `False`. Aside from not satisfying the above rule of thumb, the assumptions system uses a three-valued logic (`True`, `False`, `None`), whereas `S.true` and `S.false` represent a two-valued logic. When in doubt, use `True`.

"`S.true == True is True`."

While "`S.true is True`" is `False`, "`S.true == True`" is `True`, so if there is any doubt over whether a function or expression will return `S.true` or `True`, just use == instead of

is to do the comparison, and it will work in either case. Finally, for boolean flags, it's better to just use `if x` instead of `if x is True`. To quote PEP 8:

Don't compare boolean values to `True` or `False` using ==.

- Yes: `if greeting:`
- No: `if greeting == True:`
- Worse: `if greeting is True:`

Examples

```
>>> sympify(True)
true
>>> ~true
false
>>> ~True
-2
>>> Or(True, False)
true
```

class `diofant.logic.boolalg.`**`BooleanFalse`**

Diofant version of False, a singleton that can be accessed via S.false.

This is the Diofant version of False, for use in the logic module. The primary advantage of using false instead of False is that shorthand boolean operations like ~ and >> will work as expected on this class, whereas with False they act bitwise on 0. Functions in the logic module will return this class when they evaluate to false.

See also:

diofant.logic.boolalg.BooleanTrue (page 542)

Notes

See note in *BooleanTrue* (page 542).

Examples

```
>>> sympify(False)
false
>>> false >> false
true
>>> False >> False
0
>>> Or(True, False)
true
```

class `diofant.logic.boolalg.`**`And`**

Logical AND function.

It evaluates its arguments in order, giving False immediately if any of them are False, and True if they are all True.

Notes

The & operator is provided as a convenience, but note that its use here is different from its normal use in Python, which is bitwise and. Hence, `And(a, b)` and `a & b` will return different things if `a` and `b` are integers.

```
>>> And(x, y).subs(x, 1)
y
```

Examples

```
>>> x & y
And(x, y)
```

class `diofant.logic.boolalg.`**`Or`**

Logical OR function

It evaluates its arguments in order, giving True immediately if any of them are True, and False if they are all False.

Notes

The | operator is provided as a convenience, but note that its use here is different from its normal use in Python, which is bitwise or. Hence, `Or(a, b)` and `a | b` will return different things if `a` and `b` are integers.

```
>>> Or(x, y).subs(x, 0)
y
```

Examples

```
>>> x | y
Or(x, y)
```

class `diofant.logic.boolalg.`**`Not`**

Logical Not function (negation).

Returns True if the statement is False. Returns False if the statement is True.

Notes

The ~ operator is provided as a convenience, but note that its use here is different from its normal use in Python, which is bitwise not. In particular, `~a` and `Not(a)` will be different if `a` is an integer. Furthermore, since bools in Python subclass from `int`, `~True` is the same as `~1` which is `-2`, which has a boolean value of True. To avoid this issue, use the Diofant boolean types `true` and `false`.

```
>>> ~True
-2
>>> ~true
false
```

Examples

```
>>> from diofant.abc import A, B
>>> Not(True)
false
>>> Not(False)
true
>>> Not(And(True, False))
true
>>> Not(Or(True, False))
false
>>> Not(And(And(True, x), Or(x, False)))
Not(x)
>>> ~x
Not(x)
>>> Not(And(Or(A, B), Or(~A, ~B)))
Not(And(Or(A, B), Or(Not(A), Not(B))))
```

class diofant.logic.boolalg.**Xor**

Logical XOR (exclusive OR) function.

Returns True if an odd number of the arguments are True and the rest are False.

Returns False if an even number of the arguments are True and the rest are False.

Notes

The ^ operator is provided as a convenience, but note that its use here is different from its normal use in Python, which is bitwise xor. In particular, `a ^ b` and `Xor(a, b)` will be different if `a` and `b` are integers.

```
>>> Xor(x, y).subs(y, 0)
x
```

Examples

```
>>> Xor(True, False)
true
>>> Xor(True, True)
false
>>> Xor(True, False, True, True, False)
true
>>> Xor(True, False, True, False)
false
>>> x ^ y
Xor(x, y)
```

class diofant.logic.boolalg.**Nand**

Logical NAND function.

It evaluates its arguments in order, giving True immediately if any of them are False, and False if they are all True.

Returns True if any of the arguments are False. Returns False if all arguments are True.

Examples

```
>>> Nand(False, True)
true
>>> Nand(True, True)
false
>>> Nand(x, y)
Not(And(x, y))
```

class diofant.logic.boolalg.**Nor**

Logical NOR function.

It evaluates its arguments in order, giving False immediately if any of them are True, and True if they are all False.

Returns False if any argument is True. Returns True if all arguments are False.

Examples

```
>>> Nor(True, False)
false
>>> Nor(True, True)
false
>>> Nor(False, True)
false
>>> Nor(False, False)
true
>>> Nor(x, y)
Not(Or(x, y))
```

class diofant.logic.boolalg.**Implies**

Logical implication.

A implies B is equivalent to !A v B

Accepts two Boolean arguments; A and B. Returns False if A is True and B is False. Returns True otherwise.

Notes

The >> and << operators are provided as a convenience, but note that their use here is different from their normal use in Python, which is bit shifts. Hence, `Implies(a, b)` and `a >> b` will return different things if `a` and `b` are integers. In particular, since Python considers `True` and `False` to be integers, `True >> True` will be the same as `1 >> 1`, i.e., 0, which has a truth value of False. To avoid this issue, use the Diofant objects `true` and `false`.

```
>>> True >> False
1
>>> true >> false
false
```

Examples

```
>>> Implies(True, False)
false
>>> Implies(False, False)
true
>>> Implies(True, True)
true
>>> Implies(False, True)
true
>>> x >> y
Implies(x, y)
>>> y << x
Implies(x, y)
```

class diofant.logic.boolalg.**Equivalent**

Equivalence relation.

Equivalent(A, B) is True iff A and B are both True or both False.

Returns True if all of the arguments are logically equivalent. Returns False otherwise.

Examples

```
>>> Equivalent(False, False, False)
true
>>> Equivalent(True, False, False)
false
>>> Equivalent(x, And(x, True))
true
```

class diofant.logic.boolalg.**ITE**

If then else clause.

ITE(A, B, C) evaluates and returns the result of B if A is true else it returns the result of C.

Examples

```
>>> ITE(True, False, True)
false
>>> ITE(Or(True, False), And(True, True), Xor(True, True))
true
>>> ITE(x, y, z)
ITE(x, y, z)
>>> ITE(True, x, y)
x
>>> ITE(False, x, y)
y
>>> ITE(x, y, y)
y
```

The following functions can be used to handle Conjunctive and Disjunctive Normal forms-

diofant.logic.boolalg.**to_cnf**(*expr*, *simplify=False*)

Convert a propositional logical sentence s to conjunctive normal form. That is, of the form ((A | ~B | …) & (B | C | …) & …). If simplify is True, the expr is evaluated to its simplest CNF form.

Examples

```
>>> from diofant.abc import A, B, D
>>> to_cnf(~(A | B) | D)
And(Or(D, Not(A)), Or(D, Not(B)))
>>> to_cnf((A | B) & (A | ~A), True)
Or(A, B)
```

diofant.logic.boolalg.**to_dnf**(*expr*, *simplify=False*)

Convert a propositional logical sentence s to disjunctive normal form. That is, of the form ((A & ~B & …) | (B & C & …) | …). If simplify is True, the expr is evaluated to its simplest DNF form.

Examples

```
>>> from diofant.abc import A, B, C
>>> to_dnf(B & (A | C))
Or(And(A, B), And(B, C))
>>> to_dnf((A & B) | (A & ~B) | (B & C) | (~B & C), True)
Or(A, C)
```

diofant.logic.boolalg.**is_cnf**(*expr*)

Test whether or not an expression is in conjunctive normal form.

Examples

```
>>> from diofant.abc import A, B, C
>>> is_cnf(A | B | C)
True
>>> is_cnf(A & B & C)
True
>>> is_cnf((A & B) | C)
False
```

diofant.logic.boolalg.**is_dnf**(*expr*)

Test whether or not an expression is in disjunctive normal form.

Examples

```
>>> from diofant.abc import A, B, C
>>> is_dnf(A | B | C)
True
>>> is_dnf(A & B & C)
True
>>> is_dnf((A & B) | C)
```

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```
True
>>> is_dnf(A & (B | C))
False
```

3.8.4 Simplification and equivalence-testing

diofant.logic.boolalg.**simplify_logic**(*expr*, *form=None*, *deep=True*)

This function simplifies a boolean function to its simplified version in SOP or POS form. The return type is an Or or And object in Diofant.

Parameters expr : string or boolean expression

form : string ('cnf' or 'dnf') or None (default).

If 'cnf' or 'dnf', the simplest expression in the corresponding normal form is returned; if None, the answer is returned according to the form with fewest args (in CNF by default).

deep : boolean (default True)

indicates whether to recursively simplify any non-boolean functions contained within the input.

Examples

```
>>> b = (~x & ~y & ~z) | (~x & ~y & z)
>>> simplify_logic(b)
And(Not(x), Not(y))
```

```
>>> sympify(b)
Or(And(Not(x), Not(y), Not(z)), And(Not(x), Not(y), z))
>>> simplify_logic(_)
And(Not(x), Not(y))
```

Diofant's simplify() function can also be used to simplify logic expressions to their simplest forms.

diofant.logic.boolalg.**bool_map**(*bool1*, *bool2*)

Return the simplified version of bool1, and the mapping of variables that makes the two expressions bool1 and bool2 represent the same logical behaviour for some correspondence between the variables of each. If more than one mappings of this sort exist, one of them is returned. For example, And(x, y) is logically equivalent to And(a, b) for the mapping {x: a, y:b} or {x: b, y:a}. If no such mapping exists, return False.

Examples

```
>>> from diofant.abc import w, a, b, c, d
>>> function1 = SOPform([x, z, y], [[1, 0, 1], [0, 0, 1]])
>>> function2 = SOPform([a, b, c], [[1, 0, 1], [1, 0, 0]])
>>> bool_map(function1, function2)
(And(Not(z), y), {y: a, z: b})
```

The results are not necessarily unique, but they are canonical. Here, `(w, z)` could be `(a, d)` or `(d, a)`:

```
>>> eq =  Or(And(Not(y), w), And(Not(y), z), And(x, y))
>>> eq2 = Or(And(Not(c), a), And(Not(c), d), And(b, c))
>>> bool_map(eq, eq2)
(Or(And(Not(y), w), And(Not(y), z),
 And(x, y)), {w: a, x: b, y: c, z: d})
>>> eq = And(Xor(a, b), c, And(c, d))
>>> bool_map(eq, eq.subs(c, x))
(And(Or(Not(a), Not(b)),
 Or(a, b), c, d),
 {a: a, b: b, c: d, d: x})
```

3.8.5 Inference

This module implements some inference routines in propositional logic.

The function satisfiable will test that a given Boolean expression is satisfiable, that is, you can assign values to the variables to make the sentence $True$.

For example, the expression `x & ~x` is not satisfiable, since there are no values for `x` that make this sentence `True`. On the other hand, `(x | y) & (x | ~y) & (~x | y)` is satisfiable with both `x` and `y` being `True`.

```
>>> satisfiable(x & ~x)
False
>>> satisfiable((x | y) & (x | ~y) & (~x | y))
{x: True, y: True}
```

As you see, when a sentence is satisfiable, it returns a model that makes that sentence `True`. If it is not satisfiable it will return `False`.

`diofant.logic.inference.`**`satisfiable`**(*expr*, *algorithm='dpll2'*, *all_models=False*)

Check satisfiability of a propositional sentence. Returns a model when it succeeds. Returns {true: true} for trivially true expressions.

On setting all_models to True, if given expr is satisfiable then returns a generator of models. However, if expr is unsatisfiable then returns a generator containing the single element False.

Examples

```
>>> from diofant.abc import A, B
>>> satisfiable(A & ~B)
{A: True, B: False}
>>> satisfiable(A & ~A)
False
>>> satisfiable(True)
{true: True}
>>> next(satisfiable(A & ~A, all_models=True))
False
>>> models = satisfiable((A >> B) & B, all_models=True)
>>> next(models)
{A: False, B: True}
```

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```
>>> next(models)
{A: True, B: True}
>>> def use_models(models):
...     for model in models:
...         if model:
...             # Do something with the model.
...             return model
...         else:
...             # Given expr is unsatisfiable.
...             print("UNSAT")
>>> use_models(satisfiable(A >> ~A, all_models=True))
{A: False}
>>> use_models(satisfiable(A ^ A, all_models=True))
UNSAT
```

3.9 Domains

Here we document the various implemented ground domains. There are three types: abstract domains, concrete domains, and "implementation domains". Abstract domains cannot be (usefully) instantiated at all, and just collect together functionality shared by many other domains. Concrete domains are those meant to be instantiated and used. In some cases, there are various possible ways to implement the data type the domain provides. For example, depending on what libraries are available on the system, the integers are implemented either using the python built-in integers, or using gmpy. Note that various aliases are created automatically depending on the libraries available. As such e.g. `ZZ` always refers to the most efficient implementation of the integer ring available.

3.9.1 Abstract Domains

class `diofant.domains.domain.`**`Domain`**

Represents an abstract domain.

`abs`(*a*)

Absolute value of `a`, implies `__abs__`.

`cofactors`(*a*, *b*)

Returns GCD and cofactors of `a` and `b`.

`convert`(*element*, *base=None*)

Convert `element` to `self.dtype`.

`convert_from`(*element*, *base*)

Convert `element` to `self.dtype` given the base domain.

`frac_field`(**symbols*, ***kwargs*)

Returns a fraction field, i.e. $K(X)$.

`from_FF_gmpy`(*a*, *K0*)

Convert `ModularInteger(mpz)` to `dtype`.

`from_FF_python`(*a*, *K0*)

Convert `ModularInteger(int)` to `dtype`.

`from_FractionField`(*a*, *K0*)

Convert a rational function to `dtype`.

from_PolynomialRing(*a*, *K0*)
Convert a polynomial to dtype.

get_exact()
Returns an exact domain associated with self.

half_gcdex(*a*, *b*)
Half extended GCD of a and b.

is_negative(*a*)
Returns True if a is negative.

is_nonnegative(*a*)
Returns True if a is non-negative.

is_nonpositive(*a*)
Returns True if a is non-positive.

is_one(*a*)
Returns True if a is one.

is_positive(*a*)
Returns True if a is positive.

map(*seq*)
Rersively apply self to all elements of seq.

of_type(*element*)
Check if a is of type dtype.

poly_ring(**symbols*, ***kwargs*)
Returns a polynomial ring, i.e. $K[X]$.

unify(*K1*, *symbols=None*)
Construct a minimal domain that contains elements of self and K1.

Known domains (from smallest to largest):

- GF(p)
- ZZ
- QQ
- RR(prec, tol)
- CC(prec, tol)
- ALG(a, b, c)
- K[x, y, z]
- K(x, y, z)
- EX

class diofant.domains.field.**Field**
Represents a field domain.

div(*a*, *b*)
Division of a and b, implies __truediv__.

exquo(*a*, *b*)
Exact quotient of a and b, implies __truediv__.

gcd(a, b)
Returns GCD of a and b.

This definition of GCD over fields allows to clear denominators in $primitive()$.

```
>>> QQ.gcd(QQ(2, 3), QQ(4, 9))
2/9
>>> gcd(Rational(2, 3), Rational(4, 9))
2/9
>>> primitive(2*x/3 + Rational(4, 9))
(2/9, 3*x + 2)
```

get_field()
Returns a field associated with self.

get_ring()
Returns a ring associated with self.

lcm(a, b)
Returns LCM of a and b.

```
>>> QQ.lcm(QQ(2, 3), QQ(4, 9))
4/3
>>> lcm(Rational(2, 3), Rational(4, 9))
4/3
```

quo(a, b)
Quotient of a and b, implies `__truediv__`.

rem(a, b)
Remainder of a and b, implies nothing.

revert(a)
Returns `a**(-1)` if possible.

class diofant.domains.ring.**Ring**
Represents a ring domain.

denom(a)
Returns denominator of a.

div(a, b)
Division of a and b, implies `__divmod__`.

exquo(a, b)
Exact quotient of a and b, implies `__floordiv__`.

get_ring()
Returns a ring associated with self.

invert(a, b)
Returns inversion of `a mod b`.

numer(a)
Returns numerator of a.

quo(a, b)
Quotient of a and b, implies `__floordiv__`.

rem(a, b)
Remainder of a and b, implies `__mod__`.

revert(*a*)
Returns `a**(-1)` if possible.

class `diofant.domains.simpledomain.`**SimpleDomain**
Base class for simple domains, e.g. ZZ, QQ.

inject(**gens*)
Inject generators into this domain.

class `diofant.domains.compositedomain.`**CompositeDomain**
Base class for composite domains, e.g. ZZ[x], ZZ(X).

inject(**symbols*)
Inject generators into this domain.

3.9.2 Concrete Domains

class `diofant.domains.`**FiniteField**(*mod*, *dom*, *symmetric=True*)
General class for finite fields.

characteristic()
Return the characteristic of this domain.

from_FF_gmpy(*a*, *K0=None*)
Convert `ModularInteger(mpz)` to `dtype`.

from_FF_python(*a*, *K0=None*)
Convert `ModularInteger(int)` to `dtype`.

from_QQ_gmpy(*a*, *K0=None*)
Convert GMPY's `mpq` to `dtype`.

from_QQ_python(*a*, *K0=None*)
Convert Python's `Fraction` to `dtype`.

from_RealField(*a*, *K0*)
Convert mpmath's `mpf` to `dtype`.

from_ZZ_gmpy(*a*, *K0=None*)
Convert GMPY's `mpz` to `dtype`.

from_ZZ_python(*a*, *K0=None*)
Convert Python's `int` to `dtype`.

from_diofant(*a*)
Convert Diofant's Integer to Diofant's `Integer`.

get_field()
Returns a field associated with `self`.

to_diofant(*a*)
Convert `a` to a Diofant object.

class `diofant.domains.`**IntegerRing**
General class for integer rings.

algebraic_field(**extension*)
Returns an algebraic field, i.e. $\mathbb{Q}(\alpha, \ldots)$.

from_AlgebraicField(*a*, *K0*)
Convert a `ANP` object to `dtype`.

get_field()
Returns a field associated with `self`.

log(*a*, *b*)
Returns b-base logarithm of `a`.

class diofant.domains.**PolynomialRing**(*domain_or_ring*, *symbols=None*, *order=None*)
A class for representing multivariate polynomial rings.

factorial(*a*)
Returns factorial of a.

from_AlgebraicField(*a*, *K0*)
Convert an algebraic number to `dtype`.

from_FractionField(*a*, *K0*)
Convert a rational function to `dtype`.

from_PolynomialRing(*a*, *K0*)
Convert a polynomial to `dtype`.

from_QQ_gmpy(*a*, *K0*)
Convert a GMPY mpq object to $dtype$.

from_QQ_python(*a*, *K0*)
Convert a Python $Fraction$ object to $dtype$.

from_RealField(*a*, *K0*)
Convert a mpmath mpf object to $dtype$.

from_ZZ_gmpy(*a*, *K0*)
Convert a GMPY mpz object to $dtype$.

from_ZZ_python(*a*, *K0*)
Convert a Python int object to $dtype$.

from_diofant(*a*)
Convert Diofant's expression to $dtype$.

gcd(*a*, *b*)
Returns GCD of a and b.

gcdex(*a*, *b*)
Extended GCD of a and b.

get_field()
Returns a field associated with $self$.

is_negative(*a*)
Returns True if $LC(a)$ is negative.

is_nonnegative(*a*)
Returns True if $LC(a)$ is non-negative.

is_nonpositive(*a*)
Returns True if $LC(a)$ is non-positive.

is_positive(*a*)
Returns True if $LC(a)$ is positive.

lcm(*a*, *b*)
Returns LCM of a and b.

to_diofant(*a*)
Convert a to a Diofant object.

class diofant.domains.**RationalField**
General class for rational fields.

algebraic_field(**extension*)
Returns an algebraic field, i.e. $\mathbb{Q}(\alpha, \ldots)$.

from_AlgebraicField(*a*, *K0*)
Convert a `ANP` object to `dtype`.

class diofant.domains.**AlgebraicField**(*dom*, **ext*)
A class for representing algebraic number fields.

algebraic_field(**extension*)
Returns an algebraic field, i.e. $\mathbb{Q}(\alpha, \ldots)$.

denom(*a*)
Returns denominator of `a`.

dtype
alias of *diofant.polys.polyclasses.ANP* (page 1073)

from_QQ_gmpy(*a*, *K0*)
Convert a GMPY `mpq` object to `dtype`.

from_QQ_python(*a*, *K0*)
Convert a Python `Fraction` object to `dtype`.

from_RealField(*a*, *K0*)
Convert a mpmath `mpf` object to `dtype`.

from_ZZ_gmpy(*a*, *K0*)
Convert a GMPY `mpz` object to `dtype`.

from_ZZ_python(*a*, *K0*)
Convert a Python `int` object to `dtype`.

from_diofant(*a*)
Convert Diofant's expression to `dtype`.

get_ring()
Returns a ring associated with `self`.

is_negative(*a*)
Returns True if `a` is negative.

is_nonnegative(*a*)
Returns True if `a` is non-negative.

is_nonpositive(*a*)
Returns True if `a` is non-positive.

is_positive(*a*)
Returns True if `a` is positive.

numer(*a*)
Returns numerator of `a`.

to_diofant(*a*)
Convert `a` to a Diofant object.

class diofant.domains.**FractionField**(*domain_or_field*, *symbols=None*, *order=None*)
A class for representing multivariate rational function fields.

denom(*a*)
Returns denominator of a.

factorial(*a*)
Returns factorial of a.

from_FractionField(*a*, *K0*)
Convert a rational function to dtype.

from_PolynomialRing(*a*, *K0*)
Convert a polynomial to dtype.

from_QQ_gmpy(*a*, *K0*)
Convert a GMPY mpq object to $dtype$.

from_QQ_python(*a*, *K0*)
Convert a Python $Fraction$ object to $dtype$.

from_RealField(*a*, *K0*)
Convert a mpmath mpf object to $dtype$.

from_ZZ_gmpy(*a*, *K0*)
Convert a GMPY mpz object to $dtype$.

from_ZZ_python(*a*, *K0*)
Convert a Python int object to $dtype$.

from_diofant(*a*)
Convert Diofant's expression to $dtype$.

get_ring()
Returns a field associated with $self$.

is_negative(*a*)
Returns True if $LC(a)$ is negative.

is_nonnegative(*a*)
Returns True if $LC(a)$ is non-negative.

is_nonpositive(*a*)
Returns True if $LC(a)$ is non-positive.

is_positive(*a*)
Returns True if $LC(a)$ is positive.

numer(*a*)
Returns numerator of a.

to_diofant(*a*)
Convert a to a Diofant object.

class diofant.domains.**RealField**(*prec=53*, *dps=None*, *tol=None*)
Real numbers up to the given precision.

almosteq(*a*, *b*, *tolerance=None*)
Check if a and b are almost equal.

from_diofant(*expr*)
Convert Diofant's number to dtype.

gcd(*a*, *b*)
Returns GCD of a and b.

get_exact()
Returns an exact domain associated with self.

get_ring()
Returns a ring associated with self.

lcm(*a*, *b*)
Returns LCM of a and b.

to_diofant(*element*)
Convert element to Diofant number.

to_rational(*element*, *limit=True*)
Convert a real number to rational number.

class diofant.domains.**ExpressionDomain**
A class for arbitrary expressions.

class Expression(*ex*)
An arbitrary expression.

denom(*a*)
Returns denominator of a.

dtype
alias of *ExpressionDomain.Expression* (page 558)

from_AlgebraicField(*a*, *K0*)
Convert a ANP object to dtype.

from_ExpressionDomain(*a*, *K0*)
Convert a EX object to dtype.

from_FractionField(*a*, *K0*)
Convert a DMF object to dtype.

from_PolynomialRing(*a*, *K0*)
Convert a DMP object to dtype.

from_QQ_gmpy(*a*, *K0*)
Convert a GMPY mpq object to dtype.

from_QQ_python(*a*, *K0*)
Convert a Python Fraction object to dtype.

from_RealField(*a*, *K0*)
Convert a mpmath mpf object to dtype.

from_ZZ_gmpy(*a*, *K0*)
Convert a GMPY mpz object to dtype.

from_ZZ_python(*a*, *K0*)
Convert a Python int object to dtype.

from_diofant(*a*)
Convert Diofant's expression to dtype.

gcd(*a*, *b*)
Returns GCD of a and b.

This definition of GCD over fields allows to clear denominators in $primitive()$.

```
>>> QQ.gcd(QQ(2, 3), QQ(4, 9))
2/9
>>> gcd(Rational(2, 3), Rational(4, 9))
2/9
>>> primitive(2*x/3 + Rational(4, 9))
(2/9, 3*x + 2)
```

get_field()
Returns a field associated with `self`.

get_ring()
Returns a ring associated with `self`.

is_negative(*a*)
Returns True if `a` is negative.

is_nonnegative(*a*)
Returns True if `a` is non-negative.

is_nonpositive(*a*)
Returns True if `a` is non-positive.

is_positive(*a*)
Returns True if `a` is positive.

lcm(*a*, *b*)
Returns LCM of `a` and `b`.

```
>>> QQ.lcm(QQ(2, 3), QQ(4, 9))
4/3
>>> lcm(Rational(2, 3), Rational(4, 9))
4/3
```

numer(*a*)
Returns numerator of `a`.

to_diofant(*a*)
Convert `a` to a Diofant object.

3.9.3 Implementation Domains

class diofant.domains.**PythonFiniteField**(*mod*, *symmetric=True*)
Finite field based on Python's integers.

class diofant.domains.**GMPYFiniteField**(*mod*, *symmetric=True*)
Finite field based on GMPY integers.

class diofant.domains.**PythonIntegerRing**
Integer ring based on Python's `int` type.

class diofant.domains.**GMPYIntegerRing**
Integer ring based on GMPY's `mpz` type.

class diofant.domains.**PythonRationalField**
Rational field based on Python rational number type.

class diofant.domains.**GMPYRationalField**
Rational field based on GMPY mpq class.

3.10 Matrices

A module that handles matrices.

Includes functions for fast creating matrices like zero, one/eye, random matrix, etc.

class diofant.matrices.**Matrix**
alias of *MutableMatrix* (page 611)

3.10.1 Matrices (linear algebra)

Creating Matrices

The linear algebra module is designed to be as simple as possible. First, we import and declare our first Matrix object:

```
>>> init_printing(pretty_print=True, use_unicode=False, wrap_line=False, no_
↪global=True)
>>> M = Matrix([[1, 0, 0], [0, 0, 0]]); M
[1  0  0]
[       ]
[0  0  0]
>>> Matrix([M, (0, 0, -1)])
[1  0  0 ]
[        ]
[0  0  0 ]
[        ]
[0  0  -1]
>>> Matrix([[1, 2, 3]])
[1 2 3]
>>> Matrix([1, 2, 3])
[1]
[ ]
[2]
[ ]
[3]
```

In addition to creating a matrix from a list of appropriately-sized lists and/or matrices, Diofant also supports more advanced methods of matrix creation including a single list of values and dimension inputs:

```
>>> Matrix(2, 3, [1, 2, 3, 4, 5, 6])
[1  2  3]
[       ]
[4  5  6]
```

More interesting (and useful), is the ability to use a 2-variable function (or lambda) to create a matrix. Here we create an indicator function which is 1 on the diagonal and then use it to make the identity matrix:

```
>>> def f(i, j):
...     if i == j:
...         return 1
...     else:
...         return 0
```

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```
...
>>> Matrix(4, 4, f)
[1  0  0  0]
[          ]
[0  1  0  0]
[          ]
[0  0  1  0]
[          ]
[0  0  0  1]
```

Finally let's use `lambda` to create a 1-line matrix with 1's in the even permutation entries:

```
>>> Matrix(3, 4, lambda i, j: 1 - (i + j)%2)
[1  0  1  0]
[          ]
[0  1  0  1]
[          ]
[1  0  1  0]
```

There are also a couple of special constructors for quick matrix construction: `eye` is the identity matrix, `zeros` and `ones` for matrices of all zeros and ones, respectively, and `diag` to put matrices or elements along the diagonal:

```
>>> eye(4)
[1  0  0  0]
[          ]
[0  1  0  0]
[          ]
[0  0  1  0]
[          ]
[0  0  0  1]
>>> zeros(2)
[0  0]
[    ]
[0  0]
>>> zeros(2, 5)
[0  0  0  0  0]
[             ]
[0  0  0  0  0]
>>> ones(3)
[1  1  1]
[       ]
[1  1  1]
[       ]
[1  1  1]
>>> ones(1, 3)
[1  1  1]
>>> diag(1, Matrix([[1, 2], [3, 4]]))
[1  0  0]
[       ]
[0  1  2]
[       ]
[0  3  4]
```

Basic Manipulation

While learning to work with matrices, let's choose one where the entries are readily identifiable. One useful thing to know is that while matrices are 2-dimensional, the storage is not and so it is allowable - though one should be careful - to access the entries as if they were a 1-d list.

```
>>> M = Matrix(2, 3, [1, 2, 3, 4, 5, 6])
>>> M[4]
5
```

Now, the more standard entry access is a pair of indices which will always return the value at the corresponding row and column of the matrix:

```
>>> M[1, 2]
6
>>> M[0, 0]
1
>>> M[1, 1]
5
```

Since this is Python we're also able to slice submatrices; slices always give a matrix in return, even if the dimension is 1 x 1:

```
>>> M[0:2, 0:2]
[1  2]
[    ]
[4  5]
>>> M[2:2, 2]
[]
>>> M[:, 2]
[3]
[ ]
[6]
>>> M[:1, 2]
[3]
```

In the second example above notice that the slice 2:2 gives an empty range. Note also (in keeping with 0-based indexing of Python) the first row/column is 0.

You cannot access rows or columns that are not present unless they are in a slice:

```
>>> M[:, 10] # the 10-th column (not there)
Traceback (most recent call last):
...
IndexError: Index out of range: a[[0, 10]]
>>> M[:, 10:11] # the 10-th column (if there)
[]
>>> M[:, :10] # all columns up to the 10-th
[1  2  3]
[       ]
[4  5  6]
```

Slicing an empty matrix works as long as you use a slice for the coordinate that has no size:

```
>>> Matrix(0, 3, [])[:, 1]
[]
```

Slicing gives a copy of what is sliced, so modifications of one object do not affect the other:

```
>>> M2 = M[:, :]
>>> M2[0, 0] = 100
>>> M[0, 0] == 100
False
```

Notice that changing `M2` didn't change `M`. Since we can slice, we can also assign entries:

```
>>> M = Matrix(([1, 2, 3, 4], [5, 6, 7, 8], [9, 10, 11, 12], [13, 14, 15, 16]))
>>> M
[1   2   3   4 ]
[              ]
[5   6   7   8 ]
[              ]
[9   10  11  12]
[              ]
[13  14  15  16]
>>> M[2, 2] = M[0, 3] = 0
>>> M
[1   2   3   0 ]
[              ]
[5   6   7   8 ]
[              ]
[9   10  0   12]
[              ]
[13  14  15  16]
```

as well as assign slices:

```
>>> M = Matrix(([1, 2, 3, 4], [5, 6, 7, 8], [9, 10, 11, 12], [13, 14, 15, 16]))
>>> M[2:, 2:] = Matrix(2, 2, lambda i, j: 0)
>>> M
[1   2   3  4]
[            ]
[5   6   7  8]
[            ]
[9   10  0  0]
[            ]
[13  14  0  0]
```

All the standard arithmetic operations are supported:

```
>>> M = Matrix(([1, 2, 3], [4, 5, 6], [7, 8, 9]))
>>> M - M
[0  0  0]
[       ]
[0  0  0]
[       ]
[0  0  0]
>>> M + M
[2   4   6 ]
[          ]
[8   10  12]
[          ]
[14  16  18]
>>> M * M
[30   36   42 ]
```

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```
[              ]
[66   81   96 ]
[              ]
[102  126  150]
>>> M2 = Matrix(3, 1, [1, 5, 0])
>>> M*M2
[11]
[  ]
[29]
[  ]
[47]
>>> M**2
[30   36   42 ]
[              ]
[66   81   96 ]
[              ]
[102  126  150]
```

As well as some useful vector operations:

```
>>> M.row_del(0)
>>> M
[4  5  6]
[       ]
[7  8  9]
>>> M.col_del(1)
>>> M
[4  6]
[    ]
[7  9]
>>> v1 = Matrix([1, 2, 3])
>>> v2 = Matrix([4, 5, 6])
>>> v3 = v1.cross(v2)
>>> v1.dot(v2)
32
>>> v2.dot(v3)
0
>>> v1.dot(v3)
0
```

Recall that the `row_del()` and `col_del()` operations don't return a value - they simply change the matrix object. We can also ''glue'' together matrices of the appropriate size:

```
>>> M1 = eye(3)
>>> M2 = zeros(3, 4)
>>> M1.row_join(M2)
[1  0  0  0  0  0  0]
[                   ]
[0  1  0  0  0  0  0]
[                   ]
[0  0  1  0  0  0  0]
>>> M3 = zeros(4, 3)
>>> M1.col_join(M3)
[1  0  0]
[       ]
[0  1  0]
```

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```
[       ]
[0  0  1]
[       ]
[0  0  0]
[       ]
[0  0  0]
[       ]
[0  0  0]
[       ]
[0  0  0]
```

Operations on entries

We are not restricted to having multiplication between two matrices:

```
>>> M = eye(3)
>>> 2*M
[2  0  0]
[       ]
[0  2  0]
[       ]
[0  0  2]
>>> 3*M
[3  0  0]
[       ]
[0  3  0]
[       ]
[0  0  3]
```

but we can also apply functions to our matrix entries using `applyfunc()`. Here we'll declare a function that double any input number. Then we apply it to the 3x3 identity matrix:

```
>>> f = lambda x: 2*x
>>> eye(3).applyfunc(f)
[2  0  0]
[       ]
[0  2  0]
[       ]
[0  0  2]
```

One more useful matrix-wide entry application function is the substitution function. Let's declare a matrix with symbolic entries then substitute a value. Remember we can substitute anything - even another symbol!:

```
>>> M = eye(3) * x
>>> M
[x  0  0]
[       ]
[0  x  0]
[       ]
[0  0  x]
>>> M.subs(x, 4)
[4  0  0]
[       ]
[0  4  0]
```

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```
[       ]
[0  0  4]
>>> y = Symbol('y')
>>> M.subs(x, y)
[y  0  0]
[       ]
[0  y  0]
[       ]
[0  0  y]
```

Linear algebra

Now that we have the basics out of the way, let's see what we can do with the actual matrices. Of course, one of the first things that comes to mind is the determinant:

```
>>> M = Matrix(([1, 2, 3], [3, 6, 2], [2, 0, 1]))
>>> M.det()
-28
>>> M2 = eye(3)
>>> M2.det()
1
>>> M3 = Matrix(([1, 0, 0], [1, 0, 0], [1, 0, 0]))
>>> M3.det()
0
```

Another common operation is the inverse: In Diofant, this is computed by Gaussian elimination by default (for dense matrices) but we can specify it be done by LU decomposition as well:

```
>>> M2.inv()
[1  0  0]
[       ]
[0  1  0]
[       ]
[0  0  1]
>>> M2.inv(method="LU")
[1  0  0]
[       ]
[0  1  0]
[       ]
[0  0  1]
>>> M.inv(method="LU")
[-3/14  1/14  1/2 ]
[                 ]
[-1/28  5/28  -1/4]
[                 ]
[ 3/7   -1/7   0  ]
>>> M * M.inv(method="LU")
[1  0  0]
[       ]
[0  1  0]
[       ]
[0  0  1]
```

We can perform a QR factorization which is handy for solving systems:

```
>>> A = Matrix([[1, 1, 1], [1, 1, 3], [2, 3, 4]])
>>> Q, R = A.QRdecomposition()
>>> Q
[  ___     ___      ___ ]
[\/ 6   -\/ 3    -\/ 2  ]
[-----  -------  -------]
[  6       3        2   ]
[                       ]
[  ___     ___      ___ ]
[\/ 6   -\/ 3     \/ 2  ]
[-----  -------   ----- ]
[  6       3        2   ]
[                       ]
[  ___     ___          ]
[\/ 6    \/ 3           ]
[-----   -----      0   ]
[  3       3            ]
>>> R
[                       ]
[          ___          ]
[  ___  4*\/ 6       ___]
[\/ 6   -------  2*\/ 6 ]
[          3            ]
[                       ]
[                       ]
[          ___          ]
[        \/ 3           ]
[  0     -----      0   ]
[          3            ]
[                       ]
[                    ___]
[  0       0      \/ 2  ]
>>> Q*R
[1  1  1]
[       ]
[1  1  3]
[       ]
[2  3  4]
```

In addition to the solvers in the `solver.py` file, we can solve the system Ax=b by passing the b vector to the matrix A's LUsolve function. Here we'll cheat a little choose A and x then multiply to get b. Then we can solve for x and check that it's correct:

```
>>> A = Matrix([[2, 3, 5], [3, 6, 2], [8, 3, 6]])
>>> x = Matrix(3, 1, [3, 7, 5])
>>> b = A*x
>>> soln = A.LUsolve(b)
>>> soln
[3]
[ ]
[7]
[ ]
[5]
```

There's also a nice Gram-Schmidt orthogonalizer which will take a set of vectors and orthogonalize them with respect to another. There is an optional argument which specifies whether or not the output should also be normalized, it defaults to `False`. Let's take some vectors and orthogonalize them - one normalized and one not:

```
>>> L = [Matrix([2, 3, 5]), Matrix([3, 6, 2]), Matrix([8, 3, 6])]
>>> out1 = GramSchmidt(L)
>>> out2 = GramSchmidt(L, True)
```

Let's take a look at the vectors:

```
>>> for i in out1:
...     print(i)
...
Matrix([
[2],
[3],
[5]])
Matrix([
[ 23/19],
[ 63/19],
[-47/19]])
Matrix([
[ 1692/353],
[-1551/706],
[ -423/706]])
>>> for i in out2:
...      print(i)
...
Matrix([
[  sqrt(38)/19],
[3*sqrt(38)/38],
[5*sqrt(38)/38]])
Matrix([
[ 23*sqrt(6707)/6707],
[ 63*sqrt(6707)/6707],
[-47*sqrt(6707)/6707]])
Matrix([
[ 12*sqrt(706)/353],
[-11*sqrt(706)/706],
[ -3*sqrt(706)/706]])
```

We can spot-check their orthogonality with dot() and their normality with norm():

```
>>> out1[0].dot(out1[1])
0
>>> out1[0].dot(out1[2])
0
>>> out1[1].dot(out1[2])
0
>>> out2[0].norm()
1
>>> out2[1].norm()
1
>>> out2[2].norm()
1
```

So there is quite a bit that can be done with the module including eigenvalues, eigenvectors, nullspace calculation, cofactor expansion tools, and so on. From here one might want to look over the `matrices.py` file for all functionality.

MatrixBase Class Reference

class diofant.matrices.matrices.**MatrixBase**

C

By-element conjugation.

D

Return Dirac conjugate (if self.rows == 4).

See also:

conjugate **(page 577)** By-element conjugation

H **(page 569)** Hermite conjugation

Examples

```
>>> m = Matrix((0, 1 + I, 2, 3))
>>> m.D
Matrix([[0, 1 - I, -2, -3]])
>>> m = (eye(4) + I*eye(4))
>>> m[0, 3] = 2
>>> m.D
Matrix([
[1 - I,     0,      0,      0],
[    0, 1 - I,      0,      0],
[    0,     0, -1 + I,      0],
[    2,     0,      0, -1 + I]])
```

If the matrix does not have 4 rows an AttributeError will be raised because this property is only defined for matrices with 4 rows.

```
>>> Matrix(eye(2)).D
Traceback (most recent call last):
...
AttributeError: Matrix has no attribute D.
```

H

Return Hermite conjugate.

See also:

conjugate **(page 577)** By-element conjugation

D **(page 569)** Dirac conjugation

Examples

```
>>> m = Matrix((0, 1 + I, 2, 3))
>>> m
Matrix([
[    0],
[1 + I],
```

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```
[    2],
[    3]])
>>> m.H
Matrix([[0, 1 - I, 2, 3]])
```

LDLdecomposition()

Returns the LDL Decomposition (L, D) of matrix A, such that L * D * L.T == A This method eliminates the use of square root. Further this ensures that all the diagonal entries of L are 1. A must be a square, symmetric, positive-definite and non-singular matrix.

See also:

cholesky (page 575), *LUdecomposition* (page 570), *QRdecomposition* (page 571)

Examples

```
>>> A = Matrix(((25, 15, -5), (15, 18, 0), (-5, 0, 11)))
>>> L, D = A.LDLdecomposition()
>>> L
Matrix([
[   1,   0, 0],
[ 3/5,   1, 0],
[-1/5, 1/3, 1]])
>>> D
Matrix([
[25, 0, 0],
[ 0, 9, 0],
[ 0, 0, 9]])
>>> L * D * L.T * A.inv() == eye(A.rows)
True
```

LDLsolve(*rhs*)

Solves Ax = B using LDL decomposition, for a general square and non-singular matrix.

For a non-square matrix with rows > cols, the least squares solution is returned.

See also:

LDLdecomposition (page 570), *lower_triangular_solve* (page 593), *upper_triangular_solve* (page 602), *cholesky_solve* (page 576), *diagonal_solve* (page 578), *LUsolve* (page 571), *QRsolve* (page 572), *pinv_solve* (page 596)

Examples

```
>>> A = eye(2)*2
>>> B = Matrix([[1, 2], [3, 4]])
>>> A.LDLsolve(B) == B/2
True
```

LUdecomposition(*iszerofunc=<function _iszero>*)

Returns the decomposition LU and the row swaps p.

See also:

cholesky (page 575), *LDLdecomposition* (page 570), *QRdecomposition* (page 571), *LUdecomposition_Simple* (page 571), *LUdecompositionFF* (page 571), *LUsolve* (page 571)

Examples

```
>>> a = Matrix([[4, 3], [6, 3]])
>>> L, U, _ = a.LUdecomposition()
>>> L
Matrix([
[  1, 0],
[3/2, 1]])
>>> U
Matrix([
[4,    3],
[0, -3/2]])
```

LUdecompositionFF()

Compute a fraction-free LU decomposition.

Returns 4 matrices P, L, D, U such that PA = L D**-1 U. If the elements of the matrix belong to some integral domain I, then all elements of L, D and U are guaranteed to belong to I.

Reference

- W. Zhou & D.J. Jeffrey, "Fraction-free matrix factors: new forms for LU and QR factors". Frontiers in Computer Science in China, Vol 2, no. 1, pp. 67-80, 2008.

See also:

LUdecomposition (page 570), *LUdecomposition_Simple* (page 571), *LUsolve* (page 571)

LUdecomposition_Simple(*iszerofunc=<function _iszero>*)

Returns A comprised of L, U (L's diag entries are 1) and p which is the list of the row swaps (in order).

See also:

LUdecomposition (page 570), *LUdecompositionFF* (page 571), *LUsolve* (page 571)

LUsolve(*rhs*, *iszerofunc=<function _iszero>*)

Solve the linear system Ax = rhs for x where A = self.

This is for symbolic matrices, for real or complex ones use mpmath.lu_solve or mpmath.qr_solve.

See also:

lower_triangular_solve (page 593), *upper_triangular_solve* (page 602), *cholesky_solve* (page 576), *diagonal_solve* (page 578), *LDLsolve* (page 570), *QRsolve* (page 572), *pinv_solve* (page 596), *LUdecomposition* (page 570)

QRdecomposition()

Return Q, R where A = Q*R, Q is orthogonal and R is upper triangular.

See also:

cholesky (page 575), *LDLdecomposition* (page 570), *LUdecomposition* (page 570), *QRsolve* (page 572)

Examples

This is the example from wikipedia:

```
>>> A = Matrix([[12, -51, 4], [6, 167, -68], [-4, 24, -41]])
>>> Q, R = A.QRdecomposition()
>>> Q
Matrix([
[ 6/7, -69/175, -58/175],
[ 3/7, 158/175,   6/175],
[-2/7,    6/35,  -33/35]])
>>> R
Matrix([
[14,  21, -14],
[ 0, 175, -70],
[ 0,   0,  35]])
>>> A == Q*R
True
```

QR factorization of an identity matrix:

```
>>> A = Matrix([[1, 0, 0], [0, 1, 0], [0, 0, 1]])
>>> Q, R = A.QRdecomposition()
>>> Q
Matrix([
[1, 0, 0],
[0, 1, 0],
[0, 0, 1]])
>>> R
Matrix([
[1, 0, 0],
[0, 1, 0],
[0, 0, 1]])
```

QRsolve(*b*)

Solve the linear system 'Ax = b'.

'self' is the matrix 'A', the method argument is the vector 'b'. The method returns the solution vector 'x'. If 'b' is a matrix, the system is solved for each column of 'b' and the return value is a matrix of the same shape as 'b'.

This method is slower (approximately by a factor of 2) but more stable for floating-point arithmetic than the LUsolve method. However, LUsolve usually uses an exact arithmetic, so you don't need to use QRsolve.

This is mainly for educational purposes and symbolic matrices, for real (or complex) matrices use mpmath.qr_solve.

See also:

lower_triangular_solve (page 593), *upper_triangular_solve* (page 602), *cholesky_solve* (page 576), *diagonal_solve* (page 578), *LDLsolve* (page 570), *LUsolve* (page 571), *pinv_solve* (page 596), *QRdecomposition* (page 571)

T

Matrix transposition.

add(*b*)
Return self + b

adjoint()
Conjugate transpose or Hermitian conjugation.

adjugate(*method='berkowitz'*)
Returns the adjugate matrix.

Adjugate matrix is the transpose of the cofactor matrix.

https//en.wikipedia.org/wiki/Adjugate

See also:

cofactorMatrix (page 576), *transpose* (page 602), *berkowitz* (page 573)

atoms(**types*)
Returns the atoms that form the current object.

Examples

```
>>> Matrix([[x]])
Matrix([[x]])
>>> _.atoms()
{x}
```

berkowitz()
The Berkowitz algorithm.

Given N x N matrix with symbolic content, compute efficiently coefficients of characteristic polynomials of 'self' and all its square sub-matrices composed by removing both i-th row and column, without division in the ground domain.

This method is particularly useful for computing determinant, principal minors and characteristic polynomial, when 'self' has complicated coefficients e.g. polynomials. Semi-direct usage of this algorithm is also important in computing efficiently sub-resultant PRS.

Assuming that M is a square matrix of dimension N x N and I is N x N identity matrix, then the following following definition of characteristic polynomial is begin used:

charpoly(M) = det(t*I - M)

As a consequence, all polynomials generated by Berkowitz algorithm are monic.

```
>>> M = Matrix([[x, y, z], [1, 0, 0], [y, z, x]])
```

```
>>> p, q, r = M.berkowitz()
```

```
>>> p  # 1 x 1 M's sub-matrix
(1, -x)
```

```
>>> q  # 2 x 2 M's sub-matrix
(1, -x, -y)
```

```
>>> r  # 3 x 3 M's sub-matrix
(1, -2*x, x**2 - y*z - y, x*y - z**2)
```

For more information on the implemented algorithm refer to:

[1] S.J. Berkowitz, On computing the determinant in small parallel
time using a small number of processors, ACM, Information Processing Letters 18, 1984, pp. 147-150

[2] M. Keber, Division-Free computation of sub-resultants using
Bezout matrices, Tech. Report MPI-I-2006-1-006, Saarbrücken, 2006

See also:

berkowitz_det (page 574), *berkowitz_minors* (page 575), *berkowitz_charpoly* (page 574), *berkowitz_eigenvals* (page 574)

berkowitz_charpoly(*x=Dummy('lambda')*, *simplify=<function simplify>*)
Computes characteristic polynomial minors using Berkowitz method.

A PurePoly is returned so using different variables for x does not affect the comparison or the polynomials:

See also:

berkowitz (page 573)

Examples

```
>>> A = Matrix([[1, 3], [2, 0]])
>>> A.berkowitz_charpoly(x) == A.berkowitz_charpoly(y)
True
```

Specifying x is optional; a Dummy with name `lambda` is used by default (which looks good when pretty-printed in unicode):

```
>>> A.berkowitz_charpoly().as_expr()
_lambda**2 - _lambda - 6
```

No test is done to see that x doesn't clash with an existing symbol, so using the default (`lambda`) or your own Dummy symbol is the safest option:

```
>>> A = Matrix([[1, 2], [x, 0]])
>>> A.charpoly().as_expr()
-2*x + _lambda**2 - _lambda
>>> A.charpoly(x).as_expr()
x**2 - 3*x
```

berkowitz_det()
Computes determinant using Berkowitz method.

See also:

det (page 577), *berkowitz* (page 573)

berkowitz_eigenvals(***flags*)
Computes eigenvalues of a Matrix using Berkowitz method.

See also:

berkowitz (page 573)

berkowitz_minors()

Computes principal minors using Berkowitz method.

See also:

berkowitz (page 573)

charpoly(*x=Dummy('lambda')*, *simplify=<function simplify>*)

Computes characteristic polynomial minors using Berkowitz method.

A PurePoly is returned so using different variables for x does not affect the comparison or the polynomials:

See also:

berkowitz (page 573)

Examples

```
>>> A = Matrix([[1, 3], [2, 0]])
>>> A.berkowitz_charpoly(x) == A.berkowitz_charpoly(y)
True
```

Specifying x is optional; a Dummy with name `lambda` is used by default (which looks good when pretty-printed in unicode):

```
>>> A.berkowitz_charpoly().as_expr()
_lambda**2 - _lambda - 6
```

No test is done to see that x doesn't clash with an existing symbol, so using the default (`lambda`) or your own Dummy symbol is the safest option:

```
>>> A = Matrix([[1, 2], [x, 0]])
>>> A.charpoly().as_expr()
-2*x + _lambda**2 - _lambda
>>> A.charpoly(x).as_expr()
x**2 - 3*x
```

cholesky()

Returns the Cholesky decomposition L of a matrix A such that L * L.T = A

A must be a square, symmetric, positive-definite and non-singular matrix.

See also:

LDLdecomposition (page 570), *LUdecomposition* (page 570), *QRdecomposition* (page 571)

Examples

```
>>> A = Matrix(((25, 15, -5), (15, 18, 0), (-5, 0, 11)))
>>> A.cholesky()
Matrix([
[ 5, 0, 0],
[ 3, 3, 0],
```

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```
[-1, 1, 3]])
>>> A.cholesky() * A.cholesky().T
Matrix([
[25, 15, -5],
[15, 18,  0],
[-5,  0, 11]])
```

cholesky_solve(*rhs*)
Solves Ax = B using Cholesky decomposition, for a general square non-singular matrix. For a non-square matrix with rows > cols, the least squares solution is returned.

See also:

lower_triangular_solve (page 593), *upper_triangular_solve* (page 602), *diagonal_solve* (page 578), *LDLsolve* (page 570), *LUsolve* (page 571), *QRsolve* (page 572), *pinv_solve* (page 596)

cofactor(*i, j, method='berkowitz'*)
Calculate the cofactor of an element.

See also:

cofactorMatrix (page 576), *minorEntry* (page 593), *minorMatrix* (page 593)

cofactorMatrix(*method='berkowitz'*)
Return a matrix containing the cofactor of each element.

See also:

cofactor (page 576), *minorEntry* (page 593), *minorMatrix* (page 593), *adjugate* (page 573)

col_insert(*pos, mti*)
Insert one or more columns at the given column position.

See also:

diofant.matrices.dense.DenseMatrix.col (page 613), *diofant.matrices.sparse.SparseMatrixBase.col* (page 627), *row_insert* (page 598)

Examples

```
>>> M = zeros(3)
>>> V = ones(3, 1)
>>> M.col_insert(1, V)
Matrix([
[0, 1, 0, 0],
[0, 1, 0, 0],
[0, 1, 0, 0]])
```

col_join(*bott*)
Concatenates two matrices along self's last and bott's first row

See also:

diofant.matrices.dense.DenseMatrix.col (page 613), *diofant.matrices.sparse.SparseMatrixBase.col* (page 627), *row_join* (page 598)

Examples

```
>>> M = zeros(3)
>>> V = ones(1, 3)
>>> M.col_join(V)
Matrix([
[0, 0, 0],
[0, 0, 0],
[0, 0, 0],
[1, 1, 1]])
```

condition_number()

Returns the condition number of a matrix.

This is the maximum singular value divided by the minimum singular value

See also:

singular_values (page 599)

Examples

```
>>> A = Matrix([[1, 0, 0], [0, 10, 0], [0, 0, S.One/10]])
>>> A.condition_number()
100
```

conjugate()

By-element conjugation.

copy()

Returns the copy of a matrix.

cross(*b*)

Return the cross product of $self$ and b relaxing the condition of compatible dimensions: if each has 3 elements, a matrix of the same type and shape as $self$ will be returned. If b has the same shape as $self$ then common identities for the cross product (like $axb = -bxa$) will hold.

See also:

dot (page 579), *multiply* (page 593), *multiply_elementwise* (page 593)

det(*method='bareis'*)

Computes the matrix determinant using the method "method".

Possible values for "method": bareis ... det_bareis berkowitz ... berkowitz_det det_LU ... det_LU_decomposition

See also:

det_bareis (page 578), *berkowitz_det* (page 574), *det_LU_decomposition* (page 577)

det_LU_decomposition()

Compute matrix determinant using LU decomposition

Note that this method fails if the LU decomposition itself fails. In particular, if the matrix has no inverse this method will fail.

TODO: Implement algorithm for sparse matrices (SFF), Hong R. Lee, B.David Saunders, Fraction Free Gaussian Elimination for Sparse Matrices, In Journal of Symbolic Computation, Volume 19, Issue 5, 1995, Pages 393-402, ISSN 0747-7171, https://www.sciencedirect.com/science/article/pii/S074771718571022X.

See also:

det (page 577), *det_bareis* (page 578), *berkowitz_det* (page 574)

det_bareis()

Compute matrix determinant using Bareis' fraction-free algorithm which is an extension of the well known Gaussian elimination method. This approach is best suited for dense symbolic matrices and will result in a determinant with minimal number of fractions. It means that less term rewriting is needed on resulting formulae.

TODO: Implement algorithm for sparse matrices (SFF), Hong R. Lee, B.David Saunders, Fraction Free Gaussian Elimination for Sparse Matrices, In Journal of Symbolic Computation, Volume 19, Issue 5, 1995, Pages 393-402, ISSN 0747-7171, https://www.sciencedirect.com/science/article/pii/S074771718571022X.

See also:

det (page 577), *berkowitz_det* (page 574)

diagonal_solve(*rhs*)

Solves Ax = B efficiently, where A is a diagonal Matrix, with non-zero diagonal entries.

See also:

lower_triangular_solve (page 593), *upper_triangular_solve* (page 602), *cholesky_solve* (page 576), *LDLsolve* (page 570), *LUsolve* (page 571), *QRsolve* (page 572), *pinv_solve* (page 596)

Examples

```
>>> A = eye(2)*2
>>> B = Matrix([[1, 2], [3, 4]])
>>> A.diagonal_solve(B) == B/2
True
```

diagonalize(*reals_only=False*, *sort=False*, *normalize=False*)

Return (P, D), where D is diagonal and

D = P^-1 * M * P

where M is current matrix.

See also:

is_diagonal (page 584), *is_diagonalizable* (page 585)

Examples

```
>>> m = Matrix(3, 3, [1, 2, 0, 0, 3, 0, 2, -4, 2])
>>> m
Matrix([
[1,  2, 0],
```

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```
[0,  3, 0],
[2, -4, 2]])
>>> (P, D) = m.diagonalize()
>>> D
Matrix([
[1, 0, 0],
[0, 2, 0],
[0, 0, 3]])
>>> P
Matrix([
[-1, 0, -1],
[ 0, 0, -1],
[ 2, 1,  2]])
>>> P.inv() * m * P
Matrix([
[1, 0, 0],
[0, 2, 0],
[0, 0, 3]])
```

diff(**args*)

Calculate the derivative of each element in the matrix.

See also:

integrate (page 582), *limit* (page 593)

Examples

```
>>> M = Matrix([[x, y], [1, 0]])
>>> M.diff(x)
Matrix([
[1, 0],
[0, 0]])
```

dot(*b*)

Return the dot product of Matrix self and b relaxing the condition of compatible dimensions: if either the number of rows or columns are the same as the length of b then the dot product is returned. If self is a row or column vector, a scalar is returned. Otherwise, a list of results is returned (and in that case the number of columns in self must match the length of b).

See also:

cross (page 577), *multiply* (page 593), *multiply_elementwise* (page 593)

Examples

```
>>> M = Matrix([[1, 2, 3], [4, 5, 6], [7, 8, 9]])
>>> v = [1, 1, 1]
>>> M.row(0).dot(v)
6
>>> M.col(0).dot(v)
12
```

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```
>>> M.dot(v)
[6, 15, 24]
```

dual()

Returns the dual of a matrix, which is:

$(1/2) * levicivita(i, j, k, l) * M(k, l)$ summed over indices k and l

Since the levicivita method is anti_symmetric for any pairwise exchange of indices, the dual of a symmetric matrix is the zero matrix. Strictly speaking the dual defined here assumes that the 'matrix' M is a contravariant anti_symmetric second rank tensor, so that the dual is a covariant second rank tensor.

eigenvals(***flags*)

Return eigen values using the berkowitz_eigenvals routine.

Since the roots routine doesn't always work well with Floats, they will be replaced with Rationals before calling that routine. If this is not desired, set flag `rational` to False.

eigenvects(***flags*)

Return list of triples (eigenval, multiplicity, basis).

The flag `simplify` has two effects: 1) if bool(simplify) is True, as_content_primitive() will be used to tidy up normalization artifacts; 2) if nullspace needs simplification to compute the basis, the simplify flag will be passed on to the nullspace routine which will interpret it there.

If the matrix contains any Floats, they will be changed to Rationals for computation purposes, but the answers will be returned after being evaluated with evalf. If it is desired to removed small imaginary portions during the evalf step, pass a value for the `chop` flag.

evalf(*dps=15*, ***options*)

Apply evalf() to each element of self.

exp()

Return the exponentiation of a square matrix.

expand(*deep=True*, *modulus=None*, *power_base=True*, *power_exp=True*, *mul=True*, *log=True*, *multinomial=True*, *basic=True*, ***hints*)

Apply core.function.expand to each entry of the matrix.

Examples

```
>>> Matrix(1, 1, [x*(x+1)])
Matrix([[x*(x + 1)]])
>>> _.expand()
Matrix([[x**2 + x]])
```

extract(*rowsList*, *colsList*)

Return a submatrix by specifying a list of rows and columns. Negative indices can be given. All indices must be in the range -n <= i < n where n is the number of rows or columns.

Examples

```
>>> m = Matrix(4, 3, range(12))
>>> m
Matrix([
[0,  1,  2],
[3,  4,  5],
[6,  7,  8],
[9, 10, 11]])
>>> m.extract([0, 1, 3], [0, 1])
Matrix([
[0,  1],
[3,  4],
[9, 10]])
```

Rows or columns can be repeated:

```
>>> m.extract([0, 0, 1], [-1])
Matrix([
[2],
[2],
[5]])
```

Every other row can be taken by using range to provide the indices:

```
>>> m.extract(range(0, m.rows, 2), [-1])
Matrix([
[2],
[8]])
```

free_symbols

Returns the free symbols within the matrix.

Examples

```
>>> Matrix([[x], [1]]).free_symbols
{x}
```

get_diag_blocks()

Obtains the square sub-matrices on the main diagonal of a square matrix.

Useful for inverting symbolic matrices or solving systems of linear equations which may be decoupled by having a block diagonal structure.

Examples

```
>>> A = Matrix([[1, 3, 0, 0], [y, z*z, 0, 0], [0, 0, x, 0], [0, 0, 0, 0]])
>>> a1, a2, a3 = A.get_diag_blocks()
>>> a1
Matrix([
[1,    3],
[y, z**2]])
>>> a2
```

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```
Matrix([[x]])
>>> a3
Matrix([[0]])
```

has(**patterns*)
Test whether any subexpression matches any of the patterns.

Examples

```
>>> A = Matrix(((1, x), (0.2, 3)))
>>> A.has(x)
True
>>> A.has(y)
False
>>> A.has(Float)
True
```

classmethod hstack(**args*)
Return a matrix formed by joining args horizontally (i.e. by repeated application of row_join).

Examples

```
>>> Matrix.hstack(eye(2), 2*eye(2))
Matrix([
[1, 0, 2, 0],
[0, 1, 0, 2]])
```

integrate(**args*)
Integrate each element of the matrix.

See also:

limit (page 593), *diff* (page 579)

Examples

```
>>> M = Matrix([[x, y], [1, 0]])
>>> M.integrate(x)
Matrix([
[x**2/2, x*y],
[     x,   0]])
>>> M.integrate((x, 0, 2))
Matrix([
[2, 2*y],
[2,   0]])
```

inv(*method=None, **kwargs*)
Returns the inverse of the matrix.

Parameters method : {'GE', 'LU', 'ADJ', 'CH', 'LDL'} or None

Selects algorithm for inversion. For dense matrices available {'GE', 'LU', 'ADJ'}, default is 'GE'. For sparse: {'CH', 'LDL'}, default is 'LDL'.

Raises ValueError

If the determinant of the matrix is zero.

See also:

inverse_LU (page 583), *inverse_GE* (page 583), *inverse_ADJ* (page 583)

inv_mod(*m*)

Returns the inverse of the matrix K (mod m), if it exists.

Method to find the matrix inverse of K (mod m) implemented in this function:

- Compute $\mathrm{adj}(K) = \mathrm{cof}(K)^t$, the adjoint matrix of K.
- Compute $r = 1/\det(K) \pmod m$.
- $K^{-1} = r \cdot \mathrm{adj}(K) \pmod m$.

Examples

```
>>> A = Matrix(2, 2, [1, 2, 3, 4])
>>> A.inv_mod(5)
Matrix([
[3, 1],
[4, 2]])
>>> A.inv_mod(3)
Matrix([
[1, 1],
[0, 1]])
```

inverse_ADJ(*iszerofunc=<function _iszero>*)

Calculates the inverse using the adjugate matrix and a determinant.

See also:

diofant.matrices.matrices.MatrixBase.inv (page 582), *inverse_LU* (page 583), *inverse_GE* (page 583)

inverse_GE(*iszerofunc=<function _iszero>*)

Calculates the inverse using Gaussian elimination.

See also:

diofant.matrices.matrices.MatrixBase.inv (page 582), *inverse_LU* (page 583), *inverse_ADJ* (page 583)

inverse_LU(*iszerofunc=<function _iszero>*)

Calculates the inverse using LU decomposition.

See also:

diofant.matrices.matrices.MatrixBase.inv (page 582), *inverse_GE* (page 583), *inverse_ADJ* (page 583)

is_anti_symmetric(*simplify=True*)

Check if matrix M is an antisymmetric matrix, that is, M is a square matrix with all M[i, j] == -M[j, i].

When `simplify=True` (default), the sum M[i, j] + M[j, i] is simplified before testing to see if it is zero. By default, the Diofant simplify function is used. To use a custom function set simplify to a function that accepts a single argument which returns a simplified expression. To skip simplification, set simplify to False but note that although this will be faster, it may induce false negatives.

Examples

```
>>> m = Matrix(2, 2, [0, 1, -1, 0])
>>> m
Matrix([
[ 0, 1],
[-1, 0]])
>>> m.is_anti_symmetric()
True
>>> x, y = symbols('x y')
>>> m = Matrix(2, 3, [0, 0, x, -y, 0, 0])
>>> m
Matrix([
[ 0, 0, x],
[-y, 0, 0]])
>>> m.is_anti_symmetric()
False
```

```
>>> m = Matrix(3, 3, [0, x**2 + 2*x + 1, y,
...                   -(x + 1)**2 , 0, x*y,
...                   -y, -x*y, 0])
```

Simplification of matrix elements is done by default so even though two elements which should be equal and opposite wouldn't pass an equality test, the matrix is still reported as anti-symmetric:

```
>>> m[0, 1] == -m[1, 0]
False
>>> m.is_anti_symmetric()
True
```

If 'simplify=False' is used for the case when a Matrix is already simplified, this will speed things up. Here, we see that without simplification the matrix does not appear anti-symmetric:

```
>>> m.is_anti_symmetric(simplify=False)
False
```

But if the matrix were already expanded, then it would appear anti-symmetric and simplification in the is_anti_symmetric routine is not needed:

```
>>> m = m.expand()
>>> m.is_anti_symmetric(simplify=False)
True
```

is_diagonal()

Check if matrix is diagonal, that is matrix in which the entries outside the main diagonal are all zero.

See also:

is_lower (page 586), *is_upper* (page 589), *is_diagonalizable* (page 585), *diagonalize* (page 578)

Examples

```
>>> m = Matrix(2, 2, [1, 0, 0, 2])
>>> m
Matrix([
[1, 0],
[0, 2]])
>>> m.is_diagonal()
True
```

```
>>> m = Matrix(2, 2, [1, 1, 0, 2])
>>> m
Matrix([
[1, 1],
[0, 2]])
>>> m.is_diagonal()
False
```

```
>>> m = diag(1, 2, 3)
>>> m
Matrix([
[1, 0, 0],
[0, 2, 0],
[0, 0, 3]])
>>> m.is_diagonal()
True
```

is_diagonalizable(*reals_only=False, clear_subproducts=True*)

Check if matrix is diagonalizable.

If reals_only==True then check that diagonalized matrix consists of the only not complex values.

Some subproducts could be used further in other methods to avoid double calculations, By default (if clear_subproducts==True) they will be deleted.

See also:

is_diagonal (page 584), *diagonalize* (page 578)

Examples

```
>>> m = Matrix(3, 3, [1, 2, 0, 0, 3, 0, 2, -4, 2])
>>> m
Matrix([
[1,  2, 0],
[0,  3, 0],
[2, -4, 2]])
>>> m.is_diagonalizable()
True
>>> m = Matrix(2, 2, [0, 1, 0, 0])
>>> m
```

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```
Matrix([
[0, 1],
[0, 0]])
>>> m.is_diagonalizable()
False
>>> m = Matrix(2, 2, [0, 1, -1, 0])
>>> m
Matrix([
[ 0, 1],
[-1, 0]])
>>> m.is_diagonalizable()
True
>>> m.is_diagonalizable(True)
False
```

is_hermitian

Checks if the matrix is Hermitian.

In a Hermitian matrix element i,j is the complex conjugate of element j,i.

Examples

```
>>> a = Matrix([[1, I], [-I, 1]])
>>> a
Matrix([
[ 1, I],
[-I, 1]])
>>> a.is_hermitian
True
>>> a[0, 0] = 2*I
>>> a.is_hermitian
False
>>> a[0, 0] = x
>>> a.is_hermitian
>>> a[0, 1] = a[1, 0]*I
>>> a.is_hermitian
False
```

is_lower

Check if matrix is a lower triangular matrix. True can be returned even if the matrix is not square.

See also:

is_upper (page 589), *is_diagonal* (page 584), *is_lower_hessenberg* (page 587)

Examples

```
>>> m = Matrix(2, 2, [1, 0, 0, 1])
>>> m
Matrix([
[1, 0],
[0, 1]])
```

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```
>>> m.is_lower
True
```

```
>>> m = Matrix(4, 3, [0, 0, 0, 2, 0, 0, 1, 4 , 0, 6, 6, 5])
>>> m
Matrix([
[0, 0, 0],
[2, 0, 0],
[1, 4, 0],
[6, 6, 5]])
>>> m.is_lower
True
```

```
>>> m = Matrix(2, 2, [x**2 + y, y**2 + x, 0, x + y])
>>> m
Matrix([
[x**2 + y, x + y**2],
[       0,    x + y]])
>>> m.is_lower
False
```

is_lower_hessenberg

Checks if the matrix is in the lower-Hessenberg form.

The lower hessenberg matrix has zero entries above the first superdiagonal.

See also:

is_upper_hessenberg (page 590), *is_lower* (page 586)

Examples

```
>>> a = Matrix([[1, 2, 0, 0], [5, 2, 3, 0], [3, 4, 3, 7], [5, 6, 1, 1]])
>>> a
Matrix([
[1, 2, 0, 0],
[5, 2, 3, 0],
[3, 4, 3, 7],
[5, 6, 1, 1]])
>>> a.is_lower_hessenberg
True
```

is_nilpotent()

Checks if a matrix is nilpotent.

A matrix B is nilpotent if for some integer k, B**k is a zero matrix.

Examples

```
>>> a = Matrix([[0, 0, 0], [1, 0, 0], [1, 1, 0]])
>>> a.is_nilpotent()
True
```

```
>>> a = Matrix([[1, 0, 1], [1, 0, 0], [1, 1, 0]])
>>> a.is_nilpotent()
False
```

is_square

Checks if a matrix is square.

A matrix is square if the number of rows equals the number of columns. The empty matrix is square by definition, since the number of rows and the number of columns are both zero.

Examples

```
>>> a = Matrix([[1, 2, 3], [4, 5, 6]])
>>> b = Matrix([[1, 2, 3], [4, 5, 6], [7, 8, 9]])
>>> c = Matrix([])
>>> a.is_square
False
>>> b.is_square
True
>>> c.is_square
True
```

is_symbolic()

Checks if any elements contain Symbols.

Examples

```
>>> M = Matrix([[x, y], [1, 0]])
>>> M.is_symbolic()
True
```

is_symmetric(*simplify=True*)

Check if matrix is symmetric matrix, that is square matrix and is equal to its transpose.

By default, simplifications occur before testing symmetry. They can be skipped using 'simplify=False'; while speeding things a bit, this may however induce false negatives.

Examples

```
>>> m = Matrix(2, 2, [0, 1, 1, 2])
>>> m
Matrix([
[0, 1],
[1, 2]])
>>> m.is_symmetric()
True
```

```
>>> m = Matrix(2, 2, [0, 1, 2, 0])
>>> m
Matrix([
[0, 1],
[2, 0]])
>>> m.is_symmetric()
False
```

```
>>> m = Matrix(2, 3, [0, 0, 0, 0, 0, 0])
>>> m
Matrix([
[0, 0, 0],
[0, 0, 0]])
>>> m.is_symmetric()
False
```

```
>>> m = Matrix(3, 3, [1, x**2 + 2*x + 1, y, (x + 1)**2 , 2, 0, y, 0, 3])
>>> m
Matrix([
[         1, x**2 + 2*x + 1, y],
[(x + 1)**2,              2, 0],
[         y,              0, 3]])
>>> m.is_symmetric()
True
```

If the matrix is already simplified, you may speed-up is_symmetric() test by using 'simplify=False'.

```
>>> m.is_symmetric(simplify=False)
False
>>> m1 = m.expand()
>>> m1.is_symmetric(simplify=False)
True
```

is_upper

Check if matrix is an upper triangular matrix. True can be returned even if the matrix is not square.

See also:

is_lower (page 586), *is_diagonal* (page 584), *is_upper_hessenberg* (page 590)

Examples

```
>>> m = Matrix(2, 2, [1, 0, 0, 1])
>>> m
Matrix([
[1, 0],
[0, 1]])
>>> m.is_upper
True
```

```
>>> m = Matrix(4, 3, [5, 1, 9, 0, 4 , 6, 0, 0, 5, 0, 0, 0])
>>> m
Matrix([
```

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```
[5, 1, 9],
[0, 4, 6],
[0, 0, 5],
[0, 0, 0]])
>>> m.is_upper
True
```

```
>>> m = Matrix(2, 3, [4, 2, 5, 6, 1, 1])
>>> m
Matrix([
[4, 2, 5],
[6, 1, 1]])
>>> m.is_upper
False
```

is_upper_hessenberg

Checks if the matrix is the upper-Hessenberg form.

The upper hessenberg matrix has zero entries below the first subdiagonal.

See also:

is_lower_hessenberg (page 587), *is_upper* (page 589)

Examples

```
>>> a = Matrix([[1, 4, 2, 3], [3, 4, 1, 7], [0, 2, 3, 4], [0, 0, 1, 3]])
>>> a
Matrix([
[1, 4, 2, 3],
[3, 4, 1, 7],
[0, 2, 3, 4],
[0, 0, 1, 3]])
>>> a.is_upper_hessenberg
True
```

is_zero

Checks if a matrix is a zero matrix.

A matrix is zero if every element is zero. A matrix need not be square to be considered zero. The empty matrix is zero by the principle of vacuous truth. For a matrix that may or may not be zero (e.g. contains a symbol), this will be None

Examples

```
>>> a = Matrix([[0, 0], [0, 0]])
>>> b = zeros(3, 4)
>>> c = Matrix([[0, 1], [0, 0]])
>>> d = Matrix([])
>>> e = Matrix([[x, 0], [0, 0]])
>>> a.is_zero
True
>>> b.is_zero
```

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```
True
>>> c.is_zero
False
>>> d.is_zero
True
>>> e.is_zero
```

jacobian(*X*)

Calculates the Jacobian matrix (derivative of a vectorial function).

Parameters self : vector of expressions representing functions f_i(x_1, ..., x_n).

X : set of x_i's in order, it can be a list or a Matrix

Both self and X can be a row or a column matrix in any order

(i.e., jacobian() should always work).

See also:

diofant.matrices.dense.hessian (page 606), *diofant.matrices.dense.wronskian* (page 607)

Examples

```
>>> from diofant.abc import rho, phi
>>> X = Matrix([rho*cos(phi), rho*sin(phi), rho**2])
>>> Y = Matrix([rho, phi])
>>> X.jacobian(Y)
Matrix([
[cos(phi), -rho*sin(phi)],
[sin(phi),  rho*cos(phi)],
[   2*rho,             0]])
>>> X = Matrix([rho*cos(phi), rho*sin(phi)])
>>> X.jacobian(Y)
Matrix([
[cos(phi), -rho*sin(phi)],
[sin(phi),  rho*cos(phi)]])
```

jordan_cells(*calc_transformation=True*)

Return a list of Jordan cells of current matrix. This list shape Jordan matrix J.

If calc_transformation is specified as False, then transformation P such that

$$J = P^{-1} \cdot M \cdot P$$

will not be calculated.

See also:

jordan_form (page 592)

Notes

Calculation of transformation P is not implemented yet.

Examples

```
>>> m = Matrix(4, 4, [
...  6,  5, -2, -3,
... -3, -1,  3,  3,
...  2,  1, -2, -3,
... -1,  1,  5,  5])
```

```
>>> P, Jcells = m.jordan_cells()
>>> Jcells[0]
Matrix([
[2, 1],
[0, 2]])
>>> Jcells[1]
Matrix([
[2, 1],
[0, 2]])
```

jordan_form(*calc_transformation=True*)
Return Jordan form J of current matrix.

Also the transformation P such that

$$J = P^{-1} \cdot M \cdot P$$

and the jordan blocks forming J will be calculated.

See also:

jordan_cells (page 591)

Examples

```
>>> m = Matrix([
...         [ 6,  5, -2, -3],
...         [-3, -1,  3,  3],
...         [ 2,  1, -2, -3],
...         [-1,  1,  5,  5]])
>>> P, J = m.jordan_form()
>>> J
Matrix([
[2, 1, 0, 0],
[0, 2, 0, 0],
[0, 0, 2, 1],
[0, 0, 0, 2]])
```

key2bounds(*keys*)
Converts a key with potentially mixed types of keys (integer and slice) into a tuple of ranges and raises an error if any index is out of self's range.

See also:

key2ij (page 592)

key2ij(*key*)
Converts key into canonical form, converting integers or indexable items into valid integers for self's range or returning slices unchanged.

See also:

key2bounds (page 592)

limit(**args*)

Calculate the limit of each element in the matrix.

See also:

integrate (page 582), *diff* (page 579)

Examples

```
>>> M = Matrix([[x, y], [1, 0]])
>>> M.limit(x, 2)
Matrix([
[2, y],
[1, 0]])
```

lower_triangular_solve(*rhs*)

Solves Ax = B, where A is a lower triangular matrix.

See also:

upper_triangular_solve (page 602), *cholesky_solve* (page 576), *diagonal_solve* (page 578), *LDLsolve* (page 570), *LUsolve* (page 571), *QRsolve* (page 572), *pinv_solve* (page 596)

minorEntry(*i, j, method='berkowitz'*)

Calculate the minor of an element.

See also:

minorMatrix (page 593), *cofactor* (page 576), *cofactorMatrix* (page 576)

minorMatrix(*i, j*)

Creates the minor matrix of a given element.

See also:

minorEntry (page 593), *cofactor* (page 576), *cofactorMatrix* (page 576)

multiply(*b*)

Returns self*b

See also:

dot (page 579), *cross* (page 577), *multiply_elementwise* (page 593)

multiply_elementwise(*b*)

Return the Hadamard product (elementwise product) of A and B

See also:

cross (page 577), *dot* (page 579), *multiply* (page 593)

Examples

```
>>> A = Matrix([[0, 1, 2], [3, 4, 5]])
>>> B = Matrix([[1, 10, 100], [100, 10, 1]])
>>> A.multiply_elementwise(B)
Matrix([
[  0, 10, 200],
[300, 40,   5]])
```

n(*dps=15*, ***options*)
Apply evalf() to each element of self.

norm(*ord=None*)
Return the Norm of a Matrix or Vector. In the simplest case this is the geometric size of the vector Other norms can be specified by the ord parameter

ord	norm for matrices	norm for vectors
None	Frobenius norm	2-norm
'fro'	Frobenius norm	• does not exist
inf	–	max(abs(x))
-inf	–	min(abs(x))
1	–	as below
-1	–	as below
2	2-norm (largest sing. value)	as below
-2	smallest singular value	as below
other	• does not exist	sum(abs(x)**ord)**(1./ord)

See also:

normalized (page 594)

Examples

```
>>> x = Symbol('x', real=True)
>>> v = Matrix([cos(x), sin(x)])
>>> trigsimp(v.norm())
1
>>> v.norm(10)
(sin(x)**10 + cos(x)**10)**(1/10)
>>> A = Matrix([[1, 1], [1, 1]])
>>> A.norm(2)  # Spectral norm (max of |Ax|/|x| under 2-vector-norm)
2
>>> A.norm(-2)  # Inverse spectral norm (smallest singular value)
0
>>> A.norm()  # Frobenius Norm
2
>>> Matrix([1, -2]).norm(oo)
2
>>> Matrix([-1, 2]).norm(-oo)
1
```

normalized()

Return the normalized version of `self`.

See also:

norm (page 594)

nullspace(*simplify=False*)

Returns list of vectors (Matrix objects) that span nullspace of self

permuteBkwd(*perm*)

Permute the rows of the matrix with the given permutation in reverse.

See also:

permuteFwd (page 595)

Examples

```
>>> M = eye(3)
>>> M.permuteBkwd([[0, 1], [0, 2]])
Matrix([
[0, 1, 0],
[0, 0, 1],
[1, 0, 0]])
```

permuteFwd(*perm*)

Permute the rows of the matrix with the given permutation.

See also:

permuteBkwd (page 595)

Examples

```
>>> M = eye(3)
>>> M.permuteFwd([[0, 1], [0, 2]])
Matrix([
[0, 0, 1],
[1, 0, 0],
[0, 1, 0]])
```

pinv()

Calculate the Moore-Penrose pseudoinverse of the matrix.

The Moore-Penrose pseudoinverse exists and is unique for any matrix. If the matrix is invertible, the pseudoinverse is the same as the inverse.

See also:

diofant.matrices.matrices.MatrixBase.inv (page 582), *pinv_solve* (page 596)

References

[R415] (page 1260)

Examples

```
>>> Matrix([[1, 2, 3], [4, 5, 6]]).pinv()
Matrix([
[-17/18,  4/9],
[  -1/9,  1/9],
[ 13/18, -2/9]])
```

pinv_solve(*B*, *arbitrary_matrix=None*)
Solve Ax = B using the Moore-Penrose pseudoinverse.

There may be zero, one, or infinite solutions. If one solution exists, it will be returned. If infinite solutions exist, one will be returned based on the value of arbitrary_matrix. If no solutions exist, the least-squares solution is returned.

Parameters B : Matrix

The right hand side of the equation to be solved for. Must have the same number of rows as matrix A.

arbitrary_matrix : Matrix

If the system is underdetermined (e.g. A has more columns than rows), infinite solutions are possible, in terms of an arbitrary matrix. This parameter may be set to a specific matrix to use for that purpose; if so, it must be the same shape as x, with as many rows as matrix A has columns, and as many columns as matrix B. If left as None, an appropriate matrix containing dummy symbols in the form of `wn_m` will be used, with n and m being row and column position of each symbol.

Returns x : Matrix

The matrix that will satisfy Ax = B. Will have as many rows as matrix A has columns, and as many columns as matrix B.

See also:

lower_triangular_solve (page 593), *upper_triangular_solve* (page 602), *cholesky_solve* (page 576), *diagonal_solve* (page 578), *LDLsolve* (page 570), *LUsolve* (page 571), *QRsolve* (page 572), *pinv* (page 595)

Notes

This may return either exact solutions or least squares solutions. To determine which, check `A * A.pinv() * B == B`. It will be True if exact solutions exist, and False if only a least-squares solution exists. Be aware that the left hand side of that equation may need to be simplified to correctly compare to the right hand side.

References

[R416] (page 1260)

Examples

```
>>> A = Matrix([[1, 2, 3], [4, 5, 6]])
>>> B = Matrix([7, 8])
>>> A.pinv_solve(B)
Matrix([
[ _w0_0/6 - _w1_0/3 + _w2_0/6 - 55/18],
[-_w0_0/3 + 2*_w1_0/3 - _w2_0/3 + 1/9],
[ _w0_0/6 - _w1_0/3 + _w2_0/6 + 59/18]])
>>> A.pinv_solve(B, arbitrary_matrix=Matrix([0, 0, 0]))
Matrix([
[-55/18],
[   1/9],
[ 59/18]])
```

print_nonzero(*symb='X'*)
Shows location of non-zero entries for fast shape lookup.

Examples

```
>>> m = Matrix(2, 3, lambda i, j: i*3+j)
>>> m
Matrix([
[0, 1, 2],
[3, 4, 5]])
>>> m.print_nonzero()
[ XX]
[XXX]
>>> m = eye(4)
>>> m.print_nonzero("x")
[x   ]
[ x  ]
[  x ]
[   x]
```

project(*v*)
Return the projection of `self` onto the line containing `v`.

Examples

```
>>> V = Matrix([sqrt(3)/2, S.Half])
>>> x = Matrix([[1, 0]])
>>> V.project(x)
Matrix([[sqrt(3)/2, 0]])
>>> V.project(-x)
Matrix([[sqrt(3)/2, 0]])
```

rank(*iszerofunc=<function _iszero>, simplify=False*)
Returns the rank of a matrix

```
>>> m = Matrix([[1, 2], [x, 1 - 1/x]])
>>> m.rank()
2
```

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```
>>> n = Matrix(3, 3, range(1, 10))
>>> n.rank()
2
```

replace(*F*, *G*)

Replaces Function F in Matrix entries with Function G.

Examples

```
>>> F, G = symbols('F, G', cls=Function)
>>> M = Matrix(2, 2, lambda i, j: F(i+j)) ; M
Matrix([
[F(0), F(1)],
[F(1), F(2)]])
>>> N = M.replace(F, G)
>>> N
Matrix([
[G(0), G(1)],
[G(1), G(2)]])
```

row_insert(*pos*, *mti*)

Insert one or more rows at the given row position.

See also:

diofant.matrices.dense.DenseMatrix.row (page 614), *diofant.matrices.sparse.SparseMatrixBase.row* (page 630), *col_insert* (page 576)

Examples

```
>>> M = zeros(3)
>>> V = ones(1, 3)
>>> M.row_insert(1, V)
Matrix([
[0, 0, 0],
[1, 1, 1],
[0, 0, 0],
[0, 0, 0]])
```

row_join(*rhs*)

Concatenates two matrices along self's last and rhs's first column

See also:

diofant.matrices.dense.DenseMatrix.row (page 614), *diofant.matrices.sparse.SparseMatrixBase.row* (page 630), *col_join* (page 576)

Examples

```
>>> M = zeros(3)
>>> V = ones(3, 1)
>>> M.row_join(V)
```

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```
Matrix([
[0, 0, 0, 1],
[0, 0, 0, 1],
[0, 0, 0, 1]])
```

rref(*iszerofunc=<function _iszero>, simplify=False*)

Return reduced row-echelon form of matrix and indices of pivot vars.

To simplify elements before finding nonzero pivots set simplify=True (to use the default Diofant simplify function) or pass a custom simplify function.

Examples

```
>>> m = Matrix([[1, 2], [x, 1 - 1/x]])
>>> m.rref()
(Matrix([
[1, 0],
[0, 1]]), [0, 1])
```

shape

The shape (dimensions) of the matrix as the 2-tuple (rows, cols).

Examples

```
>>> M = zeros(2, 3)
>>> M.shape
(2, 3)
>>> M.rows
2
>>> M.cols
3
```

simplify(*ratio=1.7, measure=<function count_ops>*)

Apply simplify to each element of the matrix.

Examples

```
>>> SparseMatrix(1, 1, [x*sin(y)**2 + x*cos(y)**2])
Matrix([[x*sin(y)**2 + x*cos(y)**2]])
>>> _.simplify()
Matrix([[x]])
```

singular_values()

Compute the singular values of a Matrix

See also:

condition_number (page 577)

Examples

```
>>> x = Symbol('x', extended_real=True)
>>> A = Matrix([[0, 1, 0], [0, x, 0], [-1, 0, 0]])
>>> A.singular_values()
[sqrt(x**2 + 1), 1, 0]
```

solve_least_squares(*rhs*, *method='CH'*)

Return the least-square fit to the data.

By default the cholesky_solve routine is used (method='CH'); other methods of matrix inversion can be used.

See also:

inv (page 582)

Examples

```
>>> A = Matrix([1, 2, 3])
>>> B = Matrix([2, 3, 4])
>>> S = Matrix(A.row_join(B))
>>> S
Matrix([
[1, 2],
[2, 3],
[3, 4]])
```

If each line of S represent coefficients of Ax + By and x and y are [2, 3] then S*xy is:

```
>>> r = S*Matrix([2, 3]); r
Matrix([
[ 8],
[13],
[18]])
```

But let's add 1 to the middle value and then solve for the least-squares value of xy:

```
>>> xy = S.solve_least_squares(Matrix([8, 14, 18])); xy
Matrix([
[ 5/3],
[10/3]])
```

The error is given by S*xy - r:

```
>>> S*xy - r
Matrix([
[1/3],
[1/3],
[1/3]])
>>> _.norm().n(2)
0.58
```

If a different xy is used, the norm will be higher:

```
>>> xy += ones(2, 1)/10
>>> (S*xy - r).norm().n(2)
1.5
```

subs(**args, **kwargs*)

Return a new matrix with subs applied to each entry.

Examples

```
>>> SparseMatrix(1, 1, [x])
Matrix([[x]])
>>> _.subs(x, y)
Matrix([[y]])
>>> Matrix(_).subs(y, x)
Matrix([[x]])
```

table(*printer, rowstart='[', rowend=']', rowsep='\n', colsep=', ', align='right'*)

String form of Matrix as a table.

`printer` is the printer to use for on the elements (generally something like StrPrinter())

`rowstart` is the string used to start each row (by default '[').

`rowend` is the string used to end each row (by default ']').

`rowsep` is the string used to separate rows (by default a newline).

`colsep` is the string used to separate columns (by default ', ').

`align` defines how the elements are aligned. Must be one of 'left', 'right', or 'center'. You can also use '<', '>', and '^' to mean the same thing, respectively.

This is used by the string printer for Matrix.

Examples

```
>>> from diofant.printing.str import StrPrinter
>>> M = Matrix([[1, 2], [-33, 4]])
>>> printer = StrPrinter()
>>> M.table(printer)
'[  1, 2]\n[-33, 4]'
>>> print(M.table(printer))
[  1, 2]
[-33, 4]
>>> print(M.table(printer, rowsep=',\n'))
[  1, 2],
[-33, 4]
>>> print('[%s]' % M.table(printer, rowsep=',\n'))
[[  1, 2],
[-33, 4]]
>>> print(M.table(printer, colsep=' '))
[  1 2]
[-33 4]
>>> print(M.table(printer, align='center'))
[ 1 , 2]
```

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```
[-33, 4]
>>> print(M.table(printer, rowstart='{', rowend='}'))
{  1, 2}
{-33, 4}
```

trace()
Returns the trace of a matrix.

transpose()
Matrix transposition.

upper_triangular_solve(*rhs*)
Solves Ax = B, where A is an upper triangular matrix.

See also:

lower_triangular_solve (page 593), *cholesky_solve* (page 576), *diagonal_solve* (page 578), *LDLsolve* (page 570), *LUsolve* (page 571), *QRsolve* (page 572), *pinv_solve* (page 596)

values()
Return non-zero values of self.

vec()
Return the Matrix converted into a one column matrix by stacking columns

See also:

vech (page 602)

Examples

```
>>> m = Matrix([[1, 3], [2, 4]])
>>> m
Matrix([
[1, 3],
[2, 4]])
>>> m.vec()
Matrix([
[1],
[2],
[3],
[4]])
```

vech(*diagonal=True, check_symmetry=True*)
Return the unique elements of a symmetric Matrix as a one column matrix by stacking the elements in the lower triangle.

Arguments: diagonal – include the diagonal cells of self or not check_symmetry – checks symmetry of self but not completely reliably

See also:

vec (page 602)

Examples

```
>>> m = Matrix([[1, 2], [2, 3]])
>>> m
Matrix([
[1, 2],
[2, 3]])
>>> m.vech()
Matrix([
[1],
[2],
[3]])
>>> m.vech(diagonal=False)
Matrix([[2]])
```

classmethod vstack(**args*)
Return a matrix formed by joining args vertically (i.e. by repeated application of col_join).

Examples

```
>>> Matrix.vstack(eye(2), 2*eye(2))
Matrix([
[1, 0],
[0, 1],
[2, 0],
[0, 2]])
```

xreplace(*rule*)
Return a new matrix with xreplace applied to each entry.

Examples

```
>>> SparseMatrix(1, 1, [x])
Matrix([[x]])
>>> _.xreplace({x: y})
Matrix([[y]])
>>> Matrix(_).xreplace({y: x})
Matrix([[x]])
```

Matrix Exceptions Reference

class diofant.matrices.matrices.**MatrixError**

class diofant.matrices.matrices.**ShapeError**
Wrong matrix shape

class diofant.matrices.matrices.**NonSquareMatrixError**

Matrix Functions Reference

diofant.matrices.matrices.**classof**(*A*, *B*)

Get the type of the result when combining matrices of different types.

Currently the strategy is that immutability is contagious.

Examples

```
>>> M = Matrix([[1, 2], [3, 4]])  # a Mutable Matrix
>>> IM = ImmutableMatrix([[1, 2], [3, 4]])
>>> classof(M, IM)
<class 'diofant.matrices.immutable.ImmutableMatrix'>
```

diofant.matrices.dense.**matrix_multiply_elementwise**(*A*, *B*)

Return the Hadamard product (elementwise product) of A and B

```
>>> A = Matrix([[0, 1, 2], [3, 4, 5]])
>>> B = Matrix([[1, 10, 100], [100, 10, 1]])
>>> matrix_multiply_elementwise(A, B)
Matrix([
[  0, 10, 200],
[300, 40,   5]])
```

See also:

diofant.matrices.dense.DenseMatrix.__mul__ (page 612)

diofant.matrices.dense.**zeros**(*r*, *c=None*, *cls=None*)

Returns a matrix of zeros with r rows and c columns; if c is omitted a square matrix will be returned.

See also:

diofant.matrices.dense.ones (page 604), *diofant.matrices.dense.eye* (page 604), *diofant.matrices.dense.diag* (page 604)

diofant.matrices.dense.**ones**(*r*, *c=None*)

Returns a matrix of ones with r rows and c columns; if c is omitted a square matrix will be returned.

See also:

diofant.matrices.dense.zeros (page 604), *diofant.matrices.dense.eye* (page 604), *diofant.matrices.dense.diag* (page 604)

diofant.matrices.dense.**eye**(*n*, *cls=None*)

Create square identity matrix n x n

See also:

diofant.matrices.dense.diag (page 604), *diofant.matrices.dense.zeros* (page 604), *diofant.matrices.dense.ones* (page 604)

diofant.matrices.dense.**diag**(**values*, ***kwargs*)

Create a sparse, diagonal matrix from a list of diagonal values.

See also:

diofant.matrices.dense.eye (page 604)

Notes

When arguments are matrices they are fitted in resultant matrix.

The returned matrix is a mutable, dense matrix. To make it a different type, send the desired class for keyword `cls`.

Examples

```
>>> diag(1, 2, 3)
Matrix([
[1, 0, 0],
[0, 2, 0],
[0, 0, 3]])
>>> diag(*[1, 2, 3])
Matrix([
[1, 0, 0],
[0, 2, 0],
[0, 0, 3]])
```

The diagonal elements can be matrices; diagonal filling will continue on the diagonal from the last element of the matrix:

```
>>> a = Matrix([x, y, z])
>>> b = Matrix([[1, 2], [3, 4]])
>>> c = Matrix([[5, 6]])
>>> diag(a, 7, b, c)
Matrix([
[x, 0, 0, 0, 0, 0],
[y, 0, 0, 0, 0, 0],
[z, 0, 0, 0, 0, 0],
[0, 7, 0, 0, 0, 0],
[0, 0, 1, 2, 0, 0],
[0, 0, 3, 4, 0, 0],
[0, 0, 0, 0, 5, 6]])
```

When diagonal elements are lists, they will be treated as arguments to Matrix:

```
>>> diag([1, 2, 3], 4)
Matrix([
[1, 0],
[2, 0],
[3, 0],
[0, 4]])
>>> diag([[1, 2, 3]], 4)
Matrix([
[1, 2, 3, 0],
[0, 0, 0, 4]])
```

A given band off the diagonal can be made by padding with a vertical or horizontal "kerning" vector:

```
>>> hpad = ones(0, 2)
>>> vpad = ones(2, 0)
>>> diag(vpad, 1, 2, 3, hpad) + diag(hpad, 4, 5, 6, vpad)
Matrix([
```

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```
[0, 0, 4, 0, 0],
[0, 0, 0, 5, 0],
[1, 0, 0, 0, 6],
[0, 2, 0, 0, 0],
[0, 0, 3, 0, 0]])
```

The type is mutable by default but can be made immutable by setting the `mutable` flag to False:

```
>>> type(diag(1))
<class 'diofant.matrices.dense.MutableDenseMatrix'>
>>> type(diag(1, cls=ImmutableMatrix))
<class 'diofant.matrices.immutable.ImmutableMatrix'>
```

diofant.matrices.dense.**jordan_cell**(*eigenval, n*)

Create matrix of Jordan cell kind:

Examples

```
>>> jordan_cell(x, 4)
Matrix([
[x, 1, 0, 0],
[0, x, 1, 0],
[0, 0, x, 1],
[0, 0, 0, x]])
```

diofant.matrices.dense.**hessian**(*f, varlist, constraints=[]*)

Compute Hessian matrix for a function f wrt parameters in varlist which may be given as a sequence or a row/column vector. A list of constraints may optionally be given.

See also:

diofant.matrices.matrices.MatrixBase.jacobian (page 591), *diofant.matrices.dense.wronskian* (page 607)

References

https//en.wikipedia.org/wiki/Hessian_matrix

Examples

```
>>> f = Function('f')(x, y)
>>> g1 = Function('g')(x, y)
>>> g2 = x**2 + 3*y
>>> pprint(hessian(f, (x, y), [g1, g2]), use_unicode=False)
[                    d               d           ]
[     0        0    --(g(x, y))     --(g(x, y))  ]
[                   dx              dy           ]
[                                                ]
[     0        0        2*x              3       ]
[                                                ]
[                     2               2          ]
```

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```
[d                   d               d           ]
[--(g(x, y))  2*x   ---(f(x, y))   -----(f(x, y))]
[dx                   2            dy dx         ]
[                   dx                           ]
[                                                ]
[                     2                2         ]
[d                   d               d           ]
[--(g(x, y))   3   -----(f(x, y))   ---(f(x, y)) ]
[dy                dy dx               2         ]
[                                    dy          ]
```

diofant.matrices.dense.**GramSchmidt**(*vlist*, *orthonormal=False*)

Apply the Gram-Schmidt process to a set of vectors.

see: https//en.wikipedia.org/wiki/Gram%E2%80%93Schmidt_process

diofant.matrices.dense.**wronskian**(*functions*, *var*, *method='bareis'*)

Compute Wronskian for [] of functions

```
                 | f1       f2        ...   fn      |
                 | f1'      f2'       ...   fn'     |
                 |  .        .        .      .      |
W(f1, ..., fn) = |  .        .         .     .      |
                 |  .        .          .    .      |
                 |  (n)      (n)            (n)     |
                 | D   (f1) D   (f2)  ...  D   (fn) |
```

see: https//en.wikipedia.org/wiki/Wronskian

See also:

diofant.matrices.matrices.MatrixBase.jacobian (page 591), *diofant.matrices.dense.hessian* (page 606)

diofant.matrices.dense.**casoratian**(*seqs*, *n*, *zero=True*)

Given linear difference operator L of order 'k' and homogeneous equation Ly = 0 we want to compute kernel of L, which is a set of 'k' sequences: a(n), b(n), ... z(n).

Solutions of L are linearly independent iff their Casoratian, denoted as C(a, b, ..., z), do not vanish for n = 0.

Casoratian is defined by k x k determinant:

```
+  a(n)     b(n)     . . . z(n)     +
|  a(n+1)   b(n+1)   . . . z(n+1)   |
|    .         .     .        .     |
|    .         .       .      .     |
|    .         .         .    .     |
+  a(n+k-1) b(n+k-1) . . . z(n+k-1) +
```

It proves very useful in rsolve_hyper() where it is applied to a generating set of a recurrence to factor out linearly dependent solutions and return a basis:

```
>>> n = Symbol('n', integer=True)
```

Exponential and factorial are linearly independent:

```
>>> casoratian([2**n, factorial(n)], n) != 0
True
```

diofant.matrices.dense.**randMatrix**(*r*, *c=None*, *min=0*, *max=99*, *seed=None*, *symmetric=False*, *percent=100*)

Create random matrix with dimensions `r x c`. If `c` is omitted the matrix will be square. If `symmetric` is True the matrix must be square. If `percent` is less than 100 then only approximately the given percentage of elements will be non-zero.

Examples

```
>>> randMatrix(3)
[25, 45, 27]
[44, 54,  9]
[23, 96, 46]
>>> randMatrix(3, 2)
[87, 29]
[23, 37]
[90, 26]
>>> randMatrix(3, 3, 0, 2)
[0, 2, 0]
[2, 0, 1]
[0, 0, 1]
>>> randMatrix(3, symmetric=True)
[85, 26, 29]
[26, 71, 43]
[29, 43, 57]
>>> A = randMatrix(3, seed=1)
>>> B = randMatrix(3, seed=2)
>>> A == B
False
>>> A == randMatrix(3, seed=1)
True
>>> randMatrix(3, symmetric=True, percent=50)
[0, 68, 43]
[0, 68,  0]
[0, 91, 34]
```

Numpy Utility Functions Reference

diofant.matrices.dense.**list2numpy**(*l*, *dtype=<class 'object'>*)

Converts python list of Diofant expressions to a NumPy array.

See also:

diofant.matrices.dense.matrix2numpy (page 608)

diofant.matrices.dense.**matrix2numpy**(*m*, *dtype=<class 'object'>*)

Converts Diofant's matrix to a NumPy array.

See also:

diofant.matrices.dense.list2numpy (page 608)

diofant.matrices.dense.**symarray**(*prefix*, *shape*, ***kwargs*)

Create a numpy ndarray of symbols (as an object array).

The created symbols are named `prefix_i1_i2_`… You should thus provide a non-empty prefix if you want your symbols to be unique for different output arrays, as Diofant symbols with identical names are the same object.

Parameters prefix : string

A prefix prepended to the name of every symbol.

shape : int or tuple

Shape of the created array. If an int, the array is one-dimensional; for more than one dimension the shape must be a tuple.

****kwargs** : dict

keyword arguments passed on to Symbol

Examples

These doctests require numpy.

```
>>> symarray('', 3)
[_0 _1 _2]
```

If you want multiple symarrays to contain distinct symbols, you *must* provide unique prefixes:

```
>>> a = symarray('', 3)
>>> b = symarray('', 3)
>>> a[0] == b[0]
True
>>> a = symarray('a', 3)
>>> b = symarray('b', 3)
>>> a[0] == b[0]
False
```

Creating symarrays with a prefix:

```
>>> symarray('a', 3)
[a_0 a_1 a_2]
```

For more than one dimension, the shape must be given as a tuple:

```
>>> symarray('a', (2, 3))
[[a_0_0 a_0_1 a_0_2]
 [a_1_0 a_1_1 a_1_2]]
>>> symarray('a', (2, 3, 2))
[[[a_0_0_0 a_0_0_1]
  [a_0_1_0 a_0_1_1]
  [a_0_2_0 a_0_2_1]]

 [[a_1_0_0 a_1_0_1]
  [a_1_1_0 a_1_1_1]
  [a_1_2_0 a_1_2_1]]]
```

For setting assumptions of the underlying Symbols:

```
>>> [s.is_real for s in symarray('a', 2, real=True)]
[True, True]
```

`diofant.matrices.dense.`**`rot_axis1`**`(`*theta*`)`

Returns a rotation matrix for a rotation of theta (in radians) about the 1-axis.

See also:

diofant.matrices.dense.rot_axis2 **(page 610)** Returns a rotation matrix for a rotation of theta (in radians) about the 2-axis

diofant.matrices.dense.rot_axis3 **(page 610)** Returns a rotation matrix for a rotation of theta (in radians) about the 3-axis

Examples

A rotation of pi/3 (60 degrees):

```
>>> theta = pi/3
>>> rot_axis1(theta)
Matrix([
[1,          0,         0],
[0,        1/2, sqrt(3)/2],
[0, -sqrt(3)/2,       1/2]])
```

If we rotate by pi/2 (90 degrees):

```
>>> rot_axis1(pi/2)
Matrix([
[1,  0, 0],
[0,  0, 1],
[0, -1, 0]])
```

diofant.matrices.dense.**rot_axis2**(*theta*)

Returns a rotation matrix for a rotation of theta (in radians) about the 2-axis.

See also:

diofant.matrices.dense.rot_axis1 **(page 609)** Returns a rotation matrix for a rotation of theta (in radians) about the 1-axis

diofant.matrices.dense.rot_axis3 **(page 610)** Returns a rotation matrix for a rotation of theta (in radians) about the 3-axis

Examples

A rotation of pi/3 (60 degrees):

```
>>> theta = pi/3
>>> rot_axis2(theta)
Matrix([
[      1/2, 0, -sqrt(3)/2],
[        0, 1,          0],
[sqrt(3)/2, 0,        1/2]])
```

If we rotate by pi/2 (90 degrees):

```
>>> rot_axis2(pi/2)
Matrix([
[0, 0, -1],
[0, 1,  0],
[1, 0,  0]])
```

diofant.matrices.dense.**rot_axis3**(*theta*)

Returns a rotation matrix for a rotation of theta (in radians) about the 3-axis.

See also:

diofant.matrices.dense.rot_axis1 **(page 609)** Returns a rotation matrix for a rotation of theta (in radians) about the 1-axis

diofant.matrices.dense.rot_axis2 **(page 610)** Returns a rotation matrix for a rotation of theta (in radians) about the 2-axis

Examples

A rotation of pi/3 (60 degrees):

```
>>> theta = pi/3
>>> rot_axis3(theta)
Matrix([
[       1/2, sqrt(3)/2, 0],
[-sqrt(3)/2,       1/2, 0],
[         0,         0, 1]])
```

If we rotate by pi/2 (90 degrees):

```
>>> rot_axis3(pi/2)
Matrix([
[ 0, 1, 0],
[-1, 0, 0],
[ 0, 0, 1]])
```

diofant.matrices.matrices.**a2idx**(*j*, *n=None*)

Return integer after making positive and validating against n.

3.10.2 Dense Matrices

diofant.matrices.dense.**MutableMatrix**

alias of *diofant.matrices.dense.MutableDenseMatrix* (page 615)

Matrix Class Reference

class diofant.matrices.dense.**DenseMatrix**

__add__(*other*)

Return self + other, raising ShapeError if shapes don't match.

__eq__(*other*)

Return self==value.

__getitem__(*key*)

Return portion of self defined by key. If the key involves a slice then a list will be returned (if key is a single slice) or a matrix (if key was a tuple involving a slice).

Examples

```
>>> m = Matrix([
... [1, 2 + I],
... [3, 4    ]])
```

If the key is a tuple that doesn't involve a slice then that element is returned:

```
>>> m[1, 0]
3
```

When a tuple key involves a slice, a matrix is returned. Here, the first column is selected (all rows, column 0):

```
>>> m[:, 0]
Matrix([
[1],
[3]])
```

If the slice is not a tuple then it selects from the underlying list of elements that are arranged in row order and a list is returned if a slice is involved:

```
>>> m[0]
1
>>> m[::2]
[1, 3]
```

__mul__(*other*)
Return self*other

applyfunc(*f*)
Apply a function to each element of the matrix.

Examples

```
>>> m = Matrix(2, 2, lambda i, j: i*2+j)
>>> m
Matrix([
[0, 1],
[2, 3]])
>>> m.applyfunc(lambda i: 2*i)
Matrix([
[0, 2],
[4, 6]])
```

as_immutable()
Returns an Immutable version of this Matrix

as_mutable()
Returns a mutable version of this matrix

Examples

```
>>> X = ImmutableMatrix([[1, 2], [3, 4]])
>>> Y = X.as_mutable()
>>> Y[1, 1] = 5 # Can set values in Y
>>> Y
Matrix([
[1, 2],
[3, 5]])
```

col(*j*)

Elementary column selector.

See also:

diofant.matrices.dense.DenseMatrix.row (page 614), *diofant.matrices.dense.MutableDenseMatrix.col_op* (page 615), *diofant.matrices.dense.MutableDenseMatrix.col_swap* (page 616), *diofant.matrices.dense.MutableDenseMatrix.col_del* (page 615), *diofant.matrices.matrices.MatrixBase.col_join* (page 576), *diofant.matrices.matrices.MatrixBase.col_insert* (page 576)

Examples

```
>>> eye(2).col(0)
Matrix([
[1],
[0]])
```

equals(*other*, *failing_expression=False*)

Applies `equals` to corresponding elements of the matrices, trying to prove that the elements are equivalent, returning True if they are, False if any pair is not, and None (or the first failing expression if failing_expression is True) if it cannot be decided if the expressions are equivalent or not. This is, in general, an expensive operation.

See also:

diofant.core.expr.Expr.equals (page 65)

Examples

```
>>> A = Matrix([x*(x - 1), 0])
>>> B = Matrix([x**2 - x, 0])
>>> A == B
False
>>> A.simplify() == B.simplify()
True
>>> A.equals(B)
True
>>> A.equals(2)
False
```

classmethod eye(*n*)

Return an n x n identity matrix.

reshape(*rows, cols*)

Reshape the matrix. Total number of elements must remain the same.

Examples

```
>>> m = Matrix(2, 3, lambda i, j: 1)
>>> m
Matrix([
[1, 1, 1],
[1, 1, 1]])
>>> m.reshape(1, 6)
Matrix([[1, 1, 1, 1, 1, 1]])
>>> m.reshape(3, 2)
Matrix([
[1, 1],
[1, 1],
[1, 1]])
```

row(*i*)

Elementary row selector.

See also:

diofant.matrices.dense.DenseMatrix.col (page 613), *diofant.matrices.dense.MutableDenseMatrix.row_op* (page 618), *diofant.matrices.dense.MutableDenseMatrix.row_swap* (page 618), *diofant.matrices.dense.MutableDenseMatrix.row_del* (page 617), *diofant.matrices.matrices.MatrixBase.row_join* (page 598), *diofant.matrices.matrices.MatrixBase.row_insert* (page 598)

Examples

```
>>> eye(2).row(0)
Matrix([[1, 0]])
```

tolist()

Return the Matrix as a nested Python list.

Examples

```
>>> m = Matrix(3, 3, range(9))
>>> m
Matrix([
[0, 1, 2],
[3, 4, 5],
[6, 7, 8]])
>>> m.tolist()
[[0, 1, 2], [3, 4, 5], [6, 7, 8]]
>>> ones(3, 0).tolist()
[[], [], []]
```

When there are no rows then it will not be possible to tell how many columns were in the original matrix:

```
>>> ones(0, 3).tolist()
[]
```

classmethod zeros(*r*, *c=None*)
Return an r x c matrix of zeros, square if c is omitted.

class diofant.matrices.dense.**MutableDenseMatrix**

as_mutable()
Returns a mutable version of this matrix

Examples

```
>>> X = ImmutableMatrix([[1, 2], [3, 4]])
>>> Y = X.as_mutable()
>>> Y[1, 1] = 5 # Can set values in Y
>>> Y
Matrix([
[1, 2],
[3, 5]])
```

col_del(*i*)
Delete the given column.

See also:

diofant.matrices.dense.DenseMatrix.col (page 613), *diofant.matrices.dense.MutableDenseMatrix.row_del* (page 617)

Examples

```
>>> M = eye(3)
>>> M.col_del(1)
>>> M
Matrix([
[1, 0],
[0, 0],
[0, 1]])
```

col_op(*j*, *f*)
In-place operation on col j using two-arg functor whose args are interpreted as (self[i, j], i).

See also:

diofant.matrices.dense.DenseMatrix.col (page 613), *diofant.matrices.dense.MutableDenseMatrix.row_op* (page 618)

Examples

```
>>> M = eye(3)
>>> M.col_op(1, lambda v, i: v + 2*M[i, 0]); M
Matrix([
[1, 2, 0],
[0, 1, 0],
[0, 0, 1]])
```

col_swap(*i*, *j*)
Swap the two given columns of the matrix in-place.

See also:

diofant.matrices.dense.DenseMatrix.col (page 613), *diofant.matrices.dense.MutableDenseMatrix.row_swap* (page 618)

Examples

```
>>> M = Matrix([[1, 0], [1, 0]])
>>> M
Matrix([
[1, 0],
[1, 0]])
>>> M.col_swap(0, 1)
>>> M
Matrix([
[0, 1],
[0, 1]])
```

copyin_list(*key*, *value*)
Copy in elements from a list.

Parameters key : slice

The section of this matrix to replace.

value : iterable

The iterable to copy values from.

See also:

diofant.matrices.dense.MutableDenseMatrix.copyin_matrix (page 617)

Examples

```
>>> I = eye(3)
>>> I[:2, 0] = [1, 2] # col
>>> I
Matrix([
[1, 0, 0],
[2, 1, 0],
[0, 0, 1]])
>>> I[1, :2] = [[3, 4]]
>>> I
Matrix([
[1, 0, 0],
```

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```
[3, 4, 0],
[0, 0, 1]])
```

copyin_matrix(*key*, *value*)

Copy in values from a matrix into the given bounds.

Parameters key : slice

The section of this matrix to replace.

value : Matrix

The matrix to copy values from.

See also:

diofant.matrices.dense.MutableDenseMatrix.copyin_list (page 616)

Examples

```
>>> M = Matrix([[0, 1], [2, 3], [4, 5]])
>>> I = eye(3)
>>> I[:3, :2] = M
>>> I
Matrix([
[0, 1, 0],
[2, 3, 0],
[4, 5, 1]])
>>> I[0, 1] = M
>>> I
Matrix([
[0, 0, 1],
[2, 2, 3],
[4, 4, 5]])
```

fill(*value*)

Fill the matrix with the scalar value.

See also:

diofant.matrices.dense.zeros (page 604), *diofant.matrices.dense.ones* (page 604)

row_del(*i*)

Delete the given row.

See also:

diofant.matrices.dense.DenseMatrix.row (page 614), *diofant.matrices.dense.MutableDenseMatrix.col_del* (page 615)

Examples

```
>>> M = eye(3)
>>> M.row_del(1)
>>> M
```

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```
Matrix([
[1, 0, 0],
[0, 0, 1]])
```

row_op(*i*, *f*)
In-place operation on row i using two-arg functor whose args are interpreted as (self[i, j], j).

See also:

diofant.matrices.dense.DenseMatrix.row (page 614), *diofant.matrices.dense.MutableDenseMatrix.zip_row_op* (page 618), *diofant.matrices.dense.MutableDenseMatrix.col_op* (page 615)

Examples

```
>>> M = eye(3)
>>> M.row_op(1, lambda v, j: v + 2*M[0, j]); M
Matrix([
[1, 0, 0],
[2, 1, 0],
[0, 0, 1]])
```

row_swap(*i*, *j*)
Swap the two given rows of the matrix in-place.

See also:

diofant.matrices.dense.DenseMatrix.row (page 614), *diofant.matrices.dense.MutableDenseMatrix.col_swap* (page 616)

Examples

```
>>> M = Matrix([[0, 1], [1, 0]])
>>> M
Matrix([
[0, 1],
[1, 0]])
>>> M.row_swap(0, 1)
>>> M
Matrix([
[1, 0],
[0, 1]])
```

simplify(*ratio=1.7*, *measure=<function count_ops>*)
Applies simplify to the elements of a matrix in place.

This is a shortcut for M.applyfunc(lambda x: simplify(x, ratio, measure))

See also:

diofant.simplify.simplify.simplify (page 776)

zip_row_op(*i*, *k*, *f*)
In-place operation on row i using two-arg functor whose args are interpreted as (self[i, j], self[k, j]).

See also:

diofant.matrices.dense.DenseMatrix.row (page 614), *diofant.matrices.dense.MutableDenseMatrix.row_op* (page 618), *diofant.matrices.dense.MutableDenseMatrix.col_op* (page 615)

Examples

```
>>> M = eye(3)
>>> M.zip_row_op(1, 0, lambda v, u: v + 2*u); M
Matrix([
[1, 0, 0],
[2, 1, 0],
[0, 0, 1]])
```

ImmutableMatrix Class Reference

class diofant.matrices.immutable.**ImmutableMatrix**

Create an immutable version of a matrix.

Examples

```
>>> ImmutableMatrix(eye(3))
Matrix([
[1, 0, 0],
[0, 1, 0],
[0, 0, 1]])
>>> _[0, 0] = 42
Traceback (most recent call last):
...
TypeError: Cannot set values of ImmutableDenseMatrix
```

C

By-element conjugation.

adjoint()

Conjugate transpose or Hermitian conjugation.

as_mutable()

Returns a mutable version of this matrix

Examples

```
>>> X = ImmutableMatrix([[1, 2], [3, 4]])
>>> Y = X.as_mutable()
>>> Y[1, 1] = 5 # Can set values in Y
>>> Y
Matrix([
[1, 2],
[3, 5]])
```

conjugate()

By-element conjugation.

equals(*other*, *failing_expression=False*)

Applies equals to corresponding elements of the matrices, trying to prove that the elements are equivalent, returning True if they are, False if any pair is not, and None (or the first failing expression if failing_expression is True) if it cannot be decided if the expressions are equivalent or not. This is, in general, an expensive operation.

See also:

diofant.core.expr.Expr.equals (page 65)

Examples

```
>>> A = Matrix([x*(x - 1), 0])
>>> B = Matrix([x**2 - x, 0])
>>> A == B
False
>>> A.simplify() == B.simplify()
True
>>> A.equals(B)
True
>>> A.equals(2)
False
```

is_zero

Checks if a matrix is a zero matrix.

A matrix is zero if every element is zero. A matrix need not be square to be considered zero. The empty matrix is zero by the principle of vacuous truth. For a matrix that may or may not be zero (e.g. contains a symbol), this will be None

Examples

```
>>> a = Matrix([[0, 0], [0, 0]])
>>> b = zeros(3, 4)
>>> c = Matrix([[0, 1], [0, 0]])
>>> d = Matrix([])
>>> e = Matrix([[x, 0], [0, 0]])
>>> a.is_zero
True
>>> b.is_zero
True
>>> c.is_zero
False
>>> d.is_zero
True
>>> e.is_zero
```

shape

The shape (dimensions) of the matrix as the 2-tuple (rows, cols).

Examples

```
>>> M = zeros(2, 3)
>>> M.shape
(2, 3)
>>> M.rows
2
>>> M.cols
3
```

3.10.3 Sparse Matrices

class diofant.matrices.sparse.**MutableSparseMatrix**(**args*)

A sparse matrix (a matrix with a large number of zero elements).

See also:

diofant.matrices.dense.DenseMatrix (page 611)

Examples

```
>>> SparseMatrix(2, 2, range(4))
Matrix([
[0, 1],
[2, 3]])
>>> SparseMatrix(2, 2, {(1, 1): 2})
Matrix([
[0, 0],
[0, 2]])
```

as_mutable()

Returns a mutable version of this matrix.

Examples

```
>>> X = ImmutableMatrix([[1, 2], [3, 4]])
>>> Y = X.as_mutable()
>>> Y[1, 1] = 5 # Can set values in Y
>>> Y
Matrix([
[1, 2],
[3, 5]])
```

col_del(*k*)

Delete the given column of the matrix.

See also:

row_del (page 623)

Examples

```
>>> M = SparseMatrix([[0, 0], [0, 1]])
>>> M
Matrix([
[0, 0],
[0, 1]])
>>> M.col_del(0)
>>> M
Matrix([
[0],
[1]])
```

col_join(*other*)

Returns B augmented beneath A (row-wise joining):

```
[A]
[B]
```

Examples

```
>>> A = SparseMatrix(ones(3))
>>> A
Matrix([
[1, 1, 1],
[1, 1, 1],
[1, 1, 1]])
>>> B = SparseMatrix.eye(3)
>>> B
Matrix([
[1, 0, 0],
[0, 1, 0],
[0, 0, 1]])
>>> C = A.col_join(B); C
Matrix([
[1, 1, 1],
[1, 1, 1],
[1, 1, 1],
[1, 0, 0],
[0, 1, 0],
[0, 0, 1]])
>>> C == A.col_join(Matrix(B))
True
```

Joining along columns is the same as appending rows at the end of the matrix:

```
>>> C == A.row_insert(A.rows, Matrix(B))
True
```

col_op(*j*, *f*)

In-place operation on col j using two-arg functor whose args are interpreted as (self[i, j], i) for i in range(self.rows).

Examples

```
>>> M = SparseMatrix.eye(3)*2
>>> M[1, 0] = -1
>>> M.col_op(1, lambda v, i: v + 2*M[i, 0]); M
Matrix([
[ 2, 4, 0],
[-1, 0, 0],
[ 0, 0, 2]])
```

col_swap(*i*, *j*)
Swap, in place, columns i and j.

Examples

```
>>> S = SparseMatrix.eye(3); S[2, 1] = 2
>>> S.col_swap(1, 0); S
Matrix([
[0, 1, 0],
[1, 0, 0],
[2, 0, 1]])
```

fill(*value*)
Fill self with the given value.

Notes

Unless many values are going to be deleted (i.e. set to zero) this will create a matrix that is slower than a dense matrix in operations.

Examples

```
>>> M = SparseMatrix.zeros(3); M
Matrix([
[0, 0, 0],
[0, 0, 0],
[0, 0, 0]])
>>> M.fill(1); M
Matrix([
[1, 1, 1],
[1, 1, 1],
[1, 1, 1]])
```

row_del(*k*)
Delete the given row of the matrix.

See also:

col_del (page 621)

Examples

```
>>> M = SparseMatrix([[0, 0], [0, 1]])
>>> M
Matrix([
[0, 0],
[0, 1]])
>>> M.row_del(0)
>>> M
Matrix([[0, 1]])
```

row_join(*other*)

Returns B appended after A (column-wise augmenting):

```
[A B]
```

Examples

```
>>> A = SparseMatrix(((1, 0, 1), (0, 1, 0), (1, 1, 0)))
>>> A
Matrix([
[1, 0, 1],
[0, 1, 0],
[1, 1, 0]])
>>> B = SparseMatrix(((1, 0, 0), (0, 1, 0), (0, 0, 1)))
>>> B
Matrix([
[1, 0, 0],
[0, 1, 0],
[0, 0, 1]])
>>> C = A.row_join(B); C
Matrix([
[1, 0, 1, 1, 0, 0],
[0, 1, 0, 0, 1, 0],
[1, 1, 0, 0, 0, 1]])
>>> C == A.row_join(Matrix(B))
True
```

Joining at row ends is the same as appending columns at the end of the matrix:

```
>>> C == A.col_insert(A.cols, B)
True
```

row_op(*i*, *f*)

In-place operation on row i using two-arg functor whose args are interpreted as `(self[i, j], j)`.

See also:

diofant.matrices.sparse.SparseMatrixBase.row (page 630), *zip_row_op* (page 625), *col_op* (page 622)

Examples

```
>>> M = SparseMatrix.eye(3)*2
>>> M[0, 1] = -1
>>> M.row_op(1, lambda v, j: v + 2*M[0, j]); M
Matrix([
[2, -1, 0],
[4,  0, 0],
[0,  0, 2]])
```

row_swap(*i, j*)

Swap, in place, columns i and j.

Examples

```
>>> S = SparseMatrix.eye(3); S[2, 1] = 2
>>> S.row_swap(1, 0); S
Matrix([
[0, 1, 0],
[1, 0, 0],
[0, 2, 1]])
```

zip_row_op(*i, k, f*)

In-place operation on row `i` using two-arg functor whose args are interpreted as `(self[i, j], self[k, j])`.

See also:

diofant.matrices.sparse.SparseMatrixBase.row (page 630), *row_op* (page 624), *col_op* (page 622)

Examples

```
>>> M = SparseMatrix.eye(3)*2
>>> M[0, 1] = -1
>>> M.zip_row_op(1, 0, lambda v, u: v + 2*u); M
Matrix([
[2, -1, 0],
[4,  0, 0],
[0,  0, 2]])
```

diofant.matrices.sparse.**SparseMatrix**

alias of *diofant.matrices.sparse.MutableSparseMatrix* (page 621)

class diofant.matrices.sparse.**SparseMatrixBase**(**args*)

A sparse matrix base class.

CL

Alternate faster representation

LDLdecomposition()

Returns the LDL Decomposition (matrices `L` and `D`) of matrix `A`, such that `L * D * L.T == A`. A must be a square, symmetric, positive-definite and non-singular.

This method eliminates the use of square root and ensures that all the diagonal entries of L are 1.

Examples

```
>>> A = SparseMatrix(((25, 15, -5), (15, 18, 0), (-5, 0, 11)))
>>> L, D = A.LDLdecomposition()
>>> L
Matrix([
[   1,   0, 0],
[ 3/5,   1, 0],
[-1/5, 1/3, 1]])
>>> D
Matrix([
[25, 0, 0],
[ 0, 9, 0],
[ 0, 0, 9]])
>>> L * D * L.T == A
True
```

RL
Alternate faster representation

add(*other*)
Add two sparse matrices with dictionary representation.

See also:

multiply (page 630)

Examples

```
>>> SparseMatrix(eye(3)).add(SparseMatrix(ones(3)))
Matrix([
[2, 1, 1],
[1, 2, 1],
[1, 1, 2]])
>>> SparseMatrix(eye(3)).add(-SparseMatrix(eye(3)))
Matrix([
[0, 0, 0],
[0, 0, 0],
[0, 0, 0]])
```

Only the non-zero elements are stored, so the resulting dictionary that is used to represent the sparse matrix is empty:

```
>>> _._smat
{}
```

applyfunc(*f*)
Apply a function to each element of the matrix.

Examples

```
>>> m = SparseMatrix(2, 2, lambda i, j: i*2+j)
>>> m
Matrix([
```

(continues on next page)

(continued from previous page)

```
[0, 1],
[2, 3]])
>>> m.applyfunc(lambda i: 2*i)
Matrix([
[0, 2],
[4, 6]])
```

as_immutable()
Returns an Immutable version of this Matrix.

as_mutable()
Returns a mutable version of this matrix.

Examples

```
>>> X = ImmutableMatrix([[1, 2], [3, 4]])
>>> Y = X.as_mutable()
>>> Y[1, 1] = 5 # Can set values in Y
>>> Y
Matrix([
[1, 2],
[3, 5]])
```

cholesky()
Returns the Cholesky decomposition L of a matrix A such that L * L.T = A

A must be a square, symmetric, positive-definite and non-singular matrix

Examples

```
>>> A = SparseMatrix(((25, 15, -5), (15, 18, 0), (-5, 0, 11)))
>>> A.cholesky()
Matrix([
[ 5, 0, 0],
[ 3, 3, 0],
[-1, 1, 3]])
>>> A.cholesky() * A.cholesky().T == A
True
```

col(*j*)
Returns column j from self as a column vector.

See also:

row (page 630), *col_list* (page 627)

Examples

```
>>> a = SparseMatrix(((1, 2), (3, 4)))
>>> a.col(0)
Matrix([
[1],
[3]])
```

col_list()

Returns a column-sorted list of non-zero elements of the matrix.

See also:

diofant.matrices.sparse.MutableSparseMatrix.col_op (page 622), *row_list* (page 631)

Examples

```
>>> SparseMatrix(((1, 2), (3, 4)))
Matrix([
[1, 2],
[3, 4]])
>>> _.CL
[(0, 0, 1), (1, 0, 3), (0, 1, 2), (1, 1, 4)]
```

copy()

Returns the copy of a matrix.

extract(*rowsList, colsList*)

Return a submatrix by specifying a list of rows and columns. Negative indices can be given. All indices must be in the range -n <= i < n where n is the number of rows or columns.

Examples

```
>>> m = Matrix(4, 3, range(12))
>>> m
Matrix([
[0,  1,  2],
[3,  4,  5],
[6,  7,  8],
[9, 10, 11]])
>>> m.extract([0, 1, 3], [0, 1])
Matrix([
[0,  1],
[3,  4],
[9, 10]])
```

Rows or columns can be repeated:

```
>>> m.extract([0, 0, 1], [-1])
Matrix([
[2],
[2],
[5]])
```

Every other row can be taken by using range to provide the indices:

```
>>> m.extract(range(0, m.rows, 2), [-1])
Matrix([
[2],
[8]])
```

classmethod eye(*n*)
Return an n x n identity matrix.

has(**patterns*)
Test whether any subexpression matches any of the patterns.

Examples

```
>>> A = SparseMatrix(((1, x), (0.2, 3)))
>>> A.has(x)
True
>>> A.has(y)
False
>>> A.has(Float)
True
```

is_hermitian
Checks if the matrix is Hermitian.

In a Hermitian matrix element i,j is the complex conjugate of element j,i.

Examples

```
>>> a = SparseMatrix([[1, I], [-I, 1]])
>>> a
Matrix([
[ 1, I],
[-I, 1]])
>>> a.is_hermitian
True
>>> a[0, 0] = 2*I
>>> a.is_hermitian
False
>>> a[0, 0] = x
>>> a.is_hermitian
>>> a[0, 1] = a[1, 0]*I
>>> a.is_hermitian
False
```

is_symmetric(*simplify=True*)
Return True if self is symmetric.

Examples

```
>>> M = SparseMatrix(eye(3))
>>> M.is_symmetric()
True
>>> M[0, 2] = 1
>>> M.is_symmetric()
False
```

liupc()
Liu's algorithm, for pre-determination of the Elimination Tree of the given matrix, used in row-based symbolic Cholesky factorization.

References

Symbolic Sparse Cholesky Factorization using Elimination Trees, Jeroen Van Grondelle (1999)

Examples

```
>>> S = SparseMatrix([
... [1, 0, 3, 2],
... [0, 0, 1, 0],
... [4, 0, 0, 5],
... [0, 6, 7, 0]])
>>> S.liupc()
([[0], [], [0], [1, 2]], [4, 3, 4, 4])
```

multiply(*other*)

Fast multiplication exploiting the sparsity of the matrix.

See also:

add (page 626)

Examples

```
>>> A, B = SparseMatrix(ones(4, 3)), SparseMatrix(ones(3, 4))
>>> A.multiply(B) == 3*ones(4)
True
```

nnz()

Returns the number of non-zero elements in Matrix.

reshape(*rows, cols*)

Reshape matrix while retaining original size.

Examples

```
>>> S = SparseMatrix(4, 2, range(8))
>>> S.reshape(2, 4)
Matrix([
[0, 1, 2, 3],
[4, 5, 6, 7]])
```

row(*i*)

Returns column i from self as a row vector.

See also:

col (page 627), *row_list* (page 631)

Examples

```
>>> a = SparseMatrix(((1, 2), (3, 4)))
>>> a.row(0)
Matrix([[1, 2]])
```

row_list()

Returns a row-sorted list of non-zero elements of the matrix.

See also:

diofant.matrices.sparse.MutableSparseMatrix.row_op (page 624), *col_list* (page 627)

Examples

```
>>> a = SparseMatrix(((1, 2), (3, 4)))
>>> a
Matrix([
[1, 2],
[3, 4]])
>>> a.RL
[(0, 0, 1), (0, 1, 2), (1, 0, 3), (1, 1, 4)]
```

row_structure_symbolic_cholesky()

Symbolic cholesky factorization, for pre-determination of the non-zero structure of the Cholesky factororization.

References

Symbolic Sparse Cholesky Factorization using Elimination Trees, Jeroen Van Grondelle (1999)

Examples

```
>>> S = SparseMatrix([
... [1, 0, 3, 2],
... [0, 0, 1, 0],
... [4, 0, 0, 5],
... [0, 6, 7, 0]])
>>> S.row_structure_symbolic_cholesky()
[[0], [], [0], [1, 2]]
```

scalar_multiply(*scalar*)

Scalar element-wise multiplication

solve(*rhs*, *method='LDL'*)

Return solution to self*soln = rhs using given inversion method.

See also:

diofant.matrices.matrices.MatrixBase.inv (page 582)

solve_least_squares(*rhs*, *method='LDL'*)

Return the least-square fit to the data.

By default the cholesky_solve routine is used (method='CH'); other methods of matrix inversion can be used.

See also:

diofant.matrices.matrices.MatrixBase.inv (page 582)

Examples

```
>>> A = Matrix([1, 2, 3])
>>> B = Matrix([2, 3, 4])
>>> S = SparseMatrix(A.row_join(B))
>>> S
Matrix([
[1, 2],
[2, 3],
[3, 4]])
```

If each line of S represent coefficients of Ax + By and x and y are [2, 3] then S*xy is:

```
>>> r = S*Matrix([2, 3]); r
Matrix([
[ 8],
[13],
[18]])
```

But let's add 1 to the middle value and then solve for the least-squares value of xy:

```
>>> xy = S.solve_least_squares(Matrix([8, 14, 18])); xy
Matrix([
[ 5/3],
[10/3]])
```

The error is given by S*xy - r:

```
>>> S*xy - r
Matrix([
[1/3],
[1/3],
[1/3]])
>>> _.norm().n(2)
0.58
```

If a different xy is used, the norm will be higher:

```
>>> xy += ones(2, 1)/10
>>> (S*xy - r).norm().n(2)
1.5
```

tolist()

Convert this sparse matrix into a list of nested Python lists.

Examples

```
>>> a = SparseMatrix(((1, 2), (3, 4)))
>>> a.tolist()
[[1, 2], [3, 4]]
```

When there are no rows then it will not be possible to tell how many columns were in the original matrix:

```
>>> SparseMatrix(ones(0, 3)).tolist()
[]
```

classmethod zeros(*r*, *c=None*)
Return an r x c matrix of zeros, square if c is omitted.

ImmutableSparseMatrix Class Reference

class diofant.matrices.immutable.**ImmutableSparseMatrix**(**args*)
Create an immutable version of a sparse matrix.

Examples

```
>>> ImmutableSparseMatrix(1, 1, {})
Matrix([[0]])
>>> ImmutableSparseMatrix(eye(3))
Matrix([
[1, 0, 0],
[0, 1, 0],
[0, 0, 1]])
>>> _[0, 0] = 42
Traceback (most recent call last):
...
TypeError: Cannot set values of ImmutableSparseMatrix
>>> _.shape
(3, 3)
```

is_zero
Checks if a matrix is a zero matrix.

A matrix is zero if every element is zero. A matrix need not be square to be considered zero. The empty matrix is zero by the principle of vacuous truth. For a matrix that may or may not be zero (e.g. contains a symbol), this will be None

Examples

```
>>> a = Matrix([[0, 0], [0, 0]])
>>> b = zeros(3, 4)
>>> c = Matrix([[0, 1], [0, 0]])
>>> d = Matrix([])
>>> e = Matrix([[x, 0], [0, 0]])
>>> a.is_zero
True
```

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```
>>> b.is_zero
True
>>> c.is_zero
False
>>> d.is_zero
True
>>> e.is_zero
```

subs(**args, **kwargs*)
Return a new matrix with subs applied to each entry.

Examples

```
>>> SparseMatrix(1, 1, [x])
Matrix([[x]])
>>> _.subs(x, y)
Matrix([[y]])
>>> Matrix(_).subs(y, x)
Matrix([[x]])
```

xreplace(*rule*)
Return a new matrix with xreplace applied to each entry.

Examples

```
>>> SparseMatrix(1, 1, [x])
Matrix([[x]])
>>> _.xreplace({x: y})
Matrix([[y]])
>>> Matrix(_).xreplace({y: x})
Matrix([[x]])
```

3.10.4 Immutable Matrices

The standard Matrix class in Diofant is mutable. This is important for performance reasons but means that standard matrices can not interact well with the rest of Diofant. This is because the *Basic* (page 42) object, from which most Diofant classes inherit, is immutable.

The mission of the *ImmutableMatrix* (page 635) class is to bridge the tension between performance/mutability and safety/immutability. Immutable matrices can do almost everything that normal matrices can do but they inherit from *Basic* (page 42) and can thus interact more naturally with the rest of Diofant. *ImmutableMatrix* (page 635) also inherits from *MatrixExpr* (page 637), allowing it to interact freely with Diofant's Matrix Expression module.

You can turn any Matrix-like object into an *ImmutableMatrix* (page 635) by calling the constructor

```
>>> M = Matrix([[1, 2, 3], [4, 5, 6], [7, 8, 9]])
>>> M[1, 1] = 0
>>> IM = ImmutableMatrix(M)
>>> IM
```

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```
Matrix([
[1, 2, 3],
[4, 0, 6],
[7, 8, 9]])
>>> IM[1, 1] = 5
Traceback (most recent call last):
...
TypeError: Can not set values in Immutable Matrix. Use Matrix instead.
```

ImmutableMatrix Class Reference

class diofant.matrices.immutable.**ImmutableMatrix**

Create an immutable version of a matrix.

Examples

```
>>> ImmutableMatrix(eye(3))
Matrix([
[1, 0, 0],
[0, 1, 0],
[0, 0, 1]])
>>> _[0, 0] = 42
Traceback (most recent call last):
...
TypeError: Cannot set values of ImmutableDenseMatrix
```

C

By-element conjugation.

adjoint()

Conjugate transpose or Hermitian conjugation.

as_mutable()

Returns a mutable version of this matrix

Examples

```
>>> X = ImmutableMatrix([[1, 2], [3, 4]])
>>> Y = X.as_mutable()
>>> Y[1, 1] = 5 # Can set values in Y
>>> Y
Matrix([
[1, 2],
[3, 5]])
```

conjugate()

By-element conjugation.

equals(*other*, *failing_expression=False*)

Applies `equals` to corresponding elements of the matrices, trying to prove that the elements are equivalent, returning True if they are, False if any pair is not, and None

(or the first failing expression if failing_expression is True) if it cannot be decided if the expressions are equivalent or not. This is, in general, an expensive operation.

See also:

diofant.core.expr.Expr.equals (page 65)

Examples

```
>>> A = Matrix([x*(x - 1), 0])
>>> B = Matrix([x**2 - x, 0])
>>> A == B
False
>>> A.simplify() == B.simplify()
True
>>> A.equals(B)
True
>>> A.equals(2)
False
```

is_zero

Checks if a matrix is a zero matrix.

A matrix is zero if every element is zero. A matrix need not be square to be considered zero. The empty matrix is zero by the principle of vacuous truth. For a matrix that may or may not be zero (e.g. contains a symbol), this will be None

Examples

```
>>> a = Matrix([[0, 0], [0, 0]])
>>> b = zeros(3, 4)
>>> c = Matrix([[0, 1], [0, 0]])
>>> d = Matrix([])
>>> e = Matrix([[x, 0], [0, 0]])
>>> a.is_zero
True
>>> b.is_zero
True
>>> c.is_zero
False
>>> d.is_zero
True
>>> e.is_zero
```

shape

The shape (dimensions) of the matrix as the 2-tuple (rows, cols).

Examples

```
>>> M = zeros(2, 3)
>>> M.shape
(2, 3)
>>> M.rows
```

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```
2
>>> M.cols
3
```

3.10.5 Matrix Expressions

The Matrix expression module allows users to write down statements like

```
>>> X = MatrixSymbol('X', 3, 3)
>>> Y = MatrixSymbol('Y', 3, 3)
>>> (X.T*X).inverse()*Y
X^-1*X.T^-1*Y
```

```
>>> Matrix(X)
Matrix([
[X[0, 0], X[0, 1], X[0, 2]],
[X[1, 0], X[1, 1], X[1, 2]],
[X[2, 0], X[2, 1], X[2, 2]]])
```

```
>>> (X*Y)[1, 2]
X[1, 0]*Y[0, 2] + X[1, 1]*Y[1, 2] + X[1, 2]*Y[2, 2]
```

where X and Y are *MatrixSymbol* (page 638)'s rather than scalar symbols.

Matrix Expressions Core Reference

class diofant.matrices.expressions.**MatrixExpr**

Superclass for Matrix Expressions

MatrixExprs represent abstract matrices, linear transformations represented within a particular basis.

Examples

```
>>> A = MatrixSymbol('A', 3, 3)
>>> y = MatrixSymbol('y', 3, 1)
>>> x = (A.T*A).inverse() * A * y
```

T

Matrix transposition.

adjoint()

Compute conjugate transpose or Hermite conjugation.

See also:

diofant.functions.elementary.complexes.adjoint (page 294)

as_explicit()

Returns a dense Matrix with elements represented explicitly

Returns an object of type ImmutableMatrix.

See also:

as_mutable **(page 638)** returns mutable Matrix type

Examples

```
>>> I = Identity(3)
>>> I
I
>>> I.as_explicit()
Matrix([
[1, 0, 0],
[0, 1, 0],
[0, 0, 1]])
```

as_mutable()

Returns a dense, mutable matrix with elements represented explicitly

See also:

as_explicit **(page 637)** returns ImmutableMatrix

Examples

```
>>> I = Identity(3)
>>> I
I
>>> I.shape
(3, 3)
>>> I.as_mutable()
Matrix([
[1, 0, 0],
[0, 1, 0],
[0, 0, 1]])
```

conjugate()

Returns the complex conjugate of self.

See also:

diofant.functions.elementary.complexes.conjugate (page 295)

equals(*other*)

Test elementwise equality between matrices, potentially of different types

```
>>> Identity(3).equals(eye(3))
True
```

transpose()

Transpose self.

See also:

diofant.functions.elementary.complexes.transpose (page 297)

class diofant.matrices.expressions.**MatrixSymbol**

Symbolic representation of a Matrix object

Creates a Diofant Symbol to represent a Matrix. This matrix has a shape and can be included in Matrix Expressions

```
>>> A = MatrixSymbol('A', 3, 4) # A 3 by 4 Matrix
>>> B = MatrixSymbol('B', 4, 3) # A 4 by 3 Matrix
>>> A.shape
(3, 4)
>>> 2*A*B + Identity(3)
I + 2*A*B
```

doit(***hints*)

Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

free_symbols

Return from the atoms of self those which are free symbols.

For most expressions, all symbols are free symbols. For some classes this is not true. e.g. Integrals use Symbols for the dummy variables which are bound variables, so Integral has a method to return all symbols except those. Derivative keeps track of symbols with respect to which it will perform a derivative; those are bound variables, too, so it has its own free_symbols method.

Any other method that uses bound variables should implement a free_symbols method.

class diofant.matrices.expressions.**MatAdd**

A Sum of Matrix Expressions

MatAdd inherits from and operates like Diofant Add

```
>>> A = MatrixSymbol('A', 5, 5)
>>> B = MatrixSymbol('B', 5, 5)
>>> C = MatrixSymbol('C', 5, 5)
>>> MatAdd(A, B, C)
A + B + C
```

doit(***kwargs*)

Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

class diofant.matrices.expressions.**MatMul**
A product of matrix expressions

Examples

```
>>> A = MatrixSymbol('A', 5, 4)
>>> B = MatrixSymbol('B', 4, 3)
>>> C = MatrixSymbol('C', 3, 6)
>>> MatMul(A, B, C)
A*B*C
```

doit(***kwargs*)
Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

class diofant.matrices.expressions.**MatPow**

doit(***kwargs*)
Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

class `diofant.matrices.expressions.`**`Inverse`**

The multiplicative inverse of a matrix expression

This is a symbolic object that simply stores its argument without evaluating it. To actually compute the inverse, use the `.inverse()` method of matrices.

Examples

```
>>> A = MatrixSymbol('A', 3, 3)
>>> B = MatrixSymbol('B', 3, 3)
>>> Inverse(A)
A^-1
>>> A.inverse() == Inverse(A)
True
>>> (A*B).inverse()
B^-1*A^-1
>>> Inverse(A*B)
(A*B)^-1
```

doit(***hints*)

Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

class `diofant.matrices.expressions.`**`Transpose`**

The transpose of a matrix expression.

This is a symbolic object that simply stores its argument without evaluating it. To actually compute the transpose, use the `transpose()` function, or the `.T` attribute of matrices.

Examples

```
>>> A = MatrixSymbol('A', 3, 5)
>>> B = MatrixSymbol('B', 5, 3)
>>> Transpose(A)
A.T
>>> A.T == transpose(A) == Transpose(A)
True
>>> Transpose(A*B)
(A*B).T
>>> transpose(A*B)
B.T*A.T
```

doit(***hints*)

Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

class diofant.matrices.expressions.**Trace**

Matrix Trace

Represents the trace of a matrix expression.

```
>>> A = MatrixSymbol('A', 3, 3)
>>> Trace(A)
Trace(A)
```

See also:

trace

doit(***kwargs*)

Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

class diofant.matrices.expressions.**FunctionMatrix**

Represents a Matrix using a function (Lambda)

This class is an alternative to SparseMatrix

```
>>> i, j = symbols('i j')
>>> X = FunctionMatrix(3, 3, Lambda((i, j), i + j))
>>> Matrix(X)
Matrix([
[0, 1, 2],
[1, 2, 3],
[2, 3, 4]])
```

```
>>> Y = FunctionMatrix(1000, 1000, Lambda((i, j), i + j))
```

```
>>> isinstance(Y*Y, MatMul) # this is an expression object
True
```

```
>>> (Y**2)[10, 10] # So this is evaluated lazily
342923500
```

class diofant.matrices.expressions.**Identity**

The Matrix Identity I - multiplicative identity

```
>>> A = MatrixSymbol('A', 3, 5)
>>> I = Identity(3)
>>> I*A
A
```

conjugate()

Returns the complex conjugate of self.

See also:

diofant.functions.elementary.complexes.conjugate (page 295)

class diofant.matrices.expressions.**ZeroMatrix**

The Matrix Zero 0 - additive identity

```
>>> A = MatrixSymbol('A', 3, 5)
>>> Z = ZeroMatrix(3, 5)
>>> A+Z
A
>>> Z*A.T
0
```

conjugate()

Returns the complex conjugate of self.

See also:

diofant.functions.elementary.complexes.conjugate (page 295)

Block Matrices

Block matrices allow you to construct larger matrices out of smaller sub-blocks. They can work with *MatrixExpr* (page 637) or *ImmutableMatrix* (page 635) objects.

class diofant.matrices.expressions.blockmatrix.**BlockMatrix**

A BlockMatrix is a Matrix composed of other smaller, submatrices

The submatrices are stored in a Diofant Matrix object but accessed as part of a Matrix Expression

```
>>> n, m, l = symbols('n m l')
>>> X = MatrixSymbol('X', n, n)
>>> Y = MatrixSymbol('Y', m, m)
>>> Z = MatrixSymbol('Z', n, m)
>>> B = BlockMatrix([[X, Z], [ZeroMatrix(m, n), Y]])
>>> B
Matrix([
[X, Z],
[0, Y]])
```

```
>>> C = BlockMatrix([[Identity(n), Z]])
>>> C
Matrix([[I, Z]])
```

```
>>> block_collapse(C*B)
Matrix([[X, Z + Z*Y]])
```

equals(*other*)

Test elementwise equality between matrices, potentially of different types

```
>>> Identity(3).equals(eye(3))
True
```

transpose()

Return transpose of matrix.

Examples

```
>>> from diofant.abc import l
>>> X = MatrixSymbol('X', n, n)
>>> Y = MatrixSymbol('Y', m, m)
>>> Z = MatrixSymbol('Z', n, m)
>>> B = BlockMatrix([[X, Z], [ZeroMatrix(m, n), Y]])
>>> B.transpose()
Matrix([
[X.T,  0],
[Z.T, Y.T]])
>>> _.transpose()
Matrix([
```

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```
[X, Z],
[0, Y]])
```

class diofant.matrices.expressions.blockmatrix.**BlockDiagMatrix**

A BlockDiagMatrix is a BlockMatrix with matrices only along the diagonal

```
>>> n, m, l = symbols('n m l')
>>> X = MatrixSymbol('X', n, n)
>>> Y = MatrixSymbol('Y', m, m)
>>> BlockDiagMatrix(X, Y)
Matrix([
[X, 0],
[0, Y]])
```

diofant.matrices.expressions.blockmatrix.**block_collapse**(*expr*)

Evaluates a block matrix expression

```
>>> n, m, l = symbols('n m l')
>>> X = MatrixSymbol('X', n, n)
>>> Y = MatrixSymbol('Y', m, m)
>>> Z = MatrixSymbol('Z', n, m)
>>> B = BlockMatrix([[X, Z], [ZeroMatrix(m, n), Y]])
>>> B
Matrix([
[X, Z],
[0, Y]])
```

```
>>> C = BlockMatrix([[Identity(n), Z]])
>>> C
Matrix([[I, Z]])
```

```
>>> block_collapse(C*B)
Matrix([[X, Z + Z*Y]])
```

3.11 Polynomials

Computations with polynomials are at the core of computer algebra and having a fast and robust polynomials manipulation module is a key for building a powerful symbolic manipulation system. Diofant has a dedicated module *diofant.polys* (page 645) for computing in polynomial algebras over various coefficient domains.

There is a vast number of methods implemented, ranging from simple tools like polynomial division, to advanced concepts including Gröbner bases and multivariate factorization over algebraic number domains.

3.11.1 Basic functionality of the module

Polynomial manipulation algorithms and algebraic objects.

Introduction

This tutorial tries to give an overview of the functionality concerning polynomials within Diofant. All code examples assume:

```
>>> init_printing(pretty_print=True, use_unicode=True, wrap_line=False, no_
↪global=True)
```

Basic functionality

These functions provide different algorithms dealing with polynomials in the form of Diofant expression, like symbols, sums etc.

Division

The function *div()* (page 659) provides division of polynomials with remainder. That is, for polynomials f and g, it computes q and r, such that $f = g \cdot q + r$ and $\deg(r) < q$. For polynomials in one variables with coefficients in a field, say, the rational numbers, q and r are uniquely defined this way:

```
>>> f = 5*x**2 + 10*x + 3
>>> g = 2*x + 2

>>> q, r = div(f, g, domain='QQ')
>>> q
5·x   5
─── + ─
 2    2
>>> r
-2
>>> (q*g + r).expand()
   2
5·x  + 10·x + 3
```

As you can see, q has a non-integer coefficient. If you want to do division only in the ring of polynomials with integer coefficients, you can specify an additional parameter:

```
>>> q, r = div(f, g, domain='ZZ')
>>> q
0
>>> r
   2
5·x  + 10·x + 3
```

But be warned, that this ring is no longer Euclidean and that the degree of the remainder doesn't need to be smaller than that of f. Since 2 doesn't divide 5, $2x$ doesn't divide $5x^2$, even if the degree is smaller. But:

```
>>> g = 5*x + 1

>>> q, r = div(f, g, domain='ZZ')
>>> q
x
>>> r
```

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```
9⋅x + 3
>>> (q*g + r).expand()
   2
5⋅x  + 10⋅x + 3
```

This also works for polynomials with multiple variables:

```
>>> f = x*y + y*z
>>> g = 3*x + 3*z

>>> q, r = div(f, g, domain='QQ')
>>> q
y
─
3
>>> r
0
```

In the last examples, all of the three variables x, y and z are assumed to be variables of the polynomials. But if you have some unrelated constant as coefficient, you can specify the variables explicitly:

```
>>> a, b, c = symbols('a b c')
>>> f = a*x**2 + b*x + c
>>> g = 3*x + 2
>>> q, r = div(f, g, domain='QQ')
>>> q
a⋅x   2⋅a   b
─── - ─── + ─
 3     9    3

>>> r
4⋅a   2⋅b
─── - ─── + c
 9     3
```

GCD and LCM

With division, there is also the computation of the greatest common divisor and the least common multiple.

When the polynomials have integer coefficients, the contents' gcd is also considered:

```
>>> f = (12*x + 12)*x
>>> g = 16*x**2
>>> gcd(f, g)
4⋅x
```

But if the polynomials have rational coefficients, then the returned polynomial is monic:

```
>>> f = 3*x**2/2
>>> g = 9*x/4
>>> gcd(f, g)
x
```

It also works with multiple variables. In this case, the variables are ordered alphabetically, be default, which has influence on the leading coefficient:

```
>>> f = x*y/2 + y**2
>>> g = 3*x + 6*y

>>> gcd(f, g)
x + 2⋅y
```

The lcm is connected with the gcd and one can be computed using the other:

```
>>> f = x*y**2 + x**2*y
>>> g = x**2*y**2
>>> gcd(f, g)
x⋅y
>>> lcm(f, g)
 3  2    2  3
x ⋅y  + x ⋅y
>>> (f*g).expand()
 4  3    3  4
x ⋅y  + x ⋅y
>>> (gcd(f, g, x, y)*lcm(f, g, x, y)).expand()
 4  3    3  4
x ⋅y  + x ⋅y
```

Square-free factorization

The square-free factorization of a univariate polynomial is the product of all factors (not necessarily irreducible) of degree 1, 2 etc.:

```
>>> f = 2*x**2 + 5*x**3 + 4*x**4 + x**5

>>> sqf_list(f)
(1, [(x + 2, 1), (x, 2), (x + 1, 2)])

>>> sqf(f)
 2        2
x ⋅(x + 1) ⋅(x + 2)
```

Factorization

This function provides factorization of univariate and multivariate polynomials with rational coefficients:

```
>>> factor(x**4/2 + 5*x**3/12 - x**2/3)
 2
x ⋅(2⋅x - 1)⋅(3⋅x + 4)
──────────────────────
          12

>>> factor(x**2 + 4*x*y + 4*y**2)
         2
(x + 2⋅y)
```

Gröbner bases

Buchberger's algorithm is implemented, supporting various monomial orders:

```
>>> groebner([x**2 + 1, y**4*x + x**3], x, y, order='lex')
             ⎛⎡ 2       4    ⎤                              ⎞
GroebnerBasis⎝⎣x  + 1, y  - 1⎦, x, y, domain=ℤ, order=lex⎠


>>> groebner([x**2 + 1, y**4*x + x**3, x*y*z**3], x, y, z, order='grevlex')
             ⎛⎡ 4       3   2    ⎤                                    ⎞
GroebnerBasis⎝⎣y  - 1, z , x  + 1⎦, x, y, z, domain=ℤ, order=grevlex⎠
```

Solving Equations

We have (incomplete) methods to find the complex or even symbolic roots of polynomials and to solve some systems of polynomial equations:

```
>>> solve(x**3 + 2*x + 3, x)
⎡          ⎧               ____  ⎫  ⎧               ____  ⎫⎤
⎢          ⎪     1       ╲╱ 11 ⋅i⎪  ⎪     1       ╲╱ 11 ⋅i⎪⎥
⎢{x: -1}, ⎨x: ─ - ────────────⎬, ⎨x: ─ + ────────────⎬⎥
⎢          ⎪     2         2     ⎪  ⎪     2         2     ⎪⎥
⎣          ⎩                     ⎭  ⎩                     ⎭⎦

>>> p = Symbol('p')
>>> q = Symbol('q')

>>> solve(x**2 + p*x + q, x)
⎡⎧                __________⎫  ⎧                __________⎫⎤
⎢⎪               ╱  2       ⎪  ⎪               ╱  2       ⎪⎥
⎢⎨      p      ╲╱  p  - 4⋅q ⎬  ⎨      p      ╲╱  p  - 4⋅q ⎬⎥
⎢⎪x: - ─ - ─────────────⎪, ⎪x: - ─ + ─────────────⎪⎥
⎣⎩      2           2        ⎭  ⎩      2           2        ⎭⎦

>>> solve_poly_system([y - x, x - 5], x, y)
[{x: 5, y: 5}]

>>> solve_poly_system([y**2 - x**3 + 1, y*x], x, y)
⎡                                           ⎧          ___        ⎫  ⎧          ___␣
↪       ⎤⎤
⎢                                           ⎪     1  ╲╱ 3 ⋅i      ⎪  ⎪     1  ╲╱ 3␣
↪⋅i      ⎥⎥
⎢{x: 0, y: -i}, {x: 0, y: i}, {x: 1, y: 0}, ⎨x: - ─ - ───────, y: 0⎬, ⎨x: - ─ +␣
↪───────, y: 0⎬⎥
⎢                                           ⎪     2      2        ⎪  ⎪     2      2 ␣
↪       ⎥⎥
⎣                                           ⎩                     ⎭  ⎩             ␣
↪       ⎦⎦
```

3.11.2 Examples from Wester's Article

Introduction

In this tutorial we present examples from Wester's article concerning comparison and critique of mathematical abilities of several computer algebra systems (see *[Wester1999]* (page 1261)). All the examples are related to polynomial and algebraic computations and Diofant specific remarks were added to all of them.

Examples

All examples in this tutorial are computable, so one can just copy and paste them into a Python shell and do something useful with them. All computations were done using the following setup:

```
>>> init_printing(pretty_print=True, use_unicode=True, wrap_line=False, no_
↪global=True)

>>> var('x, y, z, s, c, n')
(x, y, z, s, c, n)
```

Simple univariate polynomial factorization

To obtain a factorization of a polynomial use `factor()` (page 668) function. By default `factor()` (page 668) returns the result in unevaluated form, so the content of the input polynomial is left unexpanded, as in the following example:

```
>>> factor(6*x - 10)
2⋅(3⋅x - 5)
```

To achieve the same effect in a more systematic way use `primitive()` (page 666) function, which returns the content and the primitive part of the input polynomial:

```
>>> primitive(6*x - 10)
(2, 3⋅x - 5)
```

Note: The content and the primitive part can be computed only over a ring. To simplify coefficients of a polynomial over a field use `monic()` (page 666).

Univariate GCD, resultant and factorization

Consider univariate polynomials f, g and h over integers:

```
>>> f = 64*x**34 - 21*x**47 - 126*x**8 - 46*x**5 - 16*x**60 - 81
>>> g = 72*x**60 - 25*x**25 - 19*x**23 - 22*x**39 - 83*x**52 + 54*x**10 + 81
>>> h = 34*x**19 - 25*x**16 + 70*x**7 + 20*x**3 - 91*x - 86
```

We can compute the greatest common divisor (GCD) of two polynomials using `gcd()` (page 665) function:

```
>>> gcd(f, g)
1
```

We see that f and g have no common factors. However, f*h and g*h have an obvious factor h:

```
>>> gcd(expand(f*h), expand(g*h)) - h
0
```

The same can be verified using the resultant of univariate polynomials:

```
>>> resultant(expand(f*h), expand(g*h))
0
```

Factorization of large univariate polynomials (of degree 120 in this case) over integers is also possible:

```
>>> factor(expand(f*g))
 ⎛    60       47       34        8       5     ⎞ ⎛    60       52      39       25   ␣
↪   23       10     ⎞
-⎝16⋅x   + 21⋅x   - 64⋅x   + 126⋅x  + 46⋅x  + 81⎠⋅⎝72⋅x   - 83⋅x - 22⋅x   - 25⋅x   -␣
↪19⋅x   + 54⋅x   + 81⎠
```

Multivariate GCD and factorization

What can be done in univariate case, can be also done for multivariate polynomials. Consider the following polynomials f, g and h in $\mathbb{Z}[x,y,z]$:

```
>>> f = 24*x*y**19*z**8 - 47*x**17*y**5*z**8 + 6*x**15*y**9*z**2 - 3*x**22 + 5
>>> g = 34*x**5*y**8*z**13 + 20*x**7*y**7*z**7 + 12*x**9*y**16*z**4 + 80*y**14*z
>>> h = 11*x**12*y**7*z**13 - 23*x**2*y**8*z**10 + 47*x**17*y**5*z**8
```

As previously, we can verify that f and g have no common factors:

```
>>> gcd(f, g)
1
```

However, f*h and g*h have an obvious factor h:

```
>>> gcd(expand(f*h), expand(g*h)) - h
0
```

Multivariate factorization of large polynomials is also possible:

```
>>> factor(expand(f*g))
    7   ⎛   9  9  3       7  6       5    12       7⎞ ⎛   22       17  5  8      15 ␣
↪9  2         19  8    ⎞
-2⋅y ⋅z⋅⎝6⋅x ⋅y ⋅z  + 10⋅x ⋅z  + 17⋅x ⋅y⋅z   + 40⋅y ⎠⋅⎝3⋅x   + 47⋅x  ⋅y ⋅z  - 6⋅x  ⋅y␣
↪⋅z  - 24⋅x⋅y  ⋅z  - 5⎠
```

Support for symbols in exponents

Polynomial manipulation functions provided by *diofant.polys* (page 645) are mostly used with integer exponents. However, it's perfectly valid to compute with symbolic exponents, e.g.:

```
>>> gcd(2*x**(n + 4) - x**(n + 2), 4*x**(n + 1) + 3*x**n)
 n
x
```

Testing if polynomials have common zeros

To test if two polynomials have a root in common we can use *resultant()* (page 661) function. The theory says that the resultant of two polynomials vanishes if there is a common zero of those polynomials. For example:

```
>>> resultant(3*x**4 + 3*x**3 + x**2 - x - 2, x**3 - 3*x**2 + x + 5)
0
```

We can visualize this fact by factoring the polynomials:

```
>>> factor(3*x**4 + 3*x**3 + x**2 - x - 2)
        ⎛   3        ⎞
(x + 1)⋅⎝3⋅x  + x - 2⎠

>>> factor(x**3 - 3*x**2 + x + 5)
        ⎛ 2          ⎞
(x + 1)⋅⎝x  - 4⋅x + 5⎠
```

In both cases we obtained the factor $x + 1$ which tells us that the common root is $x = -1$.

Normalizing simple rational functions

To remove common factors from the numerator and the denominator of a rational function the elegant way, use *cancel()* (page 671) function. For example:

```
>>> cancel((x**2 - 4)/(x**2 + 4*x + 4))
x - 2
─────
x + 2
```

Expanding expressions and factoring back

One can work easily we expressions in both expanded and factored forms. Consider a polynomial `f` in expanded form. We differentiate it and factor the result back:

```
>>> f = expand((x + 1)**20)

>>> g = diff(f, x)

>>> factor(g)
          19
20⋅(x + 1)
```

The same can be achieved in factored form:

```
>>> diff((x + 1)**20, x)
          19
20⋅(x + 1)
```

Factoring in terms of cyclotomic polynomials

Diofant can very efficiently decompose polynomials of the form $x^n \pm 1$ in terms of cyclotomic polynomials:

```
>>> factor(x**15 - 1)
        ⎛ 2        ⎞ ⎛ 4    3    2        ⎞ ⎛ 8    7    5    4    3        ⎞
(x - 1)⋅⎝x  + x + 1⎠⋅⎝x  + x  + x  + x + 1⎠⋅⎝x  - x  + x  - x  + x  - x + 1⎠
```

The original Wester's example was $x^{100} - 1$, but was truncated for readability purpose. Note that this is not a big struggle for *factor()* (page 668) to decompose polynomials of degree 1000 or greater.

Univariate factoring over Gaussian numbers

Consider a univariate polynomial `f` with integer coefficients:

```
>>> f = 4*x**4 + 8*x**3 + 77*x**2 + 18*x + 153
```

We want to obtain a factorization of `f` over Gaussian numbers. To do this we use *factor()* (page 668) as previously, but this time we set `gaussian` keyword to `True`:

```
>>> factor(f, gaussian=True)
  ⎛    3⋅i⎞ ⎛    3⋅i⎞
4⋅⎜x - ───⎟⋅⎜x + ───⎟⋅(x + 1 - 4⋅i)⋅(x + 1 + 4⋅i)
  ⎝     2 ⎠ ⎝     2 ⎠
```

As the result we got a splitting factorization of `f` with monic factors (this is a general rule when computing in a field with Diofant). The `gaussian` keyword is useful for improving code readability, however the same result can be computed using more general syntax:

```
>>> factor(f, extension=I)
  ⎛    3⋅i⎞ ⎛    3⋅i⎞
4⋅⎜x - ───⎟⋅⎜x + ───⎟⋅(x + 1 - 4⋅i)⋅(x + 1 + 4⋅i)
  ⎝     2 ⎠ ⎝     2 ⎠
```

Computing with automatic field extensions

Consider two univariate polynomials `f` and `g`:

```
>>> f = x**3 + (sqrt(2) - 2)*x**2 - (2*sqrt(2) + 3)*x - 3*sqrt(2)
>>> g = x**2 - 2
```

We would like to reduce degrees of the numerator and the denominator of a rational function `f/g`. Do do this we employ *cancel()* (page 671) function:

```
>>> cancel(f/g)
 3      2       ___  2             ___         ___
x  - 2⋅x  + ╲╱ 2 ⋅x  - 3⋅x - 2⋅╲╱ 2 ⋅x - 3⋅╲╱ 2
─────────────────────────────────────────────────
                      2
                     x  - 2
```

Unfortunately nothing interesting happened. This is because by default Diofant treats $\sqrt{2}$ as a generator, obtaining a bivariate polynomial for the numerator. To make *cancel()* (page 671) recognize algebraic properties of $\sqrt{2}$, one needs to use `extension` keyword:

```
>>> cancel(f/g, extension=True)
 2
x  - 2⋅x - 3
────────────
       ___
 x - ╲╱ 2
```

Setting `extension=True` tells *cancel()* (page 671) to find minimal algebraic number domain for the coefficients of `f/g`. The automatically inferred domain is $\mathbb{Q}(\sqrt{2})$. If one doesn't want to rely on automatic inference, the same result can be obtained by setting the `extension` keyword with an explicit algebraic number:

```
>>> cancel(f/g, extension=sqrt(2))
 2
x  - 2⋅x - 3
────────────
       ___
 x - ╲╱ 2
```

Univariate factoring over various domains

Consider a univariate polynomial `f` with integer coefficients:

```
>>> f = x**4 - 3*x**2 + 1
```

With *diofant.polys* (page 645) we can obtain factorizations of `f` over different domains, which includes:

- rationals:

```
>>> factor(f)
⎛ 2        ⎞ ⎛ 2        ⎞
⎝x  - x - 1⎠⋅⎝x  + x - 1⎠
```

- finite fields:

```
>>> factor(f, modulus=5)
       2        2
(x - 2) ⋅(x + 2)
```

- algebraic numbers:

```
>>> alg = AlgebraicNumber((sqrt(5) - 1)/2, alias='alpha')
```

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```
>>> factor(f, extension=alg)
(x - α - 1)⋅(x - α)⋅(x + α)⋅(x + α + 1)
```

Factoring polynomials into linear factors

Currently Diofant can factor polynomials into irreducibles over various domains, which can result in a splitting factorization (into linear factors). However, there is currently no systematic way to infer a splitting field (algebraic number field) automatically. In future the following syntax will be implemented:

```
>>> factor(x**3 + x**2 - 7, split=True)
Traceback (most recent call last):
...
NotImplementedError: 'split' option is not implemented yet
```

Note this is different from `extension=True`, because the later only tells how expression parsing should be done, not what should be the domain of computation. One can simulate the `split` keyword for several classes of polynomials using *solve()* (page 839) function.

Advanced factoring over finite fields

Consider a univariate polynomial `f` with integer coefficients:

```
>>> f = x**11 + x + 1
```

We can factor `f` over a large finite field F_{65537}:

```
>>> factor(f, modulus=65537)
⎛ 2        ⎞ ⎛ 9    8    6    5    3    2    ⎞
⎝x  + x + 1⎠⋅⎝x  - x  + x  - x  + x  - x  + 1⎠
```

and expand the resulting factorization back:

```
>>> expand(_)
 11
x   + x + 1
```

obtaining polynomial `f`. This was done using symmetric polynomial representation over finite fields The same thing can be done using non-symmetric representation:

```
>>> factor(f, modulus=65537, symmetric=False)
⎛ 2        ⎞ ⎛ 9          8    6          5    3          2    ⎞
⎝x  + x + 1⎠⋅⎝x  + 65536⋅x  + x  + 65536⋅x  + x  + 65536⋅x  + 1⎠
```

As with symmetric representation we can expand the factorization to get the input polynomial back. This time, however, we need to truncate coefficients of the expanded polynomial modulo 65537:

```
>>> trunc(expand(_), 65537)
 11
x   + x + 1
```

Working with expressions as polynomials

Consider a multivariate polynomial `f` in $\mathbb{Z}[x, y, z]$:

```
>>> f = expand((x - 2*y**2 + 3*z**3)**20)
```

We want to compute factorization of `f`. To do this we use `factor` as usually, however we note that the polynomial in consideration is already in expanded form, so we can tell the factorization routine to skip expanding `f`:

```
>>> factor(f, expand=False)
                 20
⎛       2      3⎞
⎝x - 2⋅y  + 3⋅z ⎠
```

The default in *diofant.polys* (page 645) is to expand all expressions given as arguments to polynomial manipulation functions and *Poly* (page 672) class. If we know that expanding is unnecessary, then by setting `expand=False` we can save quite a lot of time for complicated inputs. This can be really important when computing with expressions like:

```
>>> g = expand((sin(x) - 2*cos(y)**2 + 3*tan(z)**3)**20)

>>> factor(g, expand=False)
                                 20
⎛               2           3   ⎞
⎝-sin(x) + 2⋅cos (y) - 3⋅tan (z)⎠
```

Computing reduced Gröbner bases

To compute a reduced Gröbner basis for a set of polynomials use *groebner()* (page 671) function. The function accepts various monomial orderings, e.g.: `lex`, `grlex` and `grevlex`, or a user defined one, via `order` keyword. The `lex` ordering is the most interesting because it has elimination property, which means that if the system of polynomial equations to *groebner()* (page 671) is zero-dimensional (has finite number of solutions) the last element of the basis is a univariate polynomial. Consider the following example:

```
>>> f = expand((1 - c**2)**5 * (1 - s**2)**5 * (c**2 + s**2)**10)

>>> groebner([f, c**2 + s**2 - 1])
             ⎛⎡ 2    2       20      18       16       14      12    10⎤                 ␣
↪           ⎞
GroebnerBasis⎝⎣c  + s  - 1, c   - 5⋅c   + 10⋅c   - 10⋅c   + 5⋅c   - c  ⎦, s, c,␣
↪domain=ℤ, order=lex⎠
```

The result is an ordinary Python list, so we can easily apply a function to all its elements, for example we can factor those elements:

```
>>> list(map(factor, _))
⎡ 2    2       10        5        5⎤
⎣c  + s  - 1, c  ⋅(c - 1) ⋅(c + 1) ⎦
```

From the above we can easily find all solutions of the system of polynomial equations. Or we can use *solve()* (page 839) to achieve this in a more systematic way:

```
>>> solve([f, s**2 + c**2 - 1], c, s)
[{c: -1, s: 0}, {c: 0, s: -1}, {c: 0, s: 1}, {c: 1, s: 0}]
```

Multivariate factoring over algebraic numbers

Computing with multivariate polynomials over various domains is as simple as in univariate case. For example consider the following factorization over $\mathbb{Q}(\sqrt{-3})$:

```
>>> factor(x**3 + y**3, extension=sqrt(-3))
        ⎛      ⎛           ___  ⎞⎞ ⎛      ⎛           ___  ⎞⎞
        ⎜      ⎜  1      ╲╱ 3 ⋅i⎟⎟ ⎜      ⎜  1      ╲╱ 3 ⋅i⎟⎟
(x + y)⋅⎜x + y⋅⎜- ─ - ───────⎟⎟⋅⎜x + y⋅⎜- ─ + ───────⎟⎟
        ⎝      ⎝  2        2    ⎠⎠ ⎝      ⎝  2        2    ⎠⎠
```

Note: Currently multivariate polynomials over finite fields aren't supported.

Partial fraction decomposition

Consider a univariate rational function `f` with integer coefficients:

```
>>> f = (x**2 + 2*x + 3)/(x**3 + 4*x**2 + 5*x + 2)
```

To decompose `f` into partial fractions use *apart()* (page 716) function:

```
>>> apart(f)
  3       2        2
───── - ───── + ────────
x + 2   x + 1          2
                (x + 1)
```

To return from partial fractions to the rational function use a composition of *together()* (page 715) and *cancel()* (page 671):

```
>>> cancel(together(_))
     2
    x  + 2⋅x + 3
───────────────────
 3      2
x  + 4⋅x  + 5⋅x + 2
```

Literature

3.11.3 Polynomials Manipulation Module Reference

Basic polynomial manipulation functions

`diofant.polys.polytools.`**`poly`**`(`*expr, *gens, **args*`)`

Efficiently transform an expression into a polynomial.

Examples

```
>>> poly(x*(x**2 + x - 1)**2)
Poly(x**5 + 2*x**4 - x**3 - 2*x**2 + x, x, domain='ZZ')
```

diofant.polys.polytools.**poly_from_expr**(*expr*, **gens*, ***args*)
Construct a polynomial from an expression.

diofant.polys.polytools.**parallel_poly_from_expr**(*exprs*, **gens*, ***args*)
Construct polynomials from expressions.

diofant.polys.polytools.**degree**(*f*, **gens*, ***args*)
Return the degree of `f` in the given variable.

The degree of 0 is negative infinity.

Examples

```
>>> degree(x**2 + y*x + 1, gen=x)
2
>>> degree(x**2 + y*x + 1, gen=y)
1
>>> degree(0, x)
-oo
```

diofant.polys.polytools.**degree_list**(*f*, **gens*, ***args*)
Return a list of degrees of `f` in all variables.

Examples

```
>>> degree_list(x**2 + y*x + 1)
(2, 1)
```

diofant.polys.polytools.**LC**(*f*, **gens*, ***args*)
Return the leading coefficient of `f`.

Examples

```
>>> LC(4*x**2 + 2*x*y**2 + x*y + 3*y)
4
```

diofant.polys.polytools.**LM**(*f*, **gens*, ***args*)
Return the leading monomial of `f`.

Examples

```
>>> LM(4*x**2 + 2*x*y**2 + x*y + 3*y)
x**2
```

diofant.polys.polytools.**LT**(*f*, **gens*, ***args*)
Return the leading term of `f`.

Examples

```
>>> LT(4*x**2 + 2*x*y**2 + x*y + 3*y)
4*x**2
```

diofant.polys.polytools.**pdiv**(*f, g, *gens, **args*)
Compute polynomial pseudo-division of f and g.

Examples

```
>>> pdiv(x**2 + 1, 2*x - 4)
(2*x + 4, 20)
```

diofant.polys.polytools.**prem**(*f, g, *gens, **args*)
Compute polynomial pseudo-remainder of f and g.

Examples

```
>>> prem(x**2 + 1, 2*x - 4)
20
```

diofant.polys.polytools.**pquo**(*f, g, *gens, **args*)
Compute polynomial pseudo-quotient of f and g.

Examples

```
>>> pquo(x**2 + 1, 2*x - 4)
2*x + 4
>>> pquo(x**2 - 1, 2*x - 1)
2*x + 1
```

diofant.polys.polytools.**pexquo**(*f, g, *gens, **args*)
Compute polynomial exact pseudo-quotient of f and g.

Examples

```
>>> pexquo(x**2 - 1, 2*x - 2)
2*x + 2
```

```
>>> pexquo(x**2 + 1, 2*x - 4)
Traceback (most recent call last):
...
ExactQuotientFailed: 2*x - 4 does not divide x**2 + 1
```

diofant.polys.polytools.**div**(*f, g, *gens, **args*)
Compute polynomial division of f and g.

Examples

```
>>> div(x**2 + 1, 2*x - 4, domain=ZZ)
(0, x**2 + 1)
>>> div(x**2 + 1, 2*x - 4, domain=QQ)
(x/2 + 1, 5)
```

diofant.polys.polytools.**rem**(*f, g, *gens, **args*)
Compute polynomial remainder of f and g.

Examples

```
>>> rem(x**2 + 1, 2*x - 4, domain=ZZ)
x**2 + 1
>>> rem(x**2 + 1, 2*x - 4, domain=QQ)
5
```

diofant.polys.polytools.**quo**(*f, g, *gens, **args*)
Compute polynomial quotient of f and g.

Examples

```
>>> quo(x**2 + 1, 2*x - 4)
x/2 + 1
>>> quo(x**2 - 1, x - 1)
x + 1
```

diofant.polys.polytools.**exquo**(*f, g, *gens, **args*)
Compute polynomial exact quotient of f and g.

Examples

```
>>> exquo(x**2 - 1, x - 1)
x + 1
```

```
>>> exquo(x**2 + 1, 2*x - 4)
Traceback (most recent call last):
...
ExactQuotientFailed: 2*x - 4 does not divide x**2 + 1
```

diofant.polys.polytools.**half_gcdex**(*f, g, *gens, **args*)
Half extended Euclidean algorithm of f and g.

Returns (s, h) such that h = gcd(f, g) and s*f = h (mod g).

Examples

```
>>> half_gcdex(x**4 - 2*x**3 - 6*x**2 + 12*x + 15, x**3 + x**2 - 4*x - 4)
(-x/5 + 3/5, x + 1)
```

diofant.polys.polytools.**gcdex**(*f, g, *gens, **args*)
Extended Euclidean algorithm of f and g.

Returns (s, t, h) such that h = gcd(f, g) and s*f + t*g = h.

Examples

```
>>> gcdex(x**4 - 2*x**3 - 6*x**2 + 12*x + 15, x**3 + x**2 - 4*x - 4)
(-x/5 + 3/5, x**2/5 - 6*x/5 + 2, x + 1)
```

diofant.polys.polytools.**invert**(*f, g, *gens, **args*)
Invert f modulo g when possible.

See also:

diofant.core.numbers.mod_inverse (page 93)

Examples

```
>>> invert(x**2 - 1, 2*x - 1)
-4/3
```

```
>>> invert(x**2 - 1, x - 1)
Traceback (most recent call last):
...
NotInvertible: zero divisor
```

For more efficient inversion of Rationals, use the mod_inverse function:

```
>>> mod_inverse(3, 5)
2
>>> (Integer(2)/5).invert(Integer(7)/3)
5/2
```

diofant.polys.polytools.**subresultants**(*f, g, *gens, **args*)
Compute subresultant PRS of f and g.

Examples

```
>>> subresultants(x**2 + 1, x**2 - 1)
[x**2 + 1, x**2 - 1, -2]
```

diofant.polys.polytools.**resultant**(*f, g, *gens, **args*)
Compute resultant of f and g.

Examples

```
>>> resultant(x**2 + 1, x**2 - 1)
4
```

diofant.polys.polytools.**discriminant**(*f, *gens, **args*)
Compute discriminant of f.

Examples

```
>>> discriminant(x**2 + 2*x + 3)
-8
```

diofant.polys.dispersion.**dispersion**(*p*, *q=None*, **gens*, ***args*)

Compute the *dispersion* of polynomials.

For two polynomials $f(x)$ and $g(x)$ with $\deg f > 0$ and $\deg g > 0$ the dispersion $\operatorname{dis}(f, g)$ is defined as:

$$\begin{aligned}\operatorname{dis}(f, g) &:= \max\{J(f, g) \cup \{0\}\} \\ &= \max\{\{a \in \mathbb{N} | \gcd(f(x), g(x + a)) \neq 1\} \cup \{0\}\}\end{aligned}$$

and for a single polynomial $\operatorname{dis}(f) := \operatorname{dis}(f, f)$. Note that we make the definition $\max\{\} := -\infty$.

See also:

diofant.polys.dispersion.dispersionset (page 663)

References

[R446] (page 1261), [R447] (page 1261), [R448] (page 1261), [R449] (page 1261)

Examples

Dispersion set and dispersion of a simple polynomial:

```
>>> fp = poly((x - 3)*(x + 3), x)
>>> sorted(dispersionset(fp))
[0, 6]
>>> dispersion(fp)
6
```

Note that the definition of the dispersion is not symmetric:

```
>>> fp = poly(x**4 - 3*x**2 + 1, x)
>>> gp = fp.shift(-3)
>>> sorted(dispersionset(fp, gp))
[2, 3, 4]
>>> dispersion(fp, gp)
4
>>> sorted(dispersionset(gp, fp))
[]
>>> dispersion(gp, fp)
-oo
```

The maximum of an empty set is defined to be $-\infty$ as seen in this example.

Computing the dispersion also works over field extensions:

```
>>> fp = poly(x**2 + sqrt(5)*x - 1, x, domain='QQ<sqrt(5)>')
>>> gp = poly(x**2 + (2 + sqrt(5))*x + sqrt(5), x, domain='QQ<sqrt(5)>')
>>> sorted(dispersionset(fp, gp))
```

(continues on next page)

(continued from previous page)

```
[2]
>>> sorted(dispersionset(gp, fp))
[1, 4]
```

We can even perform the computations for polynomials having symbolic coefficients:

```
>>> from diofant.abc import a
>>> fp = poly(4*x**4 + (4*a + 8)*x**3 + (a**2 + 6*a + 4)*x**2 + (a**2 + 2*a)*x, x)
>>> sorted(dispersionset(fp))
[0, 1]
```

diofant.polys.dispersion.**dispersionset**(*p*, *q=None*, **gens*, ***args*)

Compute the *dispersion set* of two polynomials.

For two polynomials $f(x)$ and $g(x)$ with $\deg f > 0$ and $\deg g > 0$ the dispersion set $\operatorname{J}(f, g)$ is defined as:

$$\begin{aligned}\operatorname{J}(f, g) &:= \{a \in \mathbb{N}_0 | \gcd(f(x), g(x + a)) \neq 1\} \\ &= \{a \in \mathbb{N}_0 | \deg \gcd(f(x), g(x + a)) \geq 1\}\end{aligned}$$

For a single polynomial one defines $\operatorname{J}(f) := \operatorname{J}(f, f)$.

See also:

diofant.polys.dispersion.dispersion (page 662)

References

[R450] (page 1261), [R451] (page 1261), [R452] (page 1261), [R453] (page 1261)

Examples

Dispersion set and dispersion of a simple polynomial:

```
>>> fp = poly((x - 3)*(x + 3), x)
>>> sorted(dispersionset(fp))
[0, 6]
>>> dispersion(fp)
6
```

Note that the definition of the dispersion is not symmetric:

```
>>> fp = poly(x**4 - 3*x**2 + 1, x)
>>> gp = fp.shift(-3)
>>> sorted(dispersionset(fp, gp))
[2, 3, 4]
>>> dispersion(fp, gp)
4
>>> sorted(dispersionset(gp, fp))
[]
>>> dispersion(gp, fp)
-oo
```

Computing the dispersion also works over field extensions:

```
>>> fp = poly(x**2 + sqrt(5)*x - 1, x, domain='QQ<sqrt(5)>')
>>> gp = poly(x**2 + (2 + sqrt(5))*x + sqrt(5), x, domain='QQ<sqrt(5)>')
>>> sorted(dispersionset(fp, gp))
[2]
>>> sorted(dispersionset(gp, fp))
[1, 4]
```

We can even perform the computations for polynomials having symbolic coefficients:

```
>>> from diofant.abc import a
>>> fp = poly(4*x**4 + (4*a + 8)*x**3 + (a**2 + 6*a + 4)*x**2 + (a**2 + 2*a)*x, x)
>>> sorted(dispersionset(fp))
[0, 1]
```

diofant.polys.polytools.**terms_gcd**(*f*, **gens*, ***args*)

Remove GCD of terms from f.

If the deep flag is True, then the arguments of f will have terms_gcd applied to them.

If a fraction is factored out of f and f is an Add, then an unevaluated Mul will be returned so that automatic simplification does not redistribute it. The hint clear, when set to False, can be used to prevent such factoring when all coefficients are not fractions.

See also:

diofant.core.exprtools.gcd_terms (page 153), *diofant.core.exprtools.factor_terms* (page 154)

Examples

```
>>> terms_gcd(x**6*y**2 + x**3*y, x, y)
x**3*y*(x**3*y + 1)
```

The default action of polys routines is to expand the expression given to them. terms_gcd follows this behavior:

```
>>> terms_gcd((3+3*x)*(x+x*y))
3*x*(x*y + x + y + 1)
```

If this is not desired then the hint expand can be set to False. In this case the expression will be treated as though it were comprised of one or more terms:

```
>>> terms_gcd((3+3*x)*(x+x*y), expand=False)
(3*x + 3)*(x*y + x)
```

In order to traverse factors of a Mul or the arguments of other functions, the deep hint can be used:

```
>>> terms_gcd((3 + 3*x)*(x + x*y), expand=False, deep=True)
3*x*(x + 1)*(y + 1)
>>> terms_gcd(cos(x + x*y), deep=True)
cos(x*(y + 1))
```

Rationals are factored out by default:

```
>>> terms_gcd(x + y/2)
(2*x + y)/2
```

Only the y-term had a coefficient that was a fraction; if one does not want to factor out the 1/2 in cases like this, the flag `clear` can be set to False:

```
>>> terms_gcd(x + y/2, clear=False)
x + y/2
>>> terms_gcd(x*y/2 + y**2, clear=False)
y*(x/2 + y)
```

The `clear` flag is ignored if all coefficients are fractions:

```
>>> terms_gcd(x/3 + y/2, clear=False)
(2*x + 3*y)/6
```

`diofant.polys.polytools.`**`cofactors`**(*f*, *g*, **gens*, ***args*)

Compute GCD and cofactors of `f` and `g`.

Returns polynomials `(h, cff, cfg)` such that `h = gcd(f, g)`, and `cff = quo(f, h)` and `cfg = quo(g, h)` are, so called, cofactors of `f` and `g`.

Examples

```
>>> cofactors(x**2 - 1, x**2 - 3*x + 2)
(x - 1, x + 1, x - 2)
```

`diofant.polys.polytools.`**`gcd`**(*f*, *g*, **gens*, ***args*)

Compute GCD of `f` and `g`.

Examples

```
>>> gcd(x**2 - 1, x**2 - 3*x + 2)
x - 1
```

`diofant.polys.polytools.`**`gcd_list`**(*seq*, **gens*, ***args*)

Compute GCD of a list of polynomials.

Examples

```
>>> gcd_list([x**3 - 1, x**2 - 1, x**2 - 3*x + 2])
x - 1
```

`diofant.polys.polytools.`**`lcm`**(*f*, *g*, **gens*, ***args*)

Compute LCM of `f` and `g`.

Examples

```
>>> lcm(x**2 - 1, x**2 - 3*x + 2)
x**3 - 2*x**2 - x + 2
```

`diofant.polys.polytools.`**`lcm_list`**(*seq*, **gens*, ***args*)

Compute LCM of a list of polynomials.

Examples

```
>>> lcm_list([x**3 - 1, x**2 - 1, x**2 - 3*x + 2])
x**5 - x**4 - 2*x**3 - x**2 + x + 2
```

diofant.polys.polytools.**trunc**(*f*, *p*, **gens*, ***args*)
Reduce f modulo a constant p.

Examples

```
>>> trunc(2*x**3 + 3*x**2 + 5*x + 7, 3)
-x**3 - x + 1
```

diofant.polys.polytools.**monic**(*f*, **gens*, ***args*)
Divide all coefficients of f by LC(f).

Examples

```
>>> monic(3*x**2 + 4*x + 2)
x**2 + 4*x/3 + 2/3
```

diofant.polys.polytools.**content**(*f*, **gens*, ***args*)
Compute GCD of coefficients of f.

Examples

```
>>> content(6*x**2 + 8*x + 12)
2
```

diofant.polys.polytools.**primitive**(*f*, **gens*, ***args*)
Compute content and the primitive form of f.

Examples

```
>>> primitive(6*x**2 + 8*x + 12)
(2, 3*x**2 + 4*x + 6)
```

```
>>> eq = (2 + 2*x)*x + 2
```

Expansion is performed by default:

```
>>> primitive(eq)
(2, x**2 + x + 1)
```

Set expand to False to shut this off. Note that the extraction will not be recursive; use the as_content_primitive method for recursive, non-destructive Rational extraction.

```
>>> primitive(eq, expand=False)
(1, x*(2*x + 2) + 2)
```

```
>>> eq.as_content_primitive()
(2, x*(x + 1) + 1)
```

diofant.polys.polytools.**compose**(*f*, *g*, **gens*, ***args*)
Compute functional composition `f(g)`.

Examples

```
>>> compose(x**2 + x, x - 1)
x**2 - x
```

diofant.polys.polytools.**decompose**(*f*, **gens*, ***args*)
Compute functional decomposition of `f`.

Examples

```
>>> decompose(x**4 + 2*x**3 - x - 1)
[x**2 - x - 1, x**2 + x]
```

diofant.polys.polytools.**sturm**(*f*, **gens*, ***args*)
Compute Sturm sequence of `f`.

Examples

```
>>> sturm(x**3 - 2*x**2 + x - 3)
[x**3 - 2*x**2 + x - 3, 3*x**2 - 4*x + 1, 2*x/9 + 25/9, -2079/4]
```

diofant.polys.polytools.**gff_list**(*f*, **gens*, ***args*)
Compute a list of greatest factorial factors of `f`.

Examples

```
>>> f = x**5 + 2*x**4 - x**3 - 2*x**2
```

```
>>> gff_list(f)
[(x, 1), (x + 2, 4)]
```

```
>>> (ff(x, 1)*ff(x + 2, 4)).expand() == f
True
```

diofant.polys.polytools.**gff**(*f*, **gens*, ***args*)
Compute greatest factorial factorization of `f`.

diofant.polys.polytools.**sqf_norm**(*f*, **gens*, ***args*)
Compute square-free norm of `f`.

Returns `s`, `f`, `r`, such that `g(x) = f(x-sa)` and `r(x) = Norm(g(x))` is a square-free polynomial over `K`, where `a` is the algebraic extension of the ground domain.

Examples

```
>>> sqf_norm(x**2 + 1, extension=[sqrt(3)])
(1, x**2 - 2*sqrt(3)*x + 4, x**4 - 4*x**2 + 16)
```

diofant.polys.polytools.**sqf_part**(*f*, **gens*, ***args*)
Compute square-free part of f.

Examples

```
>>> sqf_part(x**3 - 3*x - 2)
x**2 - x - 2
```

diofant.polys.polytools.**sqf_list**(*f*, **gens*, ***args*)
Compute a list of square-free factors of f.

Examples

```
>>> sqf_list(2*x**5 + 16*x**4 + 50*x**3 + 76*x**2 + 56*x + 16)
(2, [(x + 1, 2), (x + 2, 3)])
```

diofant.polys.polytools.**sqf**(*f*, **gens*, ***args*)
Compute square-free factorization of f.

Examples

```
>>> sqf(2*x**5 + 16*x**4 + 50*x**3 + 76*x**2 + 56*x + 16)
2*(x + 1)**2*(x + 2)**3
```

diofant.polys.polytools.**factor_list**(*f*, **gens*, ***args*)
Compute a list of irreducible factors of f.

Examples

```
>>> factor_list(2*x**5 + 2*x**4*y + 4*x**3 + 4*x**2*y + 2*x + 2*y)
(2, [(x + y, 1), (x**2 + 1, 2)])
```

diofant.polys.polytools.**factor**(*f*, **gens*, ***args*)
Compute the factorization of expression, f, into irreducibles. (To factor an integer into primes, use factorint.)

There two modes implemented: symbolic and formal. If f is not an instance of *Poly* (page 672) and generators are not specified, then the former mode is used. Otherwise, the formal mode is used.

In symbolic mode, *factor()* (page 668) will traverse the expression tree and factor its components without any prior expansion, unless an instance of *Add* (page 106) is encountered (in this case formal factorization is used). This way *factor()* (page 668) can handle large or symbolic exponents.

By default, the factorization is computed over the rationals. To factor over other domain, e.g. an algebraic or finite field, use appropriate options: `extension`, `modulus` or `domain`.

See also:

diofant.ntheory.factor_.factorint (page 254)

Examples

```
>>> factor(2*x**5 + 2*x**4*y + 4*x**3 + 4*x**2*y + 2*x + 2*y)
2*(x + y)*(x**2 + 1)**2
```

```
>>> factor(x**2 + 1)
x**2 + 1
>>> factor(x**2 + 1, modulus=2)
(x + 1)**2
>>> factor(x**2 + 1, gaussian=True)
(x - I)*(x + I)
```

```
>>> factor(x**2 - 2, extension=sqrt(2))
(x - sqrt(2))*(x + sqrt(2))
```

```
>>> factor((x**2 - 1)/(x**2 + 4*x + 4))
(x - 1)*(x + 1)/(x + 2)**2
>>> factor((x**2 + 4*x + 4)**10000000*(x**2 + 1))
(x + 2)**20000000*(x**2 + 1)
```

By default, factor deals with an expression as a whole:

```
>>> eq = 2**(x**2 + 2*x + 1)
>>> factor(eq)
2**(x**2 + 2*x + 1)
```

If the `deep` flag is True then subexpressions will be factored:

```
>>> factor(eq, deep=True)
2**((x + 1)**2)
```

`diofant.polys.polytools.`**`intervals`**(*F, all=False, eps=None, inf=None, sup=None, strict=False, fast=False, sqf=False*)

Compute isolating intervals for roots of `f`.

Examples

```
>>> intervals(x**2 - 3)
[((-2, -1), 1), ((1, 2), 1)]
>>> intervals(x**2 - 3, eps=1e-2)
[((-26/15, -19/11), 1), ((19/11, 26/15), 1)]
```

`diofant.polys.polytools.`**`refine_root`**(*f, s, t, eps=None, steps=None, fast=False, check_sqf=False*)

Refine an isolating interval of a root to the given precision.

Examples

```
>>> refine_root(x**2 - 3, 1, 2, eps=1e-2)
(19/11, 26/15)
```

diofant.polys.polytools.**count_roots**(*f*, *inf=None*, *sup=None*)
Return the number of roots of f in [inf, sup] interval.

If one of inf or sup is complex, it will return the number of roots in the complex rectangle with corners at inf and sup.

Examples

```
>>> count_roots(x**4 - 4, -3, 3)
2
>>> count_roots(x**4 - 4, 0, 1 + 3*I)
1
```

diofant.polys.polytools.**real_roots**(*f*, *multiple=True*)
Return a list of real roots with multiplicities of f.

Examples

```
>>> real_roots(2*x**3 - 7*x**2 + 4*x + 4)
[-1/2, 2, 2]
```

diofant.polys.polytools.**nroots**(*f*, *n=15*, *maxsteps=50*, *cleanup=True*)
Compute numerical approximations of roots of f.

Examples

```
>>> nroots(x**2 - 3, n=15)
[-1.73205080756888, 1.73205080756888]
>>> nroots(x**2 - 3, n=30)
[-1.73205080756887729352744634151, 1.73205080756887729352744634151]
```

diofant.polys.polytools.**ground_roots**(*f*, **gens*, ***args*)
Compute roots of f by factorization in the ground domain.

Examples

```
>>> ground_roots(x**6 - 4*x**4 + 4*x**3 - x**2)
{0: 2, 1: 2}
```

diofant.polys.polytools.**nth_power_roots_poly**(*f*, *n*, **gens*, ***args*)
Construct a polynomial with n-th powers of roots of f.

Examples

```
>>> f = x**4 - x**2 + 1
>>> g = factor(nth_power_roots_poly(f, 2))
```

```
>>> g
(x**2 - x + 1)**2
```

```
>>> R_f = [(r**2).expand() for r in roots(f)]
>>> R_g = roots(g)
```

```
>>> set(R_f) == set(R_g)
True
```

`diofant.polys.polytools.`**`cancel`**(*f*, **gens*, ***args*)

Cancel common factors in a rational function `f`.

Examples

```
>>> A = Symbol('A', commutative=False)
```

```
>>> cancel((2*x**2 - 2)/(x**2 - 2*x + 1))
(2*x + 2)/(x - 1)
>>> cancel((sqrt(3) + sqrt(15)*A)/(sqrt(2) + sqrt(10)*A))
sqrt(6)/2
```

`diofant.polys.polytools.`**`reduced`**(*f*, *G*, **gens*, ***args*)

Reduces a polynomial `f` modulo a set of polynomials `G`.

Given a polynomial `f` and a set of polynomials `G = (g_1, ..., g_n)`, computes a set of quotients `q = (q_1, ..., q_n)` and the remainder `r` such that `f = q_1*g_1 + ... + q_n*g_n + r`, where `r` vanishes or `r` is a completely reduced polynomial with respect to `G`.

Examples

```
>>> reduced(2*x**4 + y**2 - x**2 + y**3, [x**3 - x, y**3 - y])
([2*x, 1], x**2 + y**2 + y)
```

`diofant.polys.polytools.`**`groebner`**(*F*, **gens*, ***args*)

Computes the reduced Gröbner basis for a set of polynomials.

Use the `order` argument to set the monomial ordering that will be used to compute the basis. Allowed orders are `lex`, `grlex` and `grevlex`. If no order is specified, it defaults to `lex`.

See also:

diofant.solvers.polysys.solve_poly_system (page 843)

References

[R454] (page 1261), [R455] (page 1261)

Examples

Example taken from [1].

```
>>> F = [x*y - 2*y, 2*y**2 - x**2]
```

```
>>> groebner(F, x, y, order='lex')
GroebnerBasis([x**2 - 2*y**2, x*y - 2*y, y**3 - 2*y], x, y,
              domain='ZZ', order='lex')
>>> groebner(F, x, y, order='grlex')
GroebnerBasis([y**3 - 2*y, x**2 - 2*y**2, x*y - 2*y], x, y,
              domain='ZZ', order='grlex')
>>> groebner(F, x, y, order='grevlex')
GroebnerBasis([y**3 - 2*y, x**2 - 2*y**2, x*y - 2*y], x, y,
              domain='ZZ', order='grevlex')
```

By default, an improved implementation of the Buchberger algorithm is used. Optionally, an implementation of the F5B algorithm can be used. The algorithm can be set using `method` flag or with the function `setup()` (page 1152):

```
>>> F = [x**2 - x - 1, (2*x - 1) * y - (x**10 - (1 - x)**10)]
```

```
>>> groebner(F, x, y, method='buchberger')
GroebnerBasis([x**2 - x - 1, y - 55], x, y, domain='ZZ', order='lex')
>>> groebner(F, x, y, method='f5b')
GroebnerBasis([x**2 - x - 1, y - 55], x, y, domain='ZZ', order='lex')
```

class diofant.polys.polytools.**Poly**

Generic class for representing polynomial expressions.

EC(*order=None*)

Returns the last non-zero coefficient of `self`.

Examples

```
>>> Poly(x**3 + 2*x**2 + 3*x, x).EC()
3
```

EM(*order=None*)

Returns the last non-zero monomial of `self`.

Examples

```
>>> Poly(4*x**2 + 2*x*y**2 + x*y + 3*y, x, y).EM()
x**0*y**1
```

ET(*order=None*)

Returns the last non-zero term of `self`.

Examples

```
>>> Poly(4*x**2 + 2*x*y**2 + x*y + 3*y, x, y).ET()
(x**0*y**1, 3)
```

LC(*order=None*)
Returns the leading coefficient of `self`.

Examples

```
>>> Poly(4*x**3 + 2*x**2 + 3*x, x).LC()
4
```

LM(*order=None*)
Returns the leading monomial of `self`.

The leading monomial signifies the the monomial having the highest power of the principal generator in the polynomial expression.

Examples

```
>>> Poly(4*x**2 + 2*x*y**2 + x*y + 3*y, x, y).LM()
x**2*y**0
```

LT(*order=None*)
Returns the leading term of `self`.

The leading term signifies the term having the highest power of the principal generator in the polynomial expression.

Examples

```
>>> Poly(4*x**2 + 2*x*y**2 + x*y + 3*y, x, y).LT()
(x**2*y**0, 4)
```

TC()
Returns the trailing coefficient of `self`.

Examples

```
>>> Poly(x**3 + 2*x**2 + 3*x, x).TC()
0
```

abs()
Make all coefficients in `self` positive.

Examples

```
>>> Poly(x**2 - 1, x).abs()
Poly(x**2 + 1, x, domain='ZZ')
```

add(*other*)
Add two polynomials `self` and `other`.

Examples

```
>>> Poly(x**2 + 1, x).add(Poly(x - 2, x))
Poly(x**2 + x - 1, x, domain='ZZ')
```

```
>>> Poly(x**2 + 1, x) + Poly(x - 2, x)
Poly(x**2 + x - 1, x, domain='ZZ')
```

add_ground(*coeff*)
Add an element of the ground domain to `self`.

Examples

```
>>> Poly(x + 1).add_ground(2)
Poly(x + 3, x, domain='ZZ')
```

all_coeffs()
Returns all coefficients from a univariate polynomial `self`.

Examples

```
>>> Poly(x**3 + 2*x - 1, x).all_coeffs()
[1, 0, 2, -1]
```

all_monoms()
Returns all monomials from a univariate polynomial `self`.

See also:

all_terms (page 675)

Examples

```
>>> Poly(x**3 + 2*x - 1, x).all_monoms()
[(3,), (2,), (1,), (0,)]
```

all_roots(*multiple=True, radicals=True*)
Return a list of real and complex roots with multiplicities.

Examples

```
>>> Poly(2*x**3 - 7*x**2 + 4*x + 4).all_roots()
[-1/2, 2, 2]
>>> Poly(x**3 + x + 1).all_roots()
[RootOf(x**3 + x + 1, 0), RootOf(x**3 + x + 1, 1),
 RootOf(x**3 + x + 1, 2)]
```

all_terms()
Returns all terms from a univariate polynomial `self`.

Examples

```
>>> Poly(x**3 + 2*x - 1, x).all_terms()
[((3,), 1), ((2,), 0), ((1,), 2), ((0,), -1)]
```

args
Don't mess up with the core.

Examples

```
>>> Poly(x**2 + 1, x).args
(x**2 + 1, x)
```

as_dict(*native=False*, *zero=False*)
Switch to a `dict` representation.

Examples

```
>>> Poly(x**2 + 2*x*y**2 - y, x, y).as_dict()
{(0, 1): -1, (1, 2): 2, (2, 0): 1}
```

as_expr(**gens*)
Convert a Poly instance to an Expr instance.

Examples

```
>>> f = Poly(x**2 + 2*x*y**2 - y, x, y)
```

```
>>> f.as_expr()
x**2 + 2*x*y**2 - y
>>> f.as_expr({x: 5})
10*y**2 - y + 25
>>> f.as_expr(5, 6)
379
```

cancel(*other*, *include=False*)
Cancel common factors in a rational function `self/other`.

Examples

```
>>> Poly(2*x**2 - 2, x).cancel(Poly(x**2 - 2*x + 1, x))
(1, Poly(2*x + 2, x, domain='ZZ'), Poly(x - 1, x, domain='ZZ'))
```

```
>>> Poly(2*x**2 - 2, x).cancel(Poly(x**2 - 2*x + 1, x), include=True)
(Poly(2*x + 2, x, domain='ZZ'), Poly(x - 1, x, domain='ZZ'))
```

clear_denoms(*convert=False*)
Clear denominators, but keep the ground domain.

Examples

```
>>> f = Poly(x/2 + Rational(1, 3), x, domain=QQ)
```

```
>>> f.clear_denoms()
(6, Poly(3*x + 2, x, domain='QQ'))
>>> f.clear_denoms(convert=True)
(6, Poly(3*x + 2, x, domain='ZZ'))
```

coeff(*x, n=1, right=False*)
Returns the coefficient from the term(s) containing `x**n` or None. If `n` is zero then all terms independent of `x` will be returned.

When x is noncommutative, the coeff to the left (default) or right of x can be returned. The keyword 'right' is ignored when x is commutative.

See also:

diofant.core.expr.Expr.as_coefficient (page 56), *diofant.core.expr.Expr.as_coeff_Add* (page 55), *diofant.core.expr.Expr.as_coeff_Mul* (page 55), *diofant.core.expr.Expr.as_independent* (page 58), *diofant.polys.polytools.Poly.coeff_monomial* (page 678), *diofant.polys.polytools.Poly.nth* (page 695)

Examples

You can select terms that have an explicit negative in front of them:

```
>>> (-x + 2*y).coeff(-1)
x
>>> (x - 2*y).coeff(-1)
2*y
```

You can select terms with no Rational coefficient:

```
>>> (x + 2*y).coeff(1)
x
>>> (3 + 2*x + 4*x**2).coeff(1)
0
```

You can select terms independent of x by making n=0; in this case expr.as_independent(x)[0] is returned (and 0 will be returned instead of None):

```
>>> (3 + 2*x + 4*x**2).coeff(x, 0)
3
>>> eq = ((x + 1)**3).expand() + 1
>>> eq
x**3 + 3*x**2 + 3*x + 2
>>> [eq.coeff(x, i) for i in reversed(range(4))]
[1, 3, 3, 2]
>>> eq -= 2
>>> [eq.coeff(x, i) for i in reversed(range(4))]
[1, 3, 3, 0]
```

You can select terms that have a numerical term in front of them:

```
>>> (-x - 2*y).coeff(2)
-y
>>> (x + sqrt(2)*x).coeff(sqrt(2))
x
```

The matching is exact:

```
>>> (3 + 2*x + 4*x**2).coeff(x)
2
>>> (3 + 2*x + 4*x**2).coeff(x**2)
4
>>> (3 + 2*x + 4*x**2).coeff(x**3)
0
>>> (z*(x + y)**2).coeff((x + y)**2)
z
>>> (z*(x + y)**2).coeff(x + y)
0
```

In addition, no factoring is done, so 1 + z*(1 + y) is not obtained from the following:

```
>>> (x + z*(x + x*y)).coeff(x)
1
```

If such factoring is desired, factor_terms can be used first:

```
>>> factor_terms(x + z*(x + x*y)).coeff(x)
z*(y + 1) + 1
```

```
>>> n, m, o = symbols('n m o', commutative=False)
>>> n.coeff(n)
1
>>> (3*n).coeff(n)
3
>>> (n*m + m*n*m).coeff(n) # = (1 + m)*n*m
1 + m
>>> (n*m + m*n*m).coeff(n, right=True) # = (1 + m)*n*m
m
```

If there is more than one possible coefficient 0 is returned:

```
>>> (n*m + m*n).coeff(n)
0
```

If there is only one possible coefficient, it is returned:

```
>>> (n*m + x*m*n).coeff(m*n)
x
>>> (n*m + x*m*n).coeff(m*n, right=1)
1
```

coeff_monomial(*monom*)

Returns the coefficient of monom in self if there, else None.

See also:

nth **(page 695)** more efficient query using exponents of the monomial's generators

Examples

```
>>> p = Poly(24*x*y*exp(8) + 23*x, x, y)
```

```
>>> p.coeff_monomial(x)
23
>>> p.coeff_monomial(y)
0
>>> p.coeff_monomial(x*y)
24*E**8
```

Note that Expr.coeff() behaves differently, collecting terms if possible; the Poly must be converted to an Expr to use that method, however:

```
>>> p.as_expr().coeff(x)
24*E**8*y + 23
>>> p.as_expr().coeff(y)
24*E**8*x
>>> p.as_expr().coeff(x*y)
24*E**8
```

coeffs(*order=None*)

Returns all non-zero coefficients from self in lex order.

See also:

all_coeffs (page 674), *coeff_monomial* (page 678), *nth* (page 695)

Examples

```
>>> Poly(x**3 + 2*x + 3, x).coeffs()
[1, 2, 3]
```

cofactors(*other*)

Returns the GCD of self and other and their cofactors.

For two polynomials f and g it returns polynomials (h, cff, cfg) such that h = gcd(f, g), and cff = quo(f, h) and cfg = quo(g, h) are, so called, cofactors of f and g.

Examples

```
>>> Poly(x**2 - 1, x).cofactors(Poly(x**2 - 3*x + 2, x))
(Poly(x - 1, x, domain='ZZ'),
 Poly(x + 1, x, domain='ZZ'),
 Poly(x - 2, x, domain='ZZ'))
```

compose(*other*)
Computes the functional composition of `self` and `other`.

Examples

```
>>> Poly(x**2 + x, x).compose(Poly(x - 1, x))
Poly(x**2 - x, x, domain='ZZ')
```

content()
Returns the GCD of polynomial coefficients.

Examples

```
>>> Poly(6*x**2 + 8*x + 12, x).content()
2
```

count_roots(*inf=None*, *sup=None*)
Return the number of roots of `self` in `[inf, sup]` interval.

Examples

```
>>> Poly(x**4 - 4, x).count_roots(-3, 3)
2
>>> Poly(x**4 - 4, x).count_roots(0, 1 + 3*I)
1
```

decompose()
Computes a functional decomposition of `self`.

Examples

```
>>> Poly(x**4 + 2*x**3 - x - 1, x, domain='ZZ').decompose()
[Poly(x**2 - x - 1, x, domain='ZZ'), Poly(x**2 + x, x, domain='ZZ')]
```

deflate()
Reduce degree of `self` by mapping `x_i**m` to `y_i`.

Examples

```
>>> Poly(x**6*y**2 + x**3 + 1, x, y).deflate()
((3, 2), Poly(x**2*y + x + 1, x, y, domain='ZZ'))
```

degree(*gen=0*)
Returns degree of `self` in `x_j`.

The degree of 0 is negative infinity.

Examples

```
>>> Poly(x**2 + y*x + 1, x, y).degree()
2
>>> Poly(x**2 + y*x + y, x, y).degree(y)
1
>>> Poly(0, x).degree()
-oo
```

degree_list()
Returns a list of degrees of `self`.

Examples

```
>>> Poly(x**2 + y*x + 1, x, y).degree_list()
(2, 1)
```

diff(**specs, **kwargs*)
Computes partial derivative of `self`.

Examples

```
>>> Poly(x**2 + 2*x + 1, x).diff()
Poly(2*x + 2, x, domain='ZZ')
```

```
>>> Poly(x*y**2 + x, x, y).diff((0, 0), (1, 1))
Poly(2*x*y, x, y, domain='ZZ')
```

discriminant()
Computes the discriminant of `self`.

Examples

```
>>> Poly(x**2 + 2*x + 3, x).discriminant()
-8
```

dispersion(*other=None*)
Compute the *dispersion* of polynomials.

For two polynomials $f(x)$ and $g(x)$ with $\deg f > 0$ and $\deg g > 0$ the dispersion $\operatorname{dis}(f, g)$ is defined as:

$$\begin{aligned}\operatorname{dis}(f, g) &:= \max\{J(f, g) \cup \{0\}\} \\ &= \max\{\{a \in \mathbb{N} | \gcd(f(x), g(x + a)) \neq 1\} \cup \{0\}\}\end{aligned}$$

and for a single polynomial $\operatorname{dis}(f) := \operatorname{dis}(f, f)$.

See also:

diofant.polys.polytools.Poly.dispersionset (page 681)

References

[R456] (page 1261), *[R457]* (page 1261), *[R458]* (page 1261), *[R459]* (page 1261)

Examples

```
>>> from diofant.polys.dispersion import dispersion, dispersionset
```

Dispersion set and dispersion of a simple polynomial:

```
>>> fp = Poly((x - 3)*(x + 3), x)
>>> sorted(dispersionset(fp))
[0, 6]
>>> dispersion(fp)
6
```

Note that the definition of the dispersion is not symmetric:

```
>>> fp = Poly(x**4 - 3*x**2 + 1, x)
>>> gp = fp.shift(-3)
>>> sorted(dispersionset(fp, gp))
[2, 3, 4]
>>> dispersion(fp, gp)
4
>>> sorted(dispersionset(gp, fp))
[]
>>> dispersion(gp, fp)
-oo
```

Computing the dispersion also works over field extensions:

```
>>> fp = Poly(x**2 + sqrt(5)*x - 1, x, domain='QQ<sqrt(5)>')
>>> gp = Poly(x**2 + (2 + sqrt(5))*x + sqrt(5), x, domain='QQ<sqrt(5)>')
>>> sorted(dispersionset(fp, gp))
[2]
>>> sorted(dispersionset(gp, fp))
[1, 4]
```

We can even perform the computations for polynomials having symbolic coefficients:

```
>>> from diofant.abc import a
>>> fp = Poly(4*x**4 + (4*a + 8)*x**3 + (a**2 + 6*a + 4)*x**2 + (a**2 +
↪2*a)*x, x)
>>> sorted(dispersionset(fp))
[0, 1]
```

dispersionset(*other=None*)

Compute the *dispersion set* of two polynomials.

For two polynomials $f(x)$ and $g(x)$ with $\deg f > 0$ and $\deg g > 0$ the dispersion set

$J(f, g)$ is defined as:

$$\begin{aligned} J(f, g) &:= \{a \in \mathbb{N}_0 \,|\, \gcd(f(x), g(x + a)) \neq 1\} \\ &= \{a \in \mathbb{N}_0 \,|\, \deg \gcd(f(x), g(x + a)) \geq 1\} \end{aligned}$$

For a single polynomial one defines $J(f) := J(f, f)$.

See also:

diofant.polys.polytools.Poly.dispersion (page 680)

References

[R460] (page 1261), *[R461]* (page 1261), *[R462]* (page 1261), *[R463]* (page 1261)

Examples

```
>>> from diofant.polys.dispersion import dispersion, dispersionset
```

Dispersion set and dispersion of a simple polynomial:

```
>>> fp = Poly((x - 3)*(x + 3), x)
>>> sorted(dispersionset(fp))
[0, 6]
>>> dispersion(fp)
6
```

Note that the definition of the dispersion is not symmetric:

```
>>> fp = Poly(x**4 - 3*x**2 + 1, x)
>>> gp = fp.shift(-3)
>>> sorted(dispersionset(fp, gp))
[2, 3, 4]
>>> dispersion(fp, gp)
4
>>> sorted(dispersionset(gp, fp))
[]
>>> dispersion(gp, fp)
-oo
```

Computing the dispersion also works over field extensions:

```
>>> fp = Poly(x**2 + sqrt(5)*x - 1, x, domain='QQ<sqrt(5)>')
>>> gp = Poly(x**2 + (2 + sqrt(5))*x + sqrt(5), x, domain='QQ<sqrt(5)>')
>>> sorted(dispersionset(fp, gp))
[2]
>>> sorted(dispersionset(gp, fp))
[1, 4]
```

We can even perform the computations for polynomials having symbolic coefficients:

```
>>> from diofant.abc import a
>>> fp = Poly(4*x**4 + (4*a + 8)*x**3 + (a**2 + 6*a + 4)*x**2 + (a**2 +
↪2*a)*x, x)
>>> sorted(dispersionset(fp))
[0, 1]
```

div(*other*, *auto=True*)
Polynomial division with remainder of `self` by `other`.

Examples

```
>>> Poly(x**2 + 1, x).div(Poly(2*x - 4, x))
(Poly(1/2*x + 1, x, domain='QQ'), Poly(5, x, domain='QQ'))
```

```
>>> Poly(x**2 + 1, x).div(Poly(2*x - 4, x), auto=False)
(Poly(0, x, domain='ZZ'), Poly(x**2 + 1, x, domain='ZZ'))
```

domain
Get the ground domain of `self`.

eject(**gens*)
Eject selected generators into the ground domain.

Examples

```
>>> f = Poly(x**2*y + x*y**3 + x*y + 1, x, y)
```

```
>>> f.eject(x)
Poly(x*y**3 + (x**2 + x)*y + 1, y, domain='ZZ[x]')
>>> f.eject(y)
Poly(y*x**2 + (y**3 + y)*x + 1, x, domain='ZZ[y]')
```

eval(*x*, *a=None*, *auto=True*)
Evaluate `self` at `a` in the given variable.

Examples

```
>>> Poly(x**2 + 2*x + 3, x).eval(2)
11
```

```
>>> Poly(2*x*y + 3*x + y + 2, x, y).eval(x, 2)
Poly(5*y + 8, y, domain='ZZ')
```

```
>>> f = Poly(2*x*y + 3*x + y + 2*z, x, y, z)
```

```
>>> f.eval({x: 2})
Poly(5*y + 2*z + 6, y, z, domain='ZZ')
>>> f.eval({x: 2, y: 5})
Poly(2*z + 31, z, domain='ZZ')
>>> f.eval({x: 2, y: 5, z: 7})
45
```

```
>>> f.eval((2, 5))
Poly(2*z + 31, z, domain='ZZ')
>>> f(2, 5)
Poly(2*z + 31, z, domain='ZZ')
```

exclude()
Remove unnecessary generators from `self`.

Examples

```
>>> from diofant.abc import a, b, c, d
```

```
>>> Poly(a + x, a, b, c, d, x).exclude()
Poly(a + x, a, x, domain='ZZ')
```

exquo(*other*, *auto=True*)
Computes polynomial exact quotient of `self` by `other`.

Examples

```
>>> Poly(x**2 - 1, x).exquo(Poly(x - 1, x))
Poly(x + 1, x, domain='ZZ')
```

```
>>> Poly(x**2 + 1, x).exquo(Poly(2*x - 4, x))
Traceback (most recent call last):
...
ExactQuotientFailed: 2*x - 4 does not divide x**2 + 1
```

exquo_ground(*coeff*)
Exact quotient of `self` by a an element of the ground domain.

Examples

```
>>> Poly(2*x + 4).exquo_ground(2)
Poly(x + 2, x, domain='ZZ')
```

```
>>> Poly(2*x + 3).exquo_ground(2)
Traceback (most recent call last):
...
ExactQuotientFailed: 2 does not divide 3 in ZZ
```

factor_list()
Returns a list of irreducible factors of `self`.

Examples

```
>>> f = 2*x**5 + 2*x**4*y + 4*x**3 + 4*x**2*y + 2*x + 2*y
```

```
>>> Poly(f).factor_list()
(2, [(Poly(x + y, x, y, domain='ZZ'), 1),
     (Poly(x**2 + 1, x, y, domain='ZZ'), 2)])
```

factor_list_include()
Returns a list of irreducible factors of `self`.

Examples

```
>>> f = 2*x**5 + 2*x**4*y + 4*x**3 + 4*x**2*y + 2*x + 2*y
```

```
>>> Poly(f).factor_list_include()
[(Poly(2*x + 2*y, x, y, domain='ZZ'), 1),
 (Poly(x**2 + 1, x, y, domain='ZZ'), 2)]
```

free_symbols
Free symbols of a polynomial expression.

Examples

```
>>> Poly(x**2 + 1).free_symbols
{x}
>>> Poly(x**2 + y).free_symbols
{x, y}
>>> Poly(x**2 + y, x).free_symbols
{x, y}
```

free_symbols_in_domain
Free symbols of the domain of `self`.

Examples

```
>>> Poly(x**2 + 1).free_symbols_in_domain
set()
>>> Poly(x**2 + y).free_symbols_in_domain
set()
>>> Poly(x**2 + y, x).free_symbols_in_domain
{y}
```

classmethod from_dict(*rep*, **gens*, ***args*)
Construct a polynomial from a `dict`.

classmethod from_expr(*rep*, **gens*, ***args*)
Construct a polynomial from an expression.

classmethod from_list(*rep*, **gens*, ***args*)
Construct a polynomial from a `list`.

classmethod from_poly(*rep*, **gens*, ***args*)
Construct a polynomial from a polynomial.

gcd(*other*)
Returns the polynomial GCD of `self` and `other`.

Examples

```
>>> Poly(x**2 - 1, x).gcd(Poly(x**2 - 3*x + 2, x))
Poly(x - 1, x, domain='ZZ')
```

gcdex(*other*, *auto=True*)

Extended Euclidean algorithm of `self` and `other`.

Returns `(s, t, h)` such that `h = gcd(f, g)` and `s*f + t*g = h`.

Examples

```
>>> f = x**4 - 2*x**3 - 6*x**2 + 12*x + 15
>>> g = x**3 + x**2 - 4*x - 4
```

```
>>> Poly(f).gcdex(Poly(g))
(Poly(-1/5*x + 3/5, x, domain='QQ'),
 Poly(1/5*x**2 - 6/5*x + 2, x, domain='QQ'),
 Poly(x + 1, x, domain='QQ'))
```

gen

Return the principal generator.

Examples

```
>>> Poly(x**2 + 1, x).gen
x
```

get_modulus()

Get the modulus of `self`.

Examples

```
>>> Poly(x**2 + 1, modulus=2).get_modulus()
2
```

gff_list()

Computes greatest factorial factorization of `self`.

Examples

```
>>> f = x**5 + 2*x**4 - x**3 - 2*x**2
```

```
>>> Poly(f).gff_list()
[(Poly(x, x, domain='ZZ'), 1), (Poly(x + 2, x, domain='ZZ'), 4)]
```

ground_roots()

Compute roots of `self` by factorization in the ground domain.

Examples

```
>>> Poly(x**6 - 4*x**4 + 4*x**3 - x**2).ground_roots()
{0: 2, 1: 2}
```

half_gcdex(*other*, *auto=True*)

Half extended Euclidean algorithm of `self` and `other`.

Returns `(s, h)` such that `h = gcd(f, g)` and `s*f = h (mod g)`.

Examples

```
>>> f = x**4 - 2*x**3 - 6*x**2 + 12*x + 15
>>> g = x**3 + x**2 - 4*x - 4
```

```
>>> Poly(f).half_gcdex(Poly(g))
(Poly(-1/5*x + 3/5, x, domain='QQ'), Poly(x + 1, x, domain='QQ'))
```

has_only_gens(**gens*)

Return `True` if `Poly(f, *gens)` retains ground domain.

Examples

```
>>> Poly(x*y + 1, x, y, z).has_only_gens(x, y)
True
>>> Poly(x*y + z, x, y, z).has_only_gens(x, y)
False
```

homogeneous_order()

Returns the homogeneous order of `self`.

A homogeneous polynomial is a polynomial whose all monomials with non-zero coefficients have the same total degree. This degree is the homogeneous order of `f`. If you only want to check if a polynomial is homogeneous, then use *Poly.is_homogeneous()* (page 689).

Examples

```
>>> f = Poly(x**5 + 2*x**3*y**2 + 9*x*y**4)
>>> f.homogeneous_order()
5
```

homogenize(*s*)

Returns the homogeneous polynomial of `self`.

A homogeneous polynomial is a polynomial whose all monomials with non-zero coefficients have the same total degree. If you only want to check if a polynomial is homogeneous, then use *Poly.is_homogeneous()* (page 689). If you want not only to check if a polynomial is homogeneous but also compute its homogeneous order, then use *Poly.homogeneous_order()* (page 687).

Examples

```
>>> f = Poly(x**5 + 2*x**2*y**2 + 9*x*y**3)
>>> f.homogenize(z)
Poly(x**5 + 2*x**2*y**2*z + 9*x*y**3*z, x, y, z, domain='ZZ')
```

inject(*front=False*)
Inject ground domain generators into `self`.

Examples

```
>>> f = Poly(x**2*y + x*y**3 + x*y + 1, x)
```

```
>>> f.inject()
Poly(x**2*y + x*y**3 + x*y + 1, x, y, domain='ZZ')
>>> f.inject(front=True)
Poly(y**3*x + y*x**2 + y*x + 1, y, x, domain='ZZ')
```

integrate(**specs, **args*)
Computes indefinite integral of `self`.

Examples

```
>>> Poly(x**2 + 2*x + 1, x).integrate()
Poly(1/3*x**3 + x**2 + x, x, domain='QQ')
```

```
>>> Poly(x*y**2 + x, x, y).integrate((0, 1), (1, 0))
Poly(1/2*x**2*y**2 + 1/2*x**2, x, y, domain='QQ')
```

intervals(*all=False, eps=None, inf=None, sup=None, fast=False, sqf=False*)
Compute isolating intervals for roots of `self`.

For real roots the Vincent-Akritas-Strzebonski (VAS) continued fractions method is used.

References

[R464] (page 1261), [R465] (page 1261)

Examples

```
>>> Poly(x**2 - 3, x).intervals()
[((-2, -1), 1), ((1, 2), 1)]
>>> Poly(x**2 - 3, x).intervals(eps=1e-2)
[((-26/15, -19/11), 1), ((19/11, 26/15), 1)]
```

invert(*other, auto=True*)
Invert `self` modulo `other` when possible.

Examples

```
>>> Poly(x**2 - 1, x).invert(Poly(2*x - 1, x))
Poly(-4/3, x, domain='QQ')
```

```
>>> Poly(x**2 - 1, x).invert(Poly(x - 1, x))
Traceback (most recent call last):
...
NotInvertible: zero divisor
```

is_cyclotomic

Returns True if self is a cyclotomic polynomial.

Examples

```
>>> f = x**16 + x**14 - x**10 + x**8 - x**6 + x**2 + 1
>>> Poly(f).is_cyclotomic
False
```

```
>>> g = x**16 + x**14 - x**10 - x**8 - x**6 + x**2 + 1
>>> Poly(g).is_cyclotomic
True
```

is_ground

Returns True if self is an element of the ground domain.

Examples

```
>>> Poly(x, x).is_ground
False
>>> Poly(2, x).is_ground
True
>>> Poly(y, x).is_ground
True
```

is_homogeneous

Returns True if self is a homogeneous polynomial.

A homogeneous polynomial is a polynomial whose all monomials with non-zero coefficients have the same total degree. If you want not only to check if a polynomial is homogeneous but also compute its homogeneous order, then use *Poly.homogeneous_order()* (page 687).

Examples

```
>>> Poly(x**2 + x*y, x, y).is_homogeneous
True
>>> Poly(x**3 + x*y, x, y).is_homogeneous
False
```

is_irreducible

Returns True if self has no factors over its domain.

Examples

```
>>> Poly(x**2 + x + 1, x, modulus=2).is_irreducible
True
>>> Poly(x**2 + 1, x, modulus=2).is_irreducible
False
```

is_linear
Returns True if self is linear in all its variables.

Examples

```
>>> Poly(x + y + 2, x, y).is_linear
True
>>> Poly(x*y + 2, x, y).is_linear
False
```

is_monic
Returns True if the leading coefficient of self is one.

Examples

```
>>> Poly(x + 2, x).is_monic
True
>>> Poly(2*x + 2, x).is_monic
False
```

is_monomial
Returns True if self is zero or has only one term.

Examples

```
>>> Poly(3*x**2, x).is_monomial
True
>>> Poly(3*x**2 + 1, x).is_monomial
False
```

is_multivariate
Returns True if self is a multivariate polynomial.

Examples

```
>>> Poly(x**2 + x + 1, x).is_multivariate
False
>>> Poly(x*y**2 + x*y + 1, x, y).is_multivariate
True
>>> Poly(x*y**2 + x*y + 1, x).is_multivariate
False
>>> Poly(x**2 + x + 1, x, y).is_multivariate
True
```

is_number
Returns True if 'self' has no free symbols.

It will be faster than `if not self.free_symbols`, however, since `is_number` will fail as soon as it hits a free symbol.

Examples

```
>>> x.is_number
False
>>> (2*x).is_number
False
>>> (2 + log(2)).is_number
True
>>> (2 + Integral(2, x)).is_number
False
>>> (2 + Integral(2, (x, 1, 2))).is_number
True
```

is_one
Returns `True` if `self` is a unit polynomial.

Examples

```
>>> Poly(0, x).is_one
False
>>> Poly(1, x).is_one
True
```

is_primitive
Returns `True` if GCD of the coefficients of `self` is one.

Examples

```
>>> Poly(2*x**2 + 6*x + 12, x).is_primitive
False
>>> Poly(x**2 + 3*x + 6, x).is_primitive
True
```

is_quadratic
Returns `True` if `self` is quadratic in all its variables.

Examples

```
>>> Poly(x*y + 2, x, y).is_quadratic
True
>>> Poly(x*y**2 + 2, x, y).is_quadratic
False
```

is_sqf
Returns `True` if `self` is a square-free polynomial.

Examples

```
>>> Poly(x**2 - 2*x + 1, x).is_sqf
False
>>> Poly(x**2 - 1, x).is_sqf
True
```

is_univariate
Returns True if self is a univariate polynomial.

Examples

```
>>> Poly(x**2 + x + 1, x).is_univariate
True
>>> Poly(x*y**2 + x*y + 1, x, y).is_univariate
False
>>> Poly(x*y**2 + x*y + 1, x).is_univariate
True
>>> Poly(x**2 + x + 1, x, y).is_univariate
False
```

is_zero
Returns True if self is a zero polynomial.

Examples

```
>>> Poly(0, x).is_zero
True
>>> Poly(1, x).is_zero
False
```

l1_norm()
Returns l1 norm of self.

Examples

```
>>> Poly(-x**2 + 2*x - 3, x).l1_norm()
6
```

lcm(*other*)
Returns polynomial LCM of self and other.

Examples

```
>>> Poly(x**2 - 1, x).lcm(Poly(x**2 - 3*x + 2, x))
Poly(x**3 - 2*x**2 - x + 2, x, domain='ZZ')
```

length()
Returns the number of non-zero terms in self.

Examples

```
>>> Poly(x**2 + 2*x - 1).length()
3
```

lift()

Convert algebraic coefficients to rationals.

Examples

```
>>> Poly(x**2 + I*x + 1, x, extension=I).lift()
Poly(x**4 + 3*x**2 + 1, x, domain='QQ')
```

ltrim(*gen*)

Remove dummy generators from the "left" of `self`.

Examples

```
>>> Poly(y**2 + y*z**2, x, y, z).ltrim(y)
Poly(y**2 + y*z**2, y, z, domain='ZZ')
```

max_norm()

Returns maximum norm of `self`.

Examples

```
>>> Poly(-x**2 + 2*x - 3, x).max_norm()
3
```

monic(*auto=True*)

Divides all coefficients by `LC(f)`.

Examples

```
>>> Poly(3*x**2 + 6*x + 9, x, domain=ZZ).monic()
Poly(x**2 + 2*x + 3, x, domain='QQ')
```

```
>>> Poly(3*x**2 + 4*x + 2, x, domain=ZZ).monic()
Poly(x**2 + 4/3*x + 2/3, x, domain='QQ')
```

monoms(*order=None*)

Returns all non-zero monomials from `self` in lex order.

See also:

all_monoms (page 674)

Examples

```
>>> Poly(x**2 + 2*x*y**2 + x*y + 3*y, x, y).monoms()
[(2, 0), (1, 2), (1, 1), (0, 1)]
```

mul(*other*)
Multiply two polynomials `self` and `other`.

Examples

```
>>> Poly(x**2 + 1, x).mul(Poly(x - 2, x))
Poly(x**3 - 2*x**2 + x - 2, x, domain='ZZ')
```

```
>>> Poly(x**2 + 1, x)*Poly(x - 2, x)
Poly(x**3 - 2*x**2 + x - 2, x, domain='ZZ')
```

mul_ground(*coeff*)
Multiply `self` by a an element of the ground domain.

Examples

```
>>> Poly(x + 1).mul_ground(2)
Poly(2*x + 2, x, domain='ZZ')
```

neg()
Negate all coefficients in `self`.

Examples

```
>>> Poly(x**2 - 1, x).neg()
Poly(-x**2 + 1, x, domain='ZZ')
```

```
>>> -Poly(x**2 - 1, x)
Poly(-x**2 + 1, x, domain='ZZ')
```

classmethod new(*rep, *gens*)
Construct *Poly* (page 672) instance from raw representation.

nroots(*n=15, maxsteps=50, cleanup=True*)
Compute numerical approximations of roots of `self`.

Parameters n … the number of digits to calculate

maxsteps … the maximum number of iterations to do

If the accuracy ‘n‘ cannot be reached in ‘maxsteps‘, it will raise an exception. You need to rerun with higher maxsteps.

Examples

```
>>> Poly(x**2 - 3).nroots(n=15)
[-1.73205080756888, 1.73205080756888]
>>> Poly(x**2 - 3).nroots(n=30)
[-1.73205080756887729352744634151, 1.73205080756887729352744634151]
```

nth(**N*)

Returns the n-th coefficient of f where N are the exponents of the generators in the term of interest.

See also:

coeff_monomial (page 678)

Examples

```
>>> Poly(x**3 + 2*x**2 + 3*x, x).nth(2)
2
>>> Poly(x**3 + 2*x*y**2 + y**2, x, y).nth(1, 2)
2
>>> Poly(4*sqrt(x)*y)
Poly(4*y*sqrt(x), y, sqrt(x), domain='ZZ')
>>> _.nth(1, 1)
4
```

nth_power_roots_poly(*n*)

Construct a polynomial with n-th powers of roots of `self`.

Examples

```
>>> f = Poly(x**4 - x**2 + 1)
```

```
>>> f.nth_power_roots_poly(2)
Poly(x**4 - 2*x**3 + 3*x**2 - 2*x + 1, x, domain='ZZ')
>>> f.nth_power_roots_poly(3)
Poly(x**4 + 2*x**2 + 1, x, domain='ZZ')
>>> f.nth_power_roots_poly(4)
Poly(x**4 + 2*x**3 + 3*x**2 + 2*x + 1, x, domain='ZZ')
>>> f.nth_power_roots_poly(12)
Poly(x**4 - 4*x**3 + 6*x**2 - 4*x + 1, x, domain='ZZ')
```

one

Return one polynomial with `self`'s properties.

pdiv(*other*)

Polynomial pseudo-division of `self` by `other`.

Examples

```
>>> Poly(x**2 + 1, x).pdiv(Poly(2*x - 4, x))
(Poly(2*x + 4, x, domain='ZZ'), Poly(20, x, domain='ZZ'))
```

per(*rep, gens=None, remove=None*)
Create a Poly out of the given representation.

Examples

```
>>> from diofant.polys.polyclasses import DMP
```

```
>>> a = Poly(x**2 + 1)
```

```
>>> a.per(DMP([ZZ(1), ZZ(1)], ZZ), gens=[y])
Poly(y + 1, y, domain='ZZ')
```

pexquo(*other*)
Polynomial exact pseudo-quotient of `self` by `other`.

Examples

```
>>> Poly(x**2 - 1, x).pexquo(Poly(2*x - 2, x))
Poly(2*x + 2, x, domain='ZZ')
```

```
>>> Poly(x**2 + 1, x).pexquo(Poly(2*x - 4, x))
Traceback (most recent call last):
...
ExactQuotientFailed: 2*x - 4 does not divide x**2 + 1
```

pow(*n*)
Raise `self` to a non-negative power `n`.

Examples

```
>>> Poly(x - 2, x).pow(3)
Poly(x**3 - 6*x**2 + 12*x - 8, x, domain='ZZ')
```

```
>>> Poly(x - 2, x)**3
Poly(x**3 - 6*x**2 + 12*x - 8, x, domain='ZZ')
```

pquo(*other*)
Polynomial pseudo-quotient of `self` by `other`.

Examples

```
>>> Poly(x**2 + 1, x).pquo(Poly(2*x - 4, x))
Poly(2*x + 4, x, domain='ZZ')
```

```
>>> Poly(x**2 - 1, x).pquo(Poly(2*x - 2, x))
Poly(2*x + 2, x, domain='ZZ')
```

prem(*other*)
Polynomial pseudo-remainder of `self` by `other`.

Examples

```
>>> Poly(x**2 + 1, x).prem(Poly(2*x - 4, x))
Poly(20, x, domain='ZZ')
```

primitive()

Returns the content and a primitive form of `self`.

Examples

```
>>> Poly(2*x**2 + 8*x + 12, x).primitive()
(2, Poly(x**2 + 4*x + 6, x, domain='ZZ'))
```

quo(*other*, *auto=True*)

Computes polynomial quotient of `self` by `other`.

Examples

```
>>> Poly(x**2 + 1, x).quo(Poly(2*x - 4, x))
Poly(1/2*x + 1, x, domain='QQ')
```

```
>>> Poly(x**2 - 1, x).quo(Poly(x - 1, x))
Poly(x + 1, x, domain='ZZ')
```

quo_ground(*coeff*)

Quotient of `self` by a an element of the ground domain.

Examples

```
>>> Poly(2*x + 4).quo_ground(2)
Poly(x + 2, x, domain='ZZ')
```

```
>>> Poly(2*x + 3).quo_ground(2)
Poly(x + 1, x, domain='ZZ')
```

rat_clear_denoms(*other*)

Clear denominators in a rational function `self/other`.

Examples

```
>>> f = Poly(x**2/y + 1, x)
>>> g = Poly(x**3 + y, x)
```

```
>>> p, q = f.rat_clear_denoms(g)
```

```
>>> p
Poly(x**2 + y, x, domain='ZZ[y]')
>>> q
Poly(y*x**3 + y**2, x, domain='ZZ[y]')
```

real_roots(*multiple=True*, *radicals=True*)
Return a list of real roots with multiplicities.

Examples

```
>>> Poly(2*x**3 - 7*x**2 + 4*x + 4).real_roots()
[-1/2, 2, 2]
>>> Poly(x**3 + x + 1).real_roots()
[RootOf(x**3 + x + 1, 0)]
```

refine_root(*s*, *t*, *eps=None*, *steps=None*, *fast=False*, *check_sqf=False*)
Refine an isolating interval of a root to the given precision.

Examples

```
>>> Poly(x**2 - 3, x).refine_root(1, 2, eps=1e-2)
(19/11, 26/15)
```

rem(*other*, *auto=True*)
Computes the polynomial remainder of `self` by `other`.

Examples

```
>>> Poly(x**2 + 1, x).rem(Poly(2*x - 4, x))
Poly(5, x, domain='ZZ')
```

```
>>> Poly(x**2 + 1, x).rem(Poly(2*x - 4, x), auto=False)
Poly(x**2 + 1, x, domain='ZZ')
```

reorder(**gens*, ***args*)
Efficiently apply new order of generators.

Examples

```
>>> Poly(x**2 + x*y**2, x, y).reorder(y, x)
Poly(y**2*x + x**2, y, x, domain='ZZ')
```

replace(*x*, *y=None*)
Replace `x` with `y` in generators list.

Examples

```
>>> Poly(x**2 + 1, x).replace(x, y)
Poly(y**2 + 1, y, domain='ZZ')
```

resultant(*other*, *includePRS=False*)
Computes the resultant of `self` and `other` via PRS.

If includePRS=True, it includes the subresultant PRS in the result. Because the PRS is used to calculate the resultant, this is more efficient than calling `subresultants()` (page 661) separately.

Examples

```
>>> f = Poly(x**2 + 1, x)
```

```
>>> f.resultant(Poly(x**2 - 1, x))
4
>>> f.resultant(Poly(x**2 - 1, x), includePRS=True)
(4, [Poly(x**2 + 1, x, domain='ZZ'), Poly(x**2 - 1, x, domain='ZZ'),
     Poly(-2, x, domain='ZZ')])
```

retract(*field=None, extension=None*)
Recalculate the ground domain of a polynomial.

Examples

```
>>> f = Poly(x**2 + 1, x, domain='QQ[y]')
>>> f
Poly(x**2 + 1, x, domain='QQ[y]')
```

```
>>> f.retract()
Poly(x**2 + 1, x, domain='ZZ')
>>> f.retract(field=True)
Poly(x**2 + 1, x, domain='QQ')
```

revert(*n*)
Compute `self**(-1)` mod `x**n`.

Examples

```
>>> Poly(1, x).revert(2)
Poly(1, x, domain='ZZ')
```

```
>>> Poly(1 + x, x).revert(1)
Poly(1, x, domain='ZZ')
```

```
>>> Poly(x**2 - 1, x).revert(1)
Traceback (most recent call last):
...
NotReversible: only unity is reversible in a ring
```

```
>>> Poly(1/x, x).revert(1)
Traceback (most recent call last):
...
PolynomialError: 1/x contains an element of the generators set
```

root(*index, radicals=True*)
Get an indexed root of a polynomial.

Examples

```
>>> f = Poly(2*x**3 - 7*x**2 + 4*x + 4)
```

```
>>> f.root(0)
-1/2
>>> f.root(1)
2
>>> f.root(2)
2
>>> f.root(3)
Traceback (most recent call last):
...
IndexError: root index out of [-3, 2] range, got 3
```

```
>>> Poly(x**5 + x + 1).root(0)
RootOf(x**3 - x**2 + 1, 0)
```

set_domain(*domain*)
Set the ground domain of `self`.

set_modulus(*modulus*)
Set the modulus of `self`.

Examples

```
>>> Poly(5*x**2 + 2*x - 1, x).set_modulus(2)
Poly(x**2 + 1, x, modulus=2)
```

shift(*a*)
Efficiently compute Taylor shift `f(x + a)`.

Examples

```
>>> Poly(x**2 - 2*x + 1, x).shift(2)
Poly(x**2 + 2*x + 1, x, domain='ZZ')
```

slice(*x, m, n=None*)
Take a continuous subsequence of terms of `self`.

sqf_list(*all=False*)
Returns a list of square-free factors of `self`.

Examples

```
>>> f = 2*x**5 + 16*x**4 + 50*x**3 + 76*x**2 + 56*x + 16
```

```
>>> Poly(f).sqf_list()
(2, [(Poly(x + 1, x, domain='ZZ'), 2),
     (Poly(x + 2, x, domain='ZZ'), 3)])
```

```
>>> Poly(f).sqf_list(all=True)
(2, [(Poly(1, x, domain='ZZ'), 1),
     (Poly(x + 1, x, domain='ZZ'), 2),
     (Poly(x + 2, x, domain='ZZ'), 3)])
```

sqf_list_include(*all=False*)

Returns a list of square-free factors of `self`.

Examples

```
>>> f = expand(2*(x + 1)**3*x**4)
>>> f
2*x**7 + 6*x**6 + 6*x**5 + 2*x**4
```

```
>>> Poly(f).sqf_list_include()
[(Poly(2, x, domain='ZZ'), 1),
 (Poly(x + 1, x, domain='ZZ'), 3),
 (Poly(x, x, domain='ZZ'), 4)]
```

```
>>> Poly(f).sqf_list_include(all=True)
[(Poly(2, x, domain='ZZ'), 1),
 (Poly(1, x, domain='ZZ'), 2),
 (Poly(x + 1, x, domain='ZZ'), 3),
 (Poly(x, x, domain='ZZ'), 4)]
```

sqf_norm()

Computes square-free norm of `self`.

Returns `s`, `f`, `r`, such that `g(x) = f(x-sa)` and `r(x) = Norm(g(x))` is a square-free polynomial over `K`, where `a` is the algebraic extension of the ground domain.

Examples

```
>>> s, f, r = Poly(x**2 + 1, x, extension=[sqrt(3)]).sqf_norm()
```

```
>>> s
1
>>> f
Poly(x**2 - 2*sqrt(3)*x + 4, x, domain='QQ<sqrt(3)>')
>>> r
Poly(x**4 - 4*x**2 + 16, x, domain='QQ')
```

sqf_part()

Computes square-free part of `self`.

Examples

```
>>> Poly(x**3 - 3*x - 2, x).sqf_part()
Poly(x**2 - x - 2, x, domain='ZZ')
```

sqr()

Square a polynomial `self`.

Examples

```
>>> Poly(x - 2, x).sqr()
Poly(x**2 - 4*x + 4, x, domain='ZZ')
```

```
>>> Poly(x - 2, x)**2
Poly(x**2 - 4*x + 4, x, domain='ZZ')
```

sturm(*auto=True*)
Computes the Sturm sequence of `self`.

Examples

```
>>> Poly(x**3 - 2*x**2 + x - 3, x).sturm()
[Poly(x**3 - 2*x**2 + x - 3, x, domain='QQ'),
 Poly(3*x**2 - 4*x + 1, x, domain='QQ'),
 Poly(2/9*x + 25/9, x, domain='QQ'),
 Poly(-2079/4, x, domain='QQ')]
```

sub(*other*)
Subtract two polynomials `self` and `other`.

Examples

```
>>> Poly(x**2 + 1, x).sub(Poly(x - 2, x))
Poly(x**2 - x + 3, x, domain='ZZ')
```

```
>>> Poly(x**2 + 1, x) - Poly(x - 2, x)
Poly(x**2 - x + 3, x, domain='ZZ')
```

sub_ground(*coeff*)
Subtract an element of the ground domain from `self`.

Examples

```
>>> Poly(x + 1).sub_ground(2)
Poly(x - 1, x, domain='ZZ')
```

subresultants(*other*)
Computes the subresultant PRS of `self` and `other`.

Examples

```
>>> Poly(x**2 + 1, x).subresultants(Poly(x**2 - 1, x))
[Poly(x**2 + 1, x, domain='ZZ'),
 Poly(x**2 - 1, x, domain='ZZ'),
 Poly(-2, x, domain='ZZ')]
```

terms(*order=None*)
Returns all non-zero terms from `self` in lex order.

See also:

all_terms (page 675)

Examples

```
>>> Poly(x**2 + 2*x*y**2 + x*y + 3*y, x, y).terms()
[((2, 0), 1), ((1, 2), 2), ((1, 1), 1), ((0, 1), 3)]
```

terms_gcd()
Remove GCD of terms from the polynomial `self`.

Examples

```
>>> Poly(x**6*y**2 + x**3*y, x, y).terms_gcd()
((3, 1), Poly(x**3*y + 1, x, y, domain='ZZ'))
```

termwise(*func, *gens, **args*)
Apply a function to all terms of `self`.

Examples

```
>>> def func(k, coeff):
...     k = k[0]
...     return coeff//10**(2-k)
```

```
>>> Poly(x**2 + 20*x + 400).termwise(func)
Poly(x**2 + 2*x + 4, x, domain='ZZ')
```

to_exact()
Make the ground domain exact.

Examples

```
>>> Poly(x**2 + 1.0, x, domain=RR).to_exact()
Poly(x**2 + 1, x, domain='QQ')
```

to_field()
Make the ground domain a field.

Examples

```
>>> Poly(x**2 + 1, x, domain=ZZ).to_field()
Poly(x**2 + 1, x, domain='QQ')
```

to_ring()
Make the ground domain a ring.

Examples

```
>>> Poly(x**2 + 1, domain=QQ).to_ring()
Poly(x**2 + 1, x, domain='ZZ')
```

total_degree()
Returns the total degree of `self`.

Examples

```
>>> Poly(x**2 + y*x + 1, x, y).total_degree()
2
>>> Poly(x + y**5, x, y).total_degree()
5
```

trunc(*p*)
Reduce `self` modulo a constant `p`.

Examples

```
>>> Poly(2*x**3 + 3*x**2 + 5*x + 7, x).trunc(3)
Poly(-x**3 - x + 1, x, domain='ZZ')
```

unify(*other*)
Make `self` and `other` belong to the same domain.

Examples

```
>>> f, g = Poly(x/2 + 1), Poly(2*x + 1)
```

```
>>> f
Poly(1/2*x + 1, x, domain='QQ')
>>> g
Poly(2*x + 1, x, domain='ZZ')
```

```
>>> F, G = f.unify(g)
```

```
>>> F
Poly(1/2*x + 1, x, domain='QQ')
>>> G
Poly(2*x + 1, x, domain='QQ')
```

unit
Return unit polynomial with `self`'s properties.

zero
Return zero polynomial with `self`'s properties.

class `diofant.polys.polytools.`**PurePoly**
Class for representing pure polynomials.

free_symbols
Free symbols of a polynomial.

Examples

```
>>> PurePoly(x**2 + 1).free_symbols
set()
>>> PurePoly(x**2 + y).free_symbols
set()
>>> PurePoly(x**2 + y, x).free_symbols
{y}
```

class diofant.polys.polytools.GroebnerBasis
Represents a reduced Gröbner basis.

args
Returns a tuple of arguments of 'self'.

Examples

```
>>> cot(x).args
(x,)
>>> (x*y).args
(x, y)
```

contains(*poly*)
Check if `poly` belongs the ideal generated by `self`.

Examples

```
>>> f = 2*x**3 + y**3 + 3*y
>>> G = groebner([x**2 + y**2 - 1, x*y - 2])
```

```
>>> G.contains(f)
True
>>> G.contains(f + 1)
False
```

dimension
Dimension of the ideal, generated by a Gröbner basis.

fglm(*order*)
Convert a Gröbner basis from one ordering to another.

The FGLM algorithm converts reduced Gröbner bases of zero-dimensional ideals from one ordering to another. This method is often used when it is infeasible to compute a Gröbner basis with respect to a particular ordering directly.

References

[R466] (page 1261)

Examples

```
>>> F = [x**2 - 3*y - x + 1, y**2 - 2*x + y - 1]
>>> G = groebner(F, x, y, order='grlex')
```

```
>>> list(G.fglm('lex'))
[2*x - y**2 - y + 1, y**4 + 2*y**3 - 3*y**2 - 16*y + 7]
>>> list(groebner(F, x, y, order='lex'))
[2*x - y**2 - y + 1, y**4 + 2*y**3 - 3*y**2 - 16*y + 7]
```

independent_sets
Compute independent sets for ideal, generated by a Gröbner basis.

References

[R467] (page 1261)

reduce(*expr*, *auto=True*)
Reduces a polynomial modulo a Gröbner basis.

Given a polynomial `f` and a set of polynomials `G = (g_1, ..., g_n)`, computes a set of quotients `q = (q_1, ..., q_n)` and the remainder `r` such that `f = q_1*f_1 + ... + q_n*f_n + r`, where `r` vanishes or `r` is a completely reduced polynomial with respect to `G`.

Examples

```
>>> f = 2*x**4 - x**2 + y**3 + y**2
>>> G = groebner([x**3 - x, y**3 - y])
```

```
>>> G.reduce(f)
([2*x, 1], x**2 + y**2 + y)
>>> Q, r = _
```

```
>>> expand(sum(q*g for q, g in zip(Q, G)) + r)
2*x**4 - x**2 + y**3 + y**2
>>> _ == f
True
```

Extra polynomial manipulation functions

`diofant.polys.polyfuncs.`**symmetrize**(*F*, **gens*, ***args*)
Rewrite a polynomial in terms of elementary symmetric polynomials.

A symmetric polynomial is a multivariate polynomial that remains invariant under any variable permutation, i.e., if `f = f(x_1, x_2, ..., x_n)`, then `f = f(x_{i_1}, x_{i_2}, ..., x_{i_n})`, where `(i_1, i_2, ..., i_n)` is a permutation of `(1, 2, ..., n)` (an element of the group `S_n`).

Returns a tuple of symmetric polynomials `(f1, f2, ..., fn)` such that `f = f1 + f2 + ... + fn`.

Examples

```
>>> symmetrize(x**2 + y**2)
(-2*x*y + (x + y)**2, 0)
```

```
>>> symmetrize(x**2 + y**2, formal=True)
(s1**2 - 2*s2, 0, [(s1, x + y), (s2, x*y)])
```

```
>>> symmetrize(x**2 - y**2)
(-2*x*y + (x + y)**2, -2*y**2)
```

```
>>> symmetrize(x**2 - y**2, formal=True)
(s1**2 - 2*s2, -2*y**2, [(s1, x + y), (s2, x*y)])
```

diofant.polys.polyfuncs.**horner**(*f*, **gens*, ***args*)

Rewrite a polynomial in Horner form.

Among other applications, evaluation of a polynomial at a point is optimal when it is applied using the Horner scheme ([1]).

References

[R468] (page 1261)

Examples

```
>>> from diofant.abc import a, b, c, d, e
```

```
>>> horner(9*x**4 + 8*x**3 + 7*x**2 + 6*x + 5)
x*(x*(x*(9*x + 8) + 7) + 6) + 5
```

```
>>> horner(a*x**4 + b*x**3 + c*x**2 + d*x + e)
e + x*(d + x*(c + x*(a*x + b)))
```

```
>>> f = 4*x**2*y**2 + 2*x**2*y + 2*x*y**2 + x*y
```

```
>>> horner(f, wrt=x)
x*(x*y*(4*y + 2) + y*(2*y + 1))
```

```
>>> horner(f, wrt=y)
y*(x*y*(4*x + 2) + x*(2*x + 1))
```

diofant.polys.polyfuncs.**interpolate**(*data*, *x*)

Construct an interpolating polynomial for the data points.

Examples

A list is interpreted as though it were paired with a range starting from 1:

```
>>> interpolate([1, 4, 9, 16], x)
x**2
```

This can be made explicit by giving a list of coordinates:

```
>>> interpolate([(1, 1), (2, 4), (3, 9)], x)
x**2
```

The (x, y) coordinates can also be given as keys and values of a dictionary (and the points need not be equispaced):

```
>>> interpolate([(-1, 2), (1, 2), (2, 5)], x)
x**2 + 1
>>> interpolate({-1: 2, 1: 2, 2: 5}, x)
x**2 + 1
```

`diofant.polys.polyfuncs.`**`viete`**(*f*, *roots=None*, **gens*, ***args*)

Generate Viete's formulas for `f`.

Examples

```
>>> x, a, b, c, r1, r2 = symbols('x a:c r1:3')
```

```
>>> viete(a*x**2 + b*x + c, [r1, r2], x)
[(r1 + r2, -b/a), (r1*r2, c/a)]
```

Domain constructors

`diofant.polys.constructor.`**`construct_domain`**(*obj*, ***args*)

Construct a minimal domain for the list of coefficients.

Algebraic number fields

`diofant.polys.numberfields.`**`minimal_polynomial`**(*ex*, *x=None*, ***args*)

Computes the minimal polynomial of an algebraic element.

Parameters **ex** : algebraic element expression

x : independent variable of the minimal polynomial

compose : boolean, optional

If `True` (default), the minimal polynomial of the subexpressions of `ex` are computed, then the arithmetic operations on them are performed using the resultant and factorization. Else a bottom-up algorithm is used with `groebner`. The default algorithm stalls less frequently.

polys : boolean, optional

if `True` returns a `Poly` object (the default is `False`).

domain : Domain, optional

If no ground domain is given, it will be generated automatically from the expression.

Examples

```
>>> minimal_polynomial(sqrt(2), x)
x**2 - 2
>>> minimal_polynomial(sqrt(2), x, domain=QQ.algebraic_field(sqrt(2)))
x - sqrt(2)
>>> minimal_polynomial(sqrt(2) + sqrt(3), x)
x**4 - 10*x**2 + 1
>>> minimal_polynomial(solve(x**3 + x + 3)[0][x], x)
x**3 + x + 3
>>> minimal_polynomial(sqrt(y), x)
x**2 - y
```

diofant.polys.numberfields.**minpoly**()
alias of *minimal_polynomial()* (page 708)

diofant.polys.numberfields.**minpoly_groebner**(*ex*, *x*)
Computes the minimal polynomial of an algebraic number using Gröbner bases

References

[R469] (page 1261)

Examples

```
>>> minimal_polynomial(sqrt(2) + 3*Rational(1, 3), x, compose=False)
x**2 - 2*x - 1
```

diofant.polys.numberfields.**primitive_element**(*extension*, *x=None*, ***args*)
Construct a common number field for all extensions.

References

[R470] (page 1261)

diofant.polys.numberfields.**field_isomorphism**(*a*, *b*, ***args*)
Construct an isomorphism between two number fields.

diofant.polys.numberfields.**to_number_field**(*extension*, *theta=None*, ***args*)
Express $extension$ in the field generated by $theta$.

Monomials encoded as tuples

class diofant.polys.monomials.**Monomial**(*monom*, *gens=None*)
Class representing a monomial, i.e. a product of powers.

diofant.polys.monomials.**itermonomials**(*variables*, *degree*)
Generate a set of monomials of the given total degree or less.

Given a set of variables V and a total degree N generate a set of monomials of degree at most N. The total number of monomials is huge and is given by the following formula:

$$\frac{(\#V + N)!}{\#V!N!}$$

For example if we would like to generate a dense polynomial of a total degree $N = 50$ in 5 variables, assuming that exponents and all of coefficients are 32-bit long and stored in an array we would need almost 80 GiB of memory! Fortunately most polynomials, that we will encounter, are sparse.

Examples

Consider monomials in variables x and y:

```
>>> from diofant.polys.orderings import monomial_key

>>> sorted(itermonomials([x, y], 2), key=monomial_key('grlex', [y, x]))
[1, x, y, x**2, x*y, y**2]

>>> sorted(itermonomials([x, y], 3), key=monomial_key('grlex', [y, x]))
[1, x, y, x**2, x*y, y**2, x**3, x**2*y, x*y**2, y**3]
```

diofant.polys.monomials.**monomial_count**(V, N)

Computes the number of monomials.

The number of monomials is given by the following formula:

$$\frac{(\#V + N)!}{\#V!N!}$$

where N is a total degree and V is a set of variables.

Examples

```
>>> from diofant.polys.orderings import monomial_key
```

```
>>> monomial_count(2, 2)
6
```

```
>>> M = itermonomials([x, y], 2)
```

```
>>> sorted(M, key=monomial_key('grlex', [y, x]))
[1, x, y, x**2, x*y, y**2]
>>> len(M)
6
```

Orderings of monomials

class diofant.polys.orderings.**LexOrder**

Lexicographic order of monomials.

class diofant.polys.orderings.**GradedLexOrder**

Graded lexicographic order of monomials.

class diofant.polys.orderings.**ReversedGradedLexOrder**

Reversed graded lexicographic order of monomials.

Formal manipulation of roots of polynomials

class diofant.polys.rootoftools.**RootOf**

Represents k-th root of a univariate polynomial.

Parameters f : Expr

Univariate polynomial expression.

x : Symbol or Integer

Polynomial variable or the index of the root.

index : Integer or None, optional

Index of the root. If None (default), parameter `x` is used instead as index.

radicals : bool, optional

Explicitly solve linear or quadratic polynomial equation (enabled by default).

expand : bool, optional

Expand polynomial, enabled default.

evaluate : bool or None, optional

Control automatic evaluation.

extension : bool or None, optional

If enabled, reduce input polynomial to have integer coefficients.

Examples

```
>>> RootOf(x**3 + I*x + 2, 0, extension=True)
RootOf(x**6 + 4*x**3 + x**2 + 4, 1)
```

classmethod all_roots(*poly*, *radicals=True*, *extension=None*)

Get real and complex roots of a polynomial.

args

Returns a tuple of arguments of 'self'.

Examples

```
>>> cot(x).args
(x,)
>>> (x*y).args
(x, y)
```

eval_rational(*tol*)

Returns a Rational approximation to `self` with the tolerance `tol`.

The returned instance will be at most 'tol' from the exact root.

The following example first obtains Rational approximation to 1e-7 accuracy for all roots of the 4-th order Legendre polynomial, and then evaluates it to 5 decimal digits (so all digits will be correct including rounding):

```
>>> p = legendre_poly(4, x, polys=True)
>>> roots = [r.eval_rational(Rational(1, 10)**7) for r in p.real_roots()]
>>> roots = [str(r.n(5)) for r in roots]
>>> roots
['-0.86114', '-0.33998', '0.33998', '0.86114']
```

free_symbols
Return from the atoms of self those which are free symbols.

For most expressions, all symbols are free symbols. For some classes this is not true. e.g. Integrals use Symbols for the dummy variables which are bound variables, so Integral has a method to return all symbols except those. Derivative keeps track of symbols with respect to which it will perform a derivative; those are bound variables, too, so it has its own free_symbols method.

Any other method that uses bound variables should implement a free_symbols method.

interval
Return isolation interval for the root.

is_number
Returns True if 'self' has no free symbols.

It will be faster than `if not self.free_symbols`, however, since `is_number` will fail as soon as it hits a free symbol.

Examples

```
>>> x.is_number
False
>>> (2*x).is_number
False
>>> (2 + log(2)).is_number
True
>>> (2 + Integral(2, x)).is_number
False
>>> (2 + Integral(2, (x, 1, 2))).is_number
True
```

classmethod real_roots(*poly*, *radicals=True*, *extension=None*)
Get real roots of a polynomial.

class diofant.polys.rootoftools.RootSum
Represents a sum of all roots of a univariate polynomial.

args
Returns a tuple of arguments of 'self'.

Examples

```
>>> cot(x).args
(x,)
>>> (x*y).args
(x, y)
```

doit(***hints*)

Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

free_symbols

Return from the atoms of self those which are free symbols.

For most expressions, all symbols are free symbols. For some classes this is not true. e.g. Integrals use Symbols for the dummy variables which are bound variables, so Integral has a method to return all symbols except those. Derivative keeps track of symbols with respect to which it will perform a derivative; those are bound variables, too, so it has its own free_symbols method.

Any other method that uses bound variables should implement a free_symbols method.

is_commutative

Test if self commutes with any other object wrt multiplication operation.

classmethod new(*poly, func, auto=True*)

Construct new `RootSum` instance.

Symbolic root-finding algorithms

`diofant.polys.polyroots.`**roots**(*f, *gens, **flags*)

Computes symbolic roots of a univariate polynomial.

Given a univariate polynomial f with symbolic coefficients (or a list of the polynomial's coefficients), returns a dictionary with its roots and their multiplicities.

Only roots expressible via radicals will be returned. To get a complete set of roots use RootOf class or numerical methods instead. By default cubic and quartic formulas are used in the algorithm. To disable them because of unreadable output set `cubics=False` or `quartics=False` respectively. If cubic roots are real but are expressed in terms of

complex numbers (casus irreducibilis [1]) the `trig` flag can be set to True to have the solutions returned in terms of cosine and inverse cosine functions.

To get roots from a specific domain set the `filter` flag with one of the following specifiers: Z, Q, R, I, C. By default all roots are returned (this is equivalent to setting `filter='C'`).

By default a dictionary is returned giving a compact result in case of multiple roots. However to get a list containing all those roots set the `multiple` flag to True; the list will have identical roots appearing next to each other in the result. (For a given Poly, the all_roots method will give the roots in sorted numerical order.)

References

[R471] (page 1261)

Examples

```
>>> roots(x**2 - 1, x)
{-1: 1, 1: 1}
```

```
>>> p = Poly(x**2-1, x)
>>> roots(p)
{-1: 1, 1: 1}
```

```
>>> p = Poly(x**2-y, x, y)
```

```
>>> roots(Poly(p, x))
{-sqrt(y): 1, sqrt(y): 1}
```

```
>>> roots(x**2 - y, x)
{-sqrt(y): 1, sqrt(y): 1}
```

```
>>> roots([1, 0, -1])
{-1: 1, 1: 1}
```

Special polynomials

`diofant.polys.specialpolys.`**`swinnerton_dyer_poly`**(*n*, *x=None*, ***args*)
Generates n-th Swinnerton-Dyer polynomial in x.

`diofant.polys.specialpolys.`**`interpolating_poly`**(*n*, *x*, *X='x'*, *Y='y'*)
Construct Lagrange interpolating polynomial for `n` data points.

`diofant.polys.specialpolys.`**`cyclotomic_poly`**(*n*, *x=None*, ***args*)
Generates cyclotomic polynomial of order n in x.

`diofant.polys.specialpolys.`**`symmetric_poly`**(*n*, **gens*, ***args*)
Generates symmetric polynomial of order n.

`diofant.polys.specialpolys.`**`random_poly`**(*x*, *n*, *inf*, *sup*, *domain=GMPYIntegerRing()*, *polys=False*)
Return a polynomial of degree `n` with coefficients in `[inf, sup]`.

Orthogonal polynomials

`diofant.polys.orthopolys.`**`chebyshevt_poly`**`(`*n*, *x=None*, ***args*`)`
Generates Chebyshev polynomial of the first kind of degree n in x.

`diofant.polys.orthopolys.`**`chebyshevu_poly`**`(`*n*, *x=None*, ***args*`)`
Generates Chebyshev polynomial of the second kind of degree n in x.

`diofant.polys.orthopolys.`**`gegenbauer_poly`**`(`*n*, *a*, *x=None*, ***args*`)`
Generates Gegenbauer polynomial of degree n in x.

`diofant.polys.orthopolys.`**`hermite_poly`**`(`*n*, *x=None*, ***args*`)`
Generates Hermite polynomial of degree n in x.

`diofant.polys.orthopolys.`**`jacobi_poly`**`(`*n*, *a*, *b*, *x=None*, ***args*`)`
Generates Jacobi polynomial of degree n in x.

`diofant.polys.orthopolys.`**`legendre_poly`**`(`*n*, *x=None*, ***args*`)`
Generates Legendre polynomial of degree n in x.

`diofant.polys.orthopolys.`**`laguerre_poly`**`(`*n*, *x=None*, *alpha=None*, ***args*`)`
Generates Laguerre polynomial of degree n in x.

`diofant.polys.orthopolys.`**`spherical_bessel_fn`**`(`*n*, *x=None*, ***args*`)`
Coefficients for the spherical Bessel functions.

Those are only needed in the jn() function.

The coefficients are calculated from:

fn(0, z) = 1/z fn(1, z) = 1/z**2 fn(n-1, z) + fn(n+1, z) == (2*n+1)/z * fn(n, z)

Examples

```
>>> spherical_bessel_fn(1, z)
z**(-2)
>>> spherical_bessel_fn(2, z)
-1/z + 3/z**3
>>> spherical_bessel_fn(3, z)
-6/z**2 + 15/z**4
>>> spherical_bessel_fn(4, z)
1/z - 45/z**3 + 105/z**5
```

Manipulation of rational functions

`diofant.polys.rationaltools.`**`together`**`(`*expr*, *deep=False*`)`
Denest and combine rational expressions using symbolic methods.

This function takes an expression or a container of expressions and puts it (them) together by denesting and combining rational subexpressions. No heroic measures are taken to minimize degree of the resulting numerator and denominator. To obtain completely reduced expression use *`cancel()`* (page 671). However, *`together()`* (page 715) can preserve as much as possible of the structure of the input expression in the output (no expansion is performed).

A wide variety of objects can be put together including lists, tuples, sets, relational objects, integrals and others. It is also possible to transform interior of function applications, by setting `deep` flag to `True`.

By definition, *together()* (page 715) is a complement to *apart()* (page 716), so `apart(together(expr))` should return expr unchanged. Note however, that *together()* (page 715) uses only symbolic methods, so it might be necessary to use *cancel()* (page 671) to perform algebraic simplification and minimise degree of the numerator and denominator.

Examples

```
>>> together(1/x + 1/y)
(x + y)/(x*y)
>>> together(1/x + 1/y + 1/z)
(x*y + x*z + y*z)/(x*y*z)
```

```
>>> together(1/(x*y) + 1/y**2)
(x + y)/(x*y**2)
```

```
>>> together(1/(1 + 1/x) + 1/(1 + 1/y))
(x*(y + 1) + y*(x + 1))/((x + 1)*(y + 1))
```

```
>>> together(exp(1/x + 1/y))
E**(1/y + 1/x)
>>> together(exp(1/x + 1/y), deep=True)
E**((x + y)/(x*y))
```

```
>>> together(1/exp(x) + 1/(x*exp(x)))
E**(-x)*(x + 1)/x
```

```
>>> together(1/exp(2*x) + 1/(x*exp(3*x)))
E**(-3*x)*(E**x*x + 1)/x
```

Partial fraction decomposition

`diofant.polys.partfrac.`**`apart`**(*f*, *x=None*, *full=False*, ***options*)

Compute partial fraction decomposition of a rational function.

Given a rational function `f`, computes the partial fraction decomposition of `f`. Two algorithms are available: One is based on the undertermined coefficients method, the other is Bronstein's full partial fraction decomposition algorithm.

The undetermined coefficients method (selected by `full=False`) uses polynomial factorization (and therefore accepts the same options as factor) for the denominator. Per default it works over the rational numbers, therefore decomposition of denominators with non-rational roots (e.g. irrational, complex roots) is not supported by default (see options of factor).

Bronstein's algorithm can be selected by using `full=True` and allows a decomposition of denominators with non-rational roots. A human-readable result can be obtained via `doit()` (see examples below).

See also:

apart_list (page 717), *assemble_partfrac_list* (page 719)

Examples

By default, using the undetermined coefficients method:

```
>>> apart(y/(x + 2)/(x + 1), x)
-y/(x + 2) + y/(x + 1)
```

The undetermined coefficients method does not provide a result when the denominators roots are not rational:

```
>>> apart(y/(x**2 + x + 1), x)
y/(x**2 + x + 1)
```

You can choose Bronstein's algorithm by setting `full=True`:

```
>>> apart(y/(x**2 + x + 1), x, full=True)
RootSum(_w**2 + _w + 1, Lambda(_a, (-2*y*_a/3 - y/3)/(x - _a)))
```

Calling `doit()` yields a human-readable result:

```
>>> apart(y/(x**2 + x + 1), x, full=True).doit()
(-y/3 - 2*y*(-1/2 - sqrt(3)*I/2)/3)/(x + 1/2 + sqrt(3)*I/2) + (-y/3 -
    2*y*(-1/2 + sqrt(3)*I/2)/3)/(x + 1/2 - sqrt(3)*I/2)
```

`diofant.polys.partfrac.`**`apart_list`**(*f*, *x=None*, *dummies=None*, ***options*)

Compute partial fraction decomposition of a rational function and return the result in structured form.

Given a rational function `f` compute the partial fraction decomposition of `f`. Only Bronstein's full partial fraction decomposition algorithm is supported by this method. The return value is highly structured and perfectly suited for further algorithmic treatment rather than being human-readable. The function returns a tuple holding three elements:

- The first item is the common coefficient, free of the variable x used for decomposition. (It is an element of the base field K.)
- The second item is the polynomial part of the decomposition. This can be the zero polynomial. (It is an element of $K[x]$.)
- The third part itself is a list of quadruples. Each quadruple has the following elements in this order:
  - The (not necessarily irreducible) polynomial D whose roots w_i appear in the linear denominator of a bunch of related fraction terms. (This item can also be a list of explicit roots. However, at the moment `apart_list` never returns a result this way, but the related `assemble_partfrac_list` function accepts this format as input.)
  - The numerator of the fraction, written as a function of the root w
  - The linear denominator of the fraction *excluding its power exponent*, written as a function of the root w.
  - The power to which the denominator has to be raised.

On can always rebuild a plain expression by using the function `assemble_partfrac_list`.

See also:

apart (page 716), *assemble_partfrac_list* (page 719)

References

[R472] (page 1261)

Examples

A first example:

```
>>> from diofant.abc import t
```

```
>>> f = (2*x**3 - 2*x) / (x**2 - 2*x + 1)
>>> pfd = apart_list(f)
>>> pfd
(1,
Poly(2*x + 4, x, domain='ZZ'),
[(Poly(_w - 1, _w, domain='ZZ'), Lambda(_a, 4), Lambda(_a, x - _a), 1)])
```

```
>>> assemble_partfrac_list(pfd)
2*x + 4 + 4/(x - 1)
```

Second example:

```
>>> f = (-2*x - 2*x**2) / (3*x**2 - 6*x)
>>> pfd = apart_list(f)
>>> pfd
(-1,
Poly(2/3, x, domain='QQ'),
[(Poly(_w - 2, _w, domain='ZZ'), Lambda(_a, 2), Lambda(_a, x - _a), 1)])
```

```
>>> assemble_partfrac_list(pfd)
-2/3 - 2/(x - 2)
```

Another example, showing symbolic parameters:

```
>>> pfd = apart_list(t/(x**2 + x + t), x)
>>> pfd
(1,
Poly(0, x, domain='ZZ[t]'),
[(Poly(_w**2 + _w + t, _w, domain='ZZ[t]'),
Lambda(_a, -2*t*_a/(4*t - 1) - t/(4*t - 1)),
Lambda(_a, x - _a), 1)])
```

```
>>> assemble_partfrac_list(pfd)
RootSum(t + _w**2 + _w, Lambda(_a, (-2*t*_a/(4*t - 1) - t/(4*t - 1))/(x - _a)))
```

This example is taken from Bronstein's original paper:

```
>>> f = 36 / (x**5 - 2*x**4 - 2*x**3 + 4*x**2 + x - 2)
>>> pfd = apart_list(f)
>>> pfd
(1,
Poly(0, x, domain='ZZ'),
[(Poly(_w - 2, _w, domain='ZZ'), Lambda(_a, 4), Lambda(_a, x - _a), 1),
(Poly(_w**2 - 1, _w, domain='ZZ'), Lambda(_a, -3*_a - 6), Lambda(_a, x - _a), 2),
(Poly(_w + 1, _w, domain='ZZ'), Lambda(_a, -4), Lambda(_a, x - _a), 1)])
```

```
>>> assemble_partfrac_list(pfd)
-4/(x + 1) - 3/(x + 1)**2 - 9/(x - 1)**2 + 4/(x - 2)
```

diofant.polys.partfrac.**assemble_partfrac_list**(*partial_list*)

Reassemble a full partial fraction decomposition from a structured result obtained by the function apart_list.

See also:

apart (page 716), *apart_list* (page 717)

Examples

This example is taken from Bronstein's original paper:

```
>>> f = 36 / (x**5 - 2*x**4 - 2*x**3 + 4*x**2 + x - 2)
>>> pfd = apart_list(f)
>>> pfd
(1,
Poly(0, x, domain='ZZ'),
[(Poly(_w - 2, _w, domain='ZZ'), Lambda(_a, 4), Lambda(_a, x - _a), 1),
(Poly(_w**2 - 1, _w, domain='ZZ'), Lambda(_a, -3*_a - 6), Lambda(_a, x - _a), 2),
(Poly(_w + 1, _w, domain='ZZ'), Lambda(_a, -4), Lambda(_a, x - _a), 1)])
```

```
>>> assemble_partfrac_list(pfd)
-4/(x + 1) - 3/(x + 1)**2 - 9/(x - 1)**2 + 4/(x - 2)
```

If we happen to know some roots we can provide them easily inside the structure:

```
>>> pfd = apart_list(2/(x**2-2))
>>> pfd
(1,
Poly(0, x, domain='ZZ'),
[(Poly(_w**2 - 2, _w, domain='ZZ'),
Lambda(_a, _a/2), Lambda(_a, x - _a),
1)])
```

```
>>> pfda = assemble_partfrac_list(pfd)
>>> pfda
RootSum(_w**2 - 2, Lambda(_a, _a/(x - _a)))/2
```

```
>>> pfda.doit()
-sqrt(2)/(2*(x + sqrt(2))) + sqrt(2)/(2*(x - sqrt(2)))
```

```
>>> a = Dummy("a")
>>> pfd = (1, Poly(0, x, domain='ZZ'), [([sqrt(2), -sqrt(2)], Lambda(a, a/2),
↪Lambda(a, -a + x), 1)])
```

```
>>> assemble_partfrac_list(pfd)
-sqrt(2)/(2*(x + sqrt(2))) + sqrt(2)/(2*(x - sqrt(2)))
```

Dispersion of Polynomials

`diofant.polys.dispersion.`**`dispersionset`**(*p*, *q=None*, **gens*, ***args*)

Compute the *dispersion set* of two polynomials.

For two polynomials $f(x)$ and $g(x)$ with $\deg f > 0$ and $\deg g > 0$ the dispersion set $\operatorname{J}(f, g)$ is defined as:

$$\begin{aligned}\operatorname{J}(f, g) &:= \{a \in \mathbb{N}_0 | \gcd(f(x), g(x + a)) \neq 1\} \\ &= \{a \in \mathbb{N}_0 | \deg \gcd(f(x), g(x + a)) \geq 1\}\end{aligned}$$

For a single polynomial one defines $\operatorname{J}(f) := \operatorname{J}(f, f)$.

See also:

diofant.polys.dispersion.dispersion (page 662)

References

[R473] (page 1261), *[R474]* (page 1261), *[R475]* (page 1262), *[R476]* (page 1262)

Examples

Dispersion set and dispersion of a simple polynomial:

```
>>> fp = poly((x - 3)*(x + 3), x)
>>> sorted(dispersionset(fp))
[0, 6]
>>> dispersion(fp)
6
```

Note that the definition of the dispersion is not symmetric:

```
>>> fp = poly(x**4 - 3*x**2 + 1, x)
>>> gp = fp.shift(-3)
>>> sorted(dispersionset(fp, gp))
[2, 3, 4]
>>> dispersion(fp, gp)
4
>>> sorted(dispersionset(gp, fp))
[]
>>> dispersion(gp, fp)
-oo
```

Computing the dispersion also works over field extensions:

```
>>> fp = poly(x**2 + sqrt(5)*x - 1, x, domain='QQ<sqrt(5)>')
>>> gp = poly(x**2 + (2 + sqrt(5))*x + sqrt(5), x, domain='QQ<sqrt(5)>')
>>> sorted(dispersionset(fp, gp))
[2]
>>> sorted(dispersionset(gp, fp))
[1, 4]
```

We can even perform the computations for polynomials having symbolic coefficients:

```
>>> from diofant.abc import a
>>> fp = poly(4*x**4 + (4*a + 8)*x**3 + (a**2 + 6*a + 4)*x**2 + (a**2 + 2*a)*x, x)
>>> sorted(dispersionset(fp))
[0, 1]
```

diofant.polys.dispersion.**dispersion**(*p*, *q=None*, **gens*, ***args*)

Compute the *dispersion* of polynomials.

For two polynomials $f(x)$ and $g(x)$ with $\deg f > 0$ and $\deg g > 0$ the dispersion $\operatorname{dis}(f, g)$ is defined as:

$$\begin{aligned}\operatorname{dis}(f, g) &:= \max\{J(f, g) \cup \{0\}\} \\ &= \max\{\{a \in \mathbb{N} | \gcd(f(x), g(x+a)) \neq 1\} \cup \{0\}\}\end{aligned}$$

and for a single polynomial $\operatorname{dis}(f) := \operatorname{dis}(f, f)$. Note that we make the definition $\max\{\} := -\infty$.

See also:

diofant.polys.dispersion.dispersionset (page 663)

References

[R477] (page 1262), *[R478]* (page 1262), *[R479]* (page 1262), *[R480]* (page 1262)

Examples

Dispersion set and dispersion of a simple polynomial:

```
>>> fp = poly((x - 3)*(x + 3), x)
>>> sorted(dispersionset(fp))
[0, 6]
>>> dispersion(fp)
6
```

Note that the definition of the dispersion is not symmetric:

```
>>> fp = poly(x**4 - 3*x**2 + 1, x)
>>> gp = fp.shift(-3)
>>> sorted(dispersionset(fp, gp))
[2, 3, 4]
>>> dispersion(fp, gp)
4
>>> sorted(dispersionset(gp, fp))
[]
>>> dispersion(gp, fp)
-oo
```

The maximum of an empty set is defined to be $-\infty$ as seen in this example.

Computing the dispersion also works over field extensions:

```
>>> fp = poly(x**2 + sqrt(5)*x - 1, x, domain='QQ<sqrt(5)>')
>>> gp = poly(x**2 + (2 + sqrt(5))*x + sqrt(5), x, domain='QQ<sqrt(5)>')
>>> sorted(dispersionset(fp, gp))
```

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```
[2]
>>> sorted(dispersionset(gp, fp))
[1, 4]
```

We can even perform the computations for polynomials having symbolic coefficients:

```
>>> from diofant.abc import a
>>> fp = poly(4*x**4 + (4*a + 8)*x**3 + (a**2 + 6*a + 4)*x**2 + (a**2 + 2*a)*x, x)
>>> sorted(dispersionset(fp))
[0, 1]
```

3.11.4 Literature

The following is a non-comprehensive list of publications that were used as a theoretical foundation for implementing polynomials manipulation module.

3.12 Printing

See the *Printing* (page 10) section in Tutorial for introduction into printing.

This guide documents the printing system in Diofant and how it works internally.

3.12.1 Printer Class

Printing subsystem driver

Diofant's printing system works the following way: Any expression can be passed to a designated Printer who then is responsible to return an adequate representation of that expression.

The basic concept is the following:

1. Let the object print itself if it knows how.
2. Take the best fitting method defined in the printer.
3. As fall-back use the emptyPrinter method for the printer.

Some more information how the single concepts work and who should use which:

1. The object prints itself

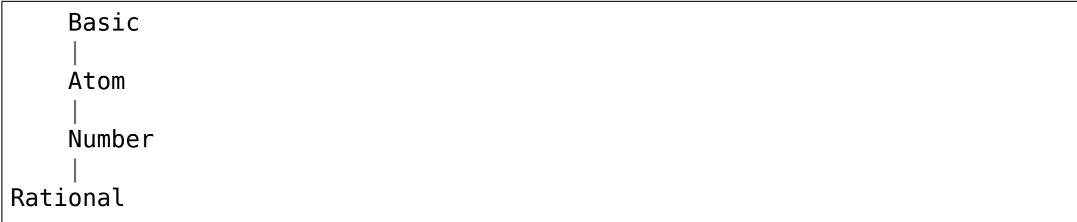
   This was the original way of doing printing in diofant. Every class had its own latex, mathml, str and repr methods, but it turned out that it is hard to produce a high quality printer, if all the methods are spread out that far. Therefore all printing code was combined into the different printers, which works great for built-in diofant objects, but not that good for user defined classes where it is inconvenient to patch the printers.

   Nevertheless, to get a fitting representation, the printers look for a specific method in every object, that will be called if it's available and is then responsible for the representation. The name of that method depends on the specific printer and is defined under Printer.printmethod.

2. Take the best fitting method defined in the printer.

The printer loops through expr classes (class + its bases), and tries to dispatch the work to _print_<EXPR_CLASS>

e.g., suppose we have the following class hierarchy:



then, for expr=Rational(…), in order to dispatch, we will try calling printer methods as shown in the figure below:

```
p._print(expr)
|
|-- p._print_Rational(expr)
|
|-- p._print_Number(expr)
|
|-- p._print_Atom(expr)
|
`-- p._print_Basic(expr)
```

if ._print_Rational method exists in the printer, then it is called, and the result is returned back.

otherwise, we proceed with trying Rational bases in the inheritance order.

3. As fall-back use the emptyPrinter method for the printer.

   As fall-back self.emptyPrinter will be called with the expression. If not defined in the Printer subclass this will be the same as str(expr).

The main class responsible for printing is `Printer` (see also its source code):

class diofant.printing.printer.**Printer**(*settings=None*)
Generic printer

Its job is to provide infrastructure for implementing new printers easily.

Basically, if you want to implement a printer, all you have to do is:

1. Subclass Printer.
2. Define Printer.printmethod in your subclass. If a object has a method with that name, this method will be used for printing.
3. In your subclass, define _print_<CLASS> methods

   For each class you want to provide printing to, define an appropriate method how to do it. For example if you want a class FOO to be printed in its own way, define _print_FOO:

```
def _print_FOO(self, e):
    ...
```

this should return how FOO instance e is printed

Also, if BAR is a subclass of FOO, `_print_FOO(bar)` will be called for instance of BAR, if no `_print_BAR` is provided. Thus, usually, we don't need to provide printing routines

for every class we want to support – only generic routine has to be provided for a set of classes.

A good example for this are functions - for example PrettyPrinter only defines _print_Function, and there is no _print_sin, _print_tan, etc...

On the other hand, a good printer will probably have to define separate routines for Symbol, Atom, Number, Integral, Limit, etc...

4. If convenient, override self.emptyPrinter

   This callable will be called to obtain printing result as a last resort, that is when no appropriate print method was found for an expression.

Examples

Here we will overload StrPrinter.

```
>>> from diofant.printing.str import StrPrinter
```

```
>>> class CustomStrPrinter(StrPrinter):
...     def _print_Derivative(self, expr):
...         return str(expr.args[0].func) + "'"*len(expr.args[1:])
>>> def mystr(e):
...     return CustomStrPrinter().doprint(e)
>>> t = Symbol('t')
>>> x = Function('x')(t)
>>> print(mystr(x.diff(t, 2)))
x''
```

printmethod = None

_print(*expr*, **args*, ***kwargs*)
Internal dispatcher

Tries the following concepts to print an expression:

1. Let the object print itself if it knows how.
2. Take the best fitting method defined in the printer.
3. As fall-back use the emptyPrinter method for the printer.

doprint(*expr*)
Returns printer's representation for expr (as a string)

classmethod set_global_settings(***settings*)
Set system-wide printing settings.

3.12.2 PrettyPrinter Class

The pretty printing subsystem is implemented in diofant.printing.pretty.pretty by the PrettyPrinter class deriving from Printer. It relies on the modules diofant.printing.pretty.stringPict, and diofant.printing.pretty.pretty_symbology for rendering nice-looking formulas.

The module stringPict provides a base class stringPict and a derived class prettyForm that ease the creation and manipulation of formulas that span across multiple lines.

The module `pretty_symbology` provides primitives to construct 2D shapes (hline, vline, etc) together with a technique to use unicode automatically when possible.

class `diofant.printing.pretty.pretty.`**PrettyPrinter**(*settings=None*)
Printer, which converts an expression into 2D ASCII-art figure.

doprint(*expr*)
Returns printer's representation for expr (as a string)

`diofant.printing.pretty.pretty.`**pprint**(*expr*, ***settings*)

`diofant.printing.pretty.pretty.`**pretty**(*expr*, ***settings*)
Returns a string containing the prettified form of expr.

For information on keyword arguments see pretty_print function.

`diofant.printing.pretty.pretty.`**pretty_print**(*expr*, ***settings*)
Prints expr in pretty form.

pprint is just a shortcut for this function.

Parameters **expr** : expression

the expression to print

wrap_line : bool, optional

line wrapping enabled/disabled, defaults to True

num_columns : int or None, optional

number of columns before line breaking (default to None which reads the terminal width), useful when using Diofant without terminal.

use_unicode : bool or None, optional

use unicode characters, such as the Greek letter pi instead of the string pi.

full_prec : bool or string, optional

use full precision. Default to "auto"

order : bool or string, optional

set to 'none' for long expressions if slow; default is None

3.12.3 CCodePrinter

This class implements C code printing (i.e. it converts Python expressions to strings of C code).

Usage:

```
>>> print_ccode(sin(x)**2 + cos(x)**2)
pow(sin(x), 2) + pow(cos(x), 2)
>>> print_ccode(2*x + cos(x), assign_to="result")
result = 2*x + cos(x);
>>> print_ccode(Abs(x**2))
fabs(pow(x, 2))
```

`diofant.printing.ccode.`**known_functions = {'Abs': [(<function <lambda> at 0x7fc460cc66a8>,**

class diofant.printing.ccode.**CCodePrinter**(*settings={}*)
A printer to convert python expressions to strings of c code

printmethod = '_ccode'

indent_code(*code*)
Accepts a string of code or a list of code lines

diofant.printing.ccode.**ccode**(*expr*, *assign_to=None*, ***settings*)
Converts an expr to a string of c code

Parameters expr : Expr

A diofant expression to be converted.

assign_to : optional

When given, the argument is used as the name of the variable to which the expression is assigned. Can be a string, `Symbol`, `MatrixSymbol`, or `Indexed` type. This is helpful in case of line-wrapping, or for expressions that generate multi-line statements.

precision : integer, optional

The precision for numbers such as pi [default=15].

user_functions : dict, optional

A dictionary where the keys are string representations of either `FunctionClass` or `UndefinedFunction` instances and the values are their desired C string representations. Alternatively, the dictionary value can be a list of tuples i.e. [(argument_test, cfunction_string)] or [(argument_test, cfunction_formater)]. See below for examples.

dereference : iterable, optional

An iterable of symbols that should be dereferenced in the printed code expression. These would be values passed by address to the function. For example, if `dereference=[a]`, the resulting code would print `(*a)` instead of `a`.

human : bool, optional

If True, the result is a single string that may contain some constant declarations for the number symbols. If False, the same information is returned in a tuple of (symbols_to_declare, not_supported_functions, code_text). [default=True].

contract: bool, optional

If True, `Indexed` instances are assumed to obey tensor contraction rules and the corresponding nested loops over indices are generated. Setting contract=False will not generate loops, instead the user is responsible to provide values for the indices in the code. [default=True].

Examples

```
>>> x, tau = symbols("x, tau")
>>> ccode((2*tau)**Rational(7, 2))
```

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```
'8*sqrt(2)*pow(tau, 7.0L/2.0L)'
>>> ccode(sin(x), assign_to="s")
's = sin(x);'
```

Simple custom printing can be defined for certain types by passing a dictionary of {"type" : "function"} to the user_functions kwarg. Alternatively, the dictionary value can be a list of tuples i.e. [(argument_test, cfunction_string)].

```
>>> custom_functions = {
...   "ceiling": "CEIL",
...   "Abs": [(lambda x: not x.is_integer, "fabs"),
...           (lambda x: x.is_integer, "ABS")],
...   "func": "f"
... }
>>> func = Function('func')
>>> ccode(func(Abs(x) + ceiling(x)), user_functions=custom_functions)
'f(fabs(x) + CEIL(x))'
```

or if the C-function takes a subset of the original arguments:

```
>>> ccode(2**x + 3**x, user_functions={'Pow': [
...   (lambda b, e: b == 2, lambda b, e: 'exp2(%s)' % e),
...   (lambda b, e: b != 2, 'pow')]})
'exp2(x) + pow(3, x)'
```

`Piecewise` expressions are converted into conditionals. If an `assign_to` variable is provided an if statement is created, otherwise the ternary operator is used. Note that if the `Piecewise` lacks a default term, represented by `(expr, True)` then an error will be thrown. This is to prevent generating an expression that may not evaluate to anything.

```
>>> expr = Piecewise((x + 1, x > 0), (x, True))
>>> print(ccode(expr, tau))
if (x > 0) {
tau = x + 1;
}
else {
tau = x;
}
```

Support for loops is provided through `Indexed` types. With `contract=True` these expressions will be turned into loops, whereas `contract=False` will just print the assignment expression that should be looped over:

```
>>> len_y = 5
>>> y = IndexedBase('y', shape=[len_y])
>>> t = IndexedBase('t', shape=[len_y])
>>> Dy = IndexedBase('Dy', shape=[len_y - 1])
>>> i = Idx('i', len_y-1)
>>> e = Eq(Dy[i], (y[i+1]-y[i])/(t[i+1]-t[i]))
>>> ccode(e.rhs, assign_to=e.lhs, contract=False)
'Dy[i] = (y[i + 1] - y[i])/(t[i + 1] - t[i]);'
```

Matrices are also supported, but a `MatrixSymbol` of the same dimensions must be provided to `assign_to`. Note that any expression that can be generated normally can also exist inside a Matrix:

```
>>> mat = Matrix([x**2, Piecewise((x + 1, x > 0), (x, True)), sin(x)])
>>> A = MatrixSymbol('A', 3, 1)
>>> print(ccode(mat, A))
A[0] = pow(x, 2);
if (x > 0) {
   A[1] = x + 1;
}
else {
   A[1] = x;
}
A[2] = sin(x);
```

diofant.printing.ccode.**print_ccode**(*expr*, ***settings*)
Prints C representation of the given expression.

3.12.4 Fortran Printing

The `fcode` function translates a diofant expression into Fortran code. The main purpose is to take away the burden of manually translating long mathematical expressions. Therefore the resulting expression should also require no (or very little) manual tweaking to make it compilable. The optional arguments of `fcode` can be used to fine-tune the behavior of `fcode` in such a way that manual changes in the result are no longer needed.

diofant.printing.fcode.**fcode**(*expr*, *assign_to=None*, ***settings*)
Converts an expr to a string of fortran code

Parameters expr : Expr

A diofant expression to be converted.

assign_to : optional

When given, the argument is used as the name of the variable to which the expression is assigned. Can be a string, `Symbol`, `MatrixSymbol`, or `Indexed` type. This is helpful in case of line-wrapping, or for expressions that generate multi-line statements.

precision : integer, optional

The precision for numbers such as pi [default=15].

user_functions : dict, optional

A dictionary where keys are `FunctionClass` instances and values are their string representations. Alternatively, the dictionary value can be a list of tuples i.e. [(argument_test, cfunction_string)]. See below for examples.

human : bool, optional

If True, the result is a single string that may contain some constant declarations for the number symbols. If False, the same information is returned in a tuple of (symbols_to_declare, not_supported_functions, code_text). [default=True].

contract: bool, optional

If True, `Indexed` instances are assumed to obey tensor contraction rules and the corresponding nested loops over indices are generated. Setting contract=False will not generate loops, instead the

user is responsible to provide values for the indices in the code. [default=True].

source_format : optional

The source format can be either 'fixed' or 'free'. [default='fixed']

standard : integer, optional

The Fortran standard to be followed. This is specified as an integer. Acceptable standards are 66, 77, 90, 95, 2003, and 2008. Default is 77. Note that currently the only distinction internally is between standards before 95, and those 95 and after. This may change later as more features are added.

Examples

```
>>> x, tau = symbols("x, tau")
>>> fcode((2*tau)**Rational(7, 2))
'      8*sqrt(2.0d0)*tau**(7.0d0/2.0d0)'
>>> fcode(sin(x), assign_to="s")
'      s = sin(x)'
```

Custom printing can be defined for certain types by passing a dictionary of "type" : "function" to the `user_functions` kwarg. Alternatively, the dictionary value can be a list of tuples i.e. [(argument_test, cfunction_string)].

```
>>> custom_functions = {
...   "ceiling": "CEIL",
...   "floor": [(lambda x: not x.is_integer, "FLOOR1"),
...             (lambda x: x.is_integer, "FLOOR2")]
... }
>>> fcode(floor(x) + ceiling(x), user_functions=custom_functions)
'      CEIL(x) + FLOOR1(x)'
```

`Piecewise` expressions are converted into conditionals. If an `assign_to` variable is provided an if statement is created, otherwise the ternary operator is used. Note that if the `Piecewise` lacks a default term, represented by `(expr, True)` then an error will be thrown. This is to prevent generating an expression that may not evaluate to anything.

```
>>> expr = Piecewise((x + 1, x > 0), (x, True))
>>> print(fcode(expr, tau))
      if (x > 0) then
         tau = x + 1
      else
         tau = x
      end if
```

Support for loops is provided through `Indexed` types. With `contract=True` these expressions will be turned into loops, whereas `contract=False` will just print the assignment expression that should be looped over:

```
>>> len_y = 5
>>> y = IndexedBase('y', shape=[len_y])
>>> t = IndexedBase('t', shape=[len_y])
>>> Dy = IndexedBase('Dy', shape=[len_y - 1])
```

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```
>>> i = Idx('i', len_y-1)
>>> e = Eq(Dy[i], (y[i+1]-y[i])/(t[i+1]-t[i]))
>>> fcode(e.rhs, assign_to=e.lhs, contract=False)
'      Dy(i) = (y(i + 1) - y(i))/(t(i + 1) - t(i))'
```

Matrices are also supported, but a `MatrixSymbol` of the same dimensions must be provided to `assign_to`. Note that any expression that can be generated normally can also exist inside a Matrix:

```
>>> mat = Matrix([x**2, Piecewise((x + 1, x > 0), (x, True)), sin(x)])
>>> A = MatrixSymbol('A', 3, 1)
>>> print(fcode(mat, A))
      A(1, 1) = x**2
         if (x > 0) then
      A(2, 1) = x + 1
         else
      A(2, 1) = x
         end if
      A(3, 1) = sin(x)
```

class diofant.printing.fcode.**FCodePrinter**(*settings={}*)

A printer to convert diofant expressions to strings of Fortran code

printmethod = '_fcode'

indent_code(*code*)

Accepts a string of code or a list of code lines

Two basic examples:

```
>>> fcode(sqrt(1-x**2))
'      sqrt(-x**2 + 1)'
>>> fcode((3 + 4*I)/(1 - conjugate(x)))
'      (cmplx(3,4))/(-conjg(x) + 1)'
```

An example where line wrapping is required:

```
>>> expr = sqrt(1 - x**2).series(x, n=20).removeO()
>>> print(fcode(expr))
      -715.0d0/65536.0d0*x**18 - 429.0d0/32768.0d0*x**16 - 33.0d0/
     @ 2048.0d0*x**14 - 21.0d0/1024.0d0*x**12 - 7.0d0/256.0d0*x**10 -
     @ 5.0d0/128.0d0*x**8 - 1.0d0/16.0d0*x**6 - 1.0d0/8.0d0*x**4 - 1.0d0
     @ /2.0d0*x**2 + 1
```

In case of line wrapping, it is handy to include the assignment so that lines are wrapped properly when the assignment part is added.

```
>>> print(fcode(expr, assign_to="var"))
      var = -715.0d0/65536.0d0*x**18 - 429.0d0/32768.0d0*x**16 - 33.0d0/
     @ 2048.0d0*x**14 - 21.0d0/1024.0d0*x**12 - 7.0d0/256.0d0*x**10 -
     @ 5.0d0/128.0d0*x**8 - 1.0d0/16.0d0*x**6 - 1.0d0/8.0d0*x**4 - 1.0d0
     @ /2.0d0*x**2 + 1
```

For piecewise functions, the `assign_to` option is mandatory:

```
>>> print(fcode(Piecewise((x, x<1), (x**2, True)), assign_to="var"))
      if (x < 1) then
```

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```
      var = x
    else
      var = x**2
    end if
```

Note that by default only top-level piecewise functions are supported due to the lack of a conditional operator in Fortran 77. Inline conditionals can be supported using the `merge` function introduced in Fortran 95 by setting of the kwarg `standard=95`:

```
>>> print(fcode(Piecewise((x, x<1), (x**2, True)), standard=95))
      merge(x, x**2, x < 1)
```

Loops are generated if there are Indexed objects in the expression. This also requires use of the assign_to option.

```
>>> A, B = map(IndexedBase, ['A', 'B'])
>>> m = Symbol('m', integer=True)
>>> i = Idx('i', m)
>>> print(fcode(2*B[i], assign_to=A[i]))
    do i = 1, m
        A(i) = 2*B(i)
    end do
```

Repeated indices in an expression with Indexed objects are interpreted as summation. For instance, code for the trace of a matrix can be generated with

```
>>> print(fcode(A[i, i], assign_to=x))
      x = 0
      do i = 1, m
          x = x + A(i, i)
      end do
```

By default, number symbols such as `pi` and `E` are detected and defined as Fortran parameters. The precision of the constants can be tuned with the precision argument. Parameter definitions are easily avoided using the `N` function.

```
>>> print(fcode(x - pi**2 - E))
      parameter (E = 2.71828182845905d0)
      parameter (pi = 3.14159265358979d0)
      x - pi**2 - E
>>> print(fcode(x - pi**2 - E, precision=25))
      parameter (E = 2.718281828459045235360287d0)
      parameter (pi = 3.141592653589793238462643d0)
      x - pi**2 - E
>>> print(fcode(N(x - pi**2, 25, strict=False)))
      x - 9.869604401089358618834491d0
```

When some functions are not part of the Fortran standard, it might be desirable to introduce the names of user-defined functions in the Fortran expression.

```
>>> print(fcode(1 - gamma(x)**2, user_functions={'gamma': 'mygamma'}))
      -mygamma(x)**2 + 1
```

However, when the user_functions argument is not provided, `fcode` attempts to use a reasonable default and adds a comment to inform the user of the issue.

```
>>> print(fcode(1 - gamma(x)**2))
C     Not supported in Fortran:
C     gamma
      -gamma(x)**2 + 1
```

By default the output is human readable code, ready for copy and paste. With the option `human=False`, the return value is suitable for post-processing with source code generators that write routines with multiple instructions. The return value is a three-tuple containing: (i) a set of number symbols that must be defined as 'Fortran parameters', (ii) a list functions that can not be translated in pure Fortran and (iii) a string of Fortran code. A few examples:

```
>>> fcode(1 - gamma(x)**2, human=False)
(set(), {gamma(x)}, '      -gamma(x)**2 + 1')
>>> fcode(1 - sin(x)**2, human=False)
(set(), set(), '      -sin(x)**2 + 1')
>>> fcode(x - pi**2, human=False)
({(pi, '3.14159265358979d0')}, set(), '      x - pi**2')
```

3.12.5 Mathematica code printing

`diofant.printing.mathematica.`**`known_functions = {'acos': [(<function <lambda> at 0x7fc460d0`**

class `diofant.printing.mathematica.`**`MCodePrinter`**(*settings={}*)

A printer to convert python expressions to strings of the Wolfram's Mathematica code

`printmethod = '_mcode'`

`doprint`(*expr*)

Returns printer's representation for expr (as a string)

`diofant.printing.mathematica.`**`mathematica_code`**(*expr*, ***settings*)

Converts an expr to a string of the Wolfram Mathematica code

Examples

```
>>> mathematica_code(sin(x).series(x).removeO())
'(1/120)*x^5 - 1/6*x^3 + x'
```

3.12.6 LambdaPrinter

This classes implements printing to strings that can be used by the *`diofant.utilities.lambdify.lambdify()`* (page 1002) function.

class `diofant.printing.lambdarepr.`**`LambdaPrinter`**(*settings=None*)

This printer converts expressions into strings that can be used by lambdify.

`printmethod = '_diofantstr'`

`diofant.printing.lambdarepr.`**`lambdarepr`**(*expr*, ***settings*)

Returns a string usable for lambdifying.

3.12.7 LatexPrinter

This class implements LaTeX printing. See `diofant.printing.latex`.

`diofant.printing.latex.`**`accepted_latex_functions = ['arcsin', 'arccos', 'arctan', 'sin', 'c`**

list() -> new empty list list(iterable) -> new list initialized from iterable's items

class `diofant.printing.latex.`**`LatexPrinter`**(*settings=None*)

`printmethod = '_latex'`

`doprint`(*expr*)

Returns printer's representation for expr (as a string)

`diofant.printing.latex.`**`latex`**(*expr*, ***settings*)

Convert the given expression to LaTeX representation.

```
>>> from diofant.abc import mu, r, tau
```

```
>>> print(latex((2*tau)**Rational(7, 2)))
8 \sqrt{2} \tau^{\frac{7}{2}}
```

Not using a print statement for printing, results in double backslashes for latex commands since that's the way Python escapes backslashes in strings.

```
>>> latex((2*tau)**Rational(7, 2))
'8 \\sqrt{2} \\tau^{\\frac{7}{2}}'
```

order: Any of the supported monomial orderings (currently "lex", "grlex", or "grevlex") and "none". This parameter does nothing for Mul objects. For very large expressions, set the 'order' keyword to 'none' if speed is a concern.

mode: Specifies how the generated code will be delimited. 'mode' can be one of 'plain', 'inline', 'equation' or 'equation*'. If 'mode' is set to 'plain', then the resulting code will not be delimited at all (this is the default). If 'mode' is set to 'inline' then inline LaTeX `$ $` will be used. If 'mode' is set to 'equation' or 'equation*', the resulting code will be enclosed in the 'equation' or 'equation*' environment (remember to import 'amsmath' for 'equation*'), unless the 'itex' option is set. In the latter case, the `$$ $$` syntax is used.

```
>>> print(latex((2*mu)**Rational(7, 2), mode='plain'))
8 \sqrt{2} \mu^{\frac{7}{2}}
```

```
>>> print(latex((2*tau)**Rational(7, 2), mode='inline'))
$8 \sqrt{2} \tau^{7 / 2}$
```

```
>>> print(latex((2*mu)**Rational(7, 2), mode='equation*'))
\begin{equation*}8 \sqrt{2} \mu^{\frac{7}{2}}\end{equation*}
```

```
>>> print(latex((2*mu)**Rational(7, 2), mode='equation'))
\begin{equation}8 \sqrt{2} \mu^{\frac{7}{2}}\end{equation}
```

itex: Specifies if itex-specific syntax is used, including emitting `$$ $$`.

```
>>> print(latex((2*mu)**Rational(7, 2), mode='equation', itex=True))
$$8 \sqrt{2} \mu^{\frac{7}{2}}$$
```

fold_frac_powers: Emit "^{p/q}" instead of "^{frac{p}{q}}" for fractional powers.

```
>>> print(latex((2*tau)**Rational(7, 2), fold_frac_powers=True))
8 \sqrt{2} \tau^{7/2}
```

fold_func_brackets: Fold function brackets where applicable.

```
>>> print(latex((2*tau)**sin(Rational(7, 2))))
\left(2 \tau\right)^{\sin{\left (\frac{7}{2} \right )}}
>>> print(latex((2*tau)**sin(Rational(7, 2)), fold_func_brackets = True))
\left(2 \tau\right)^{\sin {\frac{7}{2}}}
```

fold_short_frac: Emit "p / q" instead of "frac{p}{q}" when the denominator is simple enough (at most two terms and no powers). The default value is $True$ for inline mode, False otherwise.

```
>>> print(latex(3*x**2/y))
\frac{3 x^{2}}{y}
>>> print(latex(3*x**2/y, fold_short_frac=True))
3 x^{2} / y
```

long_frac_ratio: The allowed ratio of the width of the numerator to the width of the denominator before we start breaking off long fractions. The default value is 2.

```
>>> print(latex(Integral(r, r)/2/pi, long_frac_ratio=2))
\frac{\int r\, dr}{2 \pi}
>>> print(latex(Integral(r, r)/2/pi, long_frac_ratio=0))
\frac{1}{2 \pi} \int r\, dr
```

mul_symbol: The symbol to use for multiplication. Can be one of None, "ldot", "dot", or "times".

```
>>> print(latex((2*tau)**sin(Rational(7, 2)), mul_symbol="times"))
\left(2 \times \tau\right)^{\sin{\left (\frac{7}{2} \right )}}
```

inv_trig_style: How inverse trig functions should be displayed. Can be one of "abbreviated", "full", or "power". Defaults to "abbreviated".

```
>>> print(latex(asin(Rational(7, 2))))
\operatorname{asin}{\left (\frac{7}{2} \right )}
>>> print(latex(asin(Rational(7, 2)), inv_trig_style="full"))
\arcsin{\left (\frac{7}{2} \right )}
>>> print(latex(asin(Rational(7, 2)), inv_trig_style="power"))
\sin^{-1}{\left (\frac{7}{2} \right )}
```

mat_str: Which matrix environment string to emit. "smallmatrix", "matrix", "array", etc. Defaults to "smallmatrix" for inline mode, "matrix" for matrices of no more than 10 columns, and "array" otherwise.

```
>>> print(latex(Matrix(2, 1, [x, y])))
\left[\begin{matrix}x\\y\end{matrix}\right]
```

```
>>> print(latex(Matrix(2, 1, [x, y]), mat_str="array"))
\left[\begin{array}{c}x\\y\end{array}\right]
```

mat_delim: The delimiter to wrap around matrices. Can be one of "[", "(", or the empty string. Defaults to "[".

```
>>> print(latex(Matrix(2, 1, [x, y]), mat_delim="("))
\left(\begin{matrix}x\\y\end{matrix}\right)
```

symbol_names: Dictionary of symbols and the custom strings they should be emitted as.

```
>>> print(latex(x**2, symbol_names={x: 'x_i'}))
x_i^{2}
```

`latex` also supports the builtin container types list, tuple, and dictionary.

```
>>> print(latex([2/x, y], mode='inline'))
$\left [ 2 / x, \quad y\right ]$
```

3.12.8 MathMLPrinter

This class is responsible for MathML printing. See `diofant.printing.mathml`.

More info on mathml content: http://www.w3.org/TR/MathML2/chapter4.html

class diofant.printing.mathml.**MathMLPrinter**(*settings=None*)

Prints an expression to the MathML markup language

Whenever possible tries to use Content markup and not Presentation markup.

References

[R481] (page 1263)

printmethod = '_mathml'

doprint(*expr*)

Prints the expression as MathML.

mathml_tag(*e*)

Returns the MathML tag for an expression.

diofant.printing.mathml.**mathml**(*expr*, ***settings*)

Returns the MathML representation of expr

diofant.printing.mathml.**print_mathml**(*expr*, ***settings*)

Prints a pretty representation of the MathML code for expr

Examples

```
>>> print_mathml(x+1)
<apply>
    <plus/>
    <ci>x</ci>
    <cn>1</cn>
</apply>
```

3.12.9 PythonPrinter

This class implements Python printing. Usage:

```
>>> print_python(5*x**3 + sin(x))
x = Symbol('x')
e = 5*x**3 + sin(x)
```

3.12.10 ReprPrinter

This printer generates executable code. This code satisfies the identity `eval(srepr(expr)) == expr`.

class diofant.printing.repr.**ReprPrinter**(*settings=None*)

printmethod = '_diofantrepr'

emptyPrinter(*expr*)
The fallback printer.

reprify(*args, sep*)
Prints each item in $args$ and joins them with sep.

diofant.printing.repr.**srepr**(*expr, **settings*)
return expr in repr form

3.12.11 StrPrinter

This module generates readable representations of Diofant expressions.

class diofant.printing.str.**StrPrinter**(*settings=None*)

printmethod = '_diofantstr'

emptyPrinter(*expr*)
str(object='') -> str str(bytes_or_buffer[, encoding[, errors]]) -> str

Create a new string object from the given object. If encoding or errors is specified, then the object must expose a data buffer that will be decoded using the given encoding and error handler. Otherwise, returns the result of object.__str__() (if defined) or repr(object). encoding defaults to sys.getdefaultencoding(). errors defaults to 'strict'.

diofant.printing.str.**sstr**(*expr, **settings*)
Returns the expression as a string.

For large expressions where speed is a concern, use the setting order='none'.

Examples

```
>>> a, b = symbols('a b')
>>> sstr(Eq(a + b, 0))
'Eq(a + b, 0)'
```

diofant.printing.str.**sstrrepr**(*expr*, ***settings*)
return expr in mixed str/repr form

i.e. strings are returned in repr form with quotes, and everything else is returned in str form.

This function could be useful for hooking into sys.displayhook

3.12.12 Tree Printing

The functions in this module create a representation of an expression as a tree.

diofant.printing.tree.**pprint_nodes**(*subtrees*)
Prettyprints systems of nodes.

Examples

```
>>> print(pprint_nodes(["a", "b1\nb2", "c"]))
+-a
+-b1
| b2
+-c
```

diofant.printing.tree.**print_node**(*node*)
Returns information about the "node".

This includes class name, string representation and assumptions.

diofant.printing.tree.**tree**(*node*)
Returns a tree representation of "node" as a string.

It uses print_node() together with pprint_nodes() on node.args recursively.

See also: print_tree()

diofant.printing.tree.**print_tree**(*node*)
Prints a tree representation of "node".

Examples

```
>>> x = Symbol('x', odd=True)
>>> y = Symbol('y', even=True)
>>> print_tree(y**x)
Pow: y**x
commutative: True
+-Symbol: y
| algebraic: True
| commutative: True
| complex: True
| even: True
| extended_real: True
| finite: True
| hermitian: True
| infinite: False
| integer: True
```

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```
| irrational: False
| noninteger: False
| odd: False
| rational: True
| real: True
| transcendental: False
+-Symbol: x
  algebraic: True
  commutative: True
  complex: True
  even: False
  extended_real: True
  finite: True
  hermitian: True
  imaginary: False
  infinite: False
  integer: True
  irrational: False
  noninteger: False
  nonzero: True
  odd: True
  rational: True
  real: True
  transcendental: False
  zero: False
```

See also: tree()

3.12.13 Implementation - Helper Classes/Functions

diofant.printing.conventions.**split_super_sub**(*text*)

Split a symbol name into a name, superscripts and subscripts

The first part of the symbol name is considered to be its actual 'name', followed by super- and subscripts. Each superscript is preceded with a "^" character or by "__". Each subscript is preceded by a "_" character. The three return values are the actual name, a list with superscripts and a list with subscripts.

```
>>> split_super_sub('a_x^1')
('a', ['1'], ['x'])
>>> split_super_sub('var_sub1__sup_sub2')
('var', ['sup'], ['sub1', 'sub2'])
```

CodePrinter

This class is a base class for other classes that implement code-printing functionality, and additionally lists a number of functions that cannot be easily translated to C or Fortran.

class diofant.printing.codeprinter.**CodePrinter**(*settings=None*)

The base class for code-printing subclasses.

printmethod = '_diofantstr'

exception diofant.printing.codeprinter.**AssignmentError**

Raised if an assignment variable for a loop is missing.

Precedence

diofant.printing.precedence.**PRECEDENCE = {'Add': 40, 'And': 30, 'Atom': 1000, 'Func': 70,**
Default precedence values for some basic types.

diofant.printing.precedence.**PRECEDENCE_VALUES = {'Add': 40, 'And': 30, 'Equivalent': 10, '**
A dictionary assigning precedence values to certain classes. These values are treated like they were inherited, so not every single class has to be named here.

diofant.printing.precedence.**PRECEDENCE_FUNCTIONS = {'Float': <function precedence_Float at**
Sometimes it's not enough to assign a fixed precedence value to a class. Then a function can be inserted in this dictionary that takes an instance of this class as argument and returns the appropriate precedence value.

diofant.printing.precedence.**precedence**(*item*)
Returns the precedence of a given object.

3.12.14 Pretty-Printing Implementation Helpers

diofant.printing.pretty.pretty_symbology.**U**(*name*)
unicode character by name or None if not found

diofant.printing.pretty.pretty_symbology.**pretty_use_unicode**(*flag=None*)
Set whether pretty-printer should use unicode by default

The following two functions return the Unicode version of the inputted Greek letter.

diofant.printing.pretty.pretty_symbology.**g**(*l*)

diofant.printing.pretty.pretty_symbology.**G**(*l*)

diofant.printing.pretty.pretty_symbology.**greek_letters = ['alpha', 'beta', 'gamma', 'delta**
list() -> new empty list list(iterable) -> new list initialized from iterable's items

diofant.printing.pretty.pretty_symbology.**digit_2txt = {'0': 'ZERO', '1': 'ONE', '2': 'TWO'**

diofant.printing.pretty.pretty_symbology.**symb_2txt = {'(': 'LEFT PARENTHESIS', ')': 'RIGHT**

The following functions return the Unicode subscript/superscript version of the character.

diofant.printing.pretty.pretty_symbology.**sub = {'(': '₍', ')': '₎', '+': '₊', '-': '₋', '0**

diofant.printing.pretty.pretty_symbology.**sup = {'(': '⁽', ')': '⁾', '+': '⁺', '-': '⁻', '0**

The following functions return Unicode vertical objects.

diofant.printing.pretty.pretty_symbology.**xobj**(*symb, length*)
Construct spatial object of given length.

return: [] of equal-length strings

diofant.printing.pretty.pretty_symbology.**vobj**(*symb, height*)
Construct vertical object of a given height

see: xobj

diofant.printing.pretty.pretty_symbology.**hobj**(*symb, width*)
Construct horizontal object of a given width

see: xobj

The following constants are for rendering roots and fractions.

diofant.printing.pretty.pretty_symbology.**root = {2: '√', 3: '∛', 4: '∜'}**

diofant.printing.pretty.pretty_symbology.**VF**(*txt*)

diofant.printing.pretty.pretty_symbology.**frac = {(1, 2): '½', (1, 3): '⅓', (1, 4): '¼', (1**

The following constants/functions are for rendering atoms and symbols.

diofant.printing.pretty.pretty_symbology.**xsym**(*sym*)
: get symbology for a 'character'

diofant.printing.pretty.pretty_symbology.**atoms_table = {'EmptySet': '∅', 'Exp1': 'e', 'Ima**

diofant.printing.pretty.pretty_symbology.**pretty_atom**(*atom_name*, *default=None*)
: return pretty representation of an atom

diofant.printing.pretty.pretty_symbology.**pretty_symbol**(*symb_name*)
: return pretty representation of a symbol

diofant.printing.pretty.pretty_symbology.**annotated**(*letter*)
: Return a stylised drawing of the letter `letter`, together with information on how to put annotations (super- and subscripts to the left and to the right) on it.

See pretty.py functions _print_meijerg, _print_hyper on how to use this information.

Prettyprinter by Jurjen Bos. (I hate spammers: mail me at pietjepuk314 at the reverse of ku.oc.oohay). All objects have a method that create a "stringPict", that can be used in the str method for pretty printing.

Updates by Jason Gedge (email <my last name> at cs mun ca)

- terminal_string() method
- minor fixes and changes (mostly to prettyForm)

TODO:

- Allow left/center/right alignment options for above/below and top/center/bottom alignment options for left/right

class diofant.printing.pretty.stringpict.**stringPict**(*s*, *baseline=0*)
: An ASCII picture. The pictures are represented as a list of equal length strings.

above(**args*)
: Put pictures above this picture. Returns string, baseline arguments for stringPict. Baseline is baseline of bottom picture.

below(**args*)
: Put pictures under this picture. Returns string, baseline arguments for stringPict. Baseline is baseline of top picture

Examples

```
>>> from diofant.printing.pretty.pretty_symbology import pretty_use_unicode
>>> f = pretty_use_unicode(flag=False)
>>> print(stringPict("x+3").below(
...       stringPict.LINE, '3')[0])
x+3
---
 3
```

height()
: The height of the picture in characters.

left(**args*)
Put pictures (left to right) at left. Returns string, baseline arguments for stringPict.

static next()
Put a string of stringPicts next to each other. Returns string, baseline arguments for stringPict.

parens(*left='(', right=')', ifascii_nougly=False*)
Put parentheses around self. Returns string, baseline arguments for stringPict.

left or right can be None or empty string which means 'no paren from that side'

render(**args, **kwargs*)
Return the string form of self.

Unless the argument line_break is set to False, it will break the expression in a form that can be printed on the terminal without being broken up.

right(**args*)
Put pictures next to this one. Returns string, baseline arguments for stringPict. (Multiline) strings are allowed, and are given a baseline of 0.

Examples

```
>>> from diofant.printing.pretty.pretty_symbology import pretty_use_unicode
>>> f = pretty_use_unicode(flag=False)
>>> print(stringPict("10").right(" + ", stringPict("1\r-\r2", 1))[0])
     1
10 + -
     2
```

static stack()
Put pictures on top of each other, from top to bottom. Returns string, baseline arguments for stringPict. The baseline is the baseline of the second picture. Everything is centered. Baseline is the baseline of the second picture. Strings are allowed. The special value stringPict.LINE is a row of '-' extended to the width.

terminal_width()
Return the terminal width if possible, otherwise return 0.

width()
The width of the picture in characters.

class diofant.printing.pretty.stringpict.**prettyForm**(*s, baseline=0, binding=0, unicode=None*)
Extension of the stringPict class that knows about basic math applications, optimizing double minus signs.

"Binding" is interpreted as follows:

```
ATOM this is an atom: never needs to be parenthesized
FUNC this is a function application: parenthesize if added (?)
DIV  this is a division: make wider division if divided
POW  this is a power: only parenthesize if exponent
MUL  this is a multiplication: parenthesize if powered
ADD  this is an addition: parenthesize if multiplied or powered
NEG  this is a negative number: optimize if added, parenthesize if
     multiplied or powered
```

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```
OPEN this is an open object: parenthesize if added, multiplied, or
     powered (example: Piecewise)
```

3.12.15 dotprint

diofant.printing.dot.**dotprint**(*expr, styles=[(<class 'diofant.core.basic.Basic'>, {'color': 'blue', 'shape': 'ellipse'}), (<class 'diofant.core.expr.Expr'>, {'color': 'black'})], atom=<function <lambda>>, maxdepth=None, repeat=True, labelfunc=<class 'str'>, **kwargs*)

DOT description of a Diofant expression tree

Options are

`styles`: Styles for different classes. The default is:

```
[(Basic, {'color': 'blue', 'shape': 'ellipse'}),
(Expr, {'color': 'black'})]``
```

`atom`: Function used to determine if an arg is an atom. The default is `lambda x: not isinstance(x, Basic)`. Another good choice is `lambda x: not x.args`.

`maxdepth`: The maximum depth. The default is None, meaning no limit.

`repeat`: Whether to different nodes for separate common subexpressions. The default is True. For example, for `x + x*y` with `repeat=True`, it will have two nodes for `x` and with `repeat=False`, it will have one (warning: even if it appears twice in the same object, like Pow(x, x), it will still only appear only once. Hence, with repeat=False, the number of arrows out of an object might not equal the number of args it has).

`labelfunc`: How to label leaf nodes. The default is `str`. Another good option is `repr`. For example with `str`, the leaf nodes of `x + 1` are labeled, `x` and `1`. With `repr`, they are labeled `Symbol('x')` and `Integer(1)`.

Additional keyword arguments are included as styles for the graph.

Examples

```
>>> print(dotprint(x + 2))
digraph{

# Graph style
"bgcolor"="transparent"
"ordering"="out"
"rankdir"="TD"

#########
# Nodes #
#########

"Add(Symbol('x'), Integer(2))_()" ["color"="black", "label"="Add", "shape"=
↪"ellipse"];
```

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```
"Integer(2)_(0,)" ["color"="black", "label"="2", "shape"="ellipse"];
"Symbol('x')_(1,)" ["color"="black", "label"="x", "shape"="ellipse"];

#########
# Edges #
#########

"Add(Symbol('x'), Integer(2))_()" -> "Integer(2)_(0,)";
"Add(Symbol('x'), Integer(2))_()" -> "Symbol('x')_(1,)";
}
```

3.13 Interactive

Helper module for setting up interactive Diofant sessions.

AST transformations, provided here, could be used in IPython to reduce boilerplate while interacting with Diofant due to the Python language syntax.

`diofant.interactive.printing.`**`init_printing`**(*no_global=False*, *pretty_print=None*, ***settings*)

Initializes pretty-printer depending on the environment.

Parameters **no_global** : boolean

If True, the settings become system wide; if False, use just for this console/session.

pretty_print : boolean or None

Enable pretty printer (turned on by default for IPython, but disabled for plain Python console).

****settings** : dict

A dictionary of default settings for printers.

Notes

This function runs automatically for wildcard imports (e.g. for `from diofant import *`) in interactive sessions.

Examples

```
>>> from diofant.abc import theta
>>> sqrt(5)
sqrt(5)
>>> init_printing(pretty_print=True, no_global=True)
>>> sqrt(5)
  ___
╲╱ 5
>>> theta
θ
>>> init_printing(pretty_print=True, use_unicode=False, no_global=True)
```

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```
>>> theta
theta
>>> init_printing(pretty_print=True, order='grevlex', no_global=True)
>>> y + x + y**2 + x**2
 2    2
x  + y  + x + y
```

class diofant.interactive.session.**AutomaticSymbols**
Add missing Symbol definitions automatically.

class diofant.interactive.session.**FloatRationalizer**
Wraps all floats in a call to Rational.

class diofant.interactive.session.**IntegerDivisionWrapper**
Wrap all int divisions in a call to Rational.

3.14 Plotting

3.14.1 Introduction

The plotting module allows you to make 2-dimensional and 3-dimensional plots. Presently the plots are rendered using `matplotlib` as a backend.

The plotting module has the following functions:

- plot: Plots 2D line plots.
- plot_parametric: Plots 2D parametric plots.
- plot_implicit: Plots 2D implicit and region plots.
- plot3d: Plots 3D plots of functions in two variables.
- plot3d_parametric_line: Plots 3D line plots, defined by a parameter.
- plot3d_parametric_surface: Plots 3D parametric surface plots.

The above functions are only for convenience and ease of use. It is possible to plot any plot by passing the corresponding `Series` class to `Plot` as argument.

3.14.2 Plot Class

class diofant.plotting.plot.**Plot**(**args*, ***kwargs*)
The central class of the plotting module.

For interactive work the function `plot` is better suited.

This class permits the plotting of diofant expressions using numerous backends (matplotlib, Google charts api, etc).

The figure can contain an arbitrary number of plots of diofant expressions, lists of coordinates of points, etc. Plot has a private attribute _series that contains all data series to be plotted (expressions for lines or surfaces, lists of points, etc (all subclasses of BaseSeries)). Those data series are instances of classes not imported by `from diofant import *`.

The customization of the figure is on two levels. Global options that concern the figure as a whole (eg title, xlabel, scale, etc) and per-data series options (eg name) and aesthetics (eg. color, point shape, line type, etc.).

The difference between options and aesthetics is that an aesthetic can be a function of the coordinates (or parameters in a parametric plot). The supported values for an aesthetic are: - None (the backend uses default values) - a constant - a function of one variable (the first coordinate or parameter) - a function of two variables (the first and second coordinate or parameters) - a function of three variables (only in nonparametric 3D plots) Their implementation depends on the backend so they may not work in some backends.

If the plot is parametric and the arity of the aesthetic function permits it the aesthetic is calculated over parameters and not over coordinates. If the arity does not permit calculation over parameters the calculation is done over coordinates.

Only cartesian coordinates are supported for the moment, but you can use the parametric plots to plot in polar, spherical and cylindrical coordinates.

The arguments for the constructor Plot must be subclasses of BaseSeries.

Any global option can be specified as a keyword argument.

The global options for a figure are:

- title : str
- xlabel : str
- ylabel : str
- legend : bool
- xscale : {'linear', 'log'}
- yscale : {'linear', 'log'}
- axis : bool
- axis_center : tuple of two floats or {'center', 'auto'}
- xlim : tuple of two floats
- ylim : tuple of two floats
- aspect_ratio : tuple of two floats or {'auto'}
- autoscale : bool
- margin : float in [0, 1]

The per data series options and aesthetics are: There are none in the base series. See below for options for subclasses.

Some data series support additional aesthetics or options:

ListSeries, LineOver1DRangeSeries, Parametric2DLineSeries, Parametric3DLineSeries support the following:

Aesthetics:

- line_color : function which returns a float.

options:

- label : str

- steps : bool
- integers_only : bool

SurfaceOver2DRangeSeries, ParametricSurfaceSeries support the following:

aesthetics:

- surface_color : function which returns a float.

append(*arg*)

Adds an element from a plot's series to an existing plot.

See also:

extend (page 746)

Examples

Consider two `Plot` objects, `p1` and `p2`. To add the second plot's first series object to the first, use the `append` method, like so:

```
>>> p1 = plot(x*x)
>>> p2 = plot(x)
>>> p1.append(p2[0])
>>> print(str(p1))
Plot object containing:
[0]: cartesian line: x**2 for x over (-10.0, 10.0)
[1]: cartesian line: x for x over (-10.0, 10.0)
```

extend(*arg*)

Adds all series from another plot.

Examples

Consider two `Plot` objects, `p1` and `p2`. To add the second plot to the first, use the `extend` method, like so:

```
>>> p1 = plot(x*x)
>>> p2 = plot(x)
>>> p1.extend(p2)
>>> print(str(p1))
Plot object containing:
[0]: cartesian line: x**2 for x over (-10.0, 10.0)
[1]: cartesian line: x for x over (-10.0, 10.0)
```

3.14.3 Plotting Function Reference

`diofant.plotting.plot.`**plot**(**args, **kwargs*)

Plots a function of a single variable and returns an instance of the `Plot` class (also, see the description of the `show` keyword argument below).

The plotting uses an adaptive algorithm which samples recursively to accurately plot the plot. The adaptive algorithm uses a random point near the midpoint of two points that has to be further sampled. Hence the same plots can appear slightly different.

Single Plot.

`plot(expr, range, **kwargs)`

If the range is not specified, then a default range of (-10, 10) is used.

Multiple plots with same range.

`plot(expr1, expr2, ..., range, **kwargs)`

If the range is not specified, then a default range of (-10, 10) is used.

Multiple plots with different ranges.

`plot((expr1, range), (expr2, range), ..., **kwargs)`

Range has to be specified for every expression.

Default range may change in the future if a more advanced default range detection algorithm is implemented.

Parameters ''expr'' : Expr

Expression representing the function of single variable

''range'': tuple

(x, 0, 5), A 3-tuple denoting the range of the free variable.

''show'': Boolean, optional

The default value is set to `True`. Set show to `False` and the function will not display the plot. The returned instance of the `Plot` class can then be used to save or display the plot by calling the `save()` and `show()` methods respectively.

Arguments for ''LineOver1DRangeSeries'' class:

''adaptive'': Boolean, optional

The default value is set to True. Set adaptive to False and specify `nb_of_points` if uniform sampling is required.

''depth'': int, optional

Recursion depth of the adaptive algorithm. A depth of value `n` samples a maximum of 2^n points.

''nb_of_points'': int, optional

Used when the `adaptive` is set to False. The function is uniformly sampled at `nb_of_points` number of points.

Aesthetics options:

''line_color'': float, optional

Specifies the color for the plot. See `Plot` to see how to set color for the plots.

If there are multiple plots, then the same series series are applied to

all the plots. If you want to set these options separately, you can index

the ''Plot'' object returned and set it.

Arguments for ''Plot'' class:

''title'' : str, optional

Title of the plot. It is set to the latex representation of the expression, if the plot has only one expression.

``xlabel`` : str, optional

Label for the x-axis.

``ylabel`` : str, optional

Label for the y-axis.

``xscale``: {'linear', 'log'}, optional

Sets the scaling of the x-axis.

``yscale``: {'linear', 'log'}, optional

Sets the scaling if the y-axis.

``axis_center``: tuple of two floats

denoting the coordinates of the center or {'center', 'auto'}

``xlim`` : tuple of two floats

denoting the x-axis limits.

``ylim`` : tuple of two floats

denoting the y-axis limits.

See also:

Plot (page 744), *diofant.plotting.plot.LineOver1DRangeSeries* (page 757)

Examples

Single Plot

```
>>> print(str(plot(x**2, (x, -5, 5))))
Plot object containing:
[0]: cartesian line: x**2 for x over (-5.0, 5.0)
```

Multiple plots with single range.

```
>>> print(str(plot(x, x**2, x**3, (x, -5, 5))))
Plot object containing:
[0]: cartesian line: x for x over (-5.0, 5.0)
[1]: cartesian line: x**2 for x over (-5.0, 5.0)
[2]: cartesian line: x**3 for x over (-5.0, 5.0)
```

Multiple plots with different ranges.

```
>>> print(str(plot((x**2, (x, -6, 6)), (x, (x, -5, 5)))))
Plot object containing:
[0]: cartesian line: x**2 for x over (-6.0, 6.0)
[1]: cartesian line: x for x over (-5.0, 5.0)
```

No adaptive sampling.

```
>>> print(str(plot(x**2, adaptive=False, nb_of_points=400)))
Plot object containing:
[0]: cartesian line: x**2 for x over (-10.0, 10.0)
```

`diofant.plotting.plot.`**`plot_parametric`**(**args*, ***kwargs*)

Plots a 2D parametric plot.

The plotting uses an adaptive algorithm which samples recursively to accurately plot the plot. The adaptive algorithm uses a random point near the midpoint of two points that has to be further sampled. Hence the same plots can appear slightly different.

Single plot.

`plot_parametric(expr_x, expr_y, range, **kwargs)`

If the range is not specified, then a default range of (-10, 10) is used.

Multiple plots with same range.

`plot_parametric((expr1_x, expr1_y), (expr2_x, expr2_y), range, **kwargs)`

If the range is not specified, then a default range of (-10, 10) is used.

Multiple plots with different ranges.

`plot_parametric((expr_x, expr_y, range), ..., **kwargs)`

Range has to be specified for every expression.

Default range may change in the future if a more advanced default range detection algorithm is implemented.

Parameters ‘‘expr_x‘‘ : Expr

Expression representing the function along x.

‘‘expr_y‘‘ : Expr

Expression representing the function along y.

‘‘range‘‘: tuple

(u, 0, 5), A 3-tuple denoting the range of the parameter variable.

Arguments for ‘‘Parametric2DLineSeries‘‘ class:

‘‘adaptive‘‘: Boolean, optional

The default value is set to True. Set adaptive to False and specify `nb_of_points` if uniform sampling is required.

‘‘depth‘‘: int, optional

Recursion depth of the adaptive algorithm. A depth of value `n` samples a maximum of 2^n points.

‘‘nb_of_points‘‘: int, optional

Used when the `adaptive` is set to False. The function is uniformly sampled at `nb_of_points` number of points.

Aesthetics:

‘‘line_color‘‘: function which returns a float

Specifies the color for the plot. See `diofant.plotting.Plot` for more details.

If there are multiple plots, then the same Series arguments are applied to

all the plots. If you want to set these options separately, you can index

the returned ''Plot'' object and set it.

Arguments for ''Plot'' class:

''xlabel'' : str

Label for the x-axis.

''ylabel'' : str

Label for the y-axis.

''xscale'': {'linear', 'log'}

Sets the scaling of the x-axis.

''yscale'': {'linear', 'log'}

Sets the scaling if the y-axis.

''axis_center'': tuple of two floats

denoting the coordinates of the center or {'center', 'auto'}

''xlim'' : tuple of two floats

denoting the x-axis limits.

''ylim'' : tuple of two floats

denoting the y-axis limits.

See also:

Plot (page 744), *Parametric2DLineSeries* (page 757)

Examples

```
>>> u = symbols('u')
```

Single Parametric plot

```
>>> print(str(plot_parametric(cos(u), sin(u), (u, -5, 5))))
Plot object containing:
[0]: parametric cartesian line: (cos(u), sin(u)) for u over (-5.0, 5.0)
```

Multiple parametric plot with single range.

```
>>> print(str(plot_parametric((cos(u), sin(u)), (u, cos(u)))))
Plot object containing:
[0]: parametric cartesian line: (cos(u), sin(u)) for u over (-10.0, 10.0)
[1]: parametric cartesian line: (u, cos(u)) for u over (-10.0, 10.0)
```

Multiple parametric plots.

```
>>> print(str(plot_parametric((cos(u), sin(u), (u, -5, 5)),
...     (cos(u), u, (u, -5, 5)))))
Plot object containing:
[0]: parametric cartesian line: (cos(u), sin(u)) for u over (-5.0, 5.0)
[1]: parametric cartesian line: (cos(u), u) for u over (-5.0, 5.0)
```

`diofant.plotting.plot.`**`plot3d`**(**args*, ***kwargs*)

Plots a 3D surface plot.

Single plot.

`plot3d(expr, range_x, range_y, **kwargs)`

If the ranges are not specified, then a default range of (-10, 10) is used.

Multiple plot with the same range.

`plot3d(expr1, expr2, range_x, range_y, **kwargs)`

If the ranges are not specified, then a default range of (-10, 10) is used.

Multiple plots with different ranges.

`plot3d((expr1, range_x, range_y), (expr2, range_x, range_y), ..., **kwargs)`

Ranges have to be specified for every expression.

Default range may change in the future if a more advanced default range detection algorithm is implemented.

Parameters ``expr`` : Expr

Expression representing the function along x.

``range_x``: tuple

(x, 0, 5), A 3-tuple denoting the range of the x variable.

``range_y``: tuple

(y, 0, 5), A 3-tuple denoting the range of the y variable.

Arguments for ``SurfaceOver2DRangeSeries`` class:

``nb_of_points_x``: int

The x range is sampled uniformly at `nb_of_points_x` of points.

``nb_of_points_y``: int

The y range is sampled uniformly at `nb_of_points_y` of points.

Aesthetics:

``surface_color``: Function which returns a float

Specifies the color for the surface of the plot. See `diofant.plotting.Plot` for more details.

If there are multiple plots, then the same series arguments are applied to

all the plots. If you want to set these options separately, you can index

the returned ``Plot`` object and set it.

Arguments for ``Plot`` class:

``title`` : str

Title of the plot.

See also:

Plot (page 744), *SurfaceOver2DRangeSeries* (page 758)

Examples

Single plot

```
>>> print(str(plot3d(x*y, (x, -5, 5), (y, -5, 5))))
Plot object containing:
[0]: cartesian surface: x*y for x over (-5.0, 5.0) and y over (-5.0, 5.0)
```

Multiple plots with same range

```
>>> print(str(plot3d(x*y, -x*y, (x, -5, 5), (y, -5, 5))))
Plot object containing:
[0]: cartesian surface: x*y for x over (-5.0, 5.0) and y over (-5.0, 5.0)
[1]: cartesian surface: -x*y for x over (-5.0, 5.0) and y over (-5.0, 5.0)
```

Multiple plots with different ranges.

```
>>> print(str(plot3d((x**2 + y**2, (x, -5, 5), (y, -5, 5)),
...     (x*y, (x, -3, 3), (y, -3, 3)))))
Plot object containing:
[0]: cartesian surface: x**2 + y**2 for x over (-5.0, 5.0) and y over (-5.0, 5.0)
[1]: cartesian surface: x*y for x over (-3.0, 3.0) and y over (-3.0, 3.0)
```

diofant.plotting.plot.**plot3d_parametric_line**(**args*, ***kwargs*)

Plots a 3D parametric line plot.

Single plot.

`plot3d_parametric_line(expr_x, expr_y, expr_z, range, **kwargs)`

If the range is not specified, then a default range of (-10, 10) is used.

Multiple plots.

`plot3d_parametric_line((expr_x, expr_y, expr_z, range), ..., **kwargs)`

Ranges have to be specified for every expression.

Default range may change in the future if a more advanced default range detection algorithm is implemented.

Parameters ‘‘expr_x‘‘ : Expr

Expression representing the function along x.

‘‘expr_y‘‘ : Expr

Expression representing the function along y.

‘‘expr_z‘‘ : Expr

Expression representing the function along z.

‘‘range‘‘: tuple

`(u, 0, 5)`, A 3-tuple denoting the range of the parameter variable.

Arguments for ‘‘Parametric3DLineSeries‘‘ class.

‘‘nb_of_points‘‘: int

The range is uniformly sampled at `nb_of_points` number of points.

Aesthetics:

``line_color``: function which returns a float.

Specifies the color for the plot. See `diofant.plotting.Plot` for more details.

If there are multiple plots, then the same series arguments are applied to

all the plots. If you want to set these options separately, you can index

the returned ``Plot`` object and set it.

Arguments for ``Plot`` class.

``title`` : str

Title of the plot.

See also:

Plot (page 744), *Parametric3DLineSeries* (page 757)

Examples

```
>>> u = symbols('u')
```

Single plot.

```
>>> print(str(plot3d_parametric_line(cos(u), sin(u), u, (u, -5, 5))))
Plot object containing:
[0]: 3D parametric cartesian line: (cos(u), sin(u), u) for u over (-5.0, 5.0)
```

Multiple plots.

```
>>> print(str(plot3d_parametric_line((cos(u), sin(u), u, (u, -5, 5)),
...     (sin(u), u**2, u, (u, -5, 5)))))
Plot object containing:
[0]: 3D parametric cartesian line: (cos(u), sin(u), u) for u over (-5.0, 5.0)
[1]: 3D parametric cartesian line: (sin(u), u**2, u) for u over (-5.0, 5.0)
```

diofant.plotting.plot.**plot3d_parametric_surface**(**args*, ***kwargs*)

Plots a 3D parametric surface plot.

Single plot.

`plot3d_parametric_surface(expr_x, expr_y, expr_z, range_u, range_v, **kwargs)`

If the ranges is not specified, then a default range of (-10, 10) is used.

Multiple plots.

`plot3d_parametric_surface((expr_x, expr_y, expr_z, range_u, range_v), ..., **kwargs)`

Ranges have to be specified for every expression.

Default range may change in the future if a more advanced default range detection algorithm is implemented.

Parameters ‘‘expr_x‘‘: Expr

Expression representing the function along x.

‘‘expr_y‘‘: Expr

Expression representing the function along y.

‘‘expr_z‘‘: Expr

Expression representing the function along z.

‘‘range_u‘‘: tuple

`(u, 0, 5)`, A 3-tuple denoting the range of the u variable.

‘‘range_v‘‘: tuple

`(v, 0, 5)`, A 3-tuple denoting the range of the v variable.

Arguments for ‘‘ParametricSurfaceSeries‘‘ class:

‘‘nb_of_points_u‘‘: int

The u range is sampled uniformly at `nb_of_points_v` of points

‘‘nb_of_points_y‘‘: int

The v range is sampled uniformly at `nb_of_points_y` of points

Aesthetics:

‘‘surface_color‘‘: Function which returns a float

Specifies the color for the surface of the plot. See `diofant.plotting.Plot` for more details.

If there are multiple plots, then the same series arguments are applied for

all the plots. If you want to set these options separately, you can index

the returned ‘‘Plot‘‘ object and set it.

Arguments for ‘‘Plot‘‘ class:

‘‘title‘‘ : str

Title of the plot.

See also:

Plot (page 744), *ParametricSurfaceSeries* (page 758)

Examples

```
>>> u, v = symbols('u v')
```

Single plot.

```
>>> print(str(plot3d_parametric_surface(cos(u + v), sin(u - v), u - v,
...     (u, -5, 5), (v, -5, 5))))
Plot object containing:
[0]: parametric cartesian surface: (cos(u + v), sin(u - v), u - v) for u over (-5.
↪0, 5.0) and v over (-5.0, 5.0)
```

`diofant.plotting.plot_implicit.`**`plot_implicit`**(*expr*, *x_var=None*, *y_var=None*, ***kwargs*)

A plot function to plot implicit equations / inequalities.

Parameters ‘‘expr‘‘ : Expr

The equation / inequality that is to be plotted.

‘‘x_var‘‘ : symbol or tuple, optional

symbol to plot on x-axis or tuple giving symbol and range as `(symbol, xmin, xmax)`

‘‘y_var‘‘ : symbol or tuple, optional

symbol to plot on y-axis or tuple giving symbol and range as `(symbol, ymin, ymax)`

If neither ‘‘x_var‘‘ nor ‘‘y_var‘‘ are given then the free symbols in the expression will be assigned in the order they are sorted.

The following keyword arguments can also be used:

‘‘adaptive‘‘ : Boolean, optional

The default value is set to True. It has to be set to False if you want to use a mesh grid.

‘‘depth‘‘ : integer

The depth of recursion for adaptive mesh grid. Default value is 0. Takes value in the range (0, 4).

‘‘points‘‘ : integer

The number of points if adaptive mesh grid is not used. Default value is 200.

‘‘title‘‘ : str

The title for the plot.

‘‘xlabel‘‘ : str

The label for the x-axis

‘‘ylabel‘‘ : string

The label for the y-axis

Aesthetics options:

‘‘line_color‘‘ : float or str

Specifies the color for the plot. See `Plot` to see how to set color for the plots.

plot_implicit, by default, uses interval arithmetic to plot functions. If

the expression cannot be plotted using interval arithmetic, it defaults to

a generating a contour using a mesh grid of fixed number of points. By

setting adaptive to False, you can force plot_implicit to use the mesh

grid. The mesh grid method can be effective when adaptive plotting using

interval arithmetic, fails to plot with small line width.

Examples

Plot expressions:

Without any ranges for the symbols in the expression

```
>>> p1 = plot_implicit(Eq(x**2 + y**2, 5))
```

With the range for the symbols

```
>>> p2 = plot_implicit(Eq(x**2 + y**2, 3),
...         (x, -3, 3), (y, -3, 3))
```

With depth of recursion as argument.

```
>>> p3 = plot_implicit(Eq(x**2 + y**2, 5),
...         (x, -4, 4), (y, -4, 4), depth = 2)
```

Using mesh grid and not using adaptive meshing.

```
>>> p4 = plot_implicit(Eq(x**2 + y**2, 5),
...         (x, -5, 5), (y, -2, 2), adaptive=False)
```

Using mesh grid with number of points as input.

```
>>> p5 = plot_implicit(Eq(x**2 + y**2, 5),
...         (x, -5, 5), (y, -2, 2),
...         adaptive=False, points=400)
```

Plotting regions.

```
>>> p6 = plot_implicit(y > x**2)
```

Plotting Using boolean conjunctions.

```
>>> p7 = plot_implicit(And(y > x, y > -x))
```

When plotting an expression with a single variable (y - 1, for example), specify the x or the y variable explicitly:

```
>>> p8 = plot_implicit(y - 1, y_var=y)
>>> p9 = plot_implicit(x - 1, x_var=x)
```

3.14.4 Series Classes

class diofant.plotting.plot.**BaseSeries**

Base class for the data objects containing stuff to be plotted.

The backend should check if it supports the data series that it's given. (eg TextBackend supports only LineOver1DRange). It's the backend responsibility to know how to use the class of data series that it's given.

Some data series classes are grouped (using a class attribute like is_2Dline) according to the api they present (based only on convention). The backend is not obliged to use that api (eg. The LineOver1DRange belongs to the is_2Dline group and presents the get_points method, but the TextBackend does not use the get_points method).

class diofant.plotting.plot.**Line2DBaseSeries**

A base class for 2D lines.

- adding the label, steps and only_integers options
- making is_2Dline true
- defining get_segments and get_color_array

class diofant.plotting.plot.**LineOver1DRangeSeries**(*expr*, *var_start_end*, ***kwargs*)

Representation for a line consisting of a Diofant expression over a range.

get_segments()

Adaptively gets segments for plotting.

The adaptive sampling is done by recursively checking if three points are almost collinear. If they are not collinear, then more points are added between those points.

References

[1] **Adaptive polygonal approximation of parametric curves,** Luiz Henrique de Figueiredo.

class diofant.plotting.plot.**Parametric2DLineSeries**(*expr_x*, *expr_y*, *var_start_end*, ***kwargs*)

Representation for a line consisting of two parametric diofant expressions over a range.

get_segments()

Adaptively gets segments for plotting.

The adaptive sampling is done by recursively checking if three points are almost collinear. If they are not collinear, then more points are added between those points.

References

[1] **Adaptive polygonal approximation of parametric curves,** Luiz Henrique de Figueiredo.

class diofant.plotting.plot.**Line3DBaseSeries**

A base class for 3D lines.

Most of the stuff is derived from Line2DBaseSeries.

class diofant.plotting.plot.**Parametric3DLineSeries**(*expr_x*, *expr_y*, *expr_z*, *var_start_end*, ***kwargs*)

Representation for a 3D line consisting of two parametric diofant expressions and a range.

class diofant.plotting.plot.**SurfaceBaseSeries**

A base class for 3D surfaces.

class diofant.plotting.plot.**SurfaceOver2DRangeSeries**(*expr*, *var_start_end_x*, *var_start_end_y*, ***kwargs*)

Representation for a 3D surface consisting of a diofant expression and 2D range.

class diofant.plotting.plot.**ParametricSurfaceSeries**(*expr_x*, *expr_y*, *expr_z*, *var_start_end_u*, *var_start_end_v*, ***kwargs*)

Representation for a 3D surface consisting of three parametric diofant expressions and a range.

class diofant.plotting.plot_implicit.**ImplicitSeries**(*expr*, *var_start_end_x*, *var_start_end_y*, *has_equality*, *use_interval_math*, *depth*, *nb_of_points*, *line_color*)

Representation for Implicit plot

3.15 Series

The series module implements series expansions as a function and many related functions.

class diofant.series.limits.**Limit**

Represents a directional limit of `expr` at the point `z0`.

Parameters **expr** : Expr

algebraic expression

z : Symbol

variable of the `expr`

z0 : Expr

limit point, z_0

dir : {"+", "-", "real"}, optional

For `dir="+"` (default) it calculates the limit from the right ($z \to z_0 + 0$) and for `dir="-"` the limit from the left ($z \to z_0 - 0$). If `dir="real"`, the limit is the bidirectional real limit. For infinite `z0` (`oo` or `-oo`), the `dir` argument is determined from the direction of the infinity (i.e., `dir="-"` for `oo`).

Examples

```
>>> Limit(sin(x)/x, x, 0)
Limit(sin(x)/x, x, 0)
>>> Limit(1/x, x, 0, dir="-")
Limit(1/x, x, 0, dir='-')
```

doit(***hints*)

Evaluates limit.

Notes

First we handle some trivial cases (i.e. constant), then try Gruntz algorithm (see the *gruntz* (page 1152) module).

free_symbols

Return from the atoms of self those which are free symbols.

For most expressions, all symbols are free symbols. For some classes this is not true. e.g. Integrals use Symbols for the dummy variables which are bound variables, so Integral has a method to return all symbols except those. Derivative keeps track of symbols with respect to which it will perform a derivative; those are bound variables, too, so it has its own free_symbols method.

Any other method that uses bound variables should implement a free_symbols method.

diofant.series.limits.**limit**(*expr, z, z0, dir='+'*)

Compute the directional limit of `expr` at the point `z0`.

See also:

Limit (page 758)

Examples

```
>>> limit(sin(x)/x, x, 0)
1
>>> limit(1/x, x, 0, dir="+")
oo
>>> limit(1/x, x, 0, dir="-")
-oo
>>> limit(1/x, x, oo)
0
```

diofant.series.series.**series**(*expr, x=None, x0=0, n=6, dir='+'*)

Series expansion of `expr` in `x` around point `x0`.

See also:

diofant.core.expr.Expr.series (page 78)

diofant.series.order.**O**

alias of *diofant.series.order.Order* (page 759)

class diofant.series.order.**Order**

Represents the limiting behavior of function.

The formal definition *[R482]* (page 1263) for order symbol $O(f(x))$ (Big O) is that $g(x) \in O(f(x))$ as $x \to a$ iff

$$\lim_{x \to a} \sup \left| \frac{g(x)}{f(x)} \right| < \infty$$

Parameters expr : Expr

an expression

args : sequence of Symbol's or pairs (Symbol, Expr), optional

If only symbols are provided, i.e. no limit point are passed, then the limit point is assumed to be zero. If no symbols are passed then all symbols in the expression are used.

References

[R482] (page 1263)

Examples

The order of a function can be intuitively thought of representing all terms of powers greater than the one specified. For example, $O(x^3)$ corresponds to any terms proportional to $x^3, x^4, \ldots$ and any higher power. For a polynomial, this leaves terms proportional to x^2, x and constants.

```
>>> 1 + x + x**2 + x**3 + x**4 + O(x**3)
1 + x + x**2 + O(x**3)
```

`O(f(x))` is automatically transformed to `O(f(x).as_leading_term(x))`:

```
>>> O(x + x**2)
O(x)
>>> O(cos(x))
O(1)
```

Some arithmetic operations:

```
>>> O(x)*x
O(x**2)
>>> O(x) - O(x)
O(x)
```

The Big O symbol is a set, so we support membership test:

```
>>> x in O(x)
True
>>> O(1) in O(1, x)
True
>>> O(1, x) in O(1)
False
>>> O(x) in O(1, x)
True
>>> O(x**2) in O(x)
True
```

Limit points other then zero and multivariate Big O are also supported:

```
>>> O(x) == O(x, (x, 0))
True
>>> O(x + x**2, (x, oo))
O(x**2, (x, oo))
>>> O(cos(x), (x, pi/2))
O(x - pi/2, (x, pi/2))
```

```
>>> O(1 + x*y)
O(1, x, y)
>>> O(1 + x*y, (x, 0), (y, 0))
O(1, x, y)
>>> O(1 + x*y, (x, oo), (y, oo))
O(x*y, (x, oo), (y, oo))
```

contains(*expr*)
Membership test.

Returns Boolean or None

Return True if `expr` belongs to `self`. Return False if `self` belongs to `expr`. Return None if the inclusion relation cannot be determined.

free_symbols
Return from the atoms of self those which are free symbols.

For most expressions, all symbols are free symbols. For some classes this is not true. e.g. Integrals use Symbols for the dummy variables which are bound variables, so Integral has a method to return all symbols except those. Derivative keeps track of symbols with respect to which it will perform a derivative; those are bound variables, too, so it has its own free_symbols method.

Any other method that uses bound variables should implement a free_symbols method.

getO()
Returns the additive O(..) symbol if there is one, else None.

removeO()
Removes the additive O(..) symbol if there is one

diofant.series.residues.**residue**(*expr, x, x0*)
Finds the residue of `expr` at the point `x=x0`.

The residue is defined *[R483]* (page 1263) as the coefficient of $1/(x - x_0)$ in the power series expansion around $x = x_0$.

This notion is essential for the Residue Theorem *[R484]* (page 1263)

References

[R483] (page 1263), *[R484]* (page 1263)

Examples

```
>>> residue(1/x, x, 0)
1
>>> residue(1/x**2, x, 0)
0
>>> residue(2/sin(x), x, 0)
2
```

3.16 Sets

3.16.1 Set

class diofant.sets.sets.**Set**

The base class for any kind of set.

This is not meant to be used directly as a container of items. It does not behave like the builtin `set`; see *FiniteSet* (page 769) for that.

Real intervals are represented by the *Interval* (page 766) class and unions of sets by the *Union* (page 769) class. The empty set is represented by the *EmptySet* (page 772) class and available as a singleton as `S.EmptySet`.

boundary

The boundary or frontier of a set

A point x is on the boundary of a set S if

1. x is in the closure of S. I.e. Every neighborhood of x contains a point in S.
2. x is not in the interior of S. I.e. There does not exist an open set centered on x contained entirely within S.

There are the points on the outer rim of S. If S is open then these points need not actually be contained within S.

For example, the boundary of an interval is its start and end points. This is true regardless of whether or not the interval is open.

Examples

```
>>> Interval(0, 1).boundary
{0, 1}
>>> Interval(0, 1, True, False).boundary
{0, 1}
```

complement(*universe*)

The complement of 'self' w.r.t the given the universe.

Examples

```
>>> Interval(0, 1).complement(S.Reals)
(-oo, 0) U (1, oo)
```

```
>>> Interval(0, 1).complement(S.UniversalSet)
UniversalSet() \ [0, 1]
```

contains(*other*)

Returns True if 'other' is contained in 'self' as an element.

As a shortcut it is possible to use the 'in' operator:

Examples

```
>>> Interval(0, 1).contains(0.5)
true
>>> 0.5 in Interval(0, 1)
True
```

inf

The infimum of 'self'

Examples

```
>>> Interval(0, 1).inf
0
>>> Union(Interval(0, 1), Interval(2, 3)).inf
0
```

intersection(*other*)

Returns the intersection of 'self' and 'other'.

```
>>> Interval(1, 3).intersection(Interval(1, 2))
[1, 2]
```

is_disjoint(*other*)

Returns True if 'self' and 'other' are disjoint

References

[R485] (page 1263)

Examples

```
>>> Interval(0, 2).is_disjoint(Interval(1, 2))
False
>>> Interval(0, 2).is_disjoint(Interval(3, 4))
True
```

is_open

Test if a set is open.

A set is open if it has an empty intersection with its boundary.

See also:

boundary (page 762)

Examples

```
>>> S.Reals.is_open
True
```

is_proper_subset(*other*)
Returns True if 'self' is a proper subset of 'other'.

Examples

```
>>> Interval(0, 0.5).is_proper_subset(Interval(0, 1))
True
>>> Interval(0, 1).is_proper_subset(Interval(0, 1))
False
```

is_proper_superset(*other*)
Returns True if 'self' is a proper superset of 'other'.

Examples

```
>>> Interval(0, 1).is_proper_superset(Interval(0, 0.5))
True
>>> Interval(0, 1).is_proper_superset(Interval(0, 1))
False
```

is_subset(*other*)
Returns True if 'self' is a subset of 'other'.

Examples

```
>>> Interval(0, 0.5).is_subset(Interval(0, 1))
True
>>> Interval(0, 1).is_subset(Interval(0, 1, left_open=True))
False
```

is_superset(*other*)
Returns True if 'self' is a superset of 'other'.

Examples

```
>>> Interval(0, 0.5).is_superset(Interval(0, 1))
False
>>> Interval(0, 1).is_superset(Interval(0, 1, left_open=True))
True
```

isdisjoint(*other*)
Alias for *is_disjoint()* (page 763)

issubset(*other*)
Alias for *is_subset()* (page 764)

issuperset(*other*)
Alias for *is_superset()* (page 764)

measure
The (Lebesgue) measure of 'self'

Examples

```
>>> Interval(0, 1).measure
1
>>> Union(Interval(0, 1), Interval(2, 3)).measure
2
```

powerset()
Find the Power set of 'self'.

References

[R486] (page 1263)

Examples

```
>>> A = EmptySet()
>>> A.powerset()
{EmptySet()}
>>> A = FiniteSet(1, 2)
>>> a, b, c = FiniteSet(1), FiniteSet(2), FiniteSet(1, 2)
>>> A.powerset() == FiniteSet(a, b, c, EmptySet())
True
```

sup
The supremum of 'self'

Examples

```
>>> Interval(0, 1).sup
1
>>> Union(Interval(0, 1), Interval(2, 3)).sup
3
```

union(*other*)
Returns the union of 'self' and 'other'.

Examples

As a shortcut it is possible to use the '+' operator:

```
>>> Interval(0, 1).union(Interval(2, 3))
[0, 1] U [2, 3]
>>> Interval(0, 1) + Interval(2, 3)
[0, 1] U [2, 3]
>>> Interval(1, 2, True, True) + FiniteSet(2, 3)
(1, 2] U {3}
```

Similarly it is possible to use the '-' operator for set differences:

```
>>> Interval(0, 2) - Interval(0, 1)
(1, 2]
>>> Interval(1, 3) - FiniteSet(2)
[1, 2) U (2, 3]
```

diofant.sets.sets.**imageset**(**args*)

Image of set under transformation f.

If this function can't compute the image, it returns an unevaluated ImageSet object.

$$f(x)|x \in self$$

See also:

diofant.sets.fancysets.ImageSet (page 774)

Examples

```
>>> imageset(x, 2*x, Interval(0, 2))
[0, 4]
```

```
>>> imageset(lambda x: 2*x, Interval(0, 2))
[0, 4]
```

```
>>> imageset(Lambda(x, sin(x)), Interval(-2, 1))
ImageSet(Lambda(x, sin(x)), [-2, 1])
```

Elementary Sets

3.16.2 Interval

class diofant.sets.sets.**Interval**

Represents a real interval as a Set.

Returns an interval with end points "start" and "end".

For left_open=True (default left_open is False) the interval will be open on the left. Similarly, for right_open=True the interval will be open on the right.

Notes

- Only real end points are supported
- Interval(a, b) with a > b will return the empty set

- Use the evalf() method to turn an Interval into an mpmath 'mpi' interval instance

References

[R487] (page 1263)

Examples

```
>>> Interval(0, 1)
[0, 1]
>>> Interval(0, 1, False, True)
[0, 1)
>>> Interval.Ropen(0, 1)
[0, 1)
>>> Interval.Lopen(0, 1)
(0, 1]
>>> Interval.open(0, 1)
(0, 1)
```

```
>>> a = Symbol('a', extended_real=True)
>>> Interval(0, a)
[0, a]
```

classmethod Lopen(*a*, *b*)
Return an interval not including the left boundary.

classmethod Ropen(*a*, *b*)
Return an interval not including the right boundary.

as_relational(*x*)
Rewrite an interval in terms of inequalities and logic operators.

end
The right end point of 'self'.

This property takes the same value as the 'sup' property.

Examples

```
>>> Interval(0, 1).end
1
```

is_left_unbounded
Return True if the left endpoint is negative infinity.

is_right_unbounded
Return True if the right endpoint is positive infinity.

left
The left end point of 'self'.

This property takes the same value as the 'inf' property.

Examples

```
>>> Interval(0, 1).start
0
```

left_open
True if 'self' is left-open.

Examples

```
>>> Interval(0, 1, left_open=True).left_open
true
>>> Interval(0, 1, left_open=False).left_open
false
```

classmethod open(*a*, *b*)
Return an interval including neither boundary.

right
The right end point of 'self'.

This property takes the same value as the 'sup' property.

Examples

```
>>> Interval(0, 1).end
1
```

right_open
True if 'self' is right-open.

Examples

```
>>> Interval(0, 1, right_open=True).right_open
true
>>> Interval(0, 1, right_open=False).right_open
false
```

start
The left end point of 'self'.

This property takes the same value as the 'inf' property.

Examples

```
>>> Interval(0, 1).start
0
```

3.16.3 FiniteSet

class diofant.sets.sets.**FiniteSet**

Represents a finite set of discrete numbers

References

[R488] (page 1264)

Examples

```
>>> FiniteSet(1, 2, 3, 4)
{1, 2, 3, 4}
>>> 3 in FiniteSet(1, 2, 3, 4)
True
```

as_relational(*symbol*)

Rewrite a FiniteSet in terms of equalities and logic operators.

measure

The (Lebesgue) measure of 'self'

Examples

```
>>> Interval(0, 1).measure
1
>>> Union(Interval(0, 1), Interval(2, 3)).measure
2
```

Compound Sets

3.16.4 Union

class diofant.sets.sets.**Union**

Represents a union of sets as a *Set* (page 762).

See also:

Intersection (page 770)

References

[R489] (page 1264)

Examples

```
>>> Union(Interval(1, 2), Interval(3, 4))
[1, 2] U [3, 4]
```

The Union constructor will always try to merge overlapping intervals, if possible. For example:

```
>>> Union(Interval(1, 2), Interval(2, 3))
[1, 3]
```

as_relational(*symbol*)
Rewrite a Union in terms of equalities and logic operators.

is_iterable
bool(x) -> bool

Returns True when the argument x is true, False otherwise. The builtins True and False are the only two instances of the class bool. The class bool is a subclass of the class int, and cannot be subclassed.

static reduce()
Simplify a *Union* (page 769) using known rules

We first start with global rules like 'Merge all FiniteSets'

Then we iterate through all pairs and ask the constituent sets if they can simplify themselves with any other constituent

3.16.5 Intersection

class diofant.sets.sets.**Intersection**
Represents an intersection of sets as a *Set* (page 762).

See also:

Union (page 769)

References

[R490] (page 1264)

Examples

```
>>> Intersection(Interval(1, 3), Interval(2, 4))
[2, 3]
```

We often use the .intersect method

```
>>> Interval(1, 3).intersection(Interval(2, 4))
[2, 3]
```

as_relational(*symbol*)
Rewrite an Intersection in terms of equalities and logic operators

is_iterable
bool(x) -> bool

Returns True when the argument x is true, False otherwise. The builtins True and False are the only two instances of the class bool. The class bool is a subclass of the class int, and cannot be subclassed.

static reduce()

Simplify an intersection using known rules

We first start with global rules like 'if any empty sets return empty set' and 'distribute any unions'

Then we iterate through all pairs and ask the constituent sets if they can simplify themselves with any other constituent

3.16.6 ProductSet

class diofant.sets.sets.**ProductSet**

Represents a Cartesian Product of Sets.

Returns a Cartesian product given several sets as either an iterable or individual arguments.

Can use '*' operator on any sets for convenient shorthand.

Notes

- Passes most operations down to the argument sets
- Flattens Products of ProductSets

References

[R491] (page 1264)

Examples

```
>>> I = Interval(0, 5); S = FiniteSet(1, 2, 3)
>>> ProductSet(I, S)
[0, 5] x {1, 2, 3}
```

```
>>> (2, 2) in ProductSet(I, S)
True
```

```
>>> Interval(0, 1) * Interval(0, 1) # The unit square
[0, 1] x [0, 1]
```

```
>>> H, T = Symbol('H'), Symbol('T')
>>> coin = FiniteSet(H, T)
>>> set(coin**2)
{(H, H), (H, T), (T, H), (T, T)}
```

is_iterable

bool(x) -> bool

Returns True when the argument x is true, False otherwise. The builtins True and False are the only two instances of the class bool. The class bool is a subclass of the class int, and cannot be subclassed.

3.16.7 Complement

class diofant.sets.sets.**Complement**

Represents relative complement of a set with another set.

$A - B = \{x \in A | x \notin B\}$

See also:

Intersection (page 770), *Union* (page 769)

References

[R492] (page 1264)

Examples

```
>>> Complement(FiniteSet(0, 1, 2), FiniteSet(1))
{0, 2}
```

static reduce(B)

Simplify a *Complement* (page 772).

Singleton Sets

3.16.8 EmptySet

class diofant.sets.sets.**EmptySet**

Represents the empty set.

The empty set is available as a singleton as S.EmptySet.

See also:

UniversalSet (page 773)

References

[R493] (page 1264)

Examples

```
>>> S.EmptySet
EmptySet()
```

```
>>> Interval(1, 2).intersection(S.EmptySet)
EmptySet()
```

3.16.9 UniversalSet

class diofant.sets.sets.**UniversalSet**

Represents the set of all things.

The universal set is available as a singleton as S.UniversalSet

See also:

EmptySet (page 772)

References

[R494] (page 1264)

Examples

```
>>> S.UniversalSet
UniversalSet()
```

```
>>> Interval(1, 2).intersection(S.UniversalSet)
[1, 2]
```

Special Sets

3.16.10 Naturals

class diofant.sets.fancysets.**Naturals**

The set of natural numbers.

Represents the natural numbers (or counting numbers) which are all positive integers starting from 1. This set is also available as the Singleton, S.Naturals.

See also:

***Naturals0* (page 774)** non-negative integers

***Integers* (page 774)** also includes negative integers

Examples

```
>>> 5 in S.Naturals
True
>>> iterable = iter(S.Naturals)
>>> next(iterable)
1
>>> next(iterable)
2
>>> next(iterable)
3
>>> pprint(S.Naturals.intersection(Interval(0, 10)))
{1, 2, ..., 10}
```

3.16.11 Naturals0

class diofant.sets.fancysets.**Naturals0**
The set of natural numbers, starting from 0.

Represents the whole numbers which are all the non-negative integers, inclusive of zero.

See also:

Naturals **(page 773)** positive integers

Integers **(page 774)** also includes the negative integers

3.16.12 Integers

class diofant.sets.fancysets.**Integers**
The set of all integers.

Represents all integers: positive, negative and zero. This set is also available as the Singleton, S.Integers.

See also:

Naturals0 **(page 774)** non-negative integers

Integers **(page 774)** positive and negative integers and zero

Examples

```
>>> 5 in S.Naturals
True
>>> iterable = iter(S.Integers)
>>> next(iterable)
0
>>> next(iterable)
1
>>> next(iterable)
-1
>>> next(iterable)
2
```

```
>>> pprint(S.Integers.intersection(Interval(-4, 4)))
{-4, -3, ..., 4}
```

3.16.13 Rationals

class diofant.sets.fancysets.**Rationals**
The set of all rationals.

3.16.14 ImageSet

class diofant.sets.fancysets.**ImageSet**
Image of a set under a mathematical function.

Examples

```
>>> N = S.Naturals
>>> squares = ImageSet(Lambda(x, x**2), N) # {x**2 for x in N}
>>> 4 in squares
True
>>> 5 in squares
False
```

```
>>> FiniteSet(0, 1, 2, 3, 4, 5, 6, 7, 9, 10).intersection(squares)
{1, 4, 9}
```

```
>>> square_iterable = iter(squares)
>>> for i in range(4):
...     next(square_iterable)
1
4
9
16
```

If you want to get value for $x = 2$, 1/2 etc. (Please check whether the x value is in $base_set$ or not before passing it as args)

```
>>> squares.lamda(2)
4
>>> squares.lamda(S.One/2)
1/4
```

is_iterable

bool(x) -> bool

Returns True when the argument x is true, False otherwise. The builtins True and False are the only two instances of the class bool. The class bool is a subclass of the class int, and cannot be subclassed.

3.16.15 Range

class diofant.sets.fancysets.**Range**

Represents a range of integers.

Examples

```
>>> list(Range(5)) # 0 to 5
[0, 1, 2, 3, 4]
>>> list(Range(10, 15)) # 10 to 15
[10, 11, 12, 13, 14]
>>> list(Range(10, 20, 2)) # 10 to 20 in steps of 2
[10, 12, 14, 16, 18]
>>> list(Range(20, 10, -2)) # 20 to 10 backward in steps of 2
[12, 14, 16, 18, 20]
```

3.17 Simplify

3.17.1 simplify

diofant.simplify.simplify.**simplify**(*expr*, *ratio=1.7*, *measure=<function count_ops>*, *fu=False*)

Simplifies the given expression.

Simplification is not a well defined term and the exact strategies this function tries can change in the future versions of Diofant. If your algorithm relies on "simplification" (whatever it is), try to determine what you need exactly - is it powsimp()?, radsimp()?, together()?, logcombine()?, or something else? And use this particular function directly, because those are well defined and thus your algorithm will be robust.

Nonetheless, especially for interactive use, or when you don't know anything about the structure of the expression, simplify() tries to apply intelligent heuristics to make the input expression "simpler". For example:

```
>>> a = (x + x**2)/(x*sin(y)**2 + x*cos(y)**2)
>>> a
(x**2 + x)/(x*sin(y)**2 + x*cos(y)**2)
>>> simplify(a)
x + 1
```

Note that we could have obtained the same result by using specific simplification functions:

```
>>> trigsimp(a)
(x**2 + x)/x
>>> cancel(_)
x + 1
```

In some cases, applying *simplify()* (page 776) may actually result in some more complicated expression. The default `ratio=1.7` prevents more extreme cases: if (result length)/(input length) > ratio, then input is returned unmodified. The `measure` parameter lets you specify the function used to determine how complex an expression is. The function should take a single argument as an expression and return a number such that if expression `a` is more complex than expression `b`, then `measure(a) > measure(b)`. The default measure function is *count_ops()* (page 140), which returns the total number of operations in the expression.

For example, if `ratio=1`, `simplify` output can't be longer than input.

```
>>> root = 1/(sqrt(2)+3)
```

Since `simplify(root)` would result in a slightly longer expression, root is returned unchanged instead:

```
>>> simplify(root, ratio=1) == root
True
```

If `ratio=oo`, simplify will be applied anyway:

```
>>> count_ops(simplify(root, ratio=oo)) > count_ops(root)
True
```

Note that the shortest expression is not necessary the simplest, so setting `ratio` to 1 may not be a good idea. Heuristically, the default value `ratio=1.7` seems like a reasonable choice.

You can easily define your own measure function based on what you feel should represent the "size" or "complexity" of the input expression. Note that some choices, such as `lambda expr: len(str(expr))` may appear to be good metrics, but have other problems (in this case, the measure function may slow down simplify too much for very large expressions). If you don't know what a good metric would be, the default, `count_ops`, is a good one.

For example:

```
>>> a, b = symbols('a b', positive=True)
>>> g = log(a) + log(b) + log(a)*log(1/b)
>>> h = simplify(g)
>>> h
log(a*b**(-log(a) + 1))
>>> count_ops(g)
8
>>> count_ops(h)
5
```

So you can see that h is simpler than g using the count_ops metric. However, we may not like how `simplify` (in this case, using `logcombine`) has created the `b**(log(1/a) + 1)` term. A simple way to reduce this would be to give more weight to powers as operations in `count_ops`. We can do this by using the `visual=True` option:

```
>>> print(count_ops(g, visual=True))
2*ADD + DIV + 4*LOG + MUL
>>> print(count_ops(h, visual=True))
2*LOG + MUL + POW + SUB
```

```
>>> def my_measure(expr):
...     POW = Symbol('POW')
...     # Discourage powers by giving POW a weight of 10
...     count = count_ops(expr, visual=True).subs(POW, 10)
...     # Every other operation gets a weight of 1 (the default)
...     count = count.replace(Symbol, type(S.One))
...     return count
>>> my_measure(g)
8
>>> my_measure(h)
14
>>> 15./8 > 1.7  # 1.7 is the default ratio
True
>>> simplify(g, measure=my_measure)
-log(a)*log(b) + log(a) + log(b)
```

Note that because `simplify()` internally tries many different simplification strategies and then compares them using the measure function, we get a completely different result that is still different from the input expression by doing this.

3.17.2 separatevars

diofant.simplify.simplify.**separatevars**(*expr*, *symbols=[]*, *dict=False*, *force=False*)

Separates variables in an expression, if possible. By default, it separates with respect to all symbols in an expression and collects constant coefficients that are independent of symbols.

If dict=True then the separated terms will be returned in a dictionary keyed to their corresponding symbols. By default, all symbols in the expression will appear as keys; if symbols are provided, then all those symbols will be used as keys, and any terms in the expression containing other symbols or non-symbols will be returned keyed to the string 'coeff'. (Passing None for symbols will return the expression in a dictionary keyed to 'coeff'.)

If force=True, then bases of powers will be separated regardless of assumptions on the symbols involved.

Notes

The order of the factors is determined by Mul, so that the separated expressions may not necessarily be grouped together.

Although factoring is necessary to separate variables in some expressions, it is not necessary in all cases, so one should not count on the returned factors being factored.

Examples

```
>>> from diofant.abc import alpha
>>> separatevars((x*y)**y)
(x*y)**y
>>> separatevars((x*y)**y, force=True)
x**y*y**y
```

```
>>> e = 2*x**2*z*sin(y)+2*z*x**2
>>> separatevars(e)
2*x**2*z*(sin(y) + 1)
>>> separatevars(e, symbols=(x, y), dict=True)
{'coeff': 2*z, x: x**2, y: sin(y) + 1}
>>> separatevars(e, [x, y, alpha], dict=True)
{'coeff': 2*z, alpha: 1, x: x**2, y: sin(y) + 1}
```

If the expression is not really separable, or is only partially separable, separatevars will do the best it can to separate it by using factoring.

```
>>> separatevars(x + x*y - 3*x**2)
-x*(3*x - y - 1)
```

If the expression is not separable then expr is returned unchanged or (if dict=True) then None is returned.

```
>>> eq = 2*x + y*sin(x)
>>> separatevars(eq) == eq
True
```

(continues on next page)

(continued from previous page)

```
>>> separatevars(2*x + y*sin(x), symbols=(x, y), dict=True) == None
True
```

3.17.3 nthroot

diofant.simplify.simplify.**nthroot**(*expr, n, max_len=4, prec=15*)

compute a real nth-root of a sum of surds

Parameters expr : sum of surds

n : integer

max_len : maximum number of surds passed as constants to `nsimplify`

Notes

First `nsimplify` is used to get a candidate root; if it is not a root the minimal polynomial is computed; the answer is one of its roots.

Examples

```
>>> nthroot(90 + 34*sqrt(7), 3)
sqrt(7) + 3
```

3.17.4 besselsimp

diofant.simplify.simplify.**besselsimp**(*expr*)

Simplify bessel-type functions.

This routine tries to simplify bessel-type functions. Currently it only works on the Bessel J and I functions, however. It works by looking at all such functions in turn, and eliminating factors of "I" and "-1" (actually their polar equivalents) in front of the argument. Then, functions of half-integer order are rewritten using trigonometric functions and functions of integer order (> 1) are rewritten using functions of low order. Finally, if the expression was changed, compute factorization of the result with factor().

```
>>> from diofant.abc import nu
>>> besselsimp(besselj(nu, z*polar_lift(-1)))
E**(I*pi*nu)*besselj(nu, z)
>>> besselsimp(besseli(nu, z*polar_lift(-I)))
E**(-I*pi*nu/2)*besselj(nu, z)
>>> besselsimp(besseli(Rational(-1, 2), z))
sqrt(2)*cosh(z)/(sqrt(pi)*sqrt(z))
>>> besselsimp(z*besseli(0, z) + z*(besseli(2, z))/2 + besseli(1, z))
3*z*besseli(0, z)/2
```

3.17.5 hypersimp

diofant.simplify.simplify.**hypersimp**(*f, k*)

Given combinatorial term f(k) simplify its consecutive term ratio i.e. f(k+1)/f(k). The

input term can be composed of functions and integer sequences which have equivalent representation in terms of gamma special function.

Notes

The algorithm [R495] (page 1264) performs three basic steps:

```
1. Rewrite all functions in terms of gamma, if possible.

2. Rewrite all occurrences of gamma in terms of products
   of gamma and rising factorial with integer,  absolute
   constant exponent.

3. Perform simplification of nested fractions, powers
   and if the resulting expression is a quotient of
   polynomials, reduce their total degree.
```

If f(k) is hypergeometric then as result we arrive with a quotient of polynomials of minimal degree. Otherwise None is returned.

References

[R495] (page 1264)

3.17.6 hypersimilar

diofant.simplify.simplify.**hypersimilar**(*f*, *g*, *k*)

Returns True if 'f' and 'g' are hyper-similar.

Similarity in hypergeometric sense means that a quotient of f(k) and g(k) is a rational function in k. This procedure is useful in solving recurrence relations.

See also:

hypersimp (page 779)

3.17.7 nsimplify

diofant.simplify.simplify.**nsimplify**(*expr*, *constants=[]*, *tolerance=None*, *full=False*, *rational=None*)

Find a simple representation for a number or, if there are free symbols or if rational=True, then replace Floats with their Rational equivalents. If no change is made and rational is not False then Floats will at least be converted to Rationals.

For numerical expressions, a simple formula that numerically matches the given numerical expression is sought (and the input should be possible to evalf to a precision of at least 30 digits).

Optionally, a list of (rationally independent) constants to include in the formula may be given.

A lower tolerance may be set to find less exact matches. If no tolerance is given then the least precise value will set the tolerance (e.g. Floats default to 15 digits of precision, so would be tolerance=10**-15).

With full=True, a more extensive search is performed (this is useful to find simpler numbers when the tolerance is set low).

See also:

diofant.core.function.nfloat (page 144)

Examples

```
>>> nsimplify(4/(1+sqrt(5)), [GoldenRatio])
-2 + 2*GoldenRatio
>>> nsimplify((1/(exp(3*pi*I/5)+1)))
1/2 - I*sqrt(sqrt(5)/10 + 1/4)
>>> nsimplify(I**I, [pi])
E**(-pi/2)
>>> nsimplify(pi, tolerance=0.01)
22/7
```

3.17.8 posify

diofant.simplify.simplify.**posify**(*eq*)

Return eq (with generic symbols made positive) and a dictionary containing the mapping between the old and new symbols.

Any symbol that has positive=None will be replaced with a positive dummy symbol having the same name. This replacement will allow more symbolic processing of expressions, especially those involving powers and logarithms.

A dictionary that can be sent to subs to restore eq to its original symbols is also returned.

```
>>> posify(x + Symbol('p', positive=True) + Symbol('n', negative=True))
(n + p + _x, {_x: x})
```

```
>>> eq = 1/x
>>> log(eq).expand()
log(1/x)
>>> log(posify(eq)[0]).expand()
-log(_x)
>>> p, rep = posify(eq)
>>> log(p).expand().subs(rep)
-log(x)
```

It is possible to apply the same transformations to an iterable of expressions:

```
>>> eq = x**2 - 4
>>> solve(eq, x)
[{x: -2}, {x: 2}]
>>> eq_x, reps = posify([eq, x]); eq_x
[_x**2 - 4, _x]
>>> solve(*eq_x)
[{_x: 2}]
```

3.17.9 logcombine

diofant.simplify.simplify.**logcombine**(*expr*, *force=False*)

Takes logarithms and combines them using the following rules:

- log(x) + log(y) == log(x*y) if both are not negative
- a*log(x) == log(x**a) if x is positive and a is real

If `force` is True then the assumptions above will be assumed to hold if there is no assumption already in place on a quantity. For example, if `a` is imaginary or the argument negative, force will not perform a combination but if `a` is a symbol with no assumptions the change will take place.

See also:

posify **(page 781)** replace all symbols with symbols having positive assumptions

Examples

```
>>> from diofant.abc import a
>>> logcombine(a*log(x) + log(y) - log(z))
a*log(x) + log(y) - log(z)
>>> logcombine(a*log(x) + log(y) - log(z), force=True)
log(x**a*y/z)
>>> x, y, z = symbols('x y z', positive=True)
>>> a = Symbol('a', extended_real=True)
>>> logcombine(a*log(x) + log(y) - log(z))
log(x**a*y/z)
```

The transformation is limited to factors and/or terms that contain logs, so the result depends on the initial state of expansion:

```
>>> eq = (2 + 3*I)*log(x)
>>> logcombine(eq, force=True) == eq
True
>>> logcombine(eq.expand(), force=True)
log(x**2) + I*log(x**3)
```

3.17.10 Radsimp

radsimp

diofant.simplify.radsimp.**radsimp**(*expr*, *symbolic=True*, *max_terms=4*)

Rationalize the denominator by removing square roots.

Note: the expression returned from radsimp must be used with caution since if the denominator contains symbols, it will be possible to make substitutions that violate the assumptions of the simplification process: that for a denominator matching a + b*sqrt(c), a != +/-b*sqrt(c). (If there are no symbols, this assumptions is made valid by collecting terms of sqrt(c) so the match variable `a` does not contain `sqrt(c)`.) If you do not want the simplification to occur for symbolic denominators, set `symbolic` to False.

If there are more than `max_terms` radical terms then the expression is returned unchanged.

Examples

```
>>> from diofant.abc import a, b, c
```

```
>>> radsimp(1/(I + 1))
(1 - I)/2
>>> radsimp(1/(2 + sqrt(2)))
(-sqrt(2) + 2)/2
>>> x, y = map(Symbol, 'xy')
>>> e = ((2 + 2*sqrt(2))*x + (2 + sqrt(8))*y)/(2 + sqrt(2))
>>> radsimp(e)
sqrt(2)*(x + y)
```

No simplification beyond removal of the gcd is done. One might want to polish the result a little, however, by collecting square root terms:

```
>>> r2 = sqrt(2)
>>> r5 = sqrt(5)
>>> ans = radsimp(1/(y*r2 + x*r2 + a*r5 + b*r5))
>>> pprint(ans, use_unicode=False)
    ___       ___       ___       ___
  \/ 5 *a + \/ 5 *b - \/ 2 *x - \/ 2 *y
------------------------------------------
   2               2      2              2
5*a  + 10*a*b + 5*b  - 2*x  - 4*x*y - 2*y
```

```
>>> n, d = fraction(ans)
>>> pprint(factor_terms(signsimp(collect_sqrt(n))/d, radical=True), use_
↪unicode=False)
        ___             ___
      \/ 5 *(a + b) - \/ 2 *(x + y)
------------------------------------------
   2               2      2              2
5*a  + 10*a*b + 5*b  - 2*x  - 4*x*y - 2*y
```

If radicals in the denominator cannot be removed or there is no denominator, the original expression will be returned.

```
>>> radsimp(sqrt(2)*x + sqrt(2))
sqrt(2)*x + sqrt(2)
```

Results with symbols will not always be valid for all substitutions:

```
>>> eq = 1/(a + b*sqrt(c))
>>> eq.subs(a, b*sqrt(c))
1/(2*b*sqrt(c))
>>> radsimp(eq).subs(a, b*sqrt(c))
nan
```

If symbolic=False, symbolic denominators will not be transformed (but numeric denominators will still be processed):

```
>>> radsimp(eq, symbolic=False)
1/(a + b*sqrt(c))
```

rad_rationalize

diofant.simplify.radsimp.**rad_rationalize**(*num*, *den*)

Rationalize num/den by removing square roots in the denominator; num and den are sum of terms whose squares are rationals

Examples

```
>>> rad_rationalize(sqrt(3), 1 + sqrt(2)/3)
(-sqrt(3) + sqrt(6)/3, -7/9)
```

collect

diofant.simplify.radsimp.**collect**(*expr*, *syms*, *func=None*, *evaluate=True*, *exact=False*, *distribute_order_term=True*)

Collect additive terms of an expression.

This function collects additive terms of an expression with respect to a list of expression up to powers with rational exponents. By the term symbol here are meant arbitrary expressions, which can contain powers, products, sums etc. In other words symbol is a pattern which will be searched for in the expression's terms.

The input expression is not expanded by *collect()* (page 784), so user is expected to provide an expression is an appropriate form (for example, by using *expand()* (page 135) prior to calling this function). This makes *collect()* (page 784) more predictable as there is no magic happening behind the scenes. However, it is important to note, that powers of products are converted to products of powers using the *expand_power_base()* (page 143) function.

Parameters expr : Expr

an expression

syms : iterable of Symbol's

collected symbols

evaluate : boolean

First, if `evaluate` flag is set (by default), this function will return an expression with collected terms else it will return a dictionary with expressions up to rational powers as keys and collected coefficients as values.

See also:

collect_const (page 787), *collect_sqrt* (page 787), *rcollect* (page 786)

Examples

```
>>> from diofant.abc import a, b, c
```

This function can collect symbolic coefficients in polynomials or rational expressions. It will manage to find all integer or rational powers of collection variable:

```
>>> collect(a*x**2 + b*x**2 + a*x - b*x + c, x)
c + x**2*(a + b) + x*(a - b)
```

The same result can be achieved in dictionary form:

```
>>> d = collect(a*x**2 + b*x**2 + a*x - b*x + c, x, evaluate=False)
>>> d[x**2]
a + b
>>> d[x]
a - b
>>> d[1]
c
```

You can also work with multivariate polynomials. However, remember that this function is greedy so it will care only about a single symbol at time, in specification order:

```
>>> collect(x**2 + y*x**2 + x*y + y + a*y, [x, y])
x**2*(y + 1) + x*y + y*(a + 1)
```

Also more complicated expressions can be used as patterns:

```
>>> collect(a*sin(2*x) + b*sin(2*x), sin(2*x))
(a + b)*sin(2*x)
```

```
>>> collect(a*x*log(x) + b*(x*log(x)), x*log(x))
x*(a + b)*log(x)
```

You can use wildcards in the pattern:

```
>>> w = Wild('w1')
>>> collect(a*x**y - b*x**y, w**y)
x**y*(a - b)
```

It is also possible to work with symbolic powers, although it has more complicated behavior, because in this case power's base and symbolic part of the exponent are treated as a single symbol:

```
>>> collect(a*x**c + b*x**c, x)
a*x**c + b*x**c
>>> collect(a*x**c + b*x**c, x**c)
x**c*(a + b)
```

However if you incorporate rationals to the exponents, then you will get well known behavior:

```
>>> collect(a*x**(2*c) + b*x**(2*c), x**c)
x**(2*c)*(a + b)
```

Note also that all previously stated facts about *collect()* (page 784) function apply to the exponential function, so you can get:

```
>>> collect(a*exp(2*x) + b*exp(2*x), exp(x))
E**(2*x)*(a + b)
```

If you are interested only in collecting specific powers of some symbols then set `exact` flag in arguments:

```
>>> collect(a*x**7 + b*x**7, x, exact=True)
a*x**7 + b*x**7
>>> collect(a*x**7 + b*x**7, x**7, exact=True)
x**7*(a + b)
```

You can also apply this function to differential equations, where derivatives of arbitrary order can be collected. Note that if you collect with respect to a function or a derivative of a function, all derivatives of that function will also be collected. Use `exact=True` to prevent this from happening:

```
>>> f = f(x)
```

```
>>> collect(a*Derivative(f, x) + b*Derivative(f, x), Derivative(f, x))
(a + b)*Derivative(f(x), x)
```

```
>>> collect(a*Derivative(f, x, 2) + b*Derivative(f, x, 2), f)
(a + b)*Derivative(f(x), x, x)
```

```
>>> collect(a*Derivative(f, x, 2) + b*Derivative(f, x, 2), Derivative(f, x),␣
↪exact=True)
a*Derivative(f(x), x, x) + b*Derivative(f(x), x, x)
```

```
>>> collect(a*Derivative(f, x) + b*Derivative(f, x) + a*f + b*f, f)
f(x)*(a + b) + (a + b)*Derivative(f(x), x)
```

Or you can even match both derivative order and exponent at the same time:

```
>>> collect(a*Derivative(f, x, 2)**2 + b*Derivative(f, x, 2)**2, Derivative(f, x))
(a + b)*Derivative(f(x), x, x)**2
```

Finally, you can apply a function to each of the collected coefficients. For example you can factorize symbolic coefficients of polynomial:

```
>>> f = expand((x + a + 1)**3)
```

```
>>> collect(f, x, factor)
x**3 + 3*x**2*(a + 1) + 3*x*(a + 1)**2 + (a + 1)**3
```

`diofant.simplify.radsimp.`**`rcollect`**(*expr*, **vars*)

Recursively collect sums in an expression.

See also:

collect (page 784), *collect_const* (page 787), *collect_sqrt* (page 787)

Examples

```
>>> expr = (x**2*y + x*y + x + y)/(x + y)
```

```
>>> rcollect(expr, y)
(x + y*(x**2 + x + 1))/(x + y)
```

collect_sqrt

diofant.simplify.radsimp.**collect_sqrt**(*expr*, *evaluate=True*)

Return expr with terms having common square roots collected together. If evaluate is False a count indicating the number of sqrt-containing terms will be returned and, if non-zero, the terms of the Add will be returned, else the expression itself will be returned as a single term. If evaluate is True, the expression with any collected terms will be returned.

Note: since I = sqrt(-1), it is collected, too.

See also:

collect (page 784), *collect_const* (page 787), *rcollect* (page 786)

Examples

```
>>> from diofant.abc import a, b
```

```
>>> r2, r3, r5 = [sqrt(i) for i in [2, 3, 5]]
>>> collect_sqrt(a*r2 + b*r2)
sqrt(2)*(a + b)
>>> collect_sqrt(a*r2 + b*r2 + a*r3 + b*r3)
sqrt(2)*(a + b) + sqrt(3)*(a + b)
>>> collect_sqrt(a*r2 + b*r2 + a*r3 + b*r5)
sqrt(3)*a + sqrt(5)*b + sqrt(2)*(a + b)
```

If evaluate is False then the arguments will be sorted and returned as a list and a count of the number of sqrt-containing terms will be returned:

```
>>> collect_sqrt(a*r2 + b*r2 + a*r3 + b*r5, evaluate=False)
((sqrt(3)*a, sqrt(5)*b, sqrt(2)*(a + b)), 3)
>>> collect_sqrt(a*sqrt(2) + b, evaluate=False)
((b, sqrt(2)*a), 1)
>>> collect_sqrt(a + b, evaluate=False)
((a + b,), 0)
```

collect_const

diofant.simplify.radsimp.**collect_const**(*expr*, **vars*, ***kwargs*)

A non-greedy collection of terms with similar number coefficients in an Add expr. If vars is given then only those constants will be targeted. Although any Number can also be targeted, if this is not desired set Numbers=False and no Float or Rational will be collected.

See also:

collect (page 784), *collect_sqrt* (page 787), *rcollect* (page 786)

Examples

```
>>> from diofant.abc import a, s
>>> collect_const(sqrt(3) + sqrt(3)*(1 + sqrt(2)))
sqrt(3)*(sqrt(2) + 2)
>>> collect_const(sqrt(3)*s + sqrt(7)*s + sqrt(3) + sqrt(7))
(sqrt(3) + sqrt(7))*(s + 1)
>>> s = sqrt(2) + 2
>>> collect_const(sqrt(3)*s + sqrt(3) + sqrt(7)*s + sqrt(7))
(sqrt(2) + 3)*(sqrt(3) + sqrt(7))
>>> collect_const(sqrt(3)*s + sqrt(3) + sqrt(7)*s + sqrt(7), sqrt(3))
sqrt(7) + sqrt(3)*(sqrt(2) + 3) + sqrt(7)*(sqrt(2) + 2)
```

The collection is sign-sensitive, giving higher precedence to the unsigned values:

```
>>> collect_const(x - y - z)
x - (y + z)
>>> collect_const(-y - z)
-(y + z)
>>> collect_const(2*x - 2*y - 2*z, 2)
2*(x - y - z)
>>> collect_const(2*x - 2*y - 2*z, -2)
2*x - 2*(y + z)
```

fraction

diofant.simplify.radsimp.**fraction**(*expr*, *exact=False*)

Returns a pair with expression's numerator and denominator.

If the given expression is not a fraction then this function will return the tuple (expr, 1).

This function will not make any attempt to simplify nested fractions or to do any term rewriting at all.

If only one of the numerator/denominator pair is needed then use numer(expr) or denom(expr) functions respectively.

```
>>> fraction(x/y)
(x, y)
>>> fraction(x)
(x, 1)
```

```
>>> fraction(1/y**2)
(1, y**2)
```

```
>>> fraction(x*y/2)
(x*y, 2)
>>> fraction(Rational(1, 2))
(1, 2)
```

This function will also work fine with assumptions:

```
>>> k = Symbol('k', negative=True)
>>> fraction(x * y**k)
(x, y**(-k))
```

If we know nothing about sign of some exponent and 'exact' flag is unset, then structure this exponent's structure will be analyzed and pretty fraction will be returned:

```
>>> fraction(2*x**(-y))
(2, x**y)
```

```
>>> fraction(exp(-x))
(1, E**x)
```

```
>>> fraction(exp(-x), exact=True)
(E**(-x), 1)
```

3.17.11 Ratsimp

ratsimp

`diofant.simplify.ratsimp.`**`ratsimp`**`(`*expr*`)`

Put an expression over a common denominator, cancel and reduce.

Examples

```
>>> ratsimp(1/x + 1/y)
(x + y)/(x*y)
```

3.17.12 Trigonometric simplification

trigsimp

`diofant.simplify.trigsimp.`**`trigsimp`**`(`*expr*, ***opts*`)`

reduces expression by using known trig identities

See also:

`diofant.simplify.fu.fu` (page 790)

Notes

method: - Determine the method to use. Valid choices are ‘matching’ (default), ‘groebner’, ‘combined’, and ‘fu’. If ‘matching’, simplify the expression recursively by targeting common patterns. If ‘groebner’, apply an experimental groebner basis algorithm. In this case further options are forwarded to `trigsimp_groebner`, please refer to its docstring. If ‘combined’, first run the groebner basis algorithm with small default parameters, then run the ‘matching’ algorithm. ‘fu’ runs the collection of trigonometric transformations described by Fu, et al.

Examples

```
>>> e = 2*sin(x)**2 + 2*cos(x)**2
>>> trigsimp(e)
2
```

Simplification occurs wherever trigonometric functions are located.

```
>>> trigsimp(log(e))
log(2)
```

Using $method = "groebner"$ (or $"combined"$) might lead to greater simplification.

The old trigsimp routine can be accessed as with method 'old'.

```
>>> t = 3*tanh(x)**7 - 2/coth(x)**7
>>> trigsimp(t, method='old') == t
True
>>> trigsimp(t)
tanh(x)**7
```

futrig

`diofant.simplify.trigsimp.`**`futrig`**(*e*, ***kwargs*)

Return simplified `e` using Fu-like transformations. This is not the "Fu" algorithm. This is called by default from `trigsimp`. By default, hyperbolics subexpressions will be simplified, but this can be disabled by setting `hyper=False`.

Examples

```
>>> trigsimp(1/tan(x)**2)
tan(x)**(-2)
```

```
>>> futrig(sinh(x)/tanh(x))
cosh(x)
```

fu

`diofant.simplify.fu.`**`fu`**(*rv*, *measure=<function <lambda>>*)

Attempt to simplify expression by using transformation rules given in the algorithm by Fu et al.

fu() (page 790) will try to minimize the objective function `measure`. By default this first minimizes the number of trig terms and then minimizes the number of total operations.

References

http://rfdz.ph-noe.ac.at/fileadmin/Mathematik_Uploads/ACDCA/ DESTIME2006/DES_contribs/Fu/simplification.pdf

Examples

```
>>> from diofant.abc import a, b
```

```
>>> fu(sin(50)**2 + cos(50)**2 + sin(pi/6))
3/2
>>> fu(sqrt(6)*cos(x) + sqrt(2)*sin(x))
2*sqrt(2)*sin(x + pi/3)
```

CTR1 example

```
>>> eq = sin(x)**4 - cos(y)**2 + sin(y)**2 + 2*cos(x)**2
>>> fu(eq)
cos(x)**4 - 2*cos(y)**2 + 2
```

CTR2 example

```
>>> fu(S.Half - cos(2*x)/2)
sin(x)**2
```

CTR3 example

```
>>> fu(sin(a)*(cos(b) - sin(b)) + cos(a)*(sin(b) + cos(b)))
sqrt(2)*sin(a + b + pi/4)
```

CTR4 example

```
>>> fu(sqrt(3)*cos(x)/2 + sin(x)/2)
sin(x + pi/3)
```

Example 1

```
>>> fu(1-sin(2*x)**2/4-sin(y)**2-cos(x)**4)
-cos(x)**2 + cos(y)**2
```

Example 2

```
>>> fu(cos(4*pi/9))
sin(pi/18)
>>> fu(cos(pi/9)*cos(2*pi/9)*cos(3*pi/9)*cos(4*pi/9))
1/16
```

Example 3

```
>>> fu(tan(7*pi/18)+tan(5*pi/18)-sqrt(3)*tan(5*pi/18)*tan(7*pi/18))
-sqrt(3)
```

Objective function example

```
>>> fu(sin(x)/cos(x))  # default objective function
tan(x)
>>> fu(sin(x)/cos(x), measure=lambda x: -x.count_ops()) # maximize op count
sin(x)/cos(x)
```

3.17.13 Power simplification

powsimp

diofant.simplify.powsimp.**powsimp**(*expr*, *deep=False*, *combine='all'*, *force=False*, *measure=<function count_ops>*)
reduces expression by combining powers with similar bases and exponents.

Notes

If deep is True then powsimp() will also simplify arguments of functions. By default deep is set to False.

If force is True then bases will be combined without checking for assumptions, e.g. sqrt(x)*sqrt(y) -> sqrt(x*y) which is not true if x and y are both negative.

You can make powsimp() only combine bases or only combine exponents by changing combine='base' or combine='exp'. By default, combine='all', which does both. combine='base' will only combine:

```
 a   a          a                          2x      x
x * y  =>  (x*y)   as well as things like 2   =>  4
```

and combine='exp' will only combine

```
 a   b      (a + b)
x * x  =>  x
```

combine='exp' will strictly only combine exponents in the way that used to be automatic. Also use deep=True if you need the old behavior.

When combine='all', 'exp' is evaluated first. Consider the first example below for when there could be an ambiguity relating to this. This is done so things like the second example can be completely combined. If you want 'base' combined first, do something like powsimp(powsimp(expr, combine='base'), combine='exp').

Examples

```
>>> powsimp(x**y*x**z*y**z, combine='all')
x**(y + z)*y**z
>>> powsimp(x**y*x**z*y**z, combine='exp')
x**(y + z)*y**z
>>> powsimp(x**y*x**z*y**z, combine='base', force=True)
x**y*(x*y)**z
```

```
>>> powsimp(x**z*x**y*n**z*n**y, combine='all', force=True)
(n*x)**(y + z)
>>> powsimp(x**z*x**y*n**z*n**y, combine='exp')
n**(y + z)*x**(y + z)
>>> powsimp(x**z*x**y*n**z*n**y, combine='base', force=True)
(n*x)**y*(n*x)**z
```

```
>>> x, y = symbols('x y', positive=True)
>>> powsimp(log(exp(x)*exp(y)))
log(E**x*E**y)
>>> powsimp(log(exp(x)*exp(y)), deep=True)
x + y
```

Radicals with Mul bases will be combined if combine='exp'

```
>>> x, y = symbols('x y')
```

Two radicals are automatically joined through Mul:

```
>>> a = sqrt(x*sqrt(y))
>>> a*a**3 == a**4
True
```

But if an integer power of that radical has been autoexpanded then Mul does not join the resulting factors:

```
>>> a**4  # auto expands to a Mul, no longer a Pow
x**2*y
>>> _*a  # so Mul doesn't combine them
x**2*y*sqrt(x*sqrt(y))
>>> powsimp(_)  # but powsimp will
(x*sqrt(y))**(5/2)
>>> powsimp(x*y*a)  # but won't when doing so would violate assumptions
x*y*sqrt(x*sqrt(y))
```

powdenest

`diofant.simplify.powsimp.powdenest`(*eq*, *force=False*, *polar=False*)

Collect exponents on powers as assumptions allow.

Given `(bb**be)**e`, this can be simplified as follows:

- if `bb` is positive, or
- `e` is an integer, or
- `|be| < 1` then this simplifies to `bb**(be*e)`

Given a product of powers raised to a power, `(bb1**be1 * bb2**be2...)**e`, simplification can be done as follows:

- if e is positive, the gcd of all bei can be joined with e;
- all non-negative bb can be separated from those that are negative and their gcd can be joined with e; autosimplification already handles this separation.
- integer factors from powers that have integers in the denominator of the exponent can be removed from any term and the gcd of such integers can be joined with e

Setting `force` to True will make symbols that are not explicitly negative behave as though they are positive, resulting in more denesting.

Setting `polar` to True will do simplifications on the Riemann surface of the logarithm, also resulting in more denestings.

When there are sums of logs in exp() then a product of powers may be obtained e.g. `exp(3*(log(a) + 2*log(b)))` - > `a**3*b**6`.

Examples

```
>>> from diofant.abc import a, b
```

```
>>> powdenest((x**(2*a/3))**(3*x))
(x**(2*a/3))**(3*x)
>>> powdenest(exp(3*x*log(2)))
2**(3*x)
```

Assumptions may prevent expansion:

```
>>> powdenest(sqrt(x**2))
sqrt(x**2)
```

```
>>> p = symbols('p', positive=True)
>>> powdenest(sqrt(p**2))
p
```

No other expansion is done.

```
>>> i, j = symbols('i j', integer=True)
>>> powdenest((x**x)**(i + j))  # -X-> (x**x)**i*(x**x)**j
x**(x*(i + j))
```

But exp() will be denested by moving all non-log terms outside of the function; this may result in the collapsing of the exp to a power with a different base:

```
>>> powdenest(exp(3*y*log(x)))
x**(3*y)
>>> powdenest(exp(y*(log(a) + log(b))))
(a*b)**y
>>> powdenest(exp(3*(log(a) + log(b))))
a**3*b**3
```

If assumptions allow, symbols can also be moved to the outermost exponent:

```
>>> i = Symbol('i', integer=True)
>>> powdenest(((x**(2*i))**(3*y))**x)
((x**(2*i))**(3*y))**x
>>> powdenest(((x**(2*i))**(3*y))**x, force=True)
x**(6*i*x*y)
```

```
>>> powdenest(((x**(2*a/3))**(3*y/i))**x)
((x**(2*a/3))**(3*y/i))**x
>>> powdenest((x**(2*i)*y**(4*i))**z, force=True)
(x*y**2)**(2*i*z)
```

```
>>> n = Symbol('n', negative=True)
```

```
>>> powdenest((x**i)**y, force=True)
x**(i*y)
>>> powdenest((n**i)**x, force=True)
(n**i)**x
```

3.17.14 Combinatorial simplification

combsimp

`diofant.simplify.combsimp.combsimp(expr)`

Simplify combinatorial expressions.

This function takes as input an expression containing factorials, binomials, Pochhammer symbol and other "combinatorial" functions, and tries to minimize the number of those functions and reduce the size of their arguments.

The algorithm works by rewriting all combinatorial functions as expressions involving rising factorials (Pochhammer symbols) and applies recurrence relations and other transformations applicable to rising factorials, to reduce their arguments, possibly letting the resulting rising factorial to cancel. Rising factorials with the second argument being an integer are expanded into polynomial forms and finally all other rising factorial are rewritten in terms of more familiar functions. If the initial expression consisted of gamma functions alone, the result is expressed in terms of gamma functions. If the initial expression consists of gamma function with some other combinatorial, the result is expressed in terms of gamma functions.

If the result is expressed using gamma functions, the following three additional steps are performed:

1. Reduce the number of gammas by applying the reflection theorem gamma(x)*gamma(1-x) == pi/sin(pi*x).
2. Reduce the number of gammas by applying the multiplication theorem gamma(x)*gamma(x+1/n)*...*gamma(x+(n-1)/n) == C*gamma(n*x).
3. Reduce the number of prefactors by absorbing them into gammas, where possible.

All transformation rules can be found (or was derived from) here:

1. http://functions.wolfram.com/GammaBetaErf/Pochhammer/17/01/02/
2. http://functions.wolfram.com/GammaBetaErf/Pochhammer/27/01/0005/

Examples

```
>>> combsimp(factorial(n)/factorial(n - 3))
n*(n - 2)*(n - 1)
>>> combsimp(binomial(n+1, k+1)/binomial(n, k))
(n + 1)/(k + 1)
```

3.17.15 Square Root Denest

sqrtdenest

`diofant.simplify.sqrtdenest.sqrtdenest(expr, max_iter=3)`

Denests sqrts in an expression that contain other square roots if possible, otherwise returns the expr unchanged. This is based on the algorithms of [1].

See also:

unrad (page 796)

References

[R496] (page 1264), *[R497]* (page 1264)

Examples

```
>>> sqrtdenest(sqrt(5 + 2 * sqrt(6)))
sqrt(2) + sqrt(3)
```

unrad

`diofant.simplify.sqrtdenest.`**`unrad`**(*eq*, **syms*, ***flags*)

Remove radicals with symbolic arguments and return (eq, cov), None or raise an error:

None is returned if there are no radicals to remove.

NotImplementedError is raised if there are radicals and they cannot be removed or if the relationship between the original symbols and the change of variable needed to rewrite the system as a polynomial cannot be solved.

Otherwise the tuple, `(eq, cov)`, is returned where:

```
``eq``, ``cov``
    ``eq`` is an equation without radicals (in the symbol(s) of
    interest) whose solutions are a superset of the solutions to the
    original expression. ``eq`` might be re-written in terms of a new
    variable; the relationship to the original variables is given by
    ``cov`` which is a list containing ``v`` and ``v**p - b`` where
    ``p`` is the power needed to clear the radical and ``b`` is the
    radical now expressed as a polynomial in the symbols of interest.
    For example, for sqrt(2 - x) the tuple would be
    ``(c, c**2 - 2 + x)``. The solutions of ``eq`` will contain
    solutions to the original equation (if there are any).
```

`syms` an iterable of symbols which, if provided, will limit the focus of radical removal: only radicals with one or more of the symbols of interest will be cleared. All free symbols are used if `syms` is not set.

`flags` are used internally for communication during recursive calls. Two options are also recognized:

```
``take``, when defined, is interpreted as a single-argument function
that returns True if a given Pow should be handled.
```

Radicals can be removed from an expression if:

```
*   all bases of the radicals are the same; a change of variables is
    done in this case.
*   if all radicals appear in one term of the expression
*   there are only 4 terms with sqrt() factors or there are less than
    four terms having sqrt() factors
*   there are only two terms with radicals
```

Examples

```
>>> unrad(sqrt(x)*cbrt(x) + 2)
(x**5 - 64, [])
>>> unrad(sqrt(x) + root(x + 1, 3))
(x**3 - x**2 - 2*x - 1, [])
>>> eq = sqrt(x) + root(x, 3) - 2
>>> unrad(eq)
(_p**3 + _p**2 - 2, [_p, -x + _p**6])
```

3.17.16 Common Subexpresion Elimination

cse

`diofant.simplify.cse_main.`**`cse`**(*exprs*, *symbols=None*, *optimizations=None*, *postprocess=None*, *order='canonical'*)

Perform common subexpression elimination on an expression.

Parameters exprs : list of diofant expressions, or a single diofant expression

The expressions to reduce.

symbols : infinite iterator yielding unique Symbols

The symbols used to label the common subexpressions which are pulled out. The `numbered_symbols` generator is useful. The default is a stream of symbols of the form "x0", "x1", etc. This must be an infinite iterator.

optimizations : list of (callable, callable) pairs

The (preprocessor, postprocessor) pairs of external optimization functions. Optionally 'basic' can be passed for a set of predefined basic optimizations. Such 'basic' optimizations were used by default in old implementation, however they can be really slow on larger expressions. Now, no pre or post optimizations are made by default.

postprocess : a function which accepts the two return values of cse and

returns the desired form of output from cse, e.g. if you want the replacements reversed the function might be the following lambda: lambda r, e: return reversed(r), e

order : string, 'none' or 'canonical'

The order by which Mul and Add arguments are processed. If set to 'canonical', arguments will be canonically ordered. If set to 'none', ordering will be faster but dependent on expressions hashes, thus machine dependent and variable. For large expressions where speed is a concern, use the setting order='none'.

Returns replacements : list of (Symbol, expression) pairs

All of the common subexpressions that were replaced. Subexpressions earlier in this list might show up in subexpressions later in this list.

reduced_exprs : list of diofant expressions

The reduced expressions with all of the replacements above.

Examples

```
>>> from diofant.abc import w
>>> cse(((w + x + y + z)*(w + y + z))/(w + x)**3)
([(x0, y + z), (x1, w + x)], [(w + x0)*(x0 + x1)/x1**3])
```

Note that currently, y + z will not get substituted if -y - z is used.

```
>>> cse(((w + x + y + z)*(w - y - z))/(w + x)**3)
([(x0, w + x)], [(w - y - z)*(x0 + y + z)/x0**3])
```

List of expressions with recursive substitutions:

```
>>> m = SparseMatrix([x + y, x + y + z])
>>> cse([(x+y)**2, x + y + z, y + z, x + z + y, m])
([(x0, x + y), (x1, x0 + z)], [x0**2, x1, y + z, x1, Matrix([
[x0],
[x1]])])
```

Note: the type and mutability of input matrices is retained.

```
>>> isinstance(_[1][-1], SparseMatrix)
True
```

opt_cse

diofant.simplify.cse_main.**opt_cse**(*exprs*, *order='canonical'*)

Find optimization opportunities in Adds, Muls, Pows and negative coefficient Muls

Parameters exprs : list of diofant expressions

The expressions to optimize.

order : string, 'none' or 'canonical'

The order by which Mul and Add arguments are processed. For large expressions where speed is a concern, use the setting order='none'.

Returns opt_subs : dictionary of expression substitutions

The expression substitutions which can be useful to optimize CSE.

Examples

```
>>> opt_subs = opt_cse([x**-2])
>>> opt_subs
{x**(-2): 1/(x**2)}
```

tree_cse

diofant.simplify.cse_main.**tree_cse**(*exprs*, *symbols*, *opt_subs={}*, *order='canonical'*)

Perform raw CSE on expression tree, taking opt_subs into account.

Parameters **exprs** : list of diofant expressions

The expressions to reduce.

symbols : infinite iterator yielding unique Symbols

The symbols used to label the common subexpressions which are pulled out.

opt_subs : dictionary of expression substitutions

The expressions to be substituted before any CSE action is performed.

order : string, 'none' or 'canonical'

The order by which Mul and Add arguments are processed. For large expressions where speed is a concern, use the setting order='none'.

3.17.17 Hypergeometric Function Expansion

hyperexpand

`diofant.simplify.hyperexpand.`**`hyperexpand`**(*f*, *allow_hyper=False*, *rewrite='default'*)

Expand hypergeometric functions. If allow_hyper is True, allow partial simplification (that is a result different from input, but still containing hypergeometric functions).

Examples

```
>>> hyperexpand(hyper([], [], z))
E**z
```

Non-hyperegeometric parts of the expression and hypergeometric expressions that are not recognised are left unchanged:

```
>>> hyperexpand(1 + hyper([1, 1, 1], [], z))
hyper((1, 1, 1), (), z) + 1
```

3.17.18 Traversal Tools

use

`diofant.simplify.traversaltools.`**`use`**(*expr*, *func*, *level=0*, *args=()*, *kwargs={}*)

Use `func` to transform `expr` at the given level.

Examples

```
>>> f = (x + y)**2*x + 1
```

```
>>> use(f, expand, level=2)
x*(x**2 + 2*x*y + y**2) + 1
>>> expand(f)
x**3 + 2*x**2*y + x*y**2 + 1
```

3.17.19 EPath Tools

EPath class

class diofant.simplify.epathtools.**EPath**

Manipulate expressions using paths.

EPath grammar in EBNF notation:

```
literal   ::= /[A-Za-z_][A-Za-z_0-9]*/
number    ::= /-?\d+/
type      ::= literal
attribute ::= literal "?"
all       ::= "*"
slice     ::= "[" number? (":" number? (":" number?)?)? "]"
range     ::= all | slice
query     ::= (type | attribute) ("|" (type | attribute))*
selector  ::= range | query range?
path      ::= "/" selector ("/" selector)*
```

See also:

epath (page 801)

apply(*expr, func, args=None, kwargs=None*)

Modify parts of an expression selected by a path.

Examples

```
>>> from diofant.abc import t
```

```
>>> path = EPath("/*/[0]/Symbol")
>>> expr = [((x, 1), 2), ((3, y), z)]
```

```
>>> path.apply(expr, lambda expr: expr**2)
[((x**2, 1), 2), ((3, y**2), z)]
```

```
>>> path = EPath("/*/*/Symbol")
>>> expr = t + sin(x + 1) + cos(x + y + E)
```

```
>>> path.apply(expr, lambda expr: 2*expr)
t + sin(2*x + 1) + cos(2*x + 2*y + E)
```

select(*expr*)

Retrieve parts of an expression selected by a path.

Examples

```
>>> from diofant.abc import t
```

```
>>> path = EPath("/*/[0]/Symbol")
>>> expr = [((x, 1), 2), ((3, y), z)]
```

```
>>> path.select(expr)
[x, y]
```

```
>>> path = EPath("/*/*/Symbol")
>>> expr = t + sin(x + 1) + cos(x + y + E)
```

```
>>> path.select(expr)
[x, x, y]
```

epath

`diofant.simplify.epathtools.`**`epath`**(*path*, *expr=None*, *func=None*, *args=None*, *kwargs=None*)

Manipulate parts of an expression selected by a path.

This function allows to manipulate large nested expressions in single line of code, utilizing techniques to those applied in XML processing standards (e.g. XPath).

If `func` is `None`, *`epath()`* (page 801) retrieves elements selected by the `path`. Otherwise it applies `func` to each matching element.

Note that it is more efficient to create an EPath object and use the select and apply methods of that object, since this will compile the path string only once. This function should only be used as a convenient shortcut for interactive use.

This is the supported syntax:

- **select all: `/*`** Equivalent of `for arg in args:`.
- **select slice: `/[0]` or `/[1:5]` or `/[1:5:2]`** Supports standard Python's slice syntax.
- **select by type: `/list` or `/list|tuple`** Emulates `isinstance`.
- **select by attribute: `/__iter__?`** Emulates `hasattr`.

Parameters **path** : str | EPath

A path as a string or a compiled EPath.

expr : Basic | iterable

An expression or a container of expressions.

func : callable (optional)

A callable that will be applied to matching parts.

args : tuple (optional)

Additional positional arguments to `func`.

kwargs : dict (optional)

Additional keyword arguments to func.

Examples

```
>>> from diofant.abc import t
```

```
>>> path = "/*/[0]/Symbol"
>>> expr = [((x, 1), 2), ((3, y), z)]
```

```
>>> epath(path, expr)
[x, y]
>>> epath(path, expr, lambda expr: expr**2)
[((x**2, 1), 2), ((3, y**2), z)]
```

```
>>> path = "/*/*/Symbol"
>>> expr = t + sin(x + 1) + cos(x + y + E)
```

```
>>> epath(path, expr)
[x, x, y]
>>> epath(path, expr, lambda expr: 2*expr)
t + sin(2*x + 1) + cos(2*x + 2*y + E)
```

3.18 Stats

Diofant statistics module

Introduces a random variable type into the Diofant language.

Random variables may be declared using prebuilt functions such as Normal, Exponential, Coin, Die, etc… or built with functions like FiniteRV.

Queries on random expressions can be made using the functions

Expression	Meaning
P(condition)	Probability
E(expression)	Expected value
variance(expression)	Variance
density(expression)	Probability Density Function
sample(expression)	Produce a realization
where(condition)	Where the condition is true

3.18.1 Examples

```
>>> from diofant.stats import P, E, variance, Die, Normal
>>> X, Y = Die('X', 6), Die('Y', 6) # Define two six sided dice
>>> Z = Normal('Z', 0, 1) # Declare a Normal random variable with mean 0, std 1
>>> P(X>3) # Probability X is greater than 3
1/2
>>> E(X+Y) # Expectation of the sum of two dice
7
```

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```
>>> variance(X+Y) # Variance of the sum of two dice
35/6
>>> simplify(P(Z>1)) # Probability of Z being greater than 1
-erf(sqrt(2)/2)/2 + 1/2
```

3.18.2 Random Variable Types

Finite Types

diofant.stats.**DiscreteUniform**(*name*, *items*)

Create a Finite Random Variable representing a uniform distribution over the input set.

Returns a RandomSymbol.

Examples

```
>>> from diofant.stats import density
```

```
>>> X = DiscreteUniform('X', symbols('a b c')) # equally likely over a, b, c
>>> density(X).dict
{a: 1/3, b: 1/3, c: 1/3}
```

```
>>> Y = DiscreteUniform('Y', list(range(5))) # distribution over a range
>>> density(Y).dict
{0: 1/5, 1: 1/5, 2: 1/5, 3: 1/5, 4: 1/5}
```

diofant.stats.**Die**(*name*, *sides=6*)

Create a Finite Random Variable representing a fair die.

Returns a RandomSymbol.

```
>>> from diofant.stats import density
```

```
>>> D6 = Die('D6', 6) # Six sided Die
>>> density(D6).dict
{1: 1/6, 2: 1/6, 3: 1/6, 4: 1/6, 5: 1/6, 6: 1/6}
```

```
>>> D4 = Die('D4', 4) # Four sided Die
>>> density(D4).dict
{1: 1/4, 2: 1/4, 3: 1/4, 4: 1/4}
```

diofant.stats.**Bernoulli**(*name*, *p*, *succ=1*, *fail=0*)

Create a Finite Random Variable representing a Bernoulli process.

Returns a RandomSymbol

```
>>> from diofant.stats import density
```

```
>>> X = Bernoulli('X', Rational(3, 4)) # 1-0 Bernoulli variable, probability = 3/4
>>> density(X).dict
{0: 1/4, 1: 3/4}
```

```
>>> X = Bernoulli('X', S.Half, 'Heads', 'Tails') # A fair coin toss
>>> density(X).dict
{Heads: 1/2, Tails: 1/2}
```

diofant.stats.**Coin**(*name*, *p=Rational(1, 2)*)

Create a Finite Random Variable representing a Coin toss.

Probability p is the chance of getting "Heads." Half by default

Returns a RandomSymbol.

```
>>> from diofant.stats import density
```

```
>>> H, T = Symbol('H'), Symbol('T')
```

```
>>> C = Coin('C') # A fair coin toss
>>> density(C).dict
{H: 1/2, T: 1/2}
```

```
>>> C2 = Coin('C2', Rational(3, 5)) # An unfair coin
>>> density(C2).dict
{H: 3/5, T: 2/5}
```

diofant.stats.**Binomial**(*name*, *n*, *p*, *succ=1*, *fail=0*)

Create a Finite Random Variable representing a binomial distribution.

Returns a RandomSymbol.

Examples

```
>>> from diofant.stats import density
```

```
>>> X = Binomial('X', 4, S.Half) # Four "coin flips"
>>> density(X).dict
{0: 1/16, 1: 1/4, 2: 3/8, 3: 1/4, 4: 1/16}
```

diofant.stats.**Hypergeometric**(*name*, *N*, *m*, *n*)

Create a Finite Random Variable representing a hypergeometric distribution.

Returns a RandomSymbol.

Examples

```
>>> from diofant.stats import density
```

```
>>> X = Hypergeometric('X', 10, 5, 3) # 10 marbles, 5 white (success), 3 draws
>>> density(X).dict
{0: 1/12, 1: 5/12, 2: 5/12, 3: 1/12}
```

diofant.stats.**FiniteRV**(*name*, *density*)

Create a Finite Random Variable given a dict representing the density.

Returns a RandomSymbol.

```
>>> from diofant.stats import P, E
```

```
>>> density = {0: .1, 1: .2, 2: .3, 3: .4}
>>> X = FiniteRV('X', density)
```

```
>>> E(X)
2.00000000000000
>>> P(X >= 2)
0.700000000000000
```

Discrete Types

`diofant.stats.`**`Geometric`**`(`*name, p*`)`

Create a discrete random variable with a Geometric distribution.

The density of the Geometric distribution is given by

$$f(k) := p(1-p)^{k-1}$$

Parameters p: A probability between 0 and 1

Returns A RandomSymbol.

References

[1] https//en.wikipedia.org/wiki/Geometric_distribution [2] http://mathworld.wolfram.com/GeometricDistribution.html

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> p = S.One / 5
>>> z = Symbol("z")
```

```
>>> X = Geometric("x", p)
```

```
>>> density(X)(z)
(4/5)**(z - 1)/5
```

```
>>> E(X)
5
```

```
>>> variance(X)
20
```

`diofant.stats.`**`Poisson`**`(`*name, lamda*`)`

Create a discrete random variable with a Poisson distribution.

The density of the Poisson distribution is given by

$$f(k) := \frac{\lambda^k e^{-\lambda}}{k!}$$

Parameters **lamda: Positive number, a rate**

Returns A RandomSymbol.

References

[1] https//en.wikipedia.org/wiki/Poisson_distribution [2] http://mathworld.wolfram.com/PoissonDistribution.html

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> rate = Symbol("lambda", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Poisson("x", rate)
```

```
>>> density(X)(z)
E**(-lambda)*lambda**z/factorial(z)
```

```
>>> E(X)
lambda
```

```
>>> simplify(variance(X))
lambda
```

Continuous Types

diofant.stats.**Arcsin**(*name, a=0, b=1*)

Create a Continuous Random Variable with an arcsin distribution.

The density of the arcsin distribution is given by

$$f(x) := \frac{1}{\pi\sqrt{(x-a)(b-x)}}$$

with $x \in [a, b]$. It must hold that $-\infty < a < b < \infty$.

Parameters **a** : Real number, the left interval boundary

b : Real number, the right interval boundary

Returns A RandomSymbol.

References

[R553] (page 1264)

Examples

```
>>> from diofant.stats import density
```

```
>>> a = Symbol("a", extended_real=True)
>>> b = Symbol("b", extended_real=True)
>>> z = Symbol("z")
```

```
>>> X = Arcsin("x", a, b)
```

```
>>> density(X)(z)
1/(pi*sqrt((-a + z)*(b - z)))
```

diofant.stats.**Benini**(*name*, *alpha*, *beta*, *sigma*)

Create a Continuous Random Variable with a Benini distribution.

The density of the Benini distribution is given by

$$f(x) := e^{-\alpha \log \frac{x}{\sigma} - \beta \log^2 \left[\frac{x}{\sigma}\right]} \left(\frac{\alpha}{x} + \frac{2\beta \log \frac{x}{\sigma}}{x}\right)$$

This is a heavy-tailed distribution and is also known as the log-Rayleigh distribution.

Parameters alpha : Real number, $\alpha > 0$, a shape

beta : Real number, $\beta > 0$, a shape

sigma : Real number, $\sigma > 0$, a scale

Returns A RandomSymbol.

References

[R554] (page 1264), *[R555]* (page 1264)

Examples

```
>>> from diofant.stats import density
```

```
>>> alpha = Symbol("alpha", positive=True)
>>> beta = Symbol("beta", positive=True)
>>> sigma = Symbol("sigma", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Benini("x", alpha, beta, sigma)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
            /  z  \           2/  z  \ /                 /  z  \\
 - alpha*log|-----| - beta*log |-----| |        2*beta*log|-----||
            \sigma/            \sigma/ |alpha             \sigma/|
E                                     *|----- + -----------------|
                                       \  z             z        /
```

`diofant.stats.`**`Beta`**(*name*, *alpha*, *beta*)

Create a Continuous Random Variable with a Beta distribution.

The density of the Beta distribution is given by

$$f(x) := \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}$$

with $x \in [0,1]$.

Parameters alpha : Real number, $\alpha > 0$, a shape

beta : Real number, $\beta > 0$, a shape

Returns A RandomSymbol.

References

[R556] (page 1264), [R557] (page 1264)

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> alpha = Symbol("alpha", positive=True)
>>> beta = Symbol("beta", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Beta("x", alpha, beta)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
 alpha - 1         beta - 1
z         *(-z + 1)
---------------------------
     beta(alpha, beta)
```

```
>>> expand_func(simplify(E(X, meijerg=True)))
alpha/(alpha + beta)
```

`diofant.stats.`**`BetaPrime`**(*name*, *alpha*, *beta*)

Create a continuous random variable with a Beta prime distribution.

The density of the Beta prime distribution is given by

$$f(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$$

with $x > 0$.

Parameters alpha : Real number, $\alpha > 0$, a shape

beta : Real number, $\beta > 0$, a shape

Returns A RandomSymbol.

References

[R558] (page 1264), [R559] (page 1264)

Examples

```
>>> from diofant.stats import density
```

```
>>> alpha = Symbol("alpha", positive=True)
>>> beta = Symbol("beta", positive=True)
>>> z = Symbol("z")
```

```
>>> X = BetaPrime("x", alpha, beta)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
 alpha - 1        -alpha - beta
z         *(z + 1)
-------------------------------
       beta(alpha, beta)
```

`diofant.stats.`**`Cauchy`**`(`*name*, *x0*, *gamma*`)`

Create a continuous random variable with a Cauchy distribution.

The density of the Cauchy distribution is given by

$$f(x) := \frac{1}{\pi} \arctan\left(\frac{x - x_0}{\gamma}\right) + \frac{1}{2}$$

Parameters **x0** : Real number, the location

gamma : Real number, $\gamma > 0$, the scale

Returns A RandomSymbol.

References

[R560] (page 1264), [R561] (page 1264)

Examples

```
>>> from diofant.stats import density
```

```
>>> x0 = Symbol("x0")
>>> gamma = Symbol("gamma", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Cauchy("x", x0, gamma)
```

```
>>> density(X)(z)
1/(pi*gamma*(1 + (-x0 + z)**2/gamma**2))
```

diofant.stats.**Chi**(*name*, *k*)

Create a continuous random variable with a Chi distribution.

The density of the Chi distribution is given by

$$f(x) := \frac{2^{1-k/2}x^{k-1}e^{-x^2/2}}{\Gamma(k/2)}$$

with $x \geq 0$.

Parameters k : A positive Integer, $k > 0$, the number of degrees of freedom

Returns A RandomSymbol.

References

[R562] (page 1264), [R563] (page 1264)

Examples

```
>>> from diofant.stats import density, E, std, Chi
```

```
>>> k = Symbol("k", integer=True)
>>> z = Symbol("z")
```

```
>>> X = Chi("x", k)
```

```
>>> density(X)(z)
2**(-k/2 + 1)*E**(-z**2/2)*z**(k - 1)/gamma(k/2)
```

diofant.stats.**ChiNoncentral**(*name*, *k*, *l*)

Create a continuous random variable with a non-central Chi distribution.

The density of the non-central Chi distribution is given by

$$f(x) := \frac{e^{-(x^2+\lambda^2)/2}x^k\lambda}{(\lambda x)^{k/2}}I_{k/2-1}(\lambda x)$$

with $x \geq 0$. Here, $I_\nu(x)$ is the *modified Bessel function of the first kind* (page 381).

Parameters k : A positive Integer, $k > 0$, the number of degrees of freedom

l : Shift parameter

Returns A RandomSymbol.

References

[R564] (page 1264)

Examples

```
>>> from diofant.stats import density, E, std
```

```
>>> k = Symbol("k", integer=True)
>>> l = Symbol("l")
>>> z = Symbol("z")
```

```
>>> X = ChiNoncentral("x", k, l)
```

```
>>> density(X)(z)
E**(-l**2/2 - z**2/2)*l*z**k*(l*z)**(-k/2)*besseli(k/2 - 1, l*z)
```

diofant.stats.**ChiSquared**(*name*, *k*)

Create a continuous random variable with a Chi-squared distribution.

The density of the Chi-squared distribution is given by

$$f(x) := \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)}x^{\frac{k}{2}-1}e^{-\frac{x}{2}}$$

with $x \geq 0$.

Parameters k : A positive Integer, $k > 0$, the number of degrees of freedom

Returns A RandomSymbol.

References

[R565] (page 1264), [R566] (page 1264)

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> k = Symbol("k", integer=True, positive=True)
>>> z = Symbol("z")
```

```
>>> X = ChiSquared("x", k)
```

```
>>> density(X)(z)
2**(-k/2)*E**(-z/2)*z**(k/2 - 1)/gamma(k/2)
```

```
>>> combsimp(E(X))
k
```

```
>>> simplify(expand_func(variance(X)))
2*k
```

diofant.stats.**Dagum**(*name*, *p*, *a*, *b*)

Create a continuous random variable with a Dagum distribution.

The density of the Dagum distribution is given by

$$f(x) := \frac{ap}{x}\left(\frac{\left(\frac{x}{b}\right)^{ap}}{\left(\left(\frac{x}{b}\right)^{a}+1\right)^{p+1}}\right)$$

with $x > 0$.

Parameters p : Real number, $p > 0$, a shape

a : Real number, $a > 0$, a shape

b : Real number, $b > 0$, a scale

Returns A RandomSymbol.

References

[R567] (page 1264)

Examples

```
>>> from diofant.stats import density
```

```
>>> p = Symbol("p", positive=True)
>>> b = Symbol("b", positive=True)
>>> a = Symbol("a", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Dagum("x", p, a, b)
```

```
>>> density(X)(z)
a*p*(z/b)**(a*p)*((z/b)**a + 1)**(-p - 1)/z
```

diofant.stats.**Erlang**(*name*, *k*, *l*)

Create a continuous random variable with an Erlang distribution.

The density of the Erlang distribution is given by

$$f(x) := \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$$

with $x \in [0, \infty]$.

Parameters k : Integer

l : Real number, $\lambda > 0$, the rate

Returns A RandomSymbol.

References

[R568] (page 1264), [R569] (page 1264)

Examples

```
>>> from diofant.stats import density, cdf, E, variance
```

```
>>> k = Symbol("k", integer=True, positive=True)
>>> l = Symbol("l", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Erlang("x", k, l)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
 -l*z  k   k - 1
E    *l *z
---------------
    gamma(k)
```

```
>>> C = cdf(X, meijerg=True)(z)
>>> pprint(C, use_unicode=False)
/  k*lowergamma(k, 0)   k*lowergamma(k, l*z)
|- ------------------ + --------------------  for z >= 0
<     gamma(k + 1)          gamma(k + 1)
|
\                     0                       otherwise
```

```
>>> simplify(E(X))
k/l
```

```
>>> simplify(variance(X))
k/l**2
```

`diofant.stats.`**`Exponential`**`(`*name*, *rate*`)`

Create a continuous random variable with an Exponential distribution.

The density of the exponential distribution is given by

$$f(x) := \lambda \exp(-\lambda x)$$

with $x > 0$. Note that the expected value is $1/\lambda$.

Parameters rate : A positive Real number, $\lambda > 0$, the rate (or inverse scale/inverse mean)

Returns A RandomSymbol.

References

[R570] (page 1264), *[R571]* (page 1264)

Examples

```
>>> from diofant.stats import density, cdf, E, variance, std, skewness
```

```
>>> l = Symbol("lambda", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Exponential("x", l)
```

```
>>> density(X)(z)
E**(-lambda*z)*lambda
```

```
>>> cdf(X)(z)
Piecewise((1 - E**(-lambda*z), z >= 0), (0, true))
```

```
>>> E(X)
1/lambda
```

```
>>> variance(X)
lambda**(-2)
```

```
>>> skewness(X)
2
```

```
>>> X = Exponential('x', 10)
```

```
>>> density(X)(z)
10*E**(-10*z)
```

```
>>> E(X)
1/10
```

```
>>> std(X)
1/10
```

diofant.stats.**FDistribution**(*name*, *d1*, *d2*)

Create a continuous random variable with a F distribution.

The density of the F distribution is given by

$$f(x) := \frac{\sqrt{\frac{(d_1 x)^{d_1} d_2^{d_2}}{(d_1 x + d_2)^{d_1 + d_2}}}}{x \mathrm{B} \left(\frac{d_1}{2}, \frac{d_2}{2}\right)}$$

with $x > 0$.

Parameters d1 : $d_1 > 0$ a parameter

d2 : $d_2 > 0$ a parameter

Returns A RandomSymbol.

References

[R572] (page 1264), [R573] (page 1264)

Examples

```
>>> from diofant.stats import density
```

```
>>> d1 = Symbol("d1", positive=True)
>>> d2 = Symbol("d2", positive=True)
>>> z = Symbol("z")
```

```
>>> X = FDistribution("x", d1, d2)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
  d2
  --    ______________________________
  2    /       d1            -d1 - d2
d2  *\/  (d1*z)  *(d1*z + d2)
--------------------------------------
                   /d1  d2\
             z*beta|--, --|
                   \2   2 /
```

diofant.stats.**FisherZ**(*name*, *d1*, *d2*)

Create a Continuous Random Variable with an Fisher's Z distribution.

The density of the Fisher's Z distribution is given by

$$f(x) := \frac{2d_1^{d_1/2} d_2^{d_2/2}}{\mathrm{B}(d_1/2, d_2/2)} \frac{e^{d_1 z}}{\left(d_1 e^{2z} + d_2\right)^{\left(d_1 + d_2\right)/2}}$$

Parameters **d1** : $d_1 > 0$, degree of freedom

d2 : $d_2 > 0$, degree of freedom

Returns A RandomSymbol.

References

[R574] (page 1264), *[R575]* (page 1264)

Examples

```
>>> from diofant.stats import density
```

```
>>> d1 = Symbol("d1", positive=True)
>>> d2 = Symbol("d2", positive=True)
>>> z = Symbol("z")
```

```
>>> X = FisherZ("x", d1, d2)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
                                  d1   d2
         d1   d2               - -- - --
         --   --                 2    2
```

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```
   d1*z   2    2  / 2*z        \
2*E    *d1  *d2  *\E   *d1 + d2/
-----------------------------------------
                    /d1  d2\
               beta|--, --|
                    \2   2 /
```

diofant.stats.**Frechet**(*name*, *a*, *s=1*, *m=0*)

Create a continuous random variable with a Frechet distribution.

The density of the Frechet distribution is given by

$$f(x) := \frac{\alpha}{s}\left(\frac{x-m}{s}\right)^{-1-\alpha} e^{-(\frac{x-m}{s})^{-\alpha}}$$

with $x \geq m$.

Parameters **a** : Real number, $a \in (0, \infty)$ the shape

s : Real number, $s \in (0, \infty)$ the scale

m : Real number, $m \in (-\infty, \infty)$ the minimum

Returns A RandomSymbol.

References

[R576] (page 1264)

Examples

```
>>> from diofant.stats import density, E, std
```

```
>>> a = Symbol("a", positive=True)
>>> s = Symbol("s", positive=True)
>>> m = Symbol("m", extended_real=True)
>>> z = Symbol("z")
```

```
>>> X = Frechet("x", a, s, m)
```

```
>>> density(X)(z)
E**(-((-m + z)/s)**(-a))*a*((-m + z)/s)**(-a - 1)/s
```

diofant.stats.**Gamma**(*name*, *k*, *theta*)

Create a continuous random variable with a Gamma distribution.

The density of the Gamma distribution is given by

$$f(x) := \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}}$$

with $x \in [0, 1]$.

Parameters **k** : Real number, $k > 0$, a shape

theta : Real number, $\theta > 0$, a scale

Returns A RandomSymbol.

References

[R577] (page 1264), [R578] (page 1265)

Examples

```
>>> from diofant.stats import density, cdf, E, variance
```

```
>>> k = Symbol("k", positive=True)
>>> theta = Symbol("theta", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Gamma("x", k, theta)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
  -z
 -----
 theta      -k  k - 1
E     *theta  *z
---------------------
       gamma(k)
```

```
>>> C = cdf(X, meijerg=True)(z)
>>> pprint(C, use_unicode=False)
/                                  /     z  \
|                       k*lowergamma|k, -----|
|  k*lowergamma(k, 0)              \   theta/
<- ------------------ + ---------------------  for z >= 0
|     gamma(k + 1)           gamma(k + 1)
|
\                      0                       otherwise
```

```
>>> E(X)
theta*gamma(k + 1)/gamma(k)
```

```
>>> V = simplify(variance(X))
>>> pprint(V, use_unicode=False)
       2
k*theta
```

diofant.stats.**GammaInverse**(*name*, *a*, *b*)

Create a continuous random variable with an inverse Gamma distribution.

The density of the inverse Gamma distribution is given by

$$f(x) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha - 1} \exp\left(\frac{-\beta}{x}\right)$$

with $x > 0$.

Parameters a : Real number, $a > 0$ a shape

b : Real number, $b > 0$ a scale

Returns A RandomSymbol.

References

[R579] (page 1265)

Examples

```
>>> from diofant.stats import density, cdf, E, variance
```

```
>>> a = Symbol("a", positive=True)
>>> b = Symbol("b", positive=True)
>>> z = Symbol("z")
```

```
>>> X = GammaInverse("x", a, b)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
 -b
 ---
  z   a  -a - 1
E   *b *z
---------------
    gamma(a)
```

diofant.stats.**Kumaraswamy**(*name*, *a*, *b*)

Create a Continuous Random Variable with a Kumaraswamy distribution.

The density of the Kumaraswamy distribution is given by

$$f(x) := abx^{a-1}(1 - x^a)^{b-1}$$

with $x \in [0, 1]$.

Parameters a : Real number, $a > 0$ a shape

b : Real number, $b > 0$ a shape

Returns A RandomSymbol.

References

[R580] (page 1265)

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> a = Symbol("a", positive=True)
>>> b = Symbol("b", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Kumaraswamy("x", a, b)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
                     b - 1
     a - 1 /    a    \
a*b*z      *\- z  + 1/
```

diofant.stats.**Laplace**(*name*, *mu*, *b*)

Create a continuous random variable with a Laplace distribution.

The density of the Laplace distribution is given by

$$f(x) := \frac{1}{2b} \exp \left(-\frac{|x-\mu|}{b} \right)$$

Parameters **mu** : Real number, the location (mean)

b : Real number, $b > 0$, a scale

Returns A RandomSymbol.

References

[R581] (page 1265), *[R582]* (page 1265)

Examples

```
>>> from diofant.stats import density
```

```
>>> mu = Symbol("mu")
>>> b = Symbol("b", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Laplace("x", mu, b)
```

```
>>> density(X)(z)
E**(-Abs(mu - z)/b)/(2*b)
```

diofant.stats.**Logistic**(*name*, *mu*, *s*)

Create a continuous random variable with a logistic distribution.

The density of the logistic distribution is given by

$$f(x) := \frac{e^{-(x-\mu)/s}} {s\left(1+e^{-(x-\mu)/s}\right)^2}$$

Parameters **mu** : Real number, the location (mean)

s : Real number, $s > 0$ a scale

Returns A RandomSymbol.

References

[R583] (page 1265), *[R584]* (page 1265)

Examples

```
>>> from diofant.stats import density
```

```
>>> mu = Symbol("mu", extended_real=True)
>>> s = Symbol("s", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Logistic("x", mu, s)
```

```
>>> density(X)(z)
E**((mu - z)/s)/(s*(E**((mu - z)/s) + 1)**2)
```

diofant.stats.**LogNormal**(*name*, *mean*, *std*)

Create a continuous random variable with a log-normal distribution.

The density of the log-normal distribution is given by

$$f(x) := \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}$$

with $x \geq 0$.

Parameters mu : Real number, the log-scale

sigma : Real number, $\sigma^2 > 0$ a shape

Returns A RandomSymbol.

References

[R585] (page 1265), [R586] (page 1265)

Examples

```
>>> from diofant.stats import density
```

```
>>> mu = Symbol("mu", extended_real=True)
>>> sigma = Symbol("sigma", positive=True)
>>> z = Symbol("z")
```

```
>>> X = LogNormal("x", mu, sigma)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
                     2
       -(-mu + log(z))
       -----------------
                  2
            2*sigma
  ___
\/ 2 *E
------------------------

    2*\/ pi *sigma*z
```

```
>>> X = LogNormal('x', 0, 1) # Mean 0, standard deviation 1
```

```
>>> density(X)(z)
sqrt(2)*E**(-log(z)**2/2)/(2*sqrt(pi)*z)
```

`diofant.stats.`**`Maxwell`**`(`*name*, *a*`)`

Create a continuous random variable with a Maxwell distribution.

The density of the Maxwell distribution is given by

$$f(x) := \sqrt{\frac{2}{\pi}} \frac{x^2 e^{-x^2/(2a^2)}}{a^3}$$

with $x \geq 0$.

Parameters a : Real number, $a > 0$

Returns A RandomSymbol.

References

[R587] (page 1265), [R588] (page 1265)

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> a = Symbol("a", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Maxwell("x", a)
```

```
>>> density(X)(z)
sqrt(2)*E**(-z**2/(2*a**2))*z**2/(sqrt(pi)*a**3)
```

```
>>> E(X)
2*sqrt(2)*a/sqrt(pi)
```

```
>>> simplify(variance(X))
a**2*(-8 + 3*pi)/pi
```

`diofant.stats.`**`Nakagami`**`(`*name*, *mu*, *omega*`)`

Create a continuous random variable with a Nakagami distribution.

The density of the Nakagami distribution is given by

$$f(x) := \frac{2\mu^\mu}{\Gamma(\mu)\omega^\mu} x^{2\mu-1} \exp\left(-\frac{\mu}{\omega}x^2\right)$$

with $x > 0$.

Parameters mu : Real number, $\mu \geq \frac{1}{2}$ a shape

omega : Real number, $\omega > 0$, the spread

Returns A RandomSymbol.

References

[R589] (page 1265)

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> mu = Symbol("mu", positive=True)
>>> omega = Symbol("omega", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Nakagami("x", mu, omega)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
        2
   -mu*z
   -------
    omega    mu       -mu  2*mu - 1
2*E       *mu  *omega   *z
----------------------------------
            gamma(mu)
```

```
>>> simplify(E(X, meijerg=True))
sqrt(mu)*sqrt(omega)*gamma(mu + 1/2)/gamma(mu + 1)
```

```
>>> V = simplify(variance(X, meijerg=True))
>>> pprint(V, use_unicode=False)
                    2
         omega*gamma (mu + 1/2)
omega - -----------------------
        gamma(mu)*gamma(mu + 1)
```

`diofant.stats.Normal(name, mean, std)`

Create a continuous random variable with a Normal distribution.

The density of the Normal distribution is given by

$$f(x) := \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Parameters mu : Real number, the mean

sigma : Real number, $\sigma^2 > 0$ the variance

Returns A RandomSymbol.

References

[R590] (page 1265), [R591] (page 1265)

Examples

```
>>> from diofant.stats import density, E, std, cdf, skewness
```

```
>>> mu = Symbol("mu")
>>> sigma = Symbol("sigma", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Normal("x", mu, sigma)
```

```
>>> density(X)(z)
sqrt(2)*E**(-(-mu + z)**2/(2*sigma**2))/(2*sqrt(pi)*sigma)
```

```
>>> C = simplify(cdf(X))(z) # it needs a little more help...
>>> pprint(C, use_unicode=False)
     /   ___          \
     |\/ 2 *(mu - z)|
  erf|--------------|
     \    2*sigma   /   1
- ------------------- + -
          2             2
```

```
>>> simplify(skewness(X))
0
```

```
>>> X = Normal("x", 0, 1) # Mean 0, standard deviation 1
>>> density(X)(z)
sqrt(2)*E**(-z**2/2)/(2*sqrt(pi))
```

```
>>> E(2*X + 1)
1
```

```
>>> simplify(std(2*X + 1))
2
```

`diofant.stats.`**`Pareto`**(*name*, *xm*, *alpha*)

Create a continuous random variable with the Pareto distribution.

The density of the Pareto distribution is given by

$$f(x) := \frac{\alpha\, x_m^\alpha}{x^{\alpha+1}}$$

with $x \in [x_m, \infty]$.

Parameters xm : Real number, $x_m > 0$, a scale

alpha : Real number, $\alpha > 0$, a shape

Returns A RandomSymbol.

References

[R592] (page 1265), *[R593]* (page 1265)

Examples

```
>>> from diofant.stats import density
```

```
>>> xm = Symbol("xm", positive=True)
>>> beta = Symbol("beta", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Pareto("x", xm, beta)
```

```
>>> density(X)(z)
beta*xm**beta*z**(-beta - 1)
```

diofant.stats.**QuadraticU**(*name, a, b*)

Create a Continuous Random Variable with a U-quadratic distribution.

The density of the U-quadratic distribution is given by

$$f(x) := \alpha(x - \beta)^2$$

with $x \in [a, b]$.

Parameters a : Real number

b : Real number, $a < b$

Returns A RandomSymbol.

References

[R594] (page 1265)

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> a = Symbol("a", extended_real=True)
>>> b = Symbol("b", extended_real=True)
>>> z = Symbol("z")
```

```
>>> X = QuadraticU("x", a, b)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
/                2
|   /  a   b    \
|12*|- - - - + z|
|   \  2   2    /
<-----------------  for And(a <= z, z <= b)
|            3
|     (-a + b)
|
\        0                 otherwise
```

`diofant.stats.RaisedCosine(name, mu, s)`

Create a Continuous Random Variable with a raised cosine distribution.

The density of the raised cosine distribution is given by

$$f(x) := \frac{1}{2s}\left(1+\cos\left(\frac{x-\mu}{s}\pi\right)\right)$$

with $x \in [\mu - s, \mu + s]$.

Parameters mu : Real number

s : Real number, $s > 0$

Returns A RandomSymbol.

References

[R595] (page 1265)

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> mu = Symbol("mu", extended_real=True)
>>> s = Symbol("s", positive=True)
>>> z = Symbol("z")
```

```
>>> X = RaisedCosine("x", mu, s)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
/   /pi*(-mu + z)\
|cos|------------| + 1
|   \     s      /
<---------------------  for And(z <= mu + s, mu - s <= z)
|         2*s
|
\          0                        otherwise
```

`diofant.stats.Rayleigh(name, sigma)`

Create a continuous random variable with a Rayleigh distribution.

The density of the Rayleigh distribution is given by

$$f(x) := \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}$$

with $x > 0$.

Parameters sigma : Real number, $\sigma > 0$

Returns A RandomSymbol.

References

[R596] (page 1265), [R597] (page 1265)

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> sigma = Symbol("sigma", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Rayleigh("x", sigma)
```

```
>>> density(X)(z)
E**(-z**2/(2*sigma**2))*z/sigma**2
```

```
>>> E(X)
sqrt(2)*sqrt(pi)*sigma/2
```

```
>>> variance(X)
-pi*sigma**2/2 + 2*sigma**2
```

`diofant.stats.StudentT(name, nu)`

Create a continuous random variable with a student's t distribution.

The density of the student's t distribution is given by

$$f(x) := \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

Parameters nu : Real number, $\nu > 0$, the degrees of freedom

Returns A RandomSymbol.

References

[R598] (page 1265), [R599] (page 1265)

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> nu = Symbol("nu", positive=True)
>>> z = Symbol("z")
```

```
>>> X = StudentT("x", nu)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
           nu   1
         - -- - -
           2    2
  /     2\
  |    z |
  |1 + --|
```

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```
  \    nu/
--------------------
  ____     /     nu\
\/ nu *beta|1/2, --|
           \     2 /
```

diofant.stats.**Triangular**(*name*, *a*, *b*, *c*)

Create a continuous random variable with a triangular distribution.

The density of the triangular distribution is given by

$$f(x) := \begin{cases} 0 & \text{for } x < a, \\ \frac{2(x-a)}{(b-a)(c-a)} & \text{for } a \le x < c, \\ \frac{2}{b-a} & \text{for } x = c, \\ \frac{2(b-x)}{(b-a)(b-c)} & \text{for } c < x \le b, \\ 0 & \text{for } b < x. \end{cases}$$

Parameters **a** : Real number, $a \in (-\infty, \infty)$

b : Real number, $a < b$

c : Real number, $a \le c \le b$

Returns A RandomSymbol.

References

[R600] (page 1265), *[R601]* (page 1265)

Examples

```
>>> from diofant.stats import density, E
```

```
>>> a = Symbol("a")
>>> b = Symbol("b")
>>> c = Symbol("c")
>>> z = Symbol("z")
```

```
>>> X = Triangular("x", a, b, c)
```

```
>>> pprint(density(X)(z), use_unicode=False)
/    -2*a + 2*z
|-----------------  for And(a <= z, z < c)
|(-a + b)*(-a + c)
|
|       2
|     ------              for z = c
<     -a + b
|
|   2*b - 2*z
|----------------   for And(z <= b, c < z)
|(-a + b)*(b - c)
```

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```
|
\        0                  otherwise
```

diofant.stats.**Uniform**(*name*, *left*, *right*)

Create a continuous random variable with a uniform distribution.

The density of the uniform distribution is given by

$$f(x) := \begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

with $x \in [a, b]$.

Parameters a : Real number, $-\infty < a$ the left boundary

b : Real number, $a < b < \infty$ the right boundary

Returns A RandomSymbol.

References

[R602] (page 1265), [R603] (page 1265)

Examples

```
>>> from diofant.stats import density, cdf, E, variance, skewness
```

```
>>> a = Symbol("a", negative=True)
>>> b = Symbol("b", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Uniform("x", a, b)
```

```
>>> density(X)(z)
Piecewise((1/(-a + b), And(a <= z, z <= b)), (0, true))
```

```
>>> cdf(X)(z)
-a/(-a + b) + z/(-a + b)
```

```
>>> simplify(E(X))
a/2 + b/2
```

```
>>> simplify(variance(X))
a**2/12 - a*b/6 + b**2/12
```

diofant.stats.**UniformSum**(*name*, *n*)

Create a continuous random variable with an Irwin-Hall distribution.

The probability distribution function depends on a single parameter n which is an integer.

The density of the Irwin-Hall distribution is given by

$$f(x) := \frac{1}{(n-1)!} \sum_{k=0}^{\lfloor x \rfloor} (-1)^k \binom{n}{k} (x-k)^{n-1}$$

Parameters n : A positive Integer, $n > 0$

Returns A RandomSymbol.

References

[R604] (page 1265), [R605] (page 1265)

Examples

```
>>> from diofant.stats import density
```

```
>>> n = Symbol("n", integer=True)
>>> z = Symbol("z")
```

```
>>> X = UniformSum("x", n)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
floor(z)
  ___
  \  `
   \         k         n - 1 /n\
    )    (-1) *(z - k)      *| |
   /                         \k/
  /__,
 k = 0
-------------------------------
            (n - 1)!
```

diofant.stats.**VonMises**(*name*, *mu*, *k*)

Create a Continuous Random Variable with a von Mises distribution.

The density of the von Mises distribution is given by

$$f(x) := \frac{e^{\kappa\cos(x-\mu)}}{2\pi I_0(\kappa)}$$

with $x \in [0, 2\pi]$.

Parameters mu : Real number, measure of location

k : Real number, measure of concentration

Returns A RandomSymbol.

References

[R606] (page 1265), [R607] (page 1265)

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> mu = Symbol("mu")
>>> k = Symbol("k", positive=True)
>>> z = Symbol("z")
```

```
>>> X = VonMises("x", mu, k)
```

```
>>> D = density(X)(z)
>>> pprint(D, use_unicode=False)
   k*cos(mu - z)
  E
------------------
2*pi*besseli(0, k)
```

diofant.stats.**Weibull**(*name*, *alpha*, *beta*)

Create a continuous random variable with a Weibull distribution.

The density of the Weibull distribution is given by

$$f(x) := \begin{cases} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^{k}} & x\geq0\\ 0 & x<0 \end{cases}$$

Parameters **lambda** : Real number, $\lambda > 0$ a scale

k : Real number, $k > 0$ a shape

Returns A RandomSymbol.

References

[R608] (page 1265), *[R609]* (page 1265)

Examples

```
>>> from diofant.stats import density, E, variance
```

```
>>> l = Symbol("lambda", positive=True)
>>> k = Symbol("k", positive=True)
>>> z = Symbol("z")
```

```
>>> X = Weibull("x", l, k)
```

```
>>> density(X)(z)
E**(-(z/lambda)**k)*k*(z/lambda)**(k - 1)/lambda
```

```
>>> simplify(E(X))
lambda*gamma(1 + 1/k)
```

```
>>> simplify(variance(X))
lambda**2*(-gamma(1 + 1/k)**2 + gamma(1 + 2/k))
```

diofant.stats.**WignerSemicircle**(*name*, *R*)

Create a continuous random variable with a Wigner semicircle distribution.

The density of the Wigner semicircle distribution is given by

$$f(x) := \frac{2}{\pi R^2} \sqrt{R^2 - x^2}$$

with $x \in [-R, R]$.

Parameters R : Real number, $R > 0$, the radius

Returns A $RandomSymbol$.

References

[R610] (page 1265), [R611] (page 1265)

Examples

```
>>> from diofant.stats import density, E
```

```
>>> R = Symbol("R", positive=True)
>>> z = Symbol("z")
```

```
>>> X = WignerSemicircle("x", R)
```

```
>>> density(X)(z)
2*sqrt(R**2 - z**2)/(pi*R**2)
```

```
>>> E(X)
0
```

diofant.stats.**ContinuousRV**(*symbol*, *density*, *set=Interval(-oo, oo, true, true)*)

Create a Continuous Random Variable given the following:

– a symbol – a probability density function – set on which the pdf is valid (defaults to entire real line)

Returns a RandomSymbol.

Many common continuous random variable types are already implemented. This function should be necessary only very rarely.

Examples

```
>>> from diofant.stats import P, E
```

```
>>> x = Symbol("x")
```

```
>>> pdf = sqrt(2)*exp(-x**2/2)/(2*sqrt(pi)) # Normal distribution
>>> X = ContinuousRV(x, pdf)
```

```
>>> E(X)
0
>>> P(X>0)
1/2
```

3.18.3 Interface

diofant.stats.**P**(*condition*, *given_condition=None*, *numsamples=None*, *evaluate=True*, ***kwargs*)

Probability that a condition is true, optionally given a second condition

Parameters condition : Combination of Relationals containing RandomSymbols

The condition of which you want to compute the probability

given_condition : Combination of Relationals containing RandomSymbols

A conditional expression. P(X > 1, X > 0) is expectation of X > 1 given X > 0

numsamples : int

Enables sampling and approximates the probability with this many samples

evaluate : Bool (defaults to True)

In case of continuous systems return unevaluated integral

Examples

```
>>> from diofant.stats import P, Die
>>> X, Y = Die('X', 6), Die('Y', 6)
>>> P(X > 3)
1/2
>>> P(Eq(X, 5), X > 2) # Probability that X == 5 given that X > 2
1/4
>>> P(X > Y)
5/12
```

diofant.stats.**E**(*expr*, *condition=None*, *numsamples=None*, *evaluate=True*, ***kwargs*)

Returns the expected value of a random expression

Parameters expr : Expr containing RandomSymbols

The expression of which you want to compute the expectation value

given : Expr containing RandomSymbols

A conditional expression. E(X, X>0) is expectation of X given X > 0

numsamples : int

Enables sampling and approximates the expectation with this many samples

evalf : Bool (defaults to True)

If sampling return a number rather than a complex expression

evaluate : Bool (defaults to True)

In case of continuous systems return unevaluated integral

Examples

```
>>> from diofant.stats import E, Die
>>> X = Die('X', 6)
>>> E(X)
7/2
>>> E(2*X + 1)
8
```

```
>>> E(X, X > 3) # Expectation of X given that it is above 3
5
```

diofant.stats.**density**(*expr*, *condition=None*, *evaluate=True*, *numsamples=None*, ***kwargs*)

Probability density of a random expression, optionally given a second condition.

This density will take on different forms for different types of probability spaces. Discrete variables produce Dicts. Continuous variables produce Lambdas.

Parameters expr : Expr containing RandomSymbols

The expression of which you want to compute the density value

condition : Relational containing RandomSymbols

A conditional expression. density(X > 1, X > 0) is density of X > 1 given X > 0

numsamples : int

Enables sampling and approximates the density with this many samples

Examples

```
>>> from diofant.stats import Die, Normal
```

```
>>> x = Symbol('x')
>>> D = Die('D', 6)
>>> X = Normal(x, 0, 1)
```

```
>>> density(D).dict
{1: 1/6, 2: 1/6, 3: 1/6, 4: 1/6, 5: 1/6, 6: 1/6}
>>> density(2*D).dict
{2: 1/6, 4: 1/6, 6: 1/6, 8: 1/6, 10: 1/6, 12: 1/6}
>>> density(X)(x)
sqrt(2)*E**(-x**2/2)/(2*sqrt(pi))
```

diofant.stats.**given**(*expr*, *condition=None*, ***kwargs*)
Conditional Random Expression From a random expression and a condition on that expression creates a new probability space from the condition and returns the same expression on that conditional probability space.

Examples

```
>>> from diofant.stats import Die
>>> X = Die('X', 6)
>>> Y = given(X, X > 3)
>>> density(Y).dict
{4: 1/3, 5: 1/3, 6: 1/3}
```

Following convention, if the condition is a random symbol then that symbol is considered fixed.

```
>>> from diofant.stats import Normal
```

```
>>> X = Normal('X', 0, 1)
>>> Y = Normal('Y', 0, 1)
>>> pprint(density(X + Y, Y)(z), use_unicode=False)
                2
       -(-Y + z)
       -----------
            2
  ___
\/ 2 *E
------------------
     ____
  2*\/ pi
```

diofant.stats.**where**(*condition*, *given_condition=None*, ***kwargs*)
Returns the domain where a condition is True.

Examples

```
>>> from diofant.stats import Die, Normal
```

```
>>> D1, D2 = Die('a', 6), Die('b', 6)
>>> a, b = D1.symbol, D2.symbol
>>> X = Normal('x', 0, 1)
```

```
>>> where(X**2<1)
Domain: And(-1 < x, x < 1)
```

```
>>> where(X**2<1).set
(-1, 1)
```

```
>>> where(And(D1<=D2 , D2<3))
Domain: Or(And(Eq(a, 1), Eq(b, 1)), And(Eq(a, 1), Eq(b, 2)), And(Eq(a, 2), Eq(b,
↪2)))
```

diofant.stats.**variance**(*X*, *condition=None*, ***kwargs*)
Variance of a random expression

Expectation of (X-E(X))**2

Examples

```
>>> from diofant.stats import Die, Bernoulli
```

```
>>> X = Die('X', 6)
>>> p = Symbol('p')
>>> B = Bernoulli('B', p, 1, 0)
```

```
>>> variance(2*X)
35/3
```

```
>>> simplify(variance(B))
p*(-p + 1)
```

`diofant.stats.std`(*X*, *condition=None*, ***kwargs*)

Standard Deviation of a random expression

Square root of the Expectation of (X-E(X))**2

Examples

```
>>> from diofant.stats import Bernoulli
```

```
>>> p = Symbol('p')
>>> B = Bernoulli('B', p, 1, 0)
```

```
>>> simplify(std(B))
sqrt(p*(-p + 1))
```

`diofant.stats.sample`(*expr*, *condition=None*, ***kwargs*)

A realization of the random expression

Examples

```
>>> from diofant.stats import Die
>>> X, Y, Z = Die('X', 6), Die('Y', 6), Die('Z', 6)
```

```
>>> die_roll = sample(X + Y + Z) # A random realization of three dice
```

`diofant.stats.sample_iter`(*expr*, *condition=None*, *numsamples=oo*, ***kwargs*)

Returns an iterator of realizations from the expression given a condition

expr: Random expression to be realized condition: A conditional expression (optional) numsamples: Length of the iterator (defaults to infinity)

See also:

diofant.stats.sample (page 835), *diofant.stats.rv.sampling_P* (page 837), *diofant.stats.rv.sampling_E* (page 838)

Examples

```
>>> from diofant.stats import Normal
>>> X = Normal('X', 0, 1)
>>> expr = X*X + 3
>>> iterator = sample_iter(expr, numsamples=3)
>>> list(iterator)
[12, 4, 7]
```

3.18.4 Mechanics

Diofant Stats employs a relatively complex class hierarchy.

`RandomDomains` are a mapping of variables to possible values. For example we might say that the symbol `Symbol('x')` can take on the values $\{1, 2, 3, 4, 5, 6\}$.

class diofant.stats.rv.**RandomDomain**

A `PSpace`, or Probability Space, combines a `RandomDomain` with a density to provide probabilistic information. For example the above domain could be enhanced by a finite density `{1:1/6, 2:1/6, 3:1/6, 4:1/6, 5:1/6, 6:1/6}` to fully define the roll of a fair die named `x`.

class diofant.stats.rv.**PSpace**

A RandomSymbol represents the PSpace's symbol 'x' inside of Diofant expressions.

class diofant.stats.rv.**RandomSymbol**

The RandomDomain and PSpace classes are almost never directly instantiated. Instead they are subclassed for a variety of situations.

RandomDomains and PSpaces must be sufficiently general to represent domains and spaces of several variables with arbitrarily complex densities. This generality is often unnecessary. Instead we often build SingleDomains and SinglePSpaces to represent single, univariate events and processes such as a single die or a single normal variable.

class diofant.stats.rv.**SinglePSpace**

class diofant.stats.rv.**SingleDomain**

Another common case is to collect together a set of such univariate random variables. A collection of independent SinglePSpaces or SingleDomains can be brought together to form a ProductDomain or ProductPSpace. These objects would be useful in representing three dice rolled together for example.

class diofant.stats.rv.**ProductDomain**

class diofant.stats.rv.**ProductPSpace**

The Conditional adjective is added whenever we add a global condition to a RandomDomain or PSpace. A common example would be three independent dice where we know their sum to be greater than 12.

class diofant.stats.rv.**ConditionalDomain**

We specialize further into Finite and Continuous versions of these classes to represent finite (such as dice) and continuous (such as normals) random variables.

class diofant.stats.frv.**FiniteDomain**

class diofant.stats.frv.**FinitePSpace**

class diofant.stats.crv.**ContinuousDomain**

class diofant.stats.crv.**ContinuousPSpace**

Additionally there are a few specialized classes that implement certain common random variable types. There is for example a DiePSpace that implements SingleFinitePSpace and a NormalPSpace that implements SingleContinuousPSpace.

class diofant.stats.frv_types.**DiePSpace**

class diofant.stats.crv_types.**NormalPSpace**

RandomVariables can be extracted from these objects using the PSpace.values method.

As previously mentioned Diofant Stats employs a relatively complex class structure. Inheritance is widely used in the implementation of end-level classes. This tactic was chosen to balance between the need to allow Diofant to represent arbitrarily defined random variables and optimizing for common cases. This complicates the code but is structured to only be important to those working on extending Diofant Stats to other random variable types.

Users will not use this class structure. Instead these mechanics are exposed through variable creation functions Die, Coin, FiniteRV, Normal, Exponential, etc.... These build the appropriate SinglePSpaces and return the corresponding RandomVariable. Conditional and Product spaces are formed in the natural construction of Diofant expressions and the use of interface functions E, Given, Density, etc....

diofant.stats.**Die**()

diofant.stats.**Normal**()

There are some additional functions that may be useful. They are largely used internally.

diofant.stats.rv.**random_symbols**(*expr*)
Returns all RandomSymbols within a Diofant Expression.

diofant.stats.rv.**pspace**(*expr*)
Returns the underlying Probability Space of a random expression.

For internal use.

Examples

```
>>> from diofant.stats import Normal
>>> X = Normal('X', 0, 1)
>>> pspace(2*X + 1) == X.pspace
True
```

diofant.stats.rv.**rs_swap**(*a*, *b*)
Build a dictionary to swap RandomSymbols based on their underlying symbol.

i.e. if `X = ('x', pspace1)` and `Y = ('x', pspace2)` then `X` and `Y` match and the key, value pair `{X:Y}` will appear in the result

Inputs: collections a and b of random variables which share common symbols Output: dict mapping RVs in a to RVs in b

diofant.stats.rv.**sampling_P**(*condition*, *given_condition=None*, *numsamples=1*, *evalf=True*, ***kwargs*)
Sampling version of P

See also:

diofant.stats.P (page 832), *diofant.stats.rv.sampling_E* (page 838), *diofant.stats.rv.sampling_density* (page 838)

diofant.stats.rv.**sampling_E**(*expr*, *given_condition=None*, *numsamples=1*, *evalf=True*, ***kwargs*)

Sampling version of E

See also:

diofant.stats.P (page 832), *diofant.stats.rv.sampling_P* (page 837), *diofant.stats.rv.sampling_density* (page 838)

diofant.stats.rv.**sampling_density**(*expr*, *given_condition=None*, *numsamples=1*, ***kwargs*)

Sampling version of density

See also:

diofant.stats.density (page 833), *diofant.stats.rv.sampling_P* (page 837), *diofant.stats.rv.sampling_E* (page 838)

diofant.stats.rv.**independent**(*a*, *b*)

Independence of two random expressions

Two expressions are independent if knowledge of one does not change computations on the other.

See also:

diofant.stats.rv.dependent (page 838)

Examples

```
>>> from diofant.stats import Normal
```

```
>>> X, Y = Normal('X', 0, 1), Normal('Y', 0, 1)
>>> independent(X, Y)
True
>>> independent(2*X + Y, -Y)
False
>>> X, Y = given(Tuple(X, Y), Eq(X + Y, 3))
>>> independent(X, Y)
False
```

diofant.stats.rv.**dependent**(*a*, *b*)

Dependence of two random expressions

Two expressions are independent if knowledge of one does not change computations on the other.

See also:

diofant.stats.rv.independent (page 838)

Examples

```
>>> from diofant.stats import Normal
```

```
>>> X, Y = Normal('X', 0, 1), Normal('Y', 0, 1)
>>> dependent(X, Y)
False
>>> dependent(2*X + Y, -Y)
True
>>> X, Y = given(Tuple(X, Y), Eq(X + Y, 3))
>>> dependent(X, Y)
True
```

3.19 Solvers

A module for solving all kinds of equations.

3.19.1 Examples

```
>>> solve(x**5 + 5*x**4 + 10*x**3 + 10*x**2 + 5*x + 1, x)
[{x: -1}]
```

The *solvers* module in Diofant implements methods for solving equations and inequalities.

3.19.2 Algebraic equations

This module contain solvers for all kinds of equations, algebraic or transcendental.

`diofant.solvers.solvers.solve`(*f*, **symbols*, ***flags*)
Algebraically solves equation or system of equations.

Parameters f : Expr, Equality or iterable of above

All expressions are assumed to be equal to 0.

***symbols** : tuple

If none symbols given (empty tuple), free symbols of expressions will be used.

****flags** : dict

A dictionary of following parameters:

check [bool, optional] If False, don't do any testing of solutions. Default is True, i.e. the solutions are checked and those that doesn't satisfy given assumptions on symbols solved for or make any denominator zero - are automatically excluded.

warn [bool, optional] Show a warning if `checksol()` (page 841) could not conclude. Default is False.

simplify [bool, optional] Enable simplification (default) for all but polynomials of order 3 or greater before returning them and (if

check is not False) use the general simplify function on the solutions and the expression obtained when they are substituted into the function which should be zero.

rational [bool or None, optional] If True, recast Floats as Rational. If None (default), Floats will be recast as rationals but the answer will be recast as Floats. If the flag is False then nothing will be done to the Floats.

cubics, quartics, quintics [bool, optional] Return explicit solutions (with radicals, which can be quite long) when, respectively, cubic, quartic or quintic expressions are encountered. Default is True. If False, *RootOf* (page 711) instances will be returned instead.

See also:

***diofant.solvers.recurr.rsolve* (page 919)** solving recurrence equations

***diofant.solvers.ode.dsolve* (page 868)** solving differential equations

***diofant.solvers.inequalities.reduce_inequalities* (page 843)** solving inequalities

Notes

When an object other than a Symbol is given as a symbol, it is isolated algebraically and an implicit solution may be obtained. This is mostly provided as a convenience to save one from replacing the object with a Symbol and solving for that Symbol. It will only work if the specified object can be replaced with a Symbol using the subs method.

```
>>> solve(f(x) - x, f(x))
[{f(x): x}]
>>> solve(f(x).diff(x) - f(x) - x, f(x).diff(x))
[{Derivative(f(x), x): x + f(x)}]
```

Examples

Single equation:

```
>>> solve(x**2 - y**2)
[{x: -y}, {x: y}]
>>> solve(x**2 - 1)
[{x: -1}, {x: 1}]
```

We could restrict solutions by using assumptions:

```
>>> p = Symbol("p", positive=True)
>>> solve(p**2 - 1)
[{p: 1}]
```

Several equations:

```
>>> solve((x + 5*y - 2, -3*x + 6*y - 15))
[{x: -3, y: 1}]
>>> solve((x + 5*y - 2, -3*x + 6*y - z))
[{x: -5*z/21 + 4/7, y: z/21 + 2/7}]
```

No solution:

```
>>> solve([x + 3, x - 3])
[]
```

`diofant.solvers.solvers.`**`solve_linear`**(*f*, *x*)

Solve equation `f` wrt variable `x`.

Returns tuple

`(x, solution)`, if there is a linear solution, `(0, 1)` if `f` is independent of the symbol `x`, `(0, 0)` if solution set any denominator of `f` to zero or `(numerator, denominator)` of `f`, if it's a nonlinear expression wrt `x`.

Examples

```
>>> solve_linear(1/x - y**2, x)
(x, y**(-2))
>>> solve_linear(x**2/y**2 - 3, x)
(x**2 - 3*y**2, y**2)
>>> solve_linear(y, x)
(0, 1)
>>> solve_linear(1/(1/x - 2), x)
(0, 0)
```

`diofant.solvers.solvers.`**`minsolve_linear_system`**(*system*, **symbols*, ***flags*)

Find a particular solution to a linear system.

In particular, try to find a solution with the minimal possible number of non-zero variables. This is a very computationally hard problem.

Parameters system : Matrix

Nx(M+1) matrix, which means it has to be in augmented form.

***symbols** : list

List of M Symbol's.

****flags** : dict

A dictionary of following parameters:

quick [boolean, optional] If True, a heuristic is used. Otherwise (default) a naive algorithm with exponential complexity is used.

`diofant.solvers.solvers.`**`checksol`**(*f*, *sol*, ***flags*)

Checks whether sol is a solution of equations f.

Parameters f : Expr or iterable of Expr's

Equations to substitute solutions in.

sol : dict of Expr's

Mapping of symbols to values.

****flags** : dict

A dictionary of following parameters:

minimal [bool, optional] Do a very fast, minimal testing. Default is False.

warn [bool, optional] Show a warning if `checksol()` (page 841) could not conclude. Default is False.

simplify [bool, optional] Simplify solution before substituting into function and simplify the function before trying specific simplifications. Default is True.

force [bool, optional] Make positive all symbols without assumptions regarding sign. Default is False.

Returns bool or None

Return True, if solution satisfy all equations in `f`. Return False, if a solution doesn't satisfy any equation. Else (i.e. one or more checks are inconclusive), return None.

Examples

```
>>> checksol(x**4 - 1, {x: 1})
True
>>> checksol(x**4 - 1, {x: 0})
False
>>> checksol(x**2 + y**2 - 5**2, {x: 3, y: 4})
True
```

Systems of Polynomial Equations

Solvers of systems of polynomial equations.

`diofant.solvers.polysys.`**`solve_linear_system`**(*system*, **symbols*, ***flags*)

Solve system of linear equations.

Both under- and overdetermined systems are supported. The possible number of solutions is zero, one or infinite.

Parameters **system** : Matrix

Nx(M+1) matrix, which means it has to be in augmented form. This matrix will not be modified.

***symbols** : list

List of M Symbol's

Returns solution: dict or None

Respectively, this procedure will return None or a dictionary with solutions. In the case of underdetermined systems, all arbitrary parameters are skipped. This may cause a situation in which an empty dictionary is returned. In that case, all symbols can be assigned arbitrary values.

See also:

`diofant.matrices.matrices.MatrixBase.rref` (page 599)

Examples

Solve the following system:

```
   x + 4 y ==  2
-2 x +   y == 14
```

```
>>> system = Matrix(((1, 4, 2), (-2, 1, 14)))
>>> solve_linear_system(system, x, y)
{x: -6, y: 2}
```

A degenerate system returns an empty dictionary.

```
>>> system = Matrix(((0, 0, 0), (0, 0, 0)))
>>> solve_linear_system(system, x, y)
{}
```

diofant.solvers.polysys.**solve_poly_system**(*eqs, *gens, **args*)
Solve a system of polynomial equations.

Polynomial system may have finite number of solutions or infinitely many (positive-dimensional systems).

References

[Cox97551] (page 1265)

Examples

```
>>> solve_poly_system([x*y - 2*y, 2*y**2 - x**2], x, y)
[{x: 0, y: 0}, {x: 2, y: -sqrt(2)}, {x: 2, y: sqrt(2)}]
```

```
>>> solve_poly_system([x*y], x, y)
[{x: 0}, {y: 0}]
```

3.19.3 Inequality Solvers

Tools for solving inequalities and systems of inequalities.

diofant.solvers.inequalities.**reduce_inequalities**(*inequalities, symbols=[]*)
Reduces a system of inequalities or equations.

See also:

diofant.solvers.solvers.solve **(page 839)** solve algebraic equations

Examples

```
>>> x = Symbol('x', real=True)
>>> y = Symbol('y', real=True)
```

```
>>> reduce_inequalities(0 <= x + 3, [])
-3 <= x
>>> reduce_inequalities(0 <= x + y*2 - 1, [x])
-2*y + 1 <= x
```

3.19.4 Diophantine

Diophantine equations

The word "Diophantine" comes with the name Diophantus, a mathematician lived in the great city of Alexandria sometime around 250 AD. Often referred to as the "father of Algebra", Diophantus in his famous work "Arithmetica" presented 150 problems that marked the early beginnings of number theory, the field of study about integers and their properties. Diophantine equations play a central and an important part in number theory.

We call a "Diophantine equation" to an equation of the form, $f(x_1, x_2, \ldots x_n) = 0$ where $n \geq 2$ and $x_1, x_2, \ldots x_n$ are integer variables. If we can find n integers $a_1, a_2, \ldots a_n$ such that $x_1 = a_1, x_2 = a_2, \ldots x_n = a_n$ satisfies the above equation, we say that the equation is solvable. You can read more about Diophantine equations in[1] and[2].

Currently, following five types of Diophantine equations can be solved using *diophantine()* (page 849) and other helper functions of the Diophantine module.

- Linear Diophantine equations: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$.
- General binary quadratic equation: $ax^2 + bxy + cy^2 + dx + ey + f = 0$
- Homogeneous ternary quadratic equation: $ax^2 + by^2 + cz^2 + dxy + eyz + fzx = 0$
- Extended Pythagorean equation: $a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 = a_{n+1}x_{n+1}^2$
- General sum of squares: $x_1^2 + x_2^2 + \ldots + x_n^2 = k$

Module structure

This module contains *diophantine()* (page 849) and helper functions that are needed to solve certain Diophantine equations. It's structured in the following manner.

- *diophantine()* (page 849)
  - *diop_solve()* (page 851)
    - *classify_diop()* (page 850)
    - *diop_linear()* (page 851)
    - *diop_quadratic()* (page 852)
    - *diop_ternary_quadratic()* (page 857)
    - *diop_ternary_quadratic_normal()* (page 865)
    - *diop_general_pythagorean()* (page 858)
    - *diop_general_sum_of_squares()* (page 858)
    - *diop_general_sum_of_even_powers()* (page 859)

[1] Andreescu, Titu. Andrica, Dorin. Cucurezeanu, Ion. An Introduction to Diophantine Equations

[2] Diophantine Equation, Wolfram Mathworld, [online]. Available: http://mathworld.wolfram.com/DiophantineEquation.html

- *merge_solution()* (page 863)

When an equation is given to *diophantine()* (page 849), it factors the equation(if possible) and solves the equation given by each factor by calling *diop_solve()* (page 851) separately. Then all the results are combined using *merge_solution()* (page 863).

diop_solve() (page 851) internally uses *classify_diop()* (page 850) to find the type of the equation(and some other details) given to it and then calls the appropriate solver function based on the type returned. For example, if *classify_diop()* (page 850) returned "linear" as the type of the equation, then *diop_solve()* (page 851) calls *diop_linear()* (page 851) to solve the equation.

Each of the functions, *diop_linear()* (page 851), *diop_quadratic()* (page 852), *diop_ternary_quadratic()* (page 857), *diop_general_pythagorean()* (page 858) and *diop_general_sum_of_squares()* (page 858) solves a specific type of equations and the type can be easily guessed by it's name.

Apart from these functions, there are a considerable number of other functions in the "Diophantine Module" and all of them are listed under User functions and Internal functions.

Tutorial

First, let's import the highest API of the Diophantine module.

Before we start solving the equations, we need to define the variables.

```
>>> x, y, z, t, p, q = symbols("x, y, z, t, p, q", integer=True)
>>> t1, t2, t3, t4, t5 = symbols("t1:6", integer=True)
```

Let's start by solving the easiest type of Diophantine equations, i.e. linear Diophantine equations. Let's solve $2x + 3y = 5$. Note that although we write the equation in the above form, when we input the equation to any of the functions in Diophantine module, it needs to be in the form $eq = 0$.

```
>>> diophantine(2*x + 3*y - 5)
{(3*t_0 - 5, -2*t_0 + 5)}
```

Note that stepping one more level below the highest API, we can solve the very same equation by calling *diop_solve()* (page 851).

```
>>> from diofant.solvers.diophantine import diop_solve
>>> diop_solve(2*x + 3*y - 5)
(3*t_0 - 5, -2*t_0 + 5)
```

Note that it returns a tuple rather than a set. *diophantine()* (page 849) always return a set of tuples. But *diop_solve()* (page 851) may return a single tuple or a set of tuples depending on the type of the equation given.

We can also solve this equation by calling *diop_linear()* (page 851), which is what *diop_solve()* (page 851) calls internally.

```
>>> from diofant.solvers.diophantine import diop_linear
>>> diop_linear(2*x + 3*y - 5)
(3*t_0 - 5, -2*t_0 + 5)
```

If the given equation has no solutions then the outputs will look like below.

```
>>> diophantine(2*x + 4*y - 3)
set()
>>> diop_solve(2*x + 4*y - 3)
(None, None)
>>> diop_linear(2*x + 4*y - 3)
(None, None)
```

Note that except for the highest level API, in case of no solutions, a tuple of $None$ are returned. Size of the tuple is the same as the number of variables. Also, one can specifically set the parameter to be used in the solutions by passing a customized parameter. Consider the following example:

```
>>> m = symbols("m", integer=True)
>>> diop_solve(2*x + 3*y - 5, m)
(3*m_0 - 5, -2*m_0 + 5)
```

For linear Diophantine equations, the customized parameter is the prefix used for each free variable in the solution. Consider the following example:

```
>>> diop_solve(2*x + 3*y - 5*z + 7, m)
(m_0, m_0 + 5*m_1 - 14, m_0 + 3*m_1 - 7)
```

In the solution above, m_0 and m_1 are independent free variables.

Please note that for the moment, users can set the parameter only for linear Diophantine equations and binary quadratic equations.

Let's try solving a binary quadratic equation which is an equation with two variables and has a degree of two. Before trying to solve these equations, an idea about various cases associated with the equation would help a lot. Please refer[3] and[4] for detailed analysis of different cases and the nature of the solutions. Let us define $\Delta = b^2 - 4ac$ w.r.t. the binary quadratic $ax^2 + bxy + cy^2 + dx + ey + f = 0$.

When $\Delta < 0$, there are either no solutions or only a finite number of solutions.

```
>>> diophantine(x**2 - 4*x*y + 8*y**2 - 3*x + 7*y - 5)
{(2, 1), (5, 1)}
```

In the above equation $\Delta = (-4)^2 - 4 * 1 * 8 = -16$ and hence only a finite number of solutions exist.

When $\Delta = 0$ we might have either no solutions or parameterized solutions.

```
>>> diophantine(3*x**2 - 6*x*y + 3*y**2 - 3*x + 7*y - 5)
set()
>>> diophantine(x**2 - 4*x*y + 4*y**2 - 3*x + 7*y - 5)
{(-2*t**2 - 7*t + 10, -t**2 - 3*t + 5)}
>>> diophantine(x**2 + 2*x*y + y**2 - 3*x - 3*y)
{(t_0, -t_0), (t_0, -t_0 + 3)}
```

The most interesting case is when $\Delta > 0$ and it is not a perfect square. In this case, the equation has either no solutions or an infinte number of solutions. Consider the below cases where $\Delta = 8$.

[3] Methods to solve Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,[online], Available: https://www.alpertron.com.ar/METHODS.HTM

[4] Solving the equation ax^2+ bxy + cy^2 + dx + ey + f= 0, [online], Available: http://www.jpr2718.org/ax2p.pdf

```
>>> diophantine(x**2 - 4*x*y + 2*y**2 - 3*x + 7*y - 5)
set()
>>> n = symbols("n", integer=True)
>>> s = diophantine(x**2 -  2*y**2 - 2*x - 4*y, n)
>>> x_1, y_1 = s.pop()
>>> x_2, y_2 = s.pop()
>>> x_n = -(-2*sqrt(2) + 3)**n/2 + sqrt(2)*(-2*sqrt(2) + 3)**n/2 - sqrt(2)*(2*sqrt(2)␣
↪+ 3)**n/2 - (2*sqrt(2) + 3)**n/2 + 1
>>> x_1 == x_n or x_2 == x_n
True
>>> y_n = -sqrt(2)*(-2*sqrt(2) + 3)**n/4 + (-2*sqrt(2) + 3)**n/2 + sqrt(2)*(2*sqrt(2)␣
↪+ 3)**n/4 + (2*sqrt(2) + 3)**n/2 - 1
>>> y_1 == y_n or y_2 == y_n
True
```

Here n is an integer. Although x_n and y_n may not look like integers, substituting in specific values for n (and simplifying) shows that they are. For example consider the following example where we set n equal to 9.

```
>>> simplify(x_n.subs({n: 9}))
-9369318
```

Any binary quadratic of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can be transformed to an equivalent form $X^2 - DY^2 = N$.

```
>>> from diofant.solvers.diophantine import find_DN, diop_DN, transformation_to_DN
>>> find_DN(x**2 - 3*x*y + y**2 - 7*x + 5*y - 3)
(5, 920)
```

So, the above equation is equivalent to the equation $X^2 - 5Y^2 = 920$ after a linear transformation. If we want to find the linear transformation, we can use *transformation_to_DN()* (page 855)

```
>>> A, B = transformation_to_DN(x**2 - 3*x*y + y**2 - 7*x + 5*y - 3)
```

Here A is a 2 X 2 matrix and B is a 2 X 1 matrix such that the transformation

$$\begin{bmatrix} X \\ Y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} + B$$

gives the equation $X^2 - 5Y^2 = 920$. Values of A and B are as belows.

```
>>> A
Matrix([
[1/10, 3/10],
[   0,  1/5]])
>>> B
Matrix([
[  1/5],
[-11/5]])
```

We can solve an equation of the form $X^2 - DY^2 = N$ by passing D and N to *diop_DN()* (page 853)

```
>>> diop_DN(5, 920)
[]
```

Unfortunately, our equation has no solution.

Now let's turn to homogeneous ternary quadratic equations. These equations are of the form $ax^2 + by^2 + cz^2 + dxy + eyz + fzx = 0$. These type of equations either have infinitely many solutions or no solutions (except the obvious solution (0, 0, 0))

```
>>> diophantine(3*x**2 + 4*y**2 - 5*z**2 + 4*x*y + 6*y*z + 7*z*x)
{(0, 0, 0)}
>>> diophantine(3*x**2 + 4*y**2 - 5*z**2 + 4*x*y - 7*y*z + 7*z*x)
{(-16*p**2 + 28*p*q + 20*q**2, 3*p**2 + 38*p*q - 25*q**2, 4*p**2 - 24*p*q + 68*q**2)}
```

If you are only interested in a base solution rather than the parameterized general solution (to be more precise, one of the general solutions), you can use *diop_ternary_quadratic()* (page 857).

```
>>> from diofant.solvers.diophantine import diop_ternary_quadratic
>>> diop_ternary_quadratic(3*x**2 + 4*y**2 - 5*z**2 + 4*x*y - 7*y*z + 7*z*x)
(-4, 5, 1)
```

diop_ternary_quadratic() (page 857) first converts the given equation to an equivalent equation of the form $w^2 = AX^2 + BY^2$ and then it uses *descent()* (page 857) to solve the latter equation. You can refer to the docs of *transformation_to_normal()* (page 868) to find more on this. The equation $w^2 = AX^2 + BY^2$ can be solved more easily by using the Aforementioned *descent()* (page 857).

```
>>> from diofant.solvers.diophantine import descent
>>> descent(3, 1) # solves the equation w**2 = 3*Y**2 + Z**2
(1, 0, 1)
```

Here the solution tuple is in the order (w, Y, Z)

The extended Pythagorean equation, $a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 = a_{n+1}x_{n+1}^2$ and the general sum of squares equation, $x_1^2 + x_2^2 + \ldots + x_n^2 = k$ can also be solved using the Diophantine module.

```
>>> from diofant.abc import a, b, c, d, e, f
>>> diophantine(9*a**2 + 16*b**2 + c**2 + 49*d**2 + 4*e**2 - 25*f**2)
{(70*t1**2 + 70*t2**2 + 70*t3**2 + 70*t4**2 - 70*t5**2, 105*t1*t5, 420*t2*t5,
 60*t3*t5, 210*t4*t5, 42*t1**2 + 42*t2**2 + 42*t3**2 + 42*t4**2 + 42*t5**2)}
```

function *diop_general_pythagorean()* (page 858) can also be called directly to solve the same equation. Either you can call *diop_general_pythagorean()* (page 858) or use the high level API. For the general sum of squares, this is also true, but one advantage of calling *diop_general_sum_of_squares()* (page 858) is that you can control how many solutions are returned.

```
>>> from diofant.solvers.diophantine import diop_general_sum_of_squares
>>> eq = a**2 + b**2 + c**2 + d**2 - 18
>>> diophantine(eq)
{(0, 0, 3, 3), (0, 1, 1, 4), (1, 2, 2, 3)}
>>> diop_general_sum_of_squares(eq, 2)
{(0, 0, 3, 3), (1, 2, 2, 3)}
```

The *sum_of_squares()* (page 862) routine will providean iterator that returns solutions and one may control whether the solutions contain zeros or not (and the solutions not containing zeros are returned first):

```
>>> from diofant.solvers.diophantine import sum_of_squares
>>> sos = sum_of_squares(18, 4, zeros=True)
>>> next(sos)
(1, 2, 2, 3)
>>> next(sos)
(0, 0, 3, 3)
```

Simple Eqyptian fractions can be found with the Diophantine module, too. For example, here are the ways that one might represent 1/2 as a sum of two unit fractions:

```
>>> diophantine(Eq(1/x + 1/y, Rational(1, 2)))
{(-2, 1), (1, -2), (3, 6), (4, 4), (6, 3)}
```

To get a more thorough understanding of the Diophantine module, please refer to the following blog.

https://thilinaatsympy.wordpress.com/

References

User Functions

This function is imported into the global namespace with `from diofant import *`:

diophantine

`diofant.solvers.diophantine.`**`diophantine`**(*eq*, *param=Symbol('t', integer=True)*, *syms=None*)

Simplify the solution procedure of diophantine equation `eq` by converting it into a product of terms which should equal zero.

$(x + y)(x - y) = 0$ and $x + y = 0$ and $x - y = 0$ are solved independently and combined. Each term is solved by calling `diop_solve()`.

Output of `diophantine()` is a set of tuples. The elements of the tuple are the solutions for each variable in the the equation and are arranged according to the alphabetic ordering of the variables. e.g. For an equation with two variables, a and b, the first element of the tuple is the solution for a and the second for b.

Parameters eq : Relational or Expr

an equation (to be solved)

t : Symbol, optional

the parameter to be used in the solution.

syms : list of Symbol's, optional

which determines the order of the elements in the returned tuple.

See also:

diofant.solvers.diophantine.diop_solve (page 851)

Examples

```
>>> diophantine(x**2 - y**2)
{(t_0, -t_0), (t_0, t_0)}
```

```
>>> diophantine(x*(2*x + 3*y - z))
{(0, n1, n2), (t_0, t_1, 2*t_0 + 3*t_1)}
>>> diophantine(x**2 + 3*x*y + 4*x)
{(0, n1), (3*t_0 - 4, -t_0)}
```

And this function is imported with `from diofant.solvers.diophantine import *`:

classify_diop

`diofant.solvers.diophantine.`**`classify_diop`**(*eq*, *_dict=True*)

Helper routine used by diop_solve() to find the type of the `eq` etc.

Parameters **eq** : Expr

an expression, which is assumed to be zero.

Returns Returns a tuple containing the type of the diophantine equation along with

the variables(free symbols) and their coefficients. Variables are returned

as a list and coefficients are returned as a dict with the key being the

respective term and the constant term is keyed to Integer(1). The type

is one of the following:

- binary_quadratic
- cubic_thue
- general_pythagorean
- general_sum_of_even_powers
- general_sum_of_squares
- homogeneous_general_quadratic
- homogeneous_ternary_quadratic
- homogeneous_ternary_quadratic_normal
- inhomogeneous_general_quadratic
- inhomogeneous_ternary_quadratic
- linear
- univariate

Examples

```
>>> from diofant.abc import w, t
>>> classify_diop(4*x + 6*y - 4)
([x, y], {1: -4, x: 4, y: 6}, 'linear')
>>> classify_diop(x + 3*y -4*z + 5)
([x, y, z], {1: 5, x: 1, y: 3, z: -4}, 'linear')
>>> classify_diop(x**2 + y**2 - x*y + x + 5)
([x, y], {1: 5, x: 1, x**2: 1, y**2: 1, x*y: -1}, 'binary_quadratic')
```

Internal Functions

These functions are intended for internal use in the Diophantine module.

diop_solve

diofant.solvers.diophantine.**diop_solve**(*eq*, *param=Symbol('t', integer=True)*)

Solves the diophantine equation eq.

Unlike diophantine(), factoring of eq is not attempted. Uses classify_diop() to determine the type of the equation and calls the appropriate solver function.

Parameters eq : Expr

an expression, which is assumed to be zero.

t : Symbol, optional

a parameter, to be used in the solution.

See also:

diofant.solvers.diophantine.diophantine (page 849)

Examples

```
>>> from diofant.abc import w
>>> diop_solve(2*x + 3*y - 5)
(3*t_0 - 5, -2*t_0 + 5)
>>> diop_solve(4*x + 3*y -4*z + 5)
(t_0, 8*t_0 + 4*t_1 + 5, 7*t_0 + 3*t_1 + 5)
>>> diop_solve(x + 3*y - 4*z + w -6)
(t_0, t_0 + t_1, 6*t_0 + 5*t_1 + 4*t_2 - 6, 5*t_0 + 4*t_1 + 3*t_2 - 6)
>>> diop_solve(x**2 + y**2 - 5)
{(-1, -2), (-1, 2), (1, -2), (1, 2)}
```

diop_linear

diofant.solvers.diophantine.**diop_linear**(*eq*, *param=Symbol('t', integer=True)*)

Solves linear diophantine equations.

A linear diophantine equation is an equation of the form $a_1x_1 + a_2x_2 + .. + a_nx_n = 0$ where $a_1, a_2, ..a_n$ are integer constants and $x_1, x_2, ..x_n$ are integer variables.

Parameters eq : Expr

is a linear diophantine equation which is assumed to be zero.

param : Symbol, optional

is the parameter to be used in the solution.

See also:

diofant.solvers.diophantine.diop_quadratic (page 852), *diofant.solvers.diophantine.diop_ternary_quadratic* (page 857), *diofant.solvers.diophantine.diop_general_pythagorean* (page 858), *diofant.solvers.diophantine.diop_general_sum_of_squares* (page 858)

Examples

```
>>> diop_linear(2*x - 3*y - 5)
(3*t_0 - 5, 2*t_0 - 5)
```

Here x = -3*t_0 - 5 and y = -2*t_0 - 5

```
>>> diop_linear(2*x - 3*y - 4*z - 3)
(t_0, 2*t_0 + 4*t_1 + 3, -t_0 - 3*t_1 - 3)
```

base_solution_linear

diofant.solvers.diophantine.**base_solution_linear**(*c*, *a*, *b*, *t=None*)

Return the base solution for the linear equation, $ax + by = c$.

Used by `diop_linear()` to find the base solution of a linear Diophantine equation. If `t` is given then the parametrized solution is returned.

`base_solution_linear(c, a, b, t)`: `a`, `b`, `c` are coefficients in $ax + by = c$ and `t` is the parameter to be used in the solution.

Examples

```
>>> from diofant.abc import t
>>> base_solution_linear(5, 2, 3)  # equation 2*x + 3*y = 5
(-5, 5)
>>> base_solution_linear(0, 5, 7)  # equation 5*x + 7*y = 0
(0, 0)
>>> base_solution_linear(5, 2, 3, t)  # equation 2*x + 3*y = 5
(3*t - 5, -2*t + 5)
>>> base_solution_linear(0, 5, 7, t)  # equation 5*x + 7*y = 0
(7*t, -5*t)
```

diop_quadratic

diofant.solvers.diophantine.**diop_quadratic**(*eq*, *param=Symbol('t'*, *integer=True)*)

Solves quadratic diophantine equations.

i.e. equations of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Returns a set containing the tuples (x, y) which contains the solutions. If there are no solutions then $(None, None)$ is returned.

Parameters eq : Expr

should be a quadratic bivariate expression which is assumed to be zero.

param : Symbol, optional

is a parameter to be used in the solution.

See also:

diofant.solvers.diophantine.diop_linear (page 851), *diofant.solvers.diophantine.diop_ternary_quadratic* (page 857), *diofant.solvers.diophantine.diop_general_sum_of_squares* (page 858), *diofant.solvers.diophantine.diop_general_pythagorean* (page 858)

References

[R498] (page 1266), [R499] (page 1266)

Examples

```
>>> from diofant.abc import t
>>> diop_quadratic(x**2 + y**2 + 2*x + 2*y + 2, t)
{(-1, -1)}
```

diop_DN

diofant.solvers.diophantine.**diop_DN**(*D*, *N*, *t=Symbol('t', integer=True)*)

Solves the equation $x^2 - Dy^2 = N$.

Mainly concerned with the case $D > 0, D$ is not a perfect square, which is the same as the generalized Pell equation. The LMM algorithm [R500] (page 1266) is used to solve this equation.

Parameters D, N : Integer

correspond to D and N in the equation.

t : Symbol, optional

is the parameter to be used in the solutions.

Returns A tuple of pairs, (x, y), for each class of the solutions.

Other solutions of the class can be constructed according to the

values of `D` and `N`.

See also:

diofant.solvers.diophantine.find_DN (page 856), *diofant.solvers.diophantine.diop_bf_DN* (page 854)

References

[R500] (page 1266)

Examples

```
>>> diop_DN(13, -4)  # Solves equation x**2 - 13*y**2 = -4
[(3, 1), (393, 109), (36, 10)]
```

The output can be interpreted as follows: There are three fundamental solutions to the equation $x^2 - 13y^2 = -4$ given by (3, 1), (393, 109) and (36, 10). Each tuple is in the form (x, y), i. e solution (3, 1) means that $x = 3$ and $y = 1$.

```
>>> diop_DN(986, 1)  # Solves equation x**2 - 986*y**2 = 1
[(49299, 1570)]
```

cornacchia

`diofant.solvers.diophantine.`**`cornacchia`**(*a*, *b*, *m*)

Solves $ax^2 + by^2 = m$ where $\gcd(a, b) = 1 = gcd(a, m)$ and $a, b > 0$.

Uses the algorithm due to Cornacchia. The method only finds primitive solutions, i.e. ones with $\gcd(x, y) = 1$. So this method can't be used to find the solutions of $x^2 + y^2 = 20$ since the only solution to former is $(x, y) = (4, 2)$ and it is not primitive. When $a = b$, only the solutions with $x \leq y$ are found. For more details, see the References.

See also:

diofant.utilities.iterables.signed_permutations (page 998)

References

[R501] (page 1266), [R502] (page 1266)

Examples

```
>>> cornacchia(2, 3, 35)  # equation 2x**2 + 3y**2 = 35
{(2, 3), (4, 1)}
>>> cornacchia(1, 1, 25)  # equation x**2 + y**2 = 25
{(3, 4)}
```

diop_bf_DN

`diofant.solvers.diophantine.`**`diop_bf_DN`**(*D*, *N*, *t=Symbol('t', integer=True)*)

Uses brute force to solve the equation, $x^2 - Dy^2 = N$.

Mainly concerned with the generalized Pell equation which is the case when $D > 0, D$ is not a perfect square. For more information on the case refer [R503] (page 1266). Let (t, u) be the minimal positive solution of the equation $x^2 - Dy^2 = 1$. Then this method requires $\sqrt{\frac{|N|(t\pm1)}{2D}}$ to be small.

Parameters D, N : Integer

correspond to D and N in the equation.

t : Symbol, optional

is the parameter to be used in the solutions.

See also:

diofant.solvers.diophantine.diop_DN (page 853)

References

[R503] (page 1266)

Examples

```
>>> diop_bf_DN(13, -4)
[(3, 1), (-3, 1), (36, 10)]
>>> diop_bf_DN(986, 1)
[(49299, 1570)]
```

transformation_to_DN

diofant.solvers.diophantine.**transformation_to_DN**(*eq*)

This function transforms general quadratic, $ax^2 + bxy + cy^2 + dx + ey + f = 0$ to more easy to deal with $X^2 - DY^2 = N$ form.

This is used to solve the general quadratic equation by transforming it to the latter form. Refer [R504] (page 1266) for more detailed information on the transformation. This function returns a tuple (A, B) where A is a 2 X 2 matrix and B is a 2 X 1 matrix such that,

Transpose([x y]) = A * Transpose([X Y]) + B

Parameters eq : Expr

the quadratic expression to be transformed.

See also:

diofant.solvers.diophantine.find_DN (page 856)

References

[R504] (page 1266)

Examples

```
>>> A, B = transformation_to_DN(x**2 - 3*x*y - y**2 - 2*y + 1)
>>> A
Matrix([
[1/26, 3/26],
[   0, 1/13]])
>>> B
Matrix([
[-6/13],
[-4/13]])
```

A, B returned are such that Transpose((x y)) = A * Transpose((X Y)) + B. Substituting these values for x and y and a bit of simplifying work will give an equation of the form $x^2 - Dy^2 = N$.

```
>>> from diofant.abc import X, Y
>>> u = (A*Matrix([X, Y]) + B)[0]  # Transformation for x
>>> u
X/26 + 3*Y/26 - 6/13
>>> v = (A*Matrix([X, Y]) + B)[1]  # Transformation for y
>>> v
Y/13 - 4/13
```

Next we will substitute these formulas for x and y and do `simplify()`.

```
>>> eq = simplify((x**2 - 3*x*y - y**2 - 2*y + 1).subs({x: u, y: v}))
>>> eq
X**2/676 - Y**2/52 + 17/13
```

By multiplying the denominator appropriately, we can get a Pell equation in the standard form.

```
>>> eq * 676
X**2 - 13*Y**2 + 884
```

If only the final equation is needed, `find_DN()` can be used.

find_DN

`diofant.solvers.diophantine.find_DN(eq)`

This function returns a tuple, (D, N) of the simplified form, $x^2 - Dy^2 = N$, corresponding to the general quadratic, $ax^2 + bxy + cy^2 + dx + ey + f = 0$.

Solving the general quadratic is then equivalent to solving the equation $X^2 - DY^2 = N$ and transforming the solutions by using the transformation matrices returned by `transformation_to_DN()`.

Parameters eq : Expr

is the quadratic expression to be transformed.

See also:

diofant.solvers.diophantine.transformation_to_DN (page 855)

References

[R505] (page 1266)

Examples

```
>>> find_DN(x**2 - 3*x*y - y**2 - 2*y + 1)
(13, -884)
```

Interpretation of the output is that we get $X^2 - 13Y^2 = -884$ after transforming $x^2 - 3xy - y^2 - 2y + 1$ using the transformation returned by `transformation_to_DN()`.

diop_ternary_quadratic

diofant.solvers.diophantine.**diop_ternary_quadratic**(*eq*)

Solves the general quadratic ternary form, $ax^2 + by^2 + cz^2 + fxy + gyz + hxz = 0$.

Parameters eq : Expr

should be an homogeneous expression of degree two in three variables and it is assumed to be zero.

Returns tuple

which is a base solution for the above equation. If there are no solutions, (None, None, None) is returned.

Examples

```
>>> diop_ternary_quadratic(x**2 + 3*y**2 - z**2)
(1, 0, 1)
>>> diop_ternary_quadratic(4*x**2 + 5*y**2 - z**2)
(1, 0, 2)
>>> diop_ternary_quadratic(45*x**2 - 7*y**2 - 8*x*y - z**2)
(28, 45, 105)
>>> diop_ternary_quadratic(x**2 - 49*y**2 - z**2 + 13*z*y -8*x*y)
(9, 1, 5)
```

square_factor

diofant.solvers.diophantine.**square_factor**(*a*)

Returns an integer c s.t. $a = c^2k, \ c,k \in Z$. Here k is square free. a can be given as an integer or a dictionary of factors.

See also:

diofant.solvers.diophantine.reconstruct (page 868), *diofant.ntheory.factor_.core* (page 258)

Examples

```
>>> square_factor(24)
2
>>> square_factor(-36*3)
6
>>> square_factor(1)
1
>>> square_factor({3: 2, 2: 1, -1: 1})
3
```

descent

diofant.solvers.diophantine.**descent**(*A*, *B*)

Returns a non-trivial solution, (x, y, z), to $x^2 = Ay^2 + Bz^2$ using Lagrange's descent method with lattice-reduction. A and B are assumed to be valid for such a solution to exist.

This is faster than the normal Lagrange's descent algorithm because the Gaussian reduction is used.

References

[R506] (page 1266)

Examples

```
>>> descent(3, 1)  # x**2 = 3*y**2 + z**2
(1, 0, 1)
```

$(x, y, z) = (1, 0, 1)$ is a solution to the above equation.

```
>>> descent(41, -113)
(-16, -3, 1)
```

diop_general_pythagorean

diofant.solvers.diophantine.**diop_general_pythagorean**(*eq*, *param=Symbol('m', integer=True)*)

Solves the general pythagorean equation, $a_1^2x_1^2 + a_2^2x_2^2 + ... + a_n^2x_n^2 - a_{n+1}^2x_{n+1}^2 = 0$.

Returns a tuple which contains a parametrized solution to the equation, sorted in the same order as the input variables.

Parameters eq : Expr

is a general pythagorean equation which is assumed to be zero

param : Symbol, optional

is the base parameter used to construct other parameters by subscripting.

Examples

```
>>> from diofant.abc import a, b, c, d, e
>>> diop_general_pythagorean(a**2 + b**2 + c**2 - d**2)
(m1**2 + m2**2 - m3**2, 2*m1*m3, 2*m2*m3, m1**2 + m2**2 + m3**2)
>>> diop_general_pythagorean(9*a**2 - 4*b**2 + 16*c**2 + 25*d**2 + e**2)
(10*m1**2  + 10*m2**2  + 10*m3**2 - 10*m4**2, 15*m1**2  + 15*m2**2  + 15*m3**2  +
↪15*m4**2, 15*m1*m4, 12*m2*m4, 60*m3*m4)
```

diop_general_sum_of_squares

diofant.solvers.diophantine.**diop_general_sum_of_squares**(*eq*, *limit=1*)

Solves the equation $x_1^2 + x_2^2 + ... + x_n^2 - k = 0$.

Returns at most `limit` number of solutions.

Parameters eq : Expr

is an expression which is assumed to be zero. Also, eq should be in the form, $x_1^2 + x_2^2 + ... + x_n^2 - k = 0$.

limit : int, optional

upper limit (the default is 1) for number of solutions returned.

Notes

When $n = 3$ if $k = 4^a(8m + 7)$ for some $a, m \in Z$ then there will be no solutions. Refer *[R507]* (page 1266) for more details.

References

[R507] (page 1266)

Examples

```
>>> from diofant.abc import a, b, c, d, e, f
>>> diop_general_sum_of_squares(a**2 + b**2 + c**2 + d**2 + e**2 - 2345)
{(15, 22, 22, 24, 24)}
```

diop_general_sum_of_even_powers

`diofant.solvers.diophantine.`**`diop_general_sum_of_even_powers`**(*eq*, *limit=1*)

Solves the equation $x_1^e + x_2^e + ... + x_n^e - k = 0$ where e is an even, integer power.

Returns at most `limit` number of solutions.

Parameters eq : Expr

An expression which is assumed to be zero. Also, eq should be in the form, $x_1^e + x_2^e + ... + x_n^e - k = 0$.

limit : Expr, optional

Limit number of returned solutions. Default is 1.

See also:

diofant.solvers.diophantine.power_representation (page 861)

Examples

```
>>> from diofant.abc import a, b
>>> diop_general_sum_of_even_powers(a**4 + b**4 - (2**4 + 3**4))
{(2, 3)}
```

partition

diofant.solvers.diophantine.**partition**(*n*, *k=None*, *zeros=False*)

Returns a generator that can be used to generate partitions of an integer n.

A partition of n is a set of positive integers which add up to n. For example, partitions of 3 are 3, 1 + 2, 1 + 1 + 1. A partition is returned as a tuple. If `k` equals None, then all possible partitions are returned irrespective of their size, otherwise only the partitions of size `k` are returned. If the `zero` parameter is set to True then a suitable number of zeros are added at the end of every partition of size less than `k`.

Parameters n : int

is a positive integer

k : int, optional

is the size of the partition which is also positive integer. The default is `None`.

zeros : boolean, optional

parameter is considered only if `k` is not `None`. When the partitions are over, the last $next()$ call throws the `StopIteration` exception, so this function should always be used inside a try - except block.

Examples

```
>>> f = partition(5)
>>> next(f)
(1, 1, 1, 1, 1)
>>> next(f)
(1, 1, 1, 2)
>>> g = partition(5, 3)
>>> next(g)
(1, 1, 3)
>>> next(g)
(1, 2, 2)
>>> g = partition(5, 3, zeros=True)
>>> next(g)
(0, 0, 5)
```

sum_of_three_squares

diofant.solvers.diophantine.**sum_of_three_squares**(*n*)

Returns a 3-tuple (a, b, c) such that $a^2 + b^2 + c^2 = n$ and $a, b, c \geq 0$.

Returns None if $n = 4^a(8m+7)$ for some $a, m \in Z$. See *[R508]* (page 1266) for more details.

Parameters n : int

a non-negative integer.

See also:

diofant.solvers.diophantine.sum_of_squares (page 862)

References

[R508] (page 1266)

Examples

```
>>> sum_of_three_squares(44542)
(18, 37, 207)
```

sum_of_four_squares

`diofant.solvers.diophantine.sum_of_four_squares(n)`

Returns a 4-tuple (a, b, c, d) such that $a^2 + b^2 + c^2 + d^2 = n$.

Here $a, b, c, d \geq 0$.

Parameters n : int

is a non-negative integer.

See also:

diofant.solvers.diophantine.sum_of_squares (page 862)

References

[R509] (page 1266)

Examples

```
>>> sum_of_four_squares(3456)
(8, 8, 32, 48)
>>> sum_of_four_squares(1294585930293)
(0, 1234, 2161, 1137796)
```

power_representation

`diofant.solvers.diophantine.power_representation(n, p, k, zeros=False)`

Returns a generator for finding k-tuples of integers, $(n_1, n_2, ...n_k)$, such that $n = n_1^p + n_2^p + ...n_k^p$.

Parameters n : int

a non-negative integer

k, p : int

parameters to control representation `n` as a sum of `k`, `p`-th powers.

zeros : boolean, optional

if `True` (the default is `False`), then the solutions will contain zeros.

Examples

Represent 1729 as a sum of two cubes:

```
>>> f = power_representation(1729, 3, 2)
>>> next(f)
(9, 10)
>>> next(f)
(1, 12)
```

If the flag $zeros$ is True, the solution may contain tuples with zeros; any such solutions will be generated after the solutions without zeros:

```
>>> list(power_representation(125, 2, 3, zeros=True))
[(5, 6, 8), (3, 4, 10), (0, 5, 10), (0, 2, 11)]
```

For even p the $permute_sign$ function can be used to get all signed values:

```
>>> from diofant.utilities.iterables import permute_signs
>>> list(permute_signs((1, 12)))
[(1, 12), (-1, 12), (1, -12), (-1, -12)]
```

All possible signed permutations can also be obtained:

```
>>> from diofant.utilities.iterables import signed_permutations
>>> list(signed_permutations((1, 12)))
[(1, 12), (-1, 12), (1, -12), (-1, -12), (12, 1), (-12, 1),
 (12, -1), (-12, -1)]
```

diofant.solvers.diophantine.**sum_of_powers**()
alias of *power_representation()* (page 861)

sum_of_squares

diofant.solvers.diophantine.**sum_of_squares**(*n*, *k*, *zeros=False*)
Return a generator that yields the k-tuples of nonnegative values, the squares of which sum to n. If zeros is False (default) then the solution will not contain zeros. The nonnegative elements of a tuple are sorted.

- If k == 1 and n is square, (n,) is returned.
- If k == 2 then n can only be written as a sum of squares if every prime in the factorization of n that has the form 4*k + 3 has an even multiplicity. If n is prime then it can only be written as a sum of two squares if it is in the form 4*k + 1.
- if k == 3 then n can be written as a sum of squares if it does not have the form 4**m*(8*k + 7).
- all integers can be written as the sum of 4 squares.
- if k > 4 then n can be partitioned and each partition can be written as a sum of 4 squares; if n is not evenly divisible by 4 then n can be written as a sum of squares only if the an additional partition can be written as as sum of squares. For example, if k = 6 then n is partitioned into two parts, the first being written as a sum of 4 squares and the second being written as a sum of 2 squares – which can only be done if the contition above for k = 2 can be met, so this will automatically reject certain partitions of n.

See also:

diofant.utilities.iterables.signed_permutations (page 998)

Examples

```
>>> list(sum_of_squares(25, 2))
[(3, 4)]
>>> list(sum_of_squares(25, 2, True))
[(3, 4), (0, 5)]
>>> list(sum_of_squares(25, 4))
[(1, 2, 2, 4)]
```

merge_solution

`diofant.solvers.diophantine.merge_solution`(*var*, *var_t*, *solution*)

This is used to construct the full solution from the solutions of sub equations.

For example when solving the equation $(x - y)(x^2 + y^2 - z^2) = 0$, solutions for each of the equations $x - y = 0$ and $x^2 + y^2 - z^2$ are found independently. Solutions for $x - y = 0$ are $(x, y) = (t, t)$. But we should introduce a value for z when we output the solution for the original equation. This function converts (t, t) into (t, t, n_1) where n_1 is an integer parameter.

divisible

`diofant.solvers.diophantine.divisible`(*a*, *b*)

Returns $True$ if a is divisible by b and $False$ otherwise.

PQa

`diofant.solvers.diophantine.PQa`(P_0, Q_0, D)

Returns useful information needed to solve the Pell equation.

There are six sequences of integers defined related to the continued fraction representation of $\frac{P+\sqrt{D}}{Q}$, namely $\{P_i\}$, $\{Q_i\}$, $\{a_i\}$,$\{A_i\}$, $\{B_i\}$, $\{G_i\}$. `PQa()` Returns these values as a 6-tuple in the same order as mentioned above. Refer *[R510]* (page 1266) for more detailed information.

Parameters P_0, Q_0, D : Integer

integers corresponding to P_0, Q_0 and D in the continued fraction $\frac{P_0+\sqrt{D}}{Q_0}$. Also it's assumed that $P_0^2 == Dmod(|Q_0|)$ and D is square free.

References

[R510] (page 1266)

Examples

```
>>> pqa = PQa(13, 4, 5)  # (13 + sqrt(5))/4
>>> next(pqa)  # (P_0, Q_0, a_0, A_0, B_0, G_0)
(13, 4, 3, 3, 1, -1)
>>> next(pqa)  # (P_1, Q_1, a_1, A_1, B_1, G_1)
(-1, 1, 1, 4, 1, 3)
```

equivalent

diofant.solvers.diophantine.**equivalent**(*u*, *v*, *r*, *s*, *D*, *N*)

Returns True if two solutions (u, v) and (r, s) of $x^2 - Dy^2 = N$ belongs to the same equivalence class and False otherwise.

Two solutions (u, v) and (r, s) to the above equation fall to the same equivalence class iff both $(ur - Dvs)$ and $(us - vr)$ are divisible by N. See reference [R511] (page 1266). No test is performed to test whether (u, v) and (r, s) are actually solutions to the equation. User should take care of this.

Parameters u, v, r, s, D, N : Integer

References

[R511] (page 1266)

Examples

```
>>> equivalent(18, 5, -18, -5, 13, -1)
True
>>> equivalent(3, 1, -18, 393, 109, -4)
False
```

parametrize_ternary_quadratic

diofant.solvers.diophantine.**parametrize_ternary_quadratic**(*eq*)

Returns the parametrized general solution for the ternary quadratic equation eq which has the form $ax^2 + by^2 + cz^2 + fxy + gyz + hxz = 0$.

References

[R512] (page 1266)

Examples

```
>>> parametrize_ternary_quadratic(x**2 + y**2 - z**2)
(2*p*q, p**2 - q**2, p**2 + q**2)
```

Here p and q are two co-prime integers.

```
>>> parametrize_ternary_quadratic(3*x**2 + 2*y**2 - z**2 - 2*x*y + 5*y*z - 7*y*z)
(2*p**2 - 2*p*q - q**2, 2*p**2 + 2*p*q - q**2, 2*p**2 - 2*p*q + 3*q**2)
>>> parametrize_ternary_quadratic(124*x**2 - 30*y**2 - 7729*z**2)
(-1410*p**2 - 363263*q**2, 2700*p**2 + 30916*p*q - 695610*q**2, -60*p**2 +
↪5400*p*q + 15458*q**2)
```

diop_ternary_quadratic_normal

diofant.solvers.diophantine.**diop_ternary_quadratic_normal**(*eq*)

Solves the quadratic ternary diophantine equation, $ax^2 + by^2 + cz^2 = 0$.

Here the coefficients a, b, and c should be non zero. Otherwise the equation will be a quadratic binary or univariate equation. If solvable, returns a tuple (x, y, z) that satisifes the given equation. If the equation does not have integer solutions, $(None, None, None)$ is returned.

Examples

```
>>> diop_ternary_quadratic_normal(x**2 + 3*y**2 - z**2)
(1, 0, 1)
>>> diop_ternary_quadratic_normal(4*x**2 + 5*y**2 - z**2)
(1, 0, 2)
>>> diop_ternary_quadratic_normal(34*x**2 - 3*y**2 - 301*z**2)
(4, 9, 1)
```

ldescent

diofant.solvers.diophantine.**ldescent**(*A*, *B*)

Return a non-trivial solution to $w^2 = Ax^2 + By^2$ using Lagrange's method; return None if there is no such solution.

Here, $A \neq 0$ and $B \neq 0$ and A and B are square free. Output a tuple (w_0, x_0, y_0) which is a solution to the above equation.

References

[R513] (page 1266), [R514] (page 1266)

Examples

```
>>> ldescent(1, 1)  # w^2 = x^2 + y^2
(1, 1, 0)
>>> ldescent(4, -7)  # w^2 = 4x^2 - 7y^2
(2, -1, 0)
```

This means that $x = -1, y = 0$ and $w = 2$ is a solution to the equation $w^2 = 4x^2 - 7y^2$

```
>>> ldescent(5, -1)  # w^2 = 5x^2 - y^2
(2, 1, -1)
```

gaussian_reduce

diofant.solvers.diophantine.**gaussian_reduce**(*w*, *a*, *b*)

Returns a reduced solution (x, z) to the congruence $X^2 - aZ^2 \equiv 0 \ (mod\ b)$ so that $x^2 + |a|z^2$ is minimal.

Here w is a solution of the congruence $x^2 \equiv a \ (mod\ b)$

References

[R515] (page 1266), *[R516]* (page 1266)

holzer

diofant.solvers.diophantine.**holzer**(*x*, *y*, *z*, *a*, *b*, *c*)

Simplify the solution (x, y, z) of the equation $ax^2 + by^2 = cz^2$ with $a, b, c > 0$ and $z^2 \geq \mid ab \mid$ to a new reduced solution (x', y', z') such that $z'^2 \leq \mid ab \mid$.

The algorithm is an interpretation of Mordell's reduction as described on page 8 of Cremona and Rusin's paper *[R517]* (page 1266) and the work of Mordell in reference *[R518]* (page 1266).

References

[R517] (page 1266), *[R518]* (page 1266)

prime_as_sum_of_two_squares

diofant.solvers.diophantine.**prime_as_sum_of_two_squares**(*p*)

Represent a prime p as a unique sum of two squares; this can only be done if the prime is congruent to 1 mod 4.

See also:

diofant.solvers.diophantine.sum_of_squares (page 862)

References

[R519] (page 1266)

Examples

```
>>> prime_as_sum_of_two_squares(7)  # can't be done
>>> prime_as_sum_of_two_squares(5)
(1, 2)
```

square_factor

diofant.solvers.diophantine.**square_factor**(a)

Returns an integer c s.t. $a = c^2k, \ c, k \in Z$. Here k is square free. a can be given as an integer or a dictionary of factors.

See also:

diofant.solvers.diophantine.reconstruct (page 868), *diofant.ntheory.factor_.core* (page 258)

Examples

```
>>> square_factor(24)
2
>>> square_factor(-36*3)
6
>>> square_factor(1)
1
>>> square_factor({3: 2, 2: 1, -1: 1})
3
```

sqf_normal

diofant.solvers.diophantine.**sqf_normal**(a, b, c, *steps=False*)

Return a', b', c', the coefficients of the square-free normal form of $ax^2 + by^2 + cz^2 = 0$, where a', b', c' are pairwise prime. If $steps$ is True then also return three tuples: sq, sqf, and (a', b', c') where sq contains the square factors of a, b and c after removing the $gcd(a, b, c)$; sqf contains the values of a, b and c after removing both the $gcd(a, b, c)$ and the square factors.

The solutions for $ax^2 + by^2 + cz^2 = 0$ can be recovered from the solutions of $a'x^2 + b'y^2 + c'z^2 = 0$.

See also:

diofant.solvers.diophantine.reconstruct (page 868)

References

[R520] (page 1267)

Examples

```
>>> sqf_normal(2 * 3**2 * 5, 2 * 5 * 11, 2 * 7**2 * 11)
(11, 1, 5)
>>> sqf_normal(2 * 3**2 * 5, 2 * 5 * 11, 2 * 7**2 * 11, True)
((3, 1, 7), (5, 55, 11), (11, 1, 5))
```

reconstruct

`diofant.solvers.diophantine.reconstruct(A, B, z)`

Reconstruct the z value of an equivalent solution of $ax^2 + by^2 + cz^2$ from the z value of a solution of the square-free normal form of the equation, $a' * x^2 + b' * y^2 + c' * z^2$, where a', b' and c' are square free and $gcd(a', b', c') == 1$.

transformation_to_normal

`diofant.solvers.diophantine.transformation_to_normal(eq)`

Returns the transformation Matrix that converts a general ternary quadratic equation eq ($ax^2 + by^2 + cz^2 + dxy + eyz + fxz$) to a form without cross terms: $ax^2 + by^2 + cz^2 = 0$. This is not used in solving ternary quadratics; it is only implemented for the sake of completeness.

3.19.5 ODE

User Functions

These are functions that are imported into the global namespace with `from diofant import *`. These functions (unlike *Hint Functions* (page 876), below) are intended for use by ordinary users of Diofant.

dsolve

`diofant.solvers.ode.dsolve(eq, func=None, hint='default', simplify=True, init=None, xi=None, eta=None, x0=0, n=6, **kwargs)`

Solves any (supported) kind of ordinary differential equation and system of ordinary differential equations.

For single ordinary differential equation

It is classified under this when number of equation in `eq` is one.

Usage

`dsolve(eq, f(x), hint)` -> Solve ordinary differential equation `eq` for function `f(x)`, using method `hint`.

Details

`eq` can be any supported ordinary differential equation (see the `ode` (page 915) docstring for supported methods). This can either be an `Equality` (page 111), or an expression, which is assumed to be equal to `0`.

`f(x)` is a function of one variable whose derivatives in that variable make up the ordinary differential equation `eq`. In many cases it is not necessary to provide this; it will be autodetected (and an error raised if it couldn't be detected).

`hint` is the solving method that you want dsolve to use. Use `classify_ode(eq, f(x))` to get all of the possible hints for an ODE. The default hint, `default`, will use whatever hint is returned first by `classify_ode()` (page 871). See Hints below for more options that you can use for hint.

simplify enables simplification by *odesimp()* (page 876). See its docstring for more information. Turn this off, for example, to disable solving of solutions for func or simplification of arbitrary constants. It will still integrate with this hint. Note that the solution may contain more arbitrary constants than the order of the ODE with this option enabled.

xi and eta are the infinitesimal functions of an ordinary differential equation. They are the infinitesimals of the Lie group of point transformations for which the differential equation is invariant. The user can specify values for the infinitesimals. If nothing is specified, xi and eta are calculated using *infinitesimals()* (page 874) with the help of various heuristics.

init is the set of initial/boundary conditions for the differential equation. It should be given in the form of {f(x0): x1, f(x).diff(x).subs(x, x2): x3} and so on. For power series solutions, if no initial conditions are specified f(0) is assumed to be C0 and the power series solution is calculated about 0.

x0 is the point about which the power series solution of a differential equation is to be evaluated.

n gives the exponent of the dependent variable up to which the power series solution of a differential equation is to be evaluated.

Hints

Aside from the various solving methods, there are also some meta-hints that you can pass to *dsolve()* (page 868):

default: This uses whatever hint is returned first by *classify_ode()* (page 871). This is the default argument to *dsolve()* (page 868).

all: To make *dsolve()* (page 868) apply all relevant classification hints, use dsolve(ODE, func, hint="all"). This will return a dictionary of hint:solution terms. If a hint causes dsolve to raise the NotImplementedError, value of that hint's key will be the exception object raised. The dictionary will also include some special keys:

- order: The order of the ODE. See also *ode_order()* (page 930) in deutils.py.
- best: The simplest hint; what would be returned by best below.
- best_hint: The hint that would produce the solution given by best. If more than one hint produces the best solution, the first one in the tuple returned by *classify_ode()* (page 871) is chosen.
- default: The solution that would be returned by default. This is the one produced by the hint that appears first in the tuple returned by *classify_ode()* (page 871).

all_Integral: This is the same as all, except if a hint also has a corresponding _Integral hint, it only returns the _Integral hint. This is useful if all causes *dsolve()* (page 868) to hang because of a difficult or impossible integral. This meta-hint will also be much faster than all, because *integrate()* (page 67) is an expensive routine.

best: To have *dsolve()* (page 868) try all methods and return the simplest one. This takes into account whether the solution is solvable in the function,

whether it contains any Integral classes (i.e. unevaluatable integrals), and which one is the shortest in size.

See also the *classify_ode()* (page 871) docstring for more info on hints, and the *ode* (page 915) docstring for a list of all supported hints.

Tips

- You can declare the derivative of an unknown function this way:

```
>>> f = Function("f")(x)  # f is a function of x
>>> # f_ will be the derivative of f with respect to x
>>> f_ = Derivative(f, x)
```

- See `test_ode.py` for many tests, which serves also as a set of examples for how to use *dsolve()* (page 868).
- *dsolve()* (page 868) always returns an *Equality* (page 111) class (except for the case when the hint is `all` or `all_Integral`). If possible, it solves the solution explicitly for the function being solved for. Otherwise, it returns an implicit solution.
- Arbitrary constants are symbols named `C1`, `C2`, and so on.
- Because all solutions should be mathematically equivalent, some hints may return the exact same result for an ODE. Often, though, two different hints will return the same solution formatted differently. The two should be equivalent. Also note that sometimes the values of the arbitrary constants in two different solutions may not be the same, because one constant may have "absorbed" other constants into it.
- Do `help(ode.ode_<hintname>)` to get help more information on a specific hint, where `<hintname>` is the name of a hint without `_Integral`.

For system of ordinary differential equations

Usage

`dsolve(eq, func) -> Solve a system of ordinary differential equations eq for func being list of functions including $x(t)$, $y(t)$, $z(t)$ where number of functions in the list depends upon the number of equations provided in `eq`.

Details

`eq` can be any supported system of ordinary differential equations This can either be an *Equality* (page 111), or an expression, which is assumed to be equal to `0`.

`func` holds `x(t)` and `y(t)` being functions of one variable which together with some of their derivatives make up the system of ordinary differential equation `eq`. It is not necessary to provide this; it will be autodetected (and an error raised if it couldn't be detected).

Hints

The hints are formed by parameters returned by classify_sysode, combining them give hints name used later for forming method name.

Examples

```
>>> f = Function('f')
>>> dsolve(Derivative(f(x), x, x) + 9*f(x), f(x))
Eq(f(x), C1*sin(3*x) + C2*cos(3*x))
```

```
>>> eq = sin(x)*cos(f(x)) + cos(x)*sin(f(x))*f(x).diff(x)
>>> dsolve(eq, hint='1st_exact')
[Eq(f(x), -acos(C1/cos(x)) + 2*pi), Eq(f(x), acos(C1/cos(x)))]
>>> dsolve(eq, hint='almost_linear')
[Eq(f(x), -acos(C1/sqrt(-cos(x)**2)) + 2*pi), Eq(f(x), acos(C1/sqrt(-cos(x)**2)))]
>>> t = symbols('t')
>>> x, y = symbols('x, y', cls=Function)
>>> eq = (Eq(Derivative(x(t), t), 12*t*x(t) + 8*y(t)), Eq(Derivative(y(t), t),
↪21*x(t) + 7*t*y(t)))
>>> dsolve(eq)
[Eq(x(t), C1*x0(t) + C2*x0(t)*Integral(8*E**Integral(7*t, t)*E**Integral(12*t, t)/
↪x0(t)**2, t)),
Eq(y(t), C1*y0(t) + C2*(E**Integral(7*t, t)*E**Integral(12*t, t)/x0(t) +
y0(t)*Integral(8*E**Integral(7*t, t)*E**Integral(12*t, t)/x0(t)**2, t)))]
>>> eq = (Eq(Derivative(x(t), t), x(t)*y(t)*sin(t)), Eq(Derivative(y(t), t),
↪y(t)**2*sin(t)))
>>> C1, C2 = symbols('C1 C2')
>>> dsolve(eq)
{Eq(x(t), -E**C1/(E**C1*C2 - cos(t))), Eq(y(t), -1/(C1 - cos(t)))}
```

classify_ode

diofant.solvers.ode.**classify_ode**(*eq*, *func=None*, *dict=False*, *init=None*, ***kwargs*)

Returns a tuple of possible *dsolve()* (page 868) classifications for an ODE.

The tuple is ordered so that first item is the classification that *dsolve()* (page 868) uses to solve the ODE by default. In general, classifications at the near the beginning of the list will produce better solutions faster than those near the end, thought there are always exceptions. To make *dsolve()* (page 868) use a different classification, use `dsolve(ODE, func, hint=<classification>)`. See also the *dsolve()* (page 868) docstring for different meta-hints you can use.

If `dict` is true, *classify_ode()* (page 871) will return a dictionary of `hint:match` expression terms. This is intended for internal use by *dsolve()* (page 868). Note that because dictionaries are ordered arbitrarily, this will most likely not be in the same order as the tuple.

You can get help on different hints by executing `help(ode.ode_hintname)`, where `hintname` is the name of the hint without `_Integral`.

See *allhints* (page 876) or the *ode* (page 915) docstring for a list of all supported hints that can be returned from *classify_ode()* (page 871).

Notes

These are remarks on hint names.

`_Integral`

If a classification has `_Integral` at the end, it will return the expression with an unevaluated *Integral* (page 529) class in it. Note that a hint may do this

anyway if `integrate()` (page 522) cannot do the integral, though just using an `_Integral` will do so much faster. Indeed, an `_Integral` hint will always be faster than its corresponding hint without `_Integral` because `integrate()` (page 522) is an expensive routine. If `dsolve()` (page 868) hangs, it is probably because `integrate()` (page 67) is hanging on a tough or impossible integral. Try using an `_Integral` hint or `all_Integral` to get it return something.

Note that some hints do not have `_Integral` counterparts. This is because `integrate()` (page 522) is not used in solving the ODE for those method. For example, nth order linear homogeneous ODEs with constant coefficients do not require integration to solve, so there is no `nth_linear_homogeneous_constant_coeff_Integrate` hint. You can easily evaluate any unevaluated `Integral` (page 529)s in an expression by doing `expr.doit()`.

Ordinals

Some hints contain an ordinal such as `1st_linear`. This is to help differentiate them from other hints, as well as from other methods that may not be implemented yet. If a hint has `nth` in it, such as the `nth_linear` hints, this means that the method used to applies to ODEs of any order.

`indep` and `dep`

Some hints contain the words `indep` or `dep`. These reference the independent variable and the dependent function, respectively. For example, if an ODE is in terms of $f(x)$, then `indep` will refer to x and `dep` will refer to f.

`subs`

If a hints has the word `subs` in it, it means the the ODE is solved by substituting the expression given after the word `subs` for a single dummy variable. This is usually in terms of `indep` and `dep` as above. The substituted expression will be written only in characters allowed for names of Python objects, meaning operators will be spelled out. For example, `indep/dep` will be written as `indep_div_dep`.

`coeff`

The word `coeff` in a hint refers to the coefficients of something in the ODE, usually of the derivative terms. See the docstring for the individual methods for more info (`help(ode)`). This is contrast to `coefficients`, as in `undetermined_coefficients`, which refers to the common name of a method.

`_best`

Methods that have more than one fundamental way to solve will have a hint for each sub-method and a `_best` meta-classification. This will evaluate all hints and return the best, using the same considerations as the normal `best` meta-hint.

Examples

```
>>> f = Function('f')
>>> classify_ode(Eq(f(x).diff(x), 0), f(x))
('separable', '1st_linear', '1st_homogeneous_coeff_best',
'1st_homogeneous_coeff_subs_indep_div_dep',
'1st_homogeneous_coeff_subs_dep_div_indep',
```

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```
'1st_power_series', 'lie_group',
'nth_linear_constant_coeff_homogeneous',
'separable_Integral', '1st_linear_Integral',
'1st_homogeneous_coeff_subs_indep_div_dep_Integral',
'1st_homogeneous_coeff_subs_dep_div_indep_Integral')
>>> classify_ode(f(x).diff(x, 2) + 3*f(x).diff(x) + 2*f(x) - 4)
('nth_linear_constant_coeff_undetermined_coefficients',
'nth_linear_constant_coeff_variation_of_parameters',
'nth_linear_constant_coeff_variation_of_parameters_Integral')
```

checkodesol

`diofant.solvers.ode.checkodesol`(*ode, sol, func=None, order='auto', solve_for_func=True*)

Substitutes `sol` into `ode` and checks that the result is `0`.

This only works when `func` is one function, like $f(x)$. `sol` can be a single solution or a list of solutions. Each solution may be an *Equality* (page 111) that the solution satisfies, e.g. `Eq(f(x), C1), Eq(f(x) + C1, 0)`; or simply an *Expr* (page 54), e.g. `f(x) - C1`. In most cases it will not be necessary to explicitly identify the function, but if the function cannot be inferred from the original equation it can be supplied through the `func` argument.

If a sequence of solutions is passed, the same sort of container will be used to return the result for each solution.

It tries the following methods, in order, until it finds zero equivalence:

1. Substitute the solution for f in the original equation. This only works if `ode` is solved for f. It will attempt to solve it first unless `solve_for_func == False`.
2. Take n derivatives of the solution, where n is the order of `ode`, and check to see if that is equal to the solution. This only works on exact ODEs.
3. Take the 1st, 2nd, ..., nth derivatives of the solution, each time solving for the derivative of f of that order (this will always be possible because f is a linear operator). Then back substitute each derivative into `ode` in reverse order.

This function returns a tuple. The first item in the tuple is `True` if the substitution results in `0`, and `False` otherwise. The second item in the tuple is what the substitution results in. It should always be `0` if the first item is `True`. Note that sometimes this function will `False`, but with an expression that is identically equal to `0`, instead of returning `True`. This is because *simplify()* (page 776) cannot reduce the expression to `0`. If an expression returned by this function vanishes identically, then `sol` really is a solution to `ode`.

If this function seems to hang, it is probably because of a hard simplification.

To use this function to test, test the first item of the tuple.

Examples

```
>>> x, C1 = symbols('x C1')
>>> f = Function('f')
>>> checkodesol(f(x).diff(x), Eq(f(x), C1))
```

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```
(True, 0)
>>> assert checkodesol(f(x).diff(x), C1)[0]
>>> assert not checkodesol(f(x).diff(x), x)[0]
>>> checkodesol(f(x).diff(x, 2), x**2)
(False, 2)
```

homogeneous_order

diofant.solvers.ode.**homogeneous_order**(*eq*, **symbols*)

Returns the order n if g is homogeneous and None if it is not homogeneous.

Determines if a function is homogeneous and if so of what order. A function $f(x, y, \cdots)$ is homogeneous of order n if $f(tx, ty, \cdots) = t^n f(x, y, \cdots)$.

If the function is of two variables, $F(x, y)$, then f being homogeneous of any order is equivalent to being able to rewrite $F(x, y)$ as $G(x/y)$ or $H(y/x)$. This fact is used to solve 1st order ordinary differential equations whose coefficients are homogeneous of the same order (see the docstrings of *ode_1st_homogeneous_coeff_subs_dep_div_indep()* (page 881) and *ode_1st_homogeneous_coeff_subs_indep_div_dep()* (page 883)).

Symbols can be functions, but every argument of the function must be a symbol, and the arguments of the function that appear in the expression must match those given in the list of symbols. If a declared function appears with different arguments than given in the list of symbols, None is returned.

Examples

```
>>> f = Function('f')
>>> homogeneous_order(f(x), f(x)) is None
True
>>> homogeneous_order(f(x, y), f(y, x), x, y) is None
True
>>> homogeneous_order(f(x), f(x), x)
1
>>> homogeneous_order(x**2*f(x)/sqrt(x**2+f(x)**2), x, f(x))
2
>>> homogeneous_order(x**2+f(x), x, f(x)) is None
True
```

infinitesimals

diofant.solvers.ode.**infinitesimals**(*eq*, *func=None*, *order=None*, *hint='default'*, *match=None*)

The infinitesimal functions of an ordinary differential equation, $\xi(x, y)$ and $\eta(x, y)$, are the infinitesimals of the Lie group of point transformations for which the differential equation is invariant. So, the ODE $y' = f(x, y)$ would admit a Lie group $x^* = X(x, y; \varepsilon) = x + \varepsilon\xi(x, y)$, $y^* = Y(x, y; \varepsilon) = y + \varepsilon\eta(x, y)$ such that $(y^*)' = f(x^*, y^*)$. A change of coordinates, to $r(x, y)$ and $s(x, y)$, can be performed so this Lie group becomes the translation group, $r^* = r$ and $s^* = s + \varepsilon$. They are tangents to the coordinate curves of the new system.

Consider the transformation $(x, y) \to (X, Y)$ such that the differential equation remains invariant. ξ and η are the tangents to the transformed coordinates X and Y, at $\varepsilon = 0$.

$$\left(\frac{\partial X(x, y; \varepsilon)}{\partial \varepsilon}\right)|_{\varepsilon=0} = \xi, \left(\frac{\partial Y(x, y; \varepsilon)}{\partial \varepsilon}\right)|_{\varepsilon=0} = \eta,$$

The infinitesimals can be found by solving the following PDE:

```
>>> xi, eta, h = map(Function, ['xi', 'eta', 'h'])
>>> h = h(x, y)  # dy/dx = h
>>> eta = eta(x, y)
>>> xi = xi(x, y)
>>> genform = Eq(eta.diff(x) + (eta.diff(y) - xi.diff(x))*h
... - (xi.diff(y))*h**2 - xi*(h.diff(x)) - eta*(h.diff(y)), 0)
>>> pprint(genform, use_unicode=False)
            d               2       d                       /d              d
- eta(x, y)*--(h(x, y)) - h (x, y)*--(xi(x, y)) + h(x, y)*|--(eta(x, y)) - --(
            dy                      dy                      \dy             dx

         \            d              d
xi(x, y))| - xi(x, y)*--(h(x, y)) + --(eta(x, y)) = 0
         /            dx             dx
```

Solving the above mentioned PDE is not trivial, and can be solved only by making intelligent assumptions for ξ and η (heuristics). Once an infinitesimal is found, the attempt to find more heuristics stops. This is done to optimise the speed of solving the differential equation. If a list of all the infinitesimals is needed, `hint` should be flagged as `all`, which gives the complete list of infinitesimals. If the infinitesimals for a particular heuristic needs to be found, it can be passed as a flag to `hint`.

References

- Solving differential equations by Symmetry Groups, John Starrett, pp. 1 - pp. 14

Examples

```
>>> f = Function('f')
>>> eta = Function('eta')
>>> xi = Function('xi')
>>> eq = f(x).diff(x) - x**2*f(x)
>>> infinitesimals(eq)
[{eta(x, f(x)): E**(x**3/3), xi(x, f(x)): 0}]
```

checkinfsol

`diofant.solvers.ode.`**`checkinfsol`**(*eq, infinitesimals, func=None, order=None*)

This function is used to check if the given infinitesimals are the actual infinitesimals of the given first order differential equation. This method is specific to the Lie Group Solver of ODEs.

As of now, it simply checks, by substituting the infinitesimals in the partial differential equation.

$$\frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x}\right) * h - \frac{\partial \xi}{\partial y} * h^2 - \xi \frac{\partial h}{\partial x} - \eta \frac{\partial h}{\partial y} = 0$$

where η, and ξ are the infinitesimals and $h(x, y) = \frac{dy}{dx}$

The infinitesimals should be given in the form of a list of dicts `[{xi(x, y): inf, eta(x, y): inf}]`, corresponding to the output of the function infinitesimals. It returns a list of values of the form `[(True/False, sol)]` where `sol` is the value obtained after substituting the infinitesimals in the PDE. If it is `True`, then `sol` would be 0.

Hint Functions

These functions are intended for internal use by *dsolve()* (page 868) and others. Unlike *User Functions* (page 868), above, these are not intended for every-day use by ordinary Diofant users. Instead, functions such as *dsolve()* (page 868) should be used. Nonetheless, these functions contain useful information in their docstrings on the various ODE solving methods. For this reason, they are documented here.

allhints

`diofant.solvers.ode.`**`allhints = ('separable', '1st_exact', '1st_linear', 'Bernoulli', 'Ricc`**

This is a list of hints in the order that they should be preferred by *classify_ode()* (page 871). In general, hints earlier in the list should produce simpler solutions than those later in the list (for ODEs that fit both). For now, the order of this list is based on empirical observations by the developers of Diofant.

The hint used by *dsolve()* (page 868) for a specific ODE can be overridden (see the docstring).

In general, `_Integral` hints are grouped at the end of the list, unless there is a method that returns an unevaluable integral most of the time (which go near the end of the list anyway). `default`, `all`, `best`, and `all_Integral` meta-hints should not be included in this list, but `_best` and `_Integral` hints should be included.

odesimp

`diofant.solvers.ode.`**`odesimp`**(*eq*, *func*, *order*, *constants*, *hint*)

Simplifies ODEs, including trying to solve for `func` and running *constantsimp()* (page 878).

It may use knowledge of the type of solution that the hint returns to apply additional simplifications.

It also attempts to integrate any *Integral* (page 529)s in the expression, if the hint is not an `_Integral` hint.

This function should have no effect on expressions returned by *dsolve()* (page 868), as *dsolve()* (page 868) already calls *odesimp()* (page 876), but the individual hint functions do not call *odesimp()* (page 876) (because the *dsolve()* (page 868) wrapper does). Therefore, this function is designed for mainly internal use.

Examples

```
>>> x, u2, C1= symbols('x u2 C1')
>>> f = Function('f')
```

```
>>> eq = dsolve(x*f(x).diff(x) - f(x) - x*sin(f(x)/x), f(x),
... hint='1st_homogeneous_coeff_subs_indep_div_dep_Integral',
... simplify=False)
>>> pprint(eq, wrap_line=False, use_unicode=False)
                        x
                       ----
                       f(x)
                         /
                        |
                        |   /        1   \
                        |  -|u2 + -------|
                        |   |        /1 \|
                        |   |     sin|--||
                        |   \        \u2//
log(f(x)) = log(C1) +   |  ---------------- d(u2)
                        |          2
                        |        u2
                        |
                       /
```

```
>>> pprint(odesimp(eq, f(x), 1, {C1},
...                hint='1st_homogeneous_coeff_subs_indep_div_dep'),
...        use_unicode=False)
f(x) = 2*x*atan(C1*x)
```

constant_renumber

`diofant.solvers.ode.`**`constant_renumber`**(*expr, symbolname, startnumber, endnumber*)

Renumber arbitrary constants in `expr` to have numbers 1 through N where N is `endnumber - startnumber + 1` at most. In the process, this reorders expression terms in a standard way.

This is a simple function that goes through and renumbers any *`Symbol`* (page 79) with a name in the form `symbolname + num` where `num` is in the range from `startnumber` to `endnumber`.

Symbols are renumbered based on `.sort_key()`, so they should be numbered roughly in the order that they appear in the final, printed expression. Note that this ordering is based in part on hashes, so it can produce different results on different machines.

The structure of this function is very similar to that of *`constantsimp()`* (page 878).

Examples

```
>>> x, C0, C1, C2, C3, C4 = symbols('x C:5')
```

Only constants in the given range (inclusive) are renumbered; the renumbering always starts from 1:

```
>>> constant_renumber(C1 + C3 + C4, 'C', 1, 3)
C1 + C2 + C4
>>> constant_renumber(C0 + C1 + C3 + C4, 'C', 2, 4)
C0 + 2*C1 + C2
```

(continues on next page)

(continued from previous page)

```
>>> constant_renumber(C0 + 2*C1 + C2, 'C', 0, 1)
C1 + 3*C2
>>> pprint(C2 + C1*x + C3*x**2, use_unicode=False)
                2
C1*x + C2 + C3*x
>>> pprint(constant_renumber(C2 + C1*x + C3*x**2, 'C', 1, 3), use_unicode=False)
                2
C1 + C2*x + C3*x
```

constantsimp

diofant.solvers.ode.**constantsimp**(*expr, constants*)

Simplifies an expression with arbitrary constants in it.

This function is written specifically to work with *dsolve()* (page 868), and is not intended for general use.

Simplification is done by "absorbing" the arbitrary constants into other arbitrary constants, numbers, and symbols that they are not independent of.

The symbols must all have the same name with numbers after it, for example, `C1`, `C2`, `C3`. The `symbolname` here would be 'C', the `startnumber` would be 1, and the `endnumber` would be 3. If the arbitrary constants are independent of the variable `x`, then the independent symbol would be `x`. There is no need to specify the dependent function, such as `f(x)`, because it already has the independent symbol, `x`, in it.

Because terms are "absorbed" into arbitrary constants and because constants are renumbered after simplifying, the arbitrary constants in expr are not necessarily equal to the ones of the same name in the returned result.

If two or more arbitrary constants are added, multiplied, or raised to the power of each other, they are first absorbed together into a single arbitrary constant. Then the new constant is combined into other terms if necessary.

Absorption of constants is done with limited assistance:

1. terms of *Add* (page 106)s are collected to try join constants so $e^x(C_1 \cos(x) + C_2 \cos(x))$ will simplify to $e^x C_1 \cos(x)$;
2. powers with exponents that are *Add* (page 106)s are expanded so e^{C_1+x} will be simplified to $C_1 e^x$.

Use *constant_renumber()* (page 877) to renumber constants after simplification or else arbitrary numbers on constants may appear, e.g. $C_1 + C_3 x$.

In rare cases, a single constant can be "simplified" into two constants. Every differential equation solution should have as many arbitrary constants as the order of the differential equation. The result here will be technically correct, but it may, for example, have C_1 and C_2 in an expression, when C_1 is actually equal to C_2. Use your discretion in such situations, and also take advantage of the ability to use hints in *dsolve()* (page 868).

Examples

```
>>> C1, C2, C3, x, y = symbols('C1, C2, C3, x, y')
>>> constantsimp(2*C1*x, {C1, C2, C3})
C1*x
>>> constantsimp(C1 + 2 + x, {C1, C2, C3})
C1 + x
>>> constantsimp(C1*C2 + 2 + C2 + C3*x, {C1, C2, C3})
C1 + C3*x
```

sol_simplicity

diofant.solvers.ode.**ode_sol_simplicity**(*sol*, *func*, *trysolving=True*)

Returns an extended integer representing how simple a solution to an ODE is.

The following things are considered, in order from most simple to least:

- `sol` is solved for `func`.
- `sol` is not solved for `func`, but can be if passed to solve (e.g., a solution returned by `dsolve(ode, func, simplify=False)`.
- If `sol` is not solved for `func`, then base the result on the length of `sol`, as computed by `len(str(sol))`.
- If `sol` has any unevaluated *Integral* (page 529)s, this will automatically be considered less simple than any of the above.

This function returns an integer such that if solution A is simpler than solution B by above metric, then `ode_sol_simplicity(sola, func) < ode_sol_simplicity(solb, func)`.

Currently, the following are the numbers returned, but if the heuristic is ever improved, this may change. Only the ordering is guaranteed.

Simplicity	Return
`sol` solved for `func`	`-2`
`sol` not solved for `func` but can be	`-1`
`sol` is not solved nor solvable for `func`	`len(str(sol))`
`sol` contains an *Integral* (page 529)	`oo`

`oo` here means the Diofant infinity, which should compare greater than any integer.

If you already know *solve()* (page 839) cannot solve `sol`, you can use `trysolving=False` to skip that step, which is the only potentially slow step. For example, *dsolve()* (page 868) with the `simplify=False` flag should do this.

If `sol` is a list of solutions, if the worst solution in the list returns `oo` it returns that, otherwise it returns `len(str(sol))`, that is, the length of the string representation of the whole list.

Examples

This function is designed to be passed to `min` as the key argument, such as `min(listofsolutions, key=lambda i: ode_sol_simplicity(i, f(x)))`.

```
>>> x, C1, C2 = symbols('x, C1, C2')
>>> f = Function('f')
```

```
>>> ode_sol_simplicity(Eq(f(x), C1*x**2), f(x))
-2
>>> ode_sol_simplicity(Eq(x**2 + f(x), C1), f(x))
-1
>>> ode_sol_simplicity(Eq(f(x), C1*Integral(2*x, x)), f(x))
oo
>>> eq1 = Eq(f(x)/tan(f(x)/(2*x)), C1)
>>> eq2 = Eq(f(x)/tan(f(x)/(2*x) + f(x)), C2)
>>> [ode_sol_simplicity(eq, f(x)) for eq in [eq1, eq2]]
[28, 35]
>>> min([eq1, eq2], key=lambda i: ode_sol_simplicity(i, f(x)))
Eq(f(x)/tan(f(x)/(2*x)), C1)
```

1st_exact

diofant.solvers.ode.**ode_1st_exact**(*eq, func, order, match*)

Solves 1st order exact ordinary differential equations.

A 1st order differential equation is called exact if it is the total differential of a function. That is, the differential equation

$$P(x,y)\,\partial x + Q(x,y)\,\partial y = 0$$

is exact if there is some function $F(x,y)$ such that $P(x,y) = \partial F/\partial x$ and $Q(x,y) = \partial F/\partial y$. It can be shown that a necessary and sufficient condition for a first order ODE to be exact is that $\partial P/\partial y = \partial Q/\partial x$. Then, the solution will be as given below:

```
>>> x, y, t, x0, y0, C1= symbols('x y t x0 y0 C1')
>>> P, Q, F= map(Function, ['P', 'Q', 'F'])
>>> pprint(Eq(Eq(F(x, y), Integral(P(t, y), (t, x0, x)) +
... Integral(Q(x0, t), (t, y0, y))), C1), use_unicode=False)
            x                y
            /                /
           |                |
F(x, y) =  |  P(t, y) dt +  |  Q(x0, t) dt = C1
           |                |
          /                /
          x0               y0
```

Where the first partials of P and Q exist and are continuous in a simply connected region.

A note: Diofant currently has no way to represent inert substitution on an expression, so the hint `1st_exact_Integral` will return an integral with dy. This is supposed to represent the function that you are solving for.

References

[R521] (page 1267), [R522] (page 1267)

Examples

```
>>> f = Function('f')
>>> dsolve(cos(f(x)) - (x*sin(f(x)) - f(x)**2)*f(x).diff(x),
...        f(x), hint='1st_exact')
Eq(x*cos(f(x)) + f(x)**3/3, C1)
```

1st_homogeneous_coeff_best

diofant.solvers.ode.**ode_1st_homogeneous_coeff_best**(*eq, func, order, match*)

Returns the best solution to an ODE from the two hints `1st_homogeneous_coeff_subs_dep_div_indep` and `1st_homogeneous_coeff_subs_indep_div_dep`.

This is as determined by *ode_sol_simplicity()* (page 879).

See the *ode_1st_homogeneous_coeff_subs_indep_div_dep()* (page 883) and *ode_1st_homogeneous_coeff_subs_dep_div_indep()* (page 881) docstrings for more information on these hints. Note that there is no `ode_1st_homogeneous_coeff_best_Integral` hint.

References

[R523] (page 1267), [R524] (page 1267)

Examples

```
>>> f = Function('f')
>>> pprint(dsolve(2*x*f(x) + (x**2 + f(x)**2)*f(x).diff(x), f(x),
...               hint='1st_homogeneous_coeff_best', simplify=False),
...        use_unicode=False)
                         /    2    \
                         | 3*x     |
                      log|----- + 1|
                         | 2       |
                         \f (x)    /
log(f(x)) = log(C1) - --------------
                            3
```

1st_homogeneous_coeff_subs_dep_div_indep

diofant.solvers.ode.**ode_1st_homogeneous_coeff_subs_dep_div_indep**(*eq, func, order, match*)

Solves a 1st order differential equation with homogeneous coefficients using the substitution $u_1 = \frac{\text{<dependent variable>}}{\text{<independent variable>}}$.

This is a differential equation

$$P(x,y) + Q(x,y)dy/dx = 0$$

such that P and Q are homogeneous and of the same order. A function $F(x, y)$ is homogeneous of order n if $F(xt, yt) = t^n F(x, y)$. Equivalently, $F(x, y)$ can be rewritten as $G(y/x)$ or $H(x/y)$. See also the docstring of *homogeneous_order()* (page 874).

If the coefficients P and Q in the differential equation above are homogeneous functions of the same order, then it can be shown that the substitution $y = u_1 x$ (i.e. $u_1 = y/x$) will turn the differential equation into an equation separable in the variables x and u. If $h(u_1)$ is the function that results from making the substitution $u_1 = f(x)/x$ on $P(x, f(x))$ and $g(u_2)$ is the function that results from the substitution on $Q(x, f(x))$ in the differential equation $P(x, f(x)) + Q(x, f(x))f'(x) = 0$, then the general solution is:

```
>>> f, g, h = map(Function, ['f', 'g', 'h'])
>>> genform = g(f(x)/x) + h(f(x)/x)*f(x).diff(x)
>>> pprint(genform, use_unicode=False)
 /f(x)\    /f(x)\ d
g|----| + h|----|*--(f(x))
 \ x  /    \ x  / dx
>>> pprint(dsolve(genform, f(x),
...               hint='1st_homogeneous_coeff_subs_dep_div_indep_Integral'),
...        use_unicode=False)
               f(x)
               ----
                x
                 /
                |
                |       -h(u1)
log(x) = C1 +   |  ---------------- d(u1)
                |  u1*h(u1) + g(u1)
                |
               /
```

Where $u_1 h(u_1) + g(u_1) \ne 0$ and $x \ne 0$.

See also:

diofant.solvers.ode.ode_1st_homogeneous_coeff_best (page 881), *diofant.solvers.ode.ode_1st_homogeneous_coeff_subs_indep_div_dep* (page 883)

References

[R525] (page 1267), [R526] (page 1267)

Examples

```
>>> f = Function('f')
>>> pprint(dsolve(2*x*f(x) + (x**2 + f(x)**2)*f(x).diff(x), f(x),
...               hint='1st_homogeneous_coeff_subs_dep_div_indep',
...               simplify=False), use_unicode=False)
                      /          3   \
                      |3*f(x)   f (x)|
                   log|------ + -----|
                      |  x         3 |
                      \           x  /
log(x) = log(C1) - -------------------
                            3
```

1st_homogeneous_coeff_subs_indep_div_dep

diofant.solvers.ode.**ode_1st_homogeneous_coeff_subs_indep_div_dep**(*eq*, *func*, *order*, *match*)

Solves a 1st order differential equation with homogeneous coefficients using the substitution $u_2 = \frac{\text{<independent variable>}}{\text{<dependent variable>}}$.

This is a differential equation

$$P(x, y) + Q(x, y)dy/dx = 0$$

such that P and Q are homogeneous and of the same order. A function $F(x, y)$ is homogeneous of order n if $F(xt, yt) = t^n F(x, y)$. Equivalently, $F(x, y)$ can be rewritten as $G(y/x)$ or $H(x/y)$. See also the docstring of *homogeneous_order()* (page 874).

If the coefficients P and Q in the differential equation above are homogeneous functions of the same order, then it can be shown that the substitution $x = u_2 y$ (i.e. $u_2 = x/y$) will turn the differential equation into an equation separable in the variables y and u_2. If $h(u_2)$ is the function that results from making the substitution $u_2 = x/f(x)$ on $P(x, f(x))$ and $g(u_2)$ is the function that results from the substitution on $Q(x, f(x))$ in the differential equation $P(x, f(x)) + Q(x, f(x))f'(x) = 0$, then the general solution is:

```
>>> f, g, h = map(Function, ['f', 'g', 'h'])
>>> genform = g(x/f(x)) + h(x/f(x))*f(x).diff(x)
>>> pprint(genform, use_unicode=False)
 / x  \    / x  \ d
g|----| + h|----|*--(f(x))
 \f(x)/    \f(x)/ dx
>>> pprint(dsolve(genform, f(x),
...              hint='1st_homogeneous_coeff_subs_indep_div_dep_Integral'),
...        use_unicode=False)
          x
         ----
         f(x)
           /
          |
          |       -g(u2)
          |  ---------------- d(u2)
          |  u2*g(u2) + h(u2)
          |
         /

f(x) = E                           *C1
```

Where $u_2 g(u_2) + h(u_2) \neq 0$ and $f(x) \neq 0$.

See also:

diofant.solvers.ode.ode_1st_homogeneous_coeff_best (page 881), *diofant.solvers.ode.ode_1st_homogeneous_coeff_subs_dep_div_indep* (page 881)

References

[R527] (page 1267), [R528] (page 1267)

Examples

```
>>> f = Function('f')
>>> pprint(dsolve(2*x*f(x) + (x**2 + f(x)**2)*f(x).diff(x), f(x),
...                 hint='1st_homogeneous_coeff_subs_indep_div_dep',
...                 simplify=False), use_unicode=False)
                         /    2     \
                         | 3*x      |
                      log|----- + 1|
                         | 2        |
                         \f (x)     /
log(f(x)) = log(C1) - --------------
                            3
```

1st_linear

diofant.solvers.ode.**ode_1st_linear**(*eq, func, order, match*)

Solves 1st order linear differential equations.

These are differential equations of the form

$$dy/dx + P(x)y = Q(x).$$

These kinds of differential equations can be solved in a general way. The integrating factor $e^{\int P(x)\,dx}$ will turn the equation into a separable equation. The general solution is:

```
>>> f, P, Q = map(Function, ['f', 'P', 'Q'])
>>> genform = Eq(f(x).diff(x) + P(x)*f(x), Q(x))
>>> pprint(genform, use_unicode=False)
            d
P(x)*f(x) + --(f(x)) = Q(x)
            dx
>>> pprint(dsolve(genform, f(x), hint='1st_linear_Integral'), use_unicode=False)
                    /       /                   \
                    |       |                   |
          /         |       |    /              |
          |         |       |    |              |
        - | P(x) dx |       |    | P(x) dx      |
          |         |       |    |              |
         /          |       |   /               |
f(x) = E           *|C1 +   | E          *Q(x) dx|
                    |       |                   |
                    \      /                    /
```

References

[R529] (page 1267), [R530] (page 1267)

Examples

```
>>> f = Function('f')
>>> pprint(dsolve(Eq(x*diff(f(x), x) - f(x), x**2*sin(x)),
...                  f(x), '1st_linear'), use_unicode=False)
f(x) = x*(C1 - cos(x))
```

Bernoulli

diofant.solvers.ode.**ode_Bernoulli**(*eq, func, order, match*)

Solves Bernoulli differential equations.

These are equations of the form

$$dy/dx + P(x)y = Q(x)y^n, \, n \neq 1`.$$

The substitution $w = 1/y^{1-n}$ will transform an equation of this form into one that is linear (see the docstring of *ode_1st_linear()* (page 884)). The general solution is:

```
>>> f, P, Q = map(Function, ['f', 'P', 'Q'])
>>> genform = Eq(f(x).diff(x) + P(x)*f(x), Q(x)*f(x)**n)
>>> pprint(genform, use_unicode=False)
            d            n
P(x)*f(x) + --(f(x)) = Q(x)*f (x)
            dx
>>> pprint(dsolve(genform, f(x), hint='Bernoulli_Integral'),
...        use_unicode=False, wrap_line=False)
                                                                                1
                                                                              ----
 ↪--
                                                                              -n 
 ↪+ 1
       /                      /               /                             \\
       |                      |              |                              ||
       |             /        |              |              /               ||
       |            |         |              |             |                ||
       | -(-n + 1)* | P(x) dx |              |   (-n + 1)* | P(x) dx        ||
       |            |         |              |             |                ||
       |           /          |              |            /                 ||
f(x) = |E                    *|C1 + (n - 1)* | -E                  *Q(x) dx||
       |                      |              |                              ||
       \                      \             /                               //
```

Note that the equation is separable when $n = 1$ (see the docstring of *ode_separable()* (page 891)).

```
>>> pprint(dsolve(Eq(f(x).diff(x) + P(x)*f(x), Q(x)*f(x)), f(x),
...                  hint='separable_Integral'), use_unicode=False)
 f(x)
   /
  |                /
  |  1            |
  |  - dy = C1 +  | (-P(x) + Q(x)) dx
  |  y            |
  |              /
 /
```

References

[R531] (page 1267), [R532] (page 1267)

Examples

```
>>> f = Function('f')
```

```
>>> pprint(dsolve(Eq(x*f(x).diff(x) + f(x), log(x)*f(x)**2),
...               f(x), hint='Bernoulli'), use_unicode=False)
                1
f(x) = -------------------
         /     log(x)   1\
       x*|C1 + ------ + -|
         \       x      x/
```

Liouville

diofant.solvers.ode.**ode_Liouville**(*eq, func, order, match*)

Solves 2nd order Liouville differential equations.

The general form of a Liouville ODE is

$$\frac{d^2y}{dx^2} + g(y)\left(\frac{dy}{dx}\right)^2 + h(x)\frac{dy}{dx}.$$

The general solution is:

```
>>> f, g, h = map(Function, ['f', 'g', 'h'])
>>> genform = Eq(diff(f(x), x, x) + g(f(x))*diff(f(x), x)**2 +
...              h(x)*diff(f(x), x), 0)
>>> pprint(genform, use_unicode=False)
                  2                    2
        /d       \         d          d
g(f(x))*|--(f(x))|  + h(x)*--(f(x)) + ---(f(x)) = 0
        \dx      /         dx           2
                                      dx
>>> pprint(dsolve(genform, f(x), hint='Liouville_Integral'), use_unicode=False)
                                  f(x)
          /                     /
         |                     |
         |     /               |     /
         |    |                |    |
         |  - | h(x) dx        |    | g(y) dy
         |    |                |    |
         |   /                 |   /
C1 + C2* | E            dx +   |  E           dy = 0
         |                     |
        /                     /
```

References

[R533] (page 1267), [R534] (page 1267)

Examples

```
>>> f = Function('f')
>>> pprint(dsolve(diff(f(x), x, x) + diff(f(x), x)**2/f(x) +
...                 diff(f(x), x)/x, f(x), hint='Liouville'),
...         use_unicode=False)
            ________________           ________________
[f(x) = -\/ C1 + C2*log(x) , f(x) = \/ C1 + C2*log(x) ]
```

Riccati_special_minus2

diofant.solvers.ode.**ode_Riccati_special_minus2**(*eq, func, order, match*)

The general Riccati equation has the form

$$dy/dx = f(x)y^2 + g(x)y + h(x).$$

While it does not have a general solution [1], the "special" form, $dy/dx = ay^2 - bx^c$, does have solutions in many cases [2]. This routine returns a solution for $a(dy/dx) = by^2 + cy/x + d/x^2$ that is obtained by using a suitable change of variables to reduce it to the special form and is valid when neither a nor b are zero and either c or d is zero.

```
>>> from diofant.abc import a, b, c, d
>>> f = Function('f')
>>> y = f(x)
>>> genform = a*y.diff(x) - (b*y**2 + c*y/x + d/x**2)
>>> sol = dsolve(genform, y)
>>> pprint(sol, wrap_line=False, use_unicode=False)
        /                                  /        __________________       \\
        |           __________________     |       /                2        ||
        |          /                2      |     \/  4*b*d - (a + c)   *log(x)||
       -|a + c - \/  4*b*d - (a + c)   *tan|C1 + ----------------------------||
        \                                  \                 2*a             //
f(x) = ------------------------------------------------------------------------
                                        2*b*x
```

```
>>> checkodesol(genform, sol, order=1)[0]
True
```

References

1. https//www.maplesoft.com/support/help/Maple/view.aspx?path=odeadvisor/Riccati
2. http://eqworld.ipmnet.ru/en/solutions/ode/ode0106.pdf - http://eqworld.ipmnet.ru/en/solutions/ode/ode0123.pdf

nth_linear_constant_coeff_homogeneous

diofant.solvers.ode.**ode_nth_linear_constant_coeff_homogeneous**(*eq, func, order, match, returns='sol'*)

Solves an nth order linear homogeneous differential equation with constant coefficients.

This is an equation of the form

$$a_n f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_1 f'(x) + a_0 f(x) = 0.$$

These equations can be solved in a general manner, by taking the roots of the characteristic equation $a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$. The solution will then be the sum of $C_n x^i e^{rx}$ terms, for each where C_n is an arbitrary constant, r is a root of the characteristic equation and i is one of each from 0 to the multiplicity of the root - 1 (for example, a root 3 of multiplicity 2 would create the terms $C_1 e^{3x} + C_2 x e^{3x}$). The exponential is usually expanded for complex roots using Euler's equation $e^{Ix} = \cos(x) + I\sin(x)$. Complex roots always come in conjugate pairs in polynomials with real coefficients, so the two roots will be represented (after simplifying the constants) as $e^{ax}(C_1 \cos(bx) + C_2 \sin(bx))$.

If Diofant cannot find exact roots to the characteristic equation, a *RootOf* (page 711) instance will be return instead.

```
>>> f = Function('f')
>>> dsolve(f(x).diff(x, 5) + 10*f(x).diff(x) - 2*f(x), f(x),
... hint='nth_linear_constant_coeff_homogeneous')
...
Eq(f(x), E**(x*RootOf(_x**5 + 10*_x - 2, 0))*C1 +
E**(x*RootOf(_x**5 + 10*_x - 2, 1))*C2 +
E**(x*RootOf(_x**5 + 10*_x - 2, 2))*C3 +
E**(x*RootOf(_x**5 + 10*_x - 2, 3))*C4 +
E**(x*RootOf(_x**5 + 10*_x - 2, 4))*C5)
```

Note that because this method does not involve integration, there is no `nth_linear_constant_coeff_homogeneous_Integral` hint.

The following is for internal use:

- `returns = 'sol'` returns the solution to the ODE.
- `returns = 'list'` returns a list of linearly independent solutions, for use with non homogeneous solution methods like variation of parameters and undetermined coefficients. Note that, though the solutions should be linearly independent, this function does not explicitly check that. You can do `assert simplify(wronskian(sollist)) != 0` to check for linear independence. Also, `assert len(sollist) == order` will need to pass.
- `returns = 'both'`, return a dictionary `{'sol': <solution to ODE>, 'list': <list of linearly independent solutions>}`.

References

[R535] (page 1267), [R536] (page 1267)

Examples

```
>>> f = Function('f')
>>> pprint(dsolve(f(x).diff(x, 4) + 2*f(x).diff(x, 3) -
...               2*f(x).diff(x, 2) - 6*f(x).diff(x) + 5*f(x), f(x),
...               hint='nth_linear_constant_coeff_homogeneous'),
...        use_unicode=False)
        x               -2*x
f(x) = E *(C3 + C4*x) + E    *(C1*sin(x) + C2*cos(x))
```

nth_linear_constant_coeff_undetermined_coefficients

diofant.solvers.ode.**ode_nth_linear_constant_coeff_undetermined_coefficients**(*eq, func, order, match*)

Solves an nth order linear differential equation with constant coefficients using the method of undetermined coefficients.

This method works on differential equations of the form

$$a_n f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_1 f'(x) + a_0 f(x) = P(x),$$

where $P(x)$ is a function that has a finite number of linearly independent derivatives.

Functions that fit this requirement are finite sums functions of the form $ax^i e^{bx} \sin(cx + d)$ or $ax^i e^{bx} \cos(cx + d)$, where i is a non-negative integer and a, b, c, and d are constants. For example any polynomial in x, functions like $x^2 e^{2x}$, $x \sin(x)$, and $e^x \cos(x)$ can all be used. Products of sin's and cos's have a finite number of derivatives, because they can be expanded into $\sin(ax)$ and $\cos(bx)$ terms. However, Diofant currently cannot do that expansion, so you will need to manually rewrite the expression in terms of the above to use this method. So, for example, you will need to manually convert $\sin^2(x)$ into $(1 + \cos(2x))/2$ to properly apply the method of undetermined coefficients on it.

This method works by creating a trial function from the expression and all of its linear independent derivatives and substituting them into the original ODE. The coefficients for each term will be a system of linear equations, which are be solved for and substituted, giving the solution. If any of the trial functions are linearly dependent on the solution to the homogeneous equation, they are multiplied by sufficient x to make them linearly independent.

References

[R537] (page 1267), [R538] (page 1267)

Examples

```
>>> f = Function('f')
>>> pprint(dsolve(f(x).diff(x, 2) + 2*f(x).diff(x) + f(x) -
...              4*exp(-x)*x**2 + cos(2*x), f(x),
...              hint='nth_linear_constant_coeff_undetermined_coefficients'),
...        use_unicode=False)
                                       /             4\
         4*sin(2*x)   3*cos(2*x)    -x |            x |
f(x) = - ---------- + ---------- + E  *|C1 + C2*x + --|
             25           25           \            3 /
```

nth_linear_constant_coeff_variation_of_parameters

diofant.solvers.ode.**ode_nth_linear_constant_coeff_variation_of_parameters**(*eq, func, order, match*)

Solves an nth order linear differential equation with constant coefficients using the method of variation of parameters.

This method works on any differential equations of the form

$$f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_1 f'(x) + a_0 f(x) = P(x).$$

This method works by assuming that the particular solution takes the form

$$\sum_{x=1}^{n} c_i(x) y_i(x),$$

where y_i is the ith solution to the homogeneous equation. The solution is then solved using Wronskian's and Cramer's Rule. The particular solution is given by

$$\sum_{x=1}^{n} \left(\int \frac{W_i(x)}{W(x)} \, dx \right) y_i(x),$$

where $W(x)$ is the Wronskian of the fundamental system (the system of n linearly independent solutions to the homogeneous equation), and $W_i(x)$ is the Wronskian of the fundamental system with the ith column replaced with $[0, 0, \cdots, 0, P(x)]$.

This method is general enough to solve any nth order inhomogeneous linear differential equation with constant coefficients, but sometimes Diofant cannot simplify the Wronskian well enough to integrate it. If this method hangs, try using the `nth_linear_constant_coeff_variation_of_parameters_Integral` hint and simplifying the integrals manually. Also, prefer using `nth_linear_constant_coeff_undetermined_coefficients` when it applies, because it doesn't use integration, making it faster and more reliable.

Warning, using simplify=False with 'nth_linear_constant_coeff_variation_of_parameters' in *dsolve()* (page 868) may cause it to hang, because it will not attempt to simplify the Wronskian before integrating. It is recommended that you only use simplify=False with 'nth_linear_constant_coeff_variation_of_parameters_Integral' for this method, especially if the solution to the homogeneous equation has trigonometric functions in it.

References

[R539] (page 1267), [R540] (page 1267)

Examples

```
>>> f = Function('f')
>>> pprint(dsolve(f(x).diff(x, 3) - 3*f(x).diff(x, 2) +
...        3*f(x).diff(x) - f(x) - exp(x)*log(x), f(x),
...        hint='nth_linear_constant_coeff_variation_of_parameters'),
```

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(continued from previous page)

```
...         use_unicode=False)
          /                   3                 \
        x |                2   x *(6*log(x) - 11)|
f(x) = E *|C1 + C2*x + C3*x  + ------------------|
          \                           36        /
```

separable

diofant.solvers.ode.**ode_separable**(*eq, func, order, match*)

Solves separable 1st order differential equations.

This is any differential equation that can be written as $P(y)\frac{dy}{dx} = Q(x)$. The solution can then just be found by rearranging terms and integrating: $\int P(y)\,dy = \int Q(x)\,dx$. This hint uses *diofant.simplify.simplify.separatevars()* (page 778) as its back end, so if a separable equation is not caught by this solver, it is most likely the fault of that function. *separatevars()* (page 778) is smart enough to do most expansion and factoring necessary to convert a separable equation $F(x, y)$ into the proper form $P(x) \cdot Q(y)$. The general solution is:

```
>>> a, b, c, d, f = map(Function, ['a', 'b', 'c', 'd', 'f'])
>>> genform = Eq(a(x)*b(f(x))*f(x).diff(x), c(x)*d(f(x)))
>>> pprint(genform, use_unicode=False)
             d
a(x)*b(f(x))*--(f(x)) = c(x)*d(f(x))
             dx
>>> pprint(dsolve(genform, f(x),
...                  hint='separable_Integral'), use_unicode=False)
     f(x)
   /                  /
  |                  |
  |  b(y)            | c(x)
  |  ---- dy = C1 +  | ---- dx
  |  d(y)            | a(x)
  |                  |
 /                  /
```

References

[R541] (page 1267)

Examples

```
>>> f = Function('f')
>>> pprint(dsolve(Eq(f(x)*f(x).diff(x) + x, 3*x*f(x)**2), f(x),
...                  hint='separable', simplify=False), use_unicode=False)
   /   2       \         2
log\3*f (x) - 1/        x
---------------- = C1 + --
       6                2
```

almost_linear

diofant.solvers.ode.**ode_almost_linear**(*eq*, *func*, *order*, *match*)
Solves an almost-linear differential equation.

The general form of an almost linear differential equation is

$$f(x)g(y)y + k(x)l(y) + m(x) = 0\text{where}l'(y) = g(y).$$

This can be solved by substituting $l(y) = u(y)$. Making the given substitution reduces it to a linear differential equation of the form $u' + P(x)u + Q(x) = 0$.

The general solution is

```
>>> f, g, k, l = map(Function, ['f', 'g', 'k', 'l'])
>>> genform = Eq(f(x)*(l(y).diff(y)) + k(x)*l(y) + g(x))
>>> pprint(genform, use_unicode=False)
     d
f(x)*--(l(y)) + g(x) + k(x)*l(y) = 0
     dy
>>> pprint(dsolve(genform, hint = 'almost_linear'), use_unicode=False)
                 /     //   -y*g(x)                  \\
                 |     ||   --------     for k(x) = 0||
        -y*k(x)  |     ||     f(x)                   ||
        -------- |     ||                            ||
          f(x)   |     ||  y*k(x)                    ||
l(y) = E        *|C1 + |<  ------                    ||
                 |     ||   f(x)                     ||
                 |     ||-E      *g(x)               ||
                 |     ||--------------   otherwise  ||
                 |     ||     k(x)                   ||
                 \     \\                            //
```

See also:

diofant.solvers.ode.ode_1st_linear() (page 884)

References

- Joel Moses, "Symbolic Integration - The Stormy Decade", Communications of the ACM, Volume 14, Number 8, August 1971, pp. 558

Examples

```
>>> f = Function('f')
>>> d = f(x).diff(x)
>>> eq = x*d + x*f(x) + 1
>>> dsolve(eq, f(x), hint='almost_linear')
Eq(f(x), E**(-x)*(C1 - Ei(x)))
>>> pprint(dsolve(eq, f(x), hint='almost_linear'), use_unicode=False)
        -x
f(x) = E  *(C1 - Ei(x))
```

linear_coefficients

diofant.solvers.ode.**ode_linear_coefficients**(*eq*, *func*, *order*, *match*)

Solves a differential equation with linear coefficients.

The general form of a differential equation with linear coefficients is

$$y' + F\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right) = 0,$$

where a_1, b_1, c_1, a_2, b_2, c_2 are constants and $a_1 b_2 - a_2 b_1 \neq 0$.

This can be solved by substituting:

$$x = x' + \frac{b_2 c_1 - b_1 c_2}{a_2 b_1 - a_1 b_2}$$
$$y = y' + \frac{a_1 c_2 - a_2 c_1}{a_2 b_1 - a_1 b_2}.$$

This substitution reduces the equation to a homogeneous differential equation.

See also:

diofant.solvers.ode.ode_1st_homogeneous_coeff_best() (page 881), *diofant.solvers.ode.ode_1st_homogeneous_coeff_subs_indep_div_dep()* (page 883), *diofant.solvers.ode.ode_1st_homogeneous_coeff_subs_dep_div_indep()* (page 881)

References

- Joel Moses, "Symbolic Integration - The Stormy Decade", Communications of the ACM, Volume 14, Number 8, August 1971, pp. 558

Examples

```
>>> f = Function('f')
>>> df = f(x).diff(x)
>>> eq = (x + f(x) + 1)*df + (f(x) - 6*x + 1)
>>> dsolve(eq, hint='linear_coefficients')
[Eq(f(x), -x - sqrt(C1 + 7*x**2) - 1), Eq(f(x), -x + sqrt(C1 + 7*x**2) - 1)]
>>> pprint(dsolve(eq, hint='linear_coefficients'), use_unicode=False)
                  ___________                     ___________
                 /         2                     /         2
[f(x) = -x - \/  C1 + 7*x    - 1, f(x) = -x + \/  C1 + 7*x    - 1]
```

separable_reduced

diofant.solvers.ode.**ode_separable_reduced**(*eq*, *func*, *order*, *match*)

Solves a differential equation that can be reduced to the separable form.

The general form of this equation is

$$y' + (y/x)H(x^n y) = 0.$$

This can be solved by substituting $u(y) = x^n y$. The equation then reduces to the separable form $\frac{u'}{u(\mathrm{power} - H(u))} - \frac{1}{x} = 0$.

The general solution is:

```
>>> f, g = map(Function, ['f', 'g'])
>>> genform = f(x).diff(x) + (f(x)/x)*g(x**n*f(x))
>>> pprint(genform, use_unicode=False)
                 / n     \
d          f(x)*g\x *f(x)/
--(f(x)) + ---------------
dx                x
>>> pprint(dsolve(genform, hint='separable_reduced'), use_unicode=False)
 n
x *f(x)
  /
 |
 |         1
 |    ------------ dy = C1 + log(x)
 |    y*(n - g(y))
 |
 /
```

See also:

diofant.solvers.ode.ode_separable() (page 891)

References

- Joel Moses, "Symbolic Integration - The Stormy Decade", Communications of the ACM, Volume 14, Number 8, August 1971, pp. 558

Examples

```
>>> f = Function('f')
>>> d = f(x).diff(x)
>>> eq = (x - x**2*f(x))*d - f(x)
>>> dsolve(eq, hint='separable_reduced')
[Eq(f(x), (-sqrt(C1*x**2 + 1) + 1)/x), Eq(f(x), (sqrt(C1*x**2 + 1) + 1)/x)]
>>> pprint(dsolve(eq, hint='separable_reduced'), use_unicode=False)
               ___________                ___________
              /     2                    /     2
        - \/   C1*x  + 1  + 1          \/   C1*x  + 1  + 1
[f(x) = --------------------, f(x) = ------------------]
                 x                            x
```

lie_group

diofant.solvers.ode.**ode_lie_group**(*eq*, *func*, *order*, *match*)

This hint implements the Lie group method of solving first order differential equations. The aim is to convert the given differential equation from the given coordinate given system into another coordinate system where it becomes invariant under the one-parameter Lie group of translations. The converted ODE is quadrature and can be solved easily. It

makes use of the `diofant.solvers.ode.infinitesimals()` (page 874) function which returns the infinitesimals of the transformation.

The coordinates r and s can be found by solving the following Partial Differential Equations.

$$\xi\frac{\partial r}{\partial x} + \eta\frac{\partial r}{\partial y} = 0$$

$$\xi\frac{\partial s}{\partial x} + \eta\frac{\partial s}{\partial y} = 1$$

The differential equation becomes separable in the new coordinate system

$$\frac{ds}{dr} = \frac{\frac{\partial s}{\partial x} + h(x,y)\frac{\partial s}{\partial y}}{\frac{\partial r}{\partial x} + h(x,y)\frac{\partial r}{\partial y}}$$

After finding the solution by integration, it is then converted back to the original coordinate system by subsituting r and s in terms of x and y again.

References

[R542] (page 1267)

Examples

```
>>> f = Function('f')
>>> pprint(dsolve(f(x).diff(x) + 2*x*f(x) - x*exp(-x**2), f(x),
...               hint='lie_group'), use_unicode=False)
          2 /      2\
        -x  |     x |
f(x) = E   *|C1 + --|
            \     2 /
```

1st_power_series

`diofant.solvers.ode.ode_1st_power_series(eq, func, order, match)`

The power series solution is a method which gives the Taylor series expansion to the solution of a differential equation.

For a first order differential equation $\frac{dy}{dx} = h(x,y)$, a power series solution exists at a point $x = x_0$ if $h(x,y)$ is analytic at x_0. The solution is given by

$$y(x) = y(x_0) + \sum_{n=1}^{\infty} \frac{F_n(x_0,b)(x-x_0)^n}{n!},$$

where $y(x_0) = b$ is the value of y at the initial value of x_0. To compute the values of the $F_n(x_0,b)$ the following algorithm is followed, until the required number of terms are generated.

1. $F_1 = h(x_0,b)$
2. $F_{n+1} = \frac{\partial F_n}{\partial x} + \frac{\partial F_n}{\partial y}F_1$

References

- Travis W. Walker, Analytic power series technique for solving first-order differential equations, p.p 17, 18

Examples

```
>>> f = Function('f')
>>> eq = exp(x)*(f(x).diff(x)) - f(x)
>>> pprint(dsolve(eq, hint='1st_power_series'), use_unicode=False)
                       3       4       5
                   C1*x    C1*x    C1*x     / 6\
f(x) = C1 + C1*x - ----- + ----- + ----- + O\x /
                     6       24      60
```

2nd_power_series_ordinary

diofant.solvers.ode.**ode_2nd_power_series_ordinary**(*eq*, *func*, *order*, *match*)

Gives a power series solution to a second order homogeneous differential equation with polynomial coefficients at an ordinary point. A homogenous differential equation is of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x) = 0$$

For simplicity it is assumed that $P(x)$, $Q(x)$ and $R(x)$ are polynomials, it is sufficient that $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ exists at x_0. A recurrence relation is obtained by substituting y as $\sum_{n=0}^{\infty} a_n x^n$, in the differential equation, and equating the nth term. Using this relation various terms can be generated.

References

[R543] (page 1267), [R544] (page 1267)

Examples

```
>>> f = Function("f")
>>> eq = f(x).diff(x, 2) + f(x)
>>> pprint(dsolve(eq, hint='2nd_power_series_ordinary'), use_unicode=False)
          / 4    2    \        /   2    \
          |x    x     |        |  x     |    / 6\
f(x) = C2*|-- - -- + 1| + C1*x*|- -- + 1| + O\x /
          \24   2     /        \  6     /
```

2nd_power_series_regular

diofant.solvers.ode.**ode_2nd_power_series_regular**(*eq*, *func*, *order*, *match*)

Gives a power series solution to a second order homogeneous differential equation with

polynomial coefficients at a regular point. A second order homogenous differential equation is of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x) = 0$$

A point is said to regular singular at $x0$ if $x - x0\frac{Q(x)}{P(x)}$ and $(x - x0)^2\frac{R(x)}{P(x)}$ are analytic at $x0$. For simplicity $P(x)$, $Q(x)$ and $R(x)$ are assumed to be polynomials. The algorithm for finding the power series solutions is:

1. Try expressing $(x - x0)P(x)$ and $((x - x0)^2)Q(x)$ as power series solutions about x0. Find $p0$ and $q0$ which are the constants of the power series expansions.
2. Solve the indicial equation $f(m) = m(m - 1) + m * p0 + q0$, to obtain the roots $m1$ and $m2$ of the indicial equation.
3. If $m1 - m2$ is a non integer there exists two series solutions. If $m1 = m2$, there exists only one solution. If $m1 - m2$ is an integer, then the existence of one solution is confirmed. The other solution may or may not exist.

The power series solution is of the form $x^m\sum_{n=0}^{\infty} a_n x^n$. The coefficients are determined by the following recurrence relation. $a_n = -\frac{\sum_{k=0}^{n-1} q_{n-k} + (m+k)p_{n-k}}{f(m+n)}$. For the case in which $m1 - m2$ is an integer, it can be seen from the recurrence relation that for the lower root m, when n equals the difference of both the roots, the denominator becomes zero. So if the numerator is not equal to zero, a second series solution exists.

References

- George E. Simmons, "Differential Equations with Applications and Historical Notes", p.p 176 - 184

Examples

```
>>> f = Function("f")
>>> eq = x*(f(x).diff(x, 2)) + 2*(f(x).diff(x)) + x*f(x)
>>> pprint(dsolve(eq), use_unicode=False)
                              /    6    4    2    \
                              |   x    x    x     |
          /  4    2    \   C1*|- --- + -- - -- + 1|
          | x    x     |      \  720   24   2     /    / 6\
f(x) = C2*|--- - -- + 1| + ------------------------ + O\x /
          \120   6     /              x
```

Lie heuristics

These functions are intended for internal use of the Lie Group Solver. Nonetheless, they contain useful information in their docstrings on the algorithms implemented for the various heuristics.

abaco1_simple

`diofant.solvers.ode.`**`lie_heuristic_abaco1_simple`**(*match*, *comp=False*)

The first heuristic uses the following four sets of assumptions on ξ and η

$$\xi = 0, \eta = f(x)$$

$$\xi = 0, \eta = f(y)$$

$$\xi = f(x), \eta = 0$$

$$\xi = f(y), \eta = 0$$

The success of this heuristic is determined by algebraic factorisation. For the first assumption $\xi = 0$ and η to be a function of x, the PDE

$$\frac{\partial \eta}{\partial x} + (\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x}) * h - \frac{\partial \xi}{\partial y} * h^2 - \xi * \frac{\partial h}{\partial x} - \eta * \frac{\partial h}{\partial y} = 0$$

reduces to $f'(x) - f\frac{\partial h}{\partial y} = 0$ If $\frac{\partial h}{\partial y}$ is a function of x, then this can usually be integrated easily. A similar idea is applied to the other 3 assumptions as well.

References

- E.S Cheb-Terrab, L.G.S Duarte and L.A,C.P da Mota, Computer Algebra Solving of First Order ODEs Using Symmetry Methods, pp. 8

abaco1_product

`diofant.solvers.ode.`**`lie_heuristic_abaco1_product`**(*match*, *comp=False*)

The second heuristic uses the following two assumptions on ξ and η

$$\eta = 0, \xi = f(x) * g(y)$$

$$\eta = f(x) * g(y), \xi = 0$$

The first assumption of this heuristic holds good if $\frac{1}{h^2}\frac{\partial^2}{\partial x \partial y}\log(h)$ is separable in x and y, then the separated factors containing x is $f(x)$, and $g(y)$ is obtained by

$$e^{\int f \frac{\partial}{\partial x}\left(\frac{1}{f*h}\right) \, dy}$$

provided $f\frac{\partial}{\partial x}\left(\frac{1}{f*h}\right)$ is a function of y only.

The second assumption holds good if $\frac{dy}{dx} = h(x, y)$ is rewritten as $\frac{dy}{dx} = \frac{1}{h(y,x)}$ and the same properties of the first assumption satisifes. After obtaining $f(x)$ and $g(y)$, the coordinates are again interchanged, to get η as $f(x) * g(y)$

References

- E.S. Cheb-Terrab, A.D. Roche, Symmetries and First Order ODE Patterns, pp. 7 - pp. 8

bivariate

`diofant.solvers.ode.lie_heuristic_bivariate(match, comp=False)`

The third heuristic assumes the infinitesimals ξ and η to be bi-variate polynomials in x and y. The assumption made here for the logic below is that h is a rational function in x and y though that may not be necessary for the infinitesimals to be bivariate polynomials. The coefficients of the infinitesimals are found out by substituting them in the PDE and grouping similar terms that are polynomials and since they form a linear system, solve and check for non trivial solutions. The degree of the assumed bivariates are increased till a certain maximum value.

References

- Lie Groups and Differential Equations pp. 327 - pp. 329

chi

`diofant.solvers.ode.lie_heuristic_chi(match, comp=False)`

The aim of the fourth heuristic is to find the function $\chi(x, y)$ that satisifies the PDE $\frac{d\chi}{dx} + h\frac{d\chi}{dx} - \frac{\partial h}{\partial y}\chi = 0$.

This assumes χ to be a bivariate polynomial in x and y. By intuition, h should be a rational function in x and y. The method used here is to substitute a general binomial for χ up to a certain maximum degree is reached. The coefficients of the polynomials, are calculated by by collecting terms of the same order in x and y.

After finding χ, the next step is to use $\eta = \xi * h + \chi$, to determine ξ and η. This can be done by dividing χ by h which would give $-\xi$ as the quotient and η as the remainder.

References

- E.S Cheb-Terrab, L.G.S Duarte and L.A,C.P da Mota, Computer Algebra Solving of First Order ODEs Using Symmetry Methods, pp. 8

abaco2_similar

`diofant.solvers.ode.lie_heuristic_abaco2_similar(match, comp=False)`

This heuristic uses the following two assumptions on ξ and η

$$\eta = g(x), \xi = f(x)$$

$$\eta = f(y), \xi = g(y)$$

For the first assumption,

1. First $\frac{\frac{\partial h}{\partial y}}{\frac{\partial^2 h}{\partial yy}}$ is calculated. Let us say this value is A
2. If this is constant, then h is matched to the form $A(x) + B(x)e^{\frac{y}{C}}$ then, $\frac{e^{\int \frac{A(x)}{C} \,dx}}{B(x)}$ gives $f(x)$ and $A(x) * f(x)$ gives $g(x)$

3. Otherwise $\frac{\frac{\partial A}{\partial X}}{\frac{\partial A}{\partial Y}} = \gamma$ is calculated. If

   a] γ is a function of x alone

   b] $\frac{\gamma\frac{\partial h}{\partial y} - \gamma'(x) - \frac{\partial h}{\partial x}}{h+\gamma} = G$ is a function of x alone. then, $e^{\int G\,dx}$ gives $f(x)$ and $-\gamma * f(x)$ gives $g(x)$

The second assumption holds good if $\frac{dy}{dx} = h(x,y)$ is rewritten as $\frac{dy}{dx} = \frac{1}{h(y,x)}$ and the same properties of the first assumption satisifes. After obtaining $f(x)$ and $g(x)$, the coordinates are again interchanged, to get ξ as $f(x^*)$ and η as $g(y^*)$

References

- E.S. Cheb-Terrab, A.D. Roche, Symmetries and First Order ODE Patterns, pp. 10 - pp. 12

function_sum

`diofant.solvers.ode.`**`lie_heuristic_function_sum`**(*match*, *comp=False*)

This heuristic uses the following two assumptions on ξ and η

$$\eta = 0, \xi = f(x) + g(y)$$

$$\eta = f(x) + g(y), \xi = 0$$

The first assumption of this heuristic holds good if

$$\frac{\partial}{\partial y}[(h\frac{\partial^2}{\partial x^2}(h^{-1}))^{-1}]$$

is separable in x and y,

1. The separated factors containing y is $\frac{\partial g}{\partial y}$. From this $g(y)$ can be determined.
2. The separated factors containing x is $f''(x)$.
3. $h\frac{\partial^2}{\partial x^2}(h^{-1})$ equals $\frac{f''(x)}{f(x)+g(y)}$. From this $f(x)$ can be determined.

The second assumption holds good if $\frac{dy}{dx} = h(x,y)$ is rewritten as $\frac{dy}{dx} = \frac{1}{h(y,x)}$ and the same properties of the first assumption satisifes. After obtaining $f(x)$ and $g(y)$, the coordinates are again interchanged, to get η as $f(x) + g(y)$.

For both assumptions, the constant factors are separated among $g(y)$ and $f''(x)$, such that $f''(x)$ obtained from 3] is the same as that obtained from 2]. If not possible, then this heuristic fails.

References

- E.S. Cheb-Terrab, A.D. Roche, Symmetries and First Order ODE Patterns, pp. 7 - pp. 8

abaco2_unique_unknown

diofant.solvers.ode.**lie_heuristic_abaco2_unique_unknown**(*match*, *comp=False*)

This heuristic assumes the presence of unknown functions or known functions with non-integer powers.

1. A list of all functions and non-integer powers containing x and y
2. Loop over each element f in the list, find $\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial x}} = R$

   If it is separable in x and y, let X be the factors containing x. Then

   a] Check if $\xi = X$ and $\eta = -\frac{X}{R}$ satisfy the PDE. If yes, then return ξ and η

   b] Check if $\xi = \frac{-R}{X}$ and $\eta = -\frac{1}{X}$ satisfy the PDE. If yes, then return ξ and η

   If not, then check if

   a] $\xi = -R, \eta = 1$

   b] $\xi = 1, \eta = -\frac{1}{R}$

   are solutions.

References

- E.S. Cheb-Terrab, A.D. Roche, Symmetries and First Order ODE Patterns, pp. 10 - pp. 12

linear

diofant.solvers.ode.**lie_heuristic_linear**(*match*, *comp=False*)

This heuristic assumes

1. $\xi = ax + by + c$ and
2. $\eta = fx + gy + h$

After substituting the following assumptions in the determining PDE, it reduces to

$$f + (g - a)h - bh^2 - (ax + by + c)\frac{\partial h}{\partial x} - (fx + gy + c)\frac{\partial h}{\partial y}$$

Solving the reduced PDE obtained, using the method of characteristics, becomes impractical. The method followed is grouping similar terms and solving the system of linear equations obtained. The difference between the bivariate heuristic is that h need not be a rational function in this case.

References

- E.S. Cheb-Terrab, A.D. Roche, Symmetries and First Order ODE Patterns, pp. 10 - pp. 12

System of ODEs

These functions are intended for internal use by *dsolve()* (page 868) for system of differential equations.

system_of_odes_linear_2eq_order1_type1

`diofant.solvers.ode._linear_2eq_order1_type1(x, y, t, r, eq)`

It is classified under system of two linear homogeneous first-order constant-coefficient ordinary differential equations.

The equations which come under this type are

$$x' = ax + by,$$

$$y' = cx + dy$$

The characteristics equation is written as

$$\lambda^2 + (a + d)\lambda + ad - bc = 0$$

and its discriminant is $D = (a - d)^2 + 4bc$. There are several cases

1. Case when $ad - bc \neq 0$. The origin of coordinates, $x = y = 0$, is the only stationary point; it is - a node if $D = 0$ - a node if $D > 0$ and $ad - bc > 0$ - a saddle if $D > 0$ and $ad - bc < 0$ - a focus if $D < 0$ and $a + d \neq 0$ - a centre if $D < 0$ and $a + d \neq 0$.

1.1. If $D > 0$. The characteristic equation has two distinct real roots λ_1 and λ_2 . The general solution of the system in question is expressed as

$$x = C_1 b e^{\lambda_1 t} + C_2 b e^{\lambda_2 t}$$

$$y = C_1(\lambda_1 - a)e^{\lambda_1 t} + C_2(\lambda_2 - a)e^{\lambda_2 t}$$

where C_1 and C_2 being arbitrary constants

1.2. If $D < 0$. The characteristics equation has two conjugate roots, $\lambda_1 = \sigma + i\beta$ and $\lambda_2 = \sigma - i\beta$. The general solution of the system is given by

$$x = be^{\sigma t}(C_1 \sin(\beta t) + C_2 \cos(\beta t))$$

$$y = e^{\sigma t}([(\sigma - a)C_1 - \beta C_2] \sin(\beta t) + [\beta C_1 + (\sigma - a)C_2 \cos(\beta t)])$$

1.3. If $D = 0$ and $a \neq d$. The characteristic equation has two equal roots, $\lambda_1 = \lambda_2$. The general solution of the system is written as

$$x = 2b(C_1 + \frac{C_2}{a - d} + C_2 t)e^{\frac{a+d}{2}t}$$

$$y = [(d - a)C_1 + C_2 + (d - a)C_2 t]e^{\frac{a+d}{2}t}$$

1.4. If $D = 0$ and $a = d \neq 0$ and $b = 0$

$$x = C_1 e^{at}, y = (cC_1 t + C_2)e^{at}$$

1.5. If $D = 0$ and $a = d \neq 0$ and $c = 0$

$$x = (bC_1 t + C_2)e^{at}, y = C_1 e^{at}$$

2. Case when $ad - bc = 0$ and $a^2 + b^2 > 0$. The whole straight line $ax + by = 0$ consists of singular points. The orginal system of differential equations can be rewritten as

$$x' = ax + by, y' = k(ax + by)$$

2.1 If $a + bk \neq 0$, solution will be

$$x = bC_1 + C_2 e^{(a+bk)t}, y = -aC_1 + kC_2 e^{(a+bk)t}$$

2.2 If $a + bk = 0$, solution will be

$$x = C_1(bkt - 1) + bC_2 t, y = k^2 bC_1 t + (bk^2 t + 1)C_2$$

system_of_odes_linear_2eq_order1_type2

`diofant.solvers.ode._linear_2eq_order1_type2(x, y, t, r, eq)`

The equations of this type are

$$x' = ax + by + k1, y' = cx + dy + k2$$

The general solution of this system is given by sum of its particular solution and the general solution of the corresponding homogeneous system is obtained from type1.

1. When $ad - bc \neq 0$. The particular solution will be $x = x_0$ and $y = y_0$ where x_0 and y_0 are determined by solving linear system of equations

$$ax_0 + by_0 + k1 = 0, cx_0 + dy_0 + k2 = 0$$

2. When $ad - bc = 0$ and $a^2 + b^2 > 0$. In this case, the system of equation becomes

$$x' = ax + by + k_1, y' = k(ax + by) + k_2$$

2.1 If $\sigma = a + bk \neq 0$, particular solution is given by

$$x = b\sigma^{-1}(c_1 k - c_2)t - \sigma^{-2}(ac_1 + bc_2)$$

$$y = kx + (c_2 - c_1 k)t$$

2.2 If $\sigma = a + bk = 0$, particular solution is given by

$$x = \frac{1}{2}b(c_2 - c_1 k)t^2 + c_1 t$$

$$y = kx + (c_2 - c_1 k)t$$

system_of_odes_linear_2eq_order1_type3

`diofant.solvers.ode._linear_2eq_order1_type3(x, y, t, r, eq)`

The equations of this type of ode are

$$x' = f(t)x + g(t)y$$

$$y' = g(t)x + f(t)y$$

The solution of such equations is given by

$$x = e^F(C_1 e^G + C_2 e^{-G}), y = e^F(C_1 e^G - C_2 e^{-G})$$

where C_1 and C_2 are arbitrary constants, and

$$F = \int f(t)\,dt, G = \int g(t)\,dt$$

system_of_odes_linear_2eq_order1_type4

`diofant.solvers.ode._linear_2eq_order1_type4(x, y, t, r, eq)`

The equations of this type of ode are .

$$x' = f(t)x + g(t)y$$

$$y' = -g(t)x + f(t)y$$

The solution is given by

$$x = F(C_1 \cos(G) + C_2 \sin(G)), y = F(-C_1 \sin(G) + C_2 \cos(G))$$

where C_1 and C_2 are arbitrary constants, and

$$F = \int f(t)\,dt, G = \int g(t)\,dt$$

system_of_odes_linear_2eq_order1_type5

`diofant.solvers.ode._linear_2eq_order1_type5(x, y, t, r, eq)`

The equations of this type of ode are

$$x' = f(t)x + g(t)y$$

$$y' = ag(t)x + [f(t) + bg(t)]y$$

The transformation of

$$x = e^{\int f(t)\,dt}u, y = e^{\int f(t)\,dt}v, T = \int g(t)\,dt$$

leads to a system of constant coefficient linear differential equations

$$u'(T) = v, v'(T) = au + bv$$

system_of_odes_linear_2eq_order1_type6

`diofant.solvers.ode._linear_2eq_order1_type6(x, y, t, r, eq)`

The equations of this type of ode are .

$$x' = f(t)x + g(t)y$$

$$y' = a[f(t) + ah(t)]x + a[g(t) - h(t)]y$$

This is solved by first multiplying the first equation by $-a$ and adding it to the second equation to obtain

$$y' - ax' = -ah(t)(y - ax)$$

Setting $U = y - ax$ and integrating the equation we arrive at

$$y - ax = C_1 e^{-a\int h(t)\,dt}$$

and on substituting the value of y in first equation give rise to first order ODEs. After solving for x, we can obtain y by substituting the value of x in second equation.

system_of_odes_linear_2eq_order1_type7

`diofant.solvers.ode._linear_2eq_order1_type7(x, y, t, r, eq)`

The equations of this type of ode are .

$$x' = f(t)x + g(t)y$$

$$y' = h(t)x + p(t)y$$

Differentiating the first equation and substituting the value of y from second equation will give a second-order linear equation

$$gx'' - (fg + gp + g')x' + (fgp - g^2h + fg' - f'g)x = 0$$

This above equation can be easily integrated if following conditions are satisfied.

1. $fgp - g^2h + fg' - f'g = 0$
2. $fgp - g^2h + fg' - f'g = ag, fg + gp + g' = bg$

If first condition is satisfied then it is solved by current dsolve solver and in second case it becomes a constant coefficient differential equation which is also solved by current solver.

Otherwise if the above condition fails then, a particular solution is assumed as $x = x_0(t)$ and $y = y_0(t)$ Then the general solution is expressed as

$$x = C_1x_0(t) + C_2x_0(t)\int \frac{g(t)F(t)P(t)}{x_0^2(t)}\,dt$$

$$y = C_1y_0(t) + C_2[\frac{F(t)P(t)}{x_0(t)} + y_0(t)\int \frac{g(t)F(t)P(t)}{x_0^2(t)}\,dt]$$

where C1 and C2 are arbitrary constants and

$$F(t) = e^{\int f(t)\,dt}, P(t) = e^{\int p(t)\,dt}$$

system_of_odes_linear_2eq_order2_type1

`diofant.solvers.ode._linear_2eq_order2_type1(x, y, t, r, eq)`

System of two constant-coefficient second-order linear homogeneous differential equations

$$x'' = ax + by$$

$$y'' = cx + dy$$

The characteristic equation for above equations

$$\lambda^4 - (a + d)\lambda^2 + ad - bc = 0$$

whose discriminant is $D = (a - d)^2 + 4bc \neq 0$

1. When $ad - bc \neq 0$

1.1. If $D \neq 0$. The characteristic equation has four distinct roots, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The general solution of the system is

$$x = C_1 b e^{\lambda_1 t} + C_2 b e^{\lambda_2 t} + C_3 b e^{\lambda_3 t} + C_4 b e^{\lambda_4 t}$$

$$y = C_1 (\lambda_1^2 - a) e^{\lambda_1 t} + C_2 (\lambda_2^2 - a) e^{\lambda_2 t} + C_3 (\lambda_3^2 - a) e^{\lambda_3 t} + C_4 (\lambda_4^2 - a) e^{\lambda_4 t}$$

where $C_1, ..., C_4$ are arbitrary constants.

1.2. If $D = 0$ and $a \neq d$:

$$x = 2C_1 (bt + \frac{2bk}{a-d}) e^{\frac{kt}{2}} + 2C_2 (bt + \frac{2bk}{a-d}) e^{\frac{-kt}{2}} + 2bC_3 t e^{\frac{kt}{2}} + 2bC_4 t e^{\frac{-kt}{2}}$$

$$y = C_1 (d-a) t e^{\frac{kt}{2}} + C_2 (d-a) t e^{\frac{-kt}{2}} + C_3 [(d-a)t + 2k] e^{\frac{kt}{2}} + C_4 [(d-a)t - 2k] e^{\frac{-kt}{2}}$$

where $C_1, ..., C_4$ are arbitrary constants and $k = \sqrt{2(a+d)}$

1.3. If $D = 0$ and $a = d \neq 0$ and $b = 0$:

$$x = 2\sqrt{a} C_1 e^{\sqrt{a}t} + 2\sqrt{a} C_2 e^{-\sqrt{a}t}$$

$$y = c C_1 t e^{\sqrt{a}t} - c C_2 t e^{-\sqrt{a}t} + C_3 e^{\sqrt{a}t} + C_4 e^{-\sqrt{a}t}$$

1.4. If $D = 0$ and $a = d \neq 0$ and $c = 0$:

$$x = b C_1 t e^{\sqrt{a}t} - b C_2 t e^{-\sqrt{a}t} + C_3 e^{\sqrt{a}t} + C_4 e^{-\sqrt{a}t}$$

$$y = 2\sqrt{a} C_1 e^{\sqrt{a}t} + 2\sqrt{a} C_2 e^{-\sqrt{a}t}$$

2. When $ad - bc = 0$ and $a^2 + b^2 > 0$. Then the original system becomes

$$x'' = ax + by$$

$$y'' = k(ax + by)$$

2.1. If $a + bk \neq 0$:

$$x = C_1 e^{t\sqrt{a+bk}} + C_2 e^{-t\sqrt{a+bk}} + C_3 bt + C_4 b$$

$$y = C_1 k e^{t\sqrt{a+bk}} + C_2 k e^{-t\sqrt{a+bk}} - C_3 at - C_4 a$$

2.2. If $a + bk = 0$:

$$x = C_1 b t^3 + C_2 b t^2 + C_3 t + C_4$$

$$y = kx + 6C_1 t + 2C_2$$

system_of_odes_linear_2eq_order2_type2

diofant.solvers.ode.**_linear_2eq_order2_type2**(*x, y, t, r, eq*)

The equations in this type are

$$x'' = a_1 x + b_1 y + c_1$$

$$y'' = a_2 x + b_2 y + c_2$$

The general solution of this system is given by the sum of its particular solution and the general solution of the homogeneous system. The general solution is given by the linear system of 2 equation of order 2 and type 1

1. If $a_1 b_2 - a_2 b_1 \neq 0$. A particular solution will be $x = x_0$ and $y = y_0$ where the constants x_0 and y_0 are determined by solving the linear algebraic system

$$a_1 x_0 + b_1 y_0 + c_1 = 0, a_2 x_0 + b_2 y_0 + c_2 = 0$$

2. If $a_1 b_2 - a_2 b_1 = 0$ and $a_1^2 + b_1^2 > 0$. In this case, the system in question becomes

$$x'' = ax + by + c_1, y'' = k(ax + by) + c_2$$

2.1. If $\sigma = a + bk \neq 0$, the particular solution will be

$$x = \frac{1}{2} b\sigma^{-1}(c_1 k - c_2)t^2 - \sigma^{-2}(ac_1 + bc_2)$$

$$y = kx + \frac{1}{2}(c_2 - c_1 k)t^2$$

2.2. If $\sigma = a + bk = 0$, the particular solution will be

$$x = \frac{1}{24} b(c_2 - c_1 k)t^4 + \frac{1}{2} c_1 t^2$$

$$y = kx + \frac{1}{2}(c_2 - c_1 k)t^2$$

system_of_odes_linear_2eq_order2_type3

diofant.solvers.ode.**_linear_2eq_order2_type3**(*x, y, t, r, eq*)

These type of equation is used for describing the horizontal motion of a pendulum taking into account the Earth rotation. The solution is given with $a^2 + 4b > 0$:

$$x = C_1 \cos(\alpha t) + C_2 \sin(\alpha t) + C_3 \cos(\beta t) + C_4 \sin(\beta t)$$

$$y = -C_1 \sin(\alpha t) + C_2 \cos(\alpha t) - C_3 \sin(\beta t) + C_4 \cos(\beta t)$$

where $C_1, ..., C_4$ and

$$\alpha = \frac{1}{2} a + \frac{1}{2} \sqrt{a^2 + 4b}, \beta = \frac{1}{2} a - \frac{1}{2} \sqrt{a^2 + 4b}$$

system_of_odes_linear_2eq_order2_type5

diofant.solvers.ode.**_linear_2eq_order2_type5**(*x, y, t, r, eq*)

The equation which come under this category are

$$x'' = a(ty' - y)$$

$$y'' = b(tx' - x)$$

The transformation

$$u = tx' - x, b = ty' - y$$

leads to the first-order system

$$u' = atv, v' = btu$$

The general solution of this system is given by

If $ab > 0$:

$$u = C_1 a e^{\frac{1}{2}\sqrt{ab}t^2} + C_2 a e^{-\frac{1}{2}\sqrt{ab}t^2}$$

$$v = C_1 \sqrt{ab} e^{\frac{1}{2}\sqrt{ab}t^2} - C_2 \sqrt{ab} e^{-\frac{1}{2}\sqrt{ab}t^2}$$

If $ab < 0$:

$$u = C_1 a \cos(\frac{1}{2}\sqrt{|ab|}t^2) + C_2 a \sin(-\frac{1}{2}\sqrt{|ab|}t^2)$$

$$v = C_1 \sqrt{|ab|} \sin(\frac{1}{2}\sqrt{|ab|}t^2) + C_2 \sqrt{|ab|} \cos(-\frac{1}{2}\sqrt{|ab|}t^2)$$

where C_1 and C_2 are arbitrary constants. On substituting the value of u and v in above equations and integrating the resulting expressions, the general solution will become

$$x = C_3 t + t \int \frac{u}{t^2}\, dt, y = C_4 t + t \int \frac{u}{t^2}\, dt$$

where C_3 and C_4 are arbitrary constants.

system_of_odes_linear_2eq_order2_type6

`diofant.solvers.ode._linear_2eq_order2_type6(x, y, t, r, eq)`

The equations are

$$x'' = f(t)(a_1 x + b_1 y)$$

$$y'' = f(t)(a_2 x + b_2 y)$$

If k_1 and k_2 are roots of the quadratic equation

$$k^2 - (a_1 + b_2)k + a_1 b_2 - a_2 b_1 = 0$$

Then by multiplying appropriate constants and adding together original equations we obtain two independent equations:

$$z_1'' = k_1 f(t) z_1, z_1 = a_2 x + (k_1 - a_1)y$$

$$z_2'' = k_2 f(t) z_2, z_2 = a_2 x + (k_2 - a_1)y$$

Solving the equations will give the values of x and y after obtaining the value of z_1 and z_2 by solving the differential equation and substituting the result.

system_of_odes_linear_2eq_order2_type7

`diofant.solvers.ode._linear_2eq_order2_type7(x, y, t, r, eq)`

The equations are given as

$$x'' = f(t)(a_1x' + b_1y')$$

$$y'' = f(t)(a_2x' + b_2y')$$

If k_1 and 'k_2' are roots of the quadratic equation

$$k^2 - (a_1 + b_2)k + a_1b_2 - a_2b_1 = 0$$

Then the system can be reduced by adding together the two equations multiplied by appropriate constants give following two independent equations:

$$z_1'' = k_1f(t)z_1', z_1 = a_2x + (k_1 - a_1)y$$

$$z_2'' = k_2f(t)z_2', z_2 = a_2x + (k_2 - a_1)y$$

Integrating these and returning to the original variables, one arrives at a linear algebraic system for the unknowns x and y:

$$a_2x + (k_1 - a_1)y = C_1\int e^{k_1F(t)}\,dt + C_2$$

$$a_2x + (k_2 - a_1)y = C_3\int e^{k_2F(t)}\,dt + C_4$$

where $C_1, ..., C_4$ are arbitrary constants and $F(t) = \int f(t)\,dt$

system_of_odes_linear_2eq_order2_type8

`diofant.solvers.ode._linear_2eq_order2_type8(x, y, t, r, eq)`

The equation of this category are

$$x'' = af(t)(ty' - y)$$

$$y'' = bf(t)(tx' - x)$$

The transformation

$$u = tx' - x, v = ty' - y$$

leads to the system of first-order equations

$$u' = atf(t)v, v' = btf(t)u$$

The general solution of this system has the form

If $ab > 0$:

$$u = C_1ae^{\sqrt{ab}\int tf(t)\,dt} + C_2ae^{-\sqrt{ab}\int tf(t)\,dt}$$

$$v = C_1\sqrt{ab}e^{\sqrt{ab}\int tf(t)\,dt} - C_2\sqrt{ab}e^{-\sqrt{ab}\int tf(t)\,dt}$$

If $ab < 0$:

$$u = C_1 a \cos(\sqrt{|ab|} \int t f(t)\, dt) + C_2 a \sin(-\sqrt{|ab|} \int t f(t)\, dt)$$

$$v = C_1 \sqrt{|ab|} \sin(\sqrt{|ab|} \int t f(t)\, dt) + C_2 \sqrt{|ab|} \cos(-\sqrt{|ab|} \int t f(t)\, dt)$$

where C_1 and C_2 are arbitrary constants. On substituting the value of u and v in above equations and integrating the resulting expressions, the general solution will become

$$x = C_3 t + t \int \frac{u}{t^2}\, dt, y = C_4 t + t \int \frac{u}{t^2}\, dt$$

where C_3 and C_4 are arbitrary constants.

system_of_odes_linear_2eq_order2_type9

diofant.solvers.ode.**_linear_2eq_order2_type9**(*x*, *y*, *t*, *r*, *eq*)

$$t^2 x'' + a_1 t x' + b_1 t y' + c_1 x + d_1 y = 0$$

$$t^2 y'' + a_2 t x' + b_2 t y' + c_2 x + d_2 y = 0$$

These system of equations are euler type.

The substitution of $t = \sigma e^{\tau} (\sigma \neq 0)$ leads to the system of constant coefficient linear differential equations

$$x'' + (a_1 - 1) x' + b_1 y' + c_1 x + d_1 y = 0$$

$$y'' + a_2 x' + (b_2 - 1) y' + c_2 x + d_2 y = 0$$

The general solution of the homogeneous system of differential equations is determined by a linear combination of linearly independent particular solutions determined by the method of undetermined coefficients in the form of exponentials

$$x = A e^{\lambda t}, y = B e^{\lambda t}$$

On substituting these expressions into the original system and collecting the coefficients of the unknown A and B, one obtains

$$(\lambda^2 + (a_1 - 1)\lambda + c_1) A + (b_1 \lambda + d_1) B = 0$$

$$(a_2 \lambda + c_2) A + (\lambda^2 + (b_2 - 1)\lambda + d_2) B = 0$$

The determinant of this system must vanish for nontrivial solutions A, B to exist. This requirement results in the following characteristic equation for λ

$$(\lambda^2 + (a_1 - 1)\lambda + c_1)(\lambda^2 + (b_2 - 1)\lambda + d_2) - (b_1 \lambda + d_1)(a_2 \lambda + c_2) = 0$$

If all roots $k_1, ..., k_4$ of this equation are distinct, the general solution of the original system of the differential equations has the form

$$x = C_1 (b_1 \lambda_1 + d_1) e^{\lambda_1 t} - C_2 (b_1 \lambda_2 + d_1) e^{\lambda_2 t} - C_3 (b_1 \lambda_3 + d_1) e^{\lambda_3 t} - C_4 (b_1 \lambda_4 + d_1) e^{\lambda_4 t}$$

$$y = C_1 (\lambda_1^2 + (a_1 - 1)\lambda_1 + c_1) e^{\lambda_1 t} + C_2 (\lambda_2^2 + (a_1 - 1)\lambda_2 + c_1) e^{\lambda_2 t} + C_3 (\lambda_3^2 + (a_1 - 1)\lambda_3 + c_1) e^{\lambda_3 t} + C_4 (\lambda_4^2 + (a_1 - 1)\lambda_4$$

system_of_odes_linear_2eq_order2_type11

diofant.solvers.ode.**_linear_2eq_order2_type11**(*x*, *y*, *t*, *r*, *eq*)

The equations which comes under this type are

$$x'' = f(t)(tx' - x) + g(t)(ty' - y)$$

$$y'' = h(t)(tx' - x) + p(t)(ty' - y)$$

The transformation

$$u = tx' - x, v = ty' - y$$

leads to the linear system of first-order equations

$$u' = tf(t)u + tg(t)v, v' = th(t)u + tp(t)v$$

On substituting the value of u and v in transformed equation gives value of x and y as

$$x = C_3 t + t \int \frac{u}{t^2} \, dt, y = C_4 t + t \int \frac{v}{t^2} \, dt.$$

where C_3 and C_4 are arbitrary constants.

system_of_odes_linear_3eq_order1_type4

diofant.solvers.ode.**_linear_3eq_order1_type4**(*x*, *y*, *z*, *t*, *r*, *eq*)

Equations:

$$x' = (a_1 f(t) + g(t))x + a_2 f(t)y + a_3 f(t)z$$

$$y' = b_1 f(t)x + (b_2 f(t) + g(t))y + b_3 f(t)z$$

$$z' = c_1 f(t)x + c_2 f(t)y + (c_3 f(t) + g(t))z$$

The transformation

$$x = e^{\int g(t) \, dt} u, y = e^{\int g(t) \, dt} v, z = e^{\int g(t) \, dt} w, \tau = \int f(t) \, dt$$

leads to the system of constant coefficient linear differential equations

$$u' = a_1 u + a_2 v + a_3 w$$

$$v' = b_1 u + b_2 v + b_3 w$$

$$w' = c_1 u + c_2 v + c_3 w$$

These system of equations are solved by homogeneous linear system of constant coefficients of n equations of first order. Then substituting the value of u, v and w in transformed equation gives value of x, y and z.

system_of_odes_linear_neq_order1_type1

diofant.solvers.ode.**_linear_neq_order1_type1**(*match_*)

System of n first-order constant-coefficient linear differential equations

$$Mx' = Lx + f(t)$$

Notes

The nonhomogeneous case is not implemented yet. Mass-matrix assumed to be invertible and provided general solution uses the Jordan canonical form for $A = M^{-1}L$, see [R545] (page 1267).

References

[R545] (page 1267)

system_of_odes_nonlinear_2eq_order1_type1

diofant.solvers.ode.**_nonlinear_2eq_order1_type1**(*x*, *y*, *t*, *eq*)

Equations:

$$x' = x^n F(x, y)$$

$$y' = g(y) F(x, y)$$

Solution:

$$x = \varphi(y), \int \frac{1}{g(y) F(\varphi(y), y)} \, dy = t + C_2$$

where

if $n \neq 1$

$$\varphi = [C_1 + (1 - n) \int \frac{1}{g(y)} \, dy]^{\frac{1}{1-n}}$$

if $n = 1$

$$\varphi = C_1 e^{\int \frac{1}{g(y)} \, dy}$$

where C_1 and C_2 are arbitrary constants.

system_of_odes_nonlinear_2eq_order1_type2

diofant.solvers.ode.**_nonlinear_2eq_order1_type2**(*x*, *y*, *t*, *eq*)

Equations:

$$x' = e^{\lambda x} F(x, y)$$

$$y' = g(y) F(x, y)$$

Solution:

$$x = \varphi(y), \int \frac{1}{g(y) F(\varphi(y), y)} \, dy = t + C_2$$

where

if $\lambda \neq 0$

$$\varphi = -\frac{1}{\lambda} log(C_1 - \lambda \int \frac{1}{g(y)} \, dy)$$

if $\lambda = 0$

$$\varphi = C_1 + \int \frac{1}{g(y)}\,dy$$

where C_1 and C_2 are arbitrary constants.

system_of_odes_nonlinear_2eq_order1_type3

diofant.solvers.ode.**_nonlinear_2eq_order1_type3**(*x*, *y*, *t*, *eq*)
Autonomous system of general form

$$x' = F(x, y)$$

$$y' = G(x, y)$$

Assuming $y = y(x, C_1)$ where C_1 is an arbitrary constant is the general solution of the first-order equation

$$F(x, y)y'_x = G(x, y)$$

Then the general solution of the original system of equations has the form

$$\int \frac{1}{F(x, y(x, C_1))}\,dx = t + C_1$$

system_of_odes_nonlinear_2eq_order1_type4

diofant.solvers.ode.**_nonlinear_2eq_order1_type4**(*x*, *y*, *t*, *eq*)
Equation:

$$x' = f_1(x)g_1(y)\phi(x, y, t)$$

$$y' = f_2(x)g_2(y)\phi(x, y, t)$$

First integral:

$$\int \frac{f_2(x)}{f_1(x)}\,dx - \int \frac{g_1(y)}{g_2(y)}\,dy = C$$

where C is an arbitrary constant.

On solving the first integral for x (resp., y) and on substituting the resulting expression into either equation of the original solution, one arrives at a firs-order equation for determining y (resp., x).

system_of_odes_nonlinear_2eq_order1_type5

diofant.solvers.ode.**_nonlinear_2eq_order1_type5**(*func*, *t*, *eq*)
Clairaut system of ODEs

$$x = tx' + F(x', y')$$

$$y = ty' + G(x', y')$$

The following are solutions of the system

(i) straight lines:

$$x = C_1 t + F(C_1, C_2), y = C_2 t + G(C_1, C_2)$$

where C_1 and C_2 are arbitrary constants;

(ii) envelopes of the above lines;

(iii) continuously differentiable lines made up from segments of the lines (i) and (ii).

system_of_odes_nonlinear_3eq_order1_type1

diofant.solvers.ode.**_nonlinear_3eq_order1_type1**(*x*, *y*, *z*, *t*, *eq*)

Equations:

$$ax' = (b - c)yz, \;\; by' = (c - a)zx, \;\; cz' = (a - b)xy$$

First Integrals:

$$ax^2 + by^2 + cz^2 = C_1$$

$$a^2x^2 + b^2y^2 + c^2z^2 = C_2$$

where C_1 and C_2 are arbitrary constants. On solving the integrals for y and z and on substituting the resulting expressions into the first equation of the system, we arrives at a separable first-order equation on x. Similarly doing that for other two equations, we will arrive at first order equation on y and z too.

References

-http://eqworld.ipmnet.ru/en/solutions/sysode/sode0401.pdf

system_of_odes_nonlinear_3eq_order1_type2

diofant.solvers.ode.**_nonlinear_3eq_order1_type2**(*x*, *y*, *z*, *t*, *eq*)

Equations:

$$ax' = (b - c)yzf(x, y, z, t)$$

$$by' = (c - a)zxf(x, y, z, t)$$

$$cz' = (a - b)xyf(x, y, z, t)$$

First Integrals:

$$ax^2 + by^2 + cz^2 = C_1$$

$$a^2x^2 + b^2y^2 + c^2z^2 = C_2$$

where C_1 and C_2 are arbitrary constants. On solving the integrals for y and z and on substituting the resulting expressions into the first equation of the system, we arrives at a first-order differential equations on x. Similarly doing that for other two equations we will arrive at first order equation on y and z.

References

-http://eqworld.ipmnet.ru/en/solutions/sysode/sode0402.pdf

Information on the ode module

This module contains *dsolve()* (page 868) and different helper functions that it uses.

dsolve() (page 868) solves ordinary differential equations. See the docstring on the various functions for their uses. Note that partial differential equations support is in `pde.py`. Note that hint functions have docstrings describing their various methods, but they are intended for internal use. Use `dsolve(ode, func, hint=hint)` to solve an ODE using a specific hint. See also the docstring on *dsolve()* (page 868).

Functions in this module

These are the user functions in this module:

- *dsolve()* (page 868) - Solves ODEs.
- *classify_ode()* (page 871) - Classifies ODEs into possible hints for *dsolve()* (page 868).
- *checkodesol()* (page 873) - Checks if an equation is the solution to an ODE.
- *homogeneous_order()* (page 874) - Returns the homogeneous order of an expression.
- *infinitesimals()* (page 874) - Returns the infinitesimals of the Lie group of point transformations of an ODE, such that it is invariant.

These are the non-solver helper functions that are for internal use. The user should use the various options to *dsolve()* (page 868) to obtain the functionality provided by these functions:

- *odesimp()* (page 876) - Does all forms of ODE simplification.
- *ode_sol_simplicity()* (page 879) - A key function for comparing solutions by simplicity.
- *constantsimp()* (page 878) - Simplifies arbitrary constants.
- *constant_renumber()* (page 877) - Renumber arbitrary constants.
- *_handle_Integral()* (page 918) - Evaluate unevaluated Integrals.

See also the docstrings of these functions.

Currently implemented solver methods

The following methods are implemented for solving ordinary differential equations. See the docstrings of the various hint functions for more information on each (run `help(ode)`):

- 1st order separable differential equations.
- 1st order differential equations whose coefficients or dx and dy are functions homogeneous of the same order.
- 1st order exact differential equations.
- 1st order linear differential equations.
- 1st order Bernoulli differential equations.

- Power series solutions for first order differential equations.
- Lie Group method of solving first order differential equations.
- 2nd order Liouville differential equations.
- Power series solutions for second order differential equations at ordinary and regular singular points.
- nth order linear homogeneous differential equation with constant coefficients.
- nth order linear inhomogeneous differential equation with constant coefficients using the method of undetermined coefficients.
- nth order linear inhomogeneous differential equation with constant coefficients using the method of variation of parameters.

Philosophy behind this module

This module is designed to make it easy to add new ODE solving methods without having to mess with the solving code for other methods. The idea is that there is a *classify_ode()* (page 871) function, which takes in an ODE and tells you what hints, if any, will solve the ODE. It does this without attempting to solve the ODE, so it is fast. Each solving method is a hint, and it has its own function, named `ode_<hint>`. That function takes in the ODE and any match expression gathered by *classify_ode()* (page 871) and returns a solved result. If this result has any integrals in it, the hint function will return an unevaluated *Integral* (page 529) class. *dsolve()* (page 868), which is the user wrapper function around all of this, will then call *odesimp()* (page 876) on the result, which, among other things, will attempt to solve the equation for the dependent variable (the function we are solving for), simplify the arbitrary constants in the expression, and evaluate any integrals, if the hint allows it.

How to add new solution methods

If you have an ODE that you want *dsolve()* (page 868) to be able to solve, try to avoid adding special case code here. Instead, try finding a general method that will solve your ODE, as well as others. This way, the *ode* (page 915) module will become more robust, and unhindered by special case hacks. WolphramAlpha and Maple's DETools[odeadvisor] function are two resources you can use to classify a specific ODE. It is also better for a method to work with an nth order ODE instead of only with specific orders, if possible.

To add a new method, there are a few things that you need to do. First, you need a hint name for your method. Try to name your hint so that it is unambiguous with all other methods, including ones that may not be implemented yet. If your method uses integrals, also include a `hint_Integral` hint. If there is more than one way to solve ODEs with your method, include a hint for each one, as well as a `<hint>_best` hint. Your `ode_<hint>_best()` function should choose the best using min with `ode_sol_simplicity` as the key argument. See *ode_1st_homogeneous_coeff_best()* (page 881), for example. The function that uses your method will be called `ode_<hint>()`, so the hint must only use characters that are allowed in a Python function name (alphanumeric characters and the underscore '_' character). Include a function for every hint, except for `_Integral` hints (*dsolve()* (page 868) takes care of those automatically). Hint names should be all lowercase, unless a word is commonly capitalized (such as Integral or Bernoulli). If you have a hint that you do not want to run with `all_Integral` that doesn't have an `_Integral` counterpart (such as a best hint that would defeat the purpose of `all_Integral`), you will need to remove it manually in the *dsolve()* (page 868) code. See also the *classify_ode()* (page 871) docstring for guidelines on writing a hint name.

Determine *in general* how the solutions returned by your method compare with other methods that can potentially solve the same ODEs. Then, put your hints in the *allhints* (page 876) tuple in the order that they should be called. The ordering of this tuple determines which

hints are default. Note that exceptions are ok, because it is easy for the user to choose individual hints with *dsolve()* (page 868). In general, _Integral variants should go at the end of the list, and _best variants should go before the various hints they apply to. For example, the undetermined_coefficients hint comes before the variation_of_parameters hint because, even though variation of parameters is more general than undetermined coefficients, undetermined coefficients generally returns cleaner results for the ODEs that it can solve than variation of parameters does, and it does not require integration, so it is much faster.

Next, you need to have a match expression or a function that matches the type of the ODE, which you should put in *classify_ode()* (page 871) (if the match function is more than just a few lines, like *_undetermined_coefficients_match()* (page 918), it should go outside of *classify_ode()* (page 871)). It should match the ODE without solving for it as much as possible, so that *classify_ode()* (page 871) remains fast and is not hindered by bugs in solving code. Be sure to consider corner cases. For example, if your solution method involves dividing by something, make sure you exclude the case where that division will be 0.

In most cases, the matching of the ODE will also give you the various parts that you need to solve it. You should put that in a dictionary (.match() will do this for you), and add that as matching_hints['hint'] = matchdict in the relevant part of *classify_ode()* (page 871). *classify_ode()* (page 871) will then send this to *dsolve()* (page 868), which will send it to your function as the match argument. Your function should be named ode_<hint>(eq, func, order, match)`. If you need to send more information, put it in the ``match dictionary. For example, if you had to substitute in a dummy variable in *classify_ode()* (page 871) to match the ODE, you will need to pass it to your function using the $match$ dict to access it. You can access the independent variable using func.args[0], and the dependent variable (the function you are trying to solve for) as func.func. If, while trying to solve the ODE, you find that you cannot, raise NotImplementedError. *dsolve()* (page 868) will catch this error with the all meta-hint, rather than causing the whole routine to fail.

Add a docstring to your function that describes the method employed. Like with anything else in Diofant, you will need to add a doctest to the docstring, in addition to real tests in test_ode.py. Try to maintain consistency with the other hint functions' docstrings. Add your method to the list at the top of this docstring. Also, add your method to ode.rst in the docs/src directory, so that the Sphinx docs will pull its docstring into the main Diofant documentation. Be sure to make the Sphinx documentation by running make html from within the docs directory to verify that the docstring formats correctly.

If your solution method involves integrating, use *Integral()* (page 529) instead of *integrate()* (page 522). This allows the user to bypass hard/slow integration by using the _Integral variant of your hint. In most cases, calling *diofant.core.basic.Basic.doit()* (page 44) will integrate your solution. If this is not the case, you will need to write special code in *_handle_Integral()* (page 918). Arbitrary constants should be symbols named C1, C2, and so on. All solution methods should return an equality instance. If you need an arbitrary number of arbitrary constants, you can use constants = numbered_symbols(prefix='C', cls=Symbol, start=1). If it is possible to solve for the dependent function in a general way, do so. Otherwise, do as best as you can, but do not call solve in your ode_<hint>() function. *odesimp()* (page 876) will attempt to solve the solution for you, so you do not need to do that. Lastly, if your ODE has a common simplification that can be applied to your solutions, you can add a special case in *odesimp()* (page 876) for it. For example, solutions returned from the 1st_homogeneous_coeff hints often have many *log()* (page 322) terms, so *odesimp()* (page 876) calls *logcombine()* (page 782) on them (it also helps to write the arbitrary constant as log(C1) instead of C1 in this case). Also consider common ways that you can rearrange your solution to have *constantsimp()* (page 878) take better advantage of it. It is better to put simplification in *odesimp()* (page 876) than in your method, because it can then be turned off with the simplify flag in *dsolve()* (page 868). If you have any extraneous sim-

plification in your function, be sure to only run it using `if match.get('simplify', True):`, especially if it can be slow or if it can reduce the domain of the solution.

Finally, as with every contribution to Diofant, your method will need to be tested. Add a test for each method in `test_ode.py`. Follow the conventions there, i.e., test the solver using `dsolve(eq, f(x), hint=your_hint)`, and also test the solution using *checkodesol()* (page 873) (you can put these in a separate tests and skip/XFAIL if it runs too slow/doesn't work). Be sure to call your hint specifically in *dsolve()* (page 868), that way the test won't be broken simply by the introduction of another matching hint. If your method works for higher order (>1) ODEs, you will need to run `sol = constant_renumber(sol, 'C', 1, order)` for each solution, where `order` is the order of the ODE. This is because `constant_renumber` renumbers the arbitrary constants by printing order, which is platform dependent. Try to test every corner case of your solver, including a range of orders if it is a nth order solver, but if your solver is slow, such as if it involves hard integration, try to keep the test run time down.

Feel free to refactor existing hints to avoid duplicating code or creating inconsistencies. If you can show that your method exactly duplicates an existing method, including in the simplicity and speed of obtaining the solutions, then you can remove the old, less general method. The existing code is tested extensively in `test_ode.py`, so if anything is broken, one of those tests will surely fail.

`diofant.solvers.ode.`**`_undetermined_coefficients_match`**(*expr*, *x*)

Returns a trial function match if undetermined coefficients can be applied to `expr`, and `None` otherwise.

A trial expression can be found for an expression for use with the method of undetermined coefficients if the expression is an additive/multiplicative combination of constants, polynomials in x (the independent variable of expr), $\sin(ax + b)$, $\cos(ax + b)$, and e^{ax} terms (in other words, it has a finite number of linearly independent derivatives).

Note that you may still need to multiply each term returned here by sufficient x to make it linearly independent with the solutions to the homogeneous equation.

This is intended for internal use by `undetermined_coefficients` hints.

Diofant currently has no way to convert $\sin^n(x)\cos^m(y)$ into a sum of only $\sin(ax)$ and $\cos(bx)$ terms, so these are not implemented. So, for example, you will need to manually convert $\sin^2(x)$ into $[1 + \cos(2x)]/2$ to properly apply the method of undetermined coefficients on it.

Examples

```
>>> _undetermined_coefficients_match(9*x*exp(x) + exp(-x), x)
{'test': True, 'trialset': {E**(-x), E**x, E**x*x}}
>>> _undetermined_coefficients_match(log(x), x)
{'test': False}
```

`diofant.solvers.ode.`**`_handle_Integral`**(*expr*, *func*, *order*, *hint*)

Converts a solution with Integrals in it into an actual solution.

For most hints, this simply runs `expr.doit()`.

3.19.6 Recurrence Equations

`diofant.solvers.recurr.`**`rsolve`**(*f*, *y*, *init=None*)

Solve univariate recurrence with rational coefficients.

Given k-th order linear recurrence $\operatorname{L} y = f$, or equivalently:

$$a_k(n)y(n+k) + a_{k-1}(n)y(n+k-1) + \ldots + a_0(n)y(n) = f(n)$$

where $a_i(n)$, for $i = 0, \ldots, k$, are polynomials or rational functions in n, and f is a hypergeometric function or a sum of a fixed number of pairwise dissimilar hypergeometric terms in n, finds all solutions or returns `None`, if none were found.

Initial conditions can be given as a dictionary in two forms:

1. `{ n_0 : v_0, n_1 : v_1, ..., n_m : v_m }`
2. `{ y(n_0) : v_0, y(n_1) : v_1, ..., y(n_m) : v_m }`

or as a list `L` of values:

`L = [ v_0, v_1, ..., v_m ]`

where `L[i] = v_i`, for $i = 0, \ldots, m$, maps to $y(n_i)$.

See also:

rsolve_poly (page 919), *rsolve_ratio* (page 920), *rsolve_hyper* (page 921)

Examples

Lets consider the following recurrence:

$$(n-1)y(n+2) - (n^2 + 3n - 2)y(n+1) + 2n(n+1)y(n) = 0$$

```
>>> y = Function('y')
```

```
>>> f = (n - 1)*y(n + 2) - (n**2 + 3*n - 2)*y(n + 1) + 2*n*(n + 1)*y(n)
```

```
>>> rsolve(f, y(n))
2**n*C0 + C1*factorial(n)
```

```
>>> rsolve(f, y(n), {y(0): 0, y(1): 3})
3*2**n - 3*factorial(n)
```

`diofant.solvers.recurr.`**`rsolve_poly`**(*coeffs*, *f*, *n*, ***hints*)

Given linear recurrence operator L of order k with polynomial coefficients and inhomogeneous equation $\operatorname{L} y = f$, where f is a polynomial, we seek for all polynomial solutions over field K of characteristic zero.

The algorithm performs two basic steps:

1. Compute degree N of the general polynomial solution.
2. Find all polynomials of degree N or less of $\operatorname{L} y = f$.

There are two methods for computing the polynomial solutions. If the degree bound is relatively small, i.e. it's smaller than or equal to the order of the recurrence, then naive method of undetermined coefficients is being used. This gives system of algebraic equations with $N + 1$ unknowns.

In the other case, the algorithm performs transformation of the initial equation to an equivalent one, for which the system of algebraic equations has only r indeterminates. This method is quite sophisticated (in comparison with the naive one) and was invented together by Abramov, Bronstein and Petkovšek.

It is possible to generalize the algorithm implemented here to the case of linear q-difference and differential equations.

Lets say that we would like to compute m-th Bernoulli polynomial up to a constant. For this we can use $b(n + 1) - b(n) = mn^{m-1}$ recurrence, which has solution $b(n) = B_m + C$. For example:

```
>>> rsolve_poly([-1, 1], 4*n**3, n)
C0 + n**4 - 2*n**3 + n**2
```

References

[R546] (page 1267), [R547] (page 1267), [R548] (page 1267)

diofant.solvers.recurr.**rsolve_ratio**(*coeffs*, *f*, *n*, ***hints*)

Given linear recurrence operator L of order k with polynomial coefficients and inhomogeneous equation $\mathrm{L}\, y = f$, where f is a polynomial, we seek for all rational solutions over field K of characteristic zero.

This procedure accepts only polynomials, however if you are interested in solving recurrence with rational coefficients then use `rsolve` which will pre-process the given equation and run this procedure with polynomial arguments.

The algorithm performs two basic steps:

1. Compute polynomial $v(n)$ which can be used as universal denominator of any rational solution of equation $\mathrm{L}\, y = f$.
2. Construct new linear difference equation by substitution $y(n) = u(n)/v(n)$ and solve it for $u(n)$ finding all its polynomial solutions. Return `None` if none were found.

Algorithm implemented here is a revised version of the original Abramov's algorithm, developed in 1989. The new approach is much simpler to implement and has better overall efficiency. This method can be easily adapted to q-difference equations case.

Besides finding rational solutions alone, this functions is an important part of Hyper algorithm were it is used to find particular solution of inhomogeneous part of a recurrence.

See also:

rsolve_hyper (page 921)

References

[R549] (page 1268)

Examples

```
>>> rsolve_ratio([-2*x**3 + x**2 + 2*x - 1, 2*x**3 + x**2 - 6*x,
... - 2*x**3 - 11*x**2 - 18*x - 9, 2*x**3 + 13*x**2 + 22*x + 8], 0, x)
C2*(2*x - 3)/(2*(x**2 - 1))
```

diofant.solvers.recurr.**rsolve_hyper**(*coeffs, f, n, **hints*)

Given linear recurrence operator L of order k with polynomial coefficients and inhomogeneous equation L$y = f$ we seek for all hypergeometric solutions over field K of characteristic zero.

The inhomogeneous part can be either hypergeometric or a sum of a fixed number of pairwise dissimilar hypergeometric terms.

The algorithm performs three basic steps:

1. Group together similar hypergeometric terms in the inhomogeneous part of L$y = f$, and find particular solution using Abramov's algorithm.
2. Compute generating set of L and find basis in it, so that all solutions are linearly independent.
3. Form final solution with the number of arbitrary constants equal to dimension of basis of L.

Term $a(n)$ is hypergeometric if it is annihilated by first order linear difference equations with polynomial coefficients or, in simpler words, if consecutive term ratio is a rational function.

The output of this procedure is a linear combination of fixed number of hypergeometric terms. However the underlying method can generate larger class of solutions - D'Alembertian terms.

Note also that this method not only computes the kernel of the inhomogeneous equation, but also reduces in to a basis so that solutions generated by this procedure are linearly independent

References

[R550] (page 1268), [R551] (page 1268)

Examples

```
>>> rsolve_hyper([-1, -1, 1], 0, x)
C0*(1/2 + sqrt(5)/2)**x + C1*(-sqrt(5)/2 + 1/2)**x
```

```
>>> rsolve_hyper([-1, 1], 1 + x, x)
C0 + x*(x + 1)/2
```

3.19.7 PDE

User Functions

These are functions that are imported into the global namespace with `from diofant import *`. They are intended for user use.

pde_separate

`diofant.solvers.pde.`**`pde_separate`**(*eq*, *fun*, *sep*, *strategy='mul'*)

Separate variables in partial differential equation either by additive or multiplicative separation approach. It tries to rewrite an equation so that one of the specified variables occurs on a different side of the equation than the others.

Parameters

- **eq** – Partial differential equation
- **fun** – Original function F(x, y, z)
- **sep** – List of separated functions [X(x), u(y, z)]
- **strategy** – Separation strategy. You can choose between additive separation ('add') and multiplicative separation ('mul') which is default.

See also:

diofant.solvers.pde.pde_separate_add (page 922), *diofant.solvers.pde.pde_separate_mul* (page 923)

Examples

```
>>> from diofant.abc import t
>>> u, X, T = map(Function, 'uXT')
```

```
>>> eq = Eq(Derivative(u(x, t), x), E**(u(x, t))*Derivative(u(x, t), t))
>>> pde_separate(eq, u(x, t), [X(x), T(t)], strategy='add')
[E**(-X(x))*Derivative(X(x), x), E**T(t)*Derivative(T(t), t)]
```

```
>>> eq = Eq(Derivative(u(x, t), x, 2), Derivative(u(x, t), t, 2))
>>> pde_separate(eq, u(x, t), [X(x), T(t)], strategy='mul')
[Derivative(X(x), x, x)/X(x), Derivative(T(t), t, t)/T(t)]
```

pde_separate_add

`diofant.solvers.pde.`**`pde_separate_add`**(*eq*, *fun*, *sep*)

Helper function for searching additive separable solutions.

Consider an equation of two independent variables x, y and a dependent variable w, we look for the product of two functions depending on different arguments:

$w(x, y, z) = X(x) + y(y, z)$

Examples

```
>>> from diofant.abc import t
>>> u, X, T = map(Function, 'uXT')
```

```
>>> eq = Eq(Derivative(u(x, t), x), E**(u(x, t))*Derivative(u(x, t), t))
>>> pde_separate_add(eq, u(x, t), [X(x), T(t)])
[E**(-X(x))*Derivative(X(x), x), E**T(t)*Derivative(T(t), t)]
```

pde_separate_mul

diofant.solvers.pde.**pde_separate_mul**(*eq*, *fun*, *sep*)

Helper function for searching multiplicative separable solutions.

Consider an equation of two independent variables x, y and a dependent variable w, we look for the product of two functions depending on different arguments:

$w(x, y, z) = X(x) * u(y, z)$

Examples

```
>>> u, X, Y = map(Function, 'uXY')
```

```
>>> eq = Eq(Derivative(u(x, y), x, 2), Derivative(u(x, y), y, 2))
>>> pde_separate_mul(eq, u(x, y), [X(x), Y(y)])
[Derivative(X(x), x, x)/X(x), Derivative(Y(y), y, y)/Y(y)]
```

pdsolve

diofant.solvers.pde.**pdsolve**(*eq*, *func=None*, *hint='default'*, *dict=False*, *solvefun=None*, ***kwargs*)

Solves any (supported) kind of partial differential equation.

Usage

pdsolve(eq, f(x,y), hint) -> Solve partial differential equation eq for function f(x,y), using method hint.

Details

eq can be any supported partial differential equation (see the pde docstring for supported methods). This can either be an Equality, or an expression, which is assumed to be equal to 0.

f(x,y) is a function of two variables whose derivatives in that variable make up the partial differential equation. In many cases it is not necessary to provide this; it will be autodetected (and an error raised if it couldn't be detected).

hint is the solving method that you want pdsolve to use. Use classify_pde(eq, f(x,y)) to get all of the possible hints for a PDE. The default hint, 'default', will use whatever hint is returned first by classify_pde(). See Hints below for more options that you can use for hint.

solvefun is the convention used for arbitrary functions returned by the PDE solver. If not set by the user, it is set by default to be F.

Hints

Aside from the various solving methods, there are also some meta-hints that you can pass to pdsolve():

"default": This uses whatever hint is returned first by classify_pde(). This is the default argument to pdsolve().

"all": To make pdsolve apply all relevant classification hints, use pdsolve(PDE, func, hint="all"). This will return a dictionary of hint:solution terms. If a hint causes pdsolve to raise the NotImplementedError, value of that hint's key will be the exception object raised. The dictionary will also include some special keys:

- order: The order of the PDE. See also ode_order() in deutils.py
- default: The solution that would be returned by default. This is the one produced by the hint that appears first in the tuple returned by classify_pde().

"all_Integral": This is the same as "all", except if a hint also has a corresponding "_Integral" hint, it only returns the "_Integral" hint. This is useful if "all" causes pdsolve() to hang because of a difficult or impossible integral. This meta-hint will also be much faster than "all", because integrate() is an expensive routine.

See also the classify_pde() docstring for more info on hints, and the pde docstring for a list of all supported hints.

Tips

- You can declare the derivative of an unknown function this way:

```
>>> f = Function("f")(x, y) # f is a function of x and y
>>> # fx will be the partial derivative of f with respect to x
>>> fx = Derivative(f, x)
>>> # fy will be the partial derivative of f with respect to y
>>> fy = Derivative(f, y)
```

- See test_pde.py for many tests, which serves also as a set of examples for how to use pdsolve().
- pdsolve always returns an Equality class (except for the case when the hint is "all" or "all_Integral"). Note that it is not possible to get an explicit solution for f(x, y) as in the case of ODE's
- Do help(pde.pde_hintname) to get help more information on a specific hint

Examples

```
>>> f = Function('f')
>>> u = f(x, y)
>>> ux = u.diff(x)
>>> uy = u.diff(y)
>>> eq = Eq(1 + (2*(ux/u)) + (3*(uy/u)))
```

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```
>>> pdsolve(eq)
Eq(f(x, y), E**(-2*x/13 - 3*y/13)*F(3*x - 2*y))
```

classify_pde

diofant.solvers.pde.**classify_pde**(*eq*, *func=None*, *dict=False*, ***kwargs*)

Returns a tuple of possible pdsolve() classifications for a PDE.

The tuple is ordered so that first item is the classification that pdsolve() uses to solve the PDE by default. In general, classifications near the beginning of the list will produce better solutions faster than those near the end, though there are always exceptions. To make pdsolve use a different classification, use pdsolve(PDE, func, hint=<classification>). See also the pdsolve() docstring for different meta-hints you can use.

If `dict` is true, classify_pde() will return a dictionary of hint:match expression terms. This is intended for internal use by pdsolve(). Note that because dictionaries are ordered arbitrarily, this will most likely not be in the same order as the tuple.

You can get help on different hints by doing help(pde.pde_hintname), where hintname is the name of the hint without "_Integral".

See diofant.pde.allhints or the diofant.pde docstring for a list of all supported hints that can be returned from classify_pde.

Examples

```
>>> u = f(x, y)
>>> ux = u.diff(x)
>>> uy = u.diff(y)
>>> eq = Eq(1 + (2*(ux/u)) + (3*(uy/u)))
>>> classify_pde(eq)
('1st_linear_constant_coeff_homogeneous',)
```

checkpdesol

diofant.solvers.pde.**checkpdesol**(*pde*, *sol*, *func=None*, *solve_for_func=True*)

Checks if the given solution satisfies the partial differential equation.

pde is the partial differential equation which can be given in the form of an equation or an expression. sol is the solution for which the pde is to be checked. This can also be given in an equation or an expression form. If the function is not provided, the helper function _preprocess from deutils is used to identify the function.

If a sequence of solutions is passed, the same sort of container will be used to return the result for each solution.

The following methods are currently being implemented to check if the solution satisfies the PDE:

1. Directly substitute the solution in the PDE and check. If the solution hasn't been solved for f, then it will solve for f provided solve_for_func hasn't been set to False.

If the solution satisfies the PDE, then a tuple (True, 0) is returned. Otherwise a tuple (False, expr) where expr is the value obtained after substituting the solution in the PDE. However if a known solution returns False, it may be due to the inability of doit() to simplify it to zero.

Examples

```
>>> eq = 2*f(x, y) + 3*f(x, y).diff(x) + 4*f(x, y).diff(y)
>>> sol = pdsolve(eq)
>>> assert checkpdesol(eq, sol)[0]
>>> eq = x*f(x, y) + f(x, y).diff(x)
>>> checkpdesol(eq, sol)
(False, E**(-6*x/25 - 8*y/25)*(x*F(4*x - 3*y) - 6*F(4*x - 3*y)/25 +
↪4*Subs(Derivative(F(_xi_1), _xi_1), (_xi_1,), (4*x - 3*y,))))
```

Hint Methods

These functions are meant for internal use. However they contain useful information on the various solving methods.

pde_1st_linear_constant_coeff_homogeneous

diofant.solvers.pde.**pde_1st_linear_constant_coeff_homogeneous**(*eq, func, order, match, solvefun*)

Solves a first order linear homogeneous partial differential equation with constant coefficients.

The general form of this partial differential equation is

$$a\frac{df(x,y)}{dx} + b\frac{df(x,y)}{dy} + cf(x,y) = 0$$

where a, b and c are constants.

The general solution is of the form:

```
>>> from diofant.abc import a, b, c
>>> u = f(x, y)
>>> ux = u.diff(x)
>>> uy = u.diff(y)
>>> genform = a*ux + b*uy + c*u
>>> pprint(genform, use_unicode=False)
  d               d
a*--(f(x, y)) + b*--(f(x, y)) + c*f(x, y)
  dx              dy

>>> pprint(pdsolve(genform), use_unicode=False)
           -c*(a*x + b*y)
           ---------------
                2    2
               a  + b
f(x, y) = E               *F(-a*y + b*x)
```

References

- Viktor Grigoryan, “Partial Differential Equations” Math 124A - Fall 2010, pp.7

Examples

```
>>> pdsolve(f(x, y) + f(x, y).diff(x) + f(x, y).diff(y))
Eq(f(x, y), E**(-x/2 - y/2)*F(x - y))
>>> pprint(pdsolve(f(x, y) + f(x, y).diff(x) + f(x, y).diff(y)), use_
↪unicode=False)
             x   y
           - - - -
             2   2
f(x, y) = E        *F(x - y)
```

pde_1st_linear_constant_coeff

diofant.solvers.pde.**pde_1st_linear_constant_coeff**(*eq, func, order, match, solvefun*)

Solves a first order linear partial differential equation with constant coefficients.

The general form of this partial differential equation is

$$a\frac{df(x,y)}{dx} + b\frac{df(x,y)}{dy} + cf(x,y) = G(x,y)$$

where a, b and c are constants and $G(x, y)$ can be an arbitrary function in x and y.

The general solution of the PDE is:

```
>>> from diofant.abc import a, b, c
>>> f = Function('f')
>>> G = Function('G')
>>> u = f(x, y)
>>> ux = u.diff(x)
>>> uy = u.diff(y)
>>> genform = a*u + b*ux + c*uy - G(x, y)
>>> pprint(genform, use_unicode=False)
          d               d
a*f(x, y) + b*--(f(x, y)) + c*--(f(x, y)) - G(x, y)
          dx              dy
>>> pprint(pdsolve(genform, hint='1st_linear_constant_coeff_Integral'), use_
↪unicode=False)
          /         /          b*x + c*y
          |         |              /
          |         |             |
          |         |             |       a*xi
          |         |             |     -------
          |         |             |       2    2
          |         |             |      b  + c    /b*xi + c*eta  -b*eta + c*xi
          |         |             |     E        *G|------------, -------------
          |         |             |                |   2    2         2    2
          |  -a*xi  |             |                \  b  + c         b  + c
          | ------- |             |
          |  2    2 |            /
```

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```
             | b  + c  |
f(x, y) = |E       *|F(eta) + -----------------------------------------------
             |         |                                          2    2
             \         \                                        b  + c

        \\|
        |||
        |||
        |||
        |||
        |||
\       |||
| d(xi)|||
|       |||
/       |||
        |||
        |||
        |||
-------|||
        |||
        //|eta=-b*y + c*x, xi=b*x + c*y
```

References

- Viktor Grigoryan, "Partial Differential Equations" Math 124A - Fall 2010, pp.7

Examples

```
>>> eq = -2*f(x, y).diff(x) + 4*f(x, y).diff(y) + 5*f(x, y) - exp(x + 3*y)
>>> pdsolve(eq)
Eq(f(x, y), E**(x/2 - y)*(E**(x/2 + 4*y)/15 + F(4*x + 2*y)))
```

pde_1st_linear_variable_coeff

`diofant.solvers.pde.`**`pde_1st_linear_variable_coeff`**(*eq*, *func*, *order*, *match*, *solvefun*)

Solves a first order linear partial differential equation with variable coefficients. The general form of this partial differential equation is

$$a(x,y)\frac{df(x,y)}{dx} + a(x,y)\frac{df(x,y)}{dy} + c(x,y)f(x,y) - G(x,y)$$

where $a(x,y)$, $b(x,y)$, $c(x,y)$ and $G(x,y)$ are arbitrary functions in x and y. This PDE is converted into an ODE by making the following transformation.

1] ξ as x

2] η as the constant in the solution to the differential equation $\frac{dy}{dx} = -\frac{b}{a}$

Making the following substitutions reduces it to the linear ODE

$$a(\xi,\eta)\frac{du}{d\xi} + c(\xi,\eta)u - d(\xi,\eta) = 0$$

which can be solved using dsolve.

The general form of this PDE is:

```
>>> a, b, c, G, f= [Function(i) for i in ['a', 'b', 'c', 'G', 'f']]
>>> u = f(x, y)
>>> ux = u.diff(x)
>>> uy = u.diff(y)
>>> genform = a(x, y)*u + b(x, y)*ux + c(x, y)*uy - G(x, y)
>>> pprint(genform, use_unicode=False)
                                     d                     d
-G(x, y) + a(x, y)*f(x, y) + b(x, y)*--(f(x, y)) + c(x, y)*--(f(x, y))
                                     dx                    dy
```

References

- Viktor Grigoryan, "Partial Differential Equations" Math 124A - Fall 2010, pp.7

Examples

```
>>> f = Function('f')
>>> eq =  x*(u.diff(x)) - y*(u.diff(y)) + y**2*u - y**2
>>> pdsolve(eq)
Eq(f(x, y), E**(y**2/2)*F(x*y) + 1)
```

Information on the pde module

This module contains pdsolve() and different helper functions that it uses. It is heavily inspired by the ode module and hence the basic infrastructure remains the same.

Functions in this module

These are the user functions in this module:

- pdsolve() - Solves PDE's
- classify_pde() - Classifies PDEs into possible hints for dsolve().
- **pde_separate() - Separate variables in partial differential equation either by** additive or multiplicative separation approach.

These are the helper functions in this module:

- pde_separate_add() - Helper function for searching additive separable solutions.
- **pde_separate_mul() - Helper function for searching multiplicative** separable solutions.

Currently implemented solver methods

The following methods are implemented for solving partial differential equations. See the docstrings of the various pde_hint() functions for more information on each (run help(pde)):

- 1st order linear homogeneous partial differential equations with constant coefficients.
- 1st order linear general partial differential equations with constant coefficients.

- 1st order linear partial differential equations with variable coefficients.

3.19.8 Utilities for solving

diofant.solvers.deutils.**ode_order**(*expr*, *func*)

Returns the order of a given differential equation with respect to func.

This function is implemented recursively.

Examples

```
>>> ode_order(f(x).diff(x, 2) + f(x).diff(x)**2 +
... f(x).diff(x), f(x))
2
>>> ode_order(f(x).diff(x, 2) + g(x).diff(x, 3), f(x))
2
>>> ode_order(f(x).diff(x, 2) + g(x).diff(x, 3), g(x))
3
```

3.20 Tensors

A module to manipulate symbolic objects with indices including tensors

3.20.1 N-dim array

N-dim array module.

Four classes are provided to handle N-dim arrays, given by the combinations dense/sparse (i.e. whether to store all elements or only the non-zero ones in memory) and mutable/immutable (immutable classes are Diofant objects, but cannot change after they have been created).

Examples

The following examples show the usage of Array. This is an abbreviation for ImmutableDenseNDimArray, that is an immutable and dense N-dim array, the other classes are analogous. For mutable classes it is also possible to change element values after the object has been constructed.

Array construction can detect the shape of nested lists and tuples:

```
>>> from diofant.tensor.array import Array
>>> a1 = Array([[1, 2], [3, 4], [5, 6]])
>>> a1
[[1, 2], [3, 4], [5, 6]]
>>> a1.shape
(3, 2)
>>> a1.rank()
2
>>> a2 = Array([[[x, y], [z, x*z]], [[1, x*y], [1/x, x/y]]])
```

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```
>>> a2
[[[x, y], [z, x*z]], [[1, x*y], [1/x, x/y]]]
>>> a2.shape
(2, 2, 2)
>>> a2.rank()
3
```

Otherwise one could pass a 1-dim array followed by a shape tuple:

```
>>> m1 = Array(range(12), (3, 4))
>>> m1
[[0, 1, 2, 3], [4, 5, 6, 7], [8, 9, 10, 11]]
>>> m2 = Array(range(12), (3, 2, 2))
>>> m2
[[[0, 1], [2, 3]], [[4, 5], [6, 7]], [[8, 9], [10, 11]]]
>>> m2[1, 1, 1]
7
>>> m2.reshape(4, 3)
[[0, 1, 2], [3, 4, 5], [6, 7, 8], [9, 10, 11]]
```

Slice support:

```
>>> m2[:, 1, 1]
[3, 7, 11]
```

Elementwise derivative:

```
>>> m3 = Array([x**3, x*y, z])
>>> m3.diff(x)
[3*x**2, y, 0]
>>> m3.diff(z)
[0, 0, 1]
```

Multiplication with other Diofant expressions is applied elementwisely:

```
>>> (1+x)*m3
[x**3*(x + 1), x*y*(x + 1), z*(x + 1)]
```

To apply a function to each element of the N-dim array, use `applyfunc`:

```
>>> m3.applyfunc(lambda x: x/2)
[x**3/2, x*y/2, z/2]
```

N-dim arrays can be converted to nested lists by the `tolist()` method:

```
>>> m2.tolist()
[[[0, 1], [2, 3]], [[4, 5], [6, 7]], [[8, 9], [10, 11]]]
```

If the rank is 2, it is possible to convert them to matrices with `tomatrix()`:

```
>>> m1.tomatrix()
Matrix([
[0, 1,  2,  3],
[4, 5,  6,  7],
[8, 9, 10, 11]])
```

Products and contractions

Tensor product between arrays $A_{i_1,\ldots,i_n}$ and $B_{j_1,\ldots,j_m}$ creates the combined array $P = A \otimes B$ defined as

$$P_{i_1,\ldots,i_n,j_1,\ldots,j_m} := A_{i_1,\ldots,i_n} \cdot B_{j_1,\ldots,j_m}$$

It is available through `tensorproduct(...)`:

```
>>> from diofant.tensor.array import Array, tensorproduct
>>> from diofant.abc import t
>>> A = Array([x, y, z, t])
>>> B = Array([1, 2, 3, 4])
>>> tensorproduct(A, B)
[[x, 2*x, 3*x, 4*x], [y, 2*y, 3*y, 4*y], [z, 2*z, 3*z, 4*z],
 [t, 2*t, 3*t, 4*t]]
```

Tensor product between a rank-1 array and a matrix creates a rank-3 array:

```
>>> p1 = tensorproduct(A, eye(4))
>>> p1
[[[x, 0, 0, 0], [0, x, 0, 0], [0, 0, x, 0], [0, 0, 0, x]],
 [[y, 0, 0, 0], [0, y, 0, 0], [0, 0, y, 0], [0, 0, 0, y]],
 [[z, 0, 0, 0], [0, z, 0, 0], [0, 0, z, 0], [0, 0, 0, z]],
 [[t, 0, 0, 0], [0, t, 0, 0], [0, 0, t, 0], [0, 0, 0, t]]]
```

Now, to get back $A_0 \otimes \mathbf{1}$ one can access $p_{0,m,n}$ by slicing:

```
>>> p1[0, :, :]
[[x, 0, 0, 0], [0, x, 0, 0], [0, 0, x, 0], [0, 0, 0, x]]
```

Tensor contraction sums over the specified axes, for example contracting positions a and b means

$$A_{i_1,\ldots,i_a,\ldots,i_b,\ldots,i_n} \implies \sum_k A_{i_1,\ldots,k,\ldots,k,\ldots,i_n}$$

Remember that Python indexing is zero starting, to contract the a-th and b-th axes it is therefore necessary to specify $a-1$ and $b-1$

```
>>> from diofant.tensor.array import tensorcontraction
>>> C = Array([[x, y], [z, t]])
```

The matrix trace is equivalent to the contraction of a rank-2 array:

$A_{m,n} \implies \sum_k A_{k,k}$

```
>>> tensorcontraction(C, (0, 1))
t + x
```

Matrix product is equivalent to a tensor product of two rank-2 arrays, followed by a contraction of the 2nd and 3rd axes (in Python indexing axes number 1, 2).

$A_{m,n} \cdot B_{i,j} \implies \sum_k A_{m,k} \cdot B_{k,j}$

```
>>> D = Array([[2, 1], [0, -1]])
>>> tensorcontraction(tensorproduct(C, D), (1, 2))
[[2*x, x - y], [2*z, -t + z]]
```

One may verify that the matrix product is equivalent:

```
>>> Matrix([[x, y], [z, t]])*Matrix([[2, 1], [0, -1]])
Matrix([
[2*x,  x - y],
[2*z, -t + z]])
```

or equivalently

```
>>> C.tomatrix()*D.tomatrix()
Matrix([
[2*x,  x - y],
[2*z, -t + z]])
```

Derivatives by array

The usual derivative operation may be extended to support derivation with respect to arrays, provided that all elements in the that array are symbols or expressions suitable for derivations.

The definition of a derivative by an array is as follows: given the array $A_{i_1,\ldots,i_N}$ and the array $X_{j_1,\ldots,j_M}$ the derivative of arrays will return a new array B defined by

$B_{j_1,\ldots,j_M,i_1,\ldots,i_N} := \frac{\partial A_{i_1,\ldots,i_N}}{\partial X_{j_1,\ldots,j_M}}$

The function `derive_by_array` performs such an operation:

```
>>> from diofant.tensor.array import Array, tensorcontraction, derive_by_array
>>> from diofant.abc import t
```

With scalars, it behaves exactly as the ordinary derivative:

```
>>> derive_by_array(sin(x*y), x)
y*cos(x*y)
```

Scalar derived by an array basis:

```
>>> derive_by_array(sin(x*y), [x, y, z])
[y*cos(x*y), x*cos(x*y), 0]
```

Deriving array by an array basis: $B^{nm} := \frac{\partial A^m}{\partial x^n}$

```
>>> basis = [x, y, z]
>>> ax = derive_by_array([exp(x), sin(y*z), t], basis)
>>> ax
[[E**x, 0, 0], [0, z*cos(y*z), 0], [0, y*cos(y*z), 0]]
```

Contraction of the resulting array: $\sum_m \frac{\partial A^m}{\partial x^m}$

```
>>> tensorcontraction(ax, (0, 1))
E**x + z*cos(y*z)
```

Classes

class diofant.tensor.array.**ImmutableDenseNDimArray**

class diofant.tensor.array.**ImmutableSparseNDimArray**

class diofant.tensor.array.**MutableDenseNDimArray**

class diofant.tensor.array.**MutableSparseNDimArray**

Functions

diofant.tensor.array.**derive_by_array**(*expr*, *dx*)
Derivative by arrays. Supports both arrays and scalars.

Given the array $A_{i_1,\ldots,i_N}$ and the array $X_{j_1,\ldots,j_M}$ this function will return a new array B defined by

$B_{j_1,\ldots,j_M,i_1,\ldots,i_N} := \frac{\partial A_{i_1,\ldots,i_N}}{\partial X_{j_1,\ldots,j_M}}$

Examples

```
>>> from diofant.abc import t
>>> derive_by_array(cos(x*t), x)
-t*sin(t*x)
>>> derive_by_array(cos(x*t), [x, y, z, t])
[-t*sin(t*x), 0, 0, -x*sin(t*x)]
>>> derive_by_array([x, y**2*z], [[x, y], [z, t]])
[[[1, 0], [0, 2*y*z]], [[0, y**2], [0, 0]]]
```

diofant.tensor.array.**permutedims**(*expr*, *perm*)
Permutes the indices of an array.

Parameter specifies the permutation of the indices.

Examples

```
>>> from diofant.abc import t
>>> from diofant.tensor.array import Array
>>> a = Array([[x, y, z], [t, sin(x), 0]])
>>> a
[[x, y, z], [t, sin(x), 0]]
>>> permutedims(a, (1, 0))
[[x, t], [y, sin(x)], [z, 0]]
```

If the array is of second order, `transpose` can be used:

```
>>> transpose(a)
[[x, t], [y, sin(x)], [z, 0]]
```

Examples on higher dimensions:

```
>>> b = Array([[[1, 2], [3, 4]], [[5, 6], [7, 8]]])
>>> permutedims(b, (2, 1, 0))
[[[1, 5], [3, 7]], [[2, 6], [4, 8]]]
>>> permutedims(b, (1, 2, 0))
[[[1, 5], [2, 6]], [[3, 7], [4, 8]]]
```

`Permutation` objects are also allowed:

```
>>> permutedims(b, Permutation([1, 2, 0]))
[[[1, 5], [2, 6]], [[3, 7], [4, 8]]]
```

diofant.tensor.array.**tensorcontraction**(*array, *contraction_axes*)

Contraction of an array-like object on the specified axes.

Examples

```
>>> from diofant.tensor.array import Array
>>> tensorcontraction(eye(3), (0, 1))
3
>>> A = Array(range(18), (3, 2, 3))
>>> A
[[[0, 1, 2], [3, 4, 5]], [[6, 7, 8], [9, 10, 11]],
 [[12, 13, 14], [15, 16, 17]]]
>>> tensorcontraction(A, (0, 2))
[21, 30]
```

Matrix multiplication may be emulated with a proper combination of `tensorcontraction` and `tensorproduct`

```
>>> from diofant.abc import a, b, c, d, e, f, g, h
>>> m1 = Matrix([[a, b], [c, d]])
>>> m2 = Matrix([[e, f], [g, h]])
>>> p = tensorproduct(m1, m2)
>>> p
[[[[a*e, a*f], [a*g, a*h]], [[b*e, b*f], [b*g, b*h]]],
 [[[c*e, c*f], [c*g, c*h]], [[d*e, d*f], [d*g, d*h]]]]
>>> tensorcontraction(p, (1, 2))
[[a*e + b*g, a*f + b*h], [c*e + d*g, c*f + d*h]]
>>> m1*m2
Matrix([
[a*e + b*g, a*f + b*h],
[c*e + d*g, c*f + d*h]])
```

diofant.tensor.array.**tensorproduct**(**args*)

Tensor product among scalars or array-like objects.

Examples

```
>>> from diofant.tensor.array import Array
>>> from diofant.abc import t
>>> A = Array([[1, 2], [3, 4]])
>>> B = Array([x, y])
>>> tensorproduct(A, B)
[[[x, y], [2*x, 2*y]], [[3*x, 3*y], [4*x, 4*y]]]
>>> tensorproduct(A, x)
[[x, 2*x], [3*x, 4*x]]
>>> tensorproduct(A, B, B)
[[[[x**2, x*y], [x*y, y**2]], [[2*x**2, 2*x*y], [2*x*y, 2*y**2]]],
 [[[3*x**2, 3*x*y], [3*x*y, 3*y**2]], [[4*x**2, 4*x*y], [4*x*y, 4*y**2]]]]
```

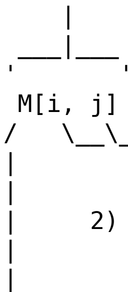
Applying this function on two matrices will result in a rank 4 array.

```
>>> m = Matrix([[x, y], [z, t]])
>>> p = tensorproduct(eye(3), m)
>>> p
[[[[x, y], [z, t]], [[0, 0], [0, 0]], [[0, 0], [0, 0]]],
 [[[0, 0], [0, 0]], [[x, y], [z, t]], [[0, 0], [0, 0]]],
 [[[0, 0], [0, 0]], [[0, 0], [0, 0]], [[x, y], [z, t]]]]
```

3.20.2 Indexed Objects

Module that defines indexed objects

The classes IndexedBase, Indexed and Idx would represent a matrix element M[i, j] as in the following graph:



```
1) The Indexed class represents the entire indexed object.
          |
       ___|___
      '       '
       M[i, j]
      /   \__\______
      |             |
      |             |
      |     2) The Idx class represent indices and each Idx can
      |        optionally contain information about its range.
      |
3) IndexedBase represents the `stem' of an indexed object, here `M'.
The stem used by itself is usually taken to represent the entire array.
```

There can be any number of indices on an Indexed object. No transformation properties are implemented in these Base objects, but implicit contraction of repeated indices is supported.

Note that the support for complicated (i.e. non-atomic) integer expressions as indices is limited. (This should be improved in future releases.)

Examples

To express the above matrix element example you would write:

```
>>> M = IndexedBase('M')
>>> i, j = symbols('i j', cls=Idx)
>>> M[i, j]
M[i, j]
```

Repeated indices in a product implies a summation, so to express a matrix-vector product in terms of Indexed objects:

```
>>> x = IndexedBase('x')
>>> M[i, j]*x[j]
x[j]*M[i, j]
```

If the indexed objects will be converted to component based arrays, e.g. with the code printers or the autowrap framework, you also need to provide (symbolic or numerical) dimensions. This can be done by passing an optional shape parameter to IndexedBase upon construction:

```
>>> dim1, dim2 = symbols('dim1 dim2', integer=True)
>>> A = IndexedBase('A', shape=(dim1, 2*dim1, dim2))
>>> A.shape
(dim1, 2*dim1, dim2)
>>> A[i, j, 3].shape
(dim1, 2*dim1, dim2)
```

If an IndexedBase object has no shape information, it is assumed that the array is as large as the ranges of its indices:

```
>>> n, m = symbols('n m', integer=True)
>>> i = Idx('i', m)
>>> j = Idx('j', n)
>>> M[i, j].shape
(m, n)
>>> M[i, j].ranges
[(0, m - 1), (0, n - 1)]
```

The above can be compared with the following:

```
>>> A[i, 2, j].shape
(dim1, 2*dim1, dim2)
>>> A[i, 2, j].ranges
[(0, m - 1), None, (0, n - 1)]
```

To analyze the structure of indexed expressions, you can use the methods get_indices() and get_contraction_structure():

```
>>> get_indices(A[i, j, j])
({i}, {})
>>> get_contraction_structure(A[i, j, j])
{(j,): {A[i, j, j]}}
```

See the appropriate docstrings for a detailed explanation of the output.

class diofant.tensor.indexed.**Idx**

Represents an integer index as an Integer or integer expression.

There are a number of ways to create an Idx object. The constructor takes two arguments:

label An integer or a symbol that labels the index.

range Optionally you can specify a range as either

- Symbol or integer: This is interpreted as a dimension. Lower and upper bounds are set to 0 and range - 1, respectively.
- tuple: The two elements are interpreted as the lower and upper bounds of the range, respectively.

Note: the Idx constructor is rather pedantic in that it only accepts integer arguments. The only exception is that you can use oo and -oo to specify an unbounded range. For all other cases, both label and bounds must be declared as integers, e.g. if n is given as an argument then n.is_integer must return True.

For convenience, if the label is given as a string it is automatically converted to an integer symbol. (Note: this conversion is not done for range or dimension arguments.)

Examples

```
>>> n, i, L, U = symbols('n i L U', integer=True)
```

If a string is given for the label an integer Symbol is created and the bounds are both None:

```
>>> idx = Idx('qwerty'); idx
qwerty
>>> idx.lower, idx.upper
(None, None)
```

Both upper and lower bounds can be specified:

```
>>> idx = Idx(i, (L, U)); idx
i
>>> idx.lower, idx.upper
(L, U)
```

When only a single bound is given it is interpreted as the dimension and the lower bound defaults to 0:

```
>>> idx = Idx(i, n); idx.lower, idx.upper
(0, n - 1)
>>> idx = Idx(i, 4); idx.lower, idx.upper
(0, 3)
>>> idx = Idx(i, oo); idx.lower, idx.upper
(0, oo)
```

The label can be a literal integer instead of a string/Symbol:

```
>>> idx = Idx(2, n); idx.lower, idx.upper
(0, n - 1)
>>> idx.label
2
```

label
Returns the label (Integer or integer expression) of the Idx object.

Examples

```
>>> Idx(2).label
2
>>> j = Symbol('j', integer=True)
>>> Idx(j).label
j
>>> Idx(j + 1).label
j + 1
```

lower
Returns the lower bound of the Index.

Examples

```
>>> Idx('j', 2).lower
0
>>> Idx('j', 5).lower
0
>>> Idx('j').lower is None
True
```

upper
Returns the upper bound of the Index.

Examples

```
>>> Idx('j', 2).upper
1
>>> Idx('j', 5).upper
4
>>> Idx('j').upper is None
True
```

exception diofant.tensor.indexed.**IndexException**

class diofant.tensor.indexed.**Indexed**
Represents a mathematical object with indices.

```
>>> i, j = symbols('i j', cls=Idx)
>>> Indexed('A', i, j)
A[i, j]
```

It is recommended that Indexed objects are created via IndexedBase:

```
>>> A = IndexedBase('A')
>>> Indexed('A', i, j) == A[i, j]
True
```

base
Returns the IndexedBase of the Indexed object.

Examples

```
>>> i, j = symbols('i j', cls=Idx)
>>> Indexed('A', i, j).base
A
>>> B = IndexedBase('B')
>>> B == B[i, j].base
True
```

indices
Returns the indices of the Indexed object.

Examples

```
>>> i, j = symbols('i j', cls=Idx)
>>> Indexed('A', i, j).indices
(i, j)
```

ranges

Returns a list of tuples with lower and upper range of each index.

If an index does not define the data members upper and lower, the corresponding slot in the list contains None instead of a tuple.

Examples

```
>>> Indexed('A', Idx('i', 2), Idx('j', 4), Idx('k', 8)).ranges
[(0, 1), (0, 3), (0, 7)]
>>> Indexed('A', Idx('i', 3), Idx('j', 3), Idx('k', 3)).ranges
[(0, 2), (0, 2), (0, 2)]
>>> x, y, z = symbols('x y z', integer=True)
>>> Indexed('A', x, y, z).ranges
[None, None, None]
```

rank

Returns the rank of the Indexed object.

Examples

```
>>> i, j, k, l, m = symbols('i:m', cls=Idx)
>>> Indexed('A', i, j).rank
2
>>> q = Indexed('A', i, j, k, l, m)
>>> q.rank
5
>>> q.rank == len(q.indices)
True
```

shape

Returns a list with dimensions of each index.

Dimensions is a property of the array, not of the indices. Still, if the IndexedBase does not define a shape attribute, it is assumed that the ranges of the indices correspond to the shape of the array.

```
>>> n, m = symbols('n m', integer=True)
>>> i = Idx('i', m)
>>> j = Idx('j', m)
>>> A = IndexedBase('A', shape=(n, n))
>>> B = IndexedBase('B')
>>> A[i, j].shape
(n, n)
>>> B[i, j].shape
(m, m)
```

class diofant.tensor.indexed.**IndexedBase**

Represent the base or stem of an indexed object

The IndexedBase class represent an array that contains elements. The main purpose of this class is to allow the convenient creation of objects of the Indexed class. The __getitem__ method of IndexedBase returns an instance of Indexed. Alone, without indices, the IndexedBase class can be used as a notation for e.g. matrix equations, resembling what you could do with the Symbol class. But, the IndexedBase class adds functionality that is not available for Symbol instances:

- An IndexedBase object can optionally store shape information. This can be used in to check array conformance and conditions for numpy broadcasting. (TODO)
- An IndexedBase object implements syntactic sugar that allows easy symbolic representation of array operations, using implicit summation of repeated indices.
- The IndexedBase object symbolizes a mathematical structure equivalent to arrays, and is recognized as such for code generation and automatic compilation and wrapping.

```
>>> A = IndexedBase('A'); A
A
>>> type(A)
<class 'diofant.tensor.indexed.IndexedBase'>
```

When an IndexedBase object receives indices, it returns an array with named axes, represented by an Indexed object:

```
>>> i, j = symbols('i j', integer=True)
>>> A[i, j, 2]
A[i, j, 2]
>>> type(A[i, j, 2])
<class 'diofant.tensor.indexed.Indexed'>
```

The IndexedBase constructor takes an optional shape argument. If given, it overrides any shape information in the indices. (But not the index ranges!)

```
>>> m, n, o, p = symbols('m n o p', integer=True)
>>> i = Idx('i', m)
>>> j = Idx('j', n)
>>> A[i, j].shape
(m, n)
>>> B = IndexedBase('B', shape=(o, p))
>>> B[i, j].shape
(o, p)
```

args

Returns the arguments used to create this IndexedBase object.

Examples

```
>>> IndexedBase('A', shape=(x, y)).args
(A, (x, y))
```

label

Returns the label of the IndexedBase object.

Examples

```
>>> IndexedBase('A', shape=(x, y)).label
A
```

shape

Returns the shape of the IndexedBase object.

Examples

```
>>> IndexedBase('A', shape=(x, y)).shape
(x, y)
```

Note: If the shape of the IndexedBase is specified, it will override any shape information given by the indices.

```
>>> A = IndexedBase('A', shape=(x, y))
>>> B = IndexedBase('B')
>>> i = Idx('i', 2)
>>> j = Idx('j', 1)
>>> A[i, j].shape
(x, y)
>>> B[i, j].shape
(2, 1)
```

3.20.3 Methods

Module with functions operating on IndexedBase, Indexed and Idx objects

- Check shape conformance
- Determine indices in resulting expression

etc.

Methods in this module could be implemented by calling methods on Expr objects instead. When things stabilize this could be a useful refactoring.

exception diofant.tensor.index_methods.**IndexConformanceException**

diofant.tensor.index_methods.**get_contraction_structure**(*expr*)

Determine dummy indices of `expr` and describe its structure

By *dummy* we mean indices that are summation indices.

The structure of the expression is determined and described as follows:

1. A conforming summation of Indexed objects is described with a dict where the keys are summation indices and the corresponding values are sets containing all terms for which the summation applies. All Add objects in the Diofant expression tree are described like this.
2. For all nodes in the Diofant expression tree that are *not* of type Add, the following applies:

   If a node discovers contractions in one of its arguments, the node itself will be stored as a key in the dict. For that key, the corresponding value is a list of dicts, each of

which is the result of a recursive call to get_contraction_structure(). The list contains only dicts for the non-trivial deeper contractions, omitting dicts with None as the one and only key.

Note: The presence of expressions among the dictionary keys indicates multiple levels of index contractions. A nested dict displays nested contractions and may itself contain dicts from a deeper level. In practical calculations the summation in the deepest nested level must be calculated first so that the outer expression can access the resulting indexed object.

Examples

```
>>> x, y, A = map(IndexedBase, ['x', 'y', 'A'])
>>> i, j, k, l = map(Idx, ['i', 'j', 'k', 'l'])
>>> get_contraction_structure(x[i]*y[i] + A[j, j])
{(i,): {x[i]*y[i]}, (j,): {A[j, j]}}
>>> get_contraction_structure(x[i]*y[j])
{None: {x[i]*y[j]}}
```

A multiplication of contracted factors results in nested dicts representing the internal contractions.

```
>>> d = get_contraction_structure(x[i, i]*y[j, j])
>>> sorted(d, key=default_sort_key)
[None, x[i, i]*y[j, j]]
```

In this case, the product has no contractions:

```
>>> d[None]
{x[i, i]*y[j, j]}
```

Factors are contracted "first":

```
>>> sorted(d[x[i, i]*y[j, j]], key=default_sort_key)
[{(i,): {x[i, i]}}, {(j,): {y[j, j]}}]
```

A parenthesized Add object is also returned as a nested dictionary. The term containing the parenthesis is a Mul with a contraction among the arguments, so it will be found as a key in the result. It stores the dictionary resulting from a recursive call on the Add expression.

```
>>> d = get_contraction_structure(x[i]*(y[i] + A[i, j]*x[j]))
>>> sorted(d, key=default_sort_key)
[(x[j]*A[i, j] + y[i])*x[i], (i,)]
>>> d[(i,)]
{(x[j]*A[i, j] + y[i])*x[i]}
>>> d[x[i]*(A[i, j]*x[j] + y[i])]
[{None: {y[i]}, (j,): {x[j]*A[i, j]}}]
```

Powers with contractions in either base or exponent will also be found as keys in the dictionary, mapping to a list of results from recursive calls:

```
>>> d = get_contraction_structure(A[j, j]**A[i, i])
>>> d[None]
{A[j, j]**A[i, i]}
>>> nested_contractions = d[A[j, j]**A[i, i]]
>>> nested_contractions[0]
{(j,): {A[j, j]}}
>>> nested_contractions[1]
{(i,): {A[i, i]}}
```

The description of the contraction structure may appear complicated when represented with a string in the above examples, but it is easy to iterate over:

```
>>> for key in d:
...     if isinstance(key, Expr):
...         continue
...     for term in d[key]:
...         if term in d:
...             # treat deepest contraction first
...             pass
...     # treat outermost contactions here
```

`diofant.tensor.index_methods.`**`get_indices`**`(`*`expr`*`)`

Determine the outer indices of expression `expr`

By *outer* we mean indices that are not summation indices. Returns a set and a dict. The set contains outer indices and the dict contains information about index symmetries.

Examples

```
>>> x, y, A = map(IndexedBase, ['x', 'y', 'A'])
>>> i, j, a, z = symbols('i j a z', integer=True)
```

The indices of the total expression is determined, Repeated indices imply a summation, for instance the trace of a matrix A:

```
>>> get_indices(A[i, i])
(set(), {})
```

In the case of many terms, the terms are required to have identical outer indices. Else an IndexConformanceException is raised.

```
>>> get_indices(x[i] + A[i, j]*y[j])
({i}, {})
```

Exceptions

An IndexConformanceException means that the terms are not compatible, e.g.

```
>>> get_indices(x[i] + y[j])
Traceback (most recent call last):
...
IndexConformanceException: Indices are not consistent: x(i) + y(j)
```

Warning: The concept of *outer* indices applies recursively, starting on the deepest level. This implies that dummies inside parenthesis are assumed to be summed first, so that the following expression is handled gracefully:

```
>>> get_indices((x[i] + A[i, j]*y[j])*x[j])
({i, j}, {})
```

This is correct and may appear convenient, but you need to be careful with this as Diofant will happily .expand() the product, if requested. The resulting expression would mix the outer `j` with the dummies inside the parenthesis, which makes it a different expression. To be on the safe side, it is best to avoid such ambiguities by using unique indices for all contractions that should be held separate.

3.20.4 Tensor

class `diofant.tensor.tensor.`**`_TensorManager`**

Class to manage tensor properties.

Notes

Tensors belong to tensor commutation groups; each group has a label `comm`; there are predefined labels:

`0` tensors commuting with any other tensor

`1` tensors anticommuting among themselves

`2` tensors not commuting, apart with those with `comm=0`

Other groups can be defined using `set_comm`; tensors in those groups commute with those with `comm=0`; by default they do not commute with any other group.

`clear`()

Clear the TensorManager.

`comm_i2symbol`(*i*)

Returns the symbol corresponding to the commutation group number.

`comm_symbols2i`(*i*)

get the commutation group number corresponding to `i`

`i` can be a symbol or a number or a string

If `i` is not already defined its commutation group number is set.

`get_comm`(*i*, *j*)

Return the commutation parameter for commutation group numbers `i, j`

see `_TensorManager.set_comm`

`set_comm`(*i*, *j*, *c*)

set the commutation parameter `c` for commutation groups `i, j`

Parameters **i, j** : symbols representing commutation groups

c : group commutation number

Notes

`i, j` can be symbols, strings or numbers, apart from `0, 1` and `2` which are reserved respectively for commuting, anticommuting tensors and tensors not commuting with any other group apart with the commuting tensors. For the remaining cases, use this method to set the commutation rules; by default `c=None`.

The group commutation number `c` is assigned in correspondence to the group commutation symbols; it can be

0 commuting

1 anticommuting

None no commutation property

Examples

`G` and `GH` do not commute with themselves and commute with each other; A is commuting.

```
>>> Lorentz = TensorIndexType('Lorentz')
>>> i0, i1, i2, i3, i4 = tensor_indices('i0:5', Lorentz)
>>> A = tensorhead('A', [Lorentz], [[1]])
>>> G = tensorhead('G', [Lorentz], [[1]], 'Gcomm')
>>> GH = tensorhead('GH', [Lorentz], [[1]], 'GHcomm')
>>> TensorManager.set_comm('Gcomm', 'GHcomm', 0)
>>> (GH(i1)*G(i0)).canon_bp()
G(i0)*GH(i1)
>>> (G(i1)*G(i0)).canon_bp()
G(i1)*G(i0)
>>> (G(i1)*A(i0)).canon_bp()
A(i0)*G(i1)
```

set_comms(**args*)

set the commutation group numbers `c` for symbols `i, j`

Parameters args : sequence of `(i, j, c)`

class diofant.tensor.tensor.**TensorIndexType**

A TensorIndexType is characterized by its name and its metric.

Parameters name : name of the tensor type

metric : metric symmetry or metric object or `None`

dim : dimension, it can be a symbol or an integer or `None`

eps_dim : dimension of the epsilon tensor

dummy_fmt : name of the head of dummy indices

Notes

The `metric` parameter can be: `metric = False` symmetric metric (in Riemannian geometry)

`metric = True` antisymmetric metric (for spinor calculus)

metric = None there is no metric

metric can be an object having name and antisym attributes.

If there is a metric the metric is used to raise and lower indices.

In the case of antisymmetric metric, the following raising and lowering conventions will be adopted:

psi(a) = g(a, b)*psi(-b); chi(-a) = chi(b)*g(-b, -a)

g(-a, b) = delta(-a, b); g(b, -a) = -delta(a, -b)

where delta(-a, b) = delta(b, -a) is the Kronecker delta (see TensorIndex for the conventions on indices).

If there is no metric it is not possible to raise or lower indices; e.g. the index of the defining representation of SU(N) is 'covariant' and the conjugate representation is 'contravariant'; for N > 2 they are linearly independent.

eps_dim is by default equal to dim, if the latter is an integer; else it can be assigned (for use in naive dimensional regularization); if eps_dim is not an integer epsilon is None.

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> Lorentz.metric
metric(Lorentz,Lorentz)
```

Examples with metric components data added, this means it is working on a fixed basis:

```
>>> Lorentz.data = [1, -1, -1, -1]
>>> print(sstr(Lorentz))
TensorIndexType(Lorentz, 0)
>>> print(sstr(Lorentz.data))
[[1 0 0 0]
[0 -1 0 0]
[0 0 -1 0]
[0 0 0 -1]]
```

Attributes

''name''	
''metric_name''	(it is 'metric' or metric.name)
''met-ric_antisym''	
''metric''	(the metric tensor)
''delta''	(Kronecker delta)
''epsilon''	(the Levi-Civita epsilon tensor)
''dim''	
''dim_eps''	
''dummy_fmt''	
''data''	(a property to add ndarray values, to work in a specified basis.)

class diofant.tensor.tensor.**TensorIndex**

Represents an abstract tensor index.

Parameters name : name of the index, or True if you want it to be automatically assigned

tensortype : TensorIndexType of the index

is_up : flag for contravariant index

Notes

Tensor indices are contracted with the Einstein summation convention.

An index can be in contravariant or in covariant form; in the latter case it is represented prepending a - to the index name.

Dummy indices have a name with head given by tensortype._dummy_fmt

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> i = TensorIndex('i', Lorentz); i
i
>>> sym1 = TensorSymmetry(*get_symmetric_group_sgs(1))
>>> S1 = TensorType([Lorentz], sym1)
>>> A, B = S1('A B')
>>> A(i)*B(-i)
A(L_0)*B(-L_0)
```

If you want the index name to be automatically assigned, just put True in the name field, it will be generated using the reserved character _ in front of its name, in order to avoid conflicts with possible existing indices:

```
>>> i0 = TensorIndex(True, Lorentz)
>>> i0
_i0
>>> i1 = TensorIndex(True, Lorentz)
>>> i1
_i1
>>> A(i0)*B(-i1)
A(_i0)*B(-_i1)
>>> A(i0)*B(-i0)
A(L_0)*B(-L_0)
```

Attributes

‘‘name‘‘	
‘‘tensortype‘‘	
‘‘is_up‘‘	

diofant.tensor.tensor.**tensor_indices**(*s*, *typ*)

Returns list of tensor indices given their names and their types

Parameters s : string of comma separated names of indices

typ : list of `TensorIndexType` of the indices

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> a, b, c, d = tensor_indices('a b c d', Lorentz)
```

class `diofant.tensor.tensor.`**`TensorSymmetry`**

Monoterm symmetry of a tensor

Parameters bsgs : tuple `(base, sgs)` BSGS of the symmetry of the tensor

See also:

diofant.combinatorics.tensor_can.get_symmetric_group_sgs (page 242)

Notes

A tensor can have an arbitrary monoterm symmetry provided by its BSGS. Multiterm symmetries, like the cyclic symmetry of the Riemann tensor, are not covered.

Examples

Define a symmetric tensor

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> sym2 = TensorSymmetry(get_symmetric_group_sgs(2))
>>> S2 = TensorType([Lorentz]*2, sym2)
>>> V = S2('V')
```

Attributes

‘‘base‘‘	(base of the BSGS)
‘‘generators‘‘	(generators of the BSGS)
‘‘rank‘‘	(rank of the tensor)

`diofant.tensor.tensor.`**`tensorsymmetry`**`(*args)`

Return a `TensorSymmetry` object.

One can represent a tensor with any monoterm slot symmetry group using a BSGS.

`args` can be a BSGS `args[0]` base `args[1]` sgs

Usually tensors are in (direct products of) representations of the symmetric group; `args` can be a list of lists representing the shapes of Young tableaux

Notes

For instance: `[[1]]` vector `[[1]*n]` symmetric tensor of rank `n` `[[n]]` antisymmetric tensor of rank `n` `[[2, 2]]` monoterm slot symmetry of the Riemann tensor `[[1],[1]]` vector*vector `[[2],[1],[1]` (antisymmetric tensor)*vector*vector

Notice that with the shape `[2, 2]` we associate only the monoterm symmetries of the Riemann tensor; this is an abuse of notation, since the shape `[2, 2]` corresponds usually to the irreducible representation characterized by the monoterm symmetries and by the cyclic symmetry.

Examples

Symmetric tensor using a Young tableau

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> sym2 = tensorsymmetry([1, 1])
>>> S2 = TensorType([Lorentz]*2, sym2)
>>> V = S2('V')
```

Symmetric tensor using a `BSGS` (base, strong generator set)

```
>>> sym2 = tensorsymmetry(*get_symmetric_group_sgs(2))
>>> S2 = TensorType([Lorentz]*2, sym2)
>>> V = S2('V')
```

class `diofant.tensor.tensor.`**`TensorType`**

Class of tensor types.

Parameters index_types : list of `TensorIndexType` of the tensor indices

symmetry : `TensorSymmetry` of the tensor

Examples

Define a symmetric tensor

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> sym2 = tensorsymmetry([1, 1])
>>> S2 = TensorType([Lorentz]*2, sym2)
>>> V = S2('V')
```

Attributes

‘‘index_types‘‘	
‘‘symmetry‘‘	
‘‘types‘‘	(list of `TensorIndexType` without repetitions)

class `diofant.tensor.tensor.`**`TensorHead`**

Tensor head of the tensor

Parameters name : name of the tensor

typ : list of TensorIndexType

comm : commutation group number

Notes

A `TensorHead` belongs to a commutation group, defined by a symbol on number `comm` (see `_TensorManager.set_comm`); tensors in a commutation group have the same commutation properties; by default `comm` is `0`, the group of the commuting tensors.

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> A = tensorhead('A', [Lorentz, Lorentz], [[1], [1]])
```

Examples with ndarray values, the components data assigned to the `TensorHead` object are assumed to be in a fully-contravariant representation. In case it is necessary to assign components data which represents the values of a non-fully covariant tensor, see the other examples.

```
>>> Lorentz.data = [1, -1, -1, -1]
>>> i0, i1 = tensor_indices('i0:2', Lorentz)
>>> A.data = [[j+2*i for j in range(4)] for i in range(4)]
```

in order to retrieve data, it is also necessary to specify abstract indices enclosed by round brackets, then numerical indices inside square brackets.

```
>>> A(i0, i1)[0, 0]
0
>>> A(i0, i1)[2, 3] == 3+2*2
True
```

Notice that square brackets create a valued tensor expression instance:

```
>>> A(i0, i1)
A(i0, i1)
```

To view the data, just type:

```
>>> print(sstr(A.data))
[[0 1 2 3]
 [2 3 4 5]
 [4 5 6 7]
 [6 7 8 9]]
```

Turning to a tensor expression, covariant indices get the corresponding components data corrected by the metric:

```
>>> print(sstr(A(i0, -i1).data))
[[0 -1 -2 -3]
 [2 -3 -4 -5]
 [4 -5 -6 -7]
 [6 -7 -8 -9]]
```

```
>>> print(sstr(A(-i0, -i1).data))
[[0 -1 -2 -3]
 [-2 3 4 5]
 [-4 5 6 7]
 [-6 7 8 9]]
```

while if all indices are contravariant, the `ndarray` remains the same

```
>>> print(sstr(A(i0, i1).data))
 [[0 1 2 3]
 [2 3 4 5]
 [4 5 6 7]
 [6 7 8 9]]
```

When all indices are contracted and components data are added to the tensor, accessing the data will return a scalar, no numpy object. In fact, numpy ndarrays are dropped to scalars if they contain only one element.

```
>>> A(i0, -i0)
A(L_0, -L_0)
>>> A(i0, -i0).data
-18
```

It is also possible to assign components data to an indexed tensor, i.e. a tensor with specified covariant and contravariant components. In this example, the covariant components data of the Electromagnetic tensor are injected into A:

```
>>> Ex, Ey, Ez, Bx, By, Bz = symbols('E_x E_y E_z B_x B_y B_z')
>>> c = symbols('c', positive=True)
```

Let's define F, an antisymmetric tensor, we have to assign an antisymmetric matrix to it, because $[[2]]$ stands for the Young tableau representation of an antisymmetric set of two elements:

```
>>> F = tensorhead('A', [Lorentz, Lorentz], [[2]])
>>> F(-i0, -i1).data = [
... [0, Ex/c, Ey/c, Ez/c],
... [-Ex/c, 0, -Bz, By],
... [-Ey/c, Bz, 0, -Bx],
... [-Ez/c, -By, Bx, 0]]
```

Now it is possible to retrieve the contravariant form of the Electromagnetic tensor:

```
>>> print(sstr(F(i0, i1).data))
[[0 -E_x/c -E_y/c -E_z/c]
 [E_x/c 0 -B_z B_y]
 [E_y/c B_z 0 -B_x]
 [E_z/c -B_y B_x 0]]
```

and the mixed contravariant-covariant form:

```
>>> print(sstr(F(i0, -i1).data))
[[0 E_x/c E_y/c E_z/c]
 [E_x/c 0 B_z -B_y]
 [E_y/c -B_z 0 B_x]
 [E_z/c B_y -B_x 0]]
```

To convert the numpy's ndarray to a diofant matrix, just cast:

```
>>> Matrix(F.data)
Matrix([
[    0, -E_x/c, -E_y/c, -E_z/c],
[E_x/c,      0,   -B_z,    B_y],
[E_y/c,    B_z,      0,   -B_x],
[E_z/c,   -B_y,    B_x,      0]])
```

Still notice, in this last example, that accessing components data from a tensor without specifying the indices is equivalent to assume that all indices are contravariant.

It is also possible to store symbolic components data inside a tensor, for example, define a four-momentum-like tensor:

```
>>> P = tensorhead('P', [Lorentz], [[1]])
>>> E, px, py, pz = symbols('E p_x p_y p_z', positive=True)
>>> P.data = [E, px, py, pz]
```

The contravariant and covariant components are, respectively:

```
>>> print(sstr(P(i0).data))
[E p_x p_y p_z]
>>> print(sstr(P(-i0).data))
[E -p_x -p_y -p_z]
```

The contraction of a 1-index tensor by itself is usually indicated by a power by two:

```
>>> P(i0)**2
E**2 - p_x**2 - p_y**2 - p_z**2
```

As the power by two is clearly identical to $P_\mu P^\mu$, it is possible to simply contract the `TensorHead` object, without specifying the indices

```
>>> P**2
E**2 - p_x**2 - p_y**2 - p_z**2
```

Attributes

“name“	
“index_types“	
“rank“	
“types“	(equal to `typ.types`)
“symmetry“	(equal to `typ.symmetry`)
“comm“	(commutation group)

`commutes_with`(*other*)
Returns `0` if `self` and `other` commute, `1` if they anticommute.

Returns `None` if `self` and `other` neither commute nor anticommute.

class `diofant.tensor.tensor.`**`TensExpr`**
Abstract base class for tensor expressions

Notes

A tensor expression is an expression formed by tensors; currently the sums of tensors are distributed.

A TensExpr can be a TensAdd or a TensMul.

TensAdd objects are put in canonical form using the Butler-Portugal algorithm for canonicalization under monoterm symmetries.

TensMul objects are formed by products of component tensors, and include a coefficient, which is a Diofant expression.

In the internal representation contracted indices are represented by (ipos1, ipos2, icomp1, icomp2), where icomp1 is the position of the component tensor with contravariant index, ipos1 is the slot which the index occupies in that component tensor.

Contracted indices are therefore nameless in the internal representation.

get_matrix()

Returns ndarray components data as a matrix, if components data are available and ndarray dimension does not exceed 2.

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> sym2 = tensorsymmetry([1]*2)
>>> S2 = TensorType([Lorentz]*2, sym2)
>>> A = S2('A')
```

The tensor A is symmetric in its indices, as can be deduced by the [1, 1] Young tableau when constructing $sym2$. One has to be careful to assign symmetric component data to A, as the symmetry properties of data are currently not checked to be compatible with the defined tensor symmetry.

```
>>> Lorentz.data = [1, -1, -1, -1]
>>> i0, i1 = tensor_indices('i0:2', Lorentz)
>>> A.data = [[j+i for j in range(4)] for i in range(4)]
>>> A(i0, i1).get_matrix()
Matrix([
[0, 1, 2, 3],
[1, 2, 3, 4],
[2, 3, 4, 5],
[3, 4, 5, 6]])
```

It is possible to perform usual operation on matrices, such as the matrix multiplication:

```
>>> A(i0, i1).get_matrix()*ones(4, 1)
Matrix([
[ 6],
[10],
[14],
[18]])
```

```
>>> del A.data
```

class diofant.tensor.tensor.**TensAdd**

Sum of tensors

Parameters free_args : list of the free indices

Notes

Sum of more than one tensor are put automatically in canonical form.

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> a, b = tensor_indices('a b', Lorentz)
>>> p, q = tensorhead('p q', [Lorentz], [[1]])
>>> t = p(a) + q(a); t
p(a) + q(a)
>>> t(b)
p(b) + q(b)
```

Examples with components data added to the tensor expression:

```
>>> Lorentz.data = [1, -1, -1, -1]
>>> a, b = tensor_indices('a, b', Lorentz)
>>> p.data = [2, 3, -2, 7]
>>> q.data = [2, 3, -2, 7]
>>> t = p(a) + q(a); t
p(a) + q(a)
>>> t(b)
p(b) + q(b)
```

The following are: 2**2 - 3**2 - 2**2 - 7**2 ==> -58

```
>>> (p(a)*p(-a)).data
-58
>>> p(a)**2
-58
```

Attributes

‘‘args‘‘	(tuple of addends)
‘‘rank‘‘	(rank of the tensor)
‘‘free_args‘‘	(list of the free indices in sorted order)

canon_bp()

canonicalize using the Butler-Portugal algorithm for canonicalization under monoterm symmetries.

contract_metric(*g*)

Raise or lower indices with the metric g

Parameters g : metric

contract_all : if True, eliminate all g which are contracted

See also:

TensorIndexType (page 946)

static from_TIDS_list(*tids_list*)

Given a list of coefficients and a list of TIDS objects, construct a TensAdd instance, equivalent to the one that would result from creating single instances of TensMul and then adding them.

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> i, j = tensor_indices('i j', Lorentz)
>>> A, B = tensorhead('A B', [Lorentz]*2, [[1]*2])
>>> eA = 3*A(i, j)
>>> eB = 2*B(j, i)
>>> t1 = eA._tids
>>> t2 = eB._tids
>>> c1 = eA.coeff
>>> c2 = eB.coeff
>>> TensAdd.from_TIDS_list([c1, c2], [t1, t2])
2*B(i, j) + 3*A(i, j)
```

If the coefficient parameter is a scalar, then it will be applied as a coefficient on all TIDS objects.

```
>>> TensAdd.from_TIDS_list(4, [t1, t2])
4*A(i, j) + 4*B(i, j)
```

fun_eval(**index_tuples*)

Return a tensor with free indices substituted according to index_tuples

Parameters index_types : list of tuples (old_index, new_index)

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> i, j, k, l = tensor_indices('i j k l', Lorentz)
>>> A, B = tensorhead('A B', [Lorentz]*2, [[1]*2])
>>> t = A(i, k)*B(-k, -j) + A(i, -j)
>>> t.fun_eval((i, k), (-j, l))
A(k, L_0)*B(l, -L_0) + A(k, l)
```

substitute_indices(**index_tuples*)

Return a tensor with free indices substituted according to index_tuples

Parameters index_types : list of tuples (old_index, new_index)

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> i, j, k, l = tensor_indices('i j k l', Lorentz)
>>> A, B = tensorhead('A B', [Lorentz]*2, [[1]*2])
```

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```
>>> t = A(i, k)*B(-k, -j); t
A(i, L_0)*B(-L_0, -j)
>>> t.substitute_indices((i, j), (j, k))
A(j, L_0)*B(-L_0, -k)
```

class diofant.tensor.tensor.**TensMul**

Product of tensors

Parameters **coeff** : Diofant coefficient of the tensor

args

Notes

`args[0]` list of `TensorHead` of the component tensors.

`args[1]` list of `(ind, ipos, icomp)` where `ind` is a free index, `ipos` is the slot position of `ind` in the `icomp`-th component tensor.

`args[2]` list of tuples representing dummy indices. `(ipos1, ipos2, icomp1, icomp2)` indicates that the contravariant dummy index is the `ipos1`-th slot position in the `icomp1`-th component tensor; the corresponding covariant index is in the `ipos2` slot position in the `icomp2`-th component tensor.

Attributes

``components``	(list of `TensorHead` of the component tensors)
``types``	(list of nonrepeated `TensorIndexType`)
``free``	(list of `(ind, ipos, icomp)`, see Notes)
``dum``	(list of `(ipos1, ipos2, icomp1, icomp2)`, see Notes)
``ext_rank``	(rank of the tensor counting the dummy indices)
``rank``	(rank of the tensor)
``coeff``	(Diofant coefficient of the tensor)
``free_args``	(list of the free indices in sorted order)
``is_canon_bp``	(True if the tensor in in canonical form)

canon_bp()

Canonicalize using the Butler-Portugal algorithm for canonicalization under monoterm symmetries.

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> m0, m1, m2 = tensor_indices('m0 m1 m2', Lorentz)
>>> A = tensorhead('A', [Lorentz]*2, [[2]])
>>> t = A(m0, -m1)*A(m1, -m0)
>>> t.canon_bp()
-A(L_0, L_1)*A(-L_0, -L_1)
>>> t = A(m0, -m1)*A(m1, -m2)*A(m2, -m0)
>>> t.canon_bp()
0
```

contract_metric(*g*)
Raise or lower indices with the metric g

Parameters g : metric

See also:

TensorIndexType (page 946)

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> m0, m1, m2 = tensor_indices('m0 m1 m2', Lorentz)
>>> g = Lorentz.metric
>>> p, q = tensorhead('p q', [Lorentz], [[1]])
>>> t = p(m0)*q(m1)*g(-m0, -m1)
>>> t.canon_bp()
metric(L_0, L_1)*p(-L_0)*q(-L_1)
>>> t.contract_metric(g).canon_bp()
p(L_0)*q(-L_0)
```

fun_eval(**index_tuples*)
Return a tensor with free indices substituted according to `index_tuples`

`index_types` list of tuples `(old_index, new_index)`

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> i, j, k, l = tensor_indices('i j k l', Lorentz)
>>> A, B = tensorhead('A B', [Lorentz]*2, [[1]*2])
>>> t = A(i, k)*B(-k, -j); t
A(i, L_0)*B(-L_0, -j)
>>> t.fun_eval((i, k), (-j, l))
A(k, L_0)*B(-L_0, l)
```

get_indices()
Returns the list of indices of the tensor

The indices are listed in the order in which they appear in the component tensors. The dummy indices are given a name which does not collide with the names of the free indices.

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> m0, m1, m2 = tensor_indices('m0 m1 m2', Lorentz)
>>> g = Lorentz.metric
>>> p, q = tensorhead('p q', [Lorentz], [[1]])
>>> t = p(m1)*g(m0, m2)
>>> t.get_indices()
[m1, m0, m2]
```

perm2tensor(*g, canon_bp=False*)
Returns the tensor corresponding to the permutation g

For further details, see the method in TIDS with the same name.

sorted_components()
Returns a tensor with sorted components calling the corresponding method in a TIDS object.

split()
Returns a list of tensors, whose product is self

Dummy indices contracted among different tensor components become free indices with the same name as the one used to represent the dummy indices.

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> a, b, c, d = tensor_indices('a b c d', Lorentz)
>>> A, B = tensorhead('A B', [Lorentz]*2, [[1]*2])
>>> t = A(a, b)*B(-b, c)
>>> t
A(a, L_0)*B(-L_0, c)
>>> t.split()
[A(a, L_0), B(-L_0, c)]
```

diofant.tensor.tensor.**canon_bp**(*p*)
Butler-Portugal canonicalization

diofant.tensor.tensor.**tensor_mul**(**a*)
product of tensors

diofant.tensor.tensor.**riemann_cyclic_replace**(*t_r*)
replace Riemann tensor with an equivalent expression

R(m,n,p,q) -> 2/3*R(m,n,p,q) - 1/3*R(m,q,n,p) + 1/3*R(m,p,n,q)

diofant.tensor.tensor.**riemann_cyclic**(*t2*)
replace each Riemann tensor with an equivalent expression satisfying the cyclic identity.

This trick is discussed in the reference guide to Cadabra.

Examples

```
>>> Lorentz = TensorIndexType('Lorentz', dummy_fmt='L')
>>> i, j, k, l = tensor_indices('i j k l', Lorentz)
>>> R = tensorhead('R', [Lorentz]*4, [[2, 2]])
>>> t = R(i, j, k, l)*(R(-i, -j, -k, -l) - 2*R(-i, -k, -j, -l))
>>> riemann_cyclic(t)
0
```

3.21 Utilities

This module contains some general purpose utilities that are used across Diofant.

3.21.1 Autowrap Module

The autowrap module works very well in tandem with the Indexed classes of the *Tensors* (page 930). Here is a simple example that shows how to setup a binary routine that calculates a matrix-vector product.

```
>>> from diofant.utilities.autowrap import autowrap
>>> A, x, y = map(IndexedBase, ['A', 'x', 'y'])
>>> m, n = symbols('m n', integer=True)
>>> i = Idx('i', m)
>>> j = Idx('j', n)
>>> instruction = Eq(y[i], A[i, j]*x[j]); instruction
Eq(y[i], x[j]*A[i, j])
```

Because the code printers treat Indexed objects with repeated indices as a summation, the above equality instance will be translated to low-level code for a matrix vector product. This is how you tell Diofant to generate the code, compile it and wrap it as a python function:

```
>>> matvec = autowrap(instruction)
```

That's it. Now let's test it with some numpy arrays. The default wrapper backend is f2py. The wrapper function it provides is set up to accept python lists, which it will silently convert to numpy arrays. So we can test the matrix vector product like this:

```
>>> M = [[0, 1],
...      [1, 0]]
>>> matvec(M, [2, 3])
[ 3.  2.]
```

Implementation details

The autowrap module is implemented with a backend consisting of CodeWrapper objects. The base class `CodeWrapper` takes care of details about module name, filenames and options. It also contains the driver routine, which runs through all steps in the correct order, and also takes care of setting up and removing the temporary working directory.

The actual compilation and wrapping is done by external resources, such as the system installed f2py command. The Cython backend runs a distutils setup script in a subprocess. Subclasses of CodeWrapper takes care of these backend-dependent details.

API Reference

Module for compiling codegen output, and wrap the binary for use in python.

This module provides a common interface for different external backends, such as f2py, fwrap, Cython, SWIG(?) etc. (Currently only f2py and Cython are implemented) The goal is to provide access to compiled binaries of acceptable performance with a one-button user interface, i.e.

```
>>> expr = ((x - y)**(25)).expand()
>>> binary_callable = autowrap(expr)
>>> binary_callable(1, 2)
-1.0
```

The callable returned from autowrap() is a binary python function, not a Diofant object. If it is desired to use the compiled function in symbolic expressions, it is better to use binary_function() which returns a Diofant Function object. The binary callable is attached as the _imp_ attribute and invoked when a numerical evaluation is requested with evalf(), or with lambdify().

```
>>> f = binary_function('f', expr)
>>> 2*f(x, y) + y
y + 2*f(x, y)
>>> (2*f(x, y) + y).evalf(2, subs={x: 1, y:2}, strict=False)
0.e-190
```

The idea is that a Diofant user will primarily be interested in working with mathematical expressions, and should not have to learn details about wrapping tools in order to evaluate expressions numerically, even if they are computationally expensive.

When is this useful?

1. For computations on large arrays, Python iterations may be too slow, and depending on the mathematical expression, it may be difficult to exploit the advanced index operations provided by NumPy.
2. For *really* long expressions that will be called repeatedly, the compiled binary should be significantly faster than Diofant's .evalf()
3. If you are generating code with the codegen utility in order to use it in another project, the automatic python wrappers let you test the binaries immediately from within Diofant.
4. To create customized ufuncs for use with numpy arrays. See *ufuncify*.

When is this module NOT the best approach?

1. If you are really concerned about speed or memory optimizations, you will probably get better results by working directly with the wrapper tools and the low level code. However, the files generated by this utility may provide a useful starting point and reference code. Temporary files will be left intact if you supply the keyword tempdir="path/to/files/".
2. If the array computation can be handled easily by numpy, and you don't need the binaries for another project.

exception diofant.utilities.autowrap.**CodeWrapError**

class diofant.utilities.autowrap.**CodeWrapper**(*generator*, *filepath=None*, *flags=[]*, *verbose=False*)

Base Class for code wrappers

class diofant.utilities.autowrap.**CythonCodeWrapper**(**args*, ***kwargs*)

Wrapper that uses Cython

dump_pyx(*routines*, *f*, *prefix*)

Write a Cython file with python wrappers

This file contains all the definitions of the routines in c code and refers to the header file.

Parameters routines : list

List of Routine instances

f : file

File-like object to write the file to

prefix : str

The filename prefix, used to refer to the proper header file. Only the basename of the prefix is used.

class diofant.utilities.autowrap.**DummyWrapper**(*generator*, *filepath=None*, *flags=[]*, *verbose=False*)

Class used for testing independent of backends

class diofant.utilities.autowrap.**F2PyCodeWrapper**(*generator*, *filepath=None*, *flags=[]*, *verbose=False*)

Wrapper that uses f2py

class diofant.utilities.autowrap.**UfuncifyCodeWrapper**(*generator*, *filepath=None*, *flags=[]*, *verbose=False*)

Wrapper for Ufuncify

dump_c(*routines*, *f*, *prefix*)

Write a C file with python wrappers

This file contains all the definitions of the routines in c code.

Parameters routines : list

List of Routine instances

f : file

File-like object to write the file to

prefix : str

The filename prefix, used to name the imported module.

diofant.utilities.autowrap.**autowrap**(*expr*, *language=None*, *backend='f2py'*, *tempdir=None*, *args=None*, *flags=None*, *verbose=False*, *helpers=None*)

Generates python callable binaries based on the math expression.

Parameters expr

The Diofant expression that should be wrapped as a binary routine.

language : string, optional

If supplied, (options: 'C' or 'F95'), specifies the language of the generated code. If `None` [default], the language is inferred based upon the specified backend.

backend : string, optional

Backend used to wrap the generated code. Either 'f2py' [default], or 'cython'.

tempdir : string, optional

Path to directory for temporary files. If this argument is supplied, the generated code and the wrapper input files are left intact in the specified path.

args : iterable, optional

An ordered iterable of symbols. Specifies the argument sequence for the function.

flags : iterable, optional

Additional option flags that will be passed to the backend.

verbose : bool, optional

If True, autowrap will not mute the command line backends. This can be helpful for debugging.

helpers : iterable, optional

Used to define auxillary expressions needed for the main expr. If the main expression needs to call a specialized function it should be put in the `helpers` iterable. Autowrap will then make sure that the compiled main expression can link to the helper routine. Items should be tuples with (<function_name>, <diofant_expression>, <arguments>). It is mandatory to supply an argument sequence to helper routines.

Examples

```
>>> expr = ((x - y + z)**(13)).expand()
>>> binary_func = autowrap(expr)
>>> binary_func(1, 4, 2)
-1.0
```

`diofant.utilities.autowrap.`**`binary_function`**(*symfunc*, *expr*, ***kwargs*)

Returns a diofant function with expr as binary implementation

This is a convenience function that automates the steps needed to autowrap the Diofant expression and attaching it to a Function object with implemented_function().

```
>>> expr = ((x - y)**(25)).expand()
>>> f = binary_function('f', expr)
>>> type(f)
<class 'diofant.core.function.UndefinedFunction'>
>>> 2*f(x, y)
2*f(x, y)
>>> f(x, y).evalf(2, subs={x: 1, y: 2})
-1.0
```

`diofant.utilities.autowrap.`**`ufuncify`**(*args*, *expr*, *language=None*, *backend='numpy'*, *tempdir=None*, *flags=None*, *verbose=False*, *helpers=None*)

Generates a binary function that supports broadcasting on numpy arrays.

Parameters args : iterable

Either a Symbol or an iterable of symbols. Specifies the argument sequence for the function.

expr

A Diofant expression that defines the element wise operation.

language : string, optional

If supplied, (options: 'C' or 'F95'), specifies the language of the generated code. If `None` [default], the language is inferred based upon the specified backend.

backend : string, optional

Backend used to wrap the generated code. Either 'numpy' [default], 'cython', or 'f2py'.

tempdir : string, optional

Path to directory for temporary files. If this argument is supplied, the generated code and the wrapper input files are left intact in the specified path.

flags : iterable, optional

Additional option flags that will be passed to the backend

verbose : bool, optional

If True, autowrap will not mute the command line backends. This can be helpful for debugging.

helpers : iterable, optional

Used to define auxillary expressions needed for the main expr. If the main expression needs to call a specialized function it should be put in the `helpers` iterable. Autowrap will then make sure that the compiled main expression can link to the helper routine. Items should be tuples with (<function_name>, <diofant_expression>, <arguments>). It is mandatory to supply an argument sequence to helper routines.

Notes

The default backend ('numpy') will create actual instances of `numpy.ufunc`. These support ndimensional broadcasting, and implicit type conversion. Use of the other backends will result in a "ufunc-like" function, which requires equal length 1-dimensional arrays for all arguments, and will not perform any type conversions.

References

[R612] (page 1268)

Examples

```
>>> import numpy as np
>>> f = ufuncify((x, y), y + x**2)
>>> type(f) is np.ufunc
True
>>> f([1, 2, 3], 2)
[  3.   6.  11.]
>>> f(np.arange(5), 3)
[  3.   4.   7.  12.  19.]
```

For the F2Py and Cython backends, inputs are required to be equal length 1-dimensional arrays. The F2Py backend will perform type conversion, but the Cython backend will error if the inputs are not of the expected type.

```
>>> f_fortran = ufuncify((x, y), y + x**2, backend='F2Py')
>>> f_fortran(1, 2)
[ 3.]
>>> f_fortran(np.array([1, 2, 3]), np.array([1.0, 2.0, 3.0]))
[  2.   6.  12.]
```

3.21.2 Codegen

This module provides functionality to generate directly compilable code from Diofant expressions. The codegen function is the user interface to the code generation functionality in Diofant. Some details of the implementation is given below for advanced users that may want to use the framework directly.

Note: The codegen callable is not in the diofant namespace automatically, to use it you must first execute

```
>>> from diofant.utilities.codegen import codegen
```

Implementation Details

Here we present the most important pieces of the internal structure, as advanced users may want to use it directly, for instance by subclassing a code generator for a specialized application. **It is very likely that you would prefer to use the codegen() function documented above.**

Basic assumptions:

- A generic Routine data structure describes the routine that must be translated into C/Fortran/… code. This data structure covers all features present in one or more of the supported languages.
- Descendants from the CodeGen class transform multiple Routine instances into compilable code. Each derived class translates into a specific language.
- In many cases, one wants a simple workflow. The friendly functions in the last part are a simple api on top of the Routine/CodeGen stuff. They are easier to use, but are less powerful.

Routine

The Routine class is a very important piece of the codegen module. Viewing the codegen utility as a translator of mathematical expressions into a set of statements in a programming language, the Routine instances are responsible for extracting and storing information about how the math can be encapsulated in a function call. Thus, it is the Routine constructor that decides what arguments the routine will need and if there should be a return value.

API Reference

module for generating C, C++, Fortran77, Fortran90 and Octave/Matlab routines that evaluate diofant expressions. This module is work in progress. Only the milestones with a '+' character in the list below have been completed.

— How is diofant.utilities.codegen different from diofant.printing.ccode? —

We considered the idea to extend the printing routines for diofant functions in such a way that it prints complete compilable code, but this leads to a few unsurmountable issues that can only be tackled with dedicated code generator:

- For C, one needs both a code and a header file, while the printing routines generate just one string. This code generator can be extended to support .pyf files for f2py.
- Diofant functions are not concerned with programming-technical issues, such as input, output and input-output arguments. Other examples are contiguous or non-contiguous arrays, including headers of other libraries such as gsl or others.
- It is highly interesting to evaluate several diofant functions in one C routine, eventually sharing common intermediate results with the help of the cse routine. This is more than just printing.
- From the programming perspective, expressions with constants should be evaluated in the code generator as much as possible. This is different for printing.

— Basic assumptions —

- A generic Routine data structure describes the routine that must be translated into C/Fortran/… code. This data structure covers all features present in one or more of the supported languages.
- Descendants from the CodeGen class transform multiple Routine instances into compilable code. Each derived class translates into a specific language.
- In many cases, one wants a simple workflow. The friendly functions in the last part are a simple api on top of the Routine/CodeGen stuff. They are easier to use, but are less powerful.

— Milestones —

- First working version with scalar input arguments, generating C code, tests
- Friendly functions that are easier to use than the rigorous Routine/CodeGen workflow.
- Integer and Real numbers as input and output
- Output arguments
- InputOutput arguments
- Sort input/output arguments properly
- Contiguous array arguments (numpy matrices)
- Also generate .pyf code for f2py (in autowrap module)
- Isolate constants and evaluate them beforehand in double precision
- Fortran 90
- Octave/Matlab
- Common Subexpression Elimination
- User defined comments in the generated code
- Optional extra include lines for libraries/objects that can eval special functions
- Test other C compilers and libraries: gcc, tcc, libtcc, gcc+gsl, …
- Contiguous array arguments (diofant matrices)

- Non-contiguous array arguments (diofant matrices)
- ccode must raise an error when it encounters something that can not be translated into c. ccode(integrate(sin(x)/x, x)) does not make sense.
- Complex numbers as input and output
- A default complex datatype
- Include extra information in the header: date, user, hostname, sha1 hash, …
- Fortran 77
- C++
- Python
- …

class diofant.utilities.codegen.**Routine**(*name*, *arguments*, *results*, *local_vars*, *global_vars*)

Generic description of evaluation routine for set of expressions.

A CodeGen class can translate instances of this class into code in a particular language. The routine specification covers all the features present in these languages. The CodeGen part must raise an exception when certain features are not present in the target language. For example, multiple return values are possible in Python, but not in C or Fortran. Another example: Fortran and Python support complex numbers, while C does not.

result_variables

Returns a list of OutputArgument, InOutArgument and Result.

If return values are present, they are at the end ot the list.

variables

Returns a set of all variables possibly used in the routine.

For routines with unnamed return values, the dummies that may or may not be used will be included in the set.

class diofant.utilities.codegen.**DataType**(*cname*, *fname*, *pyname*, *octname*)

Holds strings for a certain datatype in different languages.

diofant.utilities.codegen.**get_default_datatype**(*expr*)

Derives an appropriate datatype based on the expression.

class diofant.utilities.codegen.**Argument**(*name*, *datatype=None*, *dimensions=None*, *precision=None*)

An abstract Argument data structure: a name and a data type.

This structure is refined in the descendants below.

class diofant.utilities.codegen.**InputArgument**(*name*, *datatype=None*, *dimensions=None*, *precision=None*)

class diofant.utilities.codegen.**Result**(*expr*, *name=None*, *result_var=None*, *datatype=None*, *dimensions=None*, *precision=None*)

An expression for a return value.

The name result is used to avoid conflicts with the reserved word “return” in the python language. It is also shorter than ReturnValue.

These may or may not need a name in the destination (e.g., "return(x*y)" might return a value without ever naming it).

class diofant.utilities.codegen.**CodeGen**(*project='project'*)
Abstract class for the code generators.

dump_code(*routines*, *f*, *prefix*, *header=True*, *empty=True*)
Write the code by calling language specific methods.

The generated file contains all the definitions of the routines in low-level code and refers to the header file if appropriate.

Parameters routines : list

A list of Routine instances.

f : file-like

Where to write the file.

prefix : string

The filename prefix, used to refer to the proper header file. Only the basename of the prefix is used.

header : bool, optional

When True, a header comment is included on top of each source file. [default : True]

empty : bool, optional

When True, empty lines are included to structure the source files. [default : True]

routine(*name*, *expr*, *argument_sequence*, *global_vars*)
Creates an Routine object that is appropriate for this language.

This implementation is appropriate for at least C/Fortran. Subclasses can override this if necessary.

Here, we assume at most one return value (the l-value) which must be scalar. Additional outputs are OutputArguments (e.g., pointers on right-hand-side or pass-by-reference). Matrices are always returned via OutputArguments. If `argument_sequence` is None, arguments will be ordered alphabetically, but with all InputArguments first, and then OutputArgument and InOutArguments.

write(*routines*, *prefix*, *to_files=False*, *header=True*, *empty=True*)
Writes all the source code files for the given routines.

The generated source is returned as a list of (filename, contents) tuples, or is written to files (see below). Each filename consists of the given prefix, appended with an appropriate extension.

Parameters routines : list

A list of Routine instances to be written

prefix : string

The prefix for the output files

to_files : bool, optional

When True, the output is written to files. Otherwise, a list of (filename, contents) tuples is returned. [default: False]

header : bool, optional

When True, a header comment is included on top of each source file. [default: True]

empty : bool, optional

When True, empty lines are included to structure the source files. [default: True]

class diofant.utilities.codegen.**CCodeGen**(*project='project'*)

Generator for C code.

The .write() method inherited from CodeGen will output a code file and an interface file, <prefix>.c and <prefix>.h respectively.

dump_c(*routines*, *f*, *prefix*, *header=True*, *empty=True*)

Write the code by calling language specific methods.

The generated file contains all the definitions of the routines in low-level code and refers to the header file if appropriate.

Parameters routines : list

A list of Routine instances.

f : file-like

Where to write the file.

prefix : string

The filename prefix, used to refer to the proper header file. Only the basename of the prefix is used.

header : bool, optional

When True, a header comment is included on top of each source file. [default : True]

empty : bool, optional

When True, empty lines are included to structure the source files. [default : True]

dump_h(*routines*, *f*, *prefix*, *header=True*, *empty=True*)

Writes the C header file.

This file contains all the function declarations.

Parameters routines : list

A list of Routine instances.

f : file-like

Where to write the file.

prefix : string

The filename prefix, used to construct the include guards. Only the basename of the prefix is used.

header : bool, optional

When True, a header comment is included on top of each source file. [default : True]

empty : bool, optional

When True, empty lines are included to structure the source files. [default : True]

get_prototype(*routine*)
Returns a string for the function prototype of the routine.

If the routine has multiple result objects, an CodeGenError is raised.

See: https//en.wikipedia.org/wiki/Function_prototype

class diofant.utilities.codegen.**FCodeGen**(*project='project'*)
Generator for Fortran 95 code

The .write() method inherited from CodeGen will output a code file and an interface file, <prefix>.f90 and <prefix>.h respectively.

dump_f95(*routines, f, prefix, header=True, empty=True*)
Write the code by calling language specific methods.

The generated file contains all the definitions of the routines in low-level code and refers to the header file if appropriate.

Parameters routines : list

A list of Routine instances.

f : file-like

Where to write the file.

prefix : string

The filename prefix, used to refer to the proper header file. Only the basename of the prefix is used.

header : bool, optional

When True, a header comment is included on top of each source file. [default : True]

empty : bool, optional

When True, empty lines are included to structure the source files. [default : True]

dump_h(*routines, f, prefix, header=True, empty=True*)
Writes the interface to a header file.

This file contains all the function declarations.

Parameters routines : list

A list of Routine instances.

f : file-like

Where to write the file.

prefix : string

The filename prefix.

header : bool, optional

When True, a header comment is included on top of each source file. [default : True]

empty : bool, optional

When True, empty lines are included to structure the source files. [default : True]

get_interface(*routine*)

Returns a string for the function interface.

The routine should have a single result object, which can be None. If the routine has multiple result objects, a CodeGenError is raised.

See: https//en.wikipedia.org/wiki/Function_prototype

class diofant.utilities.codegen.**OctaveCodeGen**(*project='project'*)

Generator for Octave code.

The .write() method inherited from CodeGen will output a code file <prefix>.m.

Octave .m files usually contain one function. That function name should match the filename (`prefix`). If you pass multiple `name_expr` pairs, the latter ones are presumed to be private functions accessed by the primary function.

You should only pass inputs to `argument_sequence`: outputs are ordered according to their order in `name_expr`.

dump_m(*routines*, *f*, *prefix*, *header=True*, *empty=True*, *inline=True*)

Write the code by calling language specific methods.

The generated file contains all the definitions of the routines in low-level code and refers to the header file if appropriate.

Parameters **routines** : list

A list of Routine instances.

f : file-like

Where to write the file.

prefix : string

The filename prefix, used to refer to the proper header file. Only the basename of the prefix is used.

header : bool, optional

When True, a header comment is included on top of each source file. [default : True]

empty : bool, optional

When True, empty lines are included to structure the source files. [default : True]

routine(*name*, *expr*, *argument_sequence*, *global_vars*)

Specialized Routine creation for Octave.

diofant.utilities.codegen.**codegen**(*name_expr*, *language*, *prefix=None*, *project='project'*, *to_files=False*, *header=True*, *empty=True*, *argument_sequence=None*, *global_vars=None*)

Generate source code for expressions in a given language.

Parameters **name_expr** : tuple, or list of tuples

A single (name, expression) tuple or a list of (name, expression) tuples. Each tuple corresponds to a routine. If the expression is an equality (an instance of class Equality) the left hand side is considered an output argument. If expression is an iterable, then the routine will have multiple outputs.

language : string

A string that indicates the source code language. This is case insensitive. Currently, 'C', 'F95' and 'Octave' are supported. 'Octave' generates code compatible with both Octave and Matlab.

prefix : string, optional

A prefix for the names of the files that contain the source code. Language-dependent suffixes will be appended. If omitted, the name of the first name_expr tuple is used.

project : string, optional

A project name, used for making unique preprocessor instructions. [default: "project"]

to_files : bool, optional

When True, the code will be written to one or more files with the given prefix, otherwise strings with the names and contents of these files are returned. [default: False]

header : bool, optional

When True, a header is written on top of each source file. [default: True]

empty : bool, optional

When True, empty lines are used to structure the code. [default: True]

argument_sequence : iterable, optional

Sequence of arguments for the routine in a preferred order. A CodeGenError is raised if required arguments are missing. Redundant arguments are used without warning. If omitted, arguments will be ordered alphabetically, but with all input aguments first, and then output or in-out arguments.

global_vars : iterable, optional

Sequence of global variables used by the routine. Variables listed here will not show up as function arguments.

Examples

```
>>> [(c_name, c_code), (h_name, c_header)] = codegen(
...     ("f", x+y*z), "C", "test", header=False, empty=False)
>>> print(c_name)
test.c
>>> print(c_code)
#include "test.h"
```

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```
#include <math.h>
double f(double x, double y, double z) {
  double f_result;
  f_result = x + y*z;
  return f_result;
}
>>> print(h_name)
test.h
>>> print(c_header)
#ifndef PROJECT__TEST__H
#define PROJECT__TEST__H
double f(double x, double y, double z);
#endif
```

Another example using Equality objects to give named outputs. Here the filename (prefix) is taken from the first (name, expr) pair.

```
>>> from diofant.abc import f, g
>>> [(c_name, c_code), (h_name, c_header)] = codegen(
...      [("myfcn", x + y), ("fcn2", [Eq(f, 2*x), Eq(g, y)])],
...      "C", header=False, empty=False)
>>> print(c_name)
myfcn.c
>>> print(c_code)
#include "myfcn.h"
#include <math.h>
double myfcn(double x, double y) {
   double myfcn_result;
   myfcn_result = x + y;
   return myfcn_result;
}
void fcn2(double x, double y, double *f, double *g) {
   (*f) = 2*x;
   (*g) = y;
}
```

If the generated function(s) will be part of a larger project where various global variables have been defined, the 'global_vars' option can be used to remove the specified variables from the function signature

```
>>> [(f_name, f_code), header] = codegen(
...     ("f", x+y*z), "F95", header=False, empty=False,
...     argument_sequence=(x, y), global_vars=(z,))
>>> print(f_code)
REAL*8 function f(x, y)
implicit none
REAL*8, intent(in) :: x
REAL*8, intent(in) :: y
f = x + y*z
end function
```

diofant.utilities.codegen.**make_routine**(*name, expr, argument_sequence=None, global_vars=None, language='F95'*)

A factory that makes an appropriate Routine from an expression.

Parameters name : string

The name of this routine in the generated code.

expr : expression or list/tuple of expressions

A Diofant expression that the Routine instance will represent. If given a list or tuple of expressions, the routine will be considered to have multiple return values and/or output arguments.

argument_sequence : list or tuple, optional

List arguments for the routine in a preferred order. If omitted, the results are language dependent, for example, alphabetical order or in the same order as the given expressions.

global_vars : iterable, optional

Sequence of global variables used by the routine. Variables listed here will not show up as function arguments.

language : string, optional

Specify a target language. The Routine itself should be language-agnostic but the precise way one is created, error checking, etc depend on the language. [default: "F95"].

A decision about whether to use output arguments or return values is made

depending on both the language and the particular mathematical expressions.

For an expression of type Equality, the left hand side is typically made

into an OutputArgument (or perhaps an InOutArgument if appropriate).

Otherwise, typically, the calculated expression is made a return values of

the routine.

Examples

```
>>> from diofant.abc import f, g
>>> r = make_routine('test', [Eq(f, 2*x), Eq(g, x + y)])
>>> [arg.result_var for arg in r.results]
[]
>>> [arg.name for arg in r.arguments]
[x, y, f, g]
>>> [arg.name for arg in r.result_variables]
[f, g]
>>> r.local_vars
set()
```

Another more complicated example with a mixture of specified and automatically-assigned names. Also has Matrix output.

```
>>> r = make_routine('fcn', [x*y, Eq(f, 1), Eq(g, x + g), Matrix([[x, 2]])])
>>> [arg.result_var for arg in r.results]
[result_...]
>>> [arg.expr for arg in r.results]
[x*y]
```

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```
>>> [arg.name for arg in r.arguments]
[x, y, f, g, out_...]
```

We can examine the various arguments more closely:

```
>>> [a.name for a in r.arguments if isinstance(a, InputArgument)]
[x, y]
```

```
>>> [a.name for a in r.arguments if isinstance(a, OutputArgument)]
[f, out_...]
>>> [a.expr for a in r.arguments if isinstance(a, OutputArgument)]
[1, Matrix([[x, 2]])]
```

```
>>> [a.name for a in r.arguments if isinstance(a, InOutArgument)]
[g]
>>> [a.expr for a in r.arguments if isinstance(a, InOutArgument)]
[g + x]
```

3.21.3 Decorator

Useful utility decorators.

diofant.utilities.decorator.**conserve_mpmath_dps**(*func*)

After the function finishes, resets the value of mpmath.mp.dps to the value it had before the function was run.

diofant.utilities.decorator.**doctest_depends_on**(*exe=None*, *modules=None*, *disable_viewers=None*)

Adds metadata about the dependencies which need to be met for doctesting the docstrings of the decorated objects.

class diofant.utilities.decorator.**no_attrs_in_subclass**(*cls*, *f*)

Don't 'inherit' certain attributes from a base class

```
>>> class A:
...     x = 'test'
```

```
>>> A.x = no_attrs_in_subclass(A, A.x)
```

```
>>> class B(A):
...     pass
```

```
>>> hasattr(A, 'x')
True
>>> hasattr(B, 'x')
False
```

diofant.utilities.decorator.**public**(*obj*)

Append `obj`'s name to global `__all__` variable (call site).

By using this decorator on functions or classes you achieve the same goal as by filling `__all__` variables manually, you just don't have to repeat yourself (object's name). You also know if object is public at definition site, not at some random location (where `__all__` was set).

Note that in multiple decorator setup (in almost all cases) `@public` decorator must be applied before any other decorators, because it relies on the pointer to object's global namespace. If you apply other decorators first, `@public` may end up modifying the wrong namespace.

Examples

```
>>> __all__
Traceback (most recent call last):
...
NameError: name '__all__' is not defined
```

```
>>> @public
... def some_function():
...     pass
```

```
>>> __all__
['some_function']
```

3.21.4 Enumerative

This module includes functions and classes for enumerating and counting multiset partitions.

`diofant.utilities.enumerative.`**`multiset_partitions_taocp`**`(`*multiplicities*`)`

Enumerates partitions of a multiset.

Parameters multiplicities

list of integer multiplicities of the components of the multiset.

Yields state

Internal data structure which encodes a particular partition. This output is then usually processed by a vistor function which combines the information from this data structure with the components themselves to produce an actual partition.

Unless they wish to create their own visitor function, users will have little need to look inside this data structure. But, for reference, it is a 3-element list with components:

f is a frame array, which is used to divide pstack into parts.

lpart points to the base of the topmost part.

pstack is an array of PartComponent objects.

The `state` output offers a peek into the internal data structures of the enumeration function. The client should treat this as read-only; any modification of the data structure will cause unpredictable (and almost certainly incorrect) results. Also, the components of `state` are modified in place at each iteration. Hence, the visitor must be called at each loop iteration. Accumulating the `state` instances and processing them later will not work.

Examples

```
>>> # variables components and multiplicities represent the multiset 'abb'
>>> components = 'ab'
>>> multiplicities = [1, 2]
>>> states = multiset_partitions_taocp(multiplicities)
>>> list(list_visitor(state, components) for state in states)
[[['a', 'b', 'b']],
[['a', 'b'], ['b']],
[['a'], ['b', 'b']],
[['a'], ['b'], ['b']]]
```

`diofant.utilities.enumerative.`**`factoring_visitor`**`(`*state*, *primes*`)`

Use with multiset_partitions_taocp to enumerate the ways a number can be expressed as a product of factors. For this usage, the exponents of the prime factors of a number are arguments to the partition enumerator, while the corresponding prime factors are input here.

Examples

To enumerate the factorings of a number we can think of the elements of the partition as being the prime factors and the multiplicities as being their exponents.

```
>>> primes, multiplicities = zip(*factorint(24).items())
>>> primes
(2, 3)
>>> multiplicities
(3, 1)
>>> states = multiset_partitions_taocp(multiplicities)
>>> list(factoring_visitor(state, primes) for state in states)
[[24], [8, 3], [12, 2], [4, 6], [4, 2, 3], [6, 2, 2], [2, 2, 2, 3]]
```

`diofant.utilities.enumerative.`**`list_visitor`**`(`*state*, *components*`)`

Return a list of lists to represent the partition.

Examples

```
>>> states = multiset_partitions_taocp([1, 2, 1])
>>> s = next(states)
>>> list_visitor(s, 'abc')  # for multiset 'a b b c'
[['a', 'b', 'b', 'c']]
>>> s = next(states)
>>> list_visitor(s, [1, 2, 3])  # for multiset '1 2 2 3
[[1, 2, 2], [3]]
```

The approach of the function `multiset_partitions_taocp` is extended and generalized by the class `MultisetPartitionTraverser`.

`class` `diofant.utilities.enumerative.`**`MultisetPartitionTraverser`**

Has methods to `enumerate` and `count` the partitions of a multiset.

This implements a refactored and extended version of Knuth's algorithm 7.1.2.5M *[AOCP612]* (page 1268).

The enumeration methods of this class are generators and return data structures which can be interpreted by the same visitor functions used for the output of `multiset_partitions_taocp`.

See also:

multiset_partitions_taocp (page 976)

References

[AOCP612] (page 1268), [Factorisatio612] (page 1268), [Yorgey612] (page 1268)

Examples

```
>>> m = MultisetPartitionTraverser()
>>> m.count_partitions([4, 4, 4, 2])
127750
>>> m.count_partitions([3, 3, 3])
686
```

count_partitions(*multiplicities*)

Returns the number of partitions of a multiset whose components have the multiplicities given in `multiplicities`.

For larger counts, this method is much faster than calling one of the enumerators and counting the result. Uses dynamic programming to cut down on the number of nodes actually explored. The dictionary used in order to accelerate the counting process is stored in the `MultisetPartitionTraverser` object and persists across calls. If the the user does not expect to call `count_partitions` for any additional multisets, the object should be cleared to save memory. On the other hand, the cache built up from one count run can significantly speed up subsequent calls to `count_partitions`, so it may be advantageous not to clear the object.

Notes

If one looks at the workings of Knuth's algorithm M [AOCP615] (page 1268), it can be viewed as a traversal of a binary tree of parts. A part has (up to) two children, the left child resulting from the spread operation, and the right child from the decrement operation. The ordinary enumeration of multiset partitions is an in-order traversal of this tree, and with the partitions corresponding to paths from the root to the leaves. The mapping from paths to partitions is a little complicated, since the partition would contain only those parts which are leaves or the parents of a spread link, not those which are parents of a decrement link.

For counting purposes, it is sufficient to count leaves, and this can be done with a recursive in-order traversal. The number of leaves of a subtree rooted at a particular part is a function only of that part itself, so memoizing has the potential to speed up the counting dramatically.

This method follows a computational approach which is similar to the hypothetical memoized recursive function, but with two differences:

1. This method is iterative, borrowing its structure from the other enumerations and maintaining an explicit stack of parts which are in the process of being

counted. (There may be multisets which can be counted reasonably quickly by this implementation, but which would overflow the default Python recursion limit with a recursive implementation.)

2. Instead of using the part data structure directly, a more compact key is constructed. This saves space, but more importantly coalesces some parts which would remain separate with physical keys.

Unlike the enumeration functions, there is currently no _range version of count_partitions. If someone wants to stretch their brain, it should be possible to construct one by memoizing with a histogram of counts rather than a single count, and combining the histograms.

References

[AOCP615] (page 1268)

Examples

```
>>> m = MultisetPartitionTraverser()
>>> m.count_partitions([9, 8, 2])
288716
>>> m.count_partitions([2, 2])
9
>>> del m
```

enum_all(*multiplicities*)
Enumerate the partitions of a multiset.

See also:

multiset_partitions_taocp **(page 976)** which provides the same result as this method, but is about twice as fast. Hence, enum_all is primarily useful for testing. Also see the function for a discussion of states and visitors.

Examples

```
>>> m = MultisetPartitionTraverser()
>>> states = m.enum_all([2, 2])
>>> list(list_visitor(state, 'ab') for state in states)
[[['a', 'a', 'b', 'b']],
[['a', 'a', 'b'], ['b']],
[['a', 'a'], ['b', 'b']],
[['a', 'a'], ['b'], ['b']],
[['a', 'b', 'b'], ['a']],
[['a', 'b'], ['a', 'b']],
[['a', 'b'], ['a'], ['b']],
[['a'], ['a'], ['b', 'b']],
[['a'], ['a'], ['b'], ['b']]]
```

enum_large(*multiplicities, lb*)
Enumerate the partitions of a multiset with lb < num(parts)

Equivalent to enum_range(multiplicities, lb, sum(multiplicities))

Parameters multiplicities

list of multiplicities of the components of the multiset.

lb

Number of parts in the partition must be greater than this lower bound.

See also:

enum_all (page 979), *enum_small* (page 980), *enum_range* (page 980)

Examples

```
>>> m = MultisetPartitionTraverser()
>>> states = m.enum_large([2, 2], 2)
>>> list(list_visitor(state, 'ab') for state in states)
[[['a', 'a'], ['b'], ['b']],
[['a', 'b'], ['a'], ['b']],
[['a'], ['a'], ['b', 'b']],
[['a'], ['a'], ['b'], ['b']]]
```

enum_range(*multiplicities, lb, ub*)

Enumerate the partitions of a multiset with `lb < num(parts) <= ub`.

In particular, if partitions with exactly `k` parts are desired, call with `(multiplicities, k - 1, k)`. This method generalizes enum_all, enum_small, and enum_large.

Examples

```
>>> m = MultisetPartitionTraverser()
>>> states = m.enum_range([2, 2], 1, 2)
>>> list(list_visitor(state, 'ab') for state in states)
[[['a', 'a', 'b'], ['b']],
[['a', 'a'], ['b', 'b']],
[['a', 'b', 'b'], ['a']],
[['a', 'b'], ['a', 'b']]]
```

enum_small(*multiplicities, ub*)

Enumerate multiset partitions with no more than `ub` parts.

Equivalent to enum_range(multiplicities, 0, ub)

Parameters multiplicities

list of multiplicities of the components of the multiset.

ub

Maximum number of parts

See also:

enum_all (page 979), *enum_large* (page 979), *enum_range* (page 980)

References

[AOCP616] (page 1268)

Examples

```
>>> m = MultisetPartitionTraverser()
>>> states = m.enum_small([2, 2], 2)
>>> list(list_visitor(state, 'ab') for state in states)
[[['a', 'a', 'b', 'b']],
[['a', 'a', 'b'], ['b']],
[['a', 'a'], ['b', 'b']],
[['a', 'b', 'b'], ['a']],
[['a', 'b'], ['a', 'b']]]
```

The implementation is based, in part, on the answer given to exercise 69, in Knuth *[AOCP616]* (page 1268).

3.21.5 Iterables

variations

variations(seq, n) Returns all the variations of the list of size n.

Has an optional third argument. Must be a boolean value and makes the method return the variations with repetition if set to True, or the variations without repetition if set to False.

Examples::

```
>>> from diofant.utilities.iterables import variations
>>> list(variations([1, 2, 3], 2))
[(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)]
>>> list(variations([1, 2, 3], 2, True))
[(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)]
```

partitions

Although the combinatorics module contains Partition and IntegerPartition classes for investigation and manipulation of partitions, there are a few functions to generate partitions that can be used as low-level tools for routines: `partitions` and `multiset_partitions`. The former gives integer partitions, and the latter gives enumerated partitions of elements. There is also a routine `kbins` that will give a variety of permutations of partitions.

partitions:

```
>>> from diofant.utilities.iterables import partitions
>>> [p.copy() for s, p in partitions(7, m=2, size=True) if s == 2]
[{1: 1, 6: 1}, {2: 1, 5: 1}, {3: 1, 4: 1}]
```

multiset_partitions:

```
>>> from diofant.utilities.iterables import multiset_partitions
>>> [p for p in multiset_partitions(3, 2)]
[[[0, 1], [2]], [[0, 2], [1]], [[0], [1, 2]]]
>>> [p for p in multiset_partitions([1, 1, 1, 2], 2)]
[[[1, 1, 1], [2]], [[1, 1, 2], [1]], [[1, 1], [1, 2]]]
```

kbins:

```
>>> from diofant.utilities.iterables import kbins
>>> def show(k):
...     rv = []
...     for p in k:
...         rv.append(','.join([''.join(j) for j in p]))
...     return sorted(rv)
...
>>> show(kbins("ABCD", 2))
['A,BCD', 'AB,CD', 'ABC,D']
>>> show(kbins("ABC", 2))
['A,BC', 'AB,C']
>>> show(kbins("ABC", 2, ordered=0))  # same as multiset_partitions
['A,BC', 'AB,C', 'AC,B']
>>> show(kbins("ABC", 2, ordered=1))
['A,BC', 'A,CB',
 'B,AC', 'B,CA',
 'C,AB', 'C,BA']
>>> show(kbins("ABC", 2, ordered=10))
['A,BC', 'AB,C', 'AC,B',
 'B,AC', 'BC,A',
 'C,AB']
>>> show(kbins("ABC", 2, ordered=11))
['A,BC', 'A,CB', 'AB,C', 'AC,B',
 'B,AC', 'B,CA', 'BA,C', 'BC,A',
 'C,AB', 'C,BA', 'CA,B', 'CB,A']
```

Docstring

`diofant.utilities.iterables.`**`binary_partitions`**`(n)`

Generates the binary partition of n.

A binary partition consists only of numbers that are powers of two. Each step reduces a 2**(k+1) to 2**k and 2**k. Thus 16 is converted to 8 and 8.

References

[R618] (page 1268)

Examples

```
>>> for i in binary_partitions(5):
...     print(i)
...
[4, 1]
```

(continues on next page)

(continued from previous page)

```
[2, 2, 1]
[2, 1, 1, 1]
[1, 1, 1, 1, 1]
```

diofant.utilities.iterables.**bracelets**(*n, k*)
Wrapper to necklaces to return a free (unrestricted) necklace.

diofant.utilities.iterables.**cantor_product**(**args*)
Breadth-first (diagonal) cartesian product of iterables.

Each iterable is advanced in turn in a round-robin fashion. As usual with breadth-first, this comes at the cost of memory consumption.

```
>>> from itertools import islice, count
>>> list(islice(cantor_product(count(), count()), 9))
[(0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (1, 2), (2, 0), (2, 1), (2, 2)]
```

diofant.utilities.iterables.**capture**(*func*)
Return the printed output of func().

$func$ should be a function without arguments that produces output with print statements.

```
>>> def foo():
...     print('hello world!')
...
>>> 'hello' in capture(foo) # foo, not foo()
True
>>> capture(lambda: pprint(2/x, use_unicode=False))
'2\n-\nx\n'
```

diofant.utilities.iterables.**common_prefix**(**seqs*)
Return the subsequence that is a common start of sequences in `seqs`.

```
>>> common_prefix(list(range(3)))
[0, 1, 2]
>>> common_prefix(list(range(3)), list(range(4)))
[0, 1, 2]
>>> common_prefix([1, 2, 3], [1, 2, 5])
[1, 2]
>>> common_prefix([1, 2, 3], [1, 3, 5])
[1]
```

diofant.utilities.iterables.**common_suffix**(**seqs*)
Return the subsequence that is a common ending of sequences in `seqs`.

```
>>> common_suffix(list(range(3)))
[0, 1, 2]
>>> common_suffix(list(range(3)), list(range(4)))
[]
>>> common_suffix([1, 2, 3], [9, 2, 3])
[2, 3]
>>> common_suffix([1, 2, 3], [9, 7, 3])
[3]
```

diofant.utilities.iterables.**dict_merge**(**dicts*)
Merge dictionaries into a single dictionary.

diofant.utilities.iterables.**filter_symbols**(*iterator*, *exclude*)
Only yield elements from $iterator$ that do not occur in $exclude$.

Parameters iterator : iterable

iterator to take elements from

exclude : iterable

elements to exclude

Returns iterator : iterator

filtered iterator

diofant.utilities.iterables.**flatten**(*iterable*, *levels=None*, *cls=None*)
Recursively denest iterable containers.

```
>>> flatten([1, 2, 3])
[1, 2, 3]
>>> flatten([1, 2, [3]])
[1, 2, 3]
>>> flatten([1, [2, 3], [4, 5]])
[1, 2, 3, 4, 5]
>>> flatten([1.0, 2, (1, None)])
[1.0, 2, 1, None]
```

If you want to denest only a specified number of levels of nested containers, then set `levels` flag to the desired number of levels:

```
>>> ls = [[(-2, -1), (1, 2)], [(0, 0)]]
```

```
>>> flatten(ls, levels=1)
[(-2, -1), (1, 2), (0, 0)]
```

If cls argument is specified, it will only flatten instances of that class, for example:

```
>>> class MyOp(Basic):
...     pass
...
>>> flatten([MyOp(1, MyOp(2, 3))], cls=MyOp)
[1, 2, 3]
```

adapted from https://kogs-www.informatik.uni-hamburg.de/~meine/python_tricks

diofant.utilities.iterables.**generate_bell**(*n*)
Return permutations of [0, 1, ..., n - 1] such that each permutation differs from the last by the exchange of a single pair of neighbors. The `n!` permutations are returned as an iterator. In order to obtain the next permutation from a random starting permutation, use the `next_trotterjohnson` method of the Permutation class (which generates the same sequence in a different manner).

See also:

diofant.combinatorics.permutations.Permutation.next_trotterjohnson (page 176)

References

- https//en.wikipedia.org/wiki/Method_ringing

- https//stackoverflow.com/questions/4856615/recursive-permutation/4857018
- https://web.archive.org/web/20160324133718/http://programminggeeks.com/bell-algorithm-for-permutation/
- https//en.wikipedia.org/wiki/Steinhaus%E2%80%93Johnson%E2%80%93Trotter_algorithm
- Generating involutions, derangements, and relatives by ECO Vincent Vajnovszki, DMTCS vol 1 issue 12, 2010

Examples

```
>>> from itertools import permutations
```

This is the sort of permutation used in the ringing of physical bells, and does not produce permutations in lexicographical order. Rather, the permutations differ from each other by exactly one inversion, and the position at which the swapping occurs varies periodically in a simple fashion. Consider the first few permutations of 4 elements generated by `permutations` and `generate_bell`:

```
>>> list(permutations(range(4)))[:5]
[(0, 1, 2, 3), (0, 1, 3, 2), (0, 2, 1, 3), (0, 2, 3, 1), (0, 3, 1, 2)]
>>> list(generate_bell(4))[:5]
[(0, 1, 2, 3), (0, 1, 3, 2), (0, 3, 1, 2), (3, 0, 1, 2), (3, 0, 2, 1)]
```

Notice how the 2nd and 3rd lexicographical permutations have 3 elements out of place whereas each "bell" permutation always has only two elements out of place relative to the previous permutation (and so the signature (+/-1) of a permutation is opposite of the signature of the previous permutation).

How the position of inversion varies across the elements can be seen by tracing out where the largest number appears in the permutations:

```
>>> m = zeros(4, 24)
>>> for i, p in enumerate(generate_bell(4)):
...     m[:, i] = Matrix([j - 3 for j in list(p)])  # make largest zero
>>> m.print_nonzero('X')
[XXX  XXXXXX  XXXXXX  XXX]
[XX XX XXXX XX XXXX XX XX]
[X XXXX XX XXXX XX XXXX X]
[ XXXXXX  XXXXXX  XXXXXX ]
```

`diofant.utilities.iterables.`**`generate_derangements`**(*perm*)

Routine to generate unique derangements.

TODO: This will be rewritten to use the ECO operator approach once the permutations branch is in master.

See also:

diofant.functions.combinatorial.factorials.subfactorial (page 337)

Examples

```
>>> list(generate_derangements([0, 1, 2]))
[[1, 2, 0], [2, 0, 1]]
>>> list(generate_derangements([0, 1, 2, 3]))
[[1, 0, 3, 2], [1, 2, 3, 0], [1, 3, 0, 2], [2, 0, 3, 1],
 [2, 3, 0, 1], [2, 3, 1, 0], [3, 0, 1, 2], [3, 2, 0, 1],
 [3, 2, 1, 0]]
>>> list(generate_derangements([0, 1, 1]))
[]
```

diofant.utilities.iterables.**generate_involutions**(*n*)

Generates involutions.

An involution is a permutation that when multiplied by itself equals the identity permutation. In this implementation the involutions are generated using Fixed Points.

Alternatively, an involution can be considered as a permutation that does not contain any cycles with a length that is greater than two.

References

[R619] (page 1268)

Examples

```
>>> list(generate_involutions(3))
[(0, 1, 2), (0, 2, 1), (1, 0, 2), (2, 1, 0)]
>>> len(list(generate_involutions(4)))
10
```

diofant.utilities.iterables.**generate_oriented_forest**(*n*)

This algorithm generates oriented forests.

An oriented graph is a directed graph having no symmetric pair of directed edges. A forest is an acyclic graph, i.e., it has no cycles. A forest can also be described as a disjoint union of trees, which are graphs in which any two vertices are connected by exactly one simple path.

References

[R620] (page 1268), *[R621]* (page 1268)

Examples

```
>>> list(generate_oriented_forest(4))
[[0, 1, 2, 3], [0, 1, 2, 2], [0, 1, 2, 1], [0, 1, 2, 0],
 [0, 1, 1, 1], [0, 1, 1, 0], [0, 1, 0, 1], [0, 1, 0, 0], [0, 0, 0, 0]]
```

diofant.utilities.iterables.**group**(*seq*, *multiple=True*)

Splits a sequence into a list of lists of equal, adjacent elements.

See also:

`multiset` (page 990)

Examples

```
>>> group([1, 1, 1, 2, 2, 3])
[[1, 1, 1], [2, 2], [3]]
>>> group([1, 1, 1, 2, 2, 3], multiple=False)
[(1, 3), (2, 2), (3, 1)]
>>> group([1, 1, 3, 2, 2, 1], multiple=False)
[(1, 2), (3, 1), (2, 2), (1, 1)]
```

diofant.utilities.iterables.**has_dups**(*seq*)
Return True if there are any duplicate elements in `seq`.

Examples

```
>>> has_dups((1, 2, 1))
True
>>> has_dups(range(3))
False
>>> all(has_dups(c) is False for c in (set(), Set(), dict(), Dict()))
True
```

diofant.utilities.iterables.**has_variety**(*seq*)
Return True if there are any different elements in `seq`.

Examples

```
>>> has_variety((1, 2, 1))
True
>>> has_variety((1, 1, 1))
False
```

diofant.utilities.iterables.**ibin**(*n*, *bits=0*, *str=False*)
Return a list of length `bits` corresponding to the binary value of `n` with small bits to the right (last). If bits is omitted, the length will be the number required to represent `n`. If the bits are desired in reversed order, use the [::-1] slice of the returned list.

If a sequence of all bits-length lists starting from [0, 0,..., 0] through [1, 1, ..., 1] are desired, pass a non-integer for bits, e.g. 'all'.

If the bit *string* is desired pass `str=True`.

Examples

```
>>> ibin(2)
[1, 0]
>>> ibin(2, 4)
[0, 0, 1, 0]
>>> ibin(2, 4)[::-1]
[0, 1, 0, 0]
```

If all lists corresponding to 0 to 2**n - 1, pass a non-integer for bits:

```
>>> bits = 2
>>> for i in ibin(2, 'all'):
...     print(i)
(0, 0)
(0, 1)
(1, 0)
(1, 1)
```

If a bit string is desired of a given length, use str=True:

```
>>> n = 123
>>> bits = 10
>>> ibin(n, bits, str=True)
'0001111011'
>>> ibin(n, bits, str=True)[::-1]  # small bits left
'1101111000'
>>> list(ibin(3, 'all', str=True))
['000', '001', '010', '011', '100', '101', '110', '111']
```

`diofant.utilities.iterables.`**`kbins`**(*l*, *k*, *ordered=None*)

Return sequence `l` partitioned into `k` bins.

See also:

partitions (page 994), *multiset_partitions* (page 990)

Examples

The default is to give the items in the same order, but grouped into k partitions without any reordering:

```
>>> for p in kbins(list(range(5)), 2):
...     print(p)
...
[[0], [1, 2, 3, 4]]
[[0, 1], [2, 3, 4]]
[[0, 1, 2], [3, 4]]
[[0, 1, 2, 3], [4]]
```

The `ordered` flag which is either None (to give the simple partition of the the elements) or is a 2 digit integer indicating whether the order of the bins and the order of the items in the bins matters. Given:

```
A = [[0], [1, 2]]
B = [[1, 2], [0]]
C = [[2, 1], [0]]
D = [[0], [2, 1]]
```

the following values for `ordered` have the shown meanings:

```
00 means A == B == C == D
01 means A == B
10 means A == D
11 means A == A
```

```
>>> for ordered in [None, 0, 1, 10, 11]:
...     print('ordered = %s' % ordered)
...     for p in kbins(list(range(3)), 2, ordered=ordered):
...         print('     %s' % p)
...
ordered = None
     [[0], [1, 2]]
     [[0, 1], [2]]
ordered = 0
     [[0, 1], [2]]
     [[0, 2], [1]]
     [[0], [1, 2]]
ordered = 1
     [[0], [1, 2]]
     [[0], [2, 1]]
     [[1], [0, 2]]
     [[1], [2, 0]]
     [[2], [0, 1]]
     [[2], [1, 0]]
ordered = 10
     [[0, 1], [2]]
     [[2], [0, 1]]
     [[0, 2], [1]]
     [[1], [0, 2]]
     [[0], [1, 2]]
     [[1, 2], [0]]
ordered = 11
     [[0], [1, 2]]
     [[0, 1], [2]]
     [[0], [2, 1]]
     [[0, 2], [1]]
     [[1], [0, 2]]
     [[1, 0], [2]]
     [[1], [2, 0]]
     [[1, 2], [0]]
     [[2], [0, 1]]
     [[2, 0], [1]]
     [[2], [1, 0]]
     [[2, 1], [0]]
```

`diofant.utilities.iterables.`**`minlex`**(*seq*, *directed=True*, *is_set=False*, *small=None*)

Return a tuple where the smallest element appears first; if `directed` is True (default) then the order is preserved, otherwise the sequence will be reversed if that gives a smaller ordering.

If every element appears only once then is_set can be set to True for more efficient processing.

If the smallest element is known at the time of calling, it can be passed and the calculation of the smallest element will be omitted.

Examples

```
>>> minlex((1, 2, 0))
(0, 1, 2)
```

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```
>>> minlex((1, 0, 2))
(0, 2, 1)
>>> minlex((1, 0, 2), directed=False)
(0, 1, 2)
```

```
>>> minlex('11010011000', directed=True)
'00011010011'
>>> minlex('11010011000', directed=False)
'00011001011'
```

diofant.utilities.iterables.**multiset**(*seq*)

Return the hashable sequence in multiset form with values being the multiplicity of the item in the sequence.

See also:

group (page 986)

Examples

```
>>> multiset('mississippi')
{'i': 4, 'm': 1, 'p': 2, 's': 4}
```

diofant.utilities.iterables.**multiset_combinations**(*m, n, g=None*)

Return the unique combinations of size `n` from multiset `m`.

Examples

```
>>> from itertools import combinations
>>> [''.join(i) for i in  multiset_combinations('baby', 3)]
['abb', 'aby', 'bby']
```

```
>>> def count(f, s): return len(list(f(s, 3)))
```

The number of combinations depends on the number of letters; the number of unique combinations depends on how the letters are repeated.

```
>>> s1 = 'abracadabra'
>>> s2 = 'banana tree'
>>> count(combinations, s1), count(multiset_combinations, s1)
(165, 23)
>>> count(combinations, s2), count(multiset_combinations, s2)
(165, 54)
```

diofant.utilities.iterables.**multiset_partitions**(*multiset, m=None*)

Return unique partitions of the given multiset (in list form). If `m` is None, all multisets will be returned, otherwise only partitions with `m` parts will be returned.

If `multiset` is an integer, a range [0, 1, ..., multiset - 1] will be supplied.

Counting

The number of partitions of a set is given by the bell number:

```
>>> len(list(multiset_partitions(5))) == bell(5) == 52
True
```

The number of partitions of length k from a set of size n is given by the Stirling Number of the 2nd kind:

```
>>> def S2(n, k):
...     from diofant import Dummy, binomial, factorial, Sum
...     if k > n:
...         return 0
...     j = Dummy()
...     arg = (-1)**(k-j)*j**n*binomial(k,j)
...     return 1/factorial(k)*Sum(arg,(j,0,k)).doit()
...
>>> S2(5, 2) == len(list(multiset_partitions(5, 2))) == 15
True
```

These comments on counting apply to *sets*, not multisets.

See also:

partitions (page 994), *diofant.combinatorics.partitions.Partition* (page 155), *diofant.combinatorics.partitions.IntegerPartition* (page 157), *diofant.functions.combinatorial.numbers.nT* (page 347)

Notes

When all the elements are the same in the multiset, the order of the returned partitions is determined by the `partitions` routine. If one is counting partitions then it is better to use the `nT` function.

Examples

```
>>> list(multiset_partitions([1, 2, 3, 4], 2))
[[[1, 2, 3], [4]], [[1, 2, 4], [3]], [[1, 2], [3, 4]],
[[1, 3, 4], [2]], [[1, 3], [2, 4]], [[1, 4], [2, 3]],
[[1], [2, 3, 4]]]
>>> list(multiset_partitions([1, 2, 3, 4], 1))
[[[1, 2, 3, 4]]]
```

Only unique partitions are returned and these will be returned in a canonical order regardless of the order of the input:

```
>>> a = [1, 2, 2, 1]
>>> ans = list(multiset_partitions(a, 2))
>>> a.sort()
>>> list(multiset_partitions(a, 2)) == ans
True
>>> a = range(3, 1, -1)
>>> (list(multiset_partitions(a)) ==
...  list(multiset_partitions(sorted(a))))
True
```

If m is omitted then all partitions will be returned:

```
>>> list(multiset_partitions([1, 1, 2]))
[[[1, 1, 2]], [[1, 1], [2]], [[1, 2], [1]], [[1], [1], [2]]]
>>> list(multiset_partitions([1]*3))
[[[1, 1, 1]], [[1], [1, 1]], [[1], [1], [1]]]
```

diofant.utilities.iterables.**multiset_permutations**(*m*, *size=None*, *g=None*)
Return the unique permutations of multiset m.

Examples

```
>>> [''.join(i) for i in multiset_permutations('aab')]
['aab', 'aba', 'baa']
>>> factorial(len('banana'))
720
>>> len(list(multiset_permutations('banana')))
60
```

diofant.utilities.iterables.**necklaces**(*n*, *k*, *free=False*)
A routine to generate necklaces that may (free=True) or may not (free=False) be turned over to be viewed. The "necklaces" returned are comprised of n integers (beads) with k different values (colors). Only unique necklaces are returned.

References

http://mathworld.wolfram.com/Necklace.html

Examples

```
>>> def show(s, i):
...     return ''.join(s[j] for j in i)
```

The "unrestricted necklace" is sometimes also referred to as a "bracelet" (an object that can be turned over, a sequence that can be reversed) and the term "necklace" is used to imply a sequence that cannot be reversed. So ACB == ABC for a bracelet (rotate and reverse) while the two are different for a necklace since rotation alone cannot make the two sequences the same.

(mnemonic: Bracelets can be viewed Backwards, but Not Necklaces.)

```
>>> B = [show('ABC', i) for i in bracelets(3, 3)]
>>> N = [show('ABC', i) for i in necklaces(3, 3)]
>>> set(N) - set(B)
{'ACB'}
```

```
>>> list(necklaces(4, 2))
[(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 1),
 (0, 1, 0, 1), (0, 1, 1, 1), (1, 1, 1, 1)]
```

```
>>> [show('.o', i) for i in bracelets(4, 2)]
['....', '...o', '..oo', '.o.o', '.ooo', 'oooo']
```

`diofant.utilities.iterables.`**`numbered_symbols`**(*prefix='x'*, *cls=None*, *start=0*, *exclude=[]*, **args*, ***assumptions*)

Generate an infinite stream of Symbols consisting of a prefix and increasing subscripts provided that they do not occur in $exclude$.

Parameters prefix : str, optional

The prefix to use. By default, this function will generate symbols of the form "x0", "x1", etc.

cls : class, optional

The class to use. By default, it uses Symbol, but you can also use Wild or Dummy.

start : int, optional

The start number. By default, it is 0.

Returns sym : Symbol

The subscripted symbols.

`diofant.utilities.iterables.`**`ordered_partitions`**(*n*, *m=None*, *sort=True*)

Generates ordered partitions of integer `n`.

Parameters ‘‘m‘‘ : int or None, optional

By default (None) gives partitions of all sizes, else only those with size m. In addition, if `m` is not None then partitions are generated *in place* (see examples).

‘‘sort‘‘ : bool, optional

Controls whether partitions are returned in sorted order (default) when `m` is not None; when False, the partitions are returned as fast as possible with elements sorted, but when m|n the partitions will not be in ascending lexicographical order.

References

[R622] (page 1268), *[R623]* (page 1268)

Examples

All partitions of 5 in ascending lexicographical:

```
>>> for p in ordered_partitions(5):
...     print(p)
[1, 1, 1, 1, 1]
[1, 1, 1, 2]
[1, 1, 3]
[1, 2, 2]
[1, 4]
[2, 3]
[5]
```

Only partitions of 5 with two parts:

```
>>> for p in ordered_partitions(5, 2):
...     print(p)
[1, 4]
[2, 3]
```

When m is given, a given list objects will be used more than once for speed reasons so you will not see the correct partitions unless you make a copy of each as it is generated:

```
>>> [p for p in ordered_partitions(7, 3)]
[[1, 1, 1], [1, 1, 1], [1, 1, 1], [2, 2, 2]]
>>> [list(p) for p in ordered_partitions(7, 3)]
[[1, 1, 5], [1, 2, 4], [1, 3, 3], [2, 2, 3]]
```

When n is a multiple of m, the elements are still sorted but the partitions themselves will be *unordered* if sort is False; the default is to return them in ascending lexicographical order.

```
>>> for p in ordered_partitions(6, 2):
...     print(p)
[1, 5]
[2, 4]
[3, 3]
```

But if speed is more important than ordering, sort can be set to False:

```
>>> for p in ordered_partitions(6, 2, sort=False):
...     print(p)
[1, 5]
[3, 3]
[2, 4]
```

diofant.utilities.iterables.**partitions**(*n*, *m=None*, *k=None*, *size=False*)
Generate all partitions of positive integer, n.

Parameters ‘‘m‘‘ : integer (default gives partitions of all sizes)

limits number of parts in partition (mnemonic: m, maximum parts). Default value, None, gives partitions from 1 through n.

‘‘k‘‘ : integer (default gives partitions number from 1 through n)

limits the numbers that are kept in the partition (mnemonic: k, keys)

‘‘size‘‘ : bool (default False, only partition is returned)

when True then (M, P) is returned where M is the sum of the multiplicities and P is the generated partition.

Each partition is represented as a dictionary, mapping an integer

to the number of copies of that integer in the partition. For example,

the first partition of 4 returned is {4: 1}, “4: one of them”.

See also:

diofant.combinatorics.partitions.Partition (page 155), *diofant.combinatorics.partitions.IntegerPartition* (page 157)

Notes

Modified from Tim Peter's version [R624] (page 1268) to allow for k and m values.

References

[R624] (page 1268)

Examples

The numbers appearing in the partition (the key of the returned dict) are limited with k:

```
>>> from diofant.utilities.iterables import partitions
```

```
>>> for p in partitions(6, k=2):
...     print(p)
{2: 3}
{1: 2, 2: 2}
{1: 4, 2: 1}
{1: 6}
```

The maximum number of parts in the partition (the sum of the values in the returned dict) are limited with m:

```
>>> for p in partitions(6, m=2):
...     print(p)
...
{6: 1}
{1: 1, 5: 1}
{2: 1, 4: 1}
{3: 2}
```

Note that the _same_ dictionary object is returned each time. This is for speed: generating each partition goes quickly, taking constant time, independent of n.

```
>>> [p for p in partitions(6, k=2)]
[{1: 6}, {1: 6}, {1: 6}, {1: 6}]
```

If you want to build a list of the returned dictionaries then make a copy of them:

```
>>> [p.copy() for p in partitions(6, k=2)]
[{2: 3}, {1: 2, 2: 2}, {1: 4, 2: 1}, {1: 6}]
>>> [(M, p.copy()) for M, p in partitions(6, k=2, size=True)]
[(3, {2: 3}), (4, {1: 2, 2: 2}), (5, {1: 4, 2: 1}), (6, {1: 6})]
```

diofant.utilities.iterables.**permute_signs**(*t*)

Return iterator in which the signs of non-zero elements of t are permuted.

Examples

```
>>> list(permute_signs((0, 1, 2)))
[(0, 1, 2), (0, -1, 2), (0, 1, -2), (0, -1, -2)]
```

diofant.utilities.iterables.**postfixes**(*seq*)
Generate all postfixes of a sequence.

Examples

```
>>> list(postfixes([1,2,3,4]))
[[4], [3, 4], [2, 3, 4], [1, 2, 3, 4]]
```

diofant.utilities.iterables.**postorder_traversal**(*node*, *keys=None*)
Do a postorder traversal of a tree.

This generator recursively yields nodes that it has visited in a postorder fashion. That is, it descends through the tree depth-first to yield all of a node's children's postorder traversal before yielding the node itself.

Parameters node : diofant expression

The expression to traverse.

keys : (default None) sort key(s)

The key(s) used to sort args of Basic objects. When None, args of Basic objects are processed in arbitrary order. If key is defined, it will be passed along to ordered() as the only key(s) to use to sort the arguments; if `key` is simply True then the default keys of `ordered` will be used (node count and default_sort_key).

Yields subtree : diofant expression

All of the subtrees in the tree.

Examples

```
>>> from diofant.abc import w
```

The nodes are returned in the order that they are encountered unless key is given; simply passing key=True will guarantee that the traversal is unique.

```
>>> list(postorder_traversal(w + (x + y)*z, keys=True))
[w, z, x, y, x + y, z*(x + y), w + z*(x + y)]
```

diofant.utilities.iterables.**prefixes**(*seq*)
Generate all prefixes of a sequence.

Examples

```
>>> list(prefixes([1,2,3,4]))
[[1], [1, 2], [1, 2, 3], [1, 2, 3, 4]]
```

diofant.utilities.iterables.**reshape**(*seq*, *how*)
Reshape the sequence according to the template in `how`.

Examples

```
>>> seq = list(range(1, 9))
```

```
>>> reshape(seq, [4]) # lists of 4
[[1, 2, 3, 4], [5, 6, 7, 8]]
```

```
>>> reshape(seq, (4,)) # tuples of 4
[(1, 2, 3, 4), (5, 6, 7, 8)]
```

```
>>> reshape(seq, (2, 2)) # tuples of 4
[(1, 2, 3, 4), (5, 6, 7, 8)]
```

```
>>> reshape(seq, (2, [2])) # (i, i, [i, i])
[(1, 2, [3, 4]), (5, 6, [7, 8])]
```

```
>>> reshape(seq, ((2,), [2])) # etc....
[((1, 2), [3, 4]), ((5, 6), [7, 8])]
```

```
>>> reshape(seq, (1, [2], 1))
[(1, [2, 3], 4), (5, [6, 7], 8)]
```

```
>>> reshape(tuple(seq), ([[1], 1, (2,)],))
(([[1], 2, (3, 4)],), ([[5], 6, (7, 8)],))
```

```
>>> reshape(tuple(seq), ([1], 1, (2,)))
(([1], 2, (3, 4)), ([5], 6, (7, 8)))
```

```
>>> reshape(list(range(12)), [2, [3], {2}, (1, (3,), 1)])
[[0, 1, [2, 3, 4], {5, 6}, (7, (8, 9, 10), 11)]]
```

`diofant.utilities.iterables.`**`rotate_left`**(x, y)

Left rotates a list x by the number of steps specified in y.

Examples

```
>>> a = [0, 1, 2]
>>> rotate_left(a, 1)
[1, 2, 0]
```

`diofant.utilities.iterables.`**`rotate_right`**(x, y)

Right rotates a list x by the number of steps specified in y.

Examples

```
>>> a = [0, 1, 2]
>>> rotate_right(a, 1)
[2, 0, 1]
```

diofant.utilities.iterables.**runs**(*seq*, *op=<built-in function gt>*)

Group the sequence into lists in which successive elements all compare the same with the comparison operator, op: op(seq[i + 1], seq[i]) is True from all elements in a run.

Examples

```
>>> from operator import ge
>>> runs([0, 1, 2, 2, 1, 4, 3, 2, 2])
[[0, 1, 2], [2], [1, 4], [3], [2], [2]]
>>> runs([0, 1, 2, 2, 1, 4, 3, 2, 2], op=ge)
[[0, 1, 2, 2], [1, 4], [3], [2, 2]]
```

diofant.utilities.iterables.**sift**(*seq*, *keyfunc*)

Sift the sequence, seq into a dictionary according to keyfunc.

OUTPUT: each element in expr is stored in a list keyed to the value of keyfunc for the element.

See also:

diofant.core.compatibility.ordered (page 150)

Examples

```
>>> from collections import defaultdict
```

```
>>> sift(range(5), lambda x: x % 2) == defaultdict(int, {0: [0, 2, 4], 1: [1, 3]})
True
```

sift() returns a defaultdict() object, so any key that has no matches will give [].

```
>>> dl = sift([x], lambda x: x.is_commutative)
>>> dl == defaultdict(list, {True: [x]})
True
>>> dl[False]
[]
```

Sometimes you won't know how many keys you will get:

```
>>> sift([sqrt(x), exp(x), (y**x)**2],
...      lambda x: x.as_base_exp()[0]) == defaultdict(list,
...      {E: [exp(x)], x: [sqrt(x)], y: [y**(2*x)]})
True
```

If you need to sort the sifted items it might be better to use ordered which can economically apply multiple sort keys to a squence while sorting.

diofant.utilities.iterables.**signed_permutations**(*t*)

Return iterator in which the signs of non-zero elements of t and the order of the elements are permuted.

Examples

```
>>> list(signed_permutations((0, 1, 2)))
[(0, 1, 2), (0, -1, 2), (0, 1, -2), (0, -1, -2), (0, 2, 1),
 (0, -2, 1), (0, 2, -1), (0, -2, -1), (1, 0, 2), (-1, 0, 2),
 (1, 0, -2), (-1, 0, -2), (1, 2, 0), (-1, 2, 0), (1, -2, 0),
 (-1, -2, 0), (2, 0, 1), (-2, 0, 1), (2, 0, -1), (-2, 0, -1),
 (2, 1, 0), (-2, 1, 0), (2, -1, 0), (-2, -1, 0)]
```

`diofant.utilities.iterables.`**`subsets`**(*seq*, *k=None*, *repetition=False*)

Generates all k-subsets (combinations) from an n-element set, seq.

A k-subset of an n-element set is any subset of length exactly k. The number of k-subsets of an n-element set is given by binomial(n, k), whereas there are 2**n subsets all together. If k is None then all 2**n subsets will be returned from shortest to longest.

Examples

subsets(seq, k) will return the n!/k!/(n - k)! k-subsets (combinations) without repetition, i.e. once an item has been removed, it can no longer be "taken":

```
>>> from diofant.utilities.iterables import subsets
```

```
>>> list(subsets([1, 2], 2))
[(1, 2)]
>>> list(subsets([1, 2]))
[(), (1,), (2,), (1, 2)]
>>> list(subsets([1, 2, 3], 2))
[(1, 2), (1, 3), (2, 3)]
```

subsets(seq, k, repetition=True) will return the (n - 1 + k)!/k!/(n - 1)! combinations *with* repetition:

```
>>> list(subsets([1, 2], 2, repetition=True))
[(1, 1), (1, 2), (2, 2)]
```

If you ask for more items than are in the set you get the empty set unless you allow repetitions:

```
>>> list(subsets([0, 1], 3, repetition=False))
[]
>>> list(subsets([0, 1], 3, repetition=True))
[(0, 0, 0), (0, 0, 1), (0, 1, 1), (1, 1, 1)]
```

`diofant.utilities.iterables.`**`take`**(*iter*, *n*)

Return `n` items from `iter` iterator.

`diofant.utilities.iterables.`**`topological_sort`**(*graph*, *key=None*)

Topological sort of graph's vertices.

Parameters ''graph'' : `tuple[list, list[tuple[T, T]]`

A tuple consisting of a list of vertices and a list of edges of a graph to be sorted topologically.

''key'' : `callable[T]` (optional)

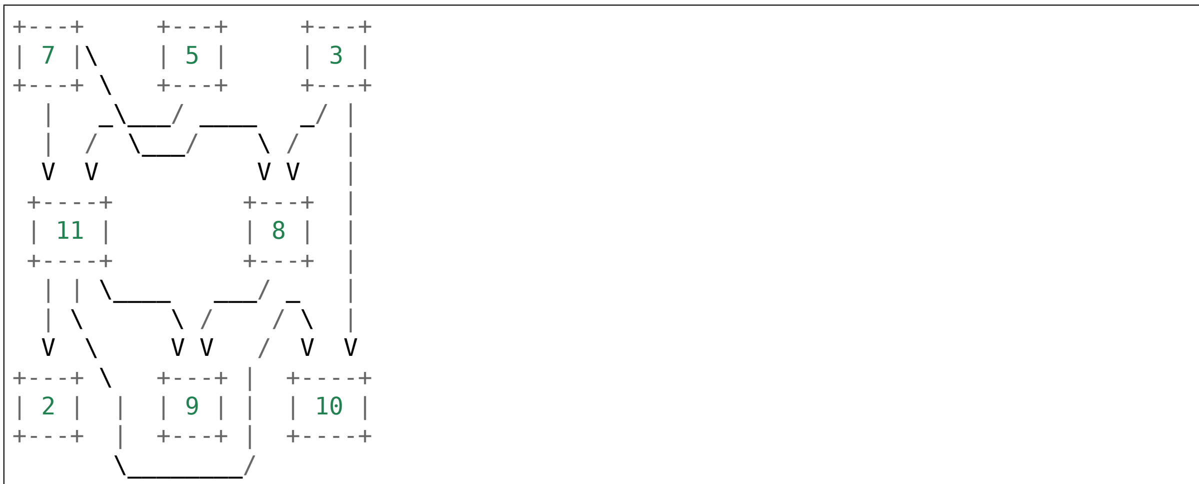
Ordering key for vertices on the same level. By default the natural (e.g. lexicographic) ordering is used (in this case the base type must implement ordering relations).

References

[R625] (page 1268)

Examples

Consider a graph:



where vertices are integers. This graph can be encoded using elementary Python's data structures as follows:

```
>>> V = [2, 3, 5, 7, 8, 9, 10, 11]
>>> E = [(7, 11), (7, 8), (5, 11), (3, 8), (3, 10),
...      (11, 2), (11, 9), (11, 10), (8, 9)]
```

To compute a topological sort for graph `(V, E)` issue:

```
>>> topological_sort((V, E))
[3, 5, 7, 8, 11, 2, 9, 10]
```

If specific tie breaking approach is needed, use `key` parameter:

```
>>> topological_sort((V, E), key=lambda v: -v)
[7, 5, 11, 3, 10, 8, 9, 2]
```

Only acyclic graphs can be sorted. If the input graph has a cycle, then ValueError will be raised:

```
>>> topological_sort((V, E + [(10, 7)]))
Traceback (most recent call last):
...
ValueError: cycle detected
```

`diofant.utilities.iterables.`**`unflatten`**(*iter*, *n=2*)

Group `iter` into tuples of length `n`. Raise an error if the length of `iter` is not a multiple of `n`.

`diofant.utilities.iterables.`**`uniq`**(*seq*, *result=None*)

Yield unique elements from `seq` as an iterator. The second parameter `result` is used internally; it is not necessary to pass anything for this.

Examples

```
>>> dat = [1, 4, 1, 5, 4, 2, 1, 2]
>>> type(uniq(dat)) in (list, tuple)
False
```

```
>>> list(uniq(dat))
[1, 4, 5, 2]
>>> list(uniq(x for x in dat))
[1, 4, 5, 2]
>>> list(uniq([[1], [2, 1], [1]]))
[[1], [2, 1]]
```

`diofant.utilities.iterables.`**`variations`**(*seq*, *n*, *repetition=False*)

Returns a generator of the n-sized variations of `seq` (size N). `repetition` controls whether items in `seq` can appear more than once;

Examples

variations(seq, n) will return N! / (N - n)! permutations without repetition of seq's elements:

```
>>> list(variations([1, 2], 2))
[(1, 2), (2, 1)]
```

variations(seq, n, True) will return the N**n permutations obtained by allowing repetition of elements:

```
>>> list(variations([1, 2], 2, repetition=True))
[(1, 1), (1, 2), (2, 1), (2, 2)]
```

If you ask for more items than are in the set you get the empty set unless you allow repetitions:

```
>>> list(variations([0, 1], 3, repetition=False))
[]
>>> list(variations([0, 1], 3, repetition=True))[:4]
[(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1)]
```

3.21.6 Lambdify

This module provides convenient functions to transform diofant expressions to lambda functions which can be used to calculate numerical values very fast.

diofant.utilities.lambdify.**implemented_function**(*symfunc*, *implementation*)

Add numerical implementation to function symfunc.

symfunc can be an UndefinedFunction instance, or a name string. In the latter case we create an UndefinedFunction instance with that name.

Be aware that this is a quick workaround, not a general method to create special symbolic functions. If you want to create a symbolic function to be used by all the machinery of Diofant you should subclass the Function class.

Parameters symfunc : str or UndefinedFunction instance

If str, then create new UndefinedFunction with this as name. If $symfunc$ is a diofant function, attach implementation to it.

implementation : callable

numerical implementation to be called by evalf() or lambdify

Returns afunc : diofant.FunctionClass instance

function with attached implementation

Examples

```
>>> f = implemented_function(Function('f'), lambda x: x+1)
>>> lam_f = lambdify(x, f(x))
>>> lam_f(4)
5
```

diofant.utilities.lambdify.**lambdastr**(*args*, *expr*, *printer=None*, *dummify=False*)

Returns a string that can be evaluated to a lambda function.

Examples

```
>>> lambdastr(x, x**2)
'lambda x: (x**2)'
>>> lambdastr((x, y, z), [z, y, x])
'lambda x,y,z: ([z, y, x])'
```

Although tuples may not appear as arguments to lambda in Python 3, lambdastr will create a lambda function that will unpack the original arguments so that nested arguments can be handled:

```
>>> lambdastr((x, (y, z)), x + y)
'lambda _0,_1: (lambda x,y,z: (x + y))(*list(__flatten_args__([_0,_1])))'
```

diofant.utilities.lambdify.**lambdify**(*args*, *expr*, *modules=None*, *printer=None*, *use_imps=True*, *dummify=True*)

Returns a lambda function for fast calculation of numerical values.

If not specified differently by the user, modules defaults to ["numpy"] if NumPy is installed, and ["math", "mpmath", "sympy"] if it isn't, that is, Diofant functions are replaced as far as possible by either numpy functions if available, and Python's standard library math, or mpmath functions otherwise. To change this behavior, the "modules" argument can be used. It accepts:

- the strings "math", "mpmath", "numpy", "numexpr", "diofant"

- any modules (e.g. math)
- dictionaries that map names of diofant functions to arbitrary functions
- lists that contain a mix of the arguments above, with higher priority given to entries appearing first.

The default behavior is to substitute all arguments in the provided expression with dummy symbols. This allows for applied functions (e.g. f(t)) to be supplied as arguments. Call the function with dummify=False if dummy substitution is unwanted (and $args$ is not a string). If you want to view the lambdified function or provide "diofant" as the module, you should probably set dummify=False.

For functions involving large array calculations, numexpr can provide a significant speedup over numpy. Please note that the available functions for numexpr are more limited than numpy but can be expanded with implemented_function and user defined subclasses of Function. If specified, numexpr may be the only option in modules. The official list of numexpr functions can be found at: https://numexpr.readthedocs.io/en/latest/user_guide.html#supported-functions

In previous releases `lambdify` replaced `Matrix` with `numpy.matrix` by default. As of release 0.7.7 `numpy.array` is the default. To get the old default behavior you must pass in `[{'ImmutableMatrix': numpy.matrix}, 'numpy']` to the `modules` kwarg.

```
>>> import numpy
>>> array2mat = [{'ImmutableMatrix': numpy.matrix}, 'numpy']
>>> f = lambdify((x, y), Matrix([x, y]), modules=array2mat)
>>> f(1, 2)
[[1]
 [2]]
```

1. Use one of the provided modules:

```
>>> f = lambdify(x, sin(x), "math")
```

Attention: Functions that are not in the math module will throw a name
error when the lambda function is evaluated! So this would be better:

```
>>> f = lambdify(x, sin(x)*gamma(x), ("math", "mpmath", "diofant"))
```

2. Use some other module:

```
>>> import numpy
>>> f = lambdify((x, y), tan(x*y), numpy)
```

Attention: There are naming differences between numpy and diofant. So if
you simply take the numpy module, e.g. diofant.atan will not be translated to numpy.arctan. Use the modified module instead by passing the string "numpy":

```
>>> f = lambdify((x, y), tan(x*y), "numpy")
>>> f(1, 2)
-2.18503986326
>>> from numpy import array
>>> f(array([1, 2, 3]), array([2, 3, 5]))
[-2.18503986 -0.29100619 -0.8559934 ]
```

3. Use a dictionary defining custom functions:

```
>>> def my_cool_function(x): return 'sin(%s) is cool' % x
>>> myfuncs = {"sin" : my_cool_function}
>>> f = lambdify(x, sin(x), myfuncs); f(1)
'sin(1) is cool'
```

Examples

```
>>> from diofant.abc import w
```

```
>>> f = lambdify(x, x**2)
>>> f(2)
4
>>> f = lambdify((x, y, z), [z, y, x])
>>> f(1, 2, 3)
[3, 2, 1]
>>> f = lambdify(x, sqrt(x))
>>> f(4)
2.0
>>> f = lambdify((x, y), sin(x*y)**2)
>>> f(0, 5)
0.0
>>> row = lambdify((x, y), Matrix((x, x + y)).T, modules='diofant')
>>> row(1, 2)
Matrix([[1, 3]])
```

Tuple arguments are handled and the lambdified function should be called with the same type of arguments as were used to create the function.:

```
>>> f = lambdify((x, (y, z)), x + y)
>>> f(1, (2, 4))
3
```

A more robust way of handling this is to always work with flattened arguments:

```
>>> args = w, (x, (y, z))
>>> vals = 1, (2, (3, 4))
>>> f = lambdify(flatten(args), w + x + y + z)
>>> f(*flatten(vals))
10
```

Functions present in $expr$ can also carry their own numerical implementations, in a callable attached to the `_imp_` attribute. Usually you attach this using the `implemented_function` factory:

```
>>> f = implemented_function(Function('f'), lambda x: x+1)
>>> func = lambdify(x, f(x))
>>> func(4)
5
```

`lambdify` always prefers `_imp_` implementations to implementations in other namespaces, unless the `use_imps` input parameter is False.

3.21.7 Memoization

diofant.utilities.memoization.**recurrence_memo**(*initial*)
Memo decorator for sequences defined by recurrence

See usage examples e.g. in the specfun/combinatorial module

3.21.8 Miscellaneous

Miscellaneous stuff that doesn't really fit anywhere else.

diofant.utilities.misc.**debug**(**args*)
Print `*args` if DIOFANT_DEBUG is True, else do nothing.

diofant.utilities.misc.**filldedent**(*s*, *w=70*)
Strips leading and trailing empty lines from a copy of s, then dedents, fills and returns it.

Empty line stripping serves to deal with docstrings like this one that start with a newline after the initial triple quote, inserting an empty line at the beginning of the string.

3.21.9 Randomised Testing

Helpers for randomized testing

diofant.utilities.randtest.**random_complex_number**(*a=2*, *b=-1*, *c=3*, *d=1*, *rational=True*)
Return a random complex number.

To reduce chance of hitting branch cuts or anything, we guarantee b <= Im z <= d, a <= Re z <= c

diofant.utilities.randtest.**verify_derivative_numerically**(*f*, *z*, *tol=1e-06*, *a=2*, *b=-1*, *c=3*, *d=1*)
Test numerically that the symbolically computed derivative of f with respect to z is correct.

This routine does not test whether there are Floats present with precision higher than 15 digits so if there are, your results may not be what you expect due to round-off errors.

Examples

```
>>> verify_derivative_numerically(sin(x), x)
true
```

diofant.utilities.randtest.**verify_numerically**(*f*, *g*, *z=None*, *tol=1e-06*, *a=2*, *b=-1*, *c=3*, *d=1*)
Test numerically that f and g agree when evaluated in the argument z.

If z is None, all symbols will be tested. This routine does not test whether there are Floats present with precision higher than 15 digits so if there are, your results may not be what you expect due to round- off errors.

Examples

```
>>> verify_numerically(sin(x)**2 + cos(x)**2, 1, x)
true
```

3.22 Parsing

3.22.1 Parsing Functions Reference

diofant.parsing.sympy_parser.**parse_expr**(*s, local_dict=None, transformations=(<function lambda_notation>, <function auto_symbol>, <function auto_number>), global_dict=None, evaluate=True*)

Converts the string `s` to a Diofant expression, in `local_dict`

Parameters s : str

The string to parse.

local_dict : dict, optional

A dictionary of local variables to use when parsing.

global_dict : dict, optional

A dictionary of global variables. By default, this is initialized with `from diofant import *`; provide this parameter to override this behavior (for instance, to parse `"Q & S"`).

transformations : tuple, optional

A tuple of transformation functions used to modify the tokens of the parsed expression before evaluation. The default transformations convert numeric literals into their Diofant equivalents, convert undefined variables into Diofant symbols.

evaluate : bool, optional

When False, the order of the arguments will remain as they were in the string and automatic simplification that would normally occur is suppressed. (see examples)

See also:

diofant.parsing.sympy_parser.stringify_expr (page 1007), *diofant.parsing.sympy_parser.eval_expr* (page 1007), *diofant.parsing.sympy_parser.standard_transformations* (page 1007), *diofant.parsing.sympy_parser.implicit_multiplication_application* (page 1008)

Examples

```
>>> parse_expr("1/2")
1/2
>>> type(_)
```

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```
<class 'diofant.core.numbers.Half'>
>>> transformations = (standard_transformations +
...                    (implicit_multiplication_application,))
>>> parse_expr("2x", transformations=transformations)
2*x
```

When evaluate=False, some automatic simplifications will not occur:

```
>>> parse_expr("2**3"), parse_expr("2**3", evaluate=False)
(8, 2**3)
```

In addition the order of the arguments will not be made canonical. This feature allows one to tell exactly how the expression was entered:

```
>>> a = parse_expr('1 + x', evaluate=False)
>>> b = parse_expr('x + 1', evaluate=0)
>>> a == b
False
>>> a.args
(1, x)
>>> b.args
(x, 1)
```

`diofant.parsing.sympy_parser.`**`stringify_expr`**(*s*, *local_dict*, *global_dict*, *transformations*)

Converts the string `s` to Python code, in `local_dict`

Generally, `parse_expr` should be used.

`diofant.parsing.sympy_parser.`**`eval_expr`**(*code*, *local_dict*, *global_dict*)

Evaluate Python code generated by `stringify_expr`.

Generally, `parse_expr` should be used.

`diofant.parsing.maxima.`**`parse_maxima`**(*str*, *globals=None*, *name_dict={}*)

`diofant.parsing.mathematica.`**`mathematica`**(*s*)

3.22.2 Parsing Transformations Reference

A transformation is a function that accepts the arguments `tokens`, `local_dict`, `global_dict` and returns a list of transformed tokens. They can be used by passing a list of functions to *`parse_expr()`* (page 1006) and are applied in the order given.

`diofant.parsing.sympy_parser.`**`standard_transformations = (<function lambda_notation>, <func`**

Standard transformations for *`parse_expr()`* (page 1006). Inserts calls to *`Symbol`* (page 79), *`Integer`* (page 89), and other Diofant datatypes.

`diofant.parsing.sympy_parser.`**`split_symbols`**(*tokens*, *local_dict*, *global_dict*)

Splits symbol names for implicit multiplication.

Intended to let expressions like `xyz` be parsed as `x*y*z`. Does not split Greek character names, so `theta` will *not* become `t*h*e*t*a`. Generally this should be used with `implicit_multiplication`.

`diofant.parsing.sympy_parser.`**`split_symbols_custom`**(*predicate*)

Creates a transformation that splits symbol names.

`predicate` should return True if the symbol name is to be split.

For instance, to retain the default behavior but avoid splitting certain symbol names, a predicate like this would work:

```
>>> def can_split(symbol):
...     if symbol not in ('list', 'of', 'unsplittable', 'names'):
...             return _token_splittable(symbol)
...     return False
...
>>> transformation = split_symbols_custom(can_split)
>>> parse_expr('unsplittable', transformations=standard_transformations +
...             (transformation, implicit_multiplication))
unsplittable
```

diofant.parsing.sympy_parser.**implicit_multiplication**(*result*, *local_dict*, *global_dict*)

Makes the multiplication operator optional in most cases.

Use this before *implicit_application()* (page 1008), otherwise expressions like `sin 2x` will be parsed as `x * sin(2)` rather than `sin(2*x)`.

Examples

```
>>> transformations = standard_transformations + (implicit_multiplication,)
>>> parse_expr('3 x y', transformations=transformations)
3*x*y
```

diofant.parsing.sympy_parser.**implicit_application**(*result*, *local_dict*, *global_dict*)

Makes parentheses optional in some cases for function calls.

Use this after *implicit_multiplication()* (page 1008), otherwise expressions like `sin 2x` will be parsed as `x * sin(2)` rather than `sin(2*x)`.

Examples

```
>>> transformations = standard_transformations + (implicit_application,)
>>> parse_expr('cot z + csc z', transformations=transformations)
cot(z) + csc(z)
```

diofant.parsing.sympy_parser.**function_exponentiation**(*tokens*, *local_dict*, *global_dict*)

Allows functions to be exponentiated, e.g. `cos**2(x)`.

Examples

```
>>> transformations = standard_transformations + (function_exponentiation,)
>>> parse_expr('sin**4(x)', transformations=transformations)
sin(x)**4
```

diofant.parsing.sympy_parser.**implicit_multiplication_application**(*result*, *local_dict*, *global_dict*)

Allows a slightly relaxed syntax.

- Parentheses for single-argument method calls are optional.
- Multiplication is implicit.
- Symbol names can be split (i.e. spaces are not needed between symbols).
- Functions can be exponentiated.

Examples

```
>>> parse_expr("10sin**2 x**2 + 3xyz + tan theta",
... transformations=(standard_transformations +
...                  (implicit_multiplication_application,)))
3*x*y*z + 10*sin(x**2)**2 + tan(theta)
```

`diofant.parsing.sympy_parser.`**`rationalize`**(*tokens*, *local_dict*, *global_dict*)
Converts floats into `Rational`. Run AFTER `auto_number`.

`diofant.parsing.sympy_parser.`**`convert_xor`**(*tokens*, *local_dict*, *global_dict*)
Treats XOR, ^, as exponentiation, **.

These are included in :data:`diofant.parsing.sympy_parser.standard_transformations` and generally don't need to be manually added by the user.

`diofant.parsing.sympy_parser.`**`auto_symbol`**(*tokens*, *local_dict*, *global_dict*)
Inserts calls to `Symbol` for undefined variables.

`diofant.parsing.sympy_parser.`**`auto_number`**(*tokens*, *local_dict*, *global_dict*)
Converts numeric literals to use Diofant equivalents.

Complex numbers use `I`; integer literals use `Integer`, float literals use `Float`, and repeating decimals use `Rational`.

3.23 Calculus

Some calculus-related methods waiting to find a better place in the Diofant modules tree.

`diofant.calculus.euler.`**`euler_equations`**(*L*, *funcs=()*, *vars=()*)
Find the Euler-Lagrange equations [R45] (page 1268) for a given Lagrangian.

Parameters L : Expr

The Lagrangian that should be a function of the functions listed in the second argument and their derivatives.

For example, in the case of two functions $f(x, y)$, $g(x, y)$ and two independent variables x, y the Lagrangian would have the form:

$$L\left(f(x,y),g(x,y),\frac{\partial f(x,y)}{\partial x},\frac{\partial f(x,y)}{\partial y},\frac{\partial g(x,y)}{\partial x},\frac{\partial g(x,y)}{\partial y},x,y\right)$$

In many cases it is not necessary to provide anything, except the Lagrangian, it will be auto-detected (and an error raised if this couldn't be done).

funcs : Function or an iterable of Functions

The functions that the Lagrangian depends on. The Euler equations are differential equations for each of these functions.

vars : Symbol or an iterable of Symbols

The Symbols that are the independent variables of the functions.

Returns eqns : list of Eq

The list of differential equations, one for each function.

References

[R45] (page 1268)

Examples

```
>>> x = Function('x')
>>> t = Symbol('t')
>>> L = (x(t).diff(t))**2/2 - x(t)**2/2
>>> euler_equations(L, x(t), t)
[Eq(-x(t) - Derivative(x(t), t, t), 0)]
>>> u = Function('u')
>>> x = Symbol('x')
>>> L = (u(t, x).diff(t))**2/2 - (u(t, x).diff(x))**2/2
>>> euler_equations(L, u(t, x), [t, x])
[Eq(-Derivative(u(t, x), t, t) + Derivative(u(t, x), x, x), 0)]
```

diofant.calculus.singularities.**singularities**(*f*, *x*)

Find singularities of real-valued function f with respect to x.

Notes

Removable singularities are not supported now.

References

[R46] (page 1268)

Examples

```
>>> singularities(1/(1 + x), x)
{-1}
```

```
>>> singularities(exp(1/x) + log(x + 1), x)
{-1, 0}
```

```
>>> singularities(exp(1/log(x + 1)), x)
{0}
```

diofant.calculus.optimization.**minimize**(*f*, **v*)

Minimizes f with respect to given variables v.

See also:

maximize (page 1011)

Examples

```
>>> minimize(x**2, x)
(0, {x: 0})
```

```
>>> minimize([x**2, x >= 1], x)
(1, {x: 1})
>>> minimize([-x**2, x >= -2, x <= 1], x)
(-4, {x: -2})
```

diofant.calculus.optimization.**maximize**(*f*, **v*)

Maximizes f with respect to given variables v.

See also:

minimize (page 1010)

3.23.1 Finite difference weights

This module implements an algorithm for efficient generation of finite difference weights for ordinary differentials of functions for derivatives from 0 (interpolation) up to arbitrary order.

The core algorithm is provided in the finite difference weight generating function (finite_diff_weights), and two convenience functions are provided for:

- **estimating a derivative (or interpolate) directly from a series of points** is also provided (`apply_finite_diff`).
- **making a finite difference approximation of a Derivative instance** (`as_finite_diff`).

diofant.calculus.finite_diff.**apply_finite_diff**(*order*, *x_list*, *y_list*, *x0=Integer(0)*)

Calculates the finite difference approximation of the derivative of requested order at x0 from points provided in x_list and y_list.

Parameters order: int

order of derivative to approximate. 0 corresponds to interpolation.

x_list: sequence

Sequence of (unique) values for the independent variable.

y_list: sequence

The function value at corresponding values for the independent variable in x_list.

x0: Number or Symbol

At what value of the independent variable the derivative should be evaluated. Defaults to Integer(0).

Returns diofant.core.add.Add or diofant.core.numbers.Number

The finite difference expression approximating the requested derivative order at x0.

See also:

diofant.calculus.finite_diff.finite_diff_weights (page 1013)

Notes

Order = 0 corresponds to interpolation. Only supply so many points you think makes sense to around x0 when extracting the derivative (the function need to be well behaved within that region). Also beware of Runge's phenomenon.

References

Fortran 90 implementation with Python interface for numerics: finitediff

Examples

```
>>> cube = lambda arg: (1.0*arg)**3
>>> xlist = range(-3, 4)
>>> apply_finite_diff(2, xlist, list(map(cube, xlist)), 2) - 12
-3.55271367880050e-15
```

we see that the example above only contain rounding errors. apply_finite_diff can also be used on more abstract objects:

```
>>> x, y = map(IndexedBase, 'xy')
>>> i = Idx('i')
>>> x_list, y_list = zip(*[(x[i + j], y[i + j]) for j in range(-1, 2)])
>>> apply_finite_diff(1, x_list, y_list, x[i])
(-1 + (x[i + 1] - x[i])/(-x[i - 1] + x[i]))*y[i]/(x[i + 1] - x[i]) +
(-x[i - 1] + x[i])*y[i + 1]/((-x[i - 1] + x[i + 1])*(x[i + 1] - x[i])) -
(x[i + 1] - x[i])*y[i - 1]/((-x[i - 1] + x[i + 1])*(-x[i - 1] + x[i]))
```

`diofant.calculus.finite_diff.`**`as_finite_diff`**(*derivative*, *points=1*, *x0=None*, *wrt=None*)

Returns an approximation of a derivative of a function in the form of a finite difference formula. The expression is a weighted sum of the function at a number of discrete values of (one of) the independent variable(s).

Parameters derivative: a Derivative instance (needs to have an variables

and expr attribute).

points: sequence or coefficient, optional

If sequence: discrete values (length >= order+1) of the independent variable used for generating the finite difference weights. If it is a coefficient, it will be used as the step-size for generating an equidistant sequence of length order+1 centered around x0. default: 1 (step-size 1)

x0: number or Symbol, optional

the value of the independent variable (wrt) at which the derivative is to be approximated. default: same as wrt

wrt: Symbol, optional

"with respect to" the variable for which the (partial) derivative is to be approximated for. If not provided it is required that the Derivative is ordinary. default: None

See also:

diofant.calculus.finite_diff.apply_finite_diff (page 1011), *diofant.calculus.finite_diff.finite_diff_weights* (page 1013)

Examples

```
>>> x, h = symbols('x h')
>>> as_finite_diff(f(x).diff(x))
-f(x - 1/2) + f(x + 1/2)
```

The default step size and number of points are 1 and `order + 1` respectively. We can change the step size by passing a symbol as a parameter:

```
>>> as_finite_diff(f(x).diff(x), h)
-f(-h/2 + x)/h + f(h/2 + x)/h
```

We can also specify the discretized values to be used in a sequence:

```
>>> as_finite_diff(f(x).diff(x), [x, x+h, x+2*h])
-3*f(x)/(2*h) + 2*f(h + x)/h - f(2*h + x)/(2*h)
```

The algorithm is not restricted to use equidistant spacing, nor do we need to make the approximation around x0, but we can get an expression estimating the derivative at an offset:

```
>>> e, sq2 = exp(1), sqrt(2)
>>> xl = [x-h, x+h, x+e*h]
>>> as_finite_diff(f(x).diff(x, 1), xl, x+h*sq2)
2*h*f(E*h + x)*((h + sqrt(2)*h)/(2*h) -
(-sqrt(2)*h + h)/(2*h))/((-h + E*h)*(h + E*h)) +
f(-h + x)*(-(-sqrt(2)*h + h)/(2*h) - (-sqrt(2)*h + E*h)/(2*h))/(h +
E*h) + f(h + x)*(-(h + sqrt(2)*h)/(2*h) + (-sqrt(2)*h +
E*h)/(2*h))/(-h + E*h)
```

Partial derivatives are also supported:

```
>>> y = Symbol('y')
>>> d2fdxdy = f(x, y).diff(x, y)
>>> as_finite_diff(d2fdxdy, wrt=x)
-f(x - 1/2, y) + f(x + 1/2, y)
```

`diofant.calculus.finite_diff.`**`finite_diff_weights`**(*order, x_list, x0=Integer(0)*)

Calculates the finite difference weights for an arbitrarily spaced one-dimensional grid (x_list) for derivatives at 'x0' of order 0, 1, ..., up to 'order' using a recursive formula. Order of accuracy is at least len(x_list) - order, if x_list is defined accurately.

Parameters order: int

Up to what derivative order weights should be calculated. 0 corresponds to interpolation.

x_list: sequence

Sequence of (unique) values for the independent variable. It is useful (but not necessary) to order x_list from nearest to farest from x0; see examples below.

x0: Number or Symbol

Root or value of the independent variable for which the finite difference weights should be generated. Defaults to Integer(0).

Returns list

A list of sublists, each corresponding to coefficients for increasing derivative order, and each containing lists of coefficients for increasing subsets of x_list.

See also:

diofant.calculus.finite_diff.apply_finite_diff (page 1011)

Notes

If weights for a finite difference approximation of 3rd order derivative is wanted, weights for 0th, 1st and 2nd order are calculated "for free", so are formulae using subsets of x_list. This is something one can take advantage of to save computational cost. Be aware that one should define x_list from nearest to farest from x_list. If not, subsets of x_list will yield poorer approximations, which might not grand an order of accuracy of len(x_list) - order.

References

[R47] (page 1268)

Examples

```
>>> res = finite_diff_weights(1, [-Rational(1, 2), Rational(1, 2), Rational(3, 2),
↪ Rational(5, 2)], 0)
>>> res
[[[1, 0, 0, 0],
  [1/2, 1/2, 0, 0],
  [3/8, 3/4, -1/8, 0],
  [5/16, 15/16, -5/16, 1/16]],
 [[0, 0, 0, 0],
  [-1, 1, 0, 0],
  [-1, 1, 0, 0],
  [-23/24, 7/8, 1/8, -1/24]]]
>>> res[0][-1]  # FD weights for 0th derivative, using full x_list
[5/16, 15/16, -5/16, 1/16]
>>> res[1][-1]  # FD weights for 1st derivative
[-23/24, 7/8, 1/8, -1/24]
>>> res[1][-2]  # FD weights for 1st derivative, using x_list[:-1]
```

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```
[-1, 1, 0, 0]
>>> res[1][-1][0]  # FD weight for 1st deriv. for x_list[0]
-23/24
>>> res[1][-1][1]  # FD weight for 1st deriv. for x_list[1], etc.
7/8
```

Each sublist contains the most accurate formula at the end. Note, that in the above example res[1][1] is the same as res[1][2]. Since res[1][2] has an order of accuracy of len(x_list[:3]) - order = 3 - 1 = 2, the same is true for res[1][1]!

```
>>> res = finite_diff_weights(1, [Integer(0), Integer(1), -Integer(1), Integer(2),
↪ -Integer(2)], 0)[1]
>>> res
[[0, 0, 0, 0, 0],
 [-1, 1, 0, 0, 0],
 [0, 1/2, -1/2, 0, 0],
 [-1/2, 1, -1/3, -1/6, 0],
 [0, 2/3, -2/3, -1/12, 1/12]]
>>> res[0]  # no approximation possible, using x_list[0] only
[0, 0, 0, 0, 0]
>>> res[1]  # classic forward step approximation
[-1, 1, 0, 0, 0]
>>> res[2]  # classic centered approximation
[0, 1/2, -1/2, 0, 0]
>>> res[3:]  # higher order approximations
[[-1/2, 1, -1/3, -1/6, 0], [0, 2/3, -2/3, -1/12, 1/12]]
```

Let us compare this to a differently defined x_list. Pay attention to foo[i][k] corresponding to the gridpoint defined by x_list[k].

```
>>> foo = finite_diff_weights(1, [-Integer(2), -Integer(1), Integer(0),
↪Integer(1), Integer(2)], 0)[1]
>>> foo
[[0, 0, 0, 0, 0],
 [-1, 1, 0, 0, 0],
 [1/2, -2, 3/2, 0, 0],
 [1/6, -1, 1/2, 1/3, 0],
 [1/12, -2/3, 0, 2/3, -1/12]]
>>> foo[1]  # not the same and of lower accuracy as res[1]!
[-1, 1, 0, 0, 0]
>>> foo[2]  # classic double backward step approximation
[1/2, -2, 3/2, 0, 0]
>>> foo[4]  # the same as res[4]
[1/12, -2/3, 0, 2/3, -1/12]
```

Note that, unless you plan on using approximations based on subsets of x_list, the order of gridpoints does not matter.

The capability to generate weights at arbitrary points can be used e.g. to minimize Runge's phenomenon by using Chebyshev nodes:

```
>>> N, (h, x) = 4, symbols('h x')
>>> x_list = [x + h*cos(i*pi/(N)) for i in range(N, -1, -1)] # chebyshev nodes
>>> x_list
[-h + x, -sqrt(2)*h/2 + x, x, sqrt(2)*h/2 + x, h + x]
>>> mycoeffs = finite_diff_weights(1, x_list, 0)[1][4]
```

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```
>>> [simplify(c) for c in  mycoeffs]
[(h**3/2 + h**2*x - 3*h*x**2 - 4*x**3)/h**4,
(-sqrt(2)*h**3 - 4*h**2*x + 3*sqrt(2)*h*x**2 + 8*x**3)/h**4,
6*x/h**2 - 8*x**3/h**4,
(sqrt(2)*h**3 - 4*h**2*x - 3*sqrt(2)*h*x**2 + 8*x**3)/h**4,
(-h**3/2 + h**2*x + 3*h*x**2 - 4*x**3)/h**4]
```

3.24 Differential Geometry

3.24.1 Introduction

3.24.2 Base Class Reference

class diofant.diffgeom.**Manifold**

Object representing a mathematical manifold.

The only role that this object plays is to keep a list of all patches defined on the manifold. It does not provide any means to study the topological characteristics of the manifold that it represents.

class diofant.diffgeom.**Patch**

Object representing a patch on a manifold.

On a manifold one can have many patches that do not always include the whole manifold. On these patches coordinate charts can be defined that permit the parametrization of any point on the patch in terms of a tuple of real numbers (the coordinates).

This object serves as a container/parent for all coordinate system charts that can be defined on the patch it represents.

Examples

Define a Manifold and a Patch on that Manifold:

```
>>> m = Manifold('M', 3)
>>> p = Patch('P', m)
>>> p in m.patches
True
```

class diofant.diffgeom.**CoordSystem**

Contains all coordinate transformation logic.

Examples

Define a Manifold and a Patch, and then define two coord systems on that patch:

```
>>> r, theta = symbols('r, theta')
>>> m = Manifold('M', 2)
>>> patch = Patch('P', m)
>>> rect = CoordSystem('rect', patch)
>>> polar = CoordSystem('polar', patch)
```

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```
>>> rect in patch.coord_systems
True
```

Connect the coordinate systems. An inverse transformation is automatically found by `solve` when possible:

```
>>> polar.connect_to(rect, [r, theta], [r*cos(theta), r*sin(theta)])
>>> polar.coord_tuple_transform_to(rect, [0, 2])
Matrix([
[0],
[0]])
>>> polar.coord_tuple_transform_to(rect, [2, pi/2])
Matrix([
[0],
[2]])
>>> rect.coord_tuple_transform_to(polar, [1, 1]).applyfunc(simplify)
Matrix([
[sqrt(2)],
[   pi/4]])
```

Calculate the jacobian of the polar to cartesian transformation:

```
>>> polar.jacobian(rect, [r, theta])
Matrix([
[cos(theta), -r*sin(theta)],
[sin(theta),  r*cos(theta)]])
```

Define a point using coordinates in one of the coordinate systems:

```
>>> p = polar.point([1, 3*pi/4])
>>> rect.point_to_coords(p)
Matrix([
[-sqrt(2)/2],
[ sqrt(2)/2]])
```

Define a basis scalar field (i.e. a coordinate function), that takes a point and returns its coordinates. It is an instance of `BaseScalarField`.

```
>>> rect.coord_function(0)(p)
-sqrt(2)/2
>>> rect.coord_function(1)(p)
sqrt(2)/2
```

Define a basis vector field (i.e. a unit vector field along the coordinate line). Vectors are also differential operators on scalar fields. It is an instance of `BaseVectorField`.

```
>>> v_x = rect.base_vector(0)
>>> x = rect.coord_function(0)
>>> v_x(x)
1
>>> v_x(v_x(x))
0
```

Define a basis oneform field:

```
>>> dx = rect.base_oneform(0)
>>> dx(v_x)
1
```

If you provide a list of names the fields will print nicely: - without provided names:

```
>>> x, v_x, dx
(rect_0, e_rect_0, drect_0)
```

- with provided names

```
>>> rect = CoordSystem('rect', patch, ['x', 'y'])
>>> rect.coord_function(0), rect.base_vector(0), rect.base_oneform(0)
(x, e_x, dx)
```

base_oneform(*coord_index*)
Return a basis 1-form field.

The basis one-form field for this coordinate system. It is also an operator on vector fields.

See also:

CoordSystem (page 1016)

base_oneforms()
Returns a list of all base oneforms.

For more details see the `base_oneform` method of this class.

base_vector(*coord_index*)
Return a basis vector field.

The basis vector field for this coordinate system. It is also an operator on scalar fields.

See also:

CoordSystem (page 1016)

base_vectors()
Returns a list of all base vectors.

For more details see the `base_vector` method of this class.

connect_to(*to_sys*, *from_coords*, *to_exprs*, *inverse=True*, *fill_in_gaps=False*)
Register the transformation used to switch to another coordinate system.

Parameters to_sys

another instance of `CoordSystem`

from_coords

list of symbols in terms of which `to_exprs` is given

to_exprs

list of the expressions of the new coordinate tuple

inverse

try to deduce and register the inverse transformation

fill_in_gaps

try to deduce other transformation that are made possible by composing the present transformation with other already registered transformation

coord_function(*coord_index*)

Return a `BaseScalarField` that takes a point and returns one of the coords.

Takes a point and returns its coordinate in this coordinate system.

See also:

CoordSystem (page 1016)

coord_functions()

Returns a list of all coordinate functions.

For more details see the `coord_function` method of this class.

coord_tuple_transform_to(*to_sys*, *coords*)

Transform `coords` to coord system `to_sys`.

See also:

CoordSystem (page 1016)

jacobian(*to_sys*, *coords*)

Return the jacobian matrix of a transformation.

point(*coords*)

Create a `Point` with coordinates given in this coord system.

See also:

CoordSystem (page 1016)

point_to_coords(*point*)

Calculate the coordinates of a point in this coord system.

See also:

CoordSystem (page 1016)

class diofant.diffgeom.**Point**(*coord_sys*, *coords*)

Point in a Manifold object.

To define a point you must supply coordinates and a coordinate system.

The usage of this object after its definition is independent of the coordinate system that was used in order to define it, however due to limitations in the simplification routines you can arrive at complicated expressions if you use inappropriate coordinate systems.

Examples

Define the boilerplate Manifold, Patch and coordinate systems:

```
>>> from diofant.diffgeom import Point
>>> r, theta = symbols('r, theta')
>>> m = Manifold('M', 2)
>>> p = Patch('P', m)
>>> rect = CoordSystem('rect', p)
```

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```
>>> polar = CoordSystem('polar', p)
>>> polar.connect_to(rect, [r, theta], [r*cos(theta), r*sin(theta)])
```

Define a point using coordinates from one of the coordinate systems:

```
>>> p = Point(polar, [r, 3*pi/4])
>>> p.coords()
Matrix([
[     r],
[3*pi/4]])
>>> p.coords(rect)
Matrix([
[-sqrt(2)*r/2],
[ sqrt(2)*r/2]])
```

coords(*to_sys=None*)

Coordinates of the point in a given coordinate system.

If `to_sys` is `None` it returns the coordinates in the system in which the point was defined.

class diofant.diffgeom.**BaseScalarField**

Base Scalar Field over a Manifold for a given Coordinate System.

A scalar field takes a point as an argument and returns a scalar.

A base scalar field of a coordinate system takes a point and returns one of the coordinates of that point in the coordinate system in question.

To define a scalar field you need to choose the coordinate system and the index of the coordinate.

The use of the scalar field after its definition is independent of the coordinate system in which it was defined, however due to limitations in the simplification routines you may arrive at more complicated expression if you use inappropriate coordinate systems.

You can build complicated scalar fields by just building up Diofant expressions containing `BaseScalarField` instances.

Examples

Define boilerplate Manifold, Patch and coordinate systems:

```
>>> r0, theta0 = symbols('r0, theta0')
>>> m = Manifold('M', 2)
>>> p = Patch('P', m)
>>> rect = CoordSystem('rect', p)
>>> polar = CoordSystem('polar', p)
>>> polar.connect_to(rect, [r0, theta0], [r0*cos(theta0), r0*sin(theta0)])
```

Point to be used as an argument for the filed:

```
>>> point = polar.point([r0, 0])
```

Examples of fields:

```
>>> fx = BaseScalarField(rect, 0)
>>> fy = BaseScalarField(rect, 1)
>>> (fx**2+fy**2).rcall(point)
r0**2
```

```
>>> g = Function('g')
>>> ftheta = BaseScalarField(polar, 1)
>>> fg = g(ftheta-pi)
>>> fg.rcall(point)
g(-pi)
```

doit(***hints*)

Evaluate objects that are not evaluated by default.

For example, limits, integrals, sums and products. All objects of this kind will be evaluated recursively, unless some species were excluded via 'hints' or unless the 'deep' hint was set to 'False'.

Examples

```
>>> 2*Integral(x, x)
2*Integral(x, x)
```

```
>>> (2*Integral(x, x)).doit()
x**2
```

```
>>> (2*Integral(x, x)).doit(deep=False)
2*Integral(x, x)
```

class diofant.diffgeom.**BaseVectorField**

Vector Field over a Manifold.

A vector field is an operator taking a scalar field and returning a directional derivative (which is also a scalar field).

A base vector field is the same type of operator, however the derivation is specifically done with respect to a chosen coordinate.

To define a base vector field you need to choose the coordinate system and the index of the coordinate.

The use of the vector field after its definition is independent of the coordinate system in which it was defined, however due to limitations in the simplification routines you may arrive at more complicated expression if you use inappropriate coordinate systems.

Examples

Use the predefined R2 manifold, setup some boilerplate.

```
>>> from diofant.diffgeom.rn import R2, R2_p, R2_r
>>> x0, y0, r0, theta0 = symbols('x0, y0, r0, theta0')
```

Points to be used as arguments for the field:

```
>>> point_p = R2_p.point([r0, theta0])
>>> point_r = R2_r.point([x0, y0])
```

Scalar field to operate on:

```
>>> g = Function('g')
>>> s_field = g(R2.x, R2.y)
>>> s_field.rcall(point_r)
g(x0, y0)
>>> s_field.rcall(point_p)
g(r0*cos(theta0), r0*sin(theta0))
```

Vector field:

```
>>> v = BaseVectorField(R2_r, 1)
>>> pprint(v(s_field), use_unicode=False)
/  d           \|
|-----(g(x, xi_2))||
\dxi_2         /|xi_2=y
>>> pprint(v(s_field).rcall(point_r).doit(), use_unicode=False)
 d
---(g(x0, y0))
dy0
>>> pprint(v(s_field).rcall(point_p).doit(), use_unicode=False)
/  d                        \|
|-----(g(r0*cos(theta0), xi_2))||
\dxi_2                      /|xi_2=r0*sin(theta0)
```

class diofant.diffgeom.**Commutator**(*v1*, *v2*)

Commutator of two vector fields.

The commutator of two vector fields v_1 and v_2 is defined as the vector field $[v_1, v_2]$ that evaluated on each scalar field f is equal to $v_1(v_2(f)) - v_2(v_1(f))$.

Examples

Use the predefined R2 manifold, setup some boilerplate.

```
>>> from diofant.diffgeom.rn import R2
```

Vector fields:

```
>>> e_x, e_y, e_r = R2.e_x, R2.e_y, R2.e_r
>>> c_xy = Commutator(e_x, e_y)
>>> c_xr = Commutator(e_x, e_r)
>>> c_xy
0
```

Unfortunately, the current code is not able to compute everything:

```
>>> c_xr
Commutator(e_x, e_r)
```

```
>>> simplify(c_xr(R2.y**2).doit())
-2*cos(theta)*y**2/(x**2 + y**2)
```

class diofant.diffgeom.**Differential**(*form_field*)
Return the differential (exterior derivative) of a form field.

The differential of a form (i.e. the exterior derivative) has a complicated definition in the general case.

The differential df of the 0-form f is defined for any vector field v as $df(v) = v(f)$.

Examples

Use the predefined R2 manifold, setup some boilerplate.

```
>>> from diofant.diffgeom.rn import R2
```

Scalar field (0-forms):

```
>>> g = Function('g')
>>> s_field = g(R2.x, R2.y)
```

Vector fields:

```
>>> e_x, e_y, = R2.e_x, R2.e_y
```

Differentials:

```
>>> dg = Differential(s_field)
>>> dg
d(g(x, y))
>>> pprint(dg(e_x), use_unicode=False)
/  d              \|
|-----(g(xi_1, y))||
\dxi_1            /|xi_1=x
>>> pprint(dg(e_y), use_unicode=False)
/  d              \|
|-----(g(x, xi_2))||
\dxi_2            /|xi_2=y
```

Applying the exterior derivative operator twice always results in:

```
>>> Differential(dg)
0
```

class diofant.diffgeom.**TensorProduct**(**args*)
Tensor product of forms.

The tensor product permits the creation of multilinear functionals (i.e. higher order tensors) out of lower order forms (e.g. 1-forms). However, the higher tensors thus created lack the interesting features provided by the other type of product, the wedge product, namely they are not antisymmetric and hence are not form fields.

Examples

Use the predefined R2 manifold, setup some boilerplate.

```
>>> from diofant.diffgeom.rn import R2
```

```
>>> TensorProduct(R2.dx, R2.dy)(R2.e_x, R2.e_y)
1
>>> TensorProduct(R2.dx, R2.dy)(R2.e_y, R2.e_x)
0
>>> TensorProduct(R2.dx, R2.x*R2.dy)(R2.x*R2.e_x, R2.e_y)
x**2
```

You can nest tensor products.

```
>>> tp1 = TensorProduct(R2.dx, R2.dy)
>>> TensorProduct(tp1, R2.dx)(R2.e_x, R2.e_y, R2.e_x)
1
```

You can make partial contraction for instance when 'raising an index'. Putting `None` in the second argument of `rcall` means that the respective position in the tensor product is left as it is.

```
>>> TP = TensorProduct
>>> metric = TP(R2.dx, R2.dx) + 3*TP(R2.dy, R2.dy)
>>> metric.rcall(R2.e_y, None)
3*dy
```

Or automatically pad the args with `None` without specifying them.

```
>>> metric.rcall(R2.e_y)
3*dy
```

class diofant.diffgeom.**WedgeProduct**(**args*)

Wedge product of forms.

In the context of integration only completely antisymmetric forms make sense. The wedge product permits the creation of such forms.

Examples

Use the predefined R2 manifold, setup some boilerplate.

```
>>> from diofant.diffgeom.rn import R2
```

```
>>> WedgeProduct(R2.dx, R2.dy)(R2.e_x, R2.e_y)
1
>>> WedgeProduct(R2.dx, R2.dy)(R2.e_y, R2.e_x)
-1
>>> WedgeProduct(R2.dx, R2.x*R2.dy)(R2.x*R2.e_x, R2.e_y)
x**2
```

You can nest wedge products.

```
>>> wp1 = WedgeProduct(R2.dx, R2.dy)
>>> WedgeProduct(wp1, R2.dx)(R2.e_x, R2.e_y, R2.e_x)
0
```

class diofant.diffgeom.**LieDerivative**(*v_field*, *expr*)

Lie derivative with respect to a vector field.

The transport operator that defines the Lie derivative is the pushforward of the field to be derived along the integral curve of the field with respect to which one derives.

Examples

```
>>> from diofant.diffgeom.rn import R2
>>> LieDerivative(R2.e_x, R2.y)
0
>>> LieDerivative(R2.e_x, R2.x)
1
>>> LieDerivative(R2.e_x, R2.e_x)
0
```

The Lie derivative of a tensor field by another tensor field is equal to their commutator:

```
>>> LieDerivative(R2.e_x, R2.e_r)
Commutator(e_x, e_r)
>>> LieDerivative(R2.e_x + R2.e_y, R2.x)
1
>>> tp = TensorProduct(R2.dx, R2.dy)
>>> LieDerivative(R2.e_x, tp)
LieDerivative(e_x, TensorProduct(dx, dy))
>>> LieDerivative(R2.e_x, tp).doit()
LieDerivative(e_x, TensorProduct(dx, dy))
```

class diofant.diffgeom.**BaseCovarDerivativeOp**(*coord_sys, index, christoffel*)
Covariant derivative operator with respect to a base vector.

Examples

```
>>> from diofant.diffgeom.rn import R2, R2_r
>>> TP = TensorProduct
>>> ch = metric_to_Christoffel_2nd(TP(R2.dx, R2.dx) + TP(R2.dy, R2.dy))
>>> ch
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
>>> cvd = BaseCovarDerivativeOp(R2_r, 0, ch)
>>> cvd(R2.x)
1
>>> cvd(R2.x*R2.e_x)
e_x
```

class diofant.diffgeom.**CovarDerivativeOp**(*wrt, christoffel*)
Covariant derivative operator.

Examples

```
>>> from diofant.diffgeom.rn import R2
>>> TP = TensorProduct
>>> ch = metric_to_Christoffel_2nd(TP(R2.dx, R2.dx) + TP(R2.dy, R2.dy))
>>> ch
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
>>> cvd = CovarDerivativeOp(R2.x*R2.e_x, ch)
>>> cvd(R2.x)
x
>>> cvd(R2.x*R2.e_x)
x*e_x
```

diofant.diffgeom.**intcurve_series**(*vector_field*, *param*, *start_point*, *n=6*, *coord_sys=None*, *coeffs=False*)

Return the series expansion for an integral curve of the field.

Integral curve is a function γ taking a parameter in R to a point in the manifold. It verifies the equation:

$V(f)\big(\gamma(t)\big) = \frac{d}{dt}f\big(\gamma(t)\big)$

where the given `vector_field` is denoted as V. This holds for any value t for the parameter and any scalar field f.

This equation can also be decomposed of a basis of coordinate functions

$V(f_i)\big(\gamma(t)\big) = \frac{d}{dt}f_i\big(\gamma(t)\big) \quad \forall i$

This function returns a series expansion of $\gamma(t)$ in terms of the coordinate system `coord_sys`. The equations and expansions are necessarily done in coordinate-system-dependent way as there is no other way to represent movement between points on the manifold (i.e. there is no such thing as a difference of points for a general manifold).

Parameters vector_field

the vector field for which an integral curve will be given

param

the argument of the function γ from R to the curve

start_point

the point which corresponds to $\gamma(0)$

n

the order to which to expand

coord_sys

the coordinate system in which to expand coeffs (default False) - if True return a list of elements of the expansion

See also:

intcurve_diffequ (page 1027)

Examples

Use the predefined R2 manifold:

```
>>> from diofant.abc import t
>>> from diofant.diffgeom.rn import R2, R2_p, R2_r
```

Specify a starting point and a vector field:

```
>>> start_point = R2_r.point([x, y])
>>> vector_field = R2_r.e_x
```

Calculate the series:

```
>>> intcurve_series(vector_field, t, start_point, n=3)
Matrix([
[t + x],
[    y]])
```

Or get the elements of the expansion in a list:

```
>>> series = intcurve_series(vector_field, t, start_point, n=3, coeffs=True)
>>> series[0]
Matrix([
[x],
[y]])
>>> series[1]
Matrix([
[t],
[0]])
>>> series[2]
Matrix([
[0],
[0]])
```

The series in the polar coordinate system:

```
>>> series = intcurve_series(vector_field, t, start_point,
...                          n=3, coord_sys=R2_p, coeffs=True)
>>> series[0]
Matrix([
[sqrt(x**2 + y**2)],
[      atan2(y, x)]])
>>> series[1]
Matrix([
[t*x/sqrt(x**2 + y**2)],
[   -t*y/(x**2 + y**2)]])
>>> series[2]
Matrix([
[t**2*(-x**2/(x**2 + y**2)**(3/2) + 1/sqrt(x**2 + y**2))/2],
[                                t**2*x*y/(x**2 + y**2)**2]])
```

diofant.diffgeom.**intcurve_diffequ**(*vector_field, param, start_point, coord_sys=None*)

Return the differential equation for an integral curve of the field.

Integral curve is a function γ taking a parameter in R to a point in the manifold. It verifies the equation:

$V(f)\big(\gamma(t)\big) = \frac{d}{dt}f\big(\gamma(t)\big)$

where the given `vector_field` is denoted as V. This holds for any value t for the parameter and any scalar field f.

This function returns the differential equation of $\gamma(t)$ in terms of the coordinate system `coord_sys`. The equations and expansions are necessarily done in coordinate-system-dependent way as there is no other way to represent movement between points on the manifold (i.e. there is no such thing as a difference of points for a general manifold).

Parameters vector_field

the vector field for which an integral curve will be given

param

the argument of the function γ from R to the curve

start_point

the point which corresponds to $\gamma(0)$

coord_sys

the coordinate system in which to give the equations

Returns a tuple of (equations, initial conditions)

See also:

intcurve_series (page 1025)

Examples

Use the predefined R2 manifold:

```
>>> from diofant.abc import t
>>> from diofant.diffgeom.rn import R2, R2_p, R2_r
```

Specify a starting point and a vector field:

```
>>> start_point = R2_r.point([0, 1])
>>> vector_field = -R2.y*R2.e_x + R2.x*R2.e_y
```

Get the equation:

```
>>> equations, init_cond = intcurve_diffequ(vector_field, t, start_point)
>>> equations
[f_1(t) + Derivative(f_0(t), t), -f_0(t) + Derivative(f_1(t), t)]
>>> init_cond
[f_0(0), f_1(0) - 1]
```

The series in the polar coordinate system:

```
>>> equations, init_cond = intcurve_diffequ(vector_field, t, start_point, R2_p)
>>> equations
[Derivative(f_0(t), t), Derivative(f_1(t), t) - 1]
>>> init_cond
[f_0(0) - 1, f_1(0) - pi/2]
```

diofant.diffgeom.**vectors_in_basis**(*expr, to_sys*)

Transform all base vectors in base vectors of a specified coord basis.

While the new base vectors are in the new coordinate system basis, any coefficients are kept in the old system.

Examples

```
>>> from diofant.diffgeom.rn import R2_r, R2_p
>>> vectors_in_basis(R2_r.e_x, R2_p)
-y*e_theta/(x**2 + y**2) + x*e_r/sqrt(x**2 + y**2)
>>> vectors_in_basis(R2_p.e_r, R2_r)
sin(theta)*e_y + cos(theta)*e_x
```

diofant.diffgeom.**twoform_to_matrix**(*expr*)

Return the matrix representing the twoform.

For the twoform w return the matrix M such that $M[i,j] = w(e_i, e_j)$, where e_i is the i-th base vector field for the coordinate system in which the expression of w is given.

Examples

```
>>> from diofant.diffgeom.rn import R2
>>> TP = TensorProduct
>>> twoform_to_matrix(TP(R2.dx, R2.dx) + TP(R2.dy, R2.dy))
Matrix([
[1, 0],
[0, 1]])
>>> twoform_to_matrix(R2.x*TP(R2.dx, R2.dx) + TP(R2.dy, R2.dy))
Matrix([
[x, 0],
[0, 1]])
>>> twoform_to_matrix(TP(R2.dx, R2.dx) + TP(R2.dy, R2.dy) - TP(R2.dx, R2.dy)/2)
Matrix([
[   1, 0],
[-1/2, 1]])
```

diofant.diffgeom.**metric_to_Christoffel_1st**(*expr*)

Return the nested list of Christoffel symbols for the given metric.

This returns the Christoffel symbol of first kind that represents the Levi-Civita connection for the given metric.

Examples

```
>>> from diofant.diffgeom.rn import R2
>>> TP = TensorProduct
>>> metric_to_Christoffel_1st(TP(R2.dx, R2.dx) + TP(R2.dy, R2.dy))
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
>>> metric_to_Christoffel_1st(R2.x*TP(R2.dx, R2.dx) + TP(R2.dy, R2.dy))
[[[1/2, 0], [0, 0]], [[0, 0], [0, 0]]]
```

diofant.diffgeom.**metric_to_Christoffel_2nd**(*expr*)

Return the nested list of Christoffel symbols for the given metric.

This returns the Christoffel symbol of second kind that represents the Levi-Civita connection for the given metric.

Examples

```
>>> from diofant.diffgeom.rn import R2
>>> TP = TensorProduct
>>> metric_to_Christoffel_2nd(TP(R2.dx, R2.dx) + TP(R2.dy, R2.dy))
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
>>> metric_to_Christoffel_2nd(R2.x*TP(R2.dx, R2.dx) + TP(R2.dy, R2.dy))
[[[1/(2*x), 0], [0, 0]], [[0, 0], [0, 0]]]
```

diofant.diffgeom.**metric_to_Riemann_components**(*expr*)

Return the components of the Riemann tensor expressed in a given basis.

Given a metric it calculates the components of the Riemann tensor in the canonical basis of the coordinate system in which the metric expression is given.

Examples

```
>>> from diofant.diffgeom.rn import R2
>>> TP = TensorProduct
>>> metric_to_Riemann_components(TP(R2.dx, R2.dx) + TP(R2.dy, R2.dy))
[[[[0, 0], [0, 0]], [[0, 0], [0, 0]]], [[[0, 0], [0, 0]], [[0, 0], [0, 0]]]]
```

```
>>> non_trivial_metric = (exp(2*R2.r)*TP(R2.dr, R2.dr) +
...                       R2.r**2*TP(R2.dtheta, R2.dtheta))
>>> non_trivial_metric
E**(2*r)*TensorProduct(dr, dr) + r**2*TensorProduct(dtheta, dtheta)
>>> riemann = metric_to_Riemann_components(non_trivial_metric)
>>> riemann[0, :, :, :]
[[[0, 0], [0, 0]], [[0, E**(-2*r)*r], [-E**(-2*r)*r, 0]]]
>>> riemann[1, :, :, :]
[[[0, -1/r], [1/r, 0]], [[0, 0], [0, 0]]]
```

diofant.diffgeom.**metric_to_Ricci_components**(*expr*)

Return the components of the Ricci tensor expressed in a given basis.

Given a metric it calculates the components of the Ricci tensor in the canonical basis of the coordinate system in which the metric expression is given.

Examples

```
>>> from diofant.diffgeom.rn import R2
>>> TP = TensorProduct
>>> metric_to_Ricci_components(TP(R2.dx, R2.dx) + TP(R2.dy, R2.dy))
[[0, 0], [0, 0]]
```

```
>>> non_trivial_metric = (exp(2*R2.r)*TP(R2.dr, R2.dr) +
...                       R2.r**2*TP(R2.dtheta, R2.dtheta))
>>> non_trivial_metric
E**(2*r)*TensorProduct(dr, dr) + r**2*TensorProduct(dtheta, dtheta)
>>> metric_to_Ricci_components(non_trivial_metric)
[[1/r, 0], [0, E**(-2*r)*r]]
```

3.25 Vectors

The vector module provides tools for basic vector math and differential calculus with respect to 3D Cartesian coordinate systems. This documentation provides an overview of all the features offered, and relevant API.

3.25.1 Guide to Vector

Introduction

This page gives a brief conceptual overview of the functionality present in *diofant.vector* (page 1032).

Vectors and Scalars

In vector math, we deal with two kinds of quantities – scalars and vectors.

A **scalar** is an entity which only has a magnitude – no direction. Examples of scalar quantities include mass, electric charge, temperature, distance, etc.

A **vector**, on the other hand, is an entity that is characterized by a magnitude and a direction. Examples of vector quantities are displacement, velocity, magnetic field, etc.

A scalar can be depicted just by a number, for e.g. a temperature of 300 K. On the other hand, vectorial quantities like acceleration are usually denoted by a vector. Given a vector $\mathbf{V}$, the magnitude of the corresponding quantity can be calculated as the magnitude of the vector itself $\Vert \mathbf{V} \Vert$, while the direction would be specified by a unit vector in the direction of the original vector, $\mathbf{\hat{V}} = \frac{\mathbf{V}}{\Vert \mathbf{V} \Vert}$.

For example, consider a displacement of $(3\mathbf{\hat{i}} + 4\mathbf{\hat{j}} + 5\mathbf{\hat{k}})$ m, where , as per standard convention, $\mathbf{\hat{i}}$, $\mathbf{\hat{j}}$ and $\mathbf{\hat{k}}$ represent unit vectors along the $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ axes respectively. Therefore, it can be concluded that the distance traveled is $\Vert 3\mathbf{\hat{i}} + 4\mathbf{\hat{j}} + 5\mathbf{\hat{k}} \Vert$ m $= 5\sqrt{2}$ m. The direction of travel is given by the unit vector $\frac{3}{5\sqrt{2}}\mathbf{\hat{i}} + \frac{4}{5\sqrt{2}}\mathbf{\hat{j}} + \frac{5}{5\sqrt{2}}\mathbf{\hat{k}}$.

Coordinate Systems

A **coordinate system** is an abstract mathematical entity used to define the notion of directions and locations in n-dimensional spaces. This module deals with 3-dimensional spaces, with the conventional X, Y and Z axes defined with respect to each coordinate system.

Each coordinate system also has a special reference point called the 'origin' defined for it. This point is used either while referring to locations in 3D space, or while calculating the coordinates of pre-defined points with respect to the system.

It is a pretty well-known concept that there is no absolute notion of location or orientation in space. Any given coordinate system defines a unique 'perspective' of quantifying positions and directions. Therefore, even if we assume that all systems deal with the same units of measurement, the expression of vectorial and scalar quantities differs according to the coordinate system a certain observer deals with.

Consider two points P and Q in space. Assuming units to be common throughtout, the distance between these points remains the same regardless of the coordinate system in which the measurements are being made. However, the 3-D coordinates of each of the two points, as well as the position vector of any of the points with respect to the other, do not. In fact, these two quantities don't make sense at all, unless they are being measured keeping in mind a certain location and orientation of the measurer (essentially the coordinate system).

Therefore, it is quite clear that the orientation and location (of the origin) of a coordinate system define the way different quantities will be expressed with respect to it. Neither of the two properties can be measured on an absolute scale, but rather with respect to another coordinate system. The orientation of one system with respect to another is measured using

the the rotation matrix, while the relative position can be quantified via the position vector of one system's origin with respect to the other.

Fields

A **field** is a vector or scalar quantity that can be specified everywhere in space as a function of position (Note that in general a field may also be dependent on time and other custom variables). Since we only deal with 3D spaces in this module, a field is defined as a function of the x, y and z coordinates corresponding to a location in the coordinate system. Here, x, y and z act as scalar variables defining the position of a general point.

For example, temperature in 3 dimensional space (a temperature field) can be written as $T(x, y, z)$ – a scalar function of the position. An example of a scalar field in electromagnetism is the electric potential.

In a similar manner, a vector field can be defined as a vectorial function of the location (x, y, z) of any point in space.

For instance, every point on the earth may be considered to be in the gravitational force field of the earth. We may specify the field by the magnitude and the direction of acceleration due to gravity (i.e. force per unit mass) $\vec{g}(x, y, z)$ at every point in space.

To give an example from electromagnetism, consider an electric potential of form $2x^2y$, a scalar field in 3D space. The corresponding conservative electric field can be computed as the gradient of the electric potential function, and expressed as $4xy\hat{\mathbf{i}} + 2x^2\hat{\mathbf{j}}$. The magnitude of this electric field can in turn be expressed as a scalar field of the form $\sqrt{4x^4 + 16x^2y^2}$.

Basic Implementation details

Package for symbolic vector algebra in 3D.

Coordinate Systems and Vectors

As of now, *diofant.vector* (page 1032) only deals with the Cartesian (also called rectangular) coordinate systems. A 3D Cartesian coordinate system can be initialized in *diofant.vector* (page 1032) as

```
>>> from diofant.vector import CoordSysCartesian
>>> N = CoordSysCartesian('N')
```

The string parameter to the constructor denotes the name assigned to the system, and will primarily be used for printing purposes.

Once a coordinate system (in essence, a `CoordSysCartesian` instance) has been defined, we can access the orthonormal unit vectors (i.e. the $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ vectors) and coordinate variables/base scalars (i.e. the $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ variables) corresponding to it. We will talk about coordinate variables in the later sections.

The basis vectors for the X, Y and Z axes can be accessed using the `i`, `j` and `k` properties respectively.

```
>>> N.i
N.i
>>> type(N.i)
<class 'diofant.vector.vector.BaseVector'>
```

As seen above, the basis vectors are all instances of a class called BaseVector.

When a BaseVector is multiplied by a scalar (essentially any Diofant Expr), we get a VectorMul - the product of a base vector and a scalar.

```
>>> 3*N.i
3*N.i
>>> type(3*N.i)
<class 'diofant.vector.vector.VectorMul'>
```

Addition of VectorMul and BaseVectors gives rise to formation of VectorAdd - except for special cases, ofcourse.

```
>>> v = 2*N.i + N.j
>>> type(v)
<class 'diofant.vector.vector.VectorAdd'>
>>> v - N.j
2*N.i
>>> type(v - N.j)
<class 'diofant.vector.vector.VectorMul'>
```

What about a zero vector? It can be accessed using the zero attribute assigned to class Vector. Since the notion of a zero vector remains the same regardless of the coordinate system in consideration, we use Vector.zero wherever such a quantity is required.

```
>>> from diofant.vector import Vector
>>> Vector.zero
0
>>> type(Vector.zero)
<class 'diofant.vector.vector.VectorZero'>
>>> N.i + Vector.zero
N.i
>>> Vector.zero == 2*Vector.zero
True
```

All the classes shown above - BaseVector, VectorMul, VectorAdd and VectorZero are subclasses of Vector.

You should never have to instantiate objects of any of the subclasses of Vector. Using the BaseVector instances assigned to a CoordSysCartesian instance and (if needed) Vector.zero as building blocks, any sort of vectorial expression can be constructed with the basic mathematical operators +, -, *. and /.

```
>>> v = N.i - 2*N.j
>>> v/3
1/3*N.i + (-2/3)*N.j
>>> v + N.k
N.i + (-2)*N.j + N.k
>>> Vector.zero/2
0
>>> (v/3)*4
4/3*N.i + (-8/3)*N.j
```

In addition to the elementary mathematical operations, the vector operations of dot and cross can also be performed on Vector.

```
>>> v1 = 2*N.i + 3*N.j - N.k
>>> v2 = N.i - 4*N.j + N.k
```

(continues on next page)

(continued from previous page)

```
>>> v1.dot(v2)
-11
>>> v1.cross(v2)
(-1)*N.i + (-3)*N.j + (-11)*N.k
>>> v2.cross(v1)
N.i + 3*N.j + 11*N.k
```

The & and ^ operators have been overloaded for the `dot` and `cross` methods respectively.

```
>>> v1 & v2
-11
>>> v1 ^ v2
(-1)*N.i + (-3)*N.j + (-11)*N.k
```

However, this is not the recommended way of performing these operations. Using the original methods makes the code clearer and easier to follow.

In addition to these operations, it is also possible to compute the outer products of `Vector` instances in *diofant.vector* (page 1032). More on that in a little bit.

Diofant operations on Vectors

The Diofant operations of `simplify`, `trigsimp`, `diff`, and `factor` work on `Vector` objects, with the standard Diofant API.

In essence, the methods work on the measure numbers(The coefficients of the basis vectors) present in the provided vectorial expression.

```
>>> from diofant.abc import a, b, c
>>> v = (a*b + a*c + b**2 + b*c)*N.i + N.j
>>> v.factor()
((a + b)*(b + c))*N.i + N.j
>>> v = (sin(a)**2 + cos(a)**2)*N.i - (2*cos(b)**2 - 1)*N.k
>>> trigsimp(v)
N.i + (-cos(2*b))*N.k
>>> v.simplify()
N.i + (-cos(2*b))*N.k
>>> diff(v, b)
(4*sin(b)*cos(b))*N.k
>>> Derivative(v, b).doit()
(4*sin(b)*cos(b))*N.k
```

`Integral` also works with `Vector` instances, similar to `Derivative`.

```
>>> v1 = a*N.i + sin(a)*N.j - N.k
>>> Integral(v1, a)
(Integral(a, a))*N.i + (Integral(sin(a), a))*N.j + (Integral(-1, a))*N.k
>>> Integral(v1, a).doit()
a**2/2*N.i + (-cos(a))*N.j + (-a)*N.k
```

Points

As mentioned before, every coordinate system corresponds to a unique origin point. Points, in general, have been implemented in *diofant.vector* (page 1032) in the form of the `Point`

class.

To access the origin of system, use the `origin` property of the `CoordSysCartesian` class.

```
>>> from diofant.vector import CoordSysCartesian
>>> N = CoordSysCartesian('N')
>>> N.origin
N.origin
>>> type(N.origin)
<class 'diofant.vector.point.Point'>
```

You can instantiate new points in space using the `locate_new` method of `Point`. The arguments include the name(string) of the new `Point`, and its position vector with respect to the 'parent' `Point`.

```
>>> from diofant.abc import a, b, c
>>> P = N.origin.locate_new('P', a*N.i + b*N.j + c*N.k)
>>> Q = P.locate_new('Q', -b*N.j)
```

Like `Vector`, a user never has to expressly instantiate an object of `Point`. This is because any location in space (albeit relative) can be pointed at by using the `origin` of a `CoordSysCartesian` as the reference, and then using `locate_new` on it and subsequent `Point` instances.

The position vector of a `Point` with respect to another `Point` can be computed using the `position_wrt` method.

```
>>> P.position_wrt(Q)
b*N.j
>>> Q.position_wrt(N.origin)
a*N.i + c*N.k
```

Additionally, it is possible to obtain the X, Y and Z coordinates of a `Point` with respect to a `CoordSysCartesian` in the form of a tuple. This is done using the `express_coordinates` method.

```
>>> Q.express_coordinates(N)
(a, 0, c)
```

Dyadics

A dyadic, or dyadic tensor, is a second-order tensor formed by the juxtaposition of pairs of vectors. Therefore, the outer products of vectors give rise to the formation of dyadics. Dyadic tensors have been implemented in *diofant.vector* (page 1032) in the `Dyadic` class.

Once again, you never have to instantiate objects of `Dyadic`. The outer products of vectors can be computed using the `outer` method of `Vector`. The `|` operator has been overloaded for `outer`.

```
>>> from diofant.vector import CoordSysCartesian
>>> N = CoordSysCartesian('N')
>>> N.i.outer(N.j)
(N.i|N.j)
>>> N.i|N.j
(N.i|N.j)
```

Similar to `Vector`, `Dyadic` also has subsequent subclasses like `BaseDyadic`, `DyadicMul`, `DyadicAdd`. As with `Vector`, a zero dyadic can be accessed from `Dyadic.zero`.

All basic mathematical operations work with `Dyadic` too.

```
>>> dyad = N.i.outer(N.k)
>>> dyad*3
3*(N.i|N.k)
>>> dyad - dyad
0
>>> dyad + 2*(N.j|N.i)
(N.i|N.k) + 2*(N.j|N.i)
```

`dot` and `cross` also work among `Dyadic` instances as well as between a `Dyadic` and `Vector` (and also vice versa) - as per the respective mathematical definitions. As with `Vector`, `&` and `^` have been overloaded for `dot` and `cross`.

```
>>> d = N.i.outer(N.j)
>>> d.dot(N.j|N.j)
(N.i|N.j)
>>> d.dot(N.i)
0
>>> d.dot(N.j)
N.i
>>> N.i.dot(d)
N.j
>>> N.k ^ d
(N.j|N.j)
```

More about Coordinate Systems

We will now look at how we can initialize new coordinate systems in *diofant.vector* (page 1032), positioned and oriented in user-defined ways with respect to already-existing systems.

Locating new systems

We already know that the `origin` property of a `CoordSysCartesian` corresponds to the `Point` instance denoting its origin reference point.

Consider a coordinate system N. Suppose we want to define a new system M, whose origin is located at $\mathbf{3\hat{i}} + \mathbf{4\hat{j}} + \mathbf{5\hat{k}}$ from N's origin. In other words, the coordinates of M's origin from N's perspective happen to be $(3, 4, 5)$. Moreover, this would also mean that the coordinates of N's origin with respect to M would be $(-3, -4, -5)$.

This can be achieved programatically as follows -

```
>>> from diofant.vector import CoordSysCartesian
>>> N = CoordSysCartesian('N')
>>> M = N.locate_new('M', 3*N.i + 4*N.j + 5*N.k)
>>> M.position_wrt(N)
3*N.i + 4*N.j + 5*N.k
>>> N.origin.express_coordinates(M)
(-3, -4, -5)
```

It is worth noting that M's orientation is the same as that of N. This means that the rotation matrix of N with respect to M, and also vice versa, is equal to the identity matrix of dimensions 3x3. The `locate_new` method initializes a `CoordSysCartesian` that is only translated in space, not re-oriented, relative to the 'parent' system.

Orienting new systems

Similar to 'locating' new systems, *diofant.vector* (page 1032) also allows for initialization of new `CoordSysCartesian` instances that are oriented in user-defined ways with respect to existing systems.

Suppose you have a coordinate system A.

```
>>> from diofant.vector import CoordSysCartesian
>>> A = CoordSysCartesian('A')
```

You want to initialize a new coordinate system B, that is rotated with respect to A's Z-axis by an angle θ.

```
>>> theta = Symbol('theta')
```

There are two ways to achieve this.

Using a method of CoordSysCartesian directly

This is the easiest, cleanest, and hence the recommended way of doing it.

```
>>> B = A.orient_new_axis('B', theta, A.k)
```

This initialzes B with the required orientation information with respect to A.

`CoordSysCartesian` provides the following direct orientation methods in its API-

1. `orient_new_axis`
2. `orient_new_body`
3. `orient_new_space`
4. `orient_new_quaternion`

Please look at the `CoordSysCartesian` class API given in the docs of this module, to know their functionality and required arguments in detail.

Using Orienter(s) and the orient_new method

You would first have to initialize an `AxisOrienter` instance for storing the rotation information.

```
>>> from diofant.vector import AxisOrienter
>>> axis_orienter = AxisOrienter(theta, A.k)
```

And then apply it using the `orient_new` method, to obtain B.

```
>>> B = A.orient_new('B', axis_orienter)
```

`orient_new` also lets you orient new systems using multiple `Orienter` instances, provided in an iterable. The rotations/orientations are applied to the new system in the order the `Orienter` instances appear in the iterable.

```
>>> from diofant.vector import BodyOrienter
>>> from diofant.abc import a, b, c
>>> body_orienter = BodyOrienter(a, b, c, 'XYZ')
>>> C = A.orient_new('C', (axis_orienter, body_orienter))
```

The *diofant.vector* (page 1032) API provides the following four `Orienter` classes for orientation purposes-

1. `AxisOrienter`
2. `BodyOrienter`
3. `SpaceOrienter`
4. `QuaternionOrienter`

Please refer to the API of the respective classes in the docs of this module to know more.

In each of the above examples, the origin of the new coordinate system coincides with the origin of the 'parent' system.

```
>>> B.position_wrt(A)
0
```

To compute the rotation matrix of any coordinate system with respect to another one, use the `rotation_matrix` method.

```
>>> B = A.orient_new_axis('B', a, A.k)
>>> B.rotation_matrix(A)
Matrix([
[ cos(a), sin(a), 0],
[-sin(a), cos(a), 0],
[      0,      0, 1]])
>>> B.rotation_matrix(B)
Matrix([
[1, 0, 0],
[0, 1, 0],
[0, 0, 1]])
```

Orienting AND Locating new systems

What if you want to initialize a new system that is not only oriented in a pre-defined way, but also translated with respect to the parent?

Each of the `orient_new_<method of orientation>` methods, as well as the `orient_new` method, support a `location` keyword argument.

If a `Vector` is supplied as the value for this `kwarg`, the new system's origin is automatically defined to be located at that position vector with respect to the parent coordinate system.

Thus, the orientation methods also act as methods to support orientation+ location of the new systems.

```
>>> C = A.orient_new_axis('C', a, A.k, location=2*A.j)
>>> C.position_wrt(A)
2*A.j
>>> from diofant.vector import express
>>> express(A.position_wrt(C), C)
(-2*sin(a))*C.i + (-2*cos(a))*C.j
```

More on the `express` function in a bit.

Expression of quantities in different coordinate systems

Vectors and Dyadics

As mentioned earlier, the same vector attains different expressions in different coordinate systems. In general, the same is true for scalar expressions and dyadic tensors.

diofant.vector (page 1032) supports the expression of vector/scalar quantities in different coordinate systems using the `express` function.

For purposes of this section, assume the following initializations-

```
>>> from diofant.vector import CoordSysCartesian, express
>>> from diofant.abc import a, b, c
>>> N = CoordSysCartesian('N')
>>> M = N.orient_new_axis('M', a, N.k)
```

`Vector` instances can be expressed in user defined systems using `express`.

```
>>> v1 = N.i + N.j + N.k
>>> express(v1, M)
(sin(a) + cos(a))*M.i + (-sin(a) + cos(a))*M.j + M.k
>>> v2 = N.i + M.j
>>> express(v2, N)
(-sin(a) + 1)*N.i + (cos(a))*N.j
```

Apart from `Vector` instances, `express` also supports reexpression of scalars (general Diofant `Expr`) and `Dyadic` objects.

`express` also accepts a second coordinate system for re-expressing `Dyadic` instances.

```
>>> d = 2*(M.i | N.j) + 3* (M.j | N.k)
>>> express(d, M)
(2*sin(a))*(M.i|M.i) + (2*cos(a))*(M.i|M.j) + 3*(M.j|M.k)
>>> express(d, M, N)
2*(M.i|N.j) + 3*(M.j|N.k)
```

Coordinate Variables

The location of a coordinate system's origin does not affect the re-expression of `BaseVector` instances. However, it does affect the way `BaseScalar` instances are expressed in different systems.

`BaseScalar` instances, are coordinate 'symbols' meant to denote the variables used in the definition of vector/scalar fields in *diofant.vector* (page 1032).

For example, consider the scalar field $\mathbf{T_N}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x} + \mathbf{y} + \mathbf{z}$ defined in system N. Thus, at a point with coordinates (a, b, c), the value of the field would be $a + b + c$. Now consider system R, whose origin is located at $(1, 2, 3)$ with respect to N (no change of orientation). A point with coordinates (a, b, c) in R has coordinates $(a + 1, b + 2, c + 3)$ in N. Therefore, the expression for $\mathbf{T_N}$ in R becomes $\mathbf{T_R}(x, y, z) = x + y + z + 6$.

Coordinate variables, if present in a vector/scalar/dyadic expression, can also be re-expressed in a given coordinate system, by setting the `variables` keyword argument of `express` to `True`.

The above mentioned example, done programatically, would look like this -

```
>>> R = N.locate_new('R', N.i + 2*N.j + 3*N.k)
>>> T_N = N.x + N.y + N.z
>>> express(T_N, R, variables=True)
R.x + R.y + R.z + 6
```

Other expression-dependent methods

The `to_matrix` method of `Vector` and `express_coordinates` method of `Point` also return different results depending on the coordinate system being provided.

```
>>> P = R.origin.locate_new('P', a*R.i + b*R.j + c*R.k)
>>> P.express_coordinates(N)
(a + 1, b + 2, c + 3)
>>> P.express_coordinates(R)
(a, b, c)
>>> v = N.i + N.j + N.k
>>> v.to_matrix(M)
Matrix([
[ sin(a) + cos(a)],
[-sin(a) + cos(a)],
[               1]])
>>> v.to_matrix(N)
Matrix([
[1],
[1],
[1]])
```

Scalar and Vector Field Functionality

Implementation in diofant.vector

Scalar and vector fields

In *diofant.vector* (page 1032), every `CoordSysCartesian` instance is assigned basis vectors corresponding to the X, Y and Z axes. These can be accessed using the properties named `i`, `j` and `k` respectively. Hence, to define a vector $\mathbf{v}$ of the form $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ with respect to a given frame $\mathbf{R}$, you would do

```
>>> from diofant.vector import CoordSysCartesian
>>> R = CoordSysCartesian('R')
>>> v = 3*R.i + 4*R.j + 5*R.k
```

Vector math and basic calculus operations with respect to vectors have already been elaborated upon in the earlier section of this module's documentation.

On the other hand, base scalars (or coordinate variables) are implemented in a special class called `BaseScalar`, and are assigned to every coordinate system, one for each axis from X, Y and Z. These coordinate variables are used to form the expressions of vector or scalar fields in 3D space. For a system `R`, the X, Y and Z `BaseScalars` instances can be accessed using the `R.x`, `R.y` and `R.z` expressions respectively.

Therefore, to generate the expression for the aforementioned electric potential field $2x^2y$, you would have to do

```
>>> from diofant.vector import CoordSysCartesian
>>> R = CoordSysCartesian('R')
>>> electric_potential = 2*R.x**2*R.y
>>> electric_potential
2*R.x**2*R.y
```

It is to be noted that BaseScalar instances can be used just like any other Diofant Symbol, except that they store the information about the coordinate system and axis they correspond to.

Scalar fields can be treated just as any other Diofant expression, for any math/calculus functionality. Hence, to differentiate the above electric potential with respect to x (i.e. R.x), you would use the diff method.

```
>>> from diofant.vector import CoordSysCartesian
>>> R = CoordSysCartesian('R')
>>> electric_potential = 2*R.x**2*R.y
>>> diff(electric_potential, R.x)
4*R.x*R.y
```

It is worth noting that having a BaseScalar in the expression implies that a 'field' changes with position, in 3D space. Technically speaking, a simple Expr with no BaseScalar s is still a field, though constant.

Like scalar fields, vector fields that vary with position can also be constructed using BaseScalar s in the measure-number expressions.

```
>>> from diofant.vector import CoordSysCartesian
>>> R = CoordSysCartesian('R')
>>> v = R.x**2*R.i + 2*R.x*R.z*R.k
```

The Del operator

The Del, or 'Nabla' operator - written as ∇ is commonly known as the vector differential operator. Depending on its usage in a mathematical expression, it may denote the gradient of a scalar field, or the divergence of a vector field, or the curl of a vector field.

Essentially, ∇ is not technically an 'operator', but a convenient mathematical notation to denote any one of the aforementioned field operations.

In *diofant.vector* (page 1032), ∇ has been implemented as the delop property of the CoordSysCartesian class. Hence, assuming C is a coordinate system, the ∇ operator corresponding to the vector differentials wrt C's coordinate variables and basis vectors would be accessible as C.delop.

Given below is an example of usage of the delop object.

```
>>> from diofant.vector import CoordSysCartesian
>>> C = CoordSysCartesian('C')
>>> gradient_field = C.delop(C.x*C.y*C.z)
>>> gradient_field
(Derivative(C.x*C.y*C.z, C.x))*C.i + (Derivative(C.x*C.y*C.z, C.y))*C.j +
↪(Derivative(C.x*C.y*C.z, C.z))*C.k
```

The above expression can be evaluated using the Diofant `doit()` routine.

```
>>> gradient_field.doit()
C.y*C.z*C.i + C.x*C.z*C.j + C.x*C.y*C.k
```

Usage of the ∇ notation in *diofant.vector* (page 1032) has been described in greater detail in the subsequent subsections.

Field operators and related functions

Here we describe some basic field-related functionality implemented in *diofant.vector* (page 1032).

Curl

A curl is a mathematical operator that describes an infinitesimal rotation of a vector in 3D space. The direction is determined by the right-hand rule (along the axis of rotation), and the magnitude is given by the magnitude of rotation.

In the 3D Cartesian system, the curl of a 3D vector $\mathbf{F}$, denoted by $\nabla \times \mathbf{F}$ is given by:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}$$

where F_x denotes the X component of vector $\mathbf{F}$.

Computing the curl of a vector field in *diofant.vector* (page 1032) can be accomplished in two ways.

One, by using the `delop` property

```
>>> from diofant.vector import CoordSysCartesian
>>> C = CoordSysCartesian('C')
>>> C.delop.cross(C.x*C.y*C.z*C.i).doit()
C.x*C.y*C.j + (-C.x*C.z)*C.k
>>> (C.delop ^ C.x*C.y*C.z*C.i).doit()
C.x*C.y*C.j + (-C.x*C.z)*C.k
```

Or by using the dedicated function

```
>>> from diofant.vector import curl
>>> curl(C.x*C.y*C.z*C.i, C)
C.x*C.y*C.j + (-C.x*C.z)*C.k
```

Divergence

Divergence is a vector operator that measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar.

The divergence operator always returns a scalar after operating on a vector.

In the 3D Cartesian system, the divergence of a 3D vector $\mathbf{F}$, denoted by $\nabla \cdot \mathbf{F}$ is given by:

$$\nabla \cdot \mathbf{F} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}$$

where U, V and W denote the X, Y and Z components of $\mathbf{F}$ respectively.

Computing the divergence of a vector field in *diofant.vector* (page 1032) can be accomplished in two ways.

One, by using the `delop` property

```
>>> from diofant.vector import CoordSysCartesian
>>> C = CoordSysCartesian('C')
>>> C.delop.dot(C.x*C.y*C.z*(C.i + C.j + C.k)).doit()
C.x*C.y + C.x*C.z + C.y*C.z
>>> (C.delop & C.x*C.y*C.z*(C.i + C.j + C.k)).doit()
C.x*C.y + C.x*C.z + C.y*C.z
```

Or by using the dedicated function

```
>>> from diofant.vector import divergence
>>> divergence(C.x*C.y*C.z*(C.i + C.j + C.k), C)
C.x*C.y + C.x*C.z + C.y*C.z
```

Gradient

Consider a scalar field $f(x, y, z)$ in 3D space. The gradient of this field is defined as the vector of the 3 partial derivatives of f with respect to x, y and z in the X, Y and Z axes respectively.

In the 3D Cartesian system, the divergence of a scalar field f, denoted by ∇f is given by -

$\nabla f = \frac{\partial f}{\partial x}\mathbf{\hat{i}} + \frac{\partial f}{\partial y}\mathbf{\hat{j}} + \frac{\partial f}{\partial z}\mathbf{\hat{k}}$

Computing the divergence of a vector field in *diofant.vector* (page 1032) can be accomplished in two ways.

One, by using the `delop` property

```
>>> from diofant.vector import CoordSysCartesian
>>> C = CoordSysCartesian('C')
>>> C.delop.gradient(C.x*C.y*C.z).doit()
C.y*C.z*C.i + C.x*C.z*C.j + C.x*C.y*C.k
>>> C.delop(C.x*C.y*C.z).doit()
C.y*C.z*C.i + C.x*C.z*C.j + C.x*C.y*C.k
```

Or by using the dedicated function

```
>>> from diofant.vector import gradient
>>> gradient(C.x*C.y*C.z, C)
C.y*C.z*C.i + C.x*C.z*C.j + C.x*C.y*C.k
```

Directional Derivative

Apart from the above three common applications of ∇, it is also possible to compute the directional derivative of a field wrt a `Vector` in *diofant.vector* (page 1032).

By definition, the directional derivative of a field $\mathbf{F}$ along a vector v at point x represents the instantaneous rate of change of $\mathbf{F}$ moving through x with the velocity v. It is represented mathematically as: $(\vec{v} \cdot \nabla)\,\mathbf{F}(x)$.

Directional derivatives of vector and scalar fields can be computed in *diofant.vector* (page 1032) using the `delop` property of `CoordSysCartesian`.

```
>>> from diofant.vector import CoordSysCartesian
>>> C = CoordSysCartesian('C')
>>> vel = C.i + C.j + C.k
>>> scalar_field = C.x*C.y*C.z
>>> vector_field = C.x*C.y*C.z*C.i
>>> (vel.dot(C.delop))(scalar_field)
C.x*C.y + C.x*C.z + C.y*C.z
>>> (vel & C.delop)(vector_field)
(C.x*C.y + C.x*C.z + C.y*C.z)*C.i
```

Conservative and Solenoidal fields

In vector calculus, a conservative field is a field that is the gradient of some scalar field. Conservative fields have the property that their line integral over any path depends only on the end-points, and is independent of the path travelled. A conservative vector field is also said to be 'irrotational', since the curl of a conservative field is always zero.

In physics, conservative fields represent forces in physical systems where energy is conserved.

To check if a vector field is conservative in *diofant.vector* (page 1032), the `is_conservative` function can be used.

```
>>> from diofant.vector import CoordSysCartesian, is_conservative
>>> R = CoordSysCartesian('R')
>>> field = R.y*R.z*R.i + R.x*R.z*R.j + R.x*R.y*R.k
>>> is_conservative(field)
True
>>> curl(field, R)
0
```

A solenoidal field, on the other hand, is a vector field whose divergence is zero at all points in space.

To check if a vector field is solenoidal in *diofant.vector* (page 1032), the `is_solenoidal` function can be used.

```
>>> from diofant.vector import CoordSysCartesian, is_solenoidal
>>> R = CoordSysCartesian('R')
>>> field = R.y*R.z*R.i + R.x*R.z*R.j + R.x*R.y*R.k
>>> is_solenoidal(field)
True
>>> divergence(field, R)
0
```

Scalar potential functions

We have previously mentioned that every conservative field can be defined as the gradient of some scalar field. This scalar field is also called the 'scalar potential field' corresponding to the aforementioned conservative field.

The `scalar_potential` function in *diofant.vector* (page 1032) calculates the scalar potential field corresponding to a given conservative vector field in 3D space - minus the extra constant of integration, of course.

Example of usage -

```
>>> from diofant.vector import CoordSysCartesian, scalar_potential
>>> R = CoordSysCartesian('R')
>>> conservative_field = 4*R.x*R.y*R.z*R.i + 2*R.x**2*R.z*R.j + 2*R.x**2*R.y*R.k
>>> scalar_potential(conservative_field, R)
2*R.x**2*R.y*R.z
```

Providing a non-conservative vector field as an argument to `scalar_potential` raises a `ValueError`.

The scalar potential difference, or simply 'potential difference', corresponding to a conservative vector field can be defined as the difference between the values of its scalar potential function at two points in space. This is useful in calculating a line integral with respect to a conservative function, since it depends only on the endpoints of the path.

This computation is performed as follows in *diofant.vector* (page 1032).

```
>>> from diofant.vector import CoordSysCartesian, Point
>>> from diofant.vector import scalar_potential_difference
>>> R = CoordSysCartesian('R')
>>> P = R.origin.locate_new('P', 1*R.i + 2*R.j + 3*R.k)
>>> vectfield = 4*R.x*R.y*R.i + 2*R.x**2*R.j
>>> scalar_potential_difference(vectfield, R, R.origin, P)
4
```

If provided with a scalar expression instead of a vector field, `scalar_potential_difference` returns the difference between the values of that scalar field at the two given points in space.

General examples of usage

This section details the solution of two basic problems in vector math/calculus using the *diofant.vector* (page 1032) package.

Quadrilateral problem

The Problem

OABC is any quadrilateral in 3D space. P is the midpoint of OA, Q is the midpoint of AB, R is the midpoint of BC and S is the midpoint of OC. Prove that PQ is parallel to SR

Solution

The solution to this problem demonstrates the usage of `Point`, and basic operations on `Vector`.

Define a coordinate system

```
>>> from diofant.vector import CoordSysCartesian
>>> Sys = CoordSysCartesian('Sys')
```

Define point O to be Sys' origin. We can do this without loss of generality

```
>>> O = Sys.origin
```

Define point A with respect to O

```
>>> a1, a2, a3 = symbols('a1 a2 a3')
>>> A = O.locate_new('A', a1*Sys.i + a2*Sys.j + a3*Sys.k)
```

Similarly define points B and C

```
>>> b1, b2, b3 = symbols('b1 b2 b3')
>>> B = O.locate_new('B', b1*Sys.i + b2*Sys.j + b3*Sys.k)
>>> c1, c2, c3 = symbols('c1 c2 c3')
>>> C = O.locate_new('C', c1*Sys.i + c2*Sys.j + c3*Sys.k)
```

P is the midpoint of OA. Lets locate it with respect to O (you could also define it with respect to A).

```
>>> P = O.locate_new('P', A.position_wrt(O) + (O.position_wrt(A) / 2))
```

Similarly define points Q, R and S as per the problem definitions.

```
>>> Q = A.locate_new('Q', B.position_wrt(A) / 2)
>>> R = B.locate_new('R', C.position_wrt(B) / 2)
>>> S = O.locate_new('R', C.position_wrt(O) / 2)
```

Now compute the vectors in the directions specified by PQ and SR.

```
>>> PQ = Q.position_wrt(P)
>>> SR = R.position_wrt(S)
```

Compute cross product

```
>>> PQ.cross(SR)
0
```

Since the cross product is a zero vector, the two vectors have to be parallel, thus proving that PQ || SR.

Third product rule for Del operator

The Problem

Prove the third rule - $\nabla \cdot (f\vec{v}) = f(\nabla \cdot \vec{v}) + \vec{v} \cdot (\nabla f)$

Solution

Start with a coordinate system

```
>>> from diofant.vector import CoordSysCartesian
>>> C = CoordSysCartesian('C')
```

The scalar field f and the measure numbers of the vector field $\vec{v}$ are all functions of the coordinate variables of the coordinate system in general. Hence, define Diofant functions that way.

```
>>> v1, v2, v3, f = symbols('v1 v2 v3 f', cls=Function)
```

v1, v2 and v3 are the X, Y and Z components of the vector field respectively.

Define the vector field as `vfield` and the scalar field as `sfield`.

```
>>> vfield = v1(C.x, C.y, C.z)*C.i + v2(C.x, C.y, C.z)*C.j + v3(C.x, C.y, C.z)*C.k
>>> ffield = f(C.x, C.y, C.z)
```

Construct the expression for the LHS of the equation using `C.delop`.

```
>>> lhs = (C.delop.dot(ffield * vfield)).doit()
```

Similarly, the RHS would be defined.

```
>>> rhs = ((vfield.dot(C.delop(ffield))) + (ffield * (C.delop.dot(vfield)))).doit()
```

Now, to prove the product rule, we would just need to equate the expanded and simplified versions of the lhs and the rhs, so that the Diofant expressions match.

```
>>> lhs.expand().simplify() == rhs.expand().doit().simplify()
True
```

Thus, the general form of the third product rule mentioned above can be proven using *diofant.vector* (page 1032).

3.25.2 Vector API

Essential Classes in diofant.vector (doctrings)

CoordSysCartesian

class diofant.vector.coordsysrect.**CoordSysCartesian**(*name*, *location=None*, *rotation_matrix=None*, *parent=None*, *vector_names=None*, *variable_names=None*, *latex_vects=None*, *pretty_vects=None*, *latex_scalars=None*, *pretty_scalars=None*)

Represents a coordinate system in 3-D space.

__init__(*name*, *location=None*, *rotation_matrix=None*, *parent=None*, *vector_names=None*, *variable_names=None*, *latex_vects=None*, *pretty_vects=None*, *latex_scalars=None*, *pretty_scalars=None*)

The orientation/location parameters are necessary if this system is being defined at a certain orientation or location wrt another.

Parameters name : str

The name of the new CoordSysCartesian instance.

location : Vector

The position vector of the new system's origin wrt the parent instance.

rotation_matrix : Diofant ImmutableMatrix

The rotation matrix of the new coordinate system with respect to the parent. In other words, the output of new_system.rotation_matrix(parent).

parent : CoordSysCartesian

The coordinate system wrt which the orientation/location (or both) is being defined.

vector_names, variable_names : iterable(optional)

Iterables of 3 strings each, with custom names for base vectors and base scalars of the new system respectively. Used for simple str printing.

locate_new(*name*, *position*, *vector_names=None*, *variable_names=None*)

Returns a CoordSysCartesian with its origin located at the given position wrt this coordinate system's origin.

Parameters name : str

The name of the new CoordSysCartesian instance.

position : Vector

The position vector of the new system's origin wrt this one.

vector_names, variable_names : iterable(optional)

Iterables of 3 strings each, with custom names for base vectors and base scalars of the new system respectively. Used for simple str printing.

Examples

```
>>> A = CoordSysCartesian('A')
>>> B = A.locate_new('B', 10 * A.i)
>>> B.origin.position_wrt(A.origin)
10*A.i
```

orient_new(*name*, *orienters*, *location=None*, *vector_names=None*, *variable_names=None*)

Creates a new CoordSysCartesian oriented in the user-specified way with respect to this system.

Please refer to the documentation of the orienter classes for more information about the orientation procedure.

Parameters name : str

The name of the new CoordSysCartesian instance.

orienters : iterable/Orienter

An Orienter or an iterable of Orienters for orienting the new coordinate system. If an Orienter is provided, it is applied to get the new system. If an iterable is provided, the orienters will be applied in the order in which they appear in the iterable.

location : Vector(optional)

The location of the new coordinate system's origin wrt this system's origin. If not specified, the origins are taken to be coincident.

vector_names, variable_names : iterable(optional)

Iterables of 3 strings each, with custom names for base vectors and base scalars of the new system respectively. Used for simple str printing.

Examples

```
>>> q0, q1, q2, q3 = symbols('q0 q1 q2 q3')
>>> N = CoordSysCartesian('N')
```

Using an AxisOrienter

```
>>> axis_orienter = AxisOrienter(q1, N.i + 2 * N.j)
>>> A = N.orient_new('A', [axis_orienter])
```

Using a BodyOrienter

```
>>> body_orienter = BodyOrienter(q1, q2, q3, '123')
>>> B = N.orient_new('B', [body_orienter])
```

Using a SpaceOrienter

```
>>> space_orienter = SpaceOrienter(q1, q2, q3, '312')
>>> C = N.orient_new('C', [space_orienter])
```

Using a QuaternionOrienter

```
>>> q_orienter = QuaternionOrienter(q0, q1, q2, q3)
>>> D = N.orient_new('D', [q_orienter])
```

orient_new_axis(*name, angle, axis, location=None, vector_names=None, variable_names=None*)

Axis rotation is a rotation about an arbitrary axis by some angle. The angle is supplied as a Diofant expr scalar, and the axis is supplied as a Vector.

Parameters **name** : string

The name of the new coordinate system

angle : Expr

The angle by which the new system is to be rotated

axis : Vector

The axis around which the rotation has to be performed

location : Vector(optional)

The location of the new coordinate system's origin wrt this system's origin. If not specified, the origins are taken to be coincident.

vector_names, variable_names : iterable(optional)

Iterables of 3 strings each, with custom names for base vectors and base scalars of the new system respectively. Used for simple str printing.

Examples

```
>>> q1 = symbols('q1')
>>> N = CoordSysCartesian('N')
>>> B = N.orient_new_axis('B', q1, N.i + 2 * N.j)
```

orient_new_body(*name, angle1, angle2, angle3, rotation_order, location=None, vector_names=None, variable_names=None*)

Body orientation takes this coordinate system through three successive simple rotations.

Body fixed rotations include both Euler Angles and Tait-Bryan Angles, see https//en.wikipedia.org/wiki/Euler_angles.

Parameters name : string

The name of the new coordinate system

angle1, angle2, angle3 : Expr

Three successive angles to rotate the coordinate system by

rotation_order : string

String defining the order of axes for rotation

location : Vector(optional)

The location of the new coordinate system's origin wrt this system's origin. If not specified, the origins are taken to be coincident.

vector_names, variable_names : iterable(optional)

Iterables of 3 strings each, with custom names for base vectors and base scalars of the new system respectively. Used for simple str printing.

Examples

```
>>> q1, q2, q3 = symbols('q1 q2 q3')
>>> N = CoordSysCartesian('N')
```

A 'Body' fixed rotation is described by three angles and three body-fixed rotation axes. To orient a coordinate system D with respect to N, each sequential rotation is always about the orthogonal unit vectors fixed to D. For example, a '123' rotation will specify rotations about N.i, then D.j, then D.k. (Initially, D.i is same as N.i) Therefore,

```
>>> D = N.orient_new_body('D', q1, q2, q3, '123')
```

is same as

```
>>> D = N.orient_new_axis('D', q1, N.i)
>>> D = D.orient_new_axis('D', q2, D.j)
>>> D = D.orient_new_axis('D', q3, D.k)
```

Acceptable rotation orders are of length 3, expressed in XYZ or 123, and cannot have a rotation about about an axis twice in a row.

```
>>> B = N.orient_new_body('B', q1, q2, q3, '123')
>>> B = N.orient_new_body('B', q1, q2, 0, 'ZXZ')
>>> B = N.orient_new_body('B', 0, 0, 0, 'XYX')
```

orient_new_quaternion(*name*, *q0*, *q1*, *q2*, *q3*, *location=None*, *vector_names=None*, *variable_names=None*)

Quaternion orientation orients the new CoordSysCartesian with Quaternions, defined as a finite rotation about lambda, a unit vector, by some amount theta.

This orientation is described by four parameters:

q0 = cos(theta/2)

q1 = lambda_x sin(theta/2)

q2 = lambda_y sin(theta/2)

q3 = lambda_z sin(theta/2)

Quaternion does not take in a rotation order.

Parameters name : string

The name of the new coordinate system

q0, q1, q2, q3 : Expr

The quaternions to rotate the coordinate system by

location : Vector(optional)

The location of the new coordinate system's origin wrt this system's origin. If not specified, the origins are taken to be coincident.

vector_names, variable_names : iterable(optional)

Iterables of 3 strings each, with custom names for base vectors and base scalars of the new system respectively. Used for simple str printing.

Examples

```
>>> q0, q1, q2, q3 = symbols('q0 q1 q2 q3')
>>> N = CoordSysCartesian('N')
>>> B = N.orient_new_quaternion('B', q0, q1, q2, q3)
```

orient_new_space(*name*, *angle1*, *angle2*, *angle3*, *rotation_order*, *location=None*, *vector_names=None*, *variable_names=None*)

Space rotation is similar to Body rotation, but the rotations are applied in the opposite order.

Parameters name : string

The name of the new coordinate system

angle1, angle2, angle3 : Expr

Three successive angles to rotate the coordinate system by

rotation_order : string

String defining the order of axes for rotation

location : Vector(optional)

The location of the new coordinate system's origin wrt this system's origin. If not specified, the origins are taken to be coincident.

vector_names, variable_names : iterable(optional)

Iterables of 3 strings each, with custom names for base vectors and base scalars of the new system respectively. Used for simple str printing.

See also:

diofant.vector.coordsysrect.CoordSysCartesian.orient_new_body **(page 1050)**
method to orient via Euler angles

Examples

```
>>> q1, q2, q3 = symbols('q1 q2 q3')
>>> N = CoordSysCartesian('N')
```

To orient a coordinate system D with respect to N, each sequential rotation is always about N's orthogonal unit vectors. For example, a '123' rotation will specify rotations about N.i, then N.j, then N.k. Therefore,

```
>>> D = N.orient_new_space('D', q1, q2, q3, '312')
```

is same as

```
>>> B = N.orient_new_axis('B', q1, N.i)
>>> C = B.orient_new_axis('C', q2, N.j)
>>> D = C.orient_new_axis('D', q3, N.k)
```

position_wrt(*other*)
Returns the position vector of the origin of this coordinate system with respect to another Point/CoordSysCartesian.

Parameters other : Point/CoordSysCartesian

If other is a Point, the position of this system's origin wrt it is returned. If its an instance of CoordSyRect, the position wrt its origin is returned.

Examples

```
>>> from diofant.vector import Point
>>> N = CoordSysCartesian('N')
>>> N1 = N.locate_new('N1', 10 * N.i)
>>> N.position_wrt(N1)
(-10)*N.i
```

rotation_matrix(*other*)
Returns the direction cosine matrix(DCM), also known as the 'rotation matrix' of this coordinate system with respect to another system.

If v_a is a vector defined in system 'A' (in matrix format) and v_b is the same vector defined in system 'B', then v_a = A.rotation_matrix(B) * v_b.

A Diofant Matrix is returned.

Parameters other : CoordSysCartesian

The system which the DCM is generated to.

Examples

```
>>> q1 = symbols('q1')
>>> N = CoordSysCartesian('N')
>>> A = N.orient_new_axis('A', q1, N.i)
>>> N.rotation_matrix(A)
Matrix([
[1,       0,        0],
[0, cos(q1), -sin(q1)],
[0, sin(q1),  cos(q1)]])
```

scalar_map(*other*)

Returns a dictionary which expresses the coordinate variables (base scalars) of this frame in terms of the variables of otherframe.

Parameters otherframe : CoordSysCartesian

The other system to map the variables to.

Examples

```
>>> A = CoordSysCartesian('A')
>>> q = Symbol('q')
>>> B = A.orient_new_axis('B', q, A.k)
>>> A.scalar_map(B)
{A.x: -sin(q)*B.y + cos(q)*B.x, A.y: sin(q)*B.x + cos(q)*B.y, A.z: B.z}
```

Vector

class diofant.vector.vector.**Vector**

Super class for all Vector classes. Ideally, neither this class nor any of its subclasses should be instantiated by the user.

components

Returns the components of this vector in the form of a Python dictionary mapping BaseVector instances to the corresponding measure numbers.

Examples

```
>>> C = CoordSysCartesian('C')
>>> v = 3*C.i + 4*C.j + 5*C.k
>>> v.components
{C.i: 3, C.j: 4, C.k: 5}
```

cross(*other*)
Returns the cross product of this Vector with another Vector or Dyadic instance. The cross product is a Vector, if 'other' is a Vector. If 'other' is a Dyadic, this returns a Dyadic instance.

Parameters other: Vector/Dyadic

The Vector or Dyadic we are crossing with.

Examples

```
>>> C = CoordSysCartesian('C')
>>> C.i.cross(C.j)
C.k
>>> C.i ^ C.i
0
>>> v = 3*C.i + 4*C.j + 5*C.k
>>> v ^ C.i
5*C.j + (-4)*C.k
>>> d = C.i.outer(C.i)
>>> C.j.cross(d)
(-1)*(C.k|C.i)
```

dot(*other*)
Returns the dot product of this Vector, either with another Vector, or a Dyadic, or a Del operator. If 'other' is a Vector, returns the dot product scalar (Diofant expression). If 'other' is a Dyadic, the dot product is returned as a Vector. If 'other' is an instance of Del, returns the directional derivate operator as a Python function. If this function is applied to a scalar expression, it returns the directional derivative of the scalar field wrt this Vector.

Parameters other: Vector/Dyadic/Del

The Vector or Dyadic we are dotting with, or a Del operator .

Examples

```
>>> C = CoordSysCartesian('C')
>>> C.i.dot(C.j)
0
>>> C.i & C.i
1
>>> v = 3*C.i + 4*C.j + 5*C.k
>>> v.dot(C.k)
5
>>> (C.i & C.delop)(C.x*C.y*C.z)
C.y*C.z
>>> d = C.i.outer(C.i)
>>> C.i.dot(d)
C.i
```

magnitude()
Returns the magnitude of this vector.

normalize()
Returns the normalized version of this vector.

outer(*other*)

Returns the outer product of this vector with another, in the form of a Dyadic instance.

Parameters other : Vector

The Vector with respect to which the outer product is to be computed.

Examples

```
>>> N = CoordSysCartesian('N')
>>> N.i.outer(N.j)
(N.i|N.j)
```

separate()

The constituents of this vector in different coordinate systems, as per its definition.

Returns a dict mapping each CoordSysCartesian to the corresponding constituent Vector.

Examples

```
>>> R1 = CoordSysCartesian('R1')
>>> R2 = CoordSysCartesian('R2')
>>> v = R1.i + R2.i
>>> v.separate()
{R1: R1.i, R2: R2.i}
```

to_matrix(*system*)

Returns the matrix form of this vector with respect to the specified coordinate system.

Parameters system : CoordSysCartesian

The system wrt which the matrix form is to be computed

Examples

```
>>> C = CoordSysCartesian('C')
>>> from diofant.abc import a, b, c
>>> v = a*C.i + b*C.j + c*C.k
>>> v.to_matrix(C)
Matrix([
[a],
[b],
[c]])
```

Dyadic

class diofant.vector.dyadic.**Dyadic**

Super class for all Dyadic-classes.

References

[R626] (page 1268), [R627] (page 1268)

components
Returns the components of this dyadic in the form of a Python dictionary mapping BaseDyadic instances to the corresponding measure numbers.

cross(*other*)
Returns the cross product between this Dyadic, and a Vector, as a Vector instance.

Parameters other : Vector

The Vector that we are crossing this Dyadic with

Examples

```
>>> from diofant.vector import CoordSysCartesian
>>> N = CoordSysCartesian('N')
>>> d = N.i.outer(N.i)
>>> d.cross(N.j)
(N.i|N.k)
```

dot(*other*)
Returns the dot product(also called inner product) of this Dyadic, with another Dyadic or Vector. If 'other' is a Dyadic, this returns a Dyadic. Else, it returns a Vector (unless an error is encountered).

Parameters other : Dyadic/Vector

The other Dyadic or Vector to take the inner product with

Examples

```
>>> from diofant.vector import CoordSysCartesian
>>> N = CoordSysCartesian('N')
>>> D1 = N.i.outer(N.j)
>>> D2 = N.j.outer(N.j)
>>> D1.dot(D2)
(N.i|N.j)
>>> D1.dot(N.j)
N.i
```

to_matrix(*system*, *second_system=None*)
Returns the matrix form of the dyadic with respect to one or two coordinate systems.

Parameters system : CoordSysCartesian

The coordinate system that the rows and columns of the matrix correspond to. If a second system is provided, this only corresponds to the rows of the matrix.

second_system : CoordSysCartesian, optional, default=None

The coordinate system that the columns of the matrix correspond to.

Examples

```
>>> from diofant.vector import CoordSysCartesian
>>> N = CoordSysCartesian('N')
>>> v = N.i + 2*N.j
>>> d = v.outer(N.i)
>>> d.to_matrix(N)
Matrix([
[1, 0, 0],
[2, 0, 0],
[0, 0, 0]])
>>> q = Symbol('q')
>>> P = N.orient_new_axis('P', q, N.k)
>>> d.to_matrix(N, P)
Matrix([
[  cos(q),   -sin(q), 0],
[2*cos(q), -2*sin(q), 0],
[       0,         0, 0]])
```

Del

class diofant.vector.deloperator.**Del**

Represents the vector differential operator, usually represented in mathematical expressions as the 'nabla' symbol.

cross(*vect*, *doit=False*)

Represents the cross product between this operator and a given vector - equal to the curl of the vector field.

Parameters vect : Vector

The vector whose curl is to be calculated.

doit : bool

If True, the result is returned after calling .doit() on each component. Else, the returned expression contains Derivative instances

Examples

```
>>> C = CoordSysCartesian('C')
>>> v = C.x*C.y*C.z * (C.i + C.j + C.k)
>>> C.delop.cross(v, doit = True)
(-C.x*C.y + C.x*C.z)*C.i + (C.x*C.y - C.y*C.z)*C.j + (-C.x*C.z + C.y*C.z)*C.k
>>> (C.delop ^ C.i).doit()
0
```

dot(*vect*, *doit=False*)

Represents the dot product between this operator and a given vector - equal to the divergence of the vector field.

Parameters vect : Vector

The vector whose divergence is to be calculated.

doit : bool

If True, the result is returned after calling .doit() on each component. Else, the returned expression contains Derivative instances

Examples

```
>>> C = CoordSysCartesian('C')
>>> C.delop.dot(C.x*C.i)
Derivative(C.x, C.x)
>>> v = C.x*C.y*C.z * (C.i + C.j + C.k)
>>> (C.delop & v).doit()
C.x*C.y + C.x*C.z + C.y*C.z
```

gradient(*scalar_field*, *doit=False*)
Returns the gradient of the given scalar field, as a Vector instance.

Parameters **scalar_field** : Diofant expression

The scalar field to calculate the gradient of.

doit : bool

If True, the result is returned after calling .doit() on each component. Else, the returned expression contains Derivative instances

Examples

```
>>> C = CoordSysCartesian('C')
>>> C.delop.gradient(9)
(Derivative(9, C.x))*C.i + (Derivative(9, C.y))*C.j + (Derivative(9, C.z))*C.k
>>> C.delop(C.x*C.y*C.z).doit()
C.y*C.z*C.i + C.x*C.z*C.j + C.x*C.y*C.k
```

Orienter classes (docstrings)

Orienter

class diofant.vector.orienters.**Orienter**
Super-class for all orienter classes.

rotation_matrix()
The rotation matrix corresponding to this orienter instance.

AxisOrienter

class diofant.vector.orienters.**AxisOrienter**(*angle*, *axis*)
Class to denote an axis orienter.

__init__(*angle*, *axis*)
Axis rotation is a rotation about an arbitrary axis by some angle. The angle is supplied as a Diofant expr scalar, and the axis is supplied as a Vector.

Parameters **angle** : Expr

The angle by which the new system is to be rotated

axis : Vector

The axis around which the rotation has to be performed

Examples

```
>>> from diofant.vector import CoordSysCartesian
>>> q1 = symbols('q1')
>>> N = CoordSysCartesian('N')
>>> orienter = AxisOrienter(q1, N.i + 2 * N.j)
>>> B = N.orient_new('B', [orienter])
```

static __new__(*angle*, *axis*)

Create and return a new object. See help(type) for accurate signature.

rotation_matrix(*system*)

The rotation matrix corresponding to this orienter instance.

Parameters system : CoordSysCartesian

The coordinate system wrt which the rotation matrix is to be computed

BodyOrienter

class diofant.vector.orienters.**BodyOrienter**(*angle1*, *angle2*, *angle3*, *rot_order*)

Class to denote a body-orienter.

__init__(*angle1*, *angle2*, *angle3*, *rot_order*)

Body orientation takes this coordinate system through three successive simple rotations.

Body fixed rotations include both Euler Angles and Tait-Bryan Angles, see https//en.wikipedia.org/wiki/Euler_angles.

Parameters angle1, angle2, angle3 : Expr

Three successive angles to rotate the coordinate system by

rotation_order : string

String defining the order of axes for rotation

Examples

```
>>> from diofant.vector import CoordSysCartesian
>>> q1, q2, q3 = symbols('q1 q2 q3')
>>> N = CoordSysCartesian('N')
```

A 'Body' fixed rotation is described by three angles and three body-fixed rotation axes. To orient a coordinate system D with respect to N, each sequential rotation is always about the orthogonal unit vectors fixed to D. For example, a '123' rotation will specify rotations about N.i, then D.j, then D.k. (Initially, D.i is same as N.i) Therefore,

```
>>> body_orienter = BodyOrienter(q1, q2, q3, '123')
>>> D = N.orient_new('D', [body_orienter])
```

is same as

```
>>> axis_orienter1 = AxisOrienter(q1, N.i)
>>> D = N.orient_new('D', [axis_orienter1])
>>> axis_orienter2 = AxisOrienter(q2, D.j)
>>> D = D.orient_new('D', [axis_orienter2])
>>> axis_orienter3 = AxisOrienter(q3, D.k)
>>> D = D.orient_new('D', [axis_orienter3])
```

Acceptable rotation orders are of length 3, expressed in XYZ or 123, and cannot have a rotation about about an axis twice in a row.

```
>>> body_orienter1 = BodyOrienter(q1, q2, q3, '123')
>>> body_orienter2 = BodyOrienter(q1, q2, 0, 'ZXZ')
>>> body_orienter3 = BodyOrienter(0, 0, 0, 'XYX')
```

static __new__(*angle1*, *angle2*, *angle3*, *rot_order*)
Create and return a new object. See help(type) for accurate signature.

SpaceOrienter

class diofant.vector.orienters.**SpaceOrienter**(*angle1*, *angle2*, *angle3*, *rot_order*)
Class to denote a space-orienter.

__init__(*angle1*, *angle2*, *angle3*, *rot_order*)
Space rotation is similar to Body rotation, but the rotations are applied in the opposite order.

Parameters **angle1, angle2, angle3** : Expr

Three successive angles to rotate the coordinate system by

rotation_order : string

String defining the order of axes for rotation

See also:

BodyOrienter **(page 1059)** Orienter to orient systems wrt Euler angles.

Examples

```
>>> from diofant.vector import CoordSysCartesian
>>> q1, q2, q3 = symbols('q1 q2 q3')
>>> N = CoordSysCartesian('N')
```

To orient a coordinate system D with respect to N, each sequential rotation is always about N's orthogonal unit vectors. For example, a '123' rotation will specify rotations about N.i, then N.j, then N.k. Therefore,

```
>>> space_orienter = SpaceOrienter(q1, q2, q3, '312')
>>> D = N.orient_new('D', [space_orienter])
```

is same as

```
>>> axis_orienter1 = AxisOrienter(q1, N.i)
>>> B = N.orient_new('B', [axis_orienter1])
>>> axis_orienter2 = AxisOrienter(q2, N.j)
>>> C = B.orient_new('C', [axis_orienter2])
>>> axis_orienter3 = AxisOrienter(q3, N.k)
>>> D = C.orient_new('C', [axis_orienter3])
```

static __new__(*angle1*, *angle2*, *angle3*, *rot_order*)
Create and return a new object. See help(type) for accurate signature.

QuaternionOrienter

class diofant.vector.orienters.**QuaternionOrienter**(*angle1*, *angle2*, *angle3*, *rot_order*)
Class to denote a quaternion-orienter.

__init__(*angle1*, *angle2*, *angle3*, *rot_order*)
Quaternion orientation orients the new CoordSysCartesian with Quaternions, defined as a finite rotation about lambda, a unit vector, by some amount theta.

This orientation is described by four parameters:

q0 = cos(theta/2)

q1 = lambda_x sin(theta/2)

q2 = lambda_y sin(theta/2)

q3 = lambda_z sin(theta/2)

Quaternion does not take in a rotation order.

Parameters q0, q1, q2, q3 : Expr

The quaternions to rotate the coordinate system by

Examples

```
>>> from diofant.vector import CoordSysCartesian
>>> q0, q1, q2, q3 = symbols('q0 q1 q2 q3')
>>> N = CoordSysCartesian('N')
>>> q_orienter = QuaternionOrienter(q0, q1, q2, q3)
>>> B = N.orient_new('B', [q_orienter])
```

static __new__(*q0*, *q1*, *q2*, *q3*)
Create and return a new object. See help(type) for accurate signature.

Essential Functions in diofant.vector (docstrings)

matrix_to_vector

diofant.vector.**matrix_to_vector**(*matrix*, *system*)
Converts a vector in matrix form to a Vector instance.

It is assumed that the elements of the Matrix represent the measure numbers of the components of the vector along basis vectors of 'system'.

Parameters matrix : Diofant Matrix, Dimensions: (3, 1)

The matrix to be converted to a vector

system : CoordSysCartesian

The coordinate system the vector is to be defined in

Examples

```
>>> m = Matrix([1, 2, 3])
>>> C = CoordSysCartesian('C')
>>> v = matrix_to_vector(m, C)
>>> v
C.i + 2*C.j + 3*C.k
>>> v.to_matrix(C) == m
True
```

express

`diofant.vector.express(expr, system, system2=None, variables=False)`

Global function for 'express' functionality.

Re-expresses a Vector, Dyadic or scalar(diofant-able) in the given coordinate system.

If 'variables' is True, then the coordinate variables (base scalars) of other coordinate systems present in the vector/scalar field or dyadic are also substituted in terms of the base scalars of the given system.

Parameters expr : Vector/Dyadic/scalar(diofant-able)

The expression to re-express in CoordSysCartesian 'system'

system: CoordSysCartesian

The coordinate system the expr is to be expressed in

system2: CoordSysCartesian

The other coordinate system required for re-expression (only for a Dyadic Expr)

variables : boolean

Specifies whether to substitute the coordinate variables present in expr, in terms of those of parameter system

Examples

```
>>> N = CoordSysCartesian('N')
>>> q = Symbol('q')
>>> B = N.orient_new_axis('B', q, N.k)
>>> express(B.i, N)
(cos(q))*N.i + (sin(q))*N.j
>>> express(N.x, B, variables=True)
-sin(q)*B.y + cos(q)*B.x
>>> d = N.i.outer(N.i)
```

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```
>>> express(d, B, N)
(cos(q))*(B.i|N.i) + (-sin(q))*(B.j|N.i)
```

curl

diofant.vector.**curl**(*vect*, *coord_sys*)
Returns the curl of a vector field computed wrt the base scalars of the given coordinate system.

Parameters vect : Vector

The vector operand

coord_sys : CoordSysCartesian

The coordinate system to calculate the curl in

Examples

```
>>> R = CoordSysCartesian('R')
>>> v1 = R.y*R.z*R.i + R.x*R.z*R.j + R.x*R.y*R.k
>>> curl(v1, R)
0
>>> v2 = R.x*R.y*R.z*R.i
>>> curl(v2, R)
R.x*R.y*R.j + (-R.x*R.z)*R.k
```

divergence

diofant.vector.**divergence**(*vect*, *coord_sys*)
Returns the divergence of a vector field computed wrt the base scalars of the given coordinate system.

Parameters vect : Vector

The vector operand

coord_sys : CoordSysCartesian

The coordinate system to calculate the divergence in

Examples

```
>>> R = CoordSysCartesian('R')
>>> v1 = R.x*R.y*R.z * (R.i+R.j+R.k)
>>> divergence(v1, R)
R.x*R.y + R.x*R.z + R.y*R.z
>>> v2 = 2*R.y*R.z*R.j
>>> divergence(v2, R)
2*R.z
```

gradient

`diofant.vector.gradient(scalar, coord_sys)`

Returns the vector gradient of a scalar field computed wrt the base scalars of the given coordinate system.

Parameters **scalar** : Diofant Expr

The scalar field to compute the gradient of

coord_sys : CoordSysCartesian

The coordinate system to calculate the gradient in

Examples

```
>>> R = CoordSysCartesian('R')
>>> s1 = R.x*R.y*R.z
>>> gradient(s1, R)
R.y*R.z*R.i + R.x*R.z*R.j + R.x*R.y*R.k
>>> s2 = 5*R.x**2*R.z
>>> gradient(s2, R)
10*R.x*R.z*R.i + 5*R.x**2*R.k
```

is_conservative

`diofant.vector.is_conservative(field)`

Checks if a field is conservative.

Parameters **field** : Vector

The field to check for conservative property

Examples

```
>>> R = CoordSysCartesian('R')
>>> is_conservative(R.y*R.z*R.i + R.x*R.z*R.j + R.x*R.y*R.k)
True
>>> is_conservative(R.z*R.j)
False
```

is_solenoidal

`diofant.vector.is_solenoidal(field)`

Checks if a field is solenoidal.

Parameters **field** : Vector

The field to check for solenoidal property

Examples

```
>>> R = CoordSysCartesian('R')
>>> is_solenoidal(R.y*R.z*R.i + R.x*R.z*R.j + R.x*R.y*R.k)
True
>>> is_solenoidal(R.y * R.j)
False
```

scalar_potential

diofant.vector.**scalar_potential**(*field*, *coord_sys*)

Returns the scalar potential function of a field in a given coordinate system (without the added integration constant).

Parameters field : Vector

The vector field whose scalar potential function is to be calculated

coord_sys : CoordSysCartesian

The coordinate system to do the calculation in

Examples

```
>>> R = CoordSysCartesian('R')
>>> scalar_potential(R.k, R) == R.z
True
>>> scalar_field = 2*R.x**2*R.y*R.z
>>> grad_field = gradient(scalar_field, R)
>>> scalar_potential(grad_field, R)
2*R.x**2*R.y*R.z
```

scalar_potential_difference

diofant.vector.**scalar_potential_difference**(*field*, *coord_sys*, *point1*, *point2*)

Returns the scalar potential difference between two points in a certain coordinate system, wrt a given field.

If a scalar field is provided, its values at the two points are considered. If a conservative vector field is provided, the values of its scalar potential function at the two points are used.

Returns (potential at point2) - (potential at point1)

The position vectors of the two Points are calculated wrt the origin of the coordinate system provided.

Parameters field : Vector/Expr

The field to calculate wrt

coord_sys : CoordSysCartesian

The coordinate system to do the calculations in

point1 : Point

The initial Point in given coordinate system

position2 : Point

The second Point in the given coordinate system

Examples

```
>>> from diofant.vector import Point
>>> R = CoordSysCartesian('R')
>>> P = R.origin.locate_new('P', R.x*R.i + R.y*R.j + R.z*R.k)
>>> vectfield = 4*R.x*R.y*R.i + 2*R.x**2*R.j
>>> scalar_potential_difference(vectfield, R, R.origin, P)
2*R.x**2*R.y
>>> Q = R.origin.locate_new('O', 3*R.i + R.j + 2*R.k)
>>> scalar_potential_difference(vectfield, R, P, Q)
-2*R.x**2*R.y + 18
```

CHAPTER

FOUR

INTERNALS

4.1 Internals of the Polynomial Manipulation Module

The implementation of the polynomials module is structured internally in “levels”. There are four levels, called L0, L1, L2 and L3. The levels three and four contain the user-facing functionality and were described in the previous section. This section focuses on levels zero and one.

Level zero provides core polynomial manipulation functionality with C-like, low-level interfaces. Level one wraps this low-level functionality into object oriented structures. These are *not* the classes seen by the user, but rather classes used internally throughout the polys module.

There is one additional complication in the implementation. This comes from the fact that all polynomial manipulations are relative to a *ground domain*. For example, when factoring a polynomial like $x^{10} - 1$, one has to decide what ring the coefficients are supposed to belong to, or less trivially, what coefficients are allowed to appear in the factorization. This choice of coefficients is called a ground domain. Typical choices include the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$ or various related rings and fields. But it is perfectly legitimate (although in this case uninteresting) to factorize over polynomial rings such as $k[Y]$, where k is some fixed field.

Thus the polynomial manipulation algorithms (both complicated ones like factoring, and simpler ones like addition or multiplication) have to rely on other code to manipulate the coefficients. In the polynomial manipulation module, such code is encapsulated in so-called “domains”. A domain is basically a factory object: it takes various representations of data, and converts them into objects with unified interface. Every object created by a domain has to implement the arithmetic operations $+$, $-$ and $\times$. Other operations are accessed through the domain, e.g. as in `ZZ.quo(ZZ(4), ZZ(2))`.

Note that there is some amount of *circularity*: the polynomial ring domains use the level one classes, the level one classes use the level zero functions, and level zero functions use domains. It is possible, in principle, but not in the current implementation, to work in rings like $k[X][Y]$. This would create even more layers. For this reason, working in the isomorphic ring $k[X, Y]$ is preferred.

4.1.1 Level One

class diofant.polys.polyclasses.**DMP**(*rep*, *dom*, *lev=None*)

Dense Multivariate Polynomials over K.

LC()

Returns the leading coefficient of `self`.

TC()
Returns the trailing coefficient of `self`.

abs()
Make all coefficients in `self` positive.

add(*other*)
Add two multivariate polynomials `self` and `other`.

add_ground(*c*)
Add an element of the ground domain to `self`.

all_coeffs()
Returns all coefficients from `self`.

all_monoms()
Returns all monomials from `self`.

all_terms()
Returns all terms from a `self`.

cancel(*other*, *include=True*)
Cancel common factors in a rational function `self/other`.

clear_denoms()
Clear denominators, but keep the ground domain.

coeffs(*order=None*)
Returns all non-zero coefficients from `self` in lex order.

cofactors(*other*)
Returns GCD of `self` and `other` and their cofactors.

compose(*other*)
Computes functional composition of `self` and `other`.

content()
Returns GCD of polynomial coefficients.

convert(*dom*)
Convert the ground domain of `self`.

count_complex_roots(*inf=None*, *sup=None*)
Return the number of complex roots of `self` in `[inf, sup]`.

count_real_roots(*inf=None*, *sup=None*)
Return the number of real roots of `self` in `[inf, sup]`.

decompose()
Computes functional decomposition of `self`.

deflate()
Reduce degree of $self$ by mapping x_i^m to y_i.

degree(*j=0*)
Returns the leading degree of `self` in `x_j`.

degree_list()
Returns a list of degrees of `self`.

diff(*m=1*, *j=0*)
Computes the `m`-th order derivative of `self` in `x_j`.

discriminant()
Computes discriminant of `self`.

div(*other*)
Polynomial division with remainder of `self` and `other`.

eject(*dom*, *front=False*)
Eject selected generators into the ground domain.

eval(*a*, *j=0*)
Evaluates `self` at the given point `a` in `x_j`.

exclude()
Remove useless generators from `self`.

Returns the removed generators and the new excluded `self`.

Examples

```
>>> DMP([[[ZZ(1)]], [[ZZ(1)], [ZZ(2)]]], ZZ).exclude()
([2], DMP([[1], [1, 2]], ZZ))
```

exquo(*other*)
Computes polynomial exact quotient of `self` and `other`.

exquo_ground(*c*)
Exact quotient of `self` by a an element of the ground domain.

factor_list()
Returns a list of irreducible factors of `self`.

factor_list_include()
Returns a list of irreducible factors of `self`.

classmethod from_dict(*rep*, *lev*, *dom*)
Construct and instance of `cls` from a `dict` representation.

classmethod from_diofant_list(*rep*, *lev*, *dom*)
Create an instance of `cls` given a list of Diofant coefficients.

classmethod from_list(*rep*, *lev*, *dom*)
Create an instance of `cls` given a list of native coefficients.

gcd(*other*)
Returns polynomial GCD of `self` and `other`.

gcdex(*other*)
Extended Euclidean algorithm, if univariate.

gff_list()
Computes greatest factorial factorization of `self`.

half_gcdex(*other*)
Half extended Euclidean algorithm, if univariate.

homogeneous_order()
Returns the homogeneous order of `self`.

homogenize(*s*)
Return homogeneous polynomial of `self`

inject(*front=False*)
Inject ground domain generators into `self`.

integrate(*m=1, j=0*)
Computes the `m`-th order indefinite integral of `self` in `x_j`.

intervals(*all=False, eps=None, inf=None, sup=None, fast=False, sqf=False*)
Compute isolating intervals for roots of `self`.

invert(*other*)
Invert `self` modulo `other`, if possible.

is_cyclotomic
Returns `True` if `self` is a cyclotomic polynomial.

is_ground
Returns `True` if `self` is an element of the ground domain.

is_homogeneous
Returns `True` if `self` is a homogeneous polynomial.

is_irreducible
Returns `True` if `self` has no factors over its domain.

is_linear
Returns `True` if `self` is linear in all its variables.

is_monic
Returns `True` if the leading coefficient of `self` is one.

is_monomial
Returns `True` if `self` is zero or has only one term.

is_one
Returns `True` if `self` is a unit polynomial.

is_primitive
Returns `True` if the GCD of the coefficients of `self` is one.

is_quadratic
Returns `True` if `self` is quadratic in all its variables.

is_sqf
Returns `True` if `self` is a square-free polynomial.

is_zero
Returns `True` if `self` is a zero polynomial.

l1_norm()
Returns l1 norm of `self`.

lcm(*other*)
Returns polynomial LCM of `self` and `other`.

lift()
Convert algebraic coefficients to rationals.

max_norm()
Returns maximum norm of `self`.

monic()
Divides all coefficients by `LC(self)`.

monoms(*order=None*)
Returns all non-zero monomials from `self` in lex order.

mul(*other*)
Multiply two multivariate polynomials `f` and `g`.

mul_ground(*c*)
Multiply `self` by a an element of the ground domain.

neg()
Negate all coefficients in `self`.

nth(**N*)
Returns the `n`-th coefficient of `self`.

pdiv(*other*)
Polynomial pseudo-division of `self` and `other`.

per(*rep, dom=None, kill=False*)
Create a DMP out of the given representation.

permute(*P*)
Returns a polynomial in $K[x_{P(1)}, ..., x_{P(n)}]$.

Examples

```
>>> DMP([[[ZZ(2)], [ZZ(1), ZZ(0)]], [[]]], ZZ).permute([1, 0, 2])
DMP([[[2], []], [[1, 0], []]], ZZ)
```

```
>>> DMP([[[ZZ(2)], [ZZ(1), ZZ(0)]], [[]]], ZZ).permute([1, 2, 0])
DMP([[[1], []], [[2, 0], []]], ZZ)
```

pexquo(*other*)
Polynomial exact pseudo-quotient of `self` and `other`.

pow(*n*)
Raise `self` to a non-negative power `n`.

pquo(*other*)
Polynomial pseudo-quotient of `self` and `other`.

prem(*other*)
Polynomial pseudo-remainder of `self` and `other`.

primitive()
Returns content and a primitive form of `self`.

quo(*other*)
Computes polynomial quotient of `self` and `other`.

quo_ground(*c*)
Quotient of `self` by a an element of the ground domain.

refine_root(*s, t, eps=None, steps=None, fast=False*)
Refine an isolating interval to the given precision.

`eps` should be a rational number.

rem(*other*)
Computes polynomial remainder of `self` and `other`.

resultant(*other*, *includePRS=False*)
Computes resultant of `self` and `other` via PRS.

revert(*n*)
Compute `self**(-1)` mod `x**n`.

shift(*a*)
Efficiently compute Taylor shift `self(x + a)`.

slice(*m*, *n*, *j=0*)
Take a continuous subsequence of terms of `self`.

sqf_list(*all=False*)
Returns a list of square-free factors of `self`.

sqf_list_include(*all=False*)
Returns a list of square-free factors of `self`.

sqf_norm()
Computes square-free norm of `self`.

sqf_part()
Computes square-free part of `self`.

sqr()
Square a multivariate polynomial `self`.

sturm()
Computes the Sturm sequence of `self`.

sub(*other*)
Subtract two multivariate polynomials `self` and `other`.

sub_ground(*c*)
Subtract an element of the ground domain from `self`.

subresultants(*other*)
Computes subresultant PRS sequence of `self` and `other`.

terms(*order=None*)
Returns all non-zero terms from `self` in lex order.

terms_gcd()
Remove GCD of terms from the polynomial `self`.

to_dict(*zero=False*)
Convert `self` to a dict representation with native coefficients.

to_diofant_dict(*zero=False*)
Convert `self` to a dict representation with Diofant coefficients.

to_exact()
Make the ground domain exact.

to_field()
Make the ground domain a field.

to_ring()
Make the ground domain a ring.

to_tuple()
Convert `self` to a tuple representation with native coefficients.

This is needed for hashing.

total_degree()
Returns the total degree of `self`.

trunc(*p*)
Reduce `self` modulo a constant `p`.

unify(*other*)
Unify representations of two multivariate polynomials.

class diofant.polys.polyclasses.**DMF**(*rep*, *dom*, *lev=None*)
Dense Multivariate Fractions over K.

add(*other*)
Add two multivariate fractions `self` and `other`.

denom()
Returns the denominator of `self`.

exquo(*other*)
Computes quotient of fractions `self` and `other`.

frac_unify(*other*)
Unify representations of two multivariate fractions.

half_per(*rep*, *kill=False*)
Create a DMP out of the given representation.

invert(*check=True*)
Computes inverse of a fraction `self`.

is_one
Returns `True` if `self` is a unit fraction.

is_zero
Returns `True` if `self` is a zero fraction.

mul(*other*)
Multiply two multivariate fractions `self` and `other`.

neg()
Negate all coefficients in `self`.

numer()
Returns the numerator of `self`.

per(*num*, *den*, *cancel=True*, *kill=False*)
Create a DMF out of the given representation.

poly_unify(*other*)
Unify a multivariate fraction and a polynomial.

pow(*n*)
Raise `self` to a non-negative power `n`.

quo(*other*)
Computes quotient of fractions `self` and `other`.

sub(*other*)
Subtract two multivariate fractions `self` and `other`.

class diofant.polys.polyclasses.**ANP**(*rep*, *mod*, *dom*)
Dense Algebraic Number Polynomials over a field.

LC()
Returns the leading coefficient of `self`.

TC()
Returns the trailing coefficient of `self`.

is_ground
Returns `True` if `self` is an element of the ground domain.

is_one
Returns `True` if `self` is a unit algebraic number.

is_zero
Returns `True` if `self` is a zero algebraic number.

pow(*n*)
Raise `self` to an integer power `n`.

to_dict()
Convert `self` to a dict representation with native coefficients.

to_diofant_dict()
Convert `self` to a dict representation with Diofant coefficients.

to_diofant_list()
Convert `self` to a list representation with Diofant coefficients.

to_list()
Convert `self` to a list representation with native coefficients.

to_tuple()
Convert `self` to a tuple representation with native coefficients.

This is needed for hashing.

unify(*other*)
Unify representations of two algebraic numbers.

4.1.2 Level Zero

Level zero contains the bulk code of the polynomial manipulation module.

Manipulation of dense, multivariate polynomials

These functions can be used to manipulate polynomials in $K[X_0, \ldots, X_u]$. Functions for manipulating multivariate polynomials in the dense representation have the prefix `dmp_`. Functions which only apply to univariate polynomials (i.e. $u = 0$) have the prefix `dup__`. The ground domain K has to be passed explicitly. For many multivariate polynomial manipulation functions also the level u, i.e. the number of generators minus one, has to be passed. (Note that, in many cases, `dup_` versions of functions are available, which may be slightly more efficient.)

Basic manipulation:

`diofant.polys.densebasic.`**dmp_LC**(*f*, *K*)
Return leading coefficient of `f`.

Examples

```
>>> dmp_LC([], ZZ)
0
>>> dmp_LC([ZZ(1), ZZ(2), ZZ(3)], ZZ)
1
```

diofant.polys.densebasic.**dmp_TC**(f, K)
Return trailing coefficient of f.

Examples

```
>>> dmp_TC([], ZZ)
0
>>> dmp_TC([ZZ(1), ZZ(2), ZZ(3)], ZZ)
3
```

diofant.polys.densebasic.**dmp_ground_LC**(f, u, K)
Return the ground leading coefficient.

Examples

```
>>> f = ZZ.map([[[1], [2, 3]]])
```

```
>>> dmp_ground_LC(f, 2, ZZ)
1
```

diofant.polys.densebasic.**dmp_ground_TC**(f, u, K)
Return the ground trailing coefficient.

Examples

```
>>> f = ZZ.map([[[1], [2, 3]]])
```

```
>>> dmp_ground_TC(f, 2, ZZ)
3
```

diofant.polys.densebasic.**dmp_true_LT**(f, u, K)
Return the leading term c * x_1**n_1 ... x_k**n_k.

Examples

```
>>> f = ZZ.map([[4], [2, 0], [3, 0, 0]])
```

```
>>> dmp_true_LT(f, 1, ZZ)
((2, 0), 4)
```

diofant.polys.densebasic.**dmp_degree**(*f*, *u*)
Return the leading degree of `f` in `x_0` in `K[X]`.

Note that the degree of 0 is negative infinity (the Diofant object -oo).

Examples

```
>>> dmp_degree([[[]]], 2)
-oo
```

```
>>> f = ZZ.map([[2], [1, 2, 3]])
```

```
>>> dmp_degree(f, 1)
1
```

diofant.polys.densebasic.**dmp_degree_in**(*f*, *j*, *u*)
Return the leading degree of `f` in `x_j` in `K[X]`.

Examples

```
>>> f = ZZ.map([[2], [1, 2, 3]])
```

```
>>> dmp_degree_in(f, 0, 1)
1
>>> dmp_degree_in(f, 1, 1)
2
```

diofant.polys.densebasic.**dmp_degree_list**(*f*, *u*)
Return a list of degrees of `f` in `K[X]`.

Examples

```
>>> f = ZZ.map([[1], [1, 2, 3]])
>>> dmp_degree_list(f, 1)
(1, 2)
```

diofant.polys.densebasic.**dmp_strip**(*f*, *u*)
Remove leading zeros from `f` in `K[X]`.

Examples

```
>>> dmp_strip([[], [0, 1, 2], [1]], 1)
[[0, 1, 2], [1]]
```

diofant.polys.densebasic.**dmp_validate**(*f*, *K=None*)
Return the number of levels in `f` and recursively strip it.

Examples

```
>>> dmp_validate([[], [0, 1, 2], [1]])
([[1, 2], [1]], 1)
```

```
>>> dmp_validate([[1], 1])
Traceback (most recent call last):
...
ValueError: invalid data structure for a multivariate polynomial
```

diofant.polys.densebasic.**dup_reverse**(*f*)
Compute `x**n * f(1/x)`, i.e.: reverse `f` in `K[x]`.

Examples

```
>>> f = ZZ.map([1, 2, 3, 0])
>>> dup_reverse(f)
[3, 2, 1]
```

diofant.polys.densebasic.**dmp_copy**(*f*, *u*)
Create a new copy of a polynomial `f` in `K[X]`.

Examples

```
>>> f = ZZ.map([[1], [1, 2]])
>>> dmp_copy(f, 1)
[[1], [1, 2]]
```

diofant.polys.densebasic.**dmp_to_tuple**(*f*, *u*)
Convert f into a nested tuple of tuples.

This is needed for hashing. This is similar to dmp_copy().

Examples

```
>>> f = ZZ.map([1, 2, 3, 0])
>>> dmp_to_tuple(f, 0)
(1, 2, 3, 0)
```

```
>>> f = ZZ.map([[1], [1, 2]])
>>> dmp_to_tuple(f, 1)
((1,), (1, 2))
```

diofant.polys.densebasic.**dmp_normal**(*f*, *u*, *K*)
Normalize a multivariate polynomial in the given domain.

Examples

```
>>> dmp_normal([[], [0, 1.5, 2]], 1, ZZ)
[[1, 2]]
```

diofant.polys.densebasic.**dmp_convert**(*f, u, K0, K1*)
Convert the ground domain of `f` from `K0` to `K1`.

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> dmp_convert([[R(1)], [R(2)]], 1, R.to_domain(), ZZ)
[[1], [2]]
>>> dmp_convert([[ZZ(1)], [ZZ(2)]], 1, ZZ, R.to_domain())
[[1], [2]]
```

diofant.polys.densebasic.**dmp_from_diofant**(*f, u, K*)
Convert the ground domain of `f` from Diofant to `K`.

Examples

```
>>> dmp_from_diofant([[Integer(1)], [Integer(2)]], 1, ZZ)
[[1], [2]]
```

diofant.polys.densebasic.**dmp_nth**(*f, n, u, K*)
Return the `n`-th coefficient of `f` in `x_0` in `K[X]`.

Examples

```
>>> f = ZZ.map([[1], [2], [3]])
>>> dmp_nth(f, 0, 1, ZZ)
[3]
>>> dmp_nth(f, 4, 1, ZZ)
[]
```

diofant.polys.densebasic.**dmp_ground_nth**(*f, N, u, K*)
Return the ground `n`-th coefficient of `f` in `K[x]`.

Examples

```
>>> f = ZZ.map([[1], [2, 3]])
>>> dmp_ground_nth(f, (0, 1), 1, ZZ)
2
```

diofant.polys.densebasic.**dmp_zero_p**(*f, u*)
Return `True` if `f` is zero in `K[X]`.

Examples

```
>>> dmp_zero_p([[[[[]]]]], 4)
True
>>> dmp_zero_p([[[[[1]]]]], 4)
False
```

diofant.polys.densebasic.**dmp_zero**(*u*)
Return a multivariate zero.

Examples

```
>>> dmp_zero(4)
[[[[[]]]]]
```

diofant.polys.densebasic.**dmp_one_p**(*f, u, K*)
Return True if f is one in K[X].

Examples

```
>>> dmp_one_p([[[ZZ(1)]]], 2, ZZ)
True
```

diofant.polys.densebasic.**dmp_one**(*u, K*)
Return a multivariate one over K.

Examples

```
>>> dmp_one(2, ZZ)
[[[1]]]
```

diofant.polys.densebasic.**dmp_ground_p**(*f, c, u*)
Return True if f is constant in K[X].

Examples

```
>>> dmp_ground_p([[[3]]], 3, 2)
True
>>> dmp_ground_p([[[4]]], None, 2)
True
```

diofant.polys.densebasic.**dmp_ground**(*c, u*)
Return a multivariate constant.

Examples

```
>>> dmp_ground(3, 5)
[[[[[[3]]]]]]
>>> dmp_ground(1, -1)
1
```

diofant.polys.densebasic.**dmp_zeros**(*n*, *u*, *K*)
Return a list of multivariate zeros.

Examples

```
>>> dmp_zeros(3, 2, ZZ)
[[[[]]], [[[]]], [[[]]]]
>>> dmp_zeros(3, -1, ZZ)
[0, 0, 0]
```

diofant.polys.densebasic.**dmp_grounds**(*c*, *n*, *u*)
Return a list of multivariate constants.

Examples

```
>>> dmp_grounds(ZZ(4), 3, 2)
[[[[4]]], [[[4]]], [[[4]]]]
>>> dmp_grounds(ZZ(4), 3, -1)
[4, 4, 4]
```

diofant.polys.densebasic.**dmp_negative_p**(*f*, *u*, *K*)
Return True if LC(f) is negative.

Examples

```
>>> dmp_negative_p([[ZZ(1)], [-ZZ(1)]], 1, ZZ)
False
>>> dmp_negative_p([[-ZZ(1)], [ZZ(1)]], 1, ZZ)
True
```

diofant.polys.densebasic.**dmp_positive_p**(*f*, *u*, *K*)
Return True if LC(f) is positive.

Examples

```
>>> dmp_positive_p([[ZZ(1)], [-ZZ(1)]], 1, ZZ)
True
>>> dmp_positive_p([[-ZZ(1)], [ZZ(1)]], 1, ZZ)
False
```

diofant.polys.densebasic.**dmp_from_dict**(*f*, *u*, *K*)
Create a K[X] polynomial from a dict.

Examples

```
>>> dmp_from_dict({(0, 0): ZZ(3), (0, 1): ZZ(2), (2, 1): ZZ(1)}, 1, ZZ)
[[1, 0], [], [2, 3]]
>>> dmp_from_dict({}, 0, ZZ)
[]
```

diofant.polys.densebasic.**dmp_to_dict**(*f*, *u*, *K=None*, *zero=False*)
Convert a `K[X]` polynomial to a `dict``.

Examples

```
>>> dmp_to_dict([[1, 0], [], [2, 3]], 1)
{(0, 0): 3, (0, 1): 2, (2, 1): 1}
```

diofant.polys.densebasic.**dmp_swap**(*f*, *i*, *j*, *u*, *K*)
Transform `K[..x_i..x_j..]` to `K[..x_j..x_i..]`.

Examples

```
>>> f = ZZ.map([[[2], [1, 0]], []])
```

```
>>> dmp_swap(f, 0, 1, 2, ZZ)
[[[2], []], [[1, 0], []]]
>>> dmp_swap(f, 1, 2, 2, ZZ)
[[[1], [2, 0]], [[]]]
>>> dmp_swap(f, 0, 2, 2, ZZ)
[[[1, 0]], [[2, 0], []]]
```

diofant.polys.densebasic.**dmp_permute**(*f*, *P*, *u*, *K*)
Return a polynomial in `K[x_{P(1)},..,x_{P(n)}]`.

Examples

```
>>> f = ZZ.map([[[2], [1, 0]], []])
```

```
>>> dmp_permute(f, [1, 0, 2], 2, ZZ)
[[[2], []], [[1, 0], []]]
>>> dmp_permute(f, [1, 2, 0], 2, ZZ)
[[[1], []], [[2, 0], []]]
```

diofant.polys.densebasic.**dmp_nest**(*f*, *l*, *K*)
Return a multivariate value nested `l`-levels.

Examples

```
>>> dmp_nest([[ZZ(1)]], 2, ZZ)
[[[[1]]]]
```

diofant.polys.densebasic.**dmp_raise**(*f*, *l*, *u*, *K*)
Return a multivariate polynomial raised l-levels.

Examples

```
>>> f = ZZ.map([[], [1, 2]])
```

```
>>> dmp_raise(f, 2, 1, ZZ)
[[[[]]], [[[1]], [[2]]]]
```

diofant.polys.densebasic.**dmp_deflate**(*f*, *u*, *K*)
Map x_i**m_i to y_i in a polynomial in K[X].

Examples

```
>>> f = ZZ.map([[1, 0, 0, 2], [], [3, 0, 0, 4]])
```

```
>>> dmp_deflate(f, 1, ZZ)
((2, 3), [[1, 2], [3, 4]])
```

diofant.polys.densebasic.**dmp_multi_deflate**(*polys*, *u*, *K*)
Map x_i**m_i to y_i in a set of polynomials in K[X].

Examples

```
>>> f = ZZ.map([[1, 0, 0, 2], [], [3, 0, 0, 4]])
>>> g = ZZ.map([[1, 0, 2], [], [3, 0, 4]])
```

```
>>> dmp_multi_deflate((f, g), 1, ZZ)
((2, 1), ([[1, 0, 0, 2], [3, 0, 0, 4]], [[1, 0, 2], [3, 0, 4]]))
```

diofant.polys.densebasic.**dmp_inflate**(*f*, *M*, *u*, *K*)
Map y_i to x_i**k_i in a polynomial in K[X].

Examples

```
>>> f = ZZ.map([[1, 2], [3, 4]])
```

```
>>> dmp_inflate(f, (2, 3), 1, ZZ)
[[1, 0, 0, 2], [], [3, 0, 0, 4]]
```

diofant.polys.densebasic.**dmp_exclude**(*f*, *u*, *K*)
Exclude useless levels from f.

Return the levels excluded, the new excluded f, and the new u.

Examples

```
>>> f = ZZ.map([[[1]], [[1], [2]]])
```

```
>>> dmp_exclude(f, 2, ZZ)
([2], [[1], [1, 2]], 1)
```

diofant.polys.densebasic.**dmp_include**(*f, J, u, K*)
Include useless levels in `f`.

Examples

```
>>> f = ZZ.map([[1], [1, 2]])
```

```
>>> dmp_include(f, [2], 1, ZZ)
[[[1]], [[1], [2]]]
```

diofant.polys.densebasic.**dmp_inject**(*f, u, K, front=False*)
Convert `f` from `K[X][Y]` to `K[X,Y]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> dmp_inject([R(1), x + 2], 0, R.to_domain())
([[[1]], [[1], [2]]], 2)
>>> dmp_inject([R(1), x + 2], 0, R.to_domain(), front=True)
([[[1]], [[1, 2]]], 2)
```

diofant.polys.densebasic.**dmp_eject**(*f, u, K, front=False*)
Convert `f` from `K[X,Y]` to `K[X][Y]`.

Examples

```
>>> dmp_eject([[[1]], [[1], [2]]], 2, ZZ['x', 'y'])
[1, x + 2]
```

diofant.polys.densebasic.**dmp_terms_gcd**(*f, u, K*)
Remove GCD of terms from `f` in `K[X]`.

Examples

```
>>> f = ZZ.map([[1, 0], [1, 0, 0], [], []])
```

```
>>> dmp_terms_gcd(f, 1, ZZ)
((2, 1), [[1], [1, 0]])
```

diofant.polys.densebasic.**dmp_list_terms**(*f, u, K, order=None*)
List all non-zero terms from `f` in the given order `order`.

Examples

```
>>> f = ZZ.map([[1, 1], [2, 3]])
```

```
>>> dmp_list_terms(f, 1, ZZ)
[((1, 1), 1), ((1, 0), 1), ((0, 1), 2), ((0, 0), 3)]
>>> dmp_list_terms(f, 1, ZZ, order='grevlex')
[((1, 1), 1), ((1, 0), 1), ((0, 1), 2), ((0, 0), 3)]
```

diofant.polys.densebasic.**dmp_apply_pairs**(*f, g, h, args, u, K*)
Apply h to pairs of coefficients of f and g.

Examples

```
>>> h = lambda x, y, z: 2*x + y - z
```

```
>>> dmp_apply_pairs([[1], [2, 3]], [[3], [2, 1]], h, [1], 1, ZZ)
[[4], [5, 6]]
```

diofant.polys.densebasic.**dmp_slice**(*f, m, n, u, K*)
Take a continuous subsequence of terms of f in K[X].

diofant.polys.densebasic.**dup_random**(*n, a, b, K, percent=None*)
Return a polynomial of degree n with coefficients in [a, b].

If percent is a natural number less than 100 then only approximately the given percentage of elements will be non-zero.

Examples

```
>>> dup_random(3, -10, 10, ZZ)
[-2, -8, 9, -4]
```

Arithmetic operations:

diofant.polys.densearith.**dmp_add_term**(*f, c, i, u, K*)
Add c(x_2..x_u)*x_0**i to f in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_add_term(x*y + 1, 2, 2)
2*x**2 + x*y + 1
```

diofant.polys.densearith.**dmp_sub_term**(*f, c, i, u, K*)
Subtract c(x_2..x_u)*x_0**i from f in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_sub_term(2*x**2 + x*y + 1, 2, 2)
x*y + 1
```

diofant.polys.densearith.**dmp_mul_term**(*f*, *c*, *i*, *u*, *K*)
Multiply f by c(x_2..x_u)*x_0**i in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_mul_term(x**2*y + x, 3*y, 2)
3*x**4*y**2 + 3*x**3*y
```

diofant.polys.densearith.**dmp_add_ground**(*f*, *c*, *u*, *K*)
Add an element of the ground domain to f.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_add_ground(x**3 + 2*x**2 + 3*x + 4, ZZ(4))
x**3 + 2*x**2 + 3*x + 8
```

diofant.polys.densearith.**dmp_sub_ground**(*f*, *c*, *u*, *K*)
Subtract an element of the ground domain from f.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_sub_ground(x**3 + 2*x**2 + 3*x + 4, ZZ(4))
x**3 + 2*x**2 + 3*x
```

diofant.polys.densearith.**dmp_mul_ground**(*f*, *c*, *u*, *K*)
Multiply f by a constant value in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_mul_ground(2*x + 2*y, ZZ(3))
6*x + 6*y
```

diofant.polys.densearith.**dmp_quo_ground**(*f*, *c*, *u*, *K*)
Quotient by a constant in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
>>> R.dmp_quo_ground(2*x**2*y + 3*x, ZZ(2))
x**2*y + x
```

```
>>> R, x, y = ring("x y", QQ)
>>> R.dmp_quo_ground(2*x**2*y + 3*x, QQ(2))
x**2*y + 3/2*x
```

diofant.polys.densearith.**dmp_exquo_ground**(*f*, *c*, *u*, *K*)
Exact quotient by a constant in K[X].

Examples

```
>>> R, x, y = ring("x y", QQ)
```

```
>>> R.dmp_exquo_ground(x**2*y + 2*x, QQ(2))
1/2*x**2*y + x
```

diofant.polys.densearith.**dup_lshift**(*f*, *n*, *K*)
Efficiently multiply f by x**n in K[x].

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> R.dup_lshift(x**2 + 1, 2)
x**4 + x**2
```

diofant.polys.densearith.**dup_rshift**(*f*, *n*, *K*)
Efficiently divide f by x**n in K[x].

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> R.dup_rshift(x**4 + x**2, 2)
x**2 + 1
>>> R.dup_rshift(x**4 + x**2 + 2, 2)
x**2 + 1
```

diofant.polys.densearith.**dmp_abs**(*f*, *u*, *K*)
Make all coefficients positive in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_abs(x**2*y - x)
x**2*y + x
```

diofant.polys.densearith.**dmp_neg**(*f*, *u*, *K*)
Negate a polynomial in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_neg(x**2*y - x)
-x**2*y + x
```

diofant.polys.densearith.**dmp_add**(*f*, *g*, *u*, *K*)
Add dense polynomials in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_add(x**2 + y, x**2*y + x)
x**2*y + x**2 + x + y
```

diofant.polys.densearith.**dmp_sub**(*f*, *g*, *u*, *K*)
Subtract dense polynomials in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_sub(x**2 + y, x**2*y + x)
-x**2*y + x**2 - x + y
```

diofant.polys.densearith.**dmp_add_mul**(*f*, *g*, *h*, *u*, *K*)
Returns `f + g*h` where `f, g, h` are in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_add_mul(x**2 + y, x, x + 2)
2*x**2 + 2*x + y
```

diofant.polys.densearith.**dmp_sub_mul**(*f*, *g*, *h*, *u*, *K*)
Returns `f - g*h` where `f, g, h` are in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_sub_mul(x**2 + y, x, x + 2)
-2*x + y
```

diofant.polys.densearith.**dmp_mul**(*f*, *g*, *u*, *K*)
Multiply dense polynomials in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_mul(x*y + 1, x)
x**2*y + x
```

diofant.polys.densearith.**dmp_sqr**(*f*, *u*, *K*)
Square dense polynomials in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_sqr(x**2 + x*y + y**2)
x**4 + 2*x**3*y + 3*x**2*y**2 + 2*x*y**3 + y**4
```

diofant.polys.densearith.**dmp_pow**(*f*, *n*, *u*, *K*)
Raise f to the n-th power in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_pow(x*y + 1, 3)
x**3*y**3 + 3*x**2*y**2 + 3*x*y + 1
```

diofant.polys.densearith.**dmp_pdiv**(*f*, *g*, *u*, *K*)
Polynomial pseudo-division in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_pdiv(x**2 + x*y, 2*x + 2)
(2*x + 2*y - 2, -4*y + 4)
```

diofant.polys.densearith.**dmp_prem**(*f*, *g*, *u*, *K*)
Polynomial pseudo-remainder in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_prem(x**2 + x*y, 2*x + 2)
-4*y + 4
```

`diofant.polys.densearith.`**`dmp_pquo`**(f, g, u, K)
Polynomial exact pseudo-quotient in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x**2 + x*y
>>> g = 2*x + 2*y
>>> h = 2*x + 2
```

```
>>> R.dmp_pquo(f, g)
2*x
```

```
>>> R.dmp_pquo(f, h)
2*x + 2*y - 2
```

`diofant.polys.densearith.`**`dmp_pexquo`**(f, g, u, K)
Polynomial pseudo-quotient in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x**2 + x*y
>>> g = 2*x + 2*y
>>> h = 2*x + 2
```

```
>>> R.dmp_pexquo(f, g)
2*x
```

```
>>> R.dmp_pexquo(f, h)
Traceback (most recent call last):
...
ExactQuotientFailed: [[2], [2]] does not divide [[1], [1, 0], []]
```

`diofant.polys.densearith.`**`dmp_rr_div`**(f, g, u, K)
Multivariate division with remainder over a ring.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_rr_div(x**2 + x*y, 2*x + 2)
(0, x**2 + x*y)
```

diofant.polys.densearith.**dmp_ff_div**(*f*, *g*, *u*, *K*)
Polynomial division with remainder over a field.

Examples

```
>>> R, x, y = ring("x y", QQ)
```

```
>>> R.dmp_ff_div(x**2 + x*y, 2*x + 2)
(1/2*x + 1/2*y - 1/2, -y + 1)
```

diofant.polys.densearith.**dmp_div**(*f*, *g*, *u*, *K*)
Polynomial division with remainder in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
>>> R.dmp_div(x**2 + x*y, 2*x + 2)
(0, x**2 + x*y)
```

```
>>> R, x, y = ring("x y", QQ)
>>> R.dmp_div(x**2 + x*y, 2*x + 2)
(1/2*x + 1/2*y - 1/2, -y + 1)
```

diofant.polys.densearith.**dmp_rem**(*f*, *g*, *u*, *K*)
Returns polynomial remainder in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
>>> R.dmp_rem(x**2 + x*y, 2*x + 2)
x**2 + x*y
```

```
>>> R, x, y = ring("x y", QQ)
>>> R.dmp_rem(x**2 + x*y, 2*x + 2)
-y + 1
```

diofant.polys.densearith.**dmp_quo**(*f*, *g*, *u*, *K*)
Returns exact polynomial quotient in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
>>> R.dmp_quo(x**2 + x*y, 2*x + 2)
0
```

```
>>> R, x, y = ring("x y", QQ)
>>> R.dmp_quo(x**2 + x*y, 2*x + 2)
1/2*x + 1/2*y - 1/2
```

diofant.polys.densearith.**dmp_exquo**(*f*, *g*, *u*, *K*)
Returns polynomial quotient in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x**2 + x*y
>>> g = x + y
>>> h = 2*x + 2
```

```
>>> R.dmp_exquo(f, g)
x
```

```
>>> R.dmp_exquo(f, h)
Traceback (most recent call last):
...
ExactQuotientFailed: [[2], [2]] does not divide [[1], [1, 0], []]
```

diofant.polys.densearith.**dmp_max_norm**(*f*, *u*, *K*)
Returns maximum norm of a polynomial in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_max_norm(2*x*y - x - 3)
3
```

diofant.polys.densearith.**dmp_l1_norm**(*f*, *u*, *K*)
Returns l1 norm of a polynomial in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_l1_norm(2*x*y - x - 3)
6
```

diofant.polys.densearith.**dmp_expand**(*polys*, *u*, *K*)
Multiply together several polynomials in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_expand([x**2 + y**2, x + 1])
x**3 + x**2 + x*y**2 + y**2
```

Further tools:

`diofant.polys.densetools.`**`dmp_integrate`**(*f*, *m*, *u*, *K*)

Computes the indefinite integral of `f` in `x_0` in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", QQ)
```

```
>>> R.dmp_integrate(x + 2*y, 1)
1/2*x**2 + 2*x*y
>>> R.dmp_integrate(x + 2*y, 2)
1/6*x**3 + x**2*y
```

`diofant.polys.densetools.`**`dmp_integrate_in`**(*f*, *m*, *j*, *u*, *K*)

Computes the indefinite integral of `f` in `x_j` in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", QQ)
```

```
>>> R.dmp_integrate_in(x + 2*y, 1, 0)
1/2*x**2 + 2*x*y
>>> R.dmp_integrate_in(x + 2*y, 1, 1)
x*y + y**2
```

`diofant.polys.densetools.`**`dmp_diff`**(*f*, *m*, *u*, *K*)

`m`-th order derivative in `x_0` of a polynomial in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x*y**2 + 2*x*y + 3*x + 2*y**2 + 3*y + 1
```

```
>>> R.dmp_diff(f, 1)
y**2 + 2*y + 3
>>> R.dmp_diff(f, 2)
0
```

`diofant.polys.densetools.`**`dmp_diff_in`**(*f*, *m*, *j*, *u*, *K*)

`m`-th order derivative in `x_j` of a polynomial in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x*y**2 + 2*x*y + 3*x + 2*y**2 + 3*y + 1
```

```
>>> R.dmp_diff_in(f, 1, 0)
y**2 + 2*y + 3
>>> R.dmp_diff_in(f, 1, 1)
2*x*y + 2*x + 4*y + 3
```

diofant.polys.densetools.**dmp_eval**(*f*, *a*, *u*, *K*)
Evaluate a polynomial at `x_0 = a` in `K[X]` using the Horner scheme.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_eval(2*x*y + 3*x + y + 2, 2)
5*y + 8
```

diofant.polys.densetools.**dmp_eval_in**(*f*, *a*, *j*, *u*, *K*)
Evaluate a polynomial at `x_j = a` in `K[X]` using the Horner scheme.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = 2*x*y + 3*x + y + 2
```

```
>>> R.dmp_eval_in(f, 2, 0)
5*y + 8
>>> R.dmp_eval_in(f, 2, 1)
7*x + 4
```

diofant.polys.densetools.**dmp_eval_tail**(*f*, *A*, *u*, *K*)
Evaluate a polynomial at `x_j = a_j, ...` in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = 2*x*y + 3*x + y + 2
```

```
>>> R.dmp_eval_tail(f, [2])
7*x + 4
>>> R.dmp_eval_tail(f, [2, 2])
18
```

diofant.polys.densetools.**dmp_diff_eval_in**(*f*, *m*, *a*, *j*, *u*, *K*)
Differentiate and evaluate a polynomial in x_j at a in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x*y**2 + 2*x*y + 3*x + 2*y**2 + 3*y + 1
```

```
>>> R.dmp_diff_eval_in(f, 1, 2, 0)
y**2 + 2*y + 3
>>> R.dmp_diff_eval_in(f, 1, 2, 1)
6*x + 11
```

diofant.polys.densetools.**dmp_trunc**(*f*, *p*, *u*, *K*)
Reduce a K[X] polynomial modulo a polynomial p in K[Y].

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = 3*x**2*y + 8*x**2 + 5*x*y + 6*x + 2*y + 3
>>> g = (y - 1).drop(x)
```

```
>>> R.dmp_trunc(f, g)
11*x**2 + 11*x + 5
```

diofant.polys.densetools.**dmp_ground_trunc**(*f*, *p*, *u*, *K*)
Reduce a K[X] polynomial modulo a constant p in K.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = 3*x**2*y + 8*x**2 + 5*x*y + 6*x + 2*y + 3
```

```
>>> R.dmp_ground_trunc(f, ZZ(3))
-x**2 - x*y - y
```

diofant.polys.densetools.**dup_monic**(*f*, *K*)
Divide all coefficients by LC(f) in K[x].

Examples

```
>>> R, x = ring("x", ZZ)
>>> R.dup_monic(3*x**2 + 6*x + 9)
x**2 + 2*x + 3
```

```
>>> R, x = ring("x", QQ)
>>> R.dup_monic(3*x**2 + 4*x + 2)
x**2 + 4/3*x + 2/3
```

diofant.polys.densetools.**dmp_ground_monic**(*f, u, K*)
Divide all coefficients by LC(f) in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
>>> f = 3*x**2*y + 6*x**2 + 3*x*y + 9*y + 3
```

```
>>> R.dmp_ground_monic(f)
x**2*y + 2*x**2 + x*y + 3*y + 1
```

```
>>> R, x, y = ring("x y", QQ)
>>> f = 3*x**2*y + 8*x**2 + 5*x*y + 6*x + 2*y + 3
```

```
>>> R.dmp_ground_monic(f)
x**2*y + 8/3*x**2 + 5/3*x*y + 2*x + 2/3*y + 1
```

diofant.polys.densetools.**dup_content**(*f, K*)
Compute the GCD of coefficients of f in K[x].

Examples

```
>>> R, x = ring("x", ZZ)
>>> f = 6*x**2 + 8*x + 12
```

```
>>> R.dup_content(f)
2
```

```
>>> R, x = ring("x", QQ)
>>> f = 6*x**2 + 8*x + 12
```

```
>>> R.dup_content(f)
2
```

diofant.polys.densetools.**dmp_ground_content**(*f, u, K*)
Compute the GCD of coefficients of f in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
>>> f = 2*x*y + 6*x + 4*y + 12
```

```
>>> R.dmp_ground_content(f)
2
```

```
>>> R, x, y = ring("x y", QQ)
>>> f = 2*x*y + 6*x + 4*y + 12
```

```
>>> R.dmp_ground_content(f)
2
```

diofant.polys.densetools.**dup_primitive**(*f*, *K*)
Compute content and the primitive form of f in K[x].

Examples

```
>>> R, x = ring("x", ZZ)
>>> f = 6*x**2 + 8*x + 12
```

```
>>> R.dup_primitive(f)
(2, 3*x**2 + 4*x + 6)
```

```
>>> R, x = ring("x", QQ)
>>> f = 6*x**2 + 8*x + 12
```

```
>>> R.dup_primitive(f)
(2, 3*x**2 + 4*x + 6)
```

diofant.polys.densetools.**dmp_ground_primitive**(*f*, *u*, *K*)
Compute content and the primitive form of f in K[X].

Examples

```
>>> R, x, y = ring("x y", ZZ)
>>> f = 2*x*y + 6*x + 4*y + 12
```

```
>>> R.dmp_ground_primitive(f)
(2, x*y + 3*x + 2*y + 6)
```

```
>>> R, x, y = ring("x y", QQ)
>>> f = 2*x*y + 6*x + 4*y + 12
```

```
>>> R.dmp_ground_primitive(f)
(2, x*y + 3*x + 2*y + 6)
```

diofant.polys.densetools.**dup_extract**(*f*, *g*, *K*)
Extract common content from a pair of polynomials in K[x].

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> R.dup_extract(6*x**2 + 12*x + 18, 4*x**2 + 8*x + 12)
(2, 3*x**2 + 6*x + 9, 2*x**2 + 4*x + 6)
```

diofant.polys.densetools.**dmp_ground_extract**(*f*, *g*, *u*, *K*)
Extract common content from a pair of polynomials in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_ground_extract(6*x*y + 12*x + 18, 4*x*y + 8*x + 12)
(2, 3*x*y + 6*x + 9, 2*x*y + 4*x + 6)
```

diofant.polys.densetools.**dup_real_imag**(*f*, *K*)
Return bivariate polynomials `f1` and `f2`, such that `f = f1 + f2*I`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dup_real_imag(x**3 + x**2 + x + 1)
(x**3 + x**2 - 3*x*y**2 + x - y**2 + 1, 3*x**2*y + 2*x*y - y**3 + y)
```

diofant.polys.densetools.**dup_mirror**(*f*, *K*)
Evaluate efficiently the composition `f(-x)` in `K[x]`.

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> R.dup_mirror(x**3 + 2*x**2 - 4*x + 2)
-x**3 + 2*x**2 + 4*x + 2
```

diofant.polys.densetools.**dup_scale**(*f*, *a*, *K*)
Evaluate efficiently composition `f(a*x)` in `K[x]`.

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> R.dup_scale(x**2 - 2*x + 1, ZZ(2))
4*x**2 - 4*x + 1
```

diofant.polys.densetools.**dup_shift**(*f*, *a*, *K*)
Evaluate efficiently Taylor shift `f(x + a)` in `K[x]`.

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> R.dup_shift(x**2 - 2*x + 1, ZZ(2))
x**2 + 2*x + 1
```

diofant.polys.densetools.**dup_transform**(*f*, *p*, *q*, *K*)
Evaluate functional transformation `q**n * f(p/q)` in `K[x]`.

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> R.dup_transform(x**2 - 2*x + 1, x**2 + 1, x - 1)
x**4 - 2*x**3 + 5*x**2 - 4*x + 4
```

diofant.polys.densetools.**dmp_compose**(*f*, *g*, *u*, *K*)
Evaluate functional composition `f(g)` in `K[X]`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_compose(x*y + 2*x + y, y)
y**2 + 3*y
```

diofant.polys.densetools.**dup_decompose**(*f*, *K*)
Computes functional decomposition of `f` in `K[x]`.

Given a univariate polynomial `f` with coefficients in a field of characteristic zero, returns list `[f_1, f_2, ..., f_n]`, where:

```
f = f_1 o f_2 o ... f_n = f_1(f_2(... f_n))
```

and `f_2, ..., f_n` are monic and homogeneous polynomials of at least second degree.

Unlike factorization, complete functional decompositions of polynomials are not unique, consider examples:

1. `f o g = f(x + b) o (g - b)`
2. `x**n o x**m = x**m o x**n`
3. `T_n o T_m = T_m o T_n`

where `T_n` and `T_m` are Chebyshev polynomials.

References

[R2] (page 1268)

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> R.dup_decompose(x**4 - 2*x**3 + x**2)
[x**2, x**2 - x]
```

diofant.polys.densetools.**dmp_lift**(*f, u, K*)
Convert algebraic coefficients to integers in K[X].

Examples

```
>>> K = QQ.algebraic_field(I)
>>> R, x = ring("x", K)
```

```
>>> f = x**2 + K([QQ(1), QQ(0)])*x + K([QQ(2), QQ(0)])
```

```
>>> R.dmp_lift(f)
x**8 + 2*x**6 + 9*x**4 - 8*x**2 + 16
```

diofant.polys.densetools.**dup_sign_variations**(*f, K*)
Compute the number of sign variations of f in K[x].

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> R.dup_sign_variations(x**4 - x**2 - x + 1)
2
```

diofant.polys.densetools.**dmp_clear_denoms**(*f, u, K0, K1=None, convert=False*)
Clear denominators, i.e. transform K_0 to K_1.

Examples

```
>>> R, x, y = ring("x y", QQ)
```

```
>>> f = x/2 + y/3 + 1
```

```
>>> R.dmp_clear_denoms(f, convert=False)
(6, 3*x + 2*y + 6)
>>> R.dmp_clear_denoms(f, convert=True)
(6, 3*x + 2*y + 6)
```

diofant.polys.densetools.**dmp_revert**(*f, g, u, K*)
Compute f**(-1) mod x**n using Newton iteration.

Examples

```
>>> R, x, y = ring("x y", QQ)
```

diofant.polys.rootisolation.**dup_sturm**(*f, K*)

Computes the Sturm sequence of f in F[x].

Given a univariate, square-free polynomial f(x) returns the associated Sturm sequence f_0(x), ..., f_n(x) defined by:

```
f_0(x), f_1(x) = f(x), f'(x)
f_n = -rem(f_{n-2}(x), f_{n-1}(x))
```

References

[R3] (page 1268)

Examples

```
>>> R, x = ring("x", QQ)
```

```
>>> R.dup_sturm(x**3 - 2*x**2 + x - 3)
[x**3 - 2*x**2 + x - 3, 3*x**2 - 4*x + 1, 2/9*x + 25/9, -2079/4]
```

Manipulation of dense, univariate polynomials with finite field coefficients

Functions in this module carry the prefix gf_, referring to the classical name "Galois Fields" for finite fields. Note that many polynomial factorization algorithms work by reduction to the finite field case, so having special implementations for this case is justified both by performance, and by the necessity of certain methods which do not even make sense over general fields.

diofant.polys.galoistools.**gf_crt**(*U, M, K=None*)

Chinese Remainder Theorem.

Given a set of integer residues u_0,...,u_n and a set of co-prime integer moduli m_0,...,m_n, returns an integer u, such that u = u_i mod m_i for i = ``0,...,n.

As an example consider a set of residues U = [49, 76, 65] and a set of moduli M = [99, 97, 95]. Then we have:

```
>>> from diofant.ntheory.modular import solve_congruence

>>> gf_crt([49, 76, 65], [99, 97, 95], ZZ)
639985
```

This is the correct result because:

```
>>> [639985 % m for m in [99, 97, 95]]
[49, 76, 65]
```

Note: this is a low-level routine with no error checking.

See also:

***diofant.ntheory.modular.crt* (page 259)** a higher level crt routine

diofant.ntheory.modular.solve_congruence (page 260)

diofant.polys.galoistools.**gf_crt1**(M, K)

First part of the Chinese Remainder Theorem.

Examples

```
>>> gf_crt1([99, 97, 95], ZZ)
(912285, [9215, 9405, 9603], [62, 24, 12])
```

diofant.polys.galoistools.**gf_crt2**(U, M, p, E, S, K)

Second part of the Chinese Remainder Theorem.

Examples

```
>>> U = [49, 76, 65]
>>> M = [99, 97, 95]
>>> p = 912285
>>> E = [9215, 9405, 9603]
>>> S = [62, 24, 12]
```

```
>>> gf_crt2(U, M, p, E, S, ZZ)
639985
```

diofant.polys.galoistools.**gf_int**(a, p)

Coerce `a mod p` to an integer in the range `[-p/2, p/2]`.

Examples

```
>>> gf_int(2, 7)
2
>>> gf_int(5, 7)
-2
```

diofant.polys.galoistools.**gf_degree**(f)

Return the leading degree of `f`.

Examples

```
>>> gf_degree([1, 1, 2, 0])
3
>>> gf_degree([])
-1
```

diofant.polys.galoistools.**gf_LC**(f, K)

Return the leading coefficient of `f`.

Examples

```
>>> gf_LC([3, 0, 1], ZZ)
3
```

diofant.polys.galoistools.**gf_TC**(*f*, *K*)
Return the trailing coefficient of f.

Examples

```
>>> gf_TC([3, 0, 1], ZZ)
1
```

diofant.polys.galoistools.**gf_strip**(*f*)
Remove leading zeros from f.

Examples

```
>>> gf_strip([0, 0, 0, 3, 0, 1])
[3, 0, 1]
```

diofant.polys.galoistools.**gf_trunc**(*f*, *p*)
Reduce all coefficients modulo p.

Examples

```
>>> gf_trunc([7, -2, 3], 5)
[2, 3, 3]
```

diofant.polys.galoistools.**gf_normal**(*f*, *p*, *K*)
Normalize all coefficients in K.

Examples

```
>>> gf_normal([5, 10, 21, -3], 5, ZZ)
[1, 2]
```

diofant.polys.galoistools.**gf_from_dict**(*f*, *p*, *K*)
Create a GF(p)[x] polynomial from a dict.

Examples

```
>>> gf_from_dict({10: ZZ(4), 4: ZZ(33), 0: ZZ(-1)}, 5, ZZ)
[4, 0, 0, 0, 0, 0, 3, 0, 0, 0, 4]
```

diofant.polys.galoistools.**gf_to_dict**(*f*, *p*, *symmetric=True*)
Convert a GF(p)[x] polynomial to a dict.

Examples

```
>>> gf_to_dict([4, 0, 0, 0, 0, 0, 3, 0, 0, 0, 4], 5)
{0: -1, 4: -2, 10: -1}
>>> gf_to_dict([4, 0, 0, 0, 0, 0, 3, 0, 0, 0, 4], 5, symmetric=False)
{0: 4, 4: 3, 10: 4}
```

diofant.polys.galoistools.**gf_from_int_poly**(*f, p*)
Create a `GF(p)[x]` polynomial from `Z[x]`.

Examples

```
>>> gf_from_int_poly([7, -2, 3], 5)
[2, 3, 3]
```

diofant.polys.galoistools.**gf_to_int_poly**(*f, p, symmetric=True*)
Convert a `GF(p)[x]` polynomial to `Z[x]`.

Examples

```
>>> gf_to_int_poly([2, 3, 3], 5)
[2, -2, -2]
>>> gf_to_int_poly([2, 3, 3], 5, symmetric=False)
[2, 3, 3]
```

diofant.polys.galoistools.**gf_neg**(*f, p, K*)
Negate a polynomial in `GF(p)[x]`.

Examples

```
>>> gf_neg([3, 2, 1, 0], 5, ZZ)
[2, 3, 4, 0]
```

diofant.polys.galoistools.**gf_add_ground**(*f, a, p, K*)
Compute `f + a` where `f` in `GF(p)[x]` and `a` in `GF(p)`.

Examples

```
>>> gf_add_ground([3, 2, 4], 2, 5, ZZ)
[3, 2, 1]
```

diofant.polys.galoistools.**gf_sub_ground**(*f, a, p, K*)
Compute `f - a` where `f` in `GF(p)[x]` and `a` in `GF(p)`.

Examples

```
>>> gf_sub_ground([3, 2, 4], 2, 5, ZZ)
[3, 2, 2]
```

diofant.polys.galoistools.**gf_mul_ground**(*f*, *a*, *p*, *K*)
Compute `f * a` where `f` in `GF(p)[x]` and `a` in `GF(p)`.

Examples

```
>>> gf_mul_ground([3, 2, 4], 2, 5, ZZ)
[1, 4, 3]
```

diofant.polys.galoistools.**gf_quo_ground**(*f*, *a*, *p*, *K*)
Compute `f/a` where `f` in `GF(p)[x]` and `a` in `GF(p)`.

Examples

```
>>> gf_quo_ground(ZZ.map([3, 2, 4]), ZZ(2), 5, ZZ)
[4, 1, 2]
```

diofant.polys.galoistools.**gf_add**(*f*, *g*, *p*, *K*)
Add polynomials in `GF(p)[x]`.

Examples

```
>>> gf_add([3, 2, 4], [2, 2, 2], 5, ZZ)
[4, 1]
```

diofant.polys.galoistools.**gf_sub**(*f*, *g*, *p*, *K*)
Subtract polynomials in `GF(p)[x]`.

Examples

```
>>> gf_sub([3, 2, 4], [2, 2, 2], 5, ZZ)
[1, 0, 2]
```

diofant.polys.galoistools.**gf_mul**(*f*, *g*, *p*, *K*)
Multiply polynomials in `GF(p)[x]`.

Examples

```
>>> gf_mul([3, 2, 4], [2, 2, 2], 5, ZZ)
[1, 0, 3, 2, 3]
```

diofant.polys.galoistools.**gf_sqr**(*f*, *p*, *K*)
Square polynomials in `GF(p)[x]`.

Examples

```
>>> gf_sqr([3, 2, 4], 5, ZZ)
[4, 2, 3, 1, 1]
```

diofant.polys.galoistools.**gf_add_mul**(*f*, *g*, *h*, *p*, *K*)
Returns `f + g*h` where `f`, `g`, `h` in `GF(p)[x]`.

Examples

```
>>> gf_add_mul([3, 2, 4], [2, 2, 2], [1, 4], 5, ZZ)
[2, 3, 2, 2]
```

diofant.polys.galoistools.**gf_sub_mul**(*f*, *g*, *h*, *p*, *K*)
Compute `f - g*h` where `f`, `g`, `h` in `GF(p)[x]`.

Examples

```
>>> gf_sub_mul([3, 2, 4], [2, 2, 2], [1, 4], 5, ZZ)
[3, 3, 2, 1]
```

diofant.polys.galoistools.**gf_expand**(*F*, *p*, *K*)
Expand results of *factor()* (page 668) in `GF(p)[x]`.

Examples

```
>>> gf_expand([([3, 2, 4], 1), ([2, 2], 2), ([3, 1], 3)], 5, ZZ)
[4, 3, 0, 3, 0, 1, 4, 1]
```

diofant.polys.galoistools.**gf_div**(*f*, *g*, *p*, *K*)
Division with remainder in `GF(p)[x]`.

Given univariate polynomials `f` and `g` with coefficients in a finite field with `p` elements, returns polynomials `q` and `r` (quotient and remainder) such that `f = q*g + r`.

Consider polynomials `x**3 + x + 1` and `x**2 + x` in GF(2):

```
>>> gf_div(ZZ.map([1, 0, 1, 1]), ZZ.map([1, 1, 0]), 2, ZZ)
([1, 1], [1])
```

As result we obtained quotient `x + 1` and remainder `1`, thus:

```
>>> gf_add_mul(ZZ.map([1]), ZZ.map([1, 1]), ZZ.map([1, 1, 0]), 2, ZZ)
[1, 0, 1, 1]
```

References

[R4] (page 1268), *[R5]* (page 1268)

diofant.polys.galoistools.**gf_rem**(*f*, *g*, *p*, *K*)
Compute polynomial remainder in `GF(p)[x]`.

Examples

```
>>> gf_rem(ZZ.map([1, 0, 1, 1]), ZZ.map([1, 1, 0]), 2, ZZ)
[1]
```

diofant.polys.galoistools.**gf_quo**(*f, g, p, K*)
Compute exact quotient in GF(p)[x].

Examples

```
>>> gf_quo(ZZ.map([1, 0, 1, 1]), ZZ.map([1, 1, 0]), 2, ZZ)
[1, 1]
>>> gf_quo(ZZ.map([1, 0, 3, 2, 3]), ZZ.map([2, 2, 2]), 5, ZZ)
[3, 2, 4]
```

diofant.polys.galoistools.**gf_exquo**(*f, g, p, K*)
Compute polynomial quotient in GF(p)[x].

Examples

```
>>> gf_exquo(ZZ.map([1, 0, 3, 2, 3]), ZZ.map([2, 2, 2]), 5, ZZ)
[3, 2, 4]
```

```
>>> gf_exquo(ZZ.map([1, 0, 1, 1]), ZZ.map([1, 1, 0]), 2, ZZ)
Traceback (most recent call last):
...
ExactQuotientFailed: [1, 1, 0] does not divide [1, 0, 1, 1]
```

diofant.polys.galoistools.**gf_lshift**(*f, n, K*)
Efficiently multiply f by x**n.

Examples

```
>>> gf_lshift([3, 2, 4], 4, ZZ)
[3, 2, 4, 0, 0, 0, 0]
```

diofant.polys.galoistools.**gf_rshift**(*f, n, K*)
Efficiently divide f by x**n.

Examples

```
>>> gf_rshift([1, 2, 3, 4, 0], 3, ZZ)
([1, 2], [3, 4, 0])
```

diofant.polys.galoistools.**gf_pow**(*f, n, p, K*)
Compute f**n in GF(p)[x] using repeated squaring.

Examples

```
>>> gf_pow([3, 2, 4], 3, 5, ZZ)
[2, 4, 4, 2, 2, 1, 4]
```

diofant.polys.galoistools.**gf_pow_mod**(*f, n, g, p, K*)
Compute `f**n` in `GF(p)[x]/(g)` using repeated squaring.

Given polynomials `f` and `g` in `GF(p)[x]` and a non-negative integer `n`, efficiently computes `f**n (mod g)` i.e. the remainder of `f**n` from division by `g`, using the repeated squaring algorithm.

References

[R6] (page 1269)

Examples

```
>>> gf_pow_mod(ZZ.map([3, 2, 4]), 3, ZZ.map([1, 1]), 5, ZZ)
[]
```

diofant.polys.galoistools.**gf_gcd**(*f, g, p, K*)
Euclidean Algorithm in `GF(p)[x]`.

Examples

```
>>> gf_gcd(ZZ.map([3, 2, 4]), ZZ.map([2, 2, 3]), 5, ZZ)
[1, 3]
```

diofant.polys.galoistools.**gf_lcm**(*f, g, p, K*)
Compute polynomial LCM in `GF(p)[x]`.

Examples

```
>>> gf_lcm(ZZ.map([3, 2, 4]), ZZ.map([2, 2, 3]), 5, ZZ)
[1, 2, 0, 4]
```

diofant.polys.galoistools.**gf_cofactors**(*f, g, p, K*)
Compute polynomial GCD and cofactors in `GF(p)[x]`.

Examples

```
>>> gf_cofactors(ZZ.map([3, 2, 4]), ZZ.map([2, 2, 3]), 5, ZZ)
([1, 3], [3, 3], [2, 1])
```

diofant.polys.galoistools.**gf_gcdex**(*f, g, p, K*)
Extended Euclidean Algorithm in `GF(p)[x]`.

Given polynomials f and g in GF(p)[x], computes polynomials s, t and h, such that h = gcd(f, g) and s*f + t*g = h. The typical application of EEA is solving polynomial diophantine equations.

Consider polynomials f = (x + 7) (x + 1), g = (x + 7) (x**2 + 1) in GF(11)[x]. Application of Extended Euclidean Algorithm gives:

```
>>> s, t, g = gf_gcdex(ZZ.map([1, 8, 7]), ZZ.map([1, 7, 1, 7]), 11, ZZ)
>>> (s, t, g)
([5, 6], [6], [1, 7])
```

As result we obtained polynomials s = 5*x + 6 and t = 6, and additionally gcd(f, g) = x + 7. This is correct because:

```
>>> S = gf_mul(s, ZZ.map([1, 8, 7]), 11, ZZ)
>>> T = gf_mul(t, ZZ.map([1, 7, 1, 7]), 11, ZZ)

>>> gf_add(S, T, 11, ZZ)
[1, 7]
```

References

[R7] (page 1269)

diofant.polys.galoistools.**gf_monic**(*f, p, K*)
Compute LC and a monic polynomial in GF(p)[x].

Examples

```
>>> gf_monic(ZZ.map([3, 2, 4]), 5, ZZ)
(3, [1, 4, 3])
```

diofant.polys.galoistools.**gf_diff**(*f, p, K*)
Differentiate polynomial in GF(p)[x].

Examples

```
>>> gf_diff([3, 2, 4], 5, ZZ)
[1, 2]
```

diofant.polys.galoistools.**gf_eval**(*f, a, p, K*)
Evaluate f(a) in GF(p) using Horner scheme.

Examples

```
>>> gf_eval([3, 2, 4], 2, 5, ZZ)
0
```

diofant.polys.galoistools.**gf_multi_eval**(*f, A, p, K*)
Evaluate f(a) for a in [a_1, ..., a_n].

Examples

```
>>> gf_multi_eval([3, 2, 4], [0, 1, 2, 3, 4], 5, ZZ)
[4, 4, 0, 2, 0]
```

`diofant.polys.galoistools.`**`gf_compose`**(f, g, p, K)

Compute polynomial composition `f(g)` in `GF(p)[x]`.

Examples

```
>>> gf_compose([3, 2, 4], [2, 2, 2], 5, ZZ)
[2, 4, 0, 3, 0]
```

`diofant.polys.galoistools.`**`gf_compose_mod`**(g, h, f, p, K)

Compute polynomial composition `g(h)` in `GF(p)[x]/(f)`.

Examples

```
>>> gf_compose_mod(ZZ.map([3, 2, 4]), ZZ.map([2, 2, 2]), ZZ.map([4, 3]), 5, ZZ)
[4]
```

`diofant.polys.galoistools.`**`gf_trace_map`**(a, b, c, n, f, p, K)

Compute polynomial trace map in `GF(p)[x]/(f)`.

Given a polynomial `f` in `GF(p)[x]`, polynomials `a`, `b`, `c` in the quotient ring `GF(p)[x]/(f)` such that `b = c**t (mod f)` for some positive power `t` of `p`, and a positive integer `n`, returns a mapping:

```
a -> a**t**n, a + a**t + a**t**2 + ... + a**t**n (mod f)
```

In factorization context, `b = x**p mod f` and `c = x mod f`. This way we can efficiently compute trace polynomials in equal degree factorization routine, much faster than with other methods, like iterated Frobenius algorithm, for large degrees.

References

[R8] (page 1269)

Examples

```
>>> gf_trace_map([1, 2], [4, 4], [1, 1], 4, [3, 2, 4], 5, ZZ)
([1, 3], [1, 3])
```

`diofant.polys.galoistools.`**`gf_random`**(n, p, K)

Generate a random polynomial in `GF(p)[x]` of degree `n`.

Examples

```
>>> gf_random(10, 5, ZZ)
[1, 2, 3, 2, 1, 1, 1, 2, 0, 4, 2]
```

diofant.polys.galoistools.**gf_irreducible**(*n*, *p*, *K*)
Generate random irreducible polynomial of degree `n` in `GF(p)[x]`.

Examples

```
>>> gf_irreducible(10, 5, ZZ)
[1, 4, 2, 2, 3, 2, 4, 1, 4, 0, 4]
```

diofant.polys.galoistools.**gf_irreducible_p**(*f*, *p*, *K*)
Test irreducibility of a polynomial `f` in `GF(p)[x]`.

Examples

```
>>> gf_irreducible_p(ZZ.map([1, 4, 2, 2, 3, 2, 4, 1, 4, 0, 4]), 5, ZZ)
True
>>> gf_irreducible_p(ZZ.map([3, 2, 4]), 5, ZZ)
False
```

diofant.polys.galoistools.**gf_sqf_p**(*f*, *p*, *K*)
Return `True` if `f` is square-free in `GF(p)[x]`.

Examples

```
>>> gf_sqf_p(ZZ.map([3, 2, 4]), 5, ZZ)
True
>>> gf_sqf_p(ZZ.map([2, 4, 4, 2, 2, 1, 4]), 5, ZZ)
False
```

diofant.polys.galoistools.**gf_sqf_part**(*f*, *p*, *K*)
Return square-free part of a `GF(p)[x]` polynomial.

Examples

```
>>> gf_sqf_part(ZZ.map([1, 1, 3, 0, 1, 0, 2, 2, 1]), 5, ZZ)
[1, 4, 3]
```

diofant.polys.galoistools.**gf_sqf_list**(*f*, *p*, *K*, *all=False*)
Return the square-free decomposition of a `GF(p)[x]` polynomial.

Given a polynomial `f` in `GF(p)[x]`, returns the leading coefficient of `f` and a square-free decomposition `f_1**e_1 f_2**e_2 ... f_k**e_k` such that all `f_i` are monic polynomials and `(f_i, f_j)` for `i != j` are co-prime and `e_1 ... e_k` are given in increasing order. All trivial terms (i.e. `f_i = 1`) aren't included in the output.

Consider polynomial `f = x**11 + 1` over `GF(11)[x]`:

```
>>> f = gf_from_dict({11: ZZ(1), 0: ZZ(1)}, 11, ZZ)
```

Note that `f'(x) = 0`:

```
>>> gf_diff(f, 11, ZZ)
[]
```

This phenomenon doesn't happen in characteristic zero. However we can still compute square-free decomposition of `f` using `gf_sqf()`:

```
>>> gf_sqf_list(f, 11, ZZ)
(1, [([1, 1], 11)])
```

We obtained factorization `f = (x + 1)**11`. This is correct because:

```
>>> gf_pow([1, 1], 11, 11, ZZ) == f
True
```

References

[R9] (page 1269)

`diofant.polys.galoistools.`**`gf_Qmatrix`**`(`*f, p, K*`)`

Calculate Berlekamp's `Q` matrix.

Examples

```
>>> gf_Qmatrix([3, 2, 4], 5, ZZ)
[[1, 0],
 [3, 4]]
```

```
>>> gf_Qmatrix([1, 0, 0, 0, 1], 5, ZZ)
[[1, 0, 0, 0],
 [0, 4, 0, 0],
 [0, 0, 1, 0],
 [0, 0, 0, 4]]
```

`diofant.polys.galoistools.`**`gf_Qbasis`**`(`*Q, p, K*`)`

Compute a basis of the kernel of `Q`.

Examples

```
>>> gf_Qbasis(gf_Qmatrix([1, 0, 0, 0, 1], 5, ZZ), 5, ZZ)
[[1, 0, 0, 0], [0, 0, 1, 0]]
```

```
>>> gf_Qbasis(gf_Qmatrix([3, 2, 4], 5, ZZ), 5, ZZ)
[[1, 0]]
```

`diofant.polys.galoistools.`**`gf_berlekamp`**`(`*f, p, K*`)`

Factor a square-free `f` in `GF(p)[x]` for small `p`.

Examples

```
>>> gf_berlekamp([1, 0, 0, 0, 1], 5, ZZ)
[[1, 0, 2], [1, 0, 3]]
```

diofant.polys.galoistools.**gf_zassenhaus**(*f, p, K*)
Factor a square-free f in GF(p)[x] for medium p.

Examples

```
>>> gf_zassenhaus(ZZ.map([1, 4, 3]), 5, ZZ)
[[1, 1], [1, 3]]
```

diofant.polys.galoistools.**gf_shoup**(*f, p, K*)
Factor a square-free f in GF(p)[x] for large p.

Examples

```
>>> gf_shoup(ZZ.map([1, 4, 3]), 5, ZZ)
[[1, 1], [1, 3]]
```

diofant.polys.galoistools.**gf_ddf_shoup**(*f, p, K*)
Kaltofen-Shoup: Deterministic Distinct Degree Factorization

Given a monic square-free polynomial f in GF(p)[x], computes partial distinct degree factorization f_1,...,f_d of f where deg(f_i) != deg(f_j) for i != j. The result is returned as a list of pairs (f_i, e_i) where deg(f_i) > 0 and e_i > 0 is an argument to the equal degree factorization routine.

This algorithm is an improved version of Zassenhaus algorithm for large deg(f) and modulus p (especially for deg(f) ~ lg(p)).

References

[R10] (page 1269), *[R11]* (page 1269), *[R12]* (page 1269)

Examples

```
>>> f = gf_from_dict({6: ZZ(1), 5: ZZ(-1), 4: ZZ(1), 3: ZZ(1), 1: ZZ(-1)}, 3, ZZ)
```

```
>>> gf_ddf_shoup(f, 3, ZZ)
[([1, 1, 0], 1), ([1, 1, 0, 1, 2], 2)]
```

diofant.polys.galoistools.**gf_factor_sqf**(*f, p, K*)
Factor a square-free polynomial f in GF(p)[x].

Examples

```
>>> gf_factor_sqf(ZZ.map([3, 2, 4]), 5, ZZ)
(3, [[1, 1], [1, 3]])
```

diofant.polys.galoistools.**gf_factor**(*f*, *p*, *K*)

Factor (non square-free) polynomials in `GF(p)[x]`.

Given a possibly non square-free polynomial `f` in `GF(p)[x]`, returns its complete factorization into irreducibles:

```
f_1(x)**e_1 f_2(x)**e_2 ... f_d(x)**e_d
```

where each `f_i` is a monic polynomial and `gcd(f_i, f_j) == 1`, for `i != j`. The result is given as a tuple consisting of the leading coefficient of `f` and a list of factors of `f` with their multiplicities.

The algorithm proceeds by first computing square-free decomposition of `f` and then iteratively factoring each of square-free factors.

Consider a non square-free polynomial `f = (7*x + 1) (x + 2)**2` in `GF(11)[x]`. We obtain its factorization into irreducibles as follows:

```
>>> gf_factor(ZZ.map([5, 2, 7, 2]), 11, ZZ)
(5, [([1, 2], 1), ([1, 8], 2)])
```

We arrived with factorization `f = 5 (x + 2) (x + 8)**2`. We didn't recover the exact form of the input polynomial because we requested to get monic factors of `f` and its leading coefficient separately.

Square-free factors of `f` can be factored into irreducibles over `GF(p)` using three very different methods:

Berlekamp efficient for very small values of `p` (usually `p < 25`)

Cantor-Zassenhaus efficient on average input and with "typical" `p`

Shoup-Kaltofen-Gathen efficient with very large inputs and modulus

If you want to use a specific factorization method, instead of the default one, set `GF_FACTOR_METHOD` with one of `berlekamp`, `zassenhaus` or `shoup` values.

References

[R13] (page 1269)

diofant.polys.galoistools.**gf_value**(*f*, *a*)

Value of polynomial 'f' at 'a' in field R.

Examples

```
>>> gf_value([1, 7, 2, 4], 11)
2204
```

diofant.polys.galoistools.**gf_csolve**(*f*, *n*)

To solve f(x) congruent 0 mod(n).

n is divided into canonical factors and f(x) cong 0 mod(p**e) will be solved for each factor. Applying the Chinese Remainder Theorem to the results returns the final answers.

References

[R14] (page 1269)

Examples

Solve [1, 1, 7] congruent 0 mod(189):

```
>>> gf_csolve([1, 1, 7], 189)
[13, 49, 76, 112, 139, 175]
```

diofant.polys.galoistools.**gf_irred_p_ben_or**(*f*, *p*, *K*)
Ben-Or's polynomial irreducibility test over finite fields.

References

[R15] (page 1269)

Examples

```
>>> gf_irred_p_ben_or(ZZ.map([1, 4, 2, 2, 3, 2, 4, 1, 4, 0, 4]), 5, ZZ)
True
>>> gf_irred_p_ben_or(ZZ.map([3, 2, 4]), 5, ZZ)
False
```

diofant.polys.galoistools.**gf_edf_shoup**(*f*, *n*, *p*, *K*)
Gathen-Shoup: Probabilistic Equal Degree Factorization

Given a monic square-free polynomial `f` in `GF(p)[x]` and integer `n` such that `n` divides `deg(f)`, returns all irreducible factors `f_1,...,f_d` of `f`, each of degree `n`. This is a complete factorization over Galois fields.

This algorithm is an improved version of Zassenhaus algorithm for large `deg(f)` and modulus `p` (especially for `deg(f) ~ lg(p)`).

References

[R16] (page 1269), *[R17]* (page 1269)

Examples

```
>>> gf_edf_shoup(ZZ.map([1, 2837, 2277]), 1, 2917, ZZ)
[[1, 852], [1, 1985]]
```

Manipulation of sparse, distributed polynomials and vectors

Dense representations quickly require infeasible amounts of storage and computation time if the number of variables increases. For this reason, there is code to manipulate polynomials in a *sparse* representation.

Sparse polynomials are represented as dictionaries.

diofant.polys.rings.**ring**(*symbols, domain, order=<diofant.polys.orderings.LexOrder object>*)

Construct a polynomial ring returning (ring, x_1, ..., x_n).

Parameters symbols : str, Symbol/Expr or sequence of str, Symbol/Expr (non-empty)

domain : *Domain* (page 551) or coercible

order : *Order* (page 1151) or coercible, optional, defaults to lex

Examples

```
>>> R, x, y, z = ring("x y z", ZZ, lex)
>>> R
Polynomial ring in x, y, z over ZZ with lex order
>>> x + y + z
x + y + z
>>> type(_)
<class 'diofant.polys.rings.PolyElement'>
```

diofant.polys.rings.**vring**(*symbols, domain, order=<diofant.polys.orderings.LexOrder object>*)

Construct a polynomial ring and inject x_1, ..., x_n into the global namespace.

Parameters symbols : str, Symbol/Expr or sequence of str, Symbol/Expr (non-empty)

domain : *Domain* (page 551) or coercible

order : *Order* (page 1151) or coercible, optional, defaults to lex

Examples

```
>>> vring("x y z", ZZ, lex)
Polynomial ring in x, y, z over ZZ with lex order
>>> x + y + z
x + y + z
>>> type(_)
<class 'diofant.polys.rings.PolyElement'>
```

diofant.polys.rings.**sring**(*exprs, *symbols, **options*)

Construct a ring deriving generators and domain from options and input expressions.

Parameters exprs : *Expr* (page 54) or sequence of *Expr* (page 54) (sympifiable)

symbols : sequence of *Symbol* (page 79)/*Expr* (page 54)

options : keyword arguments understood by *Options* (page 1152)

Examples

```
>>> x, y, z = symbols("x y z")
>>> R, f = sring(x + 2*y + 3*z)
>>> R
Polynomial ring in x, y, z over ZZ with lex order
>>> f
x + 2*y + 3*z
>>> type(_)
<class 'diofant.polys.rings.PolyElement'>
```

class diofant.polys.rings.**PolyRing**

Multivariate distributed polynomial ring.

add(**objs*)

Add a sequence of polynomials or containers of polynomials.

Examples

```
>>> R, x = ring("x", ZZ)
>>> R.add([x**2 + 2*i + 3 for i in range(4)])
4*x**2 + 24
>>> _.factor_list()
(4, [(x**2 + 6, 1)])
```

drop(**gens*)

Remove specified generators from this ring.

drop_to_ground(**gens*)

Remove specified generators from the ring and inject them into its domain.

index(*gen*)

Compute index of `gen` in `self.gens`.

monomial_basis(*i*)

Return the ith-basis element.

mul(**objs*)

Multiply a sequence of polynomials or containers of polynomials.

Examples

```
>>> R, x = ring("x", ZZ)
>>> R.mul([x**2 + 2*i + 3 for i in range(4)])
x**8 + 24*x**6 + 206*x**4 + 744*x**2 + 945
>>> _.factor_list()
(1, [(x**2 + 3, 1), (x**2 + 5, 1), (x**2 + 7, 1), (x**2 + 9, 1)])
```

class diofant.polys.rings.**PolyElement**

Element of multivariate distributed polynomial ring.

almosteq(*other, tolerance=None*)

Approximate equality test for polynomials.

cancel(*g*)

Cancel common factors in a rational function `f/g`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> (2*x**2 - 2).cancel(x**2 - 2*x + 1)
(2*x + 2, x - 1)
```

coeff(*element*)
Returns the coefficient that stands next to the given monomial.

Parameters element : PolyElement (with `is_monomial = True`) or 1

Examples

```
>>> _, x, y, z = ring("x y z", ZZ)
>>> f = 3*x**2*y - x*y*z + 7*z**3 + 23
```

```
>>> f.coeff(x**2*y)
3
>>> f.coeff(x*y)
0
>>> f.coeff(1)
23
```

coeffs(*order=None*)
Ordered list of polynomial coefficients.

Parameters order : `Order` (page 1151) or coercible, optional

Examples

```
>>> _, x, y = ring("x, y", ZZ, lex)
>>> f = x*y**7 + 2*x**2*y**3
```

```
>>> f.coeffs()
[2, 1]
>>> f.coeffs(grlex)
[1, 2]
```

compose(*x*, *a=None*)
Computes the functional composition.

const()
Returns the constant coefficient.

content()
Returns GCD of polynomial's coefficients.

copy()
Return a copy of polynomial self.

Polynomials are mutable; if one is interested in preserving a polynomial, and one plans to use inplace operations, one can copy the polynomial. This method makes a shallow copy.

Examples

```
>>> R, x, y = ring('x, y', ZZ)
>>> p = (x + y)**2
>>> p1 = p.copy()
>>> p2 = p
>>> p[R.zero_monom] = 3
>>> p
x**2 + 2*x*y + y**2 + 3
>>> p1
x**2 + 2*x*y + y**2
>>> p2
x**2 + 2*x*y + y**2 + 3
```

degree(*x=None*)
The leading degree in x or the main variable.

Note that the degree of 0 is negative infinity (the Diofant object -oo).

degrees()
A tuple containing leading degrees in all variables.

Note that the degree of 0 is negative infinity (the Diofant object -oo)

diff(*x*)
Computes partial derivative in x.

Examples

```
>>> _, x, y = ring("x y", ZZ)
>>> p = x + x**2*y**3
>>> p.diff(x)
2*x*y**3 + 1
```

div(*fv*)
Division algorithm, see [CLO] p64.

fv array of polynomials return qv, r such that self = sum(fv[i]*qv[i]) + r

All polynomials are required not to be Laurent polynomials.

Examples

```
>>> _, x, y = ring('x, y', ZZ)
>>> f = x**3
>>> f0 = x - y**2
>>> f1 = x - y
>>> qv, r = f.div((f0, f1))
>>> qv[0]
x**2 + x*y**2 + y**4
>>> qv[1]
0
>>> r
y**6
```

imul_num(*c*)
multiply inplace the polynomial self by an element in the coefficient ring, provided self is not one of the generators; else multiply not inplace

Examples

```
>>> _, x, y = ring('x, y', ZZ)
>>> p = x + y**2
>>> p1 = p.imul_num(3)
>>> p1
3*x + 3*y**2
>>> p1 is p
True
>>> p = x
>>> p1 = p.imul_num(3)
>>> p1
3*x
>>> p1 is p
False
```

itercoeffs()
Iterator over coefficients of a polynomial.

itermonoms()
Iterator over monomials of a polynomial.

iterterms()
Iterator over terms of a polynomial.

leading_expv()
Leading monomial tuple according to the monomial ordering.

Examples

```
>>> _, x, y, z = ring('x, y, z', ZZ)
>>> p = x**4 + x**3*y + x**2*z**2 + z**7
>>> p.leading_expv()
(4, 0, 0)
```

leading_monom()
Leading monomial as a polynomial element.

Examples

```
>>> _, x, y = ring('x, y', ZZ)
>>> (3*x*y + y**2).leading_monom()
x*y
```

leading_term()
Leading term as a polynomial element.

Examples

```
>>> _, x, y = ring('x, y', ZZ)
>>> (3*x*y + y**2).leading_term()
3*x*y
```

listcoeffs()
Unordered list of polynomial coefficients.

listmonoms()
Unordered list of polynomial monomials.

listterms()
Unordered list of polynomial terms.

monic()
Divides all coefficients by the leading coefficient.

monoms(*order=None*)
Ordered list of polynomial monomials.

Parameters order : *Order* (page 1151) or coercible, optional

Examples

```
>>> _, x, y = ring("x, y", ZZ, lex)
>>> f = x*y**7 + 2*x**2*y**3
```

```
>>> f.monoms()
[(2, 3), (1, 7)]
>>> f.monoms(grlex)
[(1, 7), (2, 3)]
```

primitive()
Returns content and a primitive polynomial.

square()
square of a polynomial

Examples

```
>>> _, x, y = ring('x, y', ZZ)
>>> p = x + y**2
>>> p.square()
x**2 + 2*x*y**2 + y**4
```

strip_zero()
Eliminate monomials with zero coefficient.

tail_degree(*x=None*)
The tail degree in x or the main variable.

Note that the degree of 0 is negative infinity (the Diofant object -oo)

tail_degrees()
A tuple containing tail degrees in all variables.

Note that the degree of 0 is negative infinity (the Diofant object -oo)

terms(*order=None*)
Ordered list of polynomial terms.

Parameters order : *Order* (page 1151) or coercible, optional

Examples

```
>>> _, x, y = ring("x, y", ZZ, lex)
>>> f = x*y**7 + 2*x**2*y**3
```

```
>>> f.terms()
[((2, 3), 2), ((1, 7), 1)]
>>> f.terms(grlex)
[((1, 7), 1), ((2, 3), 2)]
```

In commutative algebra, one often studies not only polynomials, but also *modules* over polynomial rings. The polynomial manipulation module provides rudimentary low-level support for finitely generated free modules. This is mainly used for Gröbner basis computations (see there), so manipulation functions are only provided to the extend needed. They carry the prefix `sdm_`. Note that in examples, the generators of the free module are called $f_1, f_2, \ldots$.

`diofant.polys.distributedmodules.`**`sdm_monomial_mul`**(M, X)
Multiply tuple `X` representing a monomial of $K[X]$ into the tuple `M` representing a monomial of F.

Examples

Multiplying xy^3 into xf_1 yields $x^2y^3f_1$:

```
>>> sdm_monomial_mul((1, 1, 0), (1, 3))
(1, 2, 3)
```

`diofant.polys.distributedmodules.`**`sdm_monomial_deg`**(M)
Return the total degree of `M`.

Examples

For example, the total degree of x^2yf_5 is 3:

```
>>> sdm_monomial_deg((5, 2, 1))
3
```

`diofant.polys.distributedmodules.`**`sdm_monomial_divides`**(A, B)
Does there exist a (polynomial) monomial X such that XA = B?

Examples

Positive examples:

In the following examples, the monomial is given in terms of x, y and the generator(s), f_1, f_2 etc. The tuple form of that monomial is used in the call to sdm_monomial_divides.

Note: the generator appears last in the expression but first in the tuple and other factors appear in the same order that they appear in the monomial expression.

$A = f_1$ divides $B = f_1$

```
>>> sdm_monomial_divides((1, 0, 0), (1, 0, 0))
True
```

$A = f_1$ divides $B = x^2 y f_1$

```
>>> sdm_monomial_divides((1, 0, 0), (1, 2, 1))
True
```

$A = xyf_5$ divides $B = x^2 y f_5$

```
>>> sdm_monomial_divides((5, 1, 1), (5, 2, 1))
True
```

Negative examples:

$A = f_1$ does not divide $B = f_2$

```
>>> sdm_monomial_divides((1, 0, 0), (2, 0, 0))
False
```

$A = xf_1$ does not divide $B = f_1$

```
>>> sdm_monomial_divides((1, 1, 0), (1, 0, 0))
False
```

$A = xy^2 f_5$ does not divide $B = yf_5$

```
>>> sdm_monomial_divides((5, 1, 2), (5, 0, 1))
False
```

diofant.polys.distributedmodules.**sdm_LC**(*f*, *K*)
Returns the leading coefficient of f.

diofant.polys.distributedmodules.**sdm_to_dict**(*f*)
Make a dictionary from a distributed polynomial.

diofant.polys.distributedmodules.**sdm_from_dict**(*d*, *O*)
Create an sdm from a dictionary.

Here O is the monomial order to use.

```
>>> dic = {(1, 1, 0): QQ(1), (1, 0, 0): QQ(2), (0, 1, 0): QQ(0)}
>>> sdm_from_dict(dic, lex)
[((1, 1, 0), 1), ((1, 0, 0), 2)]
```

diofant.polys.distributedmodules.**sdm_add**(*f*, *g*, *O*, *K*)
Add two module elements f, g.

Addition is done over the ground field K, monomials are ordered according to O.

Examples

All examples use lexicographic order.

$(xyf_1) + (f_2) = f_2 + xyf_1$

```
>>> sdm_add([((1, 1, 1), QQ(1))], [((2, 0, 0), QQ(1))], lex, QQ)
[((2, 0, 0), 1), ((1, 1, 1), 1)]
```

$(xyf_1) + (-xyf_1)$ = 0'

```
>>> sdm_add([((1, 1, 1), QQ(1))], [((1, 1, 1), QQ(-1))], lex, QQ)
[]
```

$(f_1) + (2f_1) = 3f_1$

```
>>> sdm_add([((1, 0, 0), QQ(1))], [((1, 0, 0), QQ(2))], lex, QQ)
[((1, 0, 0), 3)]
```

$(yf_1) + (xf_1) = xf_1 + yf_1$

```
>>> sdm_add([((1, 0, 1), QQ(1))], [((1, 1, 0), QQ(1))], lex, QQ)
[((1, 1, 0), 1), ((1, 0, 1), 1)]
```

diofant.polys.distributedmodules.**sdm_LM**(*f*)

Returns the leading monomial of `f`.

Only valid if $f \neq 0$.

Examples

```
>>> dic = {(1, 2, 3): QQ(1), (4, 0, 0): QQ(1), (4, 0, 1): QQ(1)}
>>> sdm_LM(sdm_from_dict(dic, lex))
(4, 0, 1)
```

diofant.polys.distributedmodules.**sdm_LT**(*f*)

Returns the leading term of `f`.

Only valid if $f \neq 0$.

Examples

```
>>> dic = {(1, 2, 3): QQ(1), (4, 0, 0): QQ(2), (4, 0, 1): QQ(3)}
>>> sdm_LT(sdm_from_dict(dic, lex))
((4, 0, 1), 3)
```

diofant.polys.distributedmodules.**sdm_mul_term**(*f, term, O, K*)

Multiply a distributed module element `f` by a (polynomial) term `term`.

Multiplication of coefficients is done over the ground field `K`, and monomials are ordered according to `O`.

Examples

$0f_1 = 0$

```
>>> sdm_mul_term([((1, 0, 0), QQ(1))], ((0, 0), QQ(0)), lex, QQ)
[]
```

$x0 = 0$

```
>>> sdm_mul_term([], ((1, 0), QQ(1)), lex, QQ)
[]
```

$(x)(f_1) = xf_1$

```
>>> sdm_mul_term([((1, 0, 0), QQ(1))], ((1, 0), QQ(1)), lex, QQ)
[((1, 1, 0), 1)]
```

$(2xy)(3xf_1 + 4yf_2) = 8xy^2f_2 + 6x^2yf_1$

```
>>> f = [((2, 0, 1), QQ(4)), ((1, 1, 0), QQ(3))]
>>> sdm_mul_term(f, ((1, 1), QQ(2)), lex, QQ)
[((2, 1, 2), 8), ((1, 2, 1), 6)]
```

`diofant.polys.distributedmodules.`**`sdm_zero`**`()`

Return the zero module element.

`diofant.polys.distributedmodules.`**`sdm_deg`**`(`*f*`)`

Degree of `f`.

This is the maximum of the degrees of all its monomials. Invalid if `f` is zero.

Examples

```
>>> sdm_deg([((1, 2, 3), 1), ((10, 0, 1), 1), ((2, 3, 4), 4)])
7
```

`diofant.polys.distributedmodules.`**`sdm_from_vector`**`(`*vec, O, K, **opts*`)`

Create an sdm from an iterable of expressions.

Coefficients are created in the ground field `K`, and terms are ordered according to monomial order `O`. Named arguments are passed on to the polys conversion code and can be used to specify for example generators.

Examples

```
>>> sdm_from_vector([x**2+y**2, 2*z], lex, QQ)
[((1, 0, 0, 1), 2), ((0, 2, 0, 0), 1), ((0, 0, 2, 0), 1)]
```

`diofant.polys.distributedmodules.`**`sdm_to_vector`**`(`*f, gens, K, n=None*`)`

Convert sdm `f` into a list of polynomial expressions.

The generators for the polynomial ring are specified via `gens`. The rank of the module is guessed, or passed via `n`. The ground field is assumed to be `K`.

Examples

```
>>> f = [((1, 0, 0, 1), QQ(2)), ((0, 2, 0, 0), QQ(1)), ((0, 0, 2, 0), QQ(1))]
>>> sdm_to_vector(f, [x, y, z], QQ)
[x**2 + y**2, 2*z]
```

Polynomial factorization algorithms

Many variants of Euclid's algorithm:

Classical remainder sequence

Let K be a field, and consider the ring $K[X]$ of polynomials in a single indeterminate X with coefficients in K. Given two elements f and g of $K[X]$ with $g \neq 0$ there are unique polynomials q and r such that $f = qg + r$ and $\deg(r) < \deg(g)$ or $r = 0$. They are denoted by $\text{quo}(f, g)$ (*quotient*) and $\text{rem}(f, g)$ (*remainder*), so we have the *division identity*

$$f = \text{quo}(f, g)g + \text{rem}(f, g).$$

It follows that every ideal I of $K[X]$ is a principal ideal, generated by any element $\neq 0$ of minimum degree (assuming I non-zero). In fact, if g is such a polynomial and f is any element of I, $\text{rem}(f, g)$ belongs to I as a linear combination of f and g, hence must be zero; therefore f is a multiple of g.

Using this result it is possible to find a greatest common divisor (gcd) of any polynomials $f, g, \ldots$ in $K[X]$. If I is the ideal formed by all linear combinations of the given polynomials with coefficients in $K[X]$, and d is its generator, then every common divisor of the polynomials also divides d. On the other hand, the given polynomials are multiples of the generator d; hence d is a gcd of the polynomials, denoted $\text{gcd}(f, g, \ldots)$.

An algorithm for the gcd of two polynomials f and g in $K[X]$ can now be obtained as follows. By the division identity, $r = \text{rem}(f, g)$ is in the ideal generated by f and g, as well as f is in the ideal generated by g and r. Hence the ideals generated by the pairs (f, g) and (g, r) are the same. Set $f_0 = f$, $f_1 = g$, and define recursively $f_i = \text{rem}(f_{i-2}, f_{i-1})$ for $i \geq 2$. The recursion ends after a finite number of steps with $f_{k+1} = 0$, since the degrees of the polynomials are strictly decreasing. By the above remark, all the pairs (f_{i-1}, f_i) generate the same ideal. In particular, the ideal generated by f and g is generated by f_k alone as $f_{k+1} = 0$. Hence $d = f_k$ is a gcd of f and g.

The sequence of polynomials f_0, $f_1, \ldots, f_k$ is called the *Euclidean polynomial remainder sequence* determined by (f, g) because of the analogy with the classical Euclidean algorithm for the gcd of natural numbers.

The algorithm may be extended to obtain an expression for d in terms of f and g by using the full division identities to write recursively each f_i as a linear combination of f and g. This leads to an equation

$$d = uf + vg \qquad (u, v \in K[X])$$

analogous to Bézout's identity in the case of integers.

diofant.polys.euclidtools.**dup_half_gcdex**(*f, g, K*)

Half extended Euclidean algorithm in $F[x]$.

Returns `(s, h)` such that `h = gcd(f, g)` and `s*f = h (mod g)`.

Examples

```
>>> R, x = ring("x", QQ)
```

```
>>> f = x**4 - 2*x**3 - 6*x**2 + 12*x + 15
>>> g = x**3 + x**2 - 4*x - 4
```

```
>>> R.dup_half_gcdex(f, g)
(-1/5*x + 3/5, x + 1)
```

diofant.polys.euclidtools.**dup_gcdex**(*f*, *g*, *K*)
Extended Euclidean algorithm in $F[x]$.

Returns `(s, t, h)` such that `h = gcd(f, g)` and `s*f + t*g = h`.

Examples

```
>>> R, x = ring("x", QQ)
```

```
>>> f = x**4 - 2*x**3 - 6*x**2 + 12*x + 15
>>> g = x**3 + x**2 - 4*x - 4
```

```
>>> R.dup_gcdex(f, g)
(-1/5*x + 3/5, 1/5*x**2 - 6/5*x + 2, x + 1)
```

Simplified remainder sequences

Assume, as is usual, that the coefficient field K is the field of fractions of an integral domain A. In this case the coefficients (numerators and denominators) of the polynomials in the Euclidean remainder sequence tend to grow very fast.

If A is a unique factorization domain, the coefficients may be reduced by cancelling common factors of numerators and denominators. Further reduction is possible noting that a gcd of polynomials in $K[X]$ is not unique: it may be multiplied by any (non-zero) constant factor.

Any polynomial f in $K[X]$ can be simplified by extracting the denominators and common factors of the numerators of its coefficients. This yields the representation $f = cF$ where $c \in K$ is the *content* of f and F is a *primitive* polynomial, i.e., a polynomial in $A[X]$ with coprime coefficients.

It is possible to start the algorithm by replacing the given polynomials f and g with their primitive parts. This will only modify $\text{rem}(f, g)$ by a constant factor. Replacing it with its primitive part and continuing recursively we obtain all the primitive parts of the polynomials in the Euclidean remainder sequence, including the primitive $\text{gcd}(f, g)$.

This sequence is the *primitive polynomial remainder sequence*. It is an example of *general polynomial remainder sequences* where the computed remainders are modified by constant multipliers (or divisors) in order to simplify the results.

diofant.polys.euclidtools.**dup_primitive_prs**(*f*, *g*, *K*)
Primitive polynomial remainder sequence (PRS) in $K[x]$.

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> f = x**8 + x**6 - 3*x**4 - 3*x**3 + 8*x**2 + 2*x - 5
>>> g = 3*x**6 + 5*x**4 - 4*x**2 - 9*x + 21
```

```
>>> prs = R.dup_primitive_prs(f, g)
```

```
>>> prs[0]
x**8 + x**6 - 3*x**4 - 3*x**3 + 8*x**2 + 2*x - 5
>>> prs[1]
3*x**6 + 5*x**4 - 4*x**2 - 9*x + 21
>>> prs[2]
-5*x**4 + x**2 - 3
>>> prs[3]
13*x**2 + 25*x - 49
>>> prs[4]
4663*x - 6150
>>> prs[5]
1
```

Subresultant sequence

The coefficients of the primitive polynomial sequence do not grow exceedingly, but the computation of the primitive parts requires extra processing effort. Besides, the method only works with fraction fields of unique factorization domains, excluding, for example, the general number fields.

Collins *[Collins67]* (page 1263) realized that the so-called *subresultant polynomials* of a pair of polynomials also form a generalized remainder sequence. The coefficients of these polynomials are expressible as determinants in the coefficients of the given polynomials. Hence (the logarithm of) their size only grows linearly. In addition, if the coefficients of the given polynomials are in the subdomain A, so are those of the subresultant polynomials. This means that the subresultant sequence is comparable to the primitive remainder sequence without relying on unique factorization in A.

To see how subresultants are associated with remainder sequences recall that all polynomials h in the sequence are linear combinations of the given polynomials f and g

$$h = uf + vg$$

with polynomials u and v in $K[X]$. Moreover, as is seen from the extended Euclidean algorithm, the degrees of u and v are relatively low, with limited growth from step to step.

Let $n = \deg(f)$, and $m = \deg(g)$, and assume $n \geq m$. If $\deg(h) = j < m$, the coefficients of the powers X^k ($k > j$) in the products uf and vg cancel each other. In particular, the products must have the same degree, say, l. Then $\deg(u) = l - n$ and $\deg(v) = l - m$ with a total of $2l - n - m + 2$ coefficients to be determined.

On the other hand, the equality $h = uf + vg$ implies that $l - j$ linear combinations of the coefficients are zero, those associated with the powers X^i ($j < i \leq l$), and one has a given non-zero value, namely the leading coefficient of h.

To satisfy these $l - j + 1$ linear equations the total number of coefficients to be determined cannot be lower than $l - j + 1$, in general. This leads to the inequality $l \geq n + m - j - 1$. Taking $l = n + m - j - 1$, we obtain $\deg(u) = m - j - 1$ and $\deg(v) = n - j - 1$.

In the case $j = 0$ the matrix of the resulting system of linear equations is the Sylvester matrix $S(f, g)$ associated to f and g, an $(n + m) \times (n + m)$ matrix with coefficients of f and g as entries.

Its determinant is the resultant $\mathrm{res}(f,g)$ of the pair (f,g). It is non-zero if and only if f and g are relatively prime.

For any j in the interval from 0 to m the matrix of the linear system is an $(n+m-2j)\times(n+m-2j)$ submatrix of the Sylvester matrix. Its determinant $s_j(f,g)$ is called the j th *scalar subresultant* of f and g.

If $s_j(f,g)$ is not zero, the associated equation $h = uf + vg$ has a unique solution where $\deg(h) = j$ and the leading coefficient of h has any given value; the one with leading coefficient $s_j(f,g)$ is the j th *subresultant polynomial* or, briefly, *subresultant* of the pair (f,g), and denoted $S_j(f,g)$. This choice guarantees that the remainining coefficients are also certain subdeterminants of the Sylvester matrix. In particular, if f and g are in $A[X]$, so is $S_j(f,g)$ as well. This construction of subresultants applies to any j between 0 and m regardless of the value of $s_j(f,g)$; if it is zero, then $\deg(S_j(f,g)) < j$.

The properties of subresultants are as follows. Let $n_0 = \deg(f)$, $n_1 = \deg(g)$, $n_2, \ldots, n_k$ be the decreasing sequence of degrees of polynomials in a remainder sequence. Let $0 \le j \le n_1$; then

- $s_j(f,g) \ne 0$ if and only if $j = n_i$ for some i.
- $S_j(f,g) \ne 0$ if and only if $j = n_i$ or $j = n_i - 1$ for some i.

Normally, $n_{i-1} - n_i = 1$ for $1 < i \le k$. If $n_{i-1} - n_i > 1$ for some i (the *abnormal* case), then $S_{n_{i-1}-1}(f,g)$ and $S_{n_i}(f,g)$ are constant multiples of each other. Hence either one could be included in the polynomial remainder sequence. The former is given by smaller determinants, so it is expected to have smaller coefficients.

Collins defined the *subresultant remainder sequence* by setting

$$f_i = S_{n_{i-1}-1}(f,g) \qquad (2 \le i \le k).$$

In the normal case, these are the same as the $S_{n_i}(f,g)$. He also derived expressions for the constants γ_i in the remainder formulas

$$\gamma_i f_i = \mathrm{rem}(f_{i-2}, f_{i-1})$$

in terms of the leading coefficients of $f_1, \ldots, f_{i-1}$, working in the field K.

Brown and Traub *[BrownTraub71]* (page 1263) later developed a recursive procedure for computing the coefficients γ_i. Their algorithm deals with elements of the domain A exclusively (assuming $f, g \in A[X]$). However, in the abnormal case there was a problem, a division in A which could only be conjectured to be exact.

This was subsequently justified by Brown *[Brown78]* (page 1263) who showed that the result of the division is, in fact, a scalar subresultant. More specifically, the constant appearing in the computation of f_i is $s_{n_{i-2}}(f,g)$ (Theorem 3). The implication of this discovery is that the scalar subresultants are computed as by-products of the algorithm, all but $s_{n_k}(f,g)$ which is not needed after finding $f_{k+1} = 0$. Completing the last step we obtain all non-zero scalar subresultants, including the last one which is the resultant if this does not vanish.

diofant.polys.euclidtools.**dmp_inner_subresultants**(*f, g, u, K*)

Subresultant PRS algorithm in $K[X]$.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = 3*x**2*y - y**3 - 4
>>> g = x**2 + x*y**3 - 9
```

```
>>> a = 3*x*y**4 + y**3 - 27*y + 4
>>> b = -3*y**10 - 12*y**7 + y**6 - 54*y**4 + 8*y**3 + 729*y**2 - 216*y + 16
```

```
>>> prs = [f, g, a, b]
>>> sres = [[1], [1], [3, 0, 0, 0, 0], [-3, 0, 0, -12, 1, 0, -54, 8, 729, -216,␣
↪16]]
```

```
>>> R.dmp_inner_subresultants(f, g) == (prs, sres)
True
```

diofant.polys.euclidtools.**dmp_subresultants**(f, g, u, K)

Computes subresultant PRS of two polynomials in $K[X]$.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = 3*x**2*y - y**3 - 4
>>> g = x**2 + x*y**3 - 9
```

```
>>> a = 3*x*y**4 + y**3 - 27*y + 4
>>> b = -3*y**10 - 12*y**7 + y**6 - 54*y**4 + 8*y**3 + 729*y**2 - 216*y + 16
```

```
>>> R.dmp_subresultants(f, g) == [f, g, a, b]
True
```

diofant.polys.euclidtools.**dmp_prs_resultant**(f, g, u, K)

Resultant algorithm in $K[X]$ using subresultant PRS.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = 3*x**2*y - y**3 - 4
>>> g = x**2 + x*y**3 - 9
```

```
>>> a = 3*x*y**4 + y**3 - 27*y + 4
>>> b = -3*y**10 - 12*y**7 + y**6 - 54*y**4 + 8*y**3 + 729*y**2 - 216*y + 16
```

```
>>> res, prs = R.dmp_prs_resultant(f, g)
```

```
>>> res == b             # resultant has n-1 variables
False
>>> res == b.drop(x)
True
>>> prs == [f, g, a, b]
True
```

diofant.polys.euclidtools.**dmp_zz_modular_resultant**(*f*, *g*, *p*, *u*, *K*)
Compute resultant of f and g modulo a prime p.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x + y + 2
>>> g = 2*x*y + x + 3
```

```
>>> R.dmp_zz_modular_resultant(f, g, 5)
-2*y**2 + 1
```

diofant.polys.euclidtools.**dmp_zz_collins_resultant**(*f*, *g*, *u*, *K*)
Collins's modular resultant algorithm in $Z[X]$.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x + y + 2
>>> g = 2*x*y + x + 3
```

```
>>> R.dmp_zz_collins_resultant(f, g)
-2*y**2 - 5*y + 1
```

diofant.polys.euclidtools.**dmp_qq_collins_resultant**(*f*, *g*, *u*, *K0*)
Collins's modular resultant algorithm in $Q[X]$.

Examples

```
>>> R, x, y = ring("x y", QQ)
```

```
>>> f = x/2 + y + QQ(2, 3)
>>> g = 2*x*y + x + 3
```

```
>>> R.dmp_qq_collins_resultant(f, g)
-2*y**2 - 7/3*y + 5/6
```

diofant.polys.euclidtools.**dmp_resultant**(*f*, *g*, *u*, *K*, *includePRS=False*)
Computes resultant of two polynomials in $K[X]$.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = 3*x**2*y - y**3 - 4
>>> g = x**2 + x*y**3 - 9
```

```
>>> R.dmp_resultant(f, g)
-3*y**10 - 12*y**7 + y**6 - 54*y**4 + 8*y**3 + 729*y**2 - 216*y + 16
```

diofant.polys.euclidtools.**dmp_discriminant**(f, u, K)

Computes discriminant of a polynomial in $K[X]$.

Examples

```
>>> R, x, y, z, t = ring("x y z t", ZZ)
```

```
>>> R.dmp_discriminant(x**2*y + x*z + t)
-4*y*t + z**2
```

diofant.polys.euclidtools.**dmp_rr_prs_gcd**(f, g, u, K)

Computes polynomial GCD using subresultants over a ring.

Returns `(h, cff, cfg)` such that `a = gcd(f, g)`, `cff = quo(f, h)`, and `cfg = quo(g, h)`.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x**2 + 2*x*y + y**2
>>> g = x**2 + x*y
```

```
>>> R.dmp_rr_prs_gcd(f, g)
(x + y, x + y, x)
```

diofant.polys.euclidtools.**dmp_ff_prs_gcd**(f, g, u, K)

Computes polynomial GCD using subresultants over a field.

Returns `(h, cff, cfg)` such that `a = gcd(f, g)`, `cff = quo(f, h)`, and `cfg = quo(g, h)`.

Examples

```
>>> R, x, y = ring("x y", QQ)
```

```
>>> f = x**2/2 + x*y + y**2/2
>>> g = x**2 + x*y
```

```
>>> R.dmp_ff_prs_gcd(f, g)
(x + y, 1/2*x + 1/2*y, x)
```

diofant.polys.euclidtools.**dmp_zz_heu_gcd**(f, g, u, K)

Heuristic polynomial GCD in $Z[X]$.

Given univariate polynomials f and g in $Z[X]$, returns their GCD and cofactors, i.e. polynomials `h`, `cff` and `cfg` such that:

```
h = gcd(f, g), cff = quo(f, h) and cfg = quo(g, h)
```

The algorithm is purely heuristic which means it may fail to compute the GCD. This will be signaled by raising an exception. In this case you will need to switch to another GCD method.

The algorithm computes the polynomial GCD by evaluating polynomials f and g at certain points and computing (fast) integer GCD of those evaluations. The polynomial GCD is recovered from the integer image by interpolation. The evaluation proces reduces f and g variable by variable into a large integer. The final step is to verify if the interpolated polynomial is the correct GCD. This gives cofactors of the input polynomials as a side effect.

References

[R18] (page 1269)

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x**2 + 2*x*y + y**2
>>> g = x**2 + x*y
```

```
>>> R.dmp_zz_heu_gcd(f, g)
(x + y, x + y, x)
```

diofant.polys.euclidtools.**dmp_qq_heu_gcd**(*f, g, u, K0*)

Heuristic polynomial GCD in $Q[X]$.

Returns (h, cff, cfg) such that a = gcd(f, g), cff = quo(f, h), and cfg = quo(g, h).

Examples

```
>>> R, x, y = ring("x y", QQ)
```

```
>>> f = x**2/4 + x*y + y**2
>>> g = x**2/2 + x*y
```

```
>>> R.dmp_qq_heu_gcd(f, g)
(x + 2*y, 1/4*x + 1/2*y, 1/2*x)
```

diofant.polys.euclidtools.**dmp_inner_gcd**(*f, g, u, K*)

Computes polynomial GCD and cofactors of f and g in $K[X]$.

Returns (h, cff, cfg) such that a = gcd(f, g), cff = quo(f, h), and cfg = quo(g, h).

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x**2 + 2*x*y + y**2
>>> g = x**2 + x*y
```

```
>>> R.dmp_inner_gcd(f, g)
(x + y, x + y, x)
```

diofant.polys.euclidtools.**dmp_gcd**(f, g, u, K)
Computes polynomial GCD of f and g in $K[X]$.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x**2 + 2*x*y + y**2
>>> g = x**2 + x*y
```

```
>>> R.dmp_gcd(f, g)
x + y
```

diofant.polys.euclidtools.**dmp_lcm**(f, g, u, K)
Computes polynomial LCM of f and g in $K[X]$.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> f = x**2 + 2*x*y + y**2
>>> g = x**2 + x*y
```

```
>>> R.dmp_lcm(f, g)
x**3 + 2*x**2*y + x*y**2
```

diofant.polys.euclidtools.**dmp_content**(f, u, K)
Returns GCD of multivariate coefficients.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_content(2*x*y + 6*x + 4*y + 12)
2*y + 6
```

diofant.polys.euclidtools.**dmp_primitive**(f, u, K)
Returns multivariate content and a primitive polynomial.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_primitive(2*x*y + 6*x + 4*y + 12)
(2*y + 6, x + 2)
```

diofant.polys.euclidtools.**dmp_cancel**(*f, g, u, K, include=True*)
Cancel common factors in a rational function f/g.

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_cancel(2*x**2 - 2, x**2 - 2*x + 1)
(2*x + 2, x - 1)
```

Polynomial factorization in characteristic zero:

diofant.polys.factortools.**dmp_trial_division**(*f, factors, u, K*)
Determine multiplicities of factors using trial division.

diofant.polys.factortools.**dmp_zz_mignotte_bound**(*f, u, K*)
Mignotte bound for multivariate polynomials in $K[X]$.

diofant.polys.factortools.**dup_zz_hensel_step**(*m, f, g, h, s, t, K*)
One step in Hensel lifting in $Z[x]$.

Given positive integer m and $Z[x]$ polynomials f, g, h, s and t such that:

```
f == g*h (mod m)
s*g + t*h == 1 (mod m)

lc(f) is not a zero divisor (mod m)
lc(h) == 1

deg(f) == deg(g) + deg(h)
deg(s) < deg(h)
deg(t) < deg(g)
```

returns polynomials G, H, S and T, such that:

```
f == G*H (mod m**2)
S*G + T**H == 1 (mod m**2)
```

References

[R19] (page 1269)

diofant.polys.factortools.**dup_zz_hensel_lift**(*p, f, f_list, l, K*)
Multifactor Hensel lifting in $Z[x]$.

Given a prime p, polynomial f over $Z[x]$ such that $lc(f)$ is a unit modulo p, monic pair-wise coprime polynomials f_i over $Z[x]$ satisfying:

```
f = lc(f) f_1 ... f_r (mod p)
```

and a positive integer l, returns a list of monic polynomials F_1, F_2, ..., F_r satisfying:

```
f = lc(f) F_1 ... F_r (mod p**l)

F_i = f_i (mod p), i = 1..r
```

References

[R20] (page 1269)

diofant.polys.factortools.**dup_zz_zassenhaus**(*f*, *K*)
Factor primitive square-free polynomials in $Z[x]$.

diofant.polys.factortools.**dup_zz_irreducible_p**(*f*, *K*)
Test irreducibility using Eisenstein's criterion.

diofant.polys.factortools.**dup_cyclotomic_p**(*f*, *K*, *irreducible=False*)
Efficiently test if `f` is a cyclotomic polynomial.

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> f = x**16 + x**14 - x**10 + x**8 - x**6 + x**2 + 1
>>> R.dup_cyclotomic_p(f)
False
```

```
>>> g = x**16 + x**14 - x**10 - x**8 - x**6 + x**2 + 1
>>> R.dup_cyclotomic_p(g)
True
```

diofant.polys.factortools.**dup_zz_cyclotomic_poly**(*n*, *K*)
Efficiently generate n-th cyclotomic polynomial.

diofant.polys.factortools.**dup_zz_cyclotomic_factor**(*f*, *K*)
Efficiently factor polynomials $x**n - 1$ and $x**n + 1$ in $Z[x]$.

Given a univariate polynomial f in $Z[x]$ returns a list of factors of f, provided that f is in the form $x**n - 1$ or $x**n + 1$ for $n >= 1$. Otherwise returns None.

Factorization is performed using using cyclotomic decomposition of f, which makes this method much faster that any other direct factorization approach (e.g. Zassenhaus's).

References

[R21] (page 1269)

diofant.polys.factortools.**dup_zz_factor_sqf**(*f*, *K*)
Factor square-free (non-primitive) polynomials in $Z[x]$.

diofant.polys.factortools.**dup_zz_factor**(*f*, *K*)

Factor (non square-free) polynomials in $Z[x]$.

Given a univariate polynomial f in $Z[x]$ computes its complete factorization $f_1, ..., f_n$ into irreducibles over integers:

```
f = content(f) f_1**k_1 ... f_n**k_n
```

The factorization is computed by reducing the input polynomial into a primitive square-free polynomial and factoring it using Zassenhaus algorithm. Trial division is used to recover the multiplicities of factors.

The result is returned as a tuple consisting of:

```
(content(f), [(f_1, k_1), ..., (f_n, k_n))
```

References

[R22] (page 1269)

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> R.dup_zz_factor(2*x**4 - 2)
(2, [(x - 1, 1), (x + 1, 1), (x**2 + 1, 1)])
```

Note that this is a complete factorization over integers, however over Gaussian integers we can factor the last term.

By default, polynomials $x**n-1$ and $x**n+1$ are factored using cyclotomic decomposition to speedup computations. To disable this behaviour set cyclotomic=False.

diofant.polys.factortools.**dmp_zz_wang_non_divisors**(*E*, *cs*, *ct*, *K*)

Wang/EEZ: Compute a set of valid divisors.

diofant.polys.factortools.**dmp_zz_wang_test_points**(*f*, *T*, *ct*, *A*, *u*, *K*)

Wang/EEZ: Test evaluation points for suitability.

diofant.polys.factortools.**dmp_zz_wang_lead_coeffs**(*f*, *T*, *cs*, *E*, *H*, *A*, *u*, *K*)

Wang/EEZ: Compute correct leading coefficients.

diofant.polys.factortools.**dmp_zz_diophantine**(*F*, *c*, *A*, *d*, *p*, *u*, *K*)

Wang/EEZ: Solve multivariate Diophantine equations.

diofant.polys.factortools.**dmp_zz_wang_hensel_lifting**(*f*, *H*, *LC*, *A*, *p*, *u*, *K*)

Wang/EEZ: Parallel Hensel lifting algorithm.

diofant.polys.factortools.**dmp_zz_wang**(*f*, *u*, *K*, *mod=None*, *seed=None*)

Factor primitive square-free polynomials in $Z[X]$.

Given a multivariate polynomial f in $Z[x_1, ..., x_n]$, which is primitive and square-free in x_1, computes factorization of f into irreducibles over integers.

The procedure is based on Wang's Enhanced Extended Zassenhaus algorithm. The algorithm works by viewing f as a univariate polynomial in $Z[x_2, ..., x_n][x_1]$, for which an evaluation mapping is computed:

```
x_2 -> a_2, ..., x_n -> a_n
```

where a_i, for $i = 2, ..., n$, are carefully chosen integers. The mapping is used to transform f into a univariate polynomial in $Z[x_1]$, which can be factored efficiently using Zassenhaus algorithm. The last step is to lift univariate factors to obtain true multivariate factors. For this purpose a parallel Hensel lifting procedure is used.

The parameter `seed` is passed to _randint and can be used to seed randint (when an integer) or (for testing purposes) can be a sequence of numbers.

References

[R23] (page 1269), *[R24]* (page 1269)

`diofant.polys.factortools.`**`dmp_zz_factor`**`(`*f*, *u*, *K*`)`

Factor (non square-free) polynomials in $Z[X]$.

Given a multivariate polynomial f in $Z[x]$ computes its complete factorization $f_1, ..., f_n$ into irreducibles over integers:

```
f = content(f) f_1**k_1 ... f_n**k_n
```

The factorization is computed by reducing the input polynomial into a primitive square-free polynomial and factoring it using Enhanced Extended Zassenhaus (EEZ) algorithm. Trial division is used to recover the multiplicities of factors.

The result is returned as a tuple consisting of:

```
(content(f), [(f_1, k_1), ..., (f_n, k_n))
```

References

[R25] (page 1269)

Examples

```
>>> R, x, y = ring("x y", ZZ)
```

```
>>> R.dmp_zz_factor(2*x**2 - 2*y**2)
(2, [(x - y, 1), (x + y, 1)])
```

`diofant.polys.factortools.`**`dmp_ext_factor`**`(`*f*, *u*, *K*`)`

Factor multivariate polynomials over algebraic number fields.

`diofant.polys.factortools.`**`dup_gf_factor`**`(`*f*, *K*`)`

Factor univariate polynomials over finite fields.

`diofant.polys.factortools.`**`dmp_factor_list`**`(`*f*, *u*, *K0*`)`

Factor polynomials into irreducibles in $K[X]$.

`diofant.polys.factortools.`**`dmp_factor_list_include`**`(`*f*, *u*, *K*`)`

Factor polynomials into irreducibles in $K[X]$.

`diofant.polys.factortools.`**`dmp_irreducible_p`**`(`*f*, *u*, *K*`)`

Returns `True` if `f` has no factors over its domain.

Gröbner basis algorithms

Gröbner bases can be used to answer many problems in computational commutative algebra. Their computation in rather complicated, and very performance-sensitive. We present here various low-level implementations of Gröbner basis computation algorithms; please see the previous section of the manual for usage.

`diofant.polys.groebnertools.`**`groebner`**`(`*seq*, *ring*, *method=None*`)`

Computes Gröbner basis for a set of polynomials in $K[X]$.

Wrapper around the (default) improved Buchberger and the other algorithms for computing Gröbner bases. The choice of algorithm can be changed via `method` argument or *`setup()`* (page 1152), where `method` can be either `buchberger` or `f5b`.

`diofant.polys.groebnertools.`**`buchberger`**`(`*f*, *ring*`)`

Computes Gröbner basis for a set of polynomials in $K[X]$.

Given a set of multivariate polynomials F, finds another set G, such that Ideal $F = Ideal G$ and G is a reduced Gröbner basis.

The resulting basis is unique and has monic generators if the ground domains is a field. Otherwise the result is non-unique but Gröbner bases over e.g. integers can be computed (if the input polynomials are monic).

Gröbner bases can be used to choose specific generators for a polynomial ideal. Because these bases are unique you can check for ideal equality by comparing the Gröbner bases. To see if one polynomial lies in an ideal, divide by the elements in the base and see if the remainder vanishes.

They can also be used to solve systems of polynomial equations as, by choosing lexicographic ordering, you can eliminate one variable at a time, provided that the ideal is zero-dimensional (finite number of solutions).

Notes

Used an improved version of Buchberger's algorithm as presented in *[R30]* (page 1269).

References

[R26] (page 1269), *[R27]* (page 1269), *[R28]* (page 1269), *[R29]* (page 1269), *[R30]* (page 1269)

`diofant.polys.groebnertools.`**`f5b`**`(`*F*, *ring*`)`

Computes a reduced Gröbner basis for the ideal generated by F.

f5b is an implementation of the F5B algorithm by Yao Sun and Dingkang Wang. Similarly to Buchberger's algorithm, the algorithm proceeds by computing critical pairs, computing the S-polynomial, reducing it and adjoining the reduced S-polynomial if it is not 0.

Unlike Buchberger's algorithm, each polynomial contains additional information, namely a signature and a number. The signature specifies the path of computation (i.e. from which polynomial in the original basis was it derived and how), the number says when the polynomial was added to the basis. With this information it is (often) possible to decide if an S-polynomial will reduce to 0 and can be discarded.

Optimizations include: Reducing the generators before computing a Gröbner basis, removing redundant critical pairs when a new polynomial enters the basis and sorting the critical pairs and the current basis.

Once a Gröbner basis has been found, it gets reduced.

References

[R31] (page 1269), *[R32]* (page 1269)

`diofant.polys.groebnertools.`**`spoly`**(*p1*, *p2*)

Compute LCM(LM(p1), LM(p2))/LM(p1)*p1 - LCM(LM(p1), LM(p2))/LM(p2)*p2.

This is the S-poly, provided p1 and p2 are monic

`diofant.polys.groebnertools.`**`red_groebner`**(*G*, *ring*)

Compute reduced Gröbner basis *[R33]* (page 1269).

Selects a subset of generators, that already generate the ideal and computes a reduced Gröbner basis for them.

References

[R33] (page 1269)

`diofant.polys.groebnertools.`**`is_groebner`**(*G*)

Check if G is a Gröbner basis.

`diofant.polys.groebnertools.`**`is_minimal`**(*G*, *ring*)

Checks if G is a minimal Gröbner basis.

`diofant.polys.fglmtools.`**`matrix_fglm`**(*F*, *ring*, *O_to*)

Converts the reduced Gröbner basis `F` of a zero-dimensional ideal w.r.t. `O_from` to a reduced Gröbner basis w.r.t. `O_to`.

References

[R34] (page 1269)

Gröbner basis algorithms for modules are also provided:

`diofant.polys.distributedmodules.`**`sdm_spoly`**(*f*, *g*, *O*, *K*, *phantom=None*)

Compute the generalized s-polynomial of `f` and `g`.

The ground field is assumed to be `K`, and monomials ordered according to `O`.

This is invalid if either of `f` or `g` is zero.

If the leading terms of f and g involve different basis elements of F, their s-poly is defined to be zero. Otherwise it is a certain linear combination of f and g in which the leading terms cancel. See [SCA, defn 2.3.6] for details.

If `phantom` is not `None`, it should be a pair of module elements on which to perform the same operation(s) as on `f` and `g`. The in this case both results are returned.

Examples

```
>>> f = [((2, 1, 1), QQ(1)), ((1, 0, 1), QQ(1))]
>>> g = [((2, 3, 0), QQ(1))]
>>> h = [((1, 2, 3), QQ(1))]
>>> sdm_spoly(f, h, lex, QQ)
[]
>>> sdm_spoly(f, g, lex, QQ)
[((1, 2, 1), 1)]
```

diofant.polys.distributedmodules.**sdm_ecart**(*f*)

Compute the ecart of `f`.

This is defined to be the difference of the total degree of f and the total degree of the leading monomial of f [SCA, defn 2.3.7].

Invalid if f is zero.

Examples

```
>>> sdm_ecart([((1, 2, 3), 1), ((1, 0, 1), 1)])
0
>>> sdm_ecart([((2, 2, 1), 1), ((1, 5, 1), 1)])
3
```

diofant.polys.distributedmodules.**sdm_nf_mora**(*f, G, O, K, phantom=None*)

Compute a weak normal form of `f` with respect to `G` and order `O`.

The ground field is assumed to be `K`, and monomials ordered according to `O`.

Weak normal forms are defined in [SCA, defn 2.3.3]. They are not unique. This function deterministically computes a weak normal form, depending on the order of G.

The most important property of a weak normal form is the following: if R is the ring associated with the monomial ordering (if the ordering is global, we just have $R = K[x_1, \ldots, x_n]$, otherwise it is a certain localization thereof), I any ideal of R and G a standard basis for I, then for any $f \in R$, we have $f \in I$ if and only if $NF(f|G) = 0$.

This is the generalized Mora algorithm for computing weak normal forms with respect to arbitrary monomial orders [SCA, algorithm 2.3.9].

If `phantom` is not `None`, it should be a pair of "phantom" arguments on which to perform the same computations as on `f`, `G`, both results are then returned.

diofant.polys.distributedmodules.**sdm_groebner**(*G, NF, O, K, extended=False*)

Compute a minimal standard basis of `G` with respect to order `O`.

The algorithm uses a normal form `NF`, for example `sdm_nf_mora`. The ground field is assumed to be `K`, and monomials ordered according to `O`.

Let N denote the submodule generated by elements of G. A standard basis for N is a subset S of N, such that $in(S) = in(N)$, where for any subset X of F, $in(X)$ denotes the submodule generated by the initial forms of elements of X. [SCA, defn 2.3.2]

A standard basis is called minimal if no subset of it is a standard basis.

One may show that standard bases are always generating sets.

Minimal standard bases are not unique. This algorithm computes a deterministic result, depending on the particular order of G.

If `extended=True`, also compute the transition matrix from the initial generators to the groebner basis. That is, return a list of coefficient vectors, expressing the elements of the groebner basis in terms of the elements of `G`.

This functions implements the "sugar" strategy, see

Giovini et al: "One sugar cube, please" OR Selection strategies in Buchberger algorithm.

4.1.3 Exceptions

These are exceptions defined by the polynomials module.

TODO sort and explain

class diofant.polys.polyerrors.**BasePolynomialError**
Base class for polynomial related exceptions.

class diofant.polys.polyerrors.**ExactQuotientFailed**(*f, g, dom=None*)

class diofant.polys.polyerrors.**OperationNotSupported**(*poly, func*)

class diofant.polys.polyerrors.**HeuristicGCDFailed**

class diofant.polys.polyerrors.**HomomorphismFailed**

class diofant.polys.polyerrors.**IsomorphismFailed**

class diofant.polys.polyerrors.**ExtraneousFactors**

class diofant.polys.polyerrors.**EvaluationFailed**

class diofant.polys.polyerrors.**RefinementFailed**

class diofant.polys.polyerrors.**CoercionFailed**

class diofant.polys.polyerrors.**NotInvertible**

class diofant.polys.polyerrors.**NotReversible**

class diofant.polys.polyerrors.**NotAlgebraic**

class diofant.polys.polyerrors.**DomainError**

class diofant.polys.polyerrors.**PolynomialError**

class diofant.polys.polyerrors.**UnificationFailed**

class diofant.polys.polyerrors.**GeneratorsNeeded**

class diofant.polys.polyerrors.**ComputationFailed**(*func, nargs, exc*)

class diofant.polys.polyerrors.**GeneratorsError**

class diofant.polys.polyerrors.**UnivariatePolynomialError**

class diofant.polys.polyerrors.**MultivariatePolynomialError**

class diofant.polys.polyerrors.**PolificationFailed**(*opt, origs, exprs, seq=False*)

class diofant.polys.polyerrors.**OptionError**

class diofant.polys.polyerrors.**FlagError**

4.1.4 Reference

Modular GCD

diofant.polys.modulargcd.**modgcd_univariate**(*f*, *g*)

Computes the GCD of two polynomials in $\mathbb{Z}[x]$ using a modular algorithm.

The algorithm computes the GCD of two univariate integer polynomials f and g by computing the GCD in $\mathbb{Z}_p[x]$ for suitable primes p and then reconstructing the coefficients with the Chinese Remainder Theorem. Trial division is only made for candidates which are very likely the desired GCD.

Parameters f : PolyElement

univariate integer polynomial

g : PolyElement

univariate integer polynomial

Returns h : PolyElement

GCD of the polynomials f and g

cff : PolyElement

cofactor of f, i.e. $\frac{f}{h}$

cfg : PolyElement

cofactor of g, i.e. $\frac{g}{h}$

References

[R35] (page 1269)

Examples

```
>>> R, x = ring("x", ZZ)
```

```
>>> f = x**5 - 1
>>> g = x - 1
```

```
>>> h, cff, cfg = modgcd_univariate(f, g)
>>> h, cff, cfg
(x - 1, x**4 + x**3 + x**2 + x + 1, 1)
```

```
>>> cff * h == f
True
>>> cfg * h == g
True
```

```
>>> f = 6*x**2 - 6
>>> g = 2*x**2 + 4*x + 2
```

```
>>> h, cff, cfg = modgcd_univariate(f, g)
>>> h, cff, cfg
(2*x + 2, 3*x - 3, x + 1)
```

```
>>> cff * h == f
True
>>> cfg * h == g
True
```

diofant.polys.modulargcd.**modgcd_bivariate**(*f*, *g*)

Computes the GCD of two polynomials in $\mathbb{Z}[x, y]$ using a modular algorithm.

The algorithm computes the GCD of two bivariate integer polynomials f and g by calculating the GCD in $\mathbb{Z}_p[x, y]$ for suitable primes p and then reconstructing the coefficients with the Chinese Remainder Theorem. To compute the bivariate GCD over $\mathbb{Z}_p$, the polynomials $f \mod p$ and $g \mod p$ are evaluated at $y = a$ for certain $a \in \mathbb{Z}_p$ and then their univariate GCD in $\mathbb{Z}_p[x]$ is computed. Interpolating those yields the bivariate GCD in $\mathbb{Z}_p[x, y]$. To verify the result in $\mathbb{Z}[x, y]$, trial division is done, but only for candidates which are very likely the desired GCD.

Parameters f : PolyElement

bivariate integer polynomial

g : PolyElement

bivariate integer polynomial

Returns h : PolyElement

GCD of the polynomials f and g

cff : PolyElement

cofactor of f, i.e. $\frac{f}{h}$

cfg : PolyElement

cofactor of g, i.e. $\frac{g}{h}$

References

[R36] (page 1269)

Examples

```
>>> R, x, y = ring("x, y", ZZ)
```

```
>>> f = x**2 - y**2
>>> g = x**2 + 2*x*y + y**2
```

```
>>> h, cff, cfg = modgcd_bivariate(f, g)
>>> h, cff, cfg
(x + y, x - y, x + y)
```

```
>>> cff * h == f
True
>>> cfg * h == g
True
```

```
>>> f = x**2*y - x**2 - 4*y + 4
>>> g = x + 2
```

```
>>> h, cff, cfg = modgcd_bivariate(f, g)
>>> h, cff, cfg
(x + 2, x*y - x - 2*y + 2, 1)
```

```
>>> cff * h == f
True
>>> cfg * h == g
True
```

diofant.polys.modulargcd.**modgcd_multivariate**(*f*, *g*)

Compute the GCD of two polynomials in $\mathbb{Z}[x_0, \ldots, x_{k-1}]$ using a modular algorithm.

The algorithm computes the GCD of two multivariate integer polynomials f and g by calculating the GCD in $\mathbb{Z}_p[x_0, \ldots, x_{k-1}]$ for suitable primes p and then reconstructing the coefficients with the Chinese Remainder Theorem. To compute the multivariate GCD over $\mathbb{Z}_p$ the recursive subroutine `_modgcd_multivariate_p` is used. To verify the result in $\mathbb{Z}[x_0, \ldots, x_{k-1}]$, trial division is done, but only for candidates which are very likely the desired GCD.

Parameters **f** : PolyElement

multivariate integer polynomial

g : PolyElement

multivariate integer polynomial

Returns **h** : PolyElement

GCD of the polynomials f and g

cff : PolyElement

cofactor of f, i.e. $\frac{f}{h}$

cfg : PolyElement

cofactor of g, i.e. $\frac{g}{h}$

See also:

_modgcd_multivariate_p (page 1147)

References

[R37] (page 1269), *[R38]* (page 1269)

Examples

```
>>> R, x, y = ring("x, y", ZZ)
```

```
>>> f = x**2 - y**2
>>> g = x**2 + 2*x*y + y**2
```

```
>>> h, cff, cfg = modgcd_multivariate(f, g)
>>> h, cff, cfg
(x + y, x - y, x + y)
```

```
>>> cff * h == f
True
>>> cfg * h == g
True
```

```
>>> R, x, y, z = ring("x, y, z", ZZ)
```

```
>>> f = x*z**2 - y*z**2
>>> g = x**2*z + z
```

```
>>> h, cff, cfg = modgcd_multivariate(f, g)
>>> h, cff, cfg
(z, x*z - y*z, x**2 + 1)
```

```
>>> cff * h == f
True
>>> cfg * h == g
True
```

diofant.polys.modulargcd.**func_field_modgcd**(*f*, *g*)

Compute the GCD of two polynomials f and g in $\mathbb{Q}(\alpha)[x_0, \ldots, x_{n-1}]$ using a modular algorithm.

The algorithm first computes the primitive associate $\check{m}_\alpha(z)$ of the minimal polynomial m_α in $\mathbb{Z}[z]$ and the primitive associates of f and g in $\mathbb{Z}[x_1, \ldots, x_{n-1}][z]/(\check{m}_\alpha)[x_0]$. Then it computes the GCD in $\mathbb{Q}(x_1, \ldots, x_{n-1})[z]/(m_\alpha(z))[x_0]$. This is done by calculating the GCD in $\mathbb{Z}_p(x_1, \ldots, x_{n-1})[z]/(\check{m}_\alpha(z))[x_0]$ for suitable primes p and then reconstructing the coefficients with the Chinese Remainder Theorem and Rational Reconstuction. The GCD over $\mathbb{Z}_p(x_1, \ldots, x_{n-1})[z]/(\check{m}_\alpha(z))[x_0]$ is computed with a recursive subroutine, which evaluates the polynomials at $x_{n-1} = a$ for suitable evaluation points $a \in \mathbb{Z}_p$ and then calls itself recursively until the ground domain does no longer contain any parameters. For $\mathbb{Z}_p[z]/(\check{m}_\alpha(z))[x_0]$ the Euclidean Algorithm is used. The results of those recursive calls are then interpolated and Rational Function Reconstruction is used to obtain the correct coefficients. The results, both in $\mathbb{Q}(x_1, \ldots, x_{n-1})[z]/(m_\alpha(z))[x_0]$ and $\mathbb{Z}_p(x_1, \ldots, x_{n-1})[z]/(\check{m}_\alpha(z))[x_0]$, are verified by a fraction free trial division.

Apart from the above GCD computation some GCDs in $\mathbb{Q}(\alpha)[x_1, \ldots, x_{n-1}]$ have to be calculated, because treating the polynomials as univariate ones can result in a spurious content of the GCD. For this `func_field_modgcd` is called recursively.

Parameters f, g : PolyElement

polynomials in $\mathbb{Q}(\alpha)[x_0, \ldots, x_{n-1}]$

Returns h : PolyElement

monic GCD of the polynomials f and g

cff : PolyElement

cofactor of f, i.e. $\frac{f}{h}$

cfg : PolyElement

cofactor of g, i.e. $\frac{g}{h}$

References

[R39] (page 1269)

Examples

```
>>> A = AlgebraicField(QQ, sqrt(2))
>>> R, x = ring('x', A)
```

```
>>> f = x**2 - 2
>>> g = x + sqrt(2)
```

```
>>> h, cff, cfg = func_field_modgcd(f, g)
```

```
>>> h == x + sqrt(2)
True
>>> cff * h == f
True
>>> cfg * h == g
True
```

```
>>> R, x, y = ring('x, y', A)
```

```
>>> f = x**2 + 2*sqrt(2)*x*y + 2*y**2
>>> g = x + sqrt(2)*y
```

```
>>> h, cff, cfg = func_field_modgcd(f, g)
```

```
>>> h == x + sqrt(2)*y
True
>>> cff * h == f
True
>>> cfg * h == g
True
```

```
>>> f = x + sqrt(2)*y
>>> g = x + y
```

```
>>> h, cff, cfg = func_field_modgcd(f, g)
```

```
>>> h == R.one
True
>>> cff * h == f
True
>>> cfg * h == g
True
```

diofant.polys.modulargcd.**_modgcd_multivariate_p**(*f*, *g*, *p*, *degbound*, *contbound*)
Compute the GCD of two polynomials in $\mathbb{Z}_p[x0, \ldots, xk-1]$.

The algorithm reduces the problem step by step by evaluating the polynomials f and g at $x_{k-1} = a$ for suitable $a \in \mathbb{Z}_p$ and then calls itself recursively to compute the GCD in $\mathbb{Z}_p[x_0, \ldots, x_{k-2}]$. If these recursive calls are successful for enough evaluation points, the GCD in k variables is interpolated, otherwise the algorithm returns `None`. Every time a GCD or a content is computed, their degrees are compared with the bounds. If a degree greater then the bound is encountered, then the current call returns `None` and a new evaluation point has to be chosen. If at some point the degree is smaller, the correspondent bound is updated and the algorithm fails.

Parameters f : PolyElement

multivariate integer polynomial with coefficients in $\mathbb{Z}_p$

g : PolyElement

multivariate integer polynomial with coefficients in $\mathbb{Z}_p$

p : Integer

prime number, modulus of f and g

degbound : list of Integer objects

`degbound[i]` is an upper bound for the degree of the GCD of f and g in the variable x_i

contbound : list of Integer objects

`contbound[i]` is an upper bound for the degree of the content of the GCD in $\mathbb{Z}_p[x_i][x_0, \ldots, x_{i-1}]$, `contbound[0]` is not used can therefore be chosen arbitrarily.

Returns h : PolyElement

GCD of the polynomials f and g or `None`

References

[R40] (page 1270), *[R41]* (page 1270)

diofant.polys.modulargcd.**trial_division**(*f*, *h*, *minpoly*, *p=None*)
Check if h divides f in $\mathbb{K}[t_1, \ldots, t_k][z]/(m_{\alpha}(z))$, where $\mathbb{K}$ is either $\mathbb{Q}$ or $\mathbb{Z}_p$.

This algorithm is based on pseudo division and does not use any fractions. By default $\mathbb{K}$ is $\mathbb{Q}$, if a prime number p is given, $\mathbb{Z}_p$ is chosen instead.

Parameters f, h : PolyElement

polynomials in $\mathbb{Z}[t_1, \ldots, t_k][x, z]$

minpoly : PolyElement

polynomial $m_\alpha(z)$ in $\mathbb{Z}[t_1, \ldots, t_k][z]$

p : Integer or None

if p is given, $\mathbb{K}$ is set to $\mathbb{Z}_p$ instead of $\mathbb{Q}$, default is `None`

Returns rem : PolyElement

remainder of $\frac{f}{h}$

References

[R42] (page 1270)

`diofant.polys.modulargcd.`**`integer_rational_reconstruction`**`(`*c*, *m*, *domain*`)`

Reconstruct a rational number $\frac{a}{b}$ from

$$c = \frac{a}{b} \mod m,$$

where c and m are integers.

The algorithm is based on the Euclidean Algorithm. In general, m is not a prime number, so it is possible that b is not invertible modulo m. In that case `None` is returned.

Parameters c : Integer

$c = \frac{a}{b} \mod m$

m : Integer

modulus, not necessarily prime

domain : IntegerRing

a, b, c are elements of `domain`

Returns frac : Rational

either $\frac{a}{b}$ in $\mathbb{Q}$ or `None`

References

[R43] (page 1270)

Manipulation of power series

Functions in this module carry the prefix `rs_`, standing for "ring series". They manipulate finite power series in the sparse representation provided by `polys.ring.ring`.

`diofant.polys.ring_series.`**`rs_trunc`**`(`*p1*, *x*, *prec*`)`

truncate the series in the `x` variable with precision `prec`, that is modulo `O(x**prec)`

Examples

```
>>> R, x = ring('x', QQ)
>>> p = x**10 + x**5 + x + 1
>>> rs_trunc(p, x, 12)
x**10 + x**5 + x + 1
>>> rs_trunc(p, x, 10)
x**5 + x + 1
```

`diofant.polys.ring_series.`**`rs_mul`**(*p1, p2, x, prec*)

product of series modulo `O(x**prec)`

`x` is the series variable or its position in the generators.

Examples

```
>>> R, x = ring('x', QQ)
>>> p1 = x**2 + 2*x + 1
>>> p2 = x + 1
>>> rs_mul(p1, p2, x, 3)
3*x**2 + 3*x + 1
```

`diofant.polys.ring_series.`**`rs_square`**(*p1, x, prec*)

square modulo `O(x**prec)`

Examples

```
>>> R, x = ring('x', QQ)
>>> p = x**2 + 2*x + 1
>>> rs_square(p, x, 3)
6*x**2 + 4*x + 1
```

`diofant.polys.ring_series.`**`rs_pow`**(*p1, n, x, prec*)

return `p1**n` modulo `O(x**prec)`

Examples

```
>>> R, x = ring('x', QQ)
>>> p = x + 1
>>> rs_pow(p, 4, x, 3)
6*x**2 + 4*x + 1
```

`diofant.polys.ring_series.`**`rs_series_inversion`**(*p, x, prec*)

multivariate series inversion `1/p` modulo `O(x**prec)`

Examples

```
>>> R, x, y = ring('x, y', QQ)
>>> rs_series_inversion(1 + x*y**2, x, 4)
-x**3*y**6 + x**2*y**4 - x*y**2 + 1
>>> rs_series_inversion(1 + x*y**2, y, 4)
-x*y**2 + 1
```

diofant.polys.ring_series.**rs_series_from_list**(*p, c, x, prec, concur=1*)

series sum `c[n]*p**n` modulo `O(x**prec)`

reduce the number of multiplication summing concurrently `ax = [1, p, p**2, .., p**(J - 1)]   s = sum(c[i]*ax[i] for i in range(r, (r + 1)*J))*p**((K - 1)*J)` with `K >= (n + 1)/J`

See also:

diofant.polys.rings.PolyElement.compose (page 1117)

Examples

```
>>> R, x = ring('x', QQ)
>>> p = x**2 + x + 1
>>> c = [1, 2, 3]
>>> rs_series_from_list(p, c, x, 4)
6*x**3 + 11*x**2 + 8*x + 6
>>> rs_trunc(1 + 2*p + 3*p**2, x, 4)
6*x**3 + 11*x**2 + 8*x + 6
>>> pc = R.from_list(list(reversed(c)))
>>> rs_trunc(pc.compose(x, p), x, 4)
6*x**3 + 11*x**2 + 8*x + 6
```

diofant.polys.ring_series.**rs_integrate**(*self, x*)

integrate p with respect to x

Examples

```
>>> R, x, y = ring('x, y', QQ)
>>> p = x + x**2*y**3
>>> rs_integrate(p, x)
1/3*x**3*y**3 + 1/2*x**2
```

diofant.polys.ring_series.**rs_log**(*p, x, prec*)

logarithm of p modulo `O(x**prec)`

Notes

truncation of `integral dx p**-1*d p/dx` is used.

Examples

```
>>> R, x = ring('x', QQ)
>>> rs_log(1 + x, x, 8)
1/7*x**7 - 1/6*x**6 + 1/5*x**5 - 1/4*x**4 + 1/3*x**3 - 1/2*x**2 + x
```

diofant.polys.ring_series.**rs_exp**(*p, x, prec*)

exponentiation of a series modulo `O(x**prec)`

Examples

```
>>> R, x = ring('x', QQ)
>>> rs_exp(x**2, x, 7)
1/6*x**6 + 1/2*x**4 + x**2 + 1
```

`diofant.polys.ring_series.`**`rs_newton`**(*p*, *x*, *prec*)
compute the truncated Newton sum of the polynomial `p`

Examples

```
>>> R, x = ring('x', QQ)
>>> p = x**2 - 2
>>> rs_newton(p, x, 5)
8*x**4 + 4*x**2 + 2
```

`diofant.polys.ring_series.`**`rs_hadamard_exp`**(*p1*, *inverse=False*)
return `sum f_i/i!*x**i` from `sum f_i*x**i`, where `x` is the first variable.

If `invers=True` return `sum f_i*i!*x**i`

Examples

```
>>> R, x = ring('x', QQ)
>>> p = 1 + x + x**2 + x**3
>>> rs_hadamard_exp(p)
1/6*x**3 + 1/2*x**2 + x + 1
```

`diofant.polys.ring_series.`**`rs_compose_add`**(*p1*, *p2*)
compute the composed sum `prod(p2(x - beta) for beta root of p1)`

References

[R44] (page 1270)

Examples

```
>>> R, x = ring('x', QQ)
>>> f = x**2 - 2
>>> g = x**2 - 3
>>> rs_compose_add(f, g)
x**4 - 10*x**2 + 1
```

4.1.5 Undocumented

Many parts of the polys module are still undocumented, and even where there is documentation it is scarce. Please contribute!

class `diofant.polys.polyoptions.`**`Order`**
`order` option to polynomial manipulation functions.

class diofant.polys.polyoptions.**Options**(*gens, args, flags=None, strict=False*)

Options manager for polynomial manipulation module.

Examples

```
>>> Options((x, y, z), {'domain': 'ZZ'})
{'auto': False, 'domain': ZZ, 'gens': (x, y, z)}
```

```
>>> build_options((x, y, z), {'domain': 'ZZ'})
{'auto': False, 'domain': ZZ, 'gens': (x, y, z)}
```

Options

- Expand — boolean option
- Gens — option
- Wrt — option
- Sort — option
- Order — option
- Field — boolean option
- Greedy — boolean option
- Domain — option
- Split — boolean option
- Gaussian — boolean option
- Extension — option
- Modulus — option
- Symmetric — boolean option
- Strict — boolean option

Flags

- Auto — boolean flag
- Frac — boolean flag
- Formal — boolean flag
- Polys — boolean flag
- Include — boolean flag
- All — boolean flag
- Gen — flag

diofant.polys.polyconfig.**setup**(*key, value=None*)

Assign a value to (or reset) a configuration item.

4.2 The Gruntz Algorithm

This section explains the basics of the algorithm [R1] (page 1270) used for computing limits. Most of the time the `limit()` (page 759) function should just work. However it is still useful to keep in mind how it is implemented in case something does not work as expected.

First we define an ordering on functions of single variable x according to how rapidly varying they at infinity. Any two functions $f(x)$ and $g(x)$ can be compared using the properties of:

$$L = \lim_{x\to\infty} \frac{\log|f(x)|}{\log|g(x)|}$$

We shall say that $f(x)$ *dominates* $g(x)$, written $f(x) \succ g(x)$, iff $L = \pm\infty$. We also say that $f(x)$ and $g(x)$ are *of the same comparability class* if neither $f(x) \succ g(x)$ nor $g(x) \succ f(x)$ and shall denote it as $f(x) \asymp g(x)$.

It is easy to show the following examples:

- $e^{e^x} \succ e^{x^2} \succ e^x \succ x \succ 42$
- $2 \asymp 3 \asymp -5$
- $x \asymp x^2 \asymp x^3 \asymp -x$
- $e^x \asymp e^{-x} \asymp e^{2x} \asymp e^{x+e^{-x}}$
- $f(x) \asymp 1/f(x)$

Using these definitions yields the following strategy for computing $\lim_{x\to\infty} f(x)$:

1. Given the function $f(x)$, we find the set of *most rapidly varying subexpressions* (MRV set) of it. All items of this set belongs to the same comparability class. Let's say it is $\{e^x, e^{2x}\}$.
2. Choose an expression ω which is positive and tends to zero and which is in the same comparability class as any element of the MRV set. Such element always exists. Then we rewrite the MRV set using ω, in our case $\{\omega^{-1}, \omega^{-2}\}$, and substitute it into $f(x)$.
3. Let $f(\omega)$ be the function which is obtained from $f(x)$ after the rewrite step above. Consider all expressions independent of ω as constants and compute the leading term of the power series of $f(\omega)$ around $\omega = 0^+$:

   $$f(\omega) = c_0\omega^{e_0} + c_1\omega^{e_1} + \ldots$$

   where $e_0 < e_1 < e_2 \ldots$
4. If the leading exponent $e_0 > 0$ then the limit is 0. If $e_0 < 0$, then the answer is $\pm\infty$ (depends on sign of c_0). Finally, if $e_0 = 0$, the limit is the limit of the leading coefficient c_0.

4.2.1 Notes

This exposition glossed over several details. For example, limits could be computed recursively (steps 1 and 4). Please address to the Gruntz thesis [R1] (page 1270) for proof of the termination (pp. 52-60).

4.2.2 References

diofant.series.gruntz.**compare**(*a*, *b*, *x*)
Determine order relation between two functons.

Returns {1, 0, -1}

Respectively, if $a(x) \succ b(x)$, $a(x) \asymp b(x)$ or $b(x) \succ a(x)$.

Examples

```
>>> x = Symbol('x', real=True, positive=True)
>>> m = Symbol('m', real=True, positive=True)
```

```
>>> compare(x, x**2, x)
0
>>> compare(1/x, x**m, x)
0
>>> compare(exp(x), exp(x**2), x)
-1
>>> compare(exp(x), x**5, x)
1
```

diofant.series.gruntz.**limitinf**(*e*, *x*)
Compute limit of the expression at the infinity.

Examples

```
>>> x = Symbol('x', real=True, positive=True)
```

```
>>> limitinf(exp(x)*(exp(1/x - exp(-x)) - exp(1/x)), x)
-1
>>> limitinf(x/log(x**(log(x**(log(2)/log(x))))), x)
oo
```

diofant.series.gruntz.**mrv**(*e*, *x*)
Calculate the MRV set of expression.

Examples

```
>>> x = Symbol('x', real=True, positive=True)
```

```
>>> mrv(log(x - log(x))/log(x), x)
{x}
>>> mrv(exp(x + exp(-x)), x)
{E**(-x), E**(x + E**(-x))}
```

diofant.series.gruntz.**mrv_leadterm**(*e*, *x*)
Compute the leading term of the series.

Returns tuple

The leading term $c_0 w^{e_0}$ of the series of e in terms of the most rapidly varying subexpression w in form of the pair `(c0, e0)` of Expr.

Examples

```
>>> x = Symbol('x', real=True, positive=True)
```

```
>>> mrv_leadterm(1/exp(-x + exp(-x)) - exp(x), x)
(-1, 0)
```

`diofant.series.gruntz.`**`mrv_max`**(*f, g, x*)

Computes the maximum of two MRV sets.

`diofant.series.gruntz.`**`rewrite`**(*e, x, w*)

Rewrites expression in terms of the most rapidly varying subexpression.

Parameters **e** : Expr

an expression

x : Symbol

variable of the e

w : Symbol

The symbol which is going to be used for substitution in place of the most rapidly varying in x subexpression.

Returns tuple

A pair: rewritten (in w) expression and $\log(w)$.

Examples

```
>>> x = Symbol('x', real=True, positive=True)
>>> m = Symbol('m', real=True, positive=True)
```

```
>>> rewrite(exp(x), x, m)
(1/m, -x)
>>> rewrite(exp(x)*log(log(exp(x))), x, m)
(log(x)/m, -x)
```

`diofant.series.gruntz.`**`sign`**(*e, x*)

Determine a sign of an expression at infinity.

Returns {1, 0, -1}

One or minus one, if $e > 0$ or $e < 0$ for x sufficiently large and zero if e is *constantly* zero for $x \to \infty$.

The result of this function is currently undefined if e changes sign arbitrarily often at infinity (e.g. $\sin(x)$).

4.3 Details on the Hypergeometric Function Expansion

This page describes how the function *hyperexpand()* (page 799) and related code work. For usage, see the documentation of the symplify module.

4.3.1 Hypergeometric Function Expansion Algorithm

This section describes the algorithm used to expand hypergeometric functions. Most of it is based on the papers *[Roach1996]* (page 1270) and *[Roach1997]* (page 1270).

Recall that the hypergeometric function is (initially) defined as

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| z \right) = \sum_{n=0}^\infty \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}. $$

It turns out that there are certain differential operators that can change the a_p and p_q parameters by integers. If a sequence of such operators is known that converts the set of indices a_r^0 and b_s^0 into a_p and b_q, then we shall say the pair a_p, b_q is reachable from a_r^0, b_s^0. Our general strategy is thus as follows: given a set a_p, b_q of parameters, try to look up an origin a_r^0, b_s^0 for which we know an expression, and then apply the sequence of differential operators to the known expression to find an expression for the Hypergeometric function we are interested in.

Notation

In the following, the symbol a will always denote a numerator parameter and the symbol b will always denote a denominator parameter. The subscripts p, q, r, s denote vectors of that length, so e.g. a_p denotes a vector of p numerator parameters. The subscripts i and j denote "running indices", so they should usually be used in conjuction with a "for all i". E.g. $a_i < 4$ for all i. Uppercase subscripts I and J denote a chosen, fixed index. So for example $a_I > 0$ is true if the inequality holds for the one index I we are currently interested in.

Incrementing and decrementing indices

Suppose $a_i \ne 0$. Set $A(a_i) = \frac{z}{a_i}\frac{\mathrm{d}}{dz}+1$. It is then easy to show that $A(a_i) {}_p F_q\left({a_p \atop b_q} \middle| z \right) = {}_p F_q\left({a_p + e_i \atop b_q} \middle| z \right)$, where e_i is the i-th unit vector. Similarly for $b_j \ne 1$ we set $B(b_j) = \frac{z}{b_j-1} \frac{\mathrm{d}}{dz} + 1$ and find $B(b_j) {}_p F_q\left({a_p \atop b_q} \middle| z \right) = {}_p F_q\left({a_p \atop b_q - e_i} \middle| z \right)$. Thus we can increment upper and decrement lower indices at will, as long as we don't go through zero. The $A(a_i)$ and $B(b_j)$ are called shift operators.

It is also easy to show that $\frac{\mathrm{d}}{dz} {}_p F_q\left({a_p \atop b_q} \middle| z \right) = \frac{a_1 \ldots a_p}{b_1 \ldots b_q} {}_p F_q\left({a_p + 1 \atop b_q + 1} \middle| z \right)$, where $a_p + 1$ is the vector $a_1 + 1, a_2 + 1, \ldots$ and similarly for $b_q + 1$. Combining this with the shift operators, we arrive at one form of the Hypergeometric differential equation: $\left[\frac{\mathrm{d}}{dz} \prod_{j=1}^q B(b_j) - \frac{a_1 \ldots a_p}{(b_1-1) \ldots (b_q-1)} \prod_{i=1}^p A(a_i) \right] {}_p F_q\left({a_p \atop b_q} \middle| z \right) = 0$. This holds if all shift operators are defined, i.e. if no $a_i = 0$ and no $b_j = 1$. Clearing denominators and multiplying through by z we arrive at the following equation: $\left[z\frac{\mathrm{d}}{dz} \prod_{j=1}^q \left(z\frac{\mathrm{d}}{dz} + b_j-1 \right) - z \prod_{i=1}^p \left(z\frac{\mathrm{d}}{\mathrm{d}z} + a_i \right) \right] {}_p F_q\left({a_p \atop b_q} \middle| z\right) = 0$. Even though our derivation does not show it, it can be checked that this equation holds whenever the ${}_p F_q$ is defined.

Notice that, under suitable conditions on a_I, b_J, each of the operators $A(a_i)$, $B(b_j)$ and $z\frac{\mathrm{d}}{\mathrm{d}z}$ can be expressed in terms of $A(a_I)$ or $B(b_J)$. Our next aim is to write the Hypergeometric

differential equation as follows: $[XA(a_I) - r]_p F_q \left({a_p \atop b_q} \middle| z \right) = 0$, for some operator X and some constant r to be determined. If $r \neq 0$, then we can write this as $\frac{-1}{r} X _p F_q \left({a_p + e_I \atop b_q} \middle| z \right) = {}_p F_q \left({a_p \atop b_q} \middle| z \right)$, and so $\frac{-1}{r} X$ undoes the shifting of $A(a_I)$, whence it will be called an inverse-shift operator.

Now $A(a_I)$ exists if $a_I \neq 0$, and then $z \frac{\mathrm{d}}{\mathrm{d}z} = a_I A(a_I) - a_I$. Observe also that all the operators $A(a_i)$, $B(b_j)$ and $z \frac{\mathrm{d}}{\mathrm{d}z}$ commute. We have $\prod_{i=1}^p \left(z \frac{\mathrm{d}}{\mathrm{d}z} + a_i \right) = \left(\prod_{i=1, i \neq I}^p \left(z \frac{\mathrm{d}}{\mathrm{d}z} + a_i \right) \right) a_I A(a_I)$, so this gives us the first half of X. The other half does not have such a nice expression. We find $z \frac{d}{dz} \prod_{j=1}^q \left(z \frac{d}{dz} + b_j - 1 \right) = (a_I A(a_I) - a_I) \prod_{j=1}^q (a_I A(a_I) - a_I + b_j - 1)$. Since the first half had no constant term, we infer $r = -a_I \prod_{j=1}^q (b_j - 1 - a_I)$.

This tells us under which conditions we can "un-shift" $A(a_I)$, namely when $a_I \neq 0$ and $r \neq 0$. Substituting $a_I - 1$ for a_I then tells us under what conditions we can decrement the index a_I. Doing a similar analysis for $B(a_J)$, we arrive at the following rules:

- An index a_I can be decremented if $a_I \neq 1$ and $a_I \neq b_j$ for all b_j.
- An index b_J can be incremented if $b_J \neq -1$ and $b_J \neq a_i$ for all a_i.

Combined with the conditions (stated above) for the existence of shift operators, we have thus established the rules of the game!

Reduction of Order

Notice that, quite trivially, if $a_I = b_J$, we have ${}_p F_q \left({a_p \atop b_q} \middle| z \right) = {}_{p-1} F_{q-1} \left({a_p^* \atop b_q^*} \middle| z \right)$, where a_p^* means a_p with a_I omitted, and similarly for b_q^*. We call this reduction of order.

In fact, we can do even better. If $a_I - b_J \in \mathbb{Z}_{>0}$, then it is easy to see that $\frac{(a_I)_n}{(b_J)_n}$ is actually a polynomial in n. It is also easy to see that $(z \frac{\mathrm{d}}{\mathrm{d}z})^k z^n = n^k z^n$. Combining these two remarks we find:

> If $a_I - b_J \in \mathbb{Z}_{>0}$, then there exists a polynomial $p(n) = p_0 + p_1 n + \ldots$ (of degree $a_I - b_J$) such that $\frac{(a_I)_n}{(b_J)_n} = p(n)$ and ${}_p F_q \left({a_p \atop b_q} \middle| z \right) = \left(p_0 + p_1 z \frac{\mathrm{d}}{\mathrm{d}z} + p_2 \left(z \frac{\mathrm{d}}{\mathrm{d}z} \right)^2 + \ldots \right) {}_{p-1} F_{q-1} \left({a_p^* \atop b_q^*} \middle| z \right)$.

Thus any set of parameters a_p, b_q is reachable from a set of parameters c_r, d_s where $c_i - d_j \in \mathbb{Z}$ implies $c_i < d_j$. Such a set of parameters c_r, d_s is called suitable. Our database of known formulae should only contain suitable origins. The reasons are twofold: firstly, working from suitable origins is easier, and secondly, a formula for a non-suitable origin can be deduced from a lower order formula, and we should put this one into the database instead.

Moving Around in the Parameter Space

It remains to investigate the following question: suppose a_p, b_q and a_p^0, b_q^0 are both suitable, and also $a_i - a_i^0 \in \mathbb{Z}$, $b_j - b_j^0 \in \mathbb{Z}$. When is a_p, b_q reachable from a_p^0, b_q^0? It is clear that we can treat all parameters independently that are incongruent mod 1. So assume that a_i and b_j are congruent to r mod 1, for all i and j. The same then follows for a_i^0 and b_j^0.

If $r \neq 0$, then any such a_p, b_q is reachable from any a_p^0, b_q^0. To see this notice that there exist constants c, c^0, congruent mod 1, such that $a_i < c < b_j$ for all i and j, and similarly $a_i^0 < c^0 < b_j^0$. If $n = c - c^0 > 0$ then we first inverse-shift all the b_j^0 n times up, and then similarly shift shift up all the a_i^0 n times. If $n < 0$ then we first inverse-shift down the a_i^0 and then shift down the b_j^0. This reduces to the case $c = c^0$. But evidently we can now shift or inverse-shift around the

a_i^0 arbitrarily so long as we keep them less than c, and similarly for the b_j^0 so long as we keep them bigger than c. Thus a_p, b_q is reachable from a_p^0, b_q^0.

If $r = 0$ then the problem is slightly more involved. WLOG no parameter is zero. We now have one additional complication: no parameter can ever move through zero. Hence a_p, b_q is reachable from a_p^0, b_q^0 if and only if the number of $a_i < 0$ equals the number of $a_i^0 < 0$, and similarly for the b_i and b_i^0. But in a suitable set of parameters, all $b_j > 0$! This is because the Hypergeometric function is undefined if one of the b_j is a non-positive integer and all a_i are smaller than the b_j. Hence the number of $b_j \leq 0$ is always zero.

We can thus associate to every suitable set of parameters a_p, b_q, where no $a_i = 0$, the following invariants:

- For every $r \in [0, 1)$ the number α_r of parameters $a_i \equiv r \pmod{1}$, and similarly the number β_r of parameters $b_i \equiv r \pmod{1}$.
- The number γ of integers a_i with $a_i < 0$.

The above reasoning shows that a_p, b_q is reachable from a_p^0, b_q^0 if and only if the invariants $\alpha_r, \beta_r, \gamma$ all agree. Thus in particular "being reachable from" is a symmetric relation on suitable parameters without zeros.

Applying the Operators

If all goes well then for a given set of parameters we find an origin in our database for which we have a nice formula. We now have to apply (potentially) many differential operators to it. If we do this blindly then the result will be very messy. This is because with Hypergeometric type functions, the derivative is usually expressed as a sum of two contiguous functions. Hence if we compute N derivatives, then the answer will involve $2N$ contiguous functions! This is clearly undesirable. In fact we know from the Hypergeometric differential equation that we need at most max$(p, q + 1)$ contiguous functions to express all derivatives.

Hence instead of differentiating blindly, we will work with a $\mathbb{C}(z)$-module basis: for an origin a_r^0, b_s^0 we either store (for particularly pretty answers) or compute a set of N functions (typically $N = \max(r, s + 1)$) with the property that the derivative of any of them is a $\mathbb{C}(z)$-linear combination of them. In formulae, we store a vector B of N functions, a matrix M and a vector C (the latter two with entries in $\mathbb{C}(z)$), with the following properties:

- ${}_rF_s\left(\begin{matrix} a_r^0 \\ b_s^0 \end{matrix} \middle| z\right) = CB$
- $z \frac{\mathrm{d}}{\mathrm{d}z} B = MB$.

Then we can compute as many derivatives as we want and we will always end up with $\mathbb{C}(z)$-linear combination of at most N special functions.

As hinted above, B, M and C can either all be stored (for particularly pretty answers) or computed from a single ${}_pF_q$ formula.

Loose Ends

This describes the bulk of the hypergeometric function algorithm. There a few further tricks, described in the hyperexpand.py source file. The extension to Meijer G-functions is also described there.

4.3.2 Meijer G-Functions of Finite Confluence

Slater's theorem essentially evaluates a G-function as a sum of residues. If all poles are simple, the resulting series can be recognised as hypergeometric series. Thus a G-function can be evaluated as a sum of Hypergeometric functions.

If the poles are not simple, the resulting series are not hypergeometric. This is known as the "confluent" or "logarithmic" case (the latter because the resulting series tend to contain logarithms). The answer depends in a complicated way on the multiplicities of various poles, and there is no accepted notation for representing it (as far as I know). However if there are only finitely many multiple poles, we can evaluate the G function as a sum of hypergeometric functions, plus finitely many extra terms. I could not find any good reference for this, which is why I work it out here.

Recall the general setup. We define

$$G(z) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s \mathrm{d}s,$$

where L is a contour starting and ending at $+\infty$, enclosing all of the poles of $\Gamma(b_j - s)$ for $j = 1, \ldots, n$ once in the negative direction, and no other poles. Also the integral is assumed absolutely convergent.

In what follows, for any complex numbers a, b, we write $a \equiv b \pmod 1$ if and only if there exists an integer k such that $a - b = k$. Thus there are double poles iff $a_i \equiv a_j \pmod 1$ for some $i \neq j \leq n$.

We now assume that whenever $b_j \equiv a_i \pmod 1$ for $i \leq m$, $j > n$ then $b_j < a_i$. This means that no quotient of the relevant gamma functions is a polynomial, and can always be achieved by "reduction of order". Fix a complex number c such that $\{b_i | b_i \equiv c \pmod 1, i \leq m\}$ is not empty. Enumerate this set as $b, b + k_1, \ldots, b + k_u$, with k_i non-negative integers. Enumerate similarly $\{a_j | a_j \equiv c \pmod 1, j > n\}$ as $b + l_1, \ldots, b + l_v$. Then $l_i > k_j$ for all i, j. For finite confluence, we need to assume $v \geq u$ for all such c.

Let $c_1, \ldots, c_w$ be distinct $\pmod 1$ and exhaust the congruence classes of the b_i. I claim

$$G(z) = -\sum_{j=1}^{w} (F_j(z) + R_j(z)),$$

where $F_j(z)$ is a hypergeometric function and $R_j(z)$ is a finite sum, both to be specified later. Indeed corresponding to every c_j there is a sequence of poles, at mostly finitely many of them multiple poles. This is where the j-th term comes from.

Hence fix again c, enumerate the relevant b_i as $b, b + k_1, \ldots, b + k_u$. We will look at the a_j corresponding to $a + l_1, \ldots, a + l_u$. The other a_i are not treated specially. The corresponding gamma functions have poles at (potentially) $s = b + r$ for $r = 0, 1, \ldots$. For $r \geq l_u$, pole of the integrand is simple. We thus set

$$R(z) = \sum_{r=0}^{l_u - 1} res_{s=r+b}.$$

We finally need to investigate the other poles. Set $r = l_u + t$, $t \geq 0$. A computation shows

$$\frac{\Gamma(k_i - l_u - t)}{\Gamma(l_i - l_u - t)} = \frac{1}{(k_i - l_u - t)_{l_i - k_i}} = \frac{(-1)^{\delta_i}}{(l_u - l_i + 1)_{\delta_i}} \frac{(l_u - l_i + 1)_t}{(l_u - k_i + 1)_t},$$

where $\delta_i = l_i - k_i$.

Also

$$\Gamma(b_j - l_u - b - t) = \frac{\Gamma(b_j - l_u - b)}{(-1)^t (l_u + b + 1 - b_j)_t},$$

$$\Gamma(1 - a_j + l_u + b + t) = \Gamma(1 - a_j + l_u + b)(1 - a_j + l_u + b)_t$$

and

$$res_{s=b+l_u+t}\Gamma(b - s) = -\frac{(-1)^{l_u+t}}{(l_u + t)!} = -\frac{(-1)^{l_u}}{l_u!}\frac{(-1)^t}{(l_u + 1)_t}.$$

Hence

$$\begin{aligned} res_{s=b+l_u+t} =& - z^{b+l_u}\frac{(-1)^{l_u}}{l_u!}\prod_{i=1}^{u}\frac{(-1)^{\delta_i}}{(l_u - k_i + 1)_{\delta_i}}\frac{\prod_{j=1}^{n}\Gamma(1 - a_j + l_u + b)\prod_{j=1}^{m}\Gamma(b_j - l_u - b)^*}{\prod_{j=n+1}^{p}\Gamma(a_j - l_u - b)^*\prod_{j=m+1}^{q}\Gamma(1 - b_j + l_u + b)} \\ &\times z^t\frac{(-1)^t}{(l_u + 1)_t}\prod_{i=1}^{u}\frac{(l_u - l_i + 1)_t}{(l_u - k_i + 1)_t}\frac{\prod_{j=1}^{n}(1 - a_j + l_u + b)_t\prod_{j=n+1}^{p}(-1)^t(l_u + b + 1 - a_j)_t^*}{\prod_{j=1}^{m}(-1)^t(l_u + b + 1 - b_j)_t^*\prod_{j=m+1}^{q}(1 - b_j + l_u + b)_t}, \end{aligned}$$

where the $*$ means to omit the terms we treated specially.

We thus arrive at

$$F(z) = C \times {}_{p+1}F_q\left(\begin{matrix} 1, (1 + l_u - l_i), (1 + l_u + b - a_i)^* \\ 1 + l_u, (1 + l_u - k_i), (1 + l_u + b - b_i)^* \end{matrix}\middle| (-1)^{p-m-n}z\right),$$

where C designates the factor in the residue independent of t. (This result can also be written in slightly simpler form by converting all the l_u etc back to $a_* - b_*$, but doing so is going to require more notation still and is not helpful for computation.)

4.3.3 Extending The Hypergeometric Tables

Adding new formulae to the tables is straightforward. At the top of the file `diofant/simplify/hyperexpand.py`, there is a function called *add_formulae()* (page 1165). Nested in it are defined two helpers, `add(ap, bq, res)` and `addb(ap, bq, B, C, M)`, as well as dummys `a`, `b`, `c`, and `z`.

The first step in adding a new formula is by using `add(ap, bq, res)`. This declares `hyper(ap, bq, z) == res`. Here `ap` and `bq` may use the dummys `a`, `b`, and `c` as free symbols. For example the well-known formula $\sum_0^\infty \frac{(-a)_n z^n}{n!} = (1 - z)^a$ is declared by the following line: `add((-a, ), (), (1-z)**a)`.

From the information provided, the matrices B, C and M will be computed, and the formula is now available when expanding hypergeometric functions. Next the test file `diofant/simplify/tests/test_hyperexpand.py` should be run, in particular the test `test_formulae`. This will test the newly added formula numerically. If it fails, there is (presumably) a typo in what was entered.

Since all newly-added formulae are probably relatively complicated, chances are that the automatically computed basis is rather suboptimal (there is no good way of testing this, other than observing very messy output). In this case the matrices B, C and M should be computed by hand. Then the helper `addb` can be used to declare a hypergeometric formula with hand-computed basis.

An example

Because this explanation so far might be very theoretical and difficult to understand, we walk through an explicit example now. We take the Fresnel function $C(z)$ which obeys the following hypergeometric representation:

$$C(z) = z \cdot {}_{1}F_{2}\left(\begin{matrix} \frac{1}{4} \\ \frac{1}{2}, \frac{5}{4} \end{matrix}\middle| -\frac{\pi^2 z^4}{16}\right) .$$

First we try to add this formula to the lookup table by using the (simpler) function `add(ap, bq, res)`. The first two arguments are simply the lists containing the parameter sets of ${}_1F_2$. The `res` argument is a little bit more complicated. We only know $C(z)$ in terms of ${}_1F_2(\ldots|f(z))$ with f a function of z, in our case

$$f(z) = -\frac{\pi^2 z^4}{16} .$$

What we need is a formula where the hypergeometric function has only z as argument ${}_1F_2(\ldots|z)$. We introduce the new complex symbol w and search for a function $g(w)$ such that

$$f(g(w)) = w$$

holds. Then we can replace every z in $C(z)$ by $g(w)$. In the case of our example the function g could look like

$$g(w) = \frac{2}{\sqrt{\pi}} \exp\left(\frac{i\pi}{4}\right) w^{\frac{1}{4}} .$$

We get these functions mainly by guessing and testing the result. Hence we proceed by computing $f(g(w))$ (and simplifying naively)

$$\begin{aligned} f(g(w)) &= -\frac{\pi^2 g(w)^4}{16} \\ &= -\frac{\pi^2 g\left(\frac{2}{\sqrt{\pi}} \exp\left(\frac{i\pi}{4}\right) w^{\frac{1}{4}}\right)^4}{16} \\ &= -\frac{\pi^2 \frac{2^4}{\sqrt{\pi}^4} \exp\left(\frac{i\pi}{4}\right)^4 w^{\frac{1}{4}4}}{16} \\ &= -\exp(i\pi)\, w \\ &= w \end{aligned}$$

and indeed get back w. (In case of branched functions we have to be aware of branch cuts. In that case we take w to be a positive real number and check the formula. If what we have found works for positive w, then just replace `exp()` (page 321) inside any branched function by `exp_polar()` (page 321) and what we get is right for *all* w.) Hence we can write the formula as

$$C(g(w)) = g(w) \cdot {}_{1}F_{2}\left(\begin{matrix} \frac{1}{4} \\ \frac{1}{2}, \frac{5}{4} \end{matrix}\middle| w\right) .$$

and trivially

$${}_{1}F_{2}\left(\begin{matrix} \frac{1}{4} \\ \frac{1}{2}, \frac{5}{4} \end{matrix}\middle| w\right) = \frac{C(g(w))}{g(w)} = \frac{C\left(\frac{2}{\sqrt{\pi}} \exp\left(\frac{i\pi}{4}\right) w^{\frac{1}{4}}\right)}{\frac{2}{\sqrt{\pi}} \exp\left(\frac{i\pi}{4}\right) w^{\frac{1}{4}}}$$

which is exactly what is needed for the third paramenter, `res`, in `add`. Finally, the whole function call to add this rule to the table looks like:

```
add([Rational(1, 4)],
    [Rational(1, 2), Rational(5, 4)],
    fresnelc(exp(pi*I/4)*root(z,4)*2/sqrt(pi)) / (exp(pi*I/4)*root(z,4)*2/sqrt(pi))
   )
```

Using this rule we will find that it works but the results are not really nice in terms of simplicity and number of special function instances included. We can obtain much better results by adding the formula to the lookup table in another way. For this we use the (more complicated) function `addb(ap, bq, B, C, M)`. The first two arguments are again the lists containing the parameter sets of ${}_1F_2$. The remaining three are the matrices mentioned earlier on this page.

We know that the $n = \max(p, q + 1)$-th derivative can be expressed as a linear combination of lower order derivatives. The matrix B contains the basis $\{B_0, B_1, \ldots\}$ and is of shape $n \times 1$. The best way to get B_i is to take the first $n = \max(p, q + 1)$ derivatives of the expression for ${}_pF_q$ and take out useful pieces. In our case we find that $n = \max(1, 2 + 1) = 3$. For computing the derivatives, we have to use the operator $z\frac{\mathrm{d}}{\mathrm{d}z}$. The first basis element B_0 is set to the expression for ${}_1F_2$ from above:

$$B_0 = \frac{\sqrt{\pi}\exp\left(-\frac{\imath\pi}{4}\right) C\left(\frac{2}{\sqrt{\pi}}\exp\left(\frac{\imath\pi}{4}\right) z^{\frac{1}{4}}\right)}{2z^{\frac{1}{4}}}$$

Next we compute $z\frac{\mathrm{d}}{\mathrm{d}z}B_0$. For this we can directly use Diofant!

```
>>> B0 = sqrt(pi)*exp(-I*pi/4)*fresnelc(2*root(z, 4)*exp(I*pi/4)/sqrt(pi))/(2*root(z,␣
↪4))
>>> z * diff(B0, z)
z*(cosh(2*sqrt(z))/(4*z) - E**(-I*pi/4)*sqrt(pi)*fresnelc(2*E**(I*pi/4)*z**(1/4)/
↪sqrt(pi))/(8*z**(5/4)))
>>> expand(_)
cosh(2*sqrt(z))/4 - E**(-I*pi/4)*sqrt(pi)*fresnelc(2*E**(I*pi/4)*z**(1/4)/sqrt(pi))/
↪(8*z**(1/4))
```

Formatting this result nicely we obtain

$$B_1' = -\frac{1}{4}\frac{\sqrt{\pi}\exp\left(-\frac{\imath\pi}{4}\right) C\left(\frac{2}{\sqrt{\pi}}\exp\left(\frac{\imath\pi}{4}\right) z^{\frac{1}{4}}\right)}{2z^{\frac{1}{4}}} + \frac{1}{4}\cosh\left(2\sqrt{z}\right)$$

Computing the second derivative we find

```
>>> B1prime = cosh(2*sqrt(z))/4 - sqrt(pi)*exp(-I*pi/4)*\
...           fresnelc(2*root(z,4)*exp(I*pi/4)/sqrt(pi))/(8*root(z,4))
>>> z * diff(B1prime, z)
z*(-cosh(2*sqrt(z))/(16*z) + sinh(2*sqrt(z))/(4*sqrt(z)) + E**(-I*pi/
↪4)*sqrt(pi)*fresnelc(2*E**(I*pi/4)*z**(1/4)/sqrt(pi))/(32*z**(5/4)))
>>> expand(_)
sqrt(z)*sinh(2*sqrt(z))/4 - cosh(2*sqrt(z))/16 + E**(-I*pi/
↪4)*sqrt(pi)*fresnelc(2*E**(I*pi/4)*z**(1/4)/sqrt(pi))/(32*z**(1/4))
```

which can be printed as

$$B_2' = \frac{1}{16}\frac{\sqrt{\pi}\exp\left(-\frac{\imath\pi}{4}\right) C\left(\frac{2}{\sqrt{\pi}}\exp\left(\frac{\imath\pi}{4}\right) z^{\frac{1}{4}}\right)}{2z^{\frac{1}{4}}} - \frac{1}{16}\cosh\left(2\sqrt{z}\right) + \frac{1}{4}\sinh\left(2\sqrt{z}\right)\sqrt{z}$$

We see the common pattern and can collect the pieces. Hence it makes sense to choose B_1

and B_2 as follows

$$B = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\pi} \exp\left(-\frac{i\pi}{4}\right) C\left(\frac{2}{\sqrt{\pi}} \exp\left(\frac{i\pi}{4}\right) z^{\frac{1}{4}}\right)}{2 z^{\frac{1}{4}}} \\ \cosh\left(2\sqrt{z}\right) \\ \sinh\left(2\sqrt{z}\right)\sqrt{z} \end{pmatrix}$$

(This is in contrast to the basis $B = (B_0, B_1', B_2')$ that would have been computed automatically if we used just `add(ap, bq, res)`.)

Because it must hold that ${}_pF_q\left(\cdots|z\right) = CB$ the entries of C are obviously

$$C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Finally we have to compute the entries of the 3×3 matrix M such that $z\frac{\mathrm{d}}{\mathrm{d}z}B = MB$ holds. This is easy. We already computed the first part $z\frac{\mathrm{d}}{\mathrm{d}z}B_0$ above. This gives us the first row of M. For the second row we have:

```
>>> B1 = cosh(2*sqrt(z))
>>> z * diff(B1, z)
sqrt(z)*sinh(2*sqrt(z))
```

and for the third one

```
>>> B2 = sinh(2*sqrt(z))*sqrt(z)
>>> expand(z * diff(B2, z))
sqrt(z)*sinh(2*sqrt(z))/2 + z*cosh(2*sqrt(z))
```

Now we have computed the entries of this matrix to be

$$M = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 \\ 0 & z & \frac{1}{2} \end{pmatrix}$$

Note that the entries of C and M should typically be rational functions in z, with rational coefficients. This is all we need to do in order to add a new formula to the lookup table for `hyperexpand`.

4.3.4 Implemented Hypergeometric Formulae

A vital part of the algorithm is a relatively large table of hypergeometric function representations. The following automatically generated list contains all the representations implemented in Diofant (of course many more are derived from them). These formulae are mostly taken from *[Luke1969]* (page 1270) and *[Prudnikov1990]* (page 1270). They are all tested numerically.

$${}_0F_0\left(\middle| z\right) = e^z$$

$${}_1F_0\left(a \middle| z\right) = (-z+1)^{-a}$$

$${}_2F_1\left(\begin{matrix} a, a-\frac{1}{2} \\ 2a \end{matrix} \middle| z\right) = 2^{2a-1}\left(\sqrt{-z+1}+1\right)^{-2a+1}$$

$${}_{2}F_{1}\left(\begin{matrix} 1, 1 \\ 2 \end{matrix}\middle| z\right) = -\frac{1}{z}\log\left(-z+1\right)$$

$${}_{2}F_{1}\left(\begin{matrix} \frac{1}{2}, 1 \\ \frac{3}{2} \end{matrix}\middle| z\right) = \frac{1}{\sqrt{z}}\operatorname{atanh}\left(\sqrt{z}\right)$$

$${}_{2}F_{1}\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} \end{matrix}\middle| z\right) = \frac{1}{\sqrt{z}}\operatorname{asin}\left(\sqrt{z}\right)$$

$${}_{2}F_{1}\left(\begin{matrix} a, a+\frac{1}{2} \\ \frac{1}{2} \end{matrix}\middle| z\right) = \frac{1}{2}\left(\sqrt{z}+1\right)^{-2a} + \frac{1}{2}\left(-\sqrt{z}+1\right)^{-2a}$$

$${}_{2}F_{1}\left(\begin{matrix} a, -a \\ \frac{1}{2} \end{matrix}\middle| z\right) = \cos\left(2a\operatorname{asin}\left(\sqrt{z}\right)\right)$$

$${}_{2}F_{1}\left(\begin{matrix} 1, 1 \\ \frac{3}{2} \end{matrix}\middle| z\right) = \frac{\operatorname{asin}\left(\sqrt{z}\right)}{\sqrt{z}\sqrt{-z+1}}$$

$${}_{2}F_{1}\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}\middle| z\right) = \frac{2K\left(z\right)}{\pi}$$

$${}_{2}F_{1}\left(\begin{matrix} -\frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}\middle| z\right) = \frac{2E\left(z\right)}{\pi}$$

$${}_{3}F_{2}\left(\begin{matrix} -\frac{1}{2}, 1, 1 \\ \frac{1}{2}, 2 \end{matrix}\middle| z\right) = -\frac{2\sqrt{z}}{3}\operatorname{atanh}\left(\sqrt{z}\right) + \frac{2}{3} - \frac{1}{3z}\log\left(-z+1\right)$$

$${}_{3}F_{2}\left(\begin{matrix} -\frac{1}{2}, 1, 1 \\ 2, 2 \end{matrix}\middle| z\right) = \left(\frac{4}{9} - \frac{16}{9z}\right)\sqrt{-z+1} + \frac{4}{3z}\log\left(\frac{1}{2}\sqrt{-z+1} + \frac{1}{2}\right) + \frac{16}{9z}$$

$${}_{1}F_{1}\left(\begin{matrix} 1 \\ b \end{matrix}\middle| z\right) = e^{z}z^{-b+1}\left(b-1\right)\gamma\left(b-1, z\right)$$

$${}_{1}F_{1}\left(\begin{matrix} a \\ 2a \end{matrix}\middle| z\right) = 4^{a-\frac{1}{2}}e^{\frac{z}{2}}z^{-a+\frac{1}{2}}I_{a-\frac{1}{2}}\left(\frac{z}{2}\right)\Gamma\left(a+\frac{1}{2}\right)$$

$${}_{1}F_{1}\left(\begin{matrix} a \\ a+1 \end{matrix}\middle| z\right) = a\left(z\operatorname{exp_{polar}}\left(i\pi\right)\right)^{-a}\gamma\left(a, z\operatorname{exp_{polar}}\left(i\pi\right)\right)$$

$${}_{1}F_{1}\left(\begin{matrix} -\frac{1}{2} \\ \frac{1}{2} \end{matrix}\middle| z\right) = e^{z} + i\sqrt{\pi}\sqrt{z}\operatorname{erf}\left(i\sqrt{z}\right)$$

$${}_{1}F_{2}\left(\begin{matrix} 1 \\ \frac{3}{4}, \frac{5}{4} \end{matrix}\middle| z\right) = \frac{e^{-\frac{i\pi}{4}}\sqrt{\pi}}{2\sqrt[4]{z}}\left(i\sinh\left(2\sqrt{z}\right)S\left(\frac{2\sqrt[4]{z}}{\sqrt{\pi}}e^{\frac{i\pi}{4}}\right) + \cosh\left(2\sqrt{z}\right)C\left(\frac{2\sqrt[4]{z}}{\sqrt{\pi}}e^{\frac{i\pi}{4}}\right)\right)$$

$${}_{2}F_{2}\left(\begin{matrix} \frac{1}{2}, a \\ \frac{3}{2}, a+1 \end{matrix}\middle| z\right) = -\frac{i\sqrt{\pi}a\sqrt{\frac{1}{z}}}{2a-1}\operatorname{erf}\left(i\sqrt{z}\right) - \frac{a\left(z\operatorname{exp_{polar}}\left(i\pi\right)\right)^{-a}}{2a-1}\gamma\left(a, z\operatorname{exp_{polar}}\left(i\pi\right)\right)$$

$${}_{2}F_{2}\left(\begin{matrix} 1, 1 \\ 2, 2 \end{matrix}\middle| z\right) = \frac{1}{z}\left(-\log\left(z\right) + \operatorname{Ei}\left(z\right)\right) - \frac{\gamma}{z}$$

$${}_{0}F_{1}\left(\begin{matrix} \\ \frac{1}{2} \end{matrix}\middle| z\right) = \cosh\left(2\sqrt{z}\right)$$

$${}_{0}F_{1}\left(\begin{matrix} \\ b \end{matrix}\middle| z\right) = z^{-\frac{b}{2}+\frac{1}{2}}I_{b-1}\left(2\sqrt{z}\right)\Gamma(b)$$

$${}_{0}F_{3}\left(\begin{matrix} \\ \frac{1}{2}, a, a+\frac{1}{2} \end{matrix}\middle| z\right) = 2^{-2a}z^{-\frac{a}{2}+\frac{1}{4}}\left(I_{2a-1}\left(4\sqrt[4]{z}\right) + J_{2a-1}\left(4\sqrt[4]{z}\right)\right)\Gamma(2a)$$

$${}_{0}F_{3}\left(\begin{matrix} \\ a, a+\frac{1}{2}, 2a \end{matrix}\middle| z\right) = \left(2\sqrt{z}\operatorname{exp_{polar}}\left(\frac{i\pi}{2}\right)\right)^{-2a+1}I_{2a-1}\left(2\sqrt{2}\sqrt[4]{z}\operatorname{exp_{polar}}\left(\frac{i\pi}{4}\right)\right)J_{2a-1}\left(2\sqrt{2}\sqrt[4]{z}\operatorname{exp_{polar}}\left(\frac{i\pi}{4}\right)\right)\Gamma^{2}(2a)$$

$${}_{1}F_{2}\left(\begin{matrix} a \\ a - \frac{1}{2}, 2a \end{matrix}\middle| z\right) = 2 \cdot 4^{a-1} z^{-a+1} I_{a-\frac{3}{2}}\left(\sqrt{z}\right) I_{a-\frac{1}{2}}\left(\sqrt{z}\right) \Gamma\left(a - \frac{1}{2}\right) \Gamma\left(a + \frac{1}{2}\right) - 4^{a-\frac{1}{2}} z^{-a+\frac{1}{2}} I^{2}_{a-\frac{1}{2}}\left(\sqrt{z}\right) \Gamma^{2}\left(a + \frac{1}{2}\right)$$

$${}_{1}F_{2}\left(\begin{matrix} \frac{1}{2} \\ b, -b+2 \end{matrix}\middle| z\right) = \frac{\pi I_{-b+1}\left(\sqrt{z}\right) I_{b-1}\left(\sqrt{z}\right)}{\sin\left(\pi b\right)} \left(-b+1\right)$$

$${}_{1}F_{2}\left(\begin{matrix} \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix}\middle| z\right) = \frac{1}{2\sqrt{z}} \operatorname{Shi}\left(2\sqrt{z}\right)$$

$${}_{1}F_{2}\left(\begin{matrix} \frac{3}{4} \\ \frac{3}{2}, \frac{7}{4} \end{matrix}\middle| z\right) = \frac{3\sqrt{\pi}}{4 z^{\frac{3}{4}}} e^{-\frac{3i}{4}\pi} S\left(\frac{2\sqrt[4]{z}}{\sqrt{\pi}} e^{\frac{i\pi}{4}}\right)$$

$${}_{1}F_{2}\left(\begin{matrix} \frac{1}{4} \\ \frac{1}{2}, \frac{5}{4} \end{matrix}\middle| z\right) = \frac{e^{-\frac{i\pi}{4}} \sqrt{\pi}}{2\sqrt[4]{z}} C\left(\frac{2\sqrt[4]{z}}{\sqrt{\pi}} e^{\frac{i\pi}{4}}\right)$$

$${}_{2}F_{3}\left(\begin{matrix} a, a + \frac{1}{2} \\ 2a, b, 2a - b + 1 \end{matrix}\middle| z\right) = \left(\frac{\sqrt{z}}{2}\right)^{-2a+1} I_{2a-b}\left(\sqrt{z}\right) I_{b-1}\left(\sqrt{z}\right) \Gamma(b) \Gamma(2a - b + 1)$$

$${}_{2}F_{3}\left(\begin{matrix} 1, 1 \\ 2, 2, \frac{3}{2} \end{matrix}\middle| z\right) = \frac{1}{z}\left(-\log\left(2\sqrt{z}\right) + \operatorname{Chi}\left(2\sqrt{z}\right)\right) - \frac{\gamma}{z}$$

$${}_{3}F_{3}\left(\begin{matrix} 1, 1, a \\ 2, 2, a + 1 \end{matrix}\middle| z\right) = -\frac{e^{z} a}{z\left(a^{2} - 2a + 1\right)} + \frac{a\left(-z\right)^{-a}}{\left(a - 1\right)^{2}} \left(\Gamma(a) - \Gamma\left(a, -z\right)\right) + \frac{a}{z\left(a^{2} - 2a + 1\right)} \left(-a + 1\right) \left(\log\left(-z\right) + \operatorname{E}_{1}\left(-z\right) + \gamma\right) +$$

`diofant.simplify.hyperexpand.`**`add_formulae`**(*formulae*)

Create our knowledge base.

4.3.5 References

4.4 Computing Integrals using Meijer G-Functions

This text aims do describe in some detail the steps (and subtleties) involved in using Meijer G-functions for computing definite and indefinite integrals. We shall ignore proofs completely.

4.4.1 Overview

The algorithm to compute $\int f(x) \mathrm{d}x$ or $\int_0^\infty f(x) \mathrm{d}x$ generally consists of three steps:

1. Rewrite the integrand using Meijer G-functions (one or sometimes two).
2. Apply an integration theorem, to get the answer (usually expressed as another G-function).
3. Expand the result in named special functions.

Step (3) is implemented in the function hyperexpand (q.v.). Steps (1) and (2) are described below. Morever, G-functions are usually branched. Thus our treatment of branched functions is described first.

Some other integrals (e.g. $\int_{-\infty}^{\infty}$) can also be computed by first recasting them into one of the above forms. There is a lot of choice involved here, and the algorithm is heuristic at best.

4.4.2 Polar Numbers and Branched Functions

Both Meijer G-Functions and Hypergeometric functions are typically branched (possible branchpoints being 0, ± 1, ∞). This is not very important when e.g. expanding a single hypergeometric function into named special functions, since sorting out the branches can be left to the human user. However this algorithm manipulates and transforms G-functions, and to do this correctly it needs at least some crude understanding of the branchings involved.

To begin, we consider the set $\mathcal{S} = \{(r, \theta) : r > 0, \theta \in \mathbb{R}\}$. We have a map $p : \mathcal{S} :\to \mathbb{C} - \{0\}, (r, \theta) \mapsto re^{i\theta}$. Decreeing this to be a local biholomorphism gives $\mathcal{S}$ both a topology and a complex structure. This Riemann Surface is usually referred to as the Riemann Surface of the logarithm, for the following reason: We can define maps $\operatorname{Exp} : \mathbb{C} \to \mathcal{S}, (x + iy) \mapsto (\exp(x), y)$ and $\operatorname{Log} : \mathcal{S} \to \mathbb{C}, (e^x, y) \mapsto x + iy$. These can both be shown to be holomorphic, and are indeed mutual inverses.

We also sometimes formally attach a point "zero" (0) to $\mathcal{S}$ and denote the resulting object $\mathcal{S}_0$. Notably there is no complex structure defined near 0. A fundamental system of neighbourhoods is given by $\{\operatorname{Exp}(z) : \Re(z) < k\}$, which at least defines a topology. Elements of $\mathcal{S}_0$ shall be called polar numbers. We further define functions $\operatorname{Arg} : \mathcal{S} \to \mathbb{R}, (r, \theta) \mapsto \theta$ and $|.| : \mathcal{S}_0 \to \mathbb{R}_{>0}, (r, \theta) \mapsto r$. These have evident meaning and are both continuous everywhere.

Using these maps many operations can be extended from $\mathbb{C}$ to $\mathcal{S}$. We define $\operatorname{Exp}(a)\operatorname{Exp}(b) = \operatorname{Exp}(a + b)$ for $a, b \in \mathbb{C}$, also for $a \in \mathcal{S}$ and $b \in \mathbb{C}$ we define $a^b = \operatorname{Exp}(b \operatorname{Log}(a))$. It can be checked easily that using these definitions, many algebraic properties holding for positive reals (e.g. $(ab)^c = a^c b^c$) which hold in $\mathbb{C}$ only for some numbers (because of branch cuts) hold indeed for all polar numbers.

As one peculiarity it should be mentioned that addition of polar numbers is not usually defined. However, formal sums of polar numbers can be used to express branching behaviour. For example, consider the functions $F(z) = \sqrt{1 + z}$ and $G(a, b) = \sqrt{a + b}$, where a, b, z are polar numbers. The general rule is that functions of a single polar variable are defined in such a way that they are continuous on circles, and agree with the usual definition for positive reals. Thus if $S(z)$ denotes the standard branch of the square root function on $\mathbb{C}$, we are forced to define

$$F(z) = \begin{cases} S(p(z)) & : |z| < 1 \\ S(p(z)) & : -\pi < \operatorname{Arg}(z) + 4\pi n \le \pi \text{ for some } n \in \mathbb{Z} \\ -S(p(z)) & : \text{else} \end{cases}.$$

(We are omitting $|z| = 1$ here, this does not matter for integration.) Finally we define $G(a, b) = \sqrt{a} F(b/a)$.

4.4.3 Representing Branched Functions on the Argand Plane

Suppose $f : \mathcal{S} \to \mathbb{C}$ is a holomorphic function. We wish to define a function F on (part of) the complex numbers $\mathbb{C}$ that represents f as closely as possible. This process is knows as "introducing branch cuts". In our situation, there is actually a canonical way of doing this (which is adhered to in all of Diofant), as follows: Introduce the "cut complex plane" $C = \mathbb{C} \setminus \mathbb{R}_{\le 0}$. Define a function $l : C \to \mathcal{S}$ via $re^{i\theta} \mapsto r \operatorname{Exp}(i\theta)$. Here $r > 0$ and $-\pi < \theta \le \pi$. Then l is holomorphic, and we define $G = f \circ l$. This called "lifting to the principal branch" throughout the Diofant documentation.

4.4.4 Table Lookups and Inverse Mellin Transforms

Suppose we are given an integrand $f(x)$ and are trying to rewrite it as a single G-function. To do this, we first split $f(x)$ into the form $x^s g(x)$ (where $g(x)$ is supposed to be simpler than $f(x)$). This is because multiplicative powers can be absorbed into the G-function later. This splitting is done by `_split_mul(f, x)`. Then we assemble a tuple of functions that occur in f (e.g. if $f(x) = e^x \cos x$, we would assemble the tuple $(\cos, \exp)$). This is done by the function `_mytype(f, x)`. Next we index a lookup table (created using `_create_lookup_table()`) with this tuple. This (hopefully) yields a list of Meijer G-function formulae involving these functions, we then pattern-match all of them. If one fits, we were successful, otherwise not and we have to try something else.

Suppose now we want to rewrite as a product of two G-functions. To do this, we (try to) find all inequivalent ways of splitting $f(x)$ into a product $f_1(x)f_2(x)$. We could try these splittings in any order, but it is often a good idea to minimise (a) the number of powers occuring in $f_i(x)$ and (b) the number of different functions occuring in $f_i(x)$. Thus given e.g. $f(x) = \sin x\, e^x \sin 2x$ we should try $f_1(x) = \sin x \, \sin 2x$, $f_2(x) = e^x$ first. All of this is done by the function `_mul_as_two_parts(f)`.

Finally, we can try a recursive Mellin transform technique. Since the Meijer G-function is defined essentially as a certain inverse mellin transform, if we want to write a function $f(x)$ as a G-function, we can compute its mellin transform $F(s)$. If $F(s)$ is in the right form, the G-function expression can be read off. This technique generalises many standard rewritings, e.g. $e^{ax}e^{bx} = e^{(a+b)x}$.

One twist is that some functions don't have mellin transforms, even though they can be written as G-functions. This is true for example for $f(x) = e^x \sin x$ (the function grows too rapidly to have a mellin transform). However if the function is recognised to be analytic, then we can try to compute the mellin-transform of $f(ax)$ for a parameter a, and deduce the G-function expression by analytic continuation. (Checking for analyticity is easy. Since we can only deal with a certain subset of functions anyway, we only have to filter out those which are not analytic.)

The function `_rewrite_single` does the table lookup and recursive mellin transform. The functions `_rewrite1` and `_rewrite2` respectively use above-mentioned helpers and `_rewrite_single` to rewrite their argument as respectively one or two G-functions.

4.4.5 Applying the Integral Theorems

If the integrand has been recast into G-functions, evaluating the integral is relatively easy. We first do some substitutions to reduce e.g. the exponent of the argument of the G-function to unity (see `_rewrite_saxena_1` and `_rewrite_saxena`, respectively, for one or two G-functions). Next we go through a list of conditions under which the integral theorem applies. It can fail for basically two reasons: either the integral does not exist, or the manipulations in deriving the theorem may not be allowed (for more details, see this *[BlogPost]* (page 1270)).

Sometimes this can be remedied by reducing the argument of the G-functions involved. For example it is clear that the G-function representing e^z is satisfies $G(\mathrm{Exp}(2\pi i)z) = G(z)$ for all $z \in \mathcal{S}$. The function `meijerg.get_period()` can be used to discover this, and the function `principal_branch(z, period)` in `functions/elementary/complexes.py` can be used to exploit the information. This is done transparently by the integration code.

4.5 The G-Function Integration Theorems

This section intends to display in detail the definite integration theorems used in the code. The following two formulae go back to Meijer (In fact he proved more general formulae; indeed in the literature formulae are usually staded in more general form. However it is very easy to deduce the general formulae from the ones we give here. It seemed best to keep the theorems as simple as possible, since they are very complicated anyway.):

1.

$$\int_0^\infty G_{p,q}^{m,n} \left.\begin{pmatrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{pmatrix}\right| \eta x \Bigg) \mathrm{d}x = \frac{\prod_{j=1}^m \Gamma(b_j+1) \prod_{j=1}^n \Gamma(-a_j)}{\eta \prod_{j=m+1}^q \Gamma(-b_j) \prod_{j=n+1}^p \Gamma(a_j+1)}$$

2.

$$\int_0^\infty G_{u,v}^{s,t} \left(\left.\begin{matrix} c_1,\dots,c_u \\ d_1,\dots,d_v \end{matrix}\right| \sigma x\right) G_{p,q}^{m,n} \left(\left.\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right| \omega x\right) \mathrm{d}x = G_{v+p,u+q}^{m+t,n+s} \left(\left.\begin{matrix} a_1,\dots,a_n,-d_1,\dots,-d_v,a_{n+1},\dots,a_p \\ b_1,\dots,b_m,-c_1,\dots,-c_u,b_{m+1},\dots,b_q \end{matrix}\right| \frac{\omega}{\sigma}\right)$$

The more interesting question is under what conditions these formulae are valid. Below we detail the conditions implemented in Diofant. They are an amalgamation of conditions found in *[Prudnikov1990]* (page 1270) and *[Luke1969]* (page 1270); please let us know if you find any errors.

4.5.1 Conditions of Convergence for Integral (1)

We can without loss of generality assume $p \le q$, since the G-functions of indices m, n, p, q and of indices n, m, q, p can be related easily (see e.g. *[Luke1969]* (page 1270), section 5.3). We introduce the following notation:

$$\xi = m + n - p$$
$$\delta = m + n - \frac{p+q}{2}$$

$$C_3 : -\Re(b_j) < 1 \text{ for } j = 1,\dots,m$$
$$0 < -\Re(a_j) \text{ for } j = 1,\dots,n$$

$$C_3^* : -\Re(b_j) < 1 \text{ for } j = 1,\dots,q$$
$$0 < -\Re(a_j) \text{ for } j = 1,\dots,p$$

$$C_4 : -\Re(\delta) + \frac{q+1-p}{2} > q - p$$

The convergence conditions will be detailed in several "cases", numbered one to five. For later use it will be helpful to separate conditions "at infinity" from conditions "at zero". By conditions "at infinity" we mean conditions that only depend on the behaviour of the integrand for large, positive values of x, whereas by conditions "at zero" we mean conditions that only depend on the behaviour of the integrand on $(0, \epsilon)$ for any $\epsilon > 0$. Since all our conditions are specified in terms of parameters of the G-functions, this distinction is not immediately visible. They are, however, of very distinct character mathematically; the conditions at infinity being in particular much harder to control.

In order for the integral theorem to be valid, conditions n "at zero" and "at infinity" both have to be fulfilled, for some n.

These are the conditions "at infinity":

1.

$$\delta > 0 \wedge |\arg(\eta)| < \delta\pi \wedge (A \vee B \vee C),$$

where

$$A = 1 \le n \wedge p < q \wedge 1 \le m$$

$$B = 1 \le p \wedge 1 \le m \wedge q = p + 1 \wedge \neg(n = 0 \wedge m = p + 1)$$

$$C = 1 \le n \wedge q = p \wedge |\arg(\eta)| \ne (\delta - 2k)\pi \text{ for } k = 0, 1, \ldots \left\lceil \frac{\delta}{2} \right\rceil.$$

2.

$$n = 0 \wedge p + 1 \le m \wedge |\arg(\eta)| < \delta\pi$$

3.

$$(p < q \wedge 1 \le m \wedge \delta > 0 \wedge |\arg(\eta)| = \delta\pi) \vee (p \le q - 2 \wedge \delta = 0 \wedge \arg(\eta) = 0)$$

4.

$$p = q \wedge \delta = 0 \wedge \arg(\eta) = 0 \wedge \eta \ne 0 \wedge \Re\left(\sum_{j=1}^{p} b_j - a_j\right) < 0$$

5.

$$\delta > 0 \wedge |\arg(\eta)| < \delta\pi$$

And these are the conditions "at zero":

1.

$$\eta \ne 0 \wedge C_3$$

2.

$$C_3$$

3.

$$C_3 \wedge C_4$$

4.

$$C_3$$

5.

$$C_3$$

4.5.2 Conditions of Convergence for Integral (2)

We introduce the following notation:

$$b^* = s + t - \frac{u+v}{2}$$

$$c^* = m + n - \frac{p+q}{2}$$

$$\rho = \sum_{j=1}^{v} d_j - \sum_{j=1}^{u} c_j + \frac{u-v}{2} + 1$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2} + 1$$

$$\phi = q - p - \frac{u-v}{2} + 1$$

$$\eta = 1 - (v-u) - \mu - \rho$$

$$\psi = \frac{\pi(q-m-n) + |\arg(\omega)|}{q-p}$$

$$\theta = \frac{\pi(v-s-t) + |\arg(\sigma)|)}{v-u}$$

$$\lambda_c = (q-p)|\omega|^{1/(q-p)} \cos\psi + (v-u)|\sigma|^{1/(v-u)} \cos\theta$$

$$\lambda_{s0}(c_1, c_2) = c_1(q-p)|\omega|^{1/(q-p)} \sin\psi + c_2(v-u)|\sigma|^{1/(v-u)} \sin\theta$$

$$\lambda_s = \begin{cases} \lambda_{s0}\left(-1,-1\right)\lambda_{s0}\left(1,1\right) & \text{for } \arg(\omega) = 0 \wedge \arg(\sigma) = 0 \\ \lambda_{s0}\left(\operatorname{sign}\left(\arg\left(\omega\right)\right),-1\right)\lambda_{s0}\left(\operatorname{sign}\left(\arg\left(\omega\right)\right),1\right) & \text{for } \arg(\omega) \neq 0 \wedge \arg(\sigma) = 0 \\ \lambda_{s0}\left(-1,\operatorname{sign}\left(\arg\left(\sigma\right)\right)\right)\lambda_{s0}\left(1,\operatorname{sign}\left(\arg\left(\sigma\right)\right)\right) & \text{for } \arg(\omega) = 0 \wedge \arg(\sigma) \neq 0) \\ \lambda_{s0}\left(\operatorname{sign}\left(\arg\left(\omega\right)\right),\operatorname{sign}\left(\arg\left(\sigma\right)\right)\right) & \text{otherwise} \end{cases}$$

$$z_0 = \frac{\omega}{\sigma} e^{-i\pi(b^*+c^*)}$$

$$z_1 = \frac{\sigma}{\omega} e^{-i\pi(b^*+c^*)}$$

The following conditions will be helpful:

$$C_1 : (a_i - b_j \notin \mathbb{Z}_{>0} \text{ for } i = 1, \ldots, n, j = 1, \ldots, m) \\ \wedge (c_i - d_j \notin \mathbb{Z}_{>0} \text{ for } i = 1, \ldots, t, j = 1, \ldots, s)$$

$$C_2 : \Re(1 + b_i + d_j) > 0 \text{ for } i = 1, \ldots, m, j = 1, \ldots, s$$

$$C_3 : \Re(a_i + c_j) < 1 \text{ for } i = 1, \ldots, n, j = 1, \ldots, t$$

$$C_4 : (p-q)\Re(c_i) - \Re(\mu) > -\frac{3}{2} \text{ for } i = 1, \ldots, t$$

$$C_5 : (p-q)\Re(1 + d_i) - \Re(\mu) > -\frac{3}{2} \text{ for } i = 1, \ldots, s$$

$$C_6 : (u-v)\Re(a_i) - \Re(\rho) > -\frac{3}{2} \text{ for } i = 1, \ldots, n$$

$$C_7 : (u-v)\Re(1 + b_i) - \Re(\rho) > -\frac{3}{2} \text{ for } i = 1, \ldots, m$$

$$C_8 : 0 < |\phi| + 2\Re\left(\left(\mu - 1\right)\left(-u + v\right) + \left(-p + q\right)\left(\rho - 1\right) + \left(-p + q\right)\left(-u + v\right)\right)$$

$$C_9 : 0 < |\phi| - 2\Re\left(\left(\mu - 1\right)\left(-u + v\right) + \left(-p + q\right)\left(\rho - 1\right) + \left(-p + q\right)\left(-u + v\right)\right)$$

$$C_{10} : \left|\arg\left(\sigma\right)\right| < \pi b^*$$

$$C_{11} : \left|\arg\left(\sigma\right)\right| = \pi b^*$$

$$C_{12} : |\arg(\omega)| < c^*\pi$$

$$C_{13} : |\arg(\omega)| = c^*\pi$$

$$C_{14}^1 : (z_0 \neq 1 \wedge |\arg(1 - z_0)| < \pi) \vee (z_0 = 1 \wedge \Re(\mu + \rho - u + v) < 1)$$

$$C_{14}^2 : (z_1 \neq 1 \wedge |\arg(1 - z_1)| < \pi) \vee (z_1 = 1 \wedge \Re(\mu + \rho - p + q) < 1)$$

$$C_{14} : \phi = 0 \wedge b^* + c^* \leq 1 \wedge (C_{14}^1 \vee C_{14}^2)$$

$$C_{15} : \lambda_c > 0 \vee (\lambda_c = 0 \wedge \lambda_s \neq 0 \wedge \Re(\eta) > -1) \vee (\lambda_c = 0 \wedge \lambda_s = 0 \wedge \Re(\eta) > 0)$$

$$C_{16} : \int_0^\infty G_{u,v}^{s,t}(\sigma x)\mathrm{d}x \text{ converges at infinity}$$

$$C_{17} : \int_0^\infty G_{p,q}^{m,n}(\omega x)\mathrm{d}x \text{ converges at infinity}$$

Note that C_{16} and C_{17} are the reason we split the convergence conditions for integral (1).

With this notation established, the implemented convergence conditions can be enumerated as follows:

1. $$mnst \neq 0 \wedge 0 < b^* \wedge 0 < c^* \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_{10} \wedge C_{12}$$

2. $$u = v \wedge b^* = 0 \wedge 0 < c^* \wedge 0 < \sigma \wedge \Re\rho < 1 \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_{12}$$

3. $$p = q \wedge u = v \wedge b^* = 0 \wedge c^* = 0 \wedge 0 < \sigma \wedge 0 < \omega \wedge \Re\mu < 1 \wedge \Re\rho < 1 \wedge \sigma \neq \omega \wedge C_1 \wedge C_2 \wedge C_3$$

4. $$p = q \wedge u = v \wedge b^* = 0 \wedge c^* = 0 \wedge 0 < \sigma \wedge 0 < \omega \wedge \Re\left(\mu + \rho\right) < 1 \wedge \omega \neq \sigma \wedge C_1 \wedge C_2 \wedge C_3$$

5. $$p = q \wedge u = v \wedge b^* = 0 \wedge c^* = 0 \wedge 0 < \sigma \wedge 0 < \omega \wedge \Re\left(\mu + \rho\right) < 1 \wedge \omega \neq \sigma \wedge C_1 \wedge C_2 \wedge C_3$$

6. $$q < p \wedge 0 < s \wedge 0 < b^* \wedge 0 \leq c^* \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_5 \wedge C_{10} \wedge C_{13}$$

7. $$p < q \wedge 0 < t \wedge 0 < b^* \wedge 0 \leq c^* \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_{10} \wedge C_{13}$$

8.

$$v < u \wedge 0 < m \wedge 0 < c^* \wedge 0 \le b^* \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_7 \wedge C_{11} \wedge C_{12}$$

9.

$$u < v \wedge 0 < n \wedge 0 < c^* \wedge 0 \le b^* \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_6 \wedge C_{11} \wedge C_{12}$$

10.

$$q < p \wedge u = v \wedge b^* = 0 \wedge 0 \le c^* \wedge 0 < \sigma \wedge \Re\rho < 1 \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_5 \wedge C_{13}$$

11.

$$p < q \wedge u = v \wedge b^* = 0 \wedge 0 \le c^* \wedge 0 < \sigma \wedge \Re\rho < 1 \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_{13}$$

12.

$$p = q \wedge v < u \wedge 0 \le b^* \wedge c^* = 0 \wedge 0 < \omega \wedge \Re\mu < 1 \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_7 \wedge C_{11}$$

13.

$$p = q \wedge u < v \wedge 0 \le b^* \wedge c^* = 0 \wedge 0 < \omega \wedge \Re\mu < 1 \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_6 \wedge C_{11}$$

14.

$$p < q \wedge v < u \wedge 0 \le b^* \wedge 0 \le c^* \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_7 \wedge C_{11} \wedge C_{13}$$

15.

$$q < p \wedge u < v \wedge 0 \le b^* \wedge 0 \le c^* \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_5 \wedge C_6 \wedge C_{11} \wedge C_{13}$$

16.

$$q < p \wedge v < u \wedge 0 \le b^* \wedge 0 \le c^* \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_5 \wedge C_7 \wedge C_8 \wedge C_{11} \wedge C_{13} \wedge C_{14}$$

17.

$$p < q \wedge u < v \wedge 0 \le b^* \wedge 0 \le c^* \wedge C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_6 \wedge C_9 \wedge C_{11} \wedge C_{13} \wedge C_{14}$$

18.

$$t = 0 \wedge 0 < s \wedge 0 < b^* \wedge 0 < \phi \wedge C_1 \wedge C_2 \wedge C_{10}$$

19.

$$s = 0 \wedge 0 < t \wedge 0 < b^* \wedge \phi < 0 \wedge C_1 \wedge C_3 \wedge C_{10}$$

20.

$$n = 0 \wedge 0 < m \wedge 0 < c^* \wedge \phi < 0 \wedge C_1 \wedge C_2 \wedge C_{12}$$

21.

$$m = 0 \wedge 0 < n \wedge 0 < c^{*} \wedge 0 < \phi \wedge C_{1} \wedge C_{3} \wedge C_{12}$$

22.

$$st = 0 \wedge 0 < b^{*} \wedge 0 < c^{*} \wedge C_{1} \wedge C_{2} \wedge C_{3} \wedge C_{10} \wedge C_{12}$$

23.

$$mn = 0 \wedge 0 < b^{*} \wedge 0 < c^{*} \wedge C_{1} \wedge C_{2} \wedge C_{3} \wedge C_{10} \wedge C_{12}$$

24.

$$p < m + n \wedge t = 0 \wedge \phi = 0 \wedge 0 < s \wedge 0 < b^{*} \wedge c^{*} < 0 \wedge |\arg(\omega)| < \pi (m + n - p + 1) \wedge C_{1} \wedge C_{2} \wedge C_{10} \wedge C_{14} \wedge C_{15}$$

25.

$$q < m + n \wedge s = 0 \wedge \phi = 0 \wedge 0 < t \wedge 0 < b^{*} \wedge c^{*} < 0 \wedge |\arg(\omega)| < \pi (m + n - q + 1) \wedge C_{1} \wedge C_{3} \wedge C_{10} \wedge C_{14} \wedge C_{15}$$

26.

$$p = q - 1 \wedge t = 0 \wedge \phi = 0 \wedge 0 < s \wedge 0 < b^{*} \wedge 0 \leq c^{*} \wedge \pi c^{*} < |\arg(\omega)| \wedge C_{1} \wedge C_{2} \wedge C_{10} \wedge C_{14} \wedge C_{15}$$

27.

$$p = q + 1 \wedge s = 0 \wedge \phi = 0 \wedge 0 < t \wedge 0 < b^{*} \wedge 0 \leq c^{*} \wedge \pi c^{*} < |\arg(\omega)| \wedge C_{1} \wedge C_{3} \wedge C_{10} \wedge C_{14} \wedge C_{15}$$

28.

$$p < q - 1 \wedge t = 0 \wedge \phi = 0 \wedge 0 < s \wedge 0 < b^{*} \wedge 0 \leq c^{*} \wedge \pi c^{*} < |\arg(\omega)| \wedge |\arg(\omega)| < \pi (m + n - p + 1) \wedge C_{1} \wedge C_{2} \wedge C_{10} \wedge$$

29.

$$q + 1 < p \wedge s = 0 \wedge \phi = 0 \wedge 0 < t \wedge 0 < b^{*} \wedge 0 \leq c^{*} \wedge \pi c^{*} < |\arg(\omega)| \wedge |\arg(\omega)| < \pi (m + n - q + 1) \wedge C_{1} \wedge C_{3} \wedge C_{10} \wedge$$

30.

$$n = 0 \wedge \phi = 0 \wedge 0 < s + t \wedge 0 < m \wedge 0 < c^{*} \wedge b^{*} < 0 \wedge |\arg(\sigma)| < \pi (s + t - u + 1) \wedge C_{1} \wedge C_{2} \wedge C_{12} \wedge C_{14} \wedge C_{15}$$

31.

$$m = 0 \wedge \phi = 0 \wedge v < s + t \wedge 0 < n \wedge 0 < c^{*} \wedge b^{*} < 0 \wedge |\arg(\sigma)| < \pi (s + t - v + 1) \wedge C_{1} \wedge C_{3} \wedge C_{12} \wedge C_{14} \wedge C_{15}$$

32.

$$n = 0 \wedge \phi = 0 \wedge u = v - 1 \wedge 0 < m \wedge 0 < c^{*} \wedge 0 \leq b^{*} \wedge \pi b^{*} < |\arg(\sigma)| \wedge |\arg(\sigma)| < \pi (b^{*} + 1) \wedge C_{1} \wedge C_{2} \wedge C_{12} \wedge C_{14} \wedge$$

33.

$$m = 0 \wedge \phi = 0 \wedge u = v + 1 \wedge 0 < n \wedge 0 < c^{*} \wedge 0 \leq b^{*} \wedge \pi b^{*} < |\arg(\sigma)| \wedge |\arg(\sigma)| < \pi (b^{*} + 1) \wedge C_{1} \wedge C_{3} \wedge C_{12} \wedge C_{14} \wedge$$

34.

$$n = 0 \wedge \phi = 0 \wedge u < v - 1 \wedge 0 < m \wedge 0 < c^{*} \wedge 0 \leq b^{*} \wedge \pi b^{*} < |\arg(\sigma)| \wedge |\arg(\sigma)| < \pi(s + t - u + 1) \wedge C_{1} \wedge C_{2} \wedge C_{12} \wedge$$

35.

$$m = 0 \wedge \phi = 0 \wedge v + 1 < u \wedge 0 < n \wedge 0 < c^{*} \wedge 0 \leq b^{*} \wedge \pi b^{*} < |\arg(\sigma)| \wedge |\arg(\sigma)| < \pi(s + t - v + 1) \wedge C_{1} \wedge C_{3} \wedge C_{12} \wedge$$

36.

$$C_{17} \wedge t = 0 \wedge u < s \wedge 0 < b^{*} \wedge C_{10} \wedge C_{1} \wedge C_{2} \wedge C_{3}$$

37.

$$C_{17} \wedge s = 0 \wedge v < t \wedge 0 < b^{*} \wedge C_{10} \wedge C_{1} \wedge C_{2} \wedge C_{3}$$

38.

$$C_{16} \wedge n = 0 \wedge p < m \wedge 0 < c^{*} \wedge C_{12} \wedge C_{1} \wedge C_{2} \wedge C_{3}$$

39.

$$C_{16} \wedge m = 0 \wedge q < n \wedge 0 < c^{*} \wedge C_{12} \wedge C_{1} \wedge C_{2} \wedge C_{3}$$

4.6 The Inverse Laplace Transform of a G-function

The inverse laplace transform of a Meijer G-function can be expressed as another G-function. This is a fairly versatile method for computing this transform. However, I could not find the details in the literature, so I work them out here. In *[Luke1969]* (page 1270), section 5.6.3, there is a formula for the inverse Laplace transform of a G-function of argument bz, and convergence conditions are also given. However, we need a formula for argument bz^a for rational a.

We are asked to compute

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{zt} G(bz^{a}) \mathrm{d}z,$$

for positive real t. Three questions arise:

1. When does this integral converge?
2. How can we compute the integral?
3. When is our computation valid?

4.6.1 How to compute the integral

We shall work formally for now. Denote by $\Delta(s)$ the product of gamma functions appearing in the definition of G, so that

$$G(z) = \frac{1}{2\pi i} \int_{L} \Delta(s) z^{s} \mathrm{d}s.$$

Thus

$$f(t) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_L e^{zt} \Delta(s) b^s z^{as} \mathrm{d}s \mathrm{d}z.$$

We interchange the order of integration to get

$$f(t) = \frac{1}{2\pi i} \int_L b^s \Delta(s) \int_{c-i\infty}^{c+i\infty} e^{zt} z^{as} \frac{\mathrm{d}z}{2\pi i} \mathrm{d}s.$$

The inner integral is easily seen to be $\frac{1}{\Gamma(-as)} \frac{1}{t^{1+as}}$. (Using Cauchy's theorem and Jordan's lemma deform the contour to run from $-\infty$ to $-\infty$, encircling 0 once in the negative sense. For as real and greater than one, this contour can be pushed onto the negative real axis and the integral is recognised as a product of a sine and a gamma function. The formula is then proved using the functional equation of the gamma function, and extended to the entire domain of convergence of the original integral by appealing to analytic continuation.) Hence we find

$$f(t) = \frac{1}{t} \frac{1}{2\pi i} \int_L \Delta(s) \frac{1}{\Gamma(-as)} \left(\frac{b}{t^a}\right)^s \mathrm{d}s,$$

which is a so-called Fox H function (of argument $\frac{b}{t^a}$). For rational a, this can be expressed as a Meijer G-function using the gamma function multiplication theorem.

4.6.2 When this computation is valid

There are a number of obstacles in this computation. Interchange of integrals is only valid if all integrals involved are absolutely convergent. In particular the inner integral has to converge. Also, for our identification of the final integral as a Fox H / Meijer G-function to be correct, the poles of the newly obtained gamma function must be separated properly.

It is easy to check that the inner integral converges absolutely for $\Re(as) < -1$. Thus the contour L has to run left of the line $\Re(as) = -1$. Under this condition, the poles of the newly-introduced gamma function are separated properly.

It remains to observe that the Meijer G-function is an analytic, unbranched function of its parameters, and of the coefficient b. Hence so is $f(t)$. Thus the final computation remains valid as long as the initial integral converges, and if there exists a changed set of parameters where the computation is valid. If we assume w.l.o.g. that $a > 0$, then the latter condition is fulfilled if G converges along contours (2) or (3) of *[Luke1969]* (page 1270), section 5.2, i.e. either $\delta >= \frac{a}{2}$ or $p \geq 1, p \geq q$.

4.6.3 When the integral exists

Using *[Luke1969]* (page 1270), section 5.10, for any given meijer G-function we can find a dominant term of the form $z^a e^{bz^c}$ (although this expression might not be the best possible, because of cancellation).

We must thus investigate

$$\lim_{T\to\infty} \int_{c-iT}^{c+iT} e^{zt} z^a e^{bz^c} \mathrm{d}z.$$

(This principal value integral is the exact statement used in the Laplace inversion theorem.) We write $z = c + i\tau$. Then $arg(z) \to \pm\frac{\pi}{2}$, and so $e^{zt} \sim e^{it\tau}$ (where $\sim$ shall always mean "asymptotically equivalent up to a positive real multiplicative constant"). Also $z^{x+iy} \sim |\tau|^x e^{iy\log|\tau|} e^{\pm xi\frac{\pi}{2}}$.

Set $\omega_\pm = be^{\pm i\Re(c)\frac{\pi}{2}}$. We have three cases:

1. $b = 0$ or $\Re(c) \le 0$. In this case the integral converges if $\Re(a) \le -1$.
2. $b \ne 0$, $\Im(c) = 0$, $\Re(c) > 0$. In this case the integral converges if $\Re(\omega_\pm) < 0$.
3. $b \ne 0$, $\Im(c) = 0$, $\Re(c) > 0$, $\Re(\omega_\pm) \le 0$, and at least one of $\Re(\omega_\pm) = 0$. Here the same condition as in (1) applies.

4.7 Implemented G-Function Formulae

An important part of the algorithm is a table expressing various functions as Meijer G-functions. This is essentially a table of Mellin Transforms in disguise. The following automatically generated table shows the formulae currently implemented in Diofant. An entry "generated" means that the corresponding G-function has a variable number of parameters. This table is intended to shrink in future, when the algorithm's capabilities of deriving new formulae improve. Of course it has to grow whenever a new class of special functions is to be dealt with.

Elementary functions:

$$a = aG_{1,1}^{1,0}\left(\begin{matrix} & 1 \\ 0 & \end{matrix} \middle| z\right) + aG_{1,1}^{0,1}\left(\begin{matrix} 1 & \\ & 0 \end{matrix} \middle| z\right)$$

$$(b + pz^q)^{-a} = \frac{b^{-a}}{\Gamma(a)}G_{1,1}^{1,1}\left(\begin{matrix} -a+1 \\ 0 \end{matrix} \middle| \frac{pz^q}{b}\right)$$

$$\frac{-b^a + (pz^q)^a}{-b + pz^q} = \frac{1}{\pi}b^{a-1}G_{2,2}^{2,2}\left(\begin{matrix} 0, a \\ 0, a \end{matrix} \middle| \frac{pz^q}{b}\right)\sin(\pi a)$$

$$\frac{-b^a + z^a}{-b + z} = \frac{1}{\pi}b^{a-1}G_{2,2}^{2,2}\left(\begin{matrix} 0, a \\ 0, a \end{matrix} \middle| \frac{z}{b}\right)\sin(\pi a)$$

$$\left(a + \sqrt{a^2 + pz^q}\right)^b = -\frac{a^b b}{2\sqrt{\pi}}G_{2,2}^{1,2}\left(\begin{matrix} \frac{b}{2} + \frac{1}{2}, \frac{b}{2} + 1 & \\ 0 & b \end{matrix} \middle| \frac{pz^q}{a^2}\right)$$

$$\left(-a + \sqrt{a^2 + pz^q}\right)^b = \frac{a^b b}{2\sqrt{\pi}}G_{2,2}^{1,2}\left(\begin{matrix} \frac{b}{2} + \frac{1}{2}, \frac{b}{2} + 1 & \\ b & 0 \end{matrix} \middle| \frac{pz^q}{a^2}\right)$$

$$\frac{\left(a + \sqrt{a^2 + pz^q}\right)^b}{\sqrt{a^2 + pz^q}} = \frac{1}{\sqrt{\pi}}a^{b-1}G_{2,2}^{1,2}\left(\begin{matrix} \frac{b}{2} + \frac{1}{2}, \frac{b}{2} & \\ 0 & b \end{matrix} \middle| \frac{pz^q}{a^2}\right)$$

$$\frac{\left(-a + \sqrt{a^2 + pz^q}\right)^b}{\sqrt{a^2 + pz^q}} = \frac{1}{\sqrt{\pi}}a^{b-1}G_{2,2}^{1,2}\left(\begin{matrix} \frac{b}{2} + \frac{1}{2}, \frac{b}{2} & \\ b & 0 \end{matrix} \middle| \frac{pz^q}{a^2}\right)$$

$$\left(\sqrt{p}z^{\frac{q}{2}} + \sqrt{a + pz^q}\right)^b = -\frac{a^{\frac{b}{2}}b}{2\sqrt{\pi}}G_{2,2}^{2,1}\left(\begin{matrix} \frac{b}{2} + 1 & -\frac{b}{2} + 1 \\ 0, \frac{1}{2} & \end{matrix} \middle| \frac{pz^q}{a}\right)$$

$$\left(-\sqrt{p}z^{\frac{q}{2}} + \sqrt{a + pz^q}\right)^b = \frac{a^{\frac{b}{2}}b}{2\sqrt{\pi}}G_{2,2}^{2,1}\left(\begin{matrix} -\frac{b}{2} + 1 & \frac{b}{2} + 1 \\ 0, \frac{1}{2} & \end{matrix} \middle| \frac{pz^q}{a}\right)$$

$$\frac{\left(\sqrt{p}z^{\frac{q}{2}} + \sqrt{a + pz^q}\right)^b}{\sqrt{a + pz^q}} = \frac{1}{\sqrt{\pi}}a^{\frac{b}{2} - \frac{1}{2}}G_{2,2}^{2,1}\left(\begin{matrix} \frac{b}{2} + \frac{1}{2} & -\frac{b}{2} + \frac{1}{2} \\ 0, \frac{1}{2} & \end{matrix} \middle| \frac{pz^q}{a}\right)$$

$$\frac{\left(-\sqrt{p}z^{\frac{q}{2}} + \sqrt{a + pz^q}\right)^b}{\sqrt{a + pz^q}} = \frac{1}{\sqrt{\pi}}a^{\frac{b}{2} - \frac{1}{2}}G_{2,2}^{2,1}\left(\begin{matrix} -\frac{b}{2} + \frac{1}{2} & \frac{b}{2} + \frac{1}{2} \\ 0, \frac{1}{2} & \end{matrix} \middle| \frac{pz^q}{a}\right)$$

$$e^{pz^q} = G_{0,1}^{1,0}\left(\begin{matrix} & \\ 0 & \end{matrix} \middle| pz^q \operatorname{exp_polar}\left(-i\pi\right)\right)$$

Functions involving $|b - pz^q|$:

$$|b - pz^q|^{-a} = 2G_{2,2}^{1,1}\left(\begin{matrix} -a+1 & -\frac{a}{2}+\frac{1}{2} \\ 0 & -\frac{a}{2}+\frac{1}{2} \end{matrix} \middle| \frac{pz^q}{b}\right) \sin\left(\frac{\pi a}{2}\right) |b|^{-a}\Gamma(-a+1), \text{ if } \Re a < 1$$

Functions involving $\operatorname{Chi}(pz^q)$:

$$\operatorname{Chi}(pz^q) = -\frac{\pi^{\frac{3}{2}}}{2}G_{2,4}^{2,0}\left(\begin{matrix} & \frac{1}{2}, 1 \\ 0, 0 & \frac{1}{2}, \frac{1}{2} \end{matrix} \middle| \frac{p^2}{4}z^{2q}\right)$$

Functions involving $\operatorname{Ci}(pz^q)$:

$$\operatorname{Ci}(pz^q) = -\frac{\sqrt{\pi}}{2}G_{1,3}^{2,0}\left(\begin{matrix} & 1 \\ 0, 0 & \frac{1}{2} \end{matrix} \middle| \frac{p^2}{4}z^{2q}\right)$$

Functions involving $\operatorname{Ei}(pz^q)$:

$$\operatorname{Ei}(pz^q) = -i\pi G_{1,1}^{1,0}\left(\begin{matrix} & 1 \\ 0 & \end{matrix} \middle| z\right) - G_{1,2}^{2,0}\left(\begin{matrix} & 1 \\ 0, 0 & \end{matrix} \middle| pz^q \operatorname{exp_polar}\left(i\pi\right)\right) - i\pi G_{1,1}^{0,1}\left(\begin{matrix} 1 & \\ & 0 \end{matrix} \middle| z\right)$$

Functions involving $\theta(-b + pz^q)$:

$$(-b + pz^q)^{a-1}\theta(-b + pz^q) = b^{a-1}G_{1,1}^{0,1}\left(\begin{matrix} a & \\ & 0 \end{matrix} \middle| \frac{pz^q}{b}\right)\Gamma(a), \text{ if } b > 0$$

$$(b - pz^q)^{a-1}\theta(b - pz^q) = b^{a-1}G_{1,1}^{1,0}\left(\begin{matrix} & a \\ 0 & \end{matrix} \middle| \frac{pz^q}{b}\right)\Gamma(a), \text{ if } b > 0$$

$$(-b + pz^q)^{a-1}\theta\left(z - \left(\frac{b}{p}\right)^{\frac{1}{q}}\right) = b^{a-1}G_{1,1}^{0,1}\left(\begin{matrix} a & \\ & 0 \end{matrix} \middle| \frac{pz^q}{b}\right)\Gamma(a), \text{ if } b > 0$$

$$(b - pz^q)^{a-1}\theta\left(-z + \left(\frac{b}{p}\right)^{\frac{1}{q}}\right) = b^{a-1}G_{1,1}^{1,0}\left(\begin{matrix} & a \\ 0 & \end{matrix} \middle| \frac{pz^q}{b}\right)\Gamma(a), \text{ if } b > 0$$

Functions involving $\operatorname{Shi}(pz^q)$:

$$\operatorname{Shi}(pz^q) = \frac{\sqrt{\pi}p}{4}z^q G_{1,3}^{1,1}\left(\begin{matrix} \frac{1}{2} & \\ 0 & -\frac{1}{2}, -\frac{1}{2} \end{matrix} \middle| \frac{p^2}{4}z^{2q}\operatorname{exp_polar}\left(i\pi\right)\right)$$

Functions involving $\operatorname{Si}(pz^q)$:

$$\operatorname{Si}(pz^q) = \frac{\sqrt{\pi}}{2}G_{1,3}^{1,1}\left(\begin{matrix} 1 & \\ \frac{1}{2} & 0, 0 \end{matrix} \middle| \frac{p^2}{4}z^{2q}\right)$$

Functions involving $I_a(pz^q)$:

$$I_a(pz^q) = \pi G_{1,3}^{1,0}\left(\begin{matrix} & \frac{a}{2}+\frac{1}{2} \\ \frac{a}{2} & -\frac{a}{2}, \frac{a}{2}+\frac{1}{2} \end{matrix} \middle| \frac{p^2}{4}z^{2q}\right)$$

Functions involving $J_a(pz^q)$:

$$J_a(pz^q) = G_{0,2}^{1,0}\left(\begin{matrix} & \\ \frac{a}{2} & -\frac{a}{2} \end{matrix} \middle| \frac{p^2}{4}z^{2q}\right)$$

Functions involving $K_a(pz^q)$:

$$K_a(pz^q) = \frac{1}{2}G_{0,2}^{2,0}\left(\begin{matrix} & \\ \frac{a}{2}, -\frac{a}{2} & \end{matrix} \middle| \frac{p^2}{4}z^{2q}\right)$$

Functions involving $Y_a\left(pz^q\right)$:

$$Y_a\left(pz^q\right) = G_{1,3}^{2,0}\left(\begin{matrix} & -\frac{a}{2} - \frac{1}{2} \\ \frac{a}{2}, -\frac{a}{2} & -\frac{a}{2} - \frac{1}{2} \end{matrix} \middle| \frac{p^2}{4} z^{2q}\right)$$

Functions involving $\cos\left(pz^q\right)$:

$$\cos\left(pz^q\right) = \sqrt{\pi} G_{0,2}^{1,0}\left(\begin{matrix} & \\ 0 & \frac{1}{2} \end{matrix} \middle| \frac{p^2}{4} z^{2q}\right)$$

Functions involving $\cosh\left(pz^q\right)$:

$$\cosh\left(pz^q\right) = \pi^{\frac{3}{2}} G_{1,3}^{1,0}\left(\begin{matrix} & \frac{1}{2} \\ 0 & \frac{1}{2}, \frac{1}{2} \end{matrix} \middle| \frac{p^2}{4} z^{2q}\right)$$

Functions involving $E\left(pz^q\right)$:

$$E\left(pz^q\right) = -\frac{1}{4} G_{2,2}^{1,2}\left(\begin{matrix} \frac{1}{2}, \frac{3}{2} & \\ 0 & 0 \end{matrix} \middle| -pz^q\right)$$

Functions involving $K\left(pz^q\right)$:

$$K\left(pz^q\right) = \frac{1}{2} G_{2,2}^{1,2}\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} & \\ 0 & 0 \end{matrix} \middle| -pz^q\right)$$

Functions involving $\operatorname{erf}\left(pz^q\right)$:

$$\operatorname{erf}\left(pz^q\right) = \frac{1}{\sqrt{\pi}} G_{1,2}^{1,1}\left(\begin{matrix} 1 & \\ \frac{1}{2} & 0 \end{matrix} \middle| p^2 z^{2q}\right)$$

Functions involving $\operatorname{erfc}\left(pz^q\right)$:

$$\operatorname{erfc}\left(pz^q\right) = \frac{1}{\sqrt{\pi}} G_{1,2}^{2,0}\left(\begin{matrix} & 1 \\ 0, \frac{1}{2} & \end{matrix} \middle| p^2 z^{2q}\right)$$

Functions involving $\operatorname{erfi}\left(pz^q\right)$:

$$\operatorname{erfi}\left(pz^q\right) = \frac{pz^q}{\sqrt{\pi}} G_{1,2}^{1,1}\left(\begin{matrix} \frac{1}{2} & \\ 0 & -\frac{1}{2} \end{matrix} \middle| -p^2 z^{2q}\right)$$

Functions involving $\operatorname{E}_a\left(pz^q\right)$:

$$\operatorname{E}_a\left(pz^q\right) = G_{1,2}^{2,0}\left(\begin{matrix} & a \\ a - 1, 0 & \end{matrix} \middle| pz^q\right)$$

Functions involving $C\left(pz^q\right)$:

$$C\left(pz^q\right) = \frac{1}{2} G_{1,3}^{1,1}\left(\begin{matrix} 1 & \\ \frac{1}{4} & 0, \frac{3}{4} \end{matrix} \middle| \frac{\pi^2 p^4}{16} z^{4q}\right)$$

Functions involving $S\left(pz^q\right)$:

$$S\left(pz^q\right) = \frac{1}{2} G_{1,3}^{1,1}\left(\begin{matrix} 1 & \\ \frac{3}{4} & 0, \frac{1}{4} \end{matrix} \middle| \frac{\pi^2 p^4}{16} z^{4q}\right)$$

Functions involving $\log\left(pz^q\right)$:

$$\log^n\left(pz^q\right) = \text{generated}$$

$$\log\left(a + pz^{q}\right) = G_{1, 1}^{1, 0}\left(\begin{matrix} & 1 \\0 & \end{matrix} \middle| {z} \right) \log{\left (a \right)} + G_{1, 1}^{0, 1}\left(\begin{matrix} 1 & \\ & 0 \end{matrix} \middle| {z} \right) \log{\left (a \right)} + G_{2, 2}^{1, 2}\left(\begin{matrix} 1, 1 & \\ 1 & 0 \end{matrix} \middle| {\frac{pz^{q}}{a}} \right)$$

$$\log\left(\left|{a - pz^{q}}\right|\right) = G_{1, 1}^{1, 0}\left(\begin{matrix} & 1 \\0 & \end{matrix} \middle| {z} \right) \log{\left (\left|{a}\right| \right)} + G_{1, 1}^{0, 1}\left(\begin{matrix} 1 & \\ & 0 \end{matrix} \middle| {z} \right) \log{\left (\left|{a}\right| \right)} + \pi G_{3, 3}^{1, 2}\left(\begin{matrix} 1, 1 & \frac{1}{2} \\ 1 & 0, \frac{1}{2} \end{matrix} \middle| {\frac{pz^{q}}{a}} \right)$$

Functions involving $\sin\left(pz^{q}\right)$:

$$\sin\left(pz^{q}\right) = \sqrt{\pi} G_{0, 2}^{1, 0}\left(\begin{matrix} & \\ \frac{1}{2} & 0 \end{matrix} \middle| {\frac{p^{2}}{4} z^{2q}} \right)$$

Functions involving $\sinh\left(pz^{q}\right)$:

$$\sinh\left(pz^{q}\right) = \pi^{\frac{3}{2}} G_{1, 3}^{1, 0}\left(\begin{matrix} & 1 \\ \frac{1}{2} & 1, 0 \end{matrix} \middle| {\frac{p^{2}}{4} z^{2q}} \right)$$

Functions involving $\theta\left(-pz^{q} + 1\right)$, $\log\left(pz^{q}\right)$:

$$\log^{n}\left(pz^{q}\right)\theta\left(-pz^{q} + 1\right) = \text{generated}$$

$$\log^{n}\left(pz^{q}\right)\theta\left(pz^{q} - 1\right) = \text{generated}$$

4.8 Numerical evaluation

4.8.1 Basics

Exact Diofant expressions can be converted to floating-point approximations (decimal numbers) using either the `.evalf()` method or the `N()` function. `N(expr, <args>)` is equivalent to `sympify(expr).evalf(<args>)`.

```
>>> N(sqrt(2)*pi)
4.44288293815837
>>> (sqrt(2)*pi).evalf()
4.44288293815837
```

By default, numerical evaluation is performed to an accuracy of 15 decimal digits. You can optionally pass a desired accuracy (which should be a positive integer) as an argument to `evalf` or `N`:

```
>>> N(sqrt(2)*pi, 5)
4.4429
>>> N(sqrt(2)*pi, 50)
4.4428829381583662470158809900606936986146216893757
```

Complex numbers are supported:

```
>>> N(1/(pi + I), 20)
0.28902548222223624241 - 0.091999668350375232456*I
```

If the expression contains symbols or for some other reason cannot be evaluated numerically, calling `.evalf()` or `N()` returns the original expression, or in some cases a partially evaluated expression. For example, when the expression is a polynomial in expanded form, the coefficients are evaluated:

```
>>> x = Symbol('x')
>>> (pi*x**2 + x/3).evalf(strict=False)
3.14159265358979*x**2 + 0.333333333333333*x
```

You can also use the standard Python functions `float()`, `complex()` to convert Diofant expressions to regular Python numbers:

```
>>> float(pi)
3.1415926535...
>>> complex(pi+E*I)
(3.1415926535...+2.7182818284...j)
```

If these functions are used, failure to evaluate the expression to an explicit number (for example if the expression contains symbols) will raise an exception.

There is essentially no upper precision limit. The following command, for example, computes the first 100,000 digits of π/e:

```
>>> N(pi/E, 100000)
1.1557273497909217179...
```

This shows digits 999,951 through 1,000,000 of pi:

```
>>> str(N(pi, 10**6))[-50:]
'95678796130331164628399634646042209010610577945815'
```

High-precision calculations can be slow. It is recommended (but entirely optional) to install gmpy2, which will significantly speed up computations such as the one above.

4.8.2 Floating-point numbers

Floating-point numbers in Diofant are instances of the class `Float`. A `Float` can be created with a custom precision as second argument:

```
>>> Float(0.1)
0.100000000000000
>>> Float(0.1, 10)
0.1000000000
>>> Float(0.125, 30)
0.125000000000000000000000000000
>>> Float(0.1, 30)
0.100000000000000005551115123126
```

As the last example shows, some Python floats are only accurate to about 15 digits as inputs, while others (those that have a denominator that is a power of 2, like 0.125 = 1/8) are exact. To create a `Float` from a high-precision decimal number, it is better to pass a string, `Rational`, or `evalf` a `Rational`:

```
>>> Float('0.1', 30)
0.100000000000000000000000000000
>>> Float(Rational(1, 10), 30)
0.100000000000000000000000000000
>>> Rational(1, 10).evalf(30)
0.100000000000000000000000000000
```

The precision of a number determines 1) the precision to use when performing arithmetic with the number, and 2) the number of digits to display when printing the number. When two numbers with different precision are used together in an arithmetic operation, the higher of the precisions is used for the result. The product of 0.1 +/- 0.001 and 3.1415 +/- 0.0001 has an uncertainty of about 0.003 and yet 5 digits of precision are shown.

```
>>> Float(0.1, 3)*Float(3.1415, 5)
0.31417
```

So the displayed precision should not be used as a model of error propagation or significance arithmetic; rather, this scheme is employed to ensure stability of numerical algorithms.

Function *N()* (page 146) (or *evalf()* (page 145) method) can be used to change the precision of existing floating-point numbers:

```
>>> N(3.5, strict=False)
3.50000000000000
>>> N(3.5, 5)
3.5000
```

However, you can "increase" precision of the *Float* (page 85) number only with it's class constructor:

```
>>> Float(3.5, 30)
3.50000000000000000000000000000
```

4.8.3 Accuracy and error handling

When the input to `N` or `evalf` is a complicated expression, numerical error propagation becomes a concern. As an example, consider the 100'th Fibonacci number and the excellent (but not exact) approximation $\varphi^{100}/\sqrt{5}$ where φ is the golden ratio. With ordinary floating-point arithmetic, subtracting these numbers from each other erroneously results in a complete cancellation:

```
>>> a, b = GoldenRatio**1000/sqrt(5), fibonacci(1000)
>>> float(a)
4.3466557686937455e+208
>>> float(b)
4.3466557686937455e+208
>>> float(a) - float(b)
0.0
```

`N` and `evalf` keep track of errors and automatically increase the precision used internally in order to obtain a correct result:

```
>>> N(fibonacci(100) - GoldenRatio**100/sqrt(5))
-5.64613129282185e-22
```

Unfortunately, numerical evaluation cannot tell an expression that is exactly zero apart from one that is merely very small. The working precision is therefore capped, by default to around 100 digits. If we try with the 1000'th Fibonacci number, the following happens:

```
>>> N(fibonacci(1000) - (GoldenRatio)**1000/sqrt(5))
Traceback (most recent call last):
...
PrecisionExhausted: ...
```

The exception indicates that N failed to achieve full accuracy. To force a higher working precision, the maxn keyword argument can be used:

```
>>> N(fibonacci(1000) - (GoldenRatio)**1000/sqrt(5), maxn=500)
-4.60123853010113e-210
```

Normally, maxn can be set very high (thousands of digits), but be aware that this may cause significant slowdown in extreme cases.

Also, you can set strict keyword argument to False to obtain imprecise answer instead of exception. For example, if we add a term so that the Fibonacci approximation becomes exact (the full form of Binet's formula), we get an expression that is exactly zero, but N does not know this:

```
>>> f = fibonacci(100) - (GoldenRatio**100 - (GoldenRatio-1)**100)/sqrt(5)
>>> N(f, strict=False)
0.e-126
>>> N(f, maxn=1000, strict=False)
0.e-1336
```

In situations where such cancellations are known to occur, the chop options is useful. This basically replaces very small numbers in the real or imaginary portions of a number with exact zeros:

```
>>> N(f, chop=True)
0
>>> N(3 + I*f, chop=True)
3.00000000000000
```

In situations where you wish to remove meaningless digits, re-evaluation or the use of the round method are useful:

```
>>> Float('.1', '')*Float('.12345', '')
0.012297
>>> ans = _
>>> N(ans, 1)
0.01
>>> ans.round(2)
0.01
```

If you are dealing with a numeric expression that contains no floats, it can be evaluated to arbitrary precision. To round the result relative to a given decimal, the round method is useful:

```
>>> v = 10*pi + cos(1)
>>> N(v)
31.9562288417661
>>> v.round(3)
31.956
```

4.8.4 Sums and integrals

Sums (in particular, infinite series) and integrals can be used like regular closed-form expressions, and support arbitrary-precision evaluation:

```
>>> var('n x')
(n, x)
>>> Sum(1/n**n, (n, 1, oo)).evalf()
1.29128599706266
>>> Integral(x**(-x), (x, 0, 1)).evalf()
1.29128599706266
>>> Sum(1/n**n, (n, 1, oo)).evalf(50)
1.2912859970626635404072825905956005414986193682745
>>> Integral(x**(-x), (x, 0, 1)).evalf(50)
1.2912859970626635404072825905956005414986193682745
>>> (Integral(exp(-x**2), (x, -oo, oo)) ** 2).evalf(30)
3.14159265358979323846264338328
```

By default, the tanh-sinh quadrature algorithm is used to evaluate integrals. This algorithm is very efficient and robust for smooth integrands (and even integrals with endpoint singularities), but may struggle with integrals that are highly oscillatory or have mid-interval discontinuities. In many cases, `evalf/N` will correctly estimate the error. With the following integral, the result is accurate but only good to four digits:

```
>>> f = abs(sin(x))
>>> Integral(abs(sin(x)), (x, 0, 4)).evalf()
Traceback (most recent call last):
...
PrecisionExhausted: ...
```

It is better to split this integral into two pieces:

```
>>> (Integral(f, (x, 0, pi)) + Integral(f, (x, pi, 4))).evalf()
2.34635637913639
```

A similar example is the following oscillatory integral:

```
>>> Integral(sin(x)/x**2, (x, 1, oo)).evalf()
Traceback (most recent call last):
...
PrecisionExhausted: ...
```

It can be dealt with much more efficiently by telling `evalf` or `N` to use an oscillatory quadrature algorithm:

```
>>> Integral(sin(x)/x**2, (x, 1, oo)).evalf(quad='osc')
0.504067061906928
>>> Integral(sin(x)/x**2, (x, 1, oo)).evalf(20, quad='osc')
0.50406706190692837199
```

Oscillatory quadrature requires an integrand containing a factor cos(ax+b) or sin(ax+b). Note that many other oscillatory integrals can be transformed to this form with a change of variables:

```
>>> init_printing(pretty_print=True, use_unicode=False,
...               wrap_line=False, no_global=True)
>>> intgrl = Integral(sin(1/x), (x, 0, 1)).transform(x, 1/x)
>>> intgrl
 oo
  /
 |
```

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```
 |  sin(x)
 |  ------ dx
 |     2
 |    x
 |
/
1
>>> N(intgrl, quad='osc')
0.504067061906928
```

Infinite series use direct summation if the series converges quickly enough. Otherwise, extrapolation methods (generally the Euler-Maclaurin formula but also Richardson extrapolation) are used to speed up convergence. This allows high-precision evaluation of slowly convergent series:

```
>>> var('k')
k
>>> Sum(1/k**2, (k, 1, oo)).evalf(strict=False)
1.64493406684823
>>> zeta(2).evalf()
1.64493406684823
>>> Sum(1/k-log(1+1/k), (k, 1, oo)).evalf()
0.577215664901533
>>> Sum(1/k-log(1+1/k), (k, 1, oo)).evalf(50)
0.57721566490153286060651209008240243104215933593992
>>> EulerGamma.evalf(50)
0.57721566490153286060651209008240243104215933593992
```

The Euler-Maclaurin formula is also used for finite series, allowing them to be approximated quickly without evaluating all terms:

```
>>> Sum(1/k, (k, 10000000, 20000000)).evalf()
0.693147255559946
```

Note that `evalf` makes some assumptions that are not always optimal. For fine-tuned control over numerical summation, it might be worthwhile to manually use the method `Sum.euler_maclaurin`.

Special optimizations are used for rational hypergeometric series (where the term is a product of polynomials, powers, factorials, binomial coefficients and the like). `N/evalf` sum series of this type very rapidly to high precision. For example, this Ramanujan formula for pi can be summed to 10,000 digits in a fraction of a second with a simple command:

```
>>> f = factorial
>>> n = Symbol('n', integer=True)
>>> R = 9801/sqrt(8)/Sum(f(4*n)*(1103+26390*n)/f(n)**4/396**(4*n),
...                      (n, 0, oo))
>>> N(R, 10000, strict=False)
3.141592653589793238462643383279502884197169399375105820974944592307...
```

4.8.5 Numerical simplification

The function `nsimplify` attempts to find a formula that is numerically equal to the given input. This feature can be used to guess an exact formula for an approximate floating-point input, or to guess a simpler formula for a complicated symbolic input. The algorithm used

by `nsimplify` is capable of identifying simple fractions, simple algebraic expressions, linear combinations of given constants, and certain elementary functional transformations of any of the preceding.

Optionally, `nsimplify` can be passed a list of constants to include (e.g. pi) and a minimum numerical tolerance. Here are some elementary examples:

```
>>> nsimplify(0.1)
1/10
>>> nsimplify(6.28, [pi], tolerance=0.01)
2*pi
>>> nsimplify(pi, tolerance=0.01)
22/7
>>> nsimplify(pi, tolerance=0.001)
355
---
113
>>> nsimplify(0.33333, tolerance=1e-4)
1/3
>>> nsimplify(2.0**(1/3.), tolerance=0.001)
635
---
504
>>> nsimplify(2.0**(1/3.), tolerance=0.001, full=True)
3 ___
\/ 2
```

Here are several more advanced examples:

```
>>> nsimplify(Float('0.130198866629986772369127970337', 30), [pi, E])
    1
----------
5*pi
---- + 2*E
 7
>>> nsimplify(cos(atan('1/3')))
    ____
3*\/ 10
--------
   10
>>> nsimplify(4/(1+sqrt(5)), [GoldenRatio])
-2 + 2*GoldenRatio
>>> nsimplify(2 + exp(2*atan('1/4')*I))
49   8*I
-- + ---
17    17
>>> nsimplify((1/(exp(3*pi*I/5)+1)))
           ___________
          /   ___
1        /  \/ 5    1
- - I*  /   ----- + -
2     \/      10    4
>>> nsimplify(I**I, [pi])
 -pi
 ----
  2
E
>>> n = Symbol('n')
```

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```
>>> nsimplify(Sum(1/n**2, (n, 1, oo)), [pi])
  2
pi
---
 6
>>> nsimplify(gamma('1/4')*gamma('3/4'), [pi])
  ___
\/ 2 *pi
```

4.9 Numeric Computation

Symbolic computer algebra systems like Diofant facilitate the construction and manipulation of mathematical expressions. Unfortunately when it comes time to evaluate these expressions on numerical data, symbolic systems often have poor performance.

Fortunately Diofant offers a number of easy-to-use hooks into other numeric systems, allowing you to create mathematical expressions in Diofant and then ship them off to the numeric system of your choice. This page documents many of the options available including the `math` library, the popular array computing package `numpy`, code generation in `Fortran` or `C`, and the use of the array compiler `Theano`.

4.9.1 Subs/evalf

Subs is the slowest but simplest option. It runs at Diofant speeds. The `.subs(...).evalf()` method can substitute a numeric value for a symbolic one and then evaluate the result within Diofant.

```
>>> expr = sin(x)/x
>>> expr.evalf(subs={x: 3.14}, strict=False)
0.000507214304613640
```

This method is slow. You should use this method production only if performance is not an issue. You can expect `.subs` to take tens of microseconds. It can be useful while prototyping or if you just want to see a value once.

4.9.2 Lambdify

The `lambdify` function translates Diofant expressions into Python functions, leveraging a variety of numerical libraries. It is used as follows:

```
>>> expr = sin(x)/x
>>> f = lambdify(x, expr)
>>> f(3.14)
0.000507214304614
```

Here lambdify makes a function that computes `f(x) = sin(x)/x`. By default lambdify relies on implementations in the `math` standard library. This numerical evaluation takes on the order of hundreds of nanoseconds, roughly two orders of magnitude faster than the `.subs` method. This is the speed difference between Diofant and raw Python.

Lambdify can leverage a variety of numerical backends. By default it uses the `math` library. However it also supports `mpmath` and most notably, `numpy`. Using the `numpy` library gives the generated function access to powerful vectorized ufuncs that are backed by compiled C code.

```
>>> expr = sin(x)/x
>>> f = lambdify(x, expr, "numpy")
```

```
>>> import numpy
>>> data = numpy.linspace(1, 10, 10000)
>>> pprint(f(data))
[ 0.84147098  0.84119981  0.84092844 ..., -0.05426074 -0.05433146
                           -0.05440211]
```

If you have array-based data this can confer a considerable speedup, on the order of 10 nano-seconds per element. Unfortunately numpy incurs some start-up time and introduces an overhead of a few microseconds.

4.9.3 uFuncify

While NumPy operations are very efficient for vectorized data they sometimes incur unnecessary costs when chained together. Consider the following operation

```
x = get_numpy_array(...)
y = sin(x)/x
```

The operators `sin` and `/` call routines that execute tight for loops in `C`. The resulting computation looks something like this

```
for(int i = 0; i < n; i++)
{
    temp[i] = sin(x[i]);
}
for(int i = i; i < n; i++)
{
    y[i] = temp[i] / x[i];
}
```

This is slightly sub-optimal because

1. We allocate an extra `temp` array
2. We walk over `x` memory twice when once would have been sufficient

A better solution would fuse both element-wise operations into a single for loop

```
for(int i = i; i < n; i++)
{
    y[i] = sin(x[i]) / x[i];
}
```

Statically compiled projects like NumPy are unable to take advantage of such optimizations. Fortunately, Diofant is able to generate efficient low-level C or Fortran code. It can then depend on projects like `Cython` or `f2py` to compile and reconnect that code back up to Python. Fortunately this process is well automated and a Diofant user wishing to make use of this code generation should call the `ufuncify` function

```
>>> expr = sin(x)/x
```

```
>>> from diofant.utilities.autowrap import ufuncify
>>> f = ufuncify((x,), expr)
```

This function `f` consumes and returns a NumPy array. Generally `ufuncify` performs at least as well as `lambdify`. If the expression is complicated then `ufuncify` often significantly outperforms the NumPy backed solution. Jensen has a good blog post on this topic.

4.9.4 Theano

Diofant has a strong connection with Theano, a mathematical array compiler. Diofant expressions can be easily translated to Theano graphs and then compiled using the Theano compiler chain.

```
>>> expr = sin(x)/x
```

```
>>> from diofant.printing.theanocode import theano_function
>>> f = theano_function([x], [expr])
```

If array broadcasting or types are desired then Theano requires this extra information

```
>>> f = theano_function([x], [expr], dims={x: 1}, dtypes={x: 'float64'})
```

Theano has a more sophisticated code generation system than Diofant's C/Fortran code printers. Among other things it handles common sub-expressions and compilation onto the GPU. Theano also supports Diofant Matrix and Matrix Expression objects.

4.9.5 So Which Should I Use?

The options here were listed in order from slowest and least dependencies to fastest and most dependencies. For example, if you have Theano installed then that will often be the best choice. If you don't have Theano but do have `f2py` then you should use `ufuncify`.

Tool	Speed	Qualities	Dependencies
subs/evalf	50us	Simple	None
lambdify	1us	Scalar functions	math
lambdify-numpy	10ns	Vector functions	numpy
ufuncify	10ns	Complex vector expressions	f2py, Cython
Theano	10ns	Many outputs, CSE, GPUs	Theano

4.10 Term rewriting

Term rewriting is a very general class of functionalities which are used to convert expressions of one type in terms of expressions of different kind. For example expanding, combining and converting expressions apply to term rewriting, and also simplification routines can be included here. Currently Diofant has several functions and basic built-in methods for performing various types of rewriting.

4.10.1 Expanding

The simplest rewrite rule is expanding expressions into a _sparse_ form. Expanding has several flavors and include expanding complex valued expressions, arithmetic expand of products and powers but also expanding functions in terms of more general functions is possible. Below are listed all currently available expand rules.

Expanding of arithmetic expressions involving products and powers:

```
>>> ((x + y)*(x - y)).expand(basic=True)
x**2 - y**2
>>> ((x + y + z)**2).expand(basic=True)
x**2 + 2*x*y + 2*x*z + y**2 + 2*y*z + z**2
```

Arithmetic expand is done by default in `expand()` so the keyword `basic` can be omitted. However you can set `basic=False` to avoid this type of expand if you use rules described below. This give complete control on what is done with the expression.

Another type of expand rule is expanding complex valued expressions and putting them into a normal form. For this `complex` keyword is used. Note that it will always perform arithmetic expand to obtain the desired normal form:

```
>>> (x + I*y).expand(complex=True)
re(x) + I*re(y) + I*im(x) - im(y)
```

```
>>> sin(x + I*y).expand(complex=True)
sin(re(x) - im(y))*cosh(re(y) + im(x)) + I*cos(re(x) - im(y))*sinh(re(y) + im(x))
```

Note also that the same behavior can be obtained by using `as_real_imag()` method. However it will return a tuple containing the real part in the first place and the imaginary part in the other. This can be also done in a two step process by using `collect` function:

```
>>> (x + I*y).as_real_imag()
(re(x) - im(y), re(y) + im(x))
```

```
>>> collect((x + I*y).expand(complex=True), I, evaluate=False)
{1: re(x) - im(y), I: re(y) + im(x)}
```

There is also possibility for expanding expressions in terms of expressions of different kind. This is very general type of expanding and usually you would use `rewrite()` to do specific type of rewrite:

```
>>> GoldenRatio.expand(func=True)
1/2 + sqrt(5)/2
```

4.10.2 Common Subexpression Detection and Collection

Before evaluating a large expression, it is often useful to identify common subexpressions, collect them and evaluate them at once. This is implemented in the *cse()* (page 797) function. Examples:

```
>>> pprint(cse(sqrt(sin(x))), use_unicode=True)
⎛    ⎡  ________⎤⎞
⎝[], ⎣╲╱ sin(x) ⎦⎠
```

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```
>>> pprint(cse(sqrt(sin(x)+5)*sqrt(sin(x)+4)), use_unicode=True)
⎛                ⎡  ________   ________⎤⎞
⎝[(x₀, sin(x))], ⎣╲╱ x₀ + 4 ⋅╲╱ x₀ + 5 ⎦⎠

>>> pprint(cse(sqrt(sin(x+1) + 5 + cos(y))*sqrt(sin(x+1) + 4 + cos(y))),
...     use_unicode=True)
⎛                             ⎡  ________   ________⎤⎞
⎝[(x₀, sin(x + 1) + cos(y))], ⎣╲╱ x₀ + 4 ⋅╲╱ x₀ + 5 ⎦⎠

>>> pprint(cse((x-y)*(z-y) + sqrt((x-y)*(z-y))), use_unicode=True)
⎛                                     ⎡  ____     ⎤⎞
⎝[(x₀, -y), (x₁, (x + x₀)⋅(x₀ + z))], ⎣╲╱ x₁  + x₁⎦⎠
```

Optimizations to be performed before and after common subexpressions elimination can be passed in the``optimizations`` optional argument. A set of predefined basic optimizations can be applied by passing `optimizations='basic'`:

```
>>> pprint(cse((x-y)*(z-y) + sqrt((x-y)*(z-y)), optimizations='basic'),
...     use_unicode=True)
⎛                          ⎡  ____     ⎤⎞
⎝[(x₀, -(x - y)⋅(y - z))], ⎣╲╱ x₀  + x₀⎦⎠
```

However, these optimizations can be very slow for large expressions. Moreover, if speed is a concern, one can pass the option `order='none'`. Order of terms will then be dependent on hashing algorithm implementation, but speed will be greatly improved.

CHAPTER

FIVE

DEVELOPER'S GUIDE

Warning: If you are new to Diofant, start with the *Tutorial* (page 3). If you are willing to contribute - it's assumed you know the Python programming language and the Git Version Control System.

5.1 Guidelines for Contributing

This project adheres to No Code of Conduct. Contributions will be judged by their technical merit. Nothing else matters.

5.1.1 Reporting Issues

When opening a new issue, please take the following steps:

1. Please search GitHub issues to avoid duplicate reports.
2. If possible, try updating to master and reproducing your issue.
3. Try to include a minimal reproducible test case as an example.
4. Include any relevant details of your local setup (i.e. Python version, installed libraries).

5.1.2 Contributing Code

All work should be submitted via Pull Requests (PR).

1. PR can be submitted as soon as there is code worth discussing.
2. Please put your work on the branch of your fork, not in the master branch. PR should generally be made against master.
3. One logical change per commit. Make good commit messages: short (<= 78 characters) one-line summary, then newline followed by verbose description of your changes. Please mention closed issues with commit message.
4. Please conform to PEP 8 and PEP 257, enable flake8 git hook to prevent badly formatted commits.
5. PR should include tests:
   (a) Bugfixes should include regression tests.
   (b) All new functionality should be tested, every new line should be covered by tests.

   (c) Optionally, provide doctests to illustrate usage. But keep in mind, doctests are not tests. Think of them as examples that happen to be tested.

6. It's good idea to be sure that **all** existing tests pass and you don't break anything, so please run:

   ```
   $ python setup.py test
   ```

7. If your change affects documentation, please build it by:

   ```
   $ python setup.py build_sphinx -W
   ```

   and check that it looks as expected.

5.1.3 License

By submitting a PR, you agree to license your code under the BSD license and LGPL.

5.2 Rosetta Stone

The Diofant project is a SymPy's fork, so it could be handy to collect here some facts about SymPy and explain historical code conventions.

First, the SymPy project was hosted in SVN repository on the Google Code and our master branch include only commits, that added after moving project on the Github. But it's not a problem for us - we keep old history on the branch sympy-svn-history. Also, you can see this history as part of master's, if you *clone our repo* (page 1) and simply do this:

```
$ git fetch origin 'refs/replace/*:refs/replace/*'
```

Please note, that we have dozens of references to SymPy issues in our codebase. Such reference must be either a direct URL of the issue, or a fully qualified reference in the Github format, like `sympy/sympy#123`. Unqualified references like `#123` or `issue 123` — are reserved for Diofant's issues. Functions for regression tests should be named like `test_sympyissue_123` and `test_diofantissue_123`, respectively.

However, in the old Git history, before commit cbdd072, please expect that `#123`, `issue #123` or `issue 123` — are references to the SymPy's issues. The whole story is a little worse, because before commit 6f68fa1 - such unqualified references assume issues on the Google Code, not Github, unless other clearly stated. SymPy issues from the Google Code were moved to the Github in March 2014 (see sympy/sympy#7235). Transfered issue numbers were shifted by 3099. I.e. `issue 123` in the history - does mean issue `sympy/sympy#3222` on Github.

5.3 Versioning and Release Procedure

We use standard Semantic Versioning numbering scheme, but adopt PEP 440 for alpha ("aN" suffix), beta ("bN") and development (".devN") releases.

Releasing a new version is done as follows:

1. Increase `__version__` in `diofant/__init__.py`.

2. Commit version update:

```
$ git commit -am 'Bump version to X.Y.Z'
```

3. Merge commit to the master branch.
4. Tag merge commit and push release tag:

```
$ git pull
$ git tag -s vX.Y.Z
$ git push origin vX.Y.Z
```

5.4 Labeling Issues and Pull Requests

Following table lists meanings of labels, including provided by Github:

Label	Description
bug	an unexpected problem or unintended behavior
wrong answer	if mathematically wrong result was obtained
duplicate	indicates similar issues or pull requests
enhancement	new feature requests (or implementation)
good first issue	indicates a good issue for first-time contributors
help wanted	indicates that a maintainer wants help here
invalid	mark that a problem is no longer relevant
question	mark support request
wontfix	indicates that work won’t continue on this issue

CHAPTER SIX

ABOUT

The Diofant project is a fork of the SymPy, started by Sergey B Kirpichev, last regular SymPy's commit is cbdd072, (22 Feb 2015). The git history goes back to 2007, when development was in svn.

The project was named after the Diophantus of Alexandria. His "Arithmetica" is one of the earliest known texts that use symbols in equations. "Diofant" is a transliteration of Диофант, from Russian.

6.1 License

Unless stated otherwise, all files in the Diofant project are licensed using the new BSD license:


```
Copyright (c) 2006-2017 SymPy Development Team,
              2013-2018 Sergey B Kirpichev

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DAMAGES (INCLUDING, BUT NOT LIMITED TO, PROCUREMENT OF SUBSTITUTE GOODS OR
SERVICES; LOSS OF USE, DATA, OR PROFITS; OR BUSINESS INTERRUPTION) HOWEVER
CAUSED AND ON ANY THEORY OF LIABILITY, WHETHER IN CONTRACT, STRICT
LIABILITY, OR TORT (INCLUDING NEGLIGENCE OR OTHERWISE) ARISING IN ANY WAY
OUT OF THE USE OF THIS SOFTWARE, EVEN IF ADVISED OF THE POSSIBILITY OF SUCH
DAMAGE.
```

6.2 SymPy Development Team

SymPy is a team project and it was developed by a lot of people. Please refer to the AUTHORS file in the SymPy repository for up-to-date list of contributors.

Please note that that list is incomplete intentionally, because some former SymPy developers don't want to be mentioned in the context of the SymPy or derived projects.

6.3 Brief History

SymPy was started by Ondřej Čertík in 2005, he wrote some code during the summer, then he wrote some more code during the summer 2006. In February 2007, Fabian Pedregosa joined the project and helped fixed many things, contributed documentation and made it alive again. 5 students (Mateusz Paprocki, Brian Jorgensen, Jason Gedge, Robert Schwarz and Chris Wu) improved SymPy incredibly during the summer 2007 as part of the Google Summer of Code (GSoC). Pearu Peterson joined the development during the summer 2007 and he has made SymPy much more competitive and fast (from 10x to 100x) by rewriting the core from scratch. Jurjen N.E. Bos has contributed pretty printing and other patches. Fredrik Johansson has wrote mpmath and contributed a lot of patches.

SymPy has participated in every GSoC since 2007. Moderate amount of SymPy's development has come from GSoC students.

In 2011, Ondřej Čertík stepped down as lead developer, with Aaron Meurer, who also started as a GSoC student, taking his place.

Ondřej Čertík is still active in the community, but is too busy with work and family to play a lead development role. Unfortunately, his remaining activity neither constructive nor productive anymore and SymPy just slowly dies now: most former contributors are inactive now or explicitly leaving this "friendly and welcoming" project.

This unfortunate situation was major reason to fork the SymPy as the Diofant project. Development in the new project will be open and public, without hidden double standards, centered about good, proved code and not project popularity counters. Here we believe that mathematical correctness is more important than political one.

CHAPTER

SEVEN

RELEASE NOTES

This section documents the changes that have been made in various versions of Diofant. Users should consult these pages to learn about new features, bug fixes and backwards incompatibilities. Developers should summarize the development work they do here in a user friendly format.

7.1 SymPy releases

For convenience, here included release notes from the old SymPy versions, up to 0.7.6. Changelog sources are taken from web archive and adapted to resemble usual Diofant's release notes.

7.1.1 SymPy 0.5.0

12 Aug 2007

- New core (from 10x to 100x speedup compared to 0.4.3).
- Multivariate functions.
- Pattern matching uses the Wild and WildFunction classes.
- Numerics module for fast arbitrary-precision numerical computations.
- Plotting module improved (colormaps, middle mouse button for zooming and more).
- sympy/modules/* was moved to sympy/* and the sympy/modules directory was deleted.

7.1.2 SymPy 0.5.1

12 Aug 2007

- importing sympy (import sympy) was made a lot faster (2.4s against 0.1s)

7.1.3 SymPy 0.5.2

20 Aug 2007

- concrete mathematics module written
- geometry module
- make the tarball conform to Debian policy

- many small bugs fixed

7.1.4 SymPy 0.5.3

8 Sep 2007

- Faster import sympy statement.
- Using the `integrate(3*t**2, (t, 0, x))` syntax again (as was in the 0.4.3 version).
- Using true division in isympy (`1/2` returns `0.5` instead of `0`, example).
- Plotting module can save images.
- Implemented extended Risch-Norman heuristic.
- Full partial fraction decomposition via `apart()`.
- Added a complete set of rewrite rules for trigonometric and hyperbolic functions.
- ComplexInfinity renamed to `zoo`.

7.1.5 SymPy 0.5.4

5 Oct 2007

- `Log` and `ApplyLog` classes were simplified to `log`, as was in the 0.4.3 version (the same for all other classes, like `sin` or `cos`).
- Limits algorithm was fixed and it works very reliably (there are some bugs in the series facility though that make some limits fail), see this post for more details.
- All functions arguments are now accessed using the `sin(x)[:]` idiom again, as in the 0.4.3 version (instead of the old `sin(x)._args` or `sin(x).args` which was briefly introduced in the 0.5.x series).

7.1.6 SymPy 0.5.5

20 Oct 2007

- `sympy.abc` module for quickly importing predefined symbols.
- Nice pretty printing when a unicode terminal is available.
- `isympy -c python` now also supports true division.
- Documentation improved (`sympy` module, `bin/isympy` and it's man page).
- A lot of problems with series expansion fixed.
- Patched pyglet to conform to Debian policy.

7.1.7 SymPy 0.5.6

30 Oct 2007

- `_sage_()` methods implemented to convert any SymPy expression to a SAGE expression.

- `isympy` fixed so that it always tries the local unpacked sympy first (the one in the directory where isympy sits) and only then the system wide installation of sympy (Debian package for example).

7.1.8 SymPy 0.5.7

17 Nov 2007

- `isympy` now uses 2D unicode pretty-printing by default.
- Convergence acceleration / extrapolation methods for series and sequences.
- SymPy was made ready to work nicely with SAGE.

7.1.9 SymPy 0.5.8

6 Dec 2007

- `_eval_apply()` method was renamed to `canonize()`.
- Added var from SAGE.
- Added more number theory functions.
- Spherical harmonics (Ylm) implemented.
- Functions interface simplified (SingleValuedFunction removed, nofargs -> nargs).
- Draw negative powers in denominator nicely.
- Integration of polynomials is 10x faster.
- `pyglet` updated to 1.0beta2.

7.1.10 SymPy 0.5.9

22 Dec 2007

- Differential solvers were polished.
- `isympy` now predefines `f` as a function.
- Matrix printing improved.
- Printing internals were documented.

7.1.11 SymPy 0.5.10

4 Jan 2008

- `view` renamed to `preview`, `pngview`, `pdfview`, `dviview` added.
- Latex printer was rewritten, `preview` uses builtin `pyglet`.
- Square root denesting implemented.
- Parser of simple Mathematica expressions added.
- TeXmacs interface written.
- Some integration fixes.

- Line width in 2D plotting can be specified.
- README was updated.
- `pyglet` and `mpmath` were updated and moved to `sympy/thirdparty`
- All `sys.path` hacks were moved to just 2 places.
- SymPy objects should work in `numpy` arrays now.
- Hand written `sympify()` parser was rewritten and simplified using Python AST.

7.1.12 SymPy 0.5.11

7 Jan 2008

- `./setup.py install` installs `pyglet` correctly now.
- `var("k")` fixed.
- Script for automatic testing of plotting in pure environment added.

7.1.13 SymPy 0.5.12

27 Jan 2008

- SymPy works with NumPy out of the box.
- `RootOf` implemented.
- Lambda support works now.
- Heuristic Risch method improved.
- `cancel()` function implemented.
- `sqrt(x)` is now equivalent to `x**(1/2)`.
- `Derivative` is now unevaluated.
- `list2numpy()` implemented.
- Series expansion of hyperbolic functions fixed.
- `sympify('lambda x: 2*x')` works, plus other fixes.
- Simple maxima parser implemented.
- `sin(x)[0]` idiom changed to `sin(x).args[0]`
- `sin(x).series(x, 5)` idiom changed to `sin(x).series(x, 0, 5)`
- Caching refactored.
- Integration of trigonometry expressions improved
- Pretty-printing for list and tuples implemented.
- Python printing implemented.
- 2D plots now don’t rotate in 3D, but translate instead.

7.1.14 SymPy 0.5.13

6 Mar 2008

- SymPy is now 2x faster in average compared to the previous release
  - first patches with 25% speedup
  - `Basic.cos` et. al. removed, use `C.cos` instead
  - `sympy.core` now uses direct imports
  - `sympifyit` decorator
  - speedup Integers creation and arithmetic
  - speedup unary operations for singleton numbers
  - remove silly slowdowns from fast-path of mul and div
  - significant speedup was achieved by reusing dummy variables
  - `is_dummy` is not an assumption anymore
  - Symbols & Wilds are cached
  - `((2+3*I)**1000).expand()` is now at least 100x faster
  - `.expand()` was made faster for cases where an expression is already expanded
  - rational powers of integers are now computed more efficiently
  - unknown assumptions are now cached as well as known assumptions
- `integrate()` can handle most of the basic integrals now
- interactive experience with isympy was improved through adding support for , () and {} to pretty-printer, and switching to it as the default ipython printer
- new `trim()` function to map all non-atomic expressions, ie. functions, derivatives and more complex objects, to symbols and remove common factors from numerator and denominator. also cancel() was improved
- `.expand()` for noncommutative symbols fixed
- bug in `(x+y+sin(x)).as_independent()` fixed
- `.subs_dict()` improved
- support for plotting geometry objects added
- bug in `.tangent_line()` of ellipse fixed
- new atan2 function and assotiated fixes for `.arg()` and expanding rational powers
- new `.coeff()` method for returning coefficient of a poly
- pretty-printer now uses unicode by default
- recognition of geometric sums were generalized
- `.is_positive` and `.is_negative` now fallback to `evalf()` when appropriate
- as the result `oo*(pi-1)` now correctly simplifies to oo
- support for objects which provide `__int__` method was added
- we finally started SymPy User's Guide
- `BasicMeths` merged into `Basic`

- cache subsystem was cleaned up – now it supports only immutable objects

7.1.15 SymPy 0.5.14

26 Apr 2008

- SymPy is now 25% faster on average compared to the previous release
  - `__eq__/__ne__/__nonzero__` returns `True/False` directly so dict lookups are not expensive anymore
  - `sum(x**i/i,i=1..400)` is now 4.8x faster
  - `isinstance(term, C.Mul)` was replaced by `term.is_Mul` and similarly for other basic classes
- Documentation was improved a lot. See http://docs.sympy.org/
- rsolve_poly & rsolve_hyper fixed
- `subs` and `subs_dict` unified to `.subs()`
- faster and more robust polynomials module
- improved `Matrix.det()`, implemented Berkowitz algorithm
- improved isympy (interactive shell for SymPy)
- pretty-printing improved
- `Rel`, `Eq`, `Ne`, `Lt`, `Le`, `Gt`, `Ge` implemented
- `Limit` class represents unevaluated limits now
- Bailey-Borwein-Plouffe algorithm (finds the nth hexidecimal digit of pi without calculating the previous digits) implemented
- solver for transcendental equations added
- `.nseries()` methods implemented (more robust/faster than .oseries)
- multivariate Lambdas implemented

7.1.16 SymPy 0.5.15

24 May 2008

- all SymPy functions support vector arguments, e.g. `sin([1, 2, 3])`
- lambdify can now use numpy/math/mpmath
- the order of lambdify arguments has changed
- all SymPy objects are pickable
- simplify improved and made more robust
- broken limit_series was removed, we now have just one limit implementation
- limits now use `.nseries()`
- `.nseries()` improved a lot
- Polys improved
- Basic kronecker delta and Levi-Civita implementation

7.1.17 SymPy 0.6.0

7 Jul 2008

- all documentation wiki pages moved to http://docs.sympy.org/
- mpmath was integrated in SymPy, numerics module removed
- mpmath can use gmpy optionally, thus calculating 1000000 digits of pi in 7.5s
- Common subexpression elimination implemented
- roots, RootsOf, RootSum implemented
- `lambdify()` now accepts Matrices
- Matrices polished and spedup
- source command implemented
- Polys were made the default polynomials in SymPy
- Add, Mul, Pow now accept evaluate=False argument

7.1.18 SymPy 0.6.1

22 Jul 2008

- almost all functions and constants can be converted to Sage
- univariate factorization algorithm was fixed
- `.evalf()` method fixed, `pi.evalf(106)` calculates 1 000 000 digits of pi
- `@threaded` decorator
- more robust solvers, polynomials and simplification
- better simplify, that makes a solver more robust
- optional compiling of functions to machine code
- msolve: solving of nonlinear equation systems using Newton's method
- `((x+y+z)**50).expand()` is now 3 times faster
- caching was removed from the Order class: 1.5x speedups in series tests

7.1.19 SymPy 0.6.2

17 Aug 2008

- SymPy is now 50% faster on average (cache:on) and 130% (cache:off) compared to previous release.
- adaptive and faster `evalf()`
- evalf: numerical summation of hypergeometric series
- evalf: fast and accurate numerical summation
- evalf: oscillatory quadrature
- integrals now support variable transformation
- we can now `integrate(f(x)·diff(f(x),x), x)`

- we can now solve `a·cos(x)=y` and `exp(x)+exp(-x)=y`
- printing system refactored
- pprint: new symbol for multiply in unicode mode(x*y -> x·y)
- pprint: matrices now look much better
- printing of dicts and sets are now more human-friendly
- latex: now supports sub- and superscripts in symbol names
- `RootSum.doit()`, now works on all roots
- Wild can now have additional predicates
- numpy-like zeros and ones functions
- `var('x,y,z')` now works
- `((x+y+z)**50).expand()` is now 4.8x faster
- big assumptions cleanup and rewrite
- access to all object attributes is now ~2.5 times faster
- we try not to let 'is_commutative' to go through (slow) assumptions path
- Add/Mul were optimized (for some cases significantly)
- isympy and sympy.interactive code were merged
- multiple inheritance removed (NoArithMeths, NoRelMeths, RelMeths, ArithMeths are gone)
- `.nseries()` is now used as default in `.series()`
- doctesting was made more robust

7.1.20 SymPy 0.6.3

19 Nov 2008

- port to python2.6 (all tests pass)
- port to jython (all tests pass except those depending on the "ast" module)
- true division fixed (all tests pass with "-Qnew" Python option)
- `http://buildbot.sympy.org` created, sympy is now regularly tested on python2.4, 2.5, 2.6 on both i386 and amd64 architectures.
- py.bench – py.test based benchmarking added
- bin/test – simple py.test like testing framework, without external dependencies, nice colored output
- most limits now work
- factorization over $Z[x]$ greatly improved
- Piecewise function added
- `nsimplify()` implemented
- `symbols` and `var` syntax unified
- C code printing

- many bugfixes

7.1.21 SymPy 0.6.4

4 Apr 2009

- robust and fast (still pure Python) multivariate factorization
- sympy works with pickle protocol 2 (thus works in ipython parallel)
- ./sympy test now uses our testing suite and it tests both regular tests and doctests
- examples directory tidied up
- more trigonometric simplifications
- polynomial roots finding and factoring vastly improved
- mpmath updated
- many bugfixes (more than 200 patches since the last release)

7.1.22 SymPy 0.6.5

17 Jul 2009

- Geometric Algebra Improvements
  - Upgrade GA module with left and right contraction operations
  - Add intersection test for the vertical segment, reimplementation of convex_hull
- Implement `series()` as function
- Core improvements
  - Refactor `Number.eval_power`
  - fix bugs in `Number.eval_power`
- Matrix improvements:
  - Improve jacobian function, introduce vec and vech
- Solver improvements:
  - solutions past linear factor found in tsolve
  - refactor sympy.solvers.guess_solve_strategy
  - Small cleanups to the ODE solver and tests
  - Fix corner case for Bernoulli equation
- Improvements on partial differential equations solvers
  - Added separation of variables for PDEs
- Expand improvements
  - Refactoring
  - `exp(x) exp(y)` is no longer automatically combined into `exp(x+y)`, use `powsimp` for that
- Documentation improvements:

  - Test also documentation under doc/
  - Added many docstrings
  - Fix Sphinx complaints/warnings/errors
  - Doctest coverage
- New logic module
  - Efficient DPLL algorithm
- LaTeX printer improvements:
  - Handle standard form in the LaTeX printer correctly
  - Latex: print_Mul fix (sympy/sympy#4381)
  - Robust printing of latex sub and superscripts
  - sorting print_Add output using a main variable
  - Matrix printing improvements
- MathML printing improvements:
  - MathML's printer extended
- Testing framework improvements
  - Make tests pass without the "py" module
- Polynomial module improvements:
  - Fixed subresultant PRS computation and `ratint()`
  - Removed old module `sympy.polynomials`
- limit fixes:
  - Compute the finite parts of the limit of a sum by direct substitution
- Test coverage script
- Code quality improvements (remove string exceptions, code quality test improvements)
- C code generation
- Update mpmath

7.1.23 SymPy 0.6.6

20 Dec 2009

- many documentation improvements, including docstrings and doctests
- new assumptions system (See assumptions documentation for more information or have a look at Fabian's blog.)
- improvements to test runner
- printing improvements (especially LaTeX, but also mathml and pretty printing)
- discriminant of polys
- block diagonal methods for matrices
- vast improvements to solving of ODEs (See ODE documentation for full details or Aaron's blog).

- logcombine function
- improvements to sets
- better trigonometric simplification
- improvements to piecewise functions
- improvements to solve() and nsolve()
- improvements to as_numer_denom()
- much better quartic and cubic polynomial rootfinding
- code refactoring and cleanup
- physics: coupled clusters and wick expansion
- matrices: symbolic QR solving
- mpmath updated
- pyglet updated
- many, many bug fixes and small improvements

7.1.24 SymPy 0.6.7

17 Mar 2010

- fix a bug where bin/test and bin/doctest would be installed into /usr/bin
- fix an example for recent matplotlib versions
- some fixes for Python 2.4 and 2.5
- try to expand integrand if integration fails
- improved isprime() (pseudoprimes were falsely reported as being prime)
- runtests.py prints less when testing documentation
- ODE code clean-up and fixes
- runtests.py is now more verbose about expected failures to avoid confusion
- update mpmath to version 0.14
- improvements to second quantization module and coupled cluster example
- implement visual `factorint()`
- implement `symarray()`: numpy array of sympy symbols
- documentation fixes and some bugfixes

7.1.25 SymPy 0.7.0

28 Jun 2011

Major Changes

- Polys
  - New internal representations of dense and sparse polynomials (see 6aecdb7, 31c9aa4)
  - Implemented algorithms for real and complex root isolation and counting (see 3acac67, 4b75dae, fa1206e, 103b928, 45c9b22, 8870c8b, b348b30)
  - Improved Gröbner bases algorithm (see ff65e9f, 891e4de, 310a585)
  - Field isomorphism algorithm (see b097b01, 08482bf)
  - Implemented efficient orthogonal polynomials (see b8fbd59)
  - Added configuration framework for polys (see 33d8cdb, 7eb81c9)
  - Function for computing minimal polynomials (see 88bf187, f800f95)
  - Function for generating Viete's formulas (see 1027408)
  - `roots()` supports more classes of polynomials (e.g. cyclotomic) (see d8c8768, 75c8d2d)
  - Added a function for recognizing cyclotomic polynomials (see b9c2a9a)
  - Added a function for computing Horner form of polynomials (see 8d235c7)
  - Added a function for computing symmetric reductions of polynomials (see 6d560f3)
  - Added generators of Swinnerton-Dyer, cyclotomic, symmetric, random and interpolating polynomials (see dad03dd, 6ccf20c, dc728d6, 2f17684, 3004db8)
  - Added a function computing isolation intervals of algebraic numbers (see 37a58f1)
  - Polynomial division (`div()`, `rem()`, `quo()`) now defaults to a field (see a72d188)
  - Added wrappers for numerical root finding algorithms (see 0d98945, f638fcf)
  - Added symbolic capabilities to `factor()`, `sqf()` and related functions (see d521c7f, 548120b, f6f74e6, b1c49cd, 3527b64)
  - `together()` was significantly improved (see dc327fe)
  - Added support for iterable containers to `gcd()` and `lcm()` (see e920870)
  - Added a function for constructing domains from coefficient containers (see a8f20e6)
  - Implemented greatest factorial factorization (see d4dbbb5)
  - Added partial fraction decomposition algorithm based on undetermined coefficient approach (see 9769d49, 496f08f)
  - `RootOf` and `RootSum` were significantly improved (see f3e432, 4c88be6, 41502d7)
  - Added support for gmpy (GNU Multiple Precision Arithmetic Library) (see 38e1683)
  - Allow to compile `sympy.polys` with Cython (see afb3886)
  - Improved configuration of variables in `Poly` (see 22c4061)
  - Added documentation based on Wester's examples (see 1c23792)
  - Irreducibility testing over finite fields (see 17e8f1f)
  - Allow symmetric and non-symmetric representations over finite fields (see 60fbff4)
  - More consistent factorization forms from `factor()` and `sqf()` (see 5df77f5)

  - Added support for automatic recognition algebraic extensions (see 7de602c)
  - Implemented Collins' modular algorithm for computing resultants (see 950969b)
  - Implemented Berlekamp's algorithm for factorization over finite fields (see 70353e9)
  - Implemented Trager's algorithm for factorization over algebraic number fields (see bd0be06)
  - Improved Wang's algorithm for efficient factorization of multivariate polynomials (see 425e225)
- Quantum
  - Symbolic, abstract dirac notation in `sympy.physics.quantum`. This includes operators, states (bras and kets), commutators, anticommutators, dagger, inner products, outer products, tensor products and Hilbert spaces
  - Symbolic quantum computing framework that is based on the general capabilities in `sympy.physics.quantum`. This includes qubits (`sympy.physics.quantum.qubit`), gates (`sympy.physics.quantum.gate`), Grover's algorithm (`sympy.physics.quantum.grover`), the quantum Fourier transform (`sympy.physics.quantum.qft`), Shor's algorithm (`sympy.physics.quantum.shor`) and circuit plotting (`sympy.physics.quantum.circuitplot`)
  - Second quantization framework that inclues creation/anihilation operators for both Fermions and Bosons and Wick's theorem for Fermions (`sympy.physics.secondquant`).
  - Symbolic quantum angular momentum (spin) algebra (`sympy.physics.quantum.spin`)
  - Hydrogen wave functions (Schroedinger) and energies (both Schroedinger and Dirac)
  - Wave functions and energies for 1D harmonic oscillator
  - Wave functions and energies for 3D spherically symmetric harmonic oscillator
  - Wigner and Clebsch Gordan coefficients
- Everything else
  - Implement symarray, providing numpy nd-arrays of symbols.
  - update mpmath to 0.16
  - Add a tensor module (see this report)
  - A lot of stuff was being imported with `from sympy import *` that shouldn't have been (like `sys`). This has been fixed.
- Assumptions:
  - Refine
  - Added predicates (see 7c0b857, 53f0e1a, d1dd6a3)
  - Added query handlers for algebraic numbers (see f3bee7a)
  - Implement a SAT solver (see this, 2d96329, acfbe75, etc)
- Concrete
  - Finalized implementation of Gosper's algorithm (see 0f187e5, 5888024)
  - Removed redundant `Sum2` and related classes (see ef1f6a7)

- Core:
  - Split `Atom` into `Atom` and `AtomicExpr` (see 965aa91)
  - Various `sympify()` improvements
  - Added functionality for action verbs (many functions can be called both as global functions and as methods e.g. `a.simplify() == simplify(a)`)
  - Improve handling of rational strings (see 053a045, sympy/sympy#4877)
  - Major changes to factoring of integers (see 273f450, sympy/sympy#5102)
  - Optimized `.has()` (see c83c9b0, sympy/sympy#5079, d86d08f)
  - Improvements to power (see c8661ef, sympy/sympy#5062)
  - Added range and lexicographic syntax to `symbols()` and `var()` (see f6452a8, 9aeb220, 957745a)
  - Added `modulus` argument to `expand()` (see 1ea5be8)
  - Allow to convert `Interval` to relational form (see 4c269fe)
  - SymPy won't manipulate minus sign of expressions any more (see 6a26941, 9c6bf0f, e9f4a0a)
  - `Real` and `.is_Real` were renamed to `Float` and `.is_Float`. `Real` and `.is_Real` still remain as deprecated shortcuts to `Float` and `is_Float` for backwards compatibility. (see abe1c49)
  - Methods coeff and as_coefficient are now non-commutative aware. (see a4ea170)
- Geometry:
  - Various improvements to Ellipse
  - Updated documentation to numpy standard
  - Polygon and Line improvements
  - Allow all geometry objects to accept a tuple as `Point` args
- Integrals:
  - Various improvements (see e.g. sympy/sympy#4871, sympy/sympy#5098, sympy/sympy#5091, sympy/sympy#5086)
- isympy
  - Fixed the `-p` switch (see e8cb04a)
  - Caching can be disabled using `-C` switch (see 0d8d748)
  - Ground types can be set using `-t` switch (see 75734f8)
  - Printing ordering can be set using `-o` switch (see fcc6b13, 4ec9dc5)
- Logic
  - implies object adheres to negative normal form
  - Create new boolean class, `logic.boolalg.Boolean`
  - Added XOR operator (^) support
  - Added If-then-else (ITE) support
  - Added the dpll algorithm

- Functions:
  - Added Piecewise, B-splines
  - Spherical Bessel function of the second kind implemented
  - Add series expansions of multivariate functions (see d4d351d)
- Matrices:
  - Add elementwise product (Hadamard product)
  - Extended QR factorization for general full ranked mxn matrices
  - Remove deprecated functions `zero()`, `zeronm()`, `one()` (see 5da0884)
  - Added cholesky and LDL factorizations, and respective solves.
  - Added functions for efficient triangular and diagonal solves.
  - `SMatrix` was renamed to `SparseMatrix` (see acd1685)
- Printing:
  - Implemented pretty printing of binomials (see 58c1dad)
  - Implemented pretty printing of Sum() (see 84f2c22, 95b4321)
  - `sympy.printing` now supports ordering of terms and factors (see 859bb33)
  - Lexicographic order is now the default. Now finally things will print as `x**2 + x + 1` instead of `1 + x + x**2`, however series still print using reversed ordering, e.g. `x - x**3/6 + O(x**5)`. You can get the old order (and other orderings) by setting the `-o` option to isympy (see 08b4932, a30c5a3)
- Series:
  - Implement a function to calculate residues, `residue()`
  - Implement nseries and lseries to handle `x0 != 0`, series should be more robust now (see 2c99999, sympy/sympy#5221 - sympy/sympy#5223)
  - Improvements to Gruntz algorithm
- Simplify:
  - Added `use()` (see 147c142)
  - `ratsimp()` now uses `cancel()` and `reduced()` (see 108fb41)
  - Implemented EPath (see 696139d, bf90689)
  - a new keyword `rational` was added to nsimplify which will replace Floats with Rational approximations. (see 053a045)
- Solvers:
  - ODE improvements (see d12a2aa, 3542041; 73fb9ac)
  - Added support for solving inequalities (see 328eaba, 8455147, f8fcaa7)
- Utilities:
  - Improve cartes, for generating the Cartesian product (see b1b10ed)
  - Added a function computing topological sort of graphs (see b2ce27b)
  - Allow to setup a customized printer in `lambdify()` (see c1ad905)
  - `flatten()` was significantly improved (see 31ed8d7)

- Major improvements to the Fortran code generator (see report, 3383aa3, 7ab2da2, etc)

Compatibility breaks

- This will be the last release of SymPy to support Python 2.4. Dropping support for Python 2.4 will let us move forward with things like supporting Python 3, and will let us use things that were introduced in Python 2.5, like with-statement context managers.
- no longer support creating matrices without brackets (see sympy/sympy#4029)
- Renamed `sum()` to `summation()` (see 3e763a8, sympy/sympy#4475, sympy/sympy#4826). This was changed so that it no longer overrides the built-in `sum()`. The unevaluated summation is still called `Sum()`.
- Renamed `abs()` to `Abs()` (see 64a12a4, sympy/sympy#4826). This was also changed so that it no longer overrides the built-in `abs()`. Note that because of `__abs__` magic, you can still do `abs(expr)` with the built-in `abs()`, and it will return `Abs(expr)`.
- Renamed `max_()` and `min_()` to now `Max()` and `Min()` (see 99a271e, sympy/sympy#5252)
- Changed behaviour of `symbols()`. `symbols('xyz')` gives now a single symbol (`'xyz'`), not three (`'x'`, `'y'` and `'z'`) (see f6452a8). Use `symbols('x,y,z')` or `symbols('x y z')` to get three symbols. The `each_char` option will still work but is being deprecated.
- Split class `Basic` into new classes `Expr`, `Boolean` (see a0ab479, 635d89c). Classes that are designed to be part of standard symbolic expressions (like `x**2*sin(x)`) should subclass from `Expr`. More generic objects that do not work in symbolic expressions but still want the basic SymPy structure like `.args` and basic methods like `.subs()` should only subclass from `Basic`.
- `as_basic()` method was renamed to `as_expr()` to reflect changes in the core (see e61819d, 80dfe91)
- Methods `as_coeff_terms` and `as_coeff_factors` were renamed to `as_coeff_mul` and `as_coeff_add`, respectively.
- Removed the `trim()` function. The function is redundant with the new polys. Use the `cancel()` function instead.
- The `assume_pos_real` option to `logcombine()` was renamed to `force` to be consistant with similar `force` options to other functions.

In addition to the more noticeable changes listed above, there have been numerous other smaller additions, improvements and bug fixes in the ~2000 commits in this release. See the git log for a full list of all changes. The command `git log sympy-0.6.7..sympy-0.7.0` will show all commits made between this release and the last. You can also see the issues closed since the last release here.

7.1.26 SymPy 0.7.1

29 Jul 2011

Major changes

- Python 2.4 is no longer supported. SymPy will not work at all in Python 2.4. If you still need to use SymPy under Python 2.4 for some reason, you will need to use SymPy 0.7.0 or earlier.
- The Pyglet plotting library is now an (optional) external dependency. Previously, we shipped a version of Pyglet with SymPy, but this was old and buggy. The plan is to eventually make the plotting in SymPy much more modular, so that it supports many backends, but this has not been done yet. For now, still only Pyglet is directly supported. Note that Pyglet is only an optional dependency and is only needed for plotting. The rest of SymPy can still be used without any dependencies (except for Python).
- `isympy` now works with the new IPython 0.11.
- mpmath has been updated to 0.17. See the corresponding mpmath release notes.
- Added a `Subs` object for representing unevaluated substitutions. This finally lets us represent derivatives evaluated at a point, i.e., `diff(f(x), x).subs(x, 0)` returns `Subs(Derivative(f(_x), _x), (_x,), (0,))`. This also means that SymPy can now correctly compute the chain rule when this functionality is required, such as with `f(g(x)).diff(x)`.
- Hypergeometric functions/Meijer G-Functions
  - Added classes `hyper()` and `meijerg()` to represent Hypergeometric and Meijer G-functions, respectively. They support numerical evaluation (using mpmath) and symbolic differentiation (not with respect to the parameters).
  - Added an algorithm for rewriting hypergeometric and meijer g-functions in terms of more familiar, named special functions. It is accessible via the function `hyperexpand()`, or also via `expand_func()`. This algorithm recognises many elementary functions, and also complete and incomplete gamma functions, bessel functions, and error functions. It can easily be extended to handle more classes of special functions.
- Sets
  - Added FiniteSet class to mimic python set behavior while also interacting with existing Intervals and Unions
  - FiniteSets and Intervals interact so that, for example `Interval(0, 10) - FiniteSet(0, 5)` produces `(0, 5) U (5, 10]`
  - FiniteSets also handle non-numerical objects so the following is possible `{1, 2, 'one', 'two', {a, b}}`
  - Added ProductSet to handle Cartesian products of sets
  - Create using the `*` operator, i.e. `twodice = FiniteSet(1, 2, 3, 4, 5, 6) * FiniteSet(1, 2, 3, 4, 5, 6)` or `square = Interval(0, 1) * Interval(0, 1)`
  - pow operator also works as expected: `R3 = Interval(-oo, oo)**3 ; (3, -5, 0) in R3 == True`
  - Subtraction, union, measurement all work taking complex intersections into account.
  - Added as_relational method to sets, producing boolean statements using `And`, `Or`, `Eq`, `Lt`, `Gt`, etc.
  - Changed `reduce_poly_inequalities` to return unions of sets rather than lists of sets

- Iterables
  - Added generating routines for integer partitions and binary partitions. The routine for integer partitions takes 3 arguments, the number itself, the maximum possible element allowed in the partitions generated and the maximum possible number of elements that will be in the partition. Binary partitions are characterized by containing only powers of two.
  - Added generating routine for multi-set partitions. Given a multiset, the algorithm implemented will generate all possible partitions of that multi-set.
  - Added generating routines for bell permutations, derangements, and involutions. A bell permutation is one in which the cycles that compose it consist of integers in a decreasing order. A derangement is a permutation such that the ith element is not at the ith position. An involution is a permutation that when multiplied by itself gives the identity permutation.
  - Added generating routine for unrestricted necklaces. An unrestricted necklace is an a-ary string of n characters, each of a possible types. These have been characterized by the parameters n and k in the routine.
  - Added generating routine for oriented forests. This is an implementation of algorithm S in TAOCP Vol 4A.
- xyz Spin bases
  - The represent, rewrite and InnerProduct logic has been improved to work between any two spin bases. This was done by utilizing the Wigner-D matrix, implemented in the WignerD class, in defining the changes between the various bases. Representing a state, i.e. `represent(JzKet(1,0), basis=Jx)`, can be used to give the vector representation of any get in any of the x/y/z bases for numerical values of j and m in the spin eigenstate. Similarly, rewriting states into different bases, i.e. `JzKet(1,0).rewrite('Jx')`, will write the states as a linear combination of elements of the given basis. Because this relies on the represent function, this only works for numerical j and m values. The inner product of two eigenstates in different bases can be evaluated, i.e. `InnerProduct(JzKet(1,0),JxKet(1,1))`. When two different bases are used, one state is rewritten into the other basis, so this requires numerical values of j and m, but innerproducts of states in the same basis can still be done symbolically.
  - The `Rotation.D` and `Rotation.d` methods, representing the Wigner-D function and the Wigner small-d function, return an instance of the WignerD class, which can be evaluated with the `doit()` method to give the corresponding matrix element of the Wigner-D matrix.
- Other changes
  - We now use MathJax in our docs. MathJax renders LaTeX math entierly in the browser using Javascript. This means that the math is much more readable than the previous png math, which uses images. MathJax is only supported on modern browsers, so LaTeX math in the docs may not work on older browsers.
  - `nroots()` now lets you set the precision of computations
  - Added support for gmpy and mpmath's types to `sympify()`
  - Fix some bugs with `lambdify()`
  - Fix a bug with as_independent and non-commutative symbols.
  - Fix a bug with collect (sympy/sympy#5615)

- Many fixes relating to porting SymPy to Python 3. Thanks to our GSoC student Vladimir Perić, this task is almost completed.
- Some people were retroactively added to the AUTHORS file.
- Added a solver for a special case of the Riccati equation in the ODE module.
- Iterated derivatives are pretty printed in a concise way.
- Fix a bug with integrating functions with multiple DiracDeltas.
- Add support for `Matrix.norm()` that works for Matrices (not just vectors).
- Improvements to the Gröbner bases algorithm.
- Plot.saveimage now supports a StringIO outfile
- Expr.as_ordered_terms now supports non lex orderings.
- diff now canonicalizes the order of differentiation symbols. This is so it can simplify expressions like `f(x, y).diff(x, y) - f(x, y).diff(y, x)`. If you want to create a Derivative object without sorting the args, you should create it explicitly with `Derivative`, so that you will get `Derivative(f(x, y), x, y) != Derivative(f(x, y), y, x)`. Note that internally, derivatives that can be computed are always computed in the order that they are given in.
- Added functions `is_sequence()` and `iterable()` for determining if something is an ordered iterable or normal iterable, respectively.
- Enabled an option in Sphinx that adds a `source` link next to each function, which links to a copy of the source code for that function.

In addition to the more noticeable changes listed above, there have been numerous other smaller additions, improvements and bug fixes in the ~300 commits in this release. See the git log for a full list of all changes. The command `git log sympy-0.7.0..sympy-0.7.1` will show all commits made between this release and the last. You can also see the issues closed since the last release here.

7.1.27 SymPy 0.7.2

16 Oct 2012

Major Changes

- Python 3 support
  - SymPy now supports Python 3. The officially supported versions are 3.2 and 3.3, but 3.1 should also work in a pinch. The Python 3-compatible tarballs will be provided separately, but it is also possible to download Python 2 code and convert it manually, via the bin/use2to3 utility. See the README for more.
- PyPy support
  - All SymPy tests pass in recent nightlies of PyPy, and so it should have full support as of the next version after 1.9.
- Combinatorics

- A new module called Combinatorics was added which is the result of a successful GSoC project. It attempts to replicate the functionality of Combinatorica and currently has full featured support for Permutations, Subsets, Gray codes and Prufer codes.
- In another GSoC project, facilities from computational group theory were added to the combinatorics module, mainly following the book "Handbook of computational group theory". Currently only permutation groups are supported. The main functionalities are: basic properties (orbits, stabilizers, random elements…), the Schreier-Sims algorithm (three implementations, in increasing speed: with Jerrum's filter, incremental, and randomized (Monte Carlo)), backtrack searching for subgroups with certain properties.

- Definite Integration
  - A new module called meijerint was added, which is also the result of a successful GSoC project. It implements a heuristic algorithm for (mainly) definite integration, similar to the one used in Mathematica. The code is automatically called by the standard integrate() function. This new algorithm allows computation of important integral transforms in many interesting cases, so helper functions for Laplace, Fourier and Mellin transforms were added as well.
- Random Variables
  - A new module called stats was added. This introduces a RandomSymbol type which can be used to model uncertainty in expressions.
- Matrix Expressions
  - A new matrix submodule named expressions was added. This introduces a MatrixSymbol type which can be used to describe a matrix without explicitly stating its entries. A new family of expression types were also added: Transpose, Inverse, Trace, and BlockMatrix. ImmutableMatrix was added so that explicitly defined matrices could interact with other SymPy expressions.
- Sets
  - A number of new sets were added including atomic sets like FiniteSet, Reals, Naturals, Integers, UniversalSet as well as compound sets like ProductSet and TransformationSet. Using these building blocks it is possible to build up a great variety of interesting sets.
- Classical Mechanics
  - A physics submodule named machanics was added which assists in formation of equations of motion for constrained multi-body systems. It is the result of 3 GSoC projects. Some nontrivial systems can be solved, and examples are provided.
- Quantum Mechanics
  - Density operator module has been added. The operator can be initialized with generic Kets or Qubits. The Density operator can also work with TensorProducts as arguments. Global methods are also added that compute entropy and fidelity of states. Trace and partial-trace operations can also be performed on these density operators.
  - To enable partial trace operations a Tr module has been added to the core library. While the functionality should remain same, this module is likely to be relocated to an alternate folder in the future. One can currently also use sympy.core.Tr to work on general trace operations, but this module is what is needed to work on trace and partial-trace operations on any sympy.physics.quantum objects.

    - The Density operators, Tr and Partial trace functionality was implemented as part of student participation in GSoC 2012.
    - Expanded angular momentum to include coupled-basis states and product-basis states. Operators can also be treated as acting on the coupled basis (default behavior) or on one component of the tensor product states. The methods for coupling and uncoupling these states can work on an arbitrary number of states. Representing, rewriting and applying states and operators between bases has been improved.
- Commutative Algebra
    - A new module `agca` was started which seeks to support computations in commutative algebra (and eventually algebraic geometry) in the style of Macaulay2 and Singular. Currently there is support for computing Gröbner bases of modules over a (generalized) polynomial ring over a field. Based on this, there are algorithms for various standard problems in commutative algebra, e.g., computing intersections of submodules, equality tests in quotient rings, etc...
- Plotting Module
    - A new plotting module has been added which uses Matplotlib as its back-end. The plotting module has functions to plot the following:
        * 2D line plots
        * 2D parametric plots.
        * 2D implicit and region plots.
        * 3D surface plots.
        * 3D parametric surface plots.
        * 3D parametric line plots.
- Differential Geometry
    - Thanks to a GSoC project the beginning of a new module covering the theory of differential geometry was started. It can be imported with `sympy.diffgeom`. It is based on "Functional Differential Geometry" by Sussman and Wisdom. Currently implemented are scalar, vector and form fields over manifolds as well as covariant and other derivatives.

Compatibility breaks

- The KroneckerDelta class was moved from `sympy/physics/quantum/kronecker.py` to `sympy/functions/special/tensor_functions.py`.
- Merged the KroneckerDelta class in `sympy/physics/secondquant.py` with the class above.
- The Dij class in `sympy/functions/special/tensor_functions.py` was replaced with KroneckerDelta.
- The errors raised for invalid `float` calls on SymPy objects were changed in order to emulate more closely the errors raised by the standard library. The `__float__` and `__complex__` methods of `Expr` are concerned with that change.
- The `solve()` function returns empty lists instead of `None` objects if no solutions were found. Idiomatic code of the form `sol = solve(...); if sol:...` will not be affected by this change.

- Piecewise no longer accepts a Set or Interval as a condition. One should explicitly specify a variable using `Set().contains(x)` to obtain a valid conditional.
- The statistics module has been deprecated in favor of the new stats module.
- `sympy/galgebra/GA.py`:
  - `set_main()` is no longer needed
  - `make_symbols()` is deprecated (use `sympy.symbols()` instead)
  - the symbols used in this package are no longer broadcast to the main program
- The classes for Infinity, NegativeInfinity, and NaN no longer subclass from Rational. Creating a Rational with 0 in the denominator will still return one of these classes, however.

Minor changes

- A new module `gaussopt` was added supporting the most basic constructions from Gaussian optics (ray tracing matrices, geometric rays and Gaussian beams).
- New classes were added to represent the following special functions: classical and generalized exponential integrals (Ei, expint), trigonometric (Si, Ci) and hyperbolic integrals (Shi, Chi), the polylogarithm (polylog) and the Lerch transcendent (lerchphi). In addition to providing all the standard sympy functionality (differentiation, numerical evaluation, rewriting …), they are supported by both the new meijerint module and the existing hypergeometric function simplification module.
- An ImmutableMatrix class was created. It has the same interface and functionality of the old Matrix but is immutable and inherits from Basic.
- A new function in `geometry.util` named `centroid` was added which will calculate the centroid of a collection of geometric entities. And the polygon module now allows triangles to be instantiated from combinations of side lengths and angles (using keywords sss, asa, sas) and defines utility functions to convert between degrees and radians.
- In `ntheory.modular` there is a function (`solve_congruence`) to solve congruences such as "What number is 2 mod 3, 3 mod 5 and 2 mod 7?"
- A utility function named `find_unit` has been added to physcis.units that allows one to find units that match a given pattern or contain a given unit.
- There have been some additions and modifications to Expr's methods:
  - Although the problem of proving that two expressions are equal is in general a difficult one (since whatever algorithm is used, there will always be an expression that will slip through the algorithm) the new method of Expr named `equals` will do its best to answer whether A equals B: A.equals(B) might given True, False or None.
  - coeff now supports a third argument `n` (which comes 2nd now, instead of `right`). This `n` is used to indicate the exponent on x which one seeks: `(x**2 + 3*x + 4).coeff(x, 1)` -> `3`. This makes it possible to extract the constant term from a polynomial: `(x**2 + 3*x + 4).coeff(x, 0)` -> `4`.
  - The method `round` has been added to round a SymPy expression to a given a number of decimal places (to the left or right of the decimal point).
- divmod is now supported for all SymPy numbers.
- In the simplify module, the algorithms for denesting of radicals (sqrtdenest) and simplifying gamma functions (in combsimp) has been significantly improved.

- The mathematica-similar `TableForm` function has been added to the printing.tableform module so one can easily generate tables with headings.
- The expand API has been updated. `expand()` now officially supports arbitrary `_eval_expand_hint()` methods on custom objects. `_eval_expand_hint()` methods are now only responsible for expanding the top-level expression. All `deep=True` related logic happens in `expand()` itself. See the docstring of `expand()` for more information and an example.
- Two options were added to `isympy` to aid in interactive usage. `isympy -a` automatically creates symbols, so that typing something like `a` will give `Symbol('a')`, even if you never typed `a = Symbol('a')` or `var('a')`. `isympy -i` automatically wraps integer literals with Integer, so that `1/2` will give `Rational(1, 2)` instead of `0.5`. `isympy -I` is the same as `isympy -a -i`. `isympy -I` makes isympy act much more like a traditional interactive computer algebra system. These both require IPython.
- The official documentation at http://docs.sympy.org/ now includes an extension that automatically hooks the documentation examples in to SymPy Live.

In addition to the more noticeable changes listed above, there have been numerous smaller additions, improvements and bug fixes in the commits in this release. See the git log for a full list of all changes. The command `git log sympy-0.7.1..sympy-0.7.2` will show all commits made between this release and the last. You can also see the issues closed since the last release here.

7.1.28 SymPy 0.7.3

13 Jul 2013

Major changes

- Integration
  - This release includes Risch integration algorithm from Aaron Meurer's 2010 Google Summer of Code project. This makes `integrate` much more powerful and much faster for the supported functions. The algorithm is called automatically from `integrate()`. For now, only transcendental elementary functions containing `exp` or `log` are supported. To access the algorithm directly, use `integrate(expr, x, risch=True)`. The algorithm has the ability to prove that integrals are nonelementary. To determine if a function is nonelementary, integrate using `risch=True`. If the resulting `Integral` class is an instance of `NonElementaryIntegral`, then it is not elementary (otherwise, that part of the algorithm has just not been implemented yet).
- ODE
  - Built basic infrastructure of the PDE module (sympy/sympy#1970)
- Theano Interaction
  - SymPy expressions can now be translated into Theano expressions for numeric evaluation. This includes most standard scalar operations (e.g. `sin`, `exp`, `gamma`, but not `beta` or `MeijerG`) and matrices. This system generally outperforms `lambdify` and `autowrap` but does require Theano to be installed.
- Matrix Expressions
  - Matrix expressions now support inference using the new assumptions system. New predicates include `invertible`, `symmetric`, `positive_definite`, `orthogonal`, ...

- New operators include `Adjoint`, `HadamardProduct`, `Determinant`, `MatrixSlice`, `DFT`. Also, preliminary support exists for factorizations like `SVD` and `LU`.

- Context manager for New Assumptions
  - Added the `with assuming(*facts)` context manager for new assumptions. See blogpost.

Compatibility breaks

- This is the last version of SymPy to support Python 2.5.
- The IPython extension, i.e., `%load_ext sympy.interactive.ipythonprinting` is deprecated. Use `from sympy import init_printing; init_printing()` instead. See sympy/sympy#7013.
- The `viewer='file'` option to `preview` without a file name is deprecated. Use `filename='name'` in addition to `viewer='file'`. See sympy/sympy#7018.
- The deprecated syntax `Symbol('x', dummy=True)`, which had been deprecated since 0.7.0, has been removed. Use `Dummy('x')` or `symbols('x', cls=Dummy)` instead. See sympy/sympy#6477.
- The deprecated `Expr` methods `as_coeff_terms` and `as_coeff_factors`, which have been deprecated in favor of `as_coeff_mul` and `as_coeff_add`, respectively (see also `as_coeff_Mul` and `as_coeff_Add`), were removed. The methods had been deprecated since SymPy 0.7.0. See sympy/sympy#6476.
- The spherical harmonics have been completely rewritten. See sympy/sympy#1510.

Minor changes

- Solvers
  - Added enhancements and improved the methods of solving exact differential equation. See sympy/sympy#1955 and sympy/sympy#1823.
  - Support for differential equations with linear coefficients and those that can be reduced to separable and linear form. See sympy/sympy#1940, sympy/sympy#1864 and sympy/sympy#1883.
  - Support for first order linear general PDE's with constant coefficients (sympy/sympy#2109).
  - Return all found independent solutions for underdetermined systems.
  - Handle recursive problems for which `y(0) = 0`.
  - Handle matrix equations.
- Integration
  - `integrate` will split out integrals into Piecewise expressions when conditions must hold for the answer to be true. For example, `integrate(x**n, x)` now gives `Piecewise((log(x), Eq(n, -1), (x**(n + 1)/(n + 1), True))` (previously it just gave `x**(n + 1)/(n + 1)`).
  - Calculate Gauss-Legendre and Gauss-Laguerre points and weights (sympy/sympy#1497).
  - Various new error and inverse error functions (sympy/sympy#1703).

- Use in heurisch for more symmetric and nicer results.
- Gruntz for expintegrals and all new erf*.
- Li, li logarithmic integrals (sympy/sympy#1708).
- Integration of li/Li by heurisch (sympy/sympy#1712).
- elliptic integrals, complete and incomplete.
- Integration of complete elliptic integrals by meijerg.
- Integration of Piecewise with symbolic conditions.
- Fixed many wrong results of DiracDelta integrals.

- Logic
  - Addition of SOPform and POSform functions to sympy.logic to generate boolean expressions from truth tables.
  - Addition of simplify_logic function and enabling `simplify()` to reduce logic expressions to their simplest forms.
  - Addition of bool_equals function to check equality of boolean expressions and return a mapping of variables from one expr to other that leads to the equality.
  - Addition of disjunctive normal form methods - to_dnf, is_dnf
- Others
  - gmpy version 2 is now supported
  - Added `is_algebraic_expr()` method (sympy/sympy#2176).
  - Many improvements to the handling of noncommutative symbols:
    * Better support in simplification functions, e.g. `factor`, `trigsimp`
    * Better integration with `Order()`
    * Better pattern matching
  - Improved pattern matching including matching the identity.
  - normalizes Jacobi polynomials
  - Quadrature rules for orthogonal polynomials in arbitrary precision (hermite, laguerre, legendre, gen_legendre, jacobi)
  - summation of harmonic numbers
  - Many improvements of the polygamma functions
  - evaluation at special arguments
  - Connections to harmonic numbers
  - structured full partial fraction decomposition (mainly interesting for developers)
  - besselsimp improvements
  - Karr summation convention
  - New spherical harmonics
  - improved minimal_polynomial using composition of algebraic numbers (sympy/sympy#2038).
  - faster integer polynomial factorization (sympy/sympy#2148).

- Euler-Descartes method for quartic equations (sympy/sympy#1947)
- algebraic operations on tensors (sympy/sympy#1700).
- tensor canonicalization (sympy/sympy#1644).
- Handle the simplification of summations and products over a KroneckerDelta.
- Implemented LaTeX printing of DiracDelta, Heaviside, KroneckerDelta and Levi-Civita, also many Matrix expressions.
- Improved LaTeX printing of fractions, Mul in general.
- IPython integration and printing issues have been ironed out.
- Stats now supports discrete distributions (e.g. `Poisson`) by relying on `Summation` objects
- Added DOT printing for visualization of expression trees
- Added information about solvability and nilpotency of named groups.

7.1.29 SymPy 0.7.4

9 Dec 2013

Major changes

- Python 3
  - SymPy now uses a single code-base for Python 2 and Python 3.
- Geometric Algebra
  - The internal representation of a multivector has been changes to more fully use the inherent capabilities of SymPy. A multivector is now represented by a linear combination of real commutative SymPy expressions and a collection of non-commutative SymPy symbols. Each non-commutative symbol represents a base in the geometric algebra of an N-dimensional vector space. The total number of non-commutative bases is `2**N - 1` (`N` of which are a basis for the vector space) which when including scalars give a dimension for the geometric algebra of `2**N`. The different products of geometric algebra are implemented as functions that take pairs of bases symbols and return a multivector for each pair of bases.
  - The LaTeX printing module for multivectors has been rewritten to simply extend the existing sympy LaTeX printing module and the sympy LaTeX module is now used to print the bases coefficients in the multivector representation instead of writing an entire LaTeX printing module from scratch.
  - The main change in the geometric algebra module from the viewpoint of the user is the interface for the gradient operator and the implementation of vector manifolds:
    - The gradient operator is now implemented as a special vector (the user can name it `grad` if they wish) so the if `F` is a multivector field all the operations of `grad` on `F` can be written `grad*F`, `F*grad`, `grad^F`, `F^grad`, `grad|F`, `F|grad`, `grad<F`, `F<grad`, `grad>F`, and `F>grad` where `**`, `^`, `|`, `<`, and `>` are the geometric product, outer product, inner product, left contraction, and right contraction, respectively.
    - The vector manifold is defined as a parametric vector field in an embedding vector space. For example a surface in a 3-dimensional space would be a vector

field as a function of two parameters. Then multivector fields can be defined on the manifold. The operations available to be performed on these fields are directional derivative, gradient, and projection. The weak point of the current manifold representation is that all fields on the manifold are represented in terms of the bases of the embedding vector space.

- Classical Cryptography, implements:
  - Affine ciphers
  - Vigenere ciphers
  - Bifid ciphers
  - Hill ciphers
  - RSA and "kid RSA"
  - linear feedback shift registers.
- Common Subexpression Elimination (CSE). Major changes have been done in cse internals resulting in a big speedup for larger expressions. Some changes reflect on the user side:
  - Adds and Muls are now recursively matched (`[w*x, w*x*y, w*x*y*z]` ǹow turns into `[(x0, w*x), (x1, x0*y)], [x0, x1, x1*z])`
  - CSE is now not performed on the non-commutative parts of multiplications (it avoids some bugs).
  - Pre and post optimizations are not performed by default anymore. The `optimizations` parameter still exists and `optimizations='basic'` can be used to apply previous default optimizations. These optimizations could really slow down cse on larger expressions and are no guarantee of better results.
  - An `order` parameter has been introduced to control whether Adds and Muls terms are ordered independently of hashing implementation. The default `order='canonical'` will independently order the terms. `order='none'` will not do any ordering (hashes order is used) and will represent a major performance improvement for really huge expressions.
  - In general, the output of cse will be slightly different from the previous implementation.
- Diophantine Equation Module. This is a new addition to SymPy as a result of a GSoC project. With the current release, following five types of equations are supported.
  - Linear Diophantine equation, $a_1x_1 + a_2x_2 + ... + a_nx_n = b$
  - General binary quadratic equation, $ax^2 + bxy + cy^2 + dx + ey + f = 0$
  - Homogeneous ternary quadratic equation, $ax^2 + by^2 + cz^2 + dxy + eyz + fzx = 0$
  - Extended Pythagorean equation, $a_1x_1^2 + a_2x_2^2 + ... + a_nx_n^2 = a_{n+1}x_{n+1}^2$
  - General sum of squares, $x_1^2 + x_2^2 + ... + x_n^2 = k$
- Unification of Sum, Product, and Integral classes
  - A new superclass has been introduced to unify the treatments of indexed expressions, such as Sum, Product, and Integral. This enforced common behavior accross the objects, and provides more robust support for a number of operations. For example, Sums and Integrals can now be factored or expanded. `S.subs()` can be used to substitute for expressions inside a Sum/Integral/Product that are independent of the

index variables, including unknown functions, for instance, `Integral(f(x), (x, 1, 3)).subs(f(x), x**2)`, while `Sum.change_index()` or `Integral.transform` are now used for other changes of summation or integration variables. Support for finite and infinite sequence products has also been restored.

  - In addition there were a number of fixes to the evaluation of nested sums and sums involving Kronecker delta functions, see sympy/sympy#7023 and sympy/sympy#7086.

- Series
  - The `Order` object used to represent the growth of a function in series expansions as a variable tend to zero can now also represent growth as a variable tend to infinity. This also fixed a number of issues with limits. See sympy/sympy#3333 and sympy/sympy#5769.
  - Division by `Order` is disallowed, see sympy/sympy#4855.
  - Addition of `Order` object is now commutative, see sympy/sympy#4279.
- Physics
  - Initial work on gamma matrices, depending on the tensor module.
- Logic
  - New objects `true` and `false` which are `Basic` versions of the Python builtins `True` and `False`.
- Other
  - Arbitrary comparisons between expressions (like `x < y`) no longer have a boolean truth value. This means code like `if x < y` or `sorted(exprs)` will raise `TypeError` if `x < y` is symbolic. A typical fix of the former is `if (x < y) is True` (assuming the `if` block should be skipped if `x < y` is symbolic), and of the latter is `sorted(exprs, key=default_sort_key)`, which will order the expressions in an arbitrary, but consistent way, even across platforms and Python versions. See sympy/sympy#5931.
  - Arbitrary comparisons between complex *numbers* (for example, `I > 1`) now raise `TypeError` as well (see sympy/sympy#2510).
  - `minimal_polynomial` now works with algebraic functions, like `minimal_polynomial(sqrt(x) + sqrt(x + 1), y)`.
  - `exp` can now act on any matrix, even those which are not diagonalizable. It is also more comfortable to call it, `exp(m)` instead of just `m.exp()`, as was required previously.
  - `sympify` now has an option `evaluate=False` that will not automatically simplify expressions like `x+x`.
  - Deep processing of `cancel` and `simplify` functions. `simplify` is now recursive through the expression tree. See e.g. sympy/sympy#7022.
  - Improved the modularity of the codebase for potential subclasses, see sympy/sympy#6751.
  - The SymPy cheatsheet was cleaned up.

Compatibility breaks

- Removed deprecated Real class and is_Real property of Basic, see sympy/sympy#4820.

- Removed deprecated 'each_char' option for `symbols()`, see sympy/sympy#5018.
- The `viewer="StringIO"` option to `preview()` has been deprecated. Use `viewer="BytesIO"` instead. See sympy/sympy#7083.
- `TransformationSet` has been renamed to `ImageSet`. Added public facing `imageset` function.

7.1.30 SymPy 0.7.5

22 Feb 2014

Major changes

- The version of mpmath included in SymPy has been updated to 0.18.
- New routines for efficiently compute the *dispersion* of a polynomial or a pair thereof.
- Fancy indexing of matrices is now provided, e.g. `A[:, [1, 2, 5]]` selects all rows and only 3 columns.
- Enumeration of multiset partitions is now based on an implementation of Algorithm 7.1.2.5M from Knuth's *The Art of Computer Programming*. The new version is much faster, and includes fast methods for enumerating only those partitions with a restricted range of sizes, and counting multiset partitions. (See the new file `sympy.utilities.enumerative.py`.)
- `distance` methods were added to Line and Ray to compute the shortest distance to them from a point.
- The `normal_lines` method was added to Ellipse to compute the lines from a point that strike the Ellipse at a normal angle.
- `inv_quick` and `det_quick` were added as functions in solvers.py to facilitate fast solution of small symbolic matrices; their use in solve has reduced greatly the time needed to solve such systems.
- `solve_univariate_inequality` has been added to sympy.solvers.inequalities.py.
- `as_set` attribute for Relationals and Booleans has been added.
- Several classes and functions strictly associated with vector calculus were moved from `sympy.physics.mechanics` to a new package `sympy.physics.vector`. (See sympy/sympy#2732, sympy/sympy#2862 and sympy/sympy#2894).
- New implementation of the Airy functions `Ai` and `Bi` and their derivatives `Ai'` and `Bi'` (called `airyai`, `airybi`, `airyaiprime` and `airybiprime`, respectively). Most of the usual features of SymPy special function are present. Notable exceptions are Gruntz limit computation helpers and meijerg special functions integration code.
- Euler-Lagrange equations (function `euler_equations`) in a new package `sympy.calculus` (sympy/sympy#2431).

Compatibility breaks

- the `submatrix` method of matrices was removed; access the functionality by providing slices or list of rows/columns to matrix directly, e.g. `A[:, [1, 2]]`.
- `Matrix([])` and `Matrix([[]])` now both return a 0x0 matrix

- `terms_gcd` no longer removes a -1.0 from expressions
- `extract_multiplicatively` will not remove a negative Number from a positive one, so `(4*x*y).extract_multiplicatively(-2*x)` will return None.
- the shape of the result from `M.cross(B)` now has the same shape as matrix M.
- The factorial of negative numbers is now `zoo` instead of `0`. This is consistent with the definition `factorial(n) = gamma(n + 1)`.
- `1/0` returns `zoo`, not `oo` (sympy/sympy#2813).
- `zoo.is_number is True` (sympy/sympy#2823).
- `oo < I` raises `TypeError`, just as for finite numbers (sympy/sympy#2734).
- `1**oo == nan` instead of `1`, better documentation for `Pow` class (sympy/sympy#2606).

Minor changes

- Some improvements to the gamma function.
- `generate_bell` now generates correct permutations for any number of elements.
- It is no longer necessary to provide `nargs` to objects subclassed from Function unless an `eval` class method is not defined. (If `eval` is defined, the number of arguments will be inferred from its signature.)
- geometric `Point` creation will be faster since simplification is done only on Floats
- Some improvements to the intersection method of the Ellipse.
- solutions from solve of equations involving multiple log terms are more robust
- `idiff` can now return higher order derivatives
- Added `to_matrix()` method to `sympy.physics.vector.Vector` and `sympy.physics.dyadic.Dyadic`. (sympy/sympy#2686).
- Printing improvements for `sympy.physics.vector` objects and mechanics printing. (See sympy/sympy#2687, sympy/sympy#2728, sympy/sympy#2772, sympy/sympy#2862 and sympy/sympy#2894).
- Functions with LaTeX symbols now print correct LaTeX. (sympy/sympy#2772).
- `init_printing` has several new options, including a flag `print_builtin` to prevent SymPy printing of basic Python types (sympy/sympy#2683), and flags to let you supply custom printers (sympy/sympy#2894).
- improvements in evaluation of `imageset` for Intervals (sympy/sympy#2723).
- Set properties to determine boundary and interior (sympy/sympy#2744).
- input to a function created by lambdify no longer needs to be flattened.

7.1.31 SymPy 0.7.6

21 Nov 2014

New features

- New module `calculus.finite_diff` for generating finite difference formulae approximating derivatives of arbitrary order on arbitrarily spaced grids.
- New module `physics.optics` for symbolic computations related to optics.
- `geometry` module now supports 3D geometry.
- Support for series expansions at a point other then `0` or `oo`. See sympy/sympy#2427.
- Rules for the intersection of integer ImageSets were added. See sympy/sympy#7587. We can now do things like `{2⋅m | m ∊ ℤ} ∩ {3⋅n | n ∊ ℤ} = {6⋅t | t ∊ ℤ}` and `{2⋅m | m ∊ ℤ} ∩ {2⋅n + 1 | n ∊ ℤ} = ∅`.
- `dsolve` module now supports system of ODEs including linear system of ODEs of 1st order for 2 and 3 equations and of 2nd order for 2 equations. It also supports homogeneous linear system of `n` equations.
- New support for continued fractions, including iterators for partial quotients and convergents, and reducing a continued fraction to a Rational or a quadratic irrational.
- Support for Egyptian fraction expansions, using several different algorithms.
- Addition of generalized linearization methods to `physics.mechanics`.
- Use an LRU cache by default instead of an unbounded one. See sympy/sympy#7464. Control cache size by the environment variable `SYMPY_CACHE_SIZE` (default is 500). `SYMPY_CACHE_SIZE=None` restores the unbounded cache.
- Added `fastcache` as an optional dependency. Requires v0.4 or newer. Control via `SYMPY_CACHE_SIZE`. May result in significant speedup. See sympy/sympy#7737.
- New experimental module `physics.unitsystems` for computation with dimensions, units and quantities gathered into systems. This opens the way to dimensional analysis and better quantity calculus. The old module `physics.units` will stay available until the new one reaches a mature state. See sympy/sympy#2628.
- New `Complement` class to represent relative complements of two sets. See sympy/sympy#7462.
- New trigonometric functions (asec, acsc), many enhancements for other trigonometric functions (see sympy/sympy#7500).
- New `Contains` class to represent the relation "is an element of" (see sympy/sympy#7989).
- The code generation tools (code printers, `codegen`, `autowrap`, and `ufuncify`) have been updated to support a wider variety of constructs, and do so in a more robust way. Major changes include added support for matrices as inputs/outputs, and more robust handling of conditional (`Piecewise`) statements.
- `ufuncify` now uses a backend that generates actual `numpy.ufuncs` by default through the use of the `numpy` C api. This allows broadcasting on *all* arguments. The previous `cython` and `f2py` backends are still accessible through the use of the `backend` kwarg.
- `CodeGen` now generates code for Octave and Matlab from SymPy expressions. This is supported by a new CodePrinter with interface `octave_code`. For example `octave_code(Matrix([[x**2, sin(pi*x*y), ceiling(x)]]))` gives the string `[x.^2 sin(pi*x.*y) ceil(x)]`.

- New general 3D vector package at `sympy.vector`. This package provides a 3D vector object with the Del, gradient, divergence, curl, and operators. It supports arbitrary rotations of Cartesian coordinate systems and arbitrary locations of points.

Compatibility breaks

- All usage of inequalities (>, >=, <, <=) on SymPy objects will now return SymPy's `S.true` or `S.false` singletons instead of Python's `True` or `False` singletons. Code that checks for, e.g., `(a < b) is True` should be changed to `(a < b) == True` or `(a < b) == S.true`. Use of `is` is not recommended here.
- The `subset()` method in `sympy.core.sets` is marked as being deprecated and will be removed in a future release (sympy/sympy#7460). Instead, the `is_subset()` method should be used.
- Previously, if you compute the series expansion at a point other than 0, the result was shifted to 0. Now SymPy returns the usual series expansion, see sympy/sympy#2427.
- In `physics.mechanics`, `KanesMethod.linearize` has a new interface. Old code should be changed to use this instead. See docstring for information.
- `physics.gaussopt` has been moved to `physics.optics.gaussopt`. You can still import it from the previous location but it may result in a deprecation warning.
- This is the last release with the bundled mpmath library. In the next release you will have to install this library from the official site.
- Previously `lambdify` would convert `Matrix` to `numpy.matrix` by default. This behavior is being deprecated, and will be completely phased out with the release of 0.7.7. To use the new behavior now set the modules kwarg to `[{'ImmutableMatrix': numpy.array}, 'numpy']`. If lambdify will be used frequently it is recommended to wrap it with a `partial` as so: `lambdify = functools.partial(lambdify, modules=[{'ImmutableMatrix': numpy.array}, 'numpy'])`. For more information see sympy/sympy#7853 and the `lambdify` docstring.
- `Set.complement` doesn't exists as an attribute anymore. Now we have a method `Set.complement(<universal_set>)` which complements the given universal set.
- Removed is_finite assumption (see sympy/sympy#7891). Use instead a combination of `is_bounded and is_nonzero` assumptions.
- is_bounded and is_unbounded assumptions were renamed to is_finite and is_infinite (see sympy/sympy#7947).
- Removed is_infinitesimal assumption (see sympy/sympy#7995).
- Removed is_real property for Sets, use `Set.is_subset(Reals)` instead (see sympy/sympy#7996).
- For generic symbol `x` (SymPy's symbols are not bounded by default), inequalities with `oo` are no longer evaluated as they were before, e.g. `x < oo` no longer evaluates to True). See sympy/sympy#7861.
- `CodeGen` has been refactored to make it easier to add other languages. The main high-level tool is still `utilities.codegen.codegen`. But if you previously used the `Routine` class directly, note its `__init__` behaviour has changed; the new `utilities.codegen.make_routine` is recommended instead and by default retains the previous C/Fortran behaviour. If needed, you can still instantiate `Routine` directly; it only does minimal sanity checking on its inputs. See sympy/sympy#8082.

- `FiniteSet([1, 2, 3, 4])` syntax not supported anymore, use `FiniteSet(1, 2, 3, 4)` instead. See sympy/sympy#7622.

Minor changes

- Updated the parsing module to allow sympification of lambda statements to their SymPy equivalent.
- Lambdify can now use `numexpr` by specifying `modules='numexpr'`.
- Use `with evaluate(False)` context manager to control automatic evaluation. E.g. `with evaluate(False): x + x` is actually `x + x`, not `2*x`.
- IndexedBase and Indexed are changed to be commutative by default.
- `sympy.core.sets` moved to `sympy.sets`.
- Changes in `sympy.sets`:
  - Infinite `Range` is now allowed. See sympy/sympy#7741.
  - `is_subset()`: The `is_subset()` method deprecates the `subset()` method. `self.is_subset(other)` checks if `self` is a subset of `other`. This is different from `self.subset(other)`, which checked if `other` is a subset of `self`.
  - `is_superset()`: A new method `is_superset()` method is now available. `self.is_superset(other)` checks if `self` is a superset of `other`.
  - `is_proper_subset` and `is_proper_superset`: Two new methods allow checking if one set is the proper subset and proper superset of another respectively. For e.g. `self.is_proper_subset(other)` and `self.is_proper_superset(other)` checks if `self` is the proper subset of `other` and if `self` is the proper superset of `other` respectively.
  - `is_disjoint()`: A new method for checking if two sets are disjoint.
  - `powerset()`: A new method `powerset()` has been added to find the power set of a set.
  - The cardinality of a `ProductSet` can be found using the `len()` function.
- Changes in `sympy.plot.plot_implicit`:
  - The `plot_implicit` function now also allows explicitly specifying the symbols to plot on the X and Y axes. If not specified, the symbols will be assigned in the order they are sorted.
  - The `plot_implicit` function also allows axes labels for the plot to be specified.
- rules for simplification of ImageSet were added sympy/sympy#7625. As a result `{x | x ∊ ℤ}` now simplifies to `ℤ` and `{sin(n) | n ∊ {tan(m) | m ∊ ℤ}}` automatically simplifies to `{sin(tan(m)) | m ∊ ℤ}`.
- coth(0) now returns Complex Infinity. See sympy/sympy#7634.
- dioptre is added to `physics.units`. See sympy/sympy#7782.
- `replace` now respects commutativity, see sympy/sympy#7752.
- The CCodePrinter gracefully handles Symbols which have string representations that match C reserved words, see sympy/sympy#8199.
- `limit` function now returns an unevaluated `Limit` instance if it can't compute given limit, see sympy/sympy#8213.

7.2 Diofant 0.8

7 Nov 2016

7.2.1 New features

- MrvAsympt algorithm to find asymptotic expansion, see *aseries()* (page 62) method and #6. Thanks to Avichal Dayal.
- *findrecur()* (page 276) method to find recurrence relations (with Sister Celine's algorithm), see #15.
- Support for *Pow* (page 100)/*log* (page 322) branch-cuts in limits, see #140.
- Added basic optimization package, see *minimize()* (page 1010) and #108.
- Cartesian product of iterables using Cantor pairing, see *cantor_product()* (page 983) and #276.
- *Rationals* (page 774) set, #255.
- New simple and robust solver for systems of linear ODEs, see #286. Thanks to Colin B. Macdonald.
- Added mutable/immutable N-dim arrays, sparse and dense, see #275.
- *dsolve()* (page 868) now support initial conditions for ODEs, see #307. Thanks to Aaron Meurer.

7.2.2 Major changes

- Depend on setuptools, see #44.
- *The Gruntz Algorithm* (page 1152) reimplemented correctly, see #68.
- Replaced `exp(x)` with `E**x` internally, see #79.
- Used *srepr()* (page 736) instead of *sstr()* (page 736) for `__repr__()` printing, see #39.
- Major cleanup for series methods, see #187.
- Depend on cachetools to implement caching, see #72 and #209.
- Assumption system (old) was validated (#316 and #334) and improved:
  - 0 now is imaginary, see #8
  - extended_real fact added, reals are finite now, see #36
  - complex are finite now, see #42.
  - added docstrings for assumption properties, see #354.

7.2.3 Compatibility breaks

- Removed physics submodule, see #23.
- Removed galgebra submodule, see #45.
- Removed pyglet plotting, see #50.

- Removed TextBackend from plotting, see #67.
- Removed SageMath support, see #84.
- Removed unify submodule, see #88.
- Removed crypto submodule, see #102.
- Removed print_gtk, see #114.
- Unbundle strategies module, see #103.
- Removed "old" argument for match/matches, see #141.
- Removed when_multiple kwarg in Piecewise, see #156.
- Support for Python 2 was removed, see #160.
- Removed core.py, see #60 and #164.
- Removed S(foo) syntax, see #115.
- Removed (new) assumptions submodule, see #122.
- Removed undocumented Symbol.__call__, see #201
- Removed categories and liealgebras submodules, see #280.
- Rename module sympy -> diofant, see #315.
- Use gmpy2, drop gmpy support, see #292.
- Removed redundant dom properties in polys, see #308.
- Removed manualintegrate function, see #279.

7.2.4 Minor changes

- Add support for bidirectional limits, see #10.
- Reimplement `cot` (page 301), see #113.
- A better implementation of `singularities()` (page 1010), see #147.
- Fix "flip" of arguments in relational expressions, see #30.
- Make Gosper code use new dispersion algorithm, see #205. Thanks to Raoul Bourquin.
- Consolidate code for solving linear systems, see #253.
- Hacks for automatic symbols and wrapping int's replaced with AST transformers, see #278 and #167.
- Build correct inhomogeneous solution in `rsolve_hyper()` (page 921), see #298.
- Evaluate matrix powers for non-diagonalizable matrices, see #275.
- Support non-orthogonal Jordan blocks, see #275.
- Make risch_integrate(x**x, x) work, see #275.
- Support CPython 3.6, see #337 and #356.

7.2.5 Developer changes

- Unbundle numpydoc, see #26.
- Deprecate AUTHORS file, all credits go to the aboutus.rst, see #87.
- Use python's `tokenize()`, see #120.
- Drop using bundled pytest fork, depend on pytest for testing, see #38, #152, #91, #48, #90, #96 and #99.
- Adopt No Code Of Conduct, see #212.
- Measure code coverage, enable codecov.io reports. See #217.
- Adopt pep8 (#2) and then flake8 (#214) for code quality testing.
- Add regression tests with DIOFANT_USE_CACHE=False #323.
- Add interface tests, see #219 and #307.
- Test for no DeprecationWarning in the codebase, see #356.

7.2.6 Issues closed

See the release milestone for complete list of issues and pull requests involved in this release.

These Sympy issues also were addressed:

- sympy/sympy#9351 order-1 series wrong with non-zero expansion point
- sympy/sympy#9034 Unicode printing problem with mixture of logs and powers
- sympy/sympy#7927 pretty print incorrect result with powers of sin
- sympy/sympy#9283 KroneckerDelta(p, 0) raises IndexError
- sympy/sympy#9274 Wrong Jordan form: complex eigenvalues w/ geo. mult. > alg. mult.
- sympy/sympy#9398 Simplify of small imaginary number yields 0
- sympy/sympy#7259 LambertW has no series expansion at x=0 (nan)
- sympy/sympy#9832 x**2 < oo returns True but x < oo un-evaluated for real x
- sympy/sympy#9053 MatMul(2, Matrix(...)).doit() doesn't do it
- sympy/sympy#9052 trace(2*A) != 2*Trace(A) because LHS still has an MatMul
- sympy/sympy#9533 Logical operators in octave_code
- sympy/sympy#9545 Mod(zoo, 0) causes RunTime Error
- sympy/sympy#9652 Fail in plot_implicit test on OSX 10.8.5
- sympy/sympy#8432 Tests fail, seems like Cython is not configured to compile with numpy correctly
- sympy/sympy#9542 codegen octave global vars should print "global foo" at top of function
- sympy/sympy#9326 Bug with Dummy
- sympy/sympy#9413 Circularity in assumptions of products
- sympy/sympy#8840 sympy fails to construct (1 + x)*x with disabled cache

- sympy/sympy#4898 Replace exp(x) with E**x internally
- sympy/sympy#10195 Simplification bug on alternating series.
- sympy/sympy#10196 reduce_inequalities error
- sympy/sympy#10198 solving abs with negative powers
- sympy/sympy#7917 Implement cot as a ReciprocalTrigonometricFunction
- sympy/sympy#8649 If t is transcendental, t**n is determined (wrongly) to be non-integer
- sympy/sympy#5641 Compatibility with py.test
- sympy/sympy#10258 Relational involving Piecewise evaluates incorrectly as True
- sympy/sympy#10268 solving inequality involving exp fails for large values
- sympy/sympy#10237 improper inequality reduction
- sympy/sympy#10255 solving a Relational involving Piecewise fails
- sympy/sympy#10290 Computing series where the free variable is not just a symbol is broken
- sympy/sympy#10304 Equality involving expression with known real part and 0 should evaluate
- sympy/sympy#9471 Wrong limit with log and constant in exponent
- sympy/sympy#9449 limit fails with "maximum recursion depth exceeded" / Python crash
- sympy/sympy#8462 Trivial bounds on binomial coefficients
- sympy/sympy#9917 O(n*sin(n) + 1, (n, oo)) returns O(n*sin(n), (n, oo))
- sympy/sympy#7383 Integration error
- sympy/sympy#7098 Incorrect expression resulting from integral evaluation
- sympy/sympy#10323 bad ceiling(sqrt(big integer))
- sympy/sympy#10326 Interval(-oo, oo) contains oo
- sympy/sympy#10095 simplify((1/(2*E))**oo) returns nan
- sympy/sympy#4187 integrate(log(x)*exp(x), (x, 0, oo)) should return -EulerGamma
- sympy/sympy#10383 det of empty matrix is 1
- sympy/sympy#10382 limit(fibonacci(n + 1)/fibonacci(n), n, oo) does not give GoldenRatio
- sympy/sympy#10388 factorial2 runs into RunTimeError for non-integer
- sympy/sympy#10391 solve((2*x + 8)*exp(-6*x), x) can't find any solution
- sympy/sympy#8241 Wrong assumption/result in a parametric limit
- sympy/sympy#3539 Symbol.__call__ should not create a Function
- sympy/sympy#7216 Limits involving branch cuts of elementary functions not handled
- sympy/sympy#10503 Series return an incorrect result
- sympy/sympy#10567 Integral(v,t).doit() differs from integrate(v,t)
- sympy/sympy#9075 sympy.limit yields incorrect result
- sympy/sympy#10610 limit(3**n*3**(-n - 1)*(n + 1)**2/n**2, n, oo) is wrong

- sympy/sympy#4173 implement maximize([x**(1/x), x>0], x)
- sympy/sympy#10803 Bad pretty printing of power of Limit
- sympy/sympy#10836 Latex generation error for .series expansion for rightarrow term
- sympy/sympy#9558 Bug with limit
- sympy/sympy#4949 solve_linear_system contains duplicate rref algorithm
- sympy/sympy#5952 Standard sets (ZZ, QQ, RR, etc.) for the sets module
- sympy/sympy#9608 Partition can't be ordered
- sympy/sympy#10961 fractional order Laguerre gives wrong result
- sympy/sympy#10976 incorrect answer for limit involving erf
- sympy/sympy#10995 acot(-x) evaluation
- sympy/sympy#11011 Scientific notation should be delimited for LaTeX
- sympy/sympy#11062 Error while simplifying equations containing csc and sec using trigsimp_groebner
- sympy/sympy#10804 1/limit(airybi(x)*root(x, 4)*exp(-2*x**(S(3)/2)/3), x, oo)**2 is wrong
- sympy/sympy#11063 Some wrong answers from rsolve
- sympy/sympy#9480 Matrix.rank() incorrect results
- sympy/sympy#10497 next(iter(S.Integers*S.Integers)) hangs (expected (0, 0), …)
- sympy/sympy#5383 Calculate limit error
- sympy/sympy#11270 Limit erroneously reported as infinity
- sympy/sympy#5172 limit() throws TypeError: an integer is required
- sympy/sympy#7055 Failures in rsolve_hyper from test_rsolve_bulk()
- sympy/sympy#11261 Recursion solver fails
- sympy/sympy#11313 Series of Derivative
- sympy/sympy#11290 1st_exact_Integral wrong result
- sympy/sympy#10761 (1/(x**-2 + x**-3)).series(x, 0) gives wrong result
- sympy/sympy#10024 Eq(Mod(x, 2*pi), 0) evaluates to False
- sympy/sympy#7985 Indexed should work with subs on a container
- sympy/sympy#9637 S.Reals - FiniteSet(n) returns EmptySet - FiniteSet(n)
- sympy/sympy#10003 P(X < -1) of ExponentialDistribution
- sympy/sympy#10052 P(X < oo) for any Continuous Distribution raises AttributeError
- sympy/sympy#10063 Integer raised to Float power does not evaluate
- sympy/sympy#10075 X.pdf(x) for Symbol x returns 0
- sympy/sympy#9823 Matrix power of identity matrix fails
- sympy/sympy#10156 do not use has() to test against self.variables when factoring Sum
- sympy/sympy#10113 imageset(lambda x: x**2/(x**2 - 4), S.Reals) returns (1, oo)
- sympy/sympy#10020 oo**I raises RunTimeError

- sympy/sympy#10240 Not(And(x>2, x<3)) does not evaluate
- sympy/sympy#8510 Differentiation of general functions
- sympy/sympy#10220 Matrix.jordan_cells() fails
- sympy/sympy#10092 subs into inequality involving RootOf raises GeneratorsNeeded
- sympy/sympy#10161 factor gives an invalid expression
- sympy/sympy#10243 Run the examples during automated testing or at release
- sympy/sympy#10274 The helpers kwarg in autowrap method is probably broken.
- sympy/sympy#10210 LaTex printing of Cycle
- sympy/sympy#9539 diophantine(6*k + 9*n + 20*m - x) gives TypeError: unsupported operand type(s) for *: 'NoneType' and 'Symbol'
- sympy/sympy#11407 Series expansion of the square root gives wrong result
- sympy/sympy#11413 Wrong result from Matrix norm
- sympy/sympy#11434 Matrix rank() produces wrong result
- sympy/sympy#11526 Different result of limit after simplify
- sympy/sympy#11553 Polynomial solve with GoldenRatio causes Traceback
- sympy/sympy#8045 make all NaN is_* properties that are now None -> False (including is_complex)
- sympy/sympy#11602 Replace dots with ldots or cdots
- sympy/sympy#4720 Initial conditions in dsolve()
- sympy/sympy#11623 Wrong groebner basis
- sympy/sympy#10292 poly cannot generically be rebuilt from its args
- sympy/sympy#6572 Remove "#doctest: +SKIP" comments on valid docstrings
- sympy/sympy#10134 Remove "raise StopIteration"
- sympy/sympy#11672 limit(Rational(-1,2)**k, k, oo) fails
- sympy/sympy#11678 Invalid limit of floating point matrix power
- sympy/sympy#11746 undesired (wrong) substition behavior in sympy?
- sympy/sympy#3904 missing docstrings in core
- sympy/sympy#3112 Asymptotic expansion
- sympy/sympy#9173 Series/limit fails unless expression is simplified first.
- sympy/sympy#9808 Complements with symbols should remain unevaluated
- sympy/sympy#9341 Cancelling very long polynomial expression
- sympy/sympy#9908 Sum(1/(n**3 - 1), (n, -oo, -2)).doit() raise UnboundLocalVariable
- sympy/sympy#6171 Limit of a piecewise function
- sympy/sympy#9276 ./bin/diagnose_imports: does it work at all?!
- sympy/sympy#10201 Solution of "first order linear non-homogeneous ODE-System" is wrong
- sympy/sympy#9057 segfault on printing Integral of phi(t)

- sympy/sympy#11159 Substitution with E fails
- sympy/sympy#2839 init_session(auto_symbols=True) and init_session(auto_int_to_Integer=True) do not work
- sympy/sympy#11081 where possible, use python fractions for Rational
- sympy/sympy#10974 solvers.py contains BOM character
- sympy/sympy#10806 LaTeX printer: Integral not surrounded in brackets
- sympy/sympy#10801 Make limit work with binomial
- sympy/sympy#9549 series expansion: (x**2 + x + 1)/(x**3 + x**2) about oo gives wrong result
- sympy/sympy#4231 add a test for complex integral from wikipedia
- sympy/sympy#8634 limit(x**n, x, -oo) is sometimes wrong
- sympy/sympy#8481 Wrong error raised trying to calculate limit of Poisson PMF
- sympy/sympy#9956 Union(Interval(-oo, oo), FiniteSet(1)) not evaluated
- sympy/sympy#9747 test_piecewise_lambdify fails locally
- sympy/sympy#7853 Deprecation of lambdify converting Matrix -> numpy.matrix
- sympy/sympy#9634 Repeated example in the docstring of hermite
- sympy/sympy#8500 Using and operator vs fuzzy_and while querying assumptions
- sympy/sympy#9192 O(y + 1) = O(1)
- sympy/sympy#7130 Definite integral returns an answer with indefinite integrals
- sympy/sympy#8514 Inverse Laplace transform of a simple function fails after updating from 0.7.5 to 0.7.6
- sympy/sympy#9334 Numexpr must be string argument to lambdify
- sympy/sympy#8229 limit((x**Rational(1,4)-2)/(sqrt(x)-4)**Rational(2, 3), x, 16) NotImplementedError
- sympy/sympy#8061 limit(4**(acos(1/(1+x**2))**2)/log(1+x, 4), x, 0) raises NotImplementedError
- sympy/sympy#7872 Substitution in Order fails
- sympy/sympy#3496 limits for complex variables
- sympy/sympy#2929 limit((x*exp(x))/(exp(x)-1), x, -oo) gives -oo
- sympy/sympy#8203 Why is oo real?
- sympy/sympy#7649 S.Zero.is_imaginary should be True?
- sympy/sympy#7256 use old assumptions in code
- sympy/sympy#6783 Get rid of confusing assumptions
- sympy/sympy#5662 AssocOp._eval_template_is_attr is wrong or misused
- sympy/sympy#5295 Document assumptions
- sympy/sympy#4856 coding style
- sympy/sympy#4555 use pyflakes to identify simple bugs in sympy and fix them
- sympy/sympy#5773 Remove the cmp_to_key() helper function

- sympy/sympy#5484 use sort_key instead of old comparison system
- sympy/sympy#8825 Can't use both weakref's & cache
- sympy/sympy#8635 limit(x**n-x**(n-k), x, oo) sometimes raises NotImplementedError
- sympy/sympy#8157 Non-informative error raised when computing limit of cos(n*pi)
- sympy/sympy#7599 Addition of expression and order term fails
- sympy/sympy#6179 wrong order in series
- sympy/sympy#5415 limit involving multi-arg function (polygamma) fails
- sympy/sympy#2865 gruntz doesn't work properly for big-O with point!=0
- sympy/sympy#5907 integrate(1/(x**2 + a**2)**2, x) is wrong if a is real
- sympy/sympy#11722 series() calculation up to O(t**k) returns invalid coefficients for t**k * log(t)
- sympy/sympy#8804 series expansion of 1/x ignores order parameter
- sympy/sympy#10728 Dummy(commutative=False).is_zero -> False

7.3 Diofant 0.9

23 Feb 2018

7.3.1 New features

- Polynomial solvers now express all available solutions with *RootOf* (page 711), see #400.
- Added *mod_inverse()* (page 93) and *invert()* (page 67), see #390. Thanks to Chris Smith.
- Support solving linear programming problems, see #283 and #461.
- Added *interval* (page 712) property for *RootOf* (page 711), see #508.
- Added AST transformation *IntegerDivisionWrapper* (page 744) to wrap integer division, see #519.
- Added AST transformation *FloatRationalizer* (page 744) to wrap `float`'s, see #538.
- Compute *independent_sets* (page 706) and dimension of the ideal, generated by Gröbner basis, see #573.
- Added *permutedims()* (page 934) and *derive_by_array()* (page 934), see #567. Thanks to Francesco Bonazzi.
- Added *is_square()* (page 262), *ordered_partitions()* (page 993), *permute_signs()* (page 995) and *signed_permutations()* (page 998), see #578. Thanks to Chris Smith.

7.3.2 Major changes

- Assumptions (old) moved from *Basic* (page 42) to *Expr* (page 54), see #311.
- *solve()* (page 839) now return `list` of `dict`'s, see #473.
- `diofant.polys.domains` module is now top-level module *domains* (page 551), see #487.

- Optionally reduce *RootOf* (page 711) instances to have polynomials with integer coefficients, see #430.
- *solve_poly_system()* (page 843) now able to solve positive-dimensional systems, see #448 and #573.
- Big update of the *diophantine* (page 844) module with a lot of bugfixes, see #578. Thanks to Chris Smith.

7.3.3 Compatibility breaks

- Removed `assumption0` property, see #382.
- *check_assumptions()* (page 41) moved to *assumptions* (page 40), see #387.
- Removed `nsolve()` function, see #387.
- *is_comparable* (page 68) and *is_hypergeometric()* (page 71) moved to *Expr* (page 54), see #391.
- Removed `solve_triangulated()` and `solve_biquadratic()` functions, *solve_poly_system()* (page 843) now use `dict` as output, see #389 and #448.
- Removed support for solving undetermined coefficients in *solve()* (page 839), see #389.
- Removed `intersect()` alias for *intersection()* (page 763), see #396.
- Removed `interactive_traversal()`, see #395.
- Removed `xring()` and `xfield()`, see #403.
- Removed `jscode` submodule and `TableForm` class, see #403.
- Removed `agca` submodule of *polys* (page 645), see #404.
- Removed `pager_print()` and `print_fcode()`, see #411.
- Disallow "increase" precision of *Float* (page 85)'s with *evalf()* (page 145), see #380.
- Removed `experimental_lambdify()` and `intervalmath` module from plotting package, see #384.
- Removed *solve()* (page 839) flags `set`, `manual`, `minimal`, `implicit`, `particular`, `quick`, `exclude`, `force` and `numerical` see #426, #554 and #549.
- Removed support for inequalities in *solve()* (page 839), please use *reduce_inequalities()* (page 843) instead, see #426.
- Removed `get_domain()` method of *Poly* (page 672), use *domain* (page 683) property instead, see #479.
- Renamed `prec` argument of *Float* (page 85) to `dps`, see #510.
- Removed `as_content_primitive()` method of *Basic* (page 42), see #529.
- Removed `canonical_variables()` property to *canonical_variables()* (page 63), see #534.
- Removed `group` option of *find()* (page 44), which now return a `dict`, see #529.
- Removed support for Python 3.4, see #543.
- Second argument of *checksol()* (page 841) must be a `dict`. See #549.
- Removed `solve_undetermined_coeffs()` function, see #554.

- Make `matches()` method for *Basic* (page 42) - private, see #557.
- Removed *replace()* (page 46) flags `simultaneous` and `map`, see #557.
- Make `strict=True` - default for *evalf()* (page 145), see #537.
- Removed `I` property of the *MatrixExpr* (page 637), see #577.
- Removed `isolate()` function, see #585.
- *gcd()* (page 665) and *lcm()* (page 665) now are two-arg functions, see #585.
- Removed `is_zero_dimensional()` function and *GroebnerBasis* (page 705)'s property of the same name, use *dimension* (page 705) instead, see #573.
- Removed `MonomialOps` class, see #586.
- Renamed `n` argiment of *evalf()* (page 145) to `dps`, see #596.
- Return representation of elements via primitive in *primitive_element()* (page 709) (former `ex=True` format), see #597.
- Removed `pprint_try_use_unicode()` function, see #605.

7.3.4 Minor changes

- New integration heuristics for integrals with *Abs* (page 293), see #321.
- Support unevaluated *RootOf* (page 711), see #400.
- Sorting of symbolic quadratic roots now same as in *RootOf* (page 711) for numerical coefficients, see #400.
- Improve printing of Mathematica code, see #400, #433, #438, #519, #553 and #571.
- Support simple first-order DAE with *dsolve()* (page 868) helper *ode_lie_group()* (page 894), see #413.
- Added support for limits of relational expressions, see #414.
- Make *MatrixSymbol* (page 638) truly atomic, see #415.
- Support rewriting *Min* (page 324) and *Max* (page 325) as *Piecewise* (page 323), see #426.
- *minimal_polynomial()* (page 708) fixed to support generic *AlgebraicNumber* (page 91)'s, see #433 and #438.
- *AlgebraicNumber* (page 91) now support arithmetic operations, see #428 and #485.
- Support rewrite *RootOf* (page 711) via radicals, see #563.
- Export set singletons, see #577.
- Correct implementation of the trial method (uses Gröbner bases) in *primitive_element()* (page 709), see #608 and #609.
- Support (not in *RootOf* (page 711) yet) of root isolation for polynomials over Gaussian rationals, see #606.
- 100% test coverage for *matrices* (page 560), *domains* (page 551), *logic* (page 540), *parsing* (page 1006) and *printing* (page 722) modules. Overall test coverage is 96%.

7.3.5 Developer changes

- Enabled docstring testing with flake8, see #408.
- Use only relative imports in the codebase, see #421.
- Enabled flake8-comprehensions plugin, see #420.
- Imports are sorted with isort, see #520.
- Depend on hypothesis, see #547.
- Depend on pytest-xdist, see #551.
- Depend on pytest-timeout, see #608.

7.3.6 Issues closed

See the release milestone for complete list of issues and pull requests involved in this release.

These Sympy issues also were addressed:

- sympy/sympy#11879 Strange output from common limit used in elementary calculus
- sympy/sympy#11884 Addition with Order gives wrong result
- sympy/sympy#11045 integrate(1/(x*sqrt(x**2-1)), (x, 1, 2)) Sympy latest version AttributeError: 'Or' object has no attribute 'lts'
- sympy/sympy#7165 integrate(abs(y - x**2), (y,0,2)) raises ValueError: gamma function pole
- sympy/sympy#8733 integrate(abs(x+1), (x, 0, 1)) raises gamma function pole error
- sympy/sympy#8430 integrate(abs(x), (x, 0, 1)) does not simplify
- sympy/sympy#12005 Subs._eval_derivative doubles derivatives
- sympy/sympy#11799 Something wrong with the Riemann tensor?
- sympy/sympy#12018 solution not found by Sum and gosper_sum
- sympy/sympy#5649 Bug with AlgebraicNumber.__eq__
- sympy/sympy#11538 Bug in solve maybe
- sympy/sympy#12081 integrate(x**(-S(3)/2)*exp(-x), (x, 0, oo)) encounters Runtime Error
- sympy/sympy#7214 Move old assumptions from Basic to Expr
- sympy/sympy#4678 Have solve() return RootOf when it can't solve equations
- sympy/sympy#7789 Poly(...).all_roots fails for general quadratic equation
- sympy/sympy#8255 roots_quadratic should return roots in same order as Poly.all_roots(radicals=False)
- sympy/sympy#7138 How to solve system of differential equations with symbolic solution?
- sympy/sympy#11691 Test failing with matplotlib 2.0.0
- sympy/sympy#7457 TypeError when using both multiprocessing and gmpy
- sympy/sympy#12115 Cannot access imported submodules in sympy.core
- sympy/sympy#4315 series expansion of piecewise fails

- sympy/sympy#6807 atoms does not work correctly in the otherwise case of Piecewise
- sympy/sympy#12114 solve() leads to ZeroDivisionError: polynomial division
- sympy/sympy#5169 All elements of .args should be Basic
- sympy/sympy#6249 Problems with MatrixSymbol and simplifying functions
- sympy/sympy#6426 test_args.py should also test rebuilability
- sympy/sympy#11461 NameError: name 'Ne' is not defined plotting real_root((log(x/(x-2))), 3)
- sympy/sympy#10925 plot doesn't work with Piecewise
- sympy/sympy#12180 Confusing output from sympy.solve
- sympy/sympy#5786 factor(extension=[I]) gives wrong results
- sympy/sympy#9607 factor - incorrect result
- sympy/sympy#8754 Problem factoring trivial polynomial
- sympy/sympy#8697 rsolve fails to find solutions to some higer order recurrence relations
- sympy/sympy#8694 Match fail
- sympy/sympy#8710 geometry's encloses method fails for non-polygons
- sympy/sympy#10337 bad Boolean args not rejected
- sympy/sympy#9447 sets.Complement fails on certain Unions
- sympy/sympy#10305 Complement Of Universal Subsets
- sympy/sympy#10413 ascii pprint of ProductSet uses non-ascii multiplication symbol
- sympy/sympy#10414 pprint(Union, use_unicode=False) raises error (but str(Union) works)
- sympy/sympy#10375 lambdify on sympy.Min does not work with NumPy
- sympy/sympy#10433 Dict does not accept collections.defaultdict
- sympy/sympy#9044 pretty printing: Trace could be improved (and LaTeX)
- sympy/sympy#10445 Improper integral does not evaluate
- sympy/sympy#10379 dsolve() converts floats to integers/rationals
- sympy/sympy#10633 Eq(True, False) doesn't evaluate
- sympy/sympy#7163 integrate((sign(x - 1) - sign(x - 2))*cos(x), x) raises TypeError: doit() got an unexpected keyword argument 'manual'
- sympy/sympy#11881 ZeroDivisionError: pole in hypergeometric series random test failure
- sympy/sympy#11801 Exception when printing Symbol('')
- sympy/sympy#11911 typo in docs of printing
- sympy/sympy#10489 Mathematical Symbol does not seem to serialize correctly LaTeX printer
- sympy/sympy#10336 nsimplify problems with oo and inf
- sympy/sympy#12345 nonlinsolve (solve_biquadratic) gives no solution with radical
- sympy/sympy#12375 sympy.series() is broken?

- sympy/sympy#5514 Poly(x, x) * I != I * Poly(x, x)
- sympy/sympy#12398 Limits With abs in certain cases remains unevaluated
- sympy/sympy#12400 polytool.poly() can't raise polynomial to negative power?
- sympy/sympy#12221 Issue with definite piecewise integration
- sympy/sympy#12522 BooleanTrue and Boolean False should have simplify method
- sympy/sympy#12555 limit((3**x + 2 * x**10) / (x**10 + E**x), x, -oo) gives 0 instead of 2
- sympy/sympy#12569 problem with polygamma or im
- sympy/sympy#12578 Taylor expansion wrong (likely because of wrong substitution at point of evaluation?)
- sympy/sympy#12582 Can't solve integrate(abs(x**2-3*x), (x, -15, 15))
- sympy/sympy#12747 Missing constant coefficient in Taylor series of degree 1
- sympy/sympy#12769 Slow limit() calculation?!
- sympy/sympy#12942 Remove x**1.0 == x hack from core
- sympy/sympy#12238 match can take a long time (possibly forever)
- sympy/sympy#4269 ordering of classes
- sympy/sympy#13081 Some comparisons between rational and irrational numbers are incorrect
- sympy/sympy#13078 Return NotImplemented, not False, upon rich comparison with unknown type
- sympy/sympy#13098 sympy.floor() sometimes returns the wrong answer
- sympy/sympy#13312 SymPy does not evaluate integrals of exponentials with symbolic parameter and limit
- sympy/sympy#13111 Don't use "is" to compare classes
- sympy/sympy#10488 integrate(x/(a*x+b), x) gives wrong answer
- sympy/sympy#9706 Interval(-oo, 0).closure hangs
- sympy/sympy#10740 Add a test for Interval(..) in Interval(..) == False
- sympy/sympy#10592 zeta(0, n) where n is negative is wrong
- sympy/sympy#7858 Nth root mod giving wrong solutions
- sympy/sympy#5412 N(oo*I) returns wrong result
- sympy/sympy#10710 Any dict-like object in expr.subs
- sympy/sympy#10810 Implemented function gives ValueError when constructing float expression in sympy 1.0
- sympy/sympy#10867 Getting KeyError while solving ode : dsolve(Eq(g(x).diff(x).diff(x) , (x-2)**2 +(x-3)**3), g(x))
- sympy/sympy#10782 condition_number() for empty matrices giving ValueError
- sympy/sympy#10719 eigenvals of empty matrix raises IndexError
- sympy/sympy#10680 unable to get unevaluated Integral object for integrate (x**log (x**log (x**log(x))) , x) .

- sympy/sympy#10701 Is the empty matrix nilpotent? IndexError: Index out of range: a[0]
- sympy/sympy#10770 Adding a row or a column to an empty matrix
- sympy/sympy#10773 sympify evaluates Div Operation in case of Unary Operator when evaluate = False
- sympy/sympy#13332 limit(): AttributeError: 'NoneType' object has no attribute 'rewrite'
- sympy/sympy#13382 Incorrect Result for limit(n*(((n+1)**2+1)/((n)**2+1)-1), n ,oo)
- sympy/sympy#13403 Incorrect Result for limit(n*(-1 + (n + log(n + 1) + 1)/(n + log(n))), n ,oo)
- sympy/sympy#13416 Incorrect Result for limit((-n**3*log(n)**3 + (n - 1)*(n + 1)**2*log(n + 1)**3)/(n**2*log(n)**3), n ,oo)
- sympy/sympy#13462 Bug in sympy.limit()
- sympy/sympy#13501 Incorrect integral of a rational function with a symbolic coefficient
- sympy/sympy#13536 TypeError for integration from infinity to a positive value
- sympy/sympy#13545 Poly loses modulus after arithemetic
- sympy/sympy#13460 Integration of certain cubic rational functions is incorrect
- sympy/sympy#13071 meijerg.is_number is wrong
- sympy/sympy#13575 limit(acos(erfi(x)), x, 1) causes recursion error
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- sympy/sympy#11378 S.Reals should be accessible as just "Reals"
- sympy/sympy#10999 diop: holzer error
- sympy/sympy#11000 diop: power_representation
- sympy/sympy#11026 diophantine(x**3+y**3-2) -> KeyError instead of {(1, 1)}
- sympy/sympy#8943 diophantine misses trivial solution
- sympy/sympy#11016 diop: sum of squares needs to try more options to satisfy conditions
- sympy/sympy#9538 diophantine() doesn't let you specify the variable order
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- sympy/sympy#13853 Why does the expansion of polylog(1, z) have exp_polar(-I*pi)?
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1. x > y > z
2. (x > y) and (y > z)
3. (GreaterThanObject) and (y > z)
4. (GreaterThanObject.__bool__()) and (y > z)
5. TypeError

Because of the "and" added at step 2, the statement gets turned into a weak ternary statement, and the first object's __bool__ method will raise TypeError. Thus, creating a chained inequality is not possible.

> In Python, there is no way to override the `and` operator, or to control how it short circuits, so it is impossible to make something like `x > y > z` work. There was a PEP to change this, **PEP 335**, but it was officially closed in March, 2012.

[R117] For more information, see these two bug reports:

"Separate boolean and symbolic relationals" Issue sympy/sympy#4986

"It right 0 < x < 1 ?" Issue sympy/sympy#6059

[R118] This implementation detail is that Python provides no reliable method to determine that a chained inequality is being built. Chained comparison operators are evaluated pairwise, using "and" logic (see https://docs.python.org/3/reference/expressions.html#

not-in). This is done in an efficient way, so that each object being compared is only evaluated once and the comparison can short-circuit. For example, `1 > 2 > 3` is evaluated by Python as `(1 > 2) and (2 > 3)`. The `and` operator coerces each side into a bool, returning the object itself when it short-circuits. The bool of the –Than operators will raise TypeError on purpose, because Diofant cannot determine the mathematical ordering of symbolic expressions. Thus, if we were to compute `x > y > z`, with `x`, `y`, and `z` being Symbols, Python converts the statement (roughly) into these steps:

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[R119] For more information, see these two bug reports:

"Separate boolean and symbolic relationals" Issue sympy/sympy#4986

"It right 0 < x < 1 ?" Issue sympy/sympy#6059

[R120] This implementation detail is that Python provides no reliable method to determine that a chained inequality is being built. Chained comparison operators are evaluated pairwise, using "and" logic (see https://docs.python.org/3/reference/expressions.html#not-in). This is done in an efficient way, so that each object being compared is only evaluated once and the comparison can short-circuit. For example, `1 > 2 > 3` is evaluated by Python as `(1 > 2) and (2 > 3)`. The `and` operator coerces each side into a bool, returning the object itself when it short-circuits. The bool of the –Than operators will raise TypeError on purpose, because Diofant cannot determine the mathematical ordering of symbolic expressions. Thus, if we were to compute `x > y > z`, with `x`, `y`, and `z` being Symbols, Python converts the statement (roughly) into these steps:

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[R121] For more information, see these two bug reports:

"Separate boolean and symbolic relationals" Issue sympy/sympy#4986

"It right 0 < x < 1 ?" Issue sympy/sympy#6059

[R122] This implementation detail is that Python provides no reliable method to determine that a chained inequality is being built. Chained comparison operators are evaluated pairwise, using "and" logic (see https://docs.python.org/3/reference/expressions.html#not-in). This is done in an efficient way, so that each object being compared is only evaluated once and the comparison can short-circuit. For example, `1 > 2 > 3` is evaluated by Python as `(1 > 2) and (2 > 3)`. The `and` operator coerces each side into a bool, returning the object itself when it short-circuits. The bool of the –Than operators will raise TypeError on purpose, because Diofant cannot determine the mathematical ordering of symbolic expressions. Thus, if we were to compute `x > y > z`, with `x`, `y`, and `z` being Symbols, Python converts the statement (roughly) into these steps:

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2. (x > y) and (y > z)
3. (GreaterThanObject) and (y > z)
4. (GreaterThanObject.__bool__()) and (y > z)
5. TypeError

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