1. Identification issue

Assume that the distribution of y conditional on a vector of exogenous variable x and an unobserved disturbance term u^0 is a Poisson distribution such that

$$\begin{aligned} \mathcal{D}(y \quad | \quad x, u^0) &= \text{Poisson } (\lambda^0), \\ \text{where } \lambda^0 \quad &= \quad \exp(x'\beta^0 + u^0). \end{aligned} \tag{1.1}$$

To simplify notation, we write

$$\mu^0 = \exp(x'\beta^0) \tag{1.2}$$

and thus

$$\lambda^0 = \mu^0 \exp u^0$$

It is important to note that the regression part should not be "with a constant term", otherwise we run into an identification problem.

To see this, we first assume that

$$E(u^0 \mid x) = 0$$

and

$$V((u^0 \mid x) = \sigma^{02}$$

It follows that

$$\begin{aligned} E(y \quad | \quad x) = E(\lambda^0 \mid x) \\ &= \quad E(\mu^0 \exp u^0 \mid x) \\ &= \quad E(\mu^0 \mid x) E\left[\exp(u^0 \mid x)\right] \\ &= \quad \mu^0 E\left[\exp(u^0)\right] \end{aligned} \tag{1.3}$$

and thus,

$$E(y \mid x) = \exp(x'\beta^0) E\left[\exp(u^0)\right] \tag{1.4}$$

Assume now, that the regression $x'\beta^0$ has a constant term and may be written as

$$x'\beta^0 = \alpha^0 + x^{*\prime}\gamma^0. \tag{1.5}$$

In that case, we have

$$E(y \mid x) = \exp(x^{*\prime}\gamma^0) \exp(\alpha^0) E\left[\exp(u^0)\right] \tag{1.6}$$

It follows from (1.6) that we cannot separately identify the constant term α^0 and $E\left[\exp(u^0)\right]$ as only the product $\exp(\alpha^0)E\left[\exp(u^0)\right]$ is identified. Therefore, in order to identify the remaining regression vector of parameters γ^0 and the parameter $E\left[\exp(u^0)\right]$, we need to assume that $\alpha^0 = 0$, that is to say that the regression $x'\beta^0$ does not contain a constant term.

Specialising this result in the Poisson log-normal distribution, we have

$$\mathcal{D}(u^0) = \mathcal{N}(0, \sigma^{02}),$$

so that

$$\mathcal{D}(\exp(u^0)) = \mathcal{LN}(\mu^0, \sigma^{02}),$$

and thus

$$E\left[\exp(u^0)\right] = \exp(\sigma^{02}/2).$$

Therefore, the assumption $\alpha^0 = 0$, that is to say that the regression $x'\beta^0$ does not contain a constant term, is necessary to identify the regression vector of parameters γ^0 and the variance σ^{02}.