NOTE ON THE FIML ESTIMATION OF THE THREE-EQUATIONS MODEL WITH MIXED POISSON AND QUALITATIVE VARIABLES

ALBERTO HOLLY

1. Notations and preliminary results

Consider a two-dimensional random vector $X = (X_1, X_2)'$ such that

$$\mathcal{D}(X) = \mathcal{N}(0, R) \tag{1.1}$$

where

$$R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \qquad (\rho^2 < 1). \tag{1.2}$$

Let $\Psi(\rho, a, b)$ be the function defined by

$$\Psi(\rho, a, b) = \frac{1}{2\pi} \int_0^{\rho} \exp\left(-\frac{1}{2}\frac{a^2 + b^2 - 2tab}{1 - t^2}\right) \frac{dt}{\sqrt{1 - t^2}} \tag{1.3}$$

Hereafter we use the standard notation $\Phi(c) = \int_{-\infty}^{c} \varphi(x)dx$ and $\varphi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

It has been shown in Lazard Holly and Holly (2002) (see also Huguenin (2004) and Huguenin, Pelgrin and Holly (2013)) that

$$P\{X_1 \geqq c_1; X_2 \geqq c_2\} = [1 - \Phi(c_1)]\,[1 - \Phi(c_2)] + \Psi(\rho, c_1, c_2) \tag{1.4}$$

and

$$\begin{aligned} P\{X_1 &\geqq c_1; X_2 < c_2\} = [1 - \Phi(c_1)]\,\Phi(c_2) - \Psi(\rho, c_1, c_2) &(1.5)\\ P\{X_1 &< c_1; X_2 \geqq c_2\} = \Phi(c_1)\,[1 - \Phi(c_2)] - \Psi(\rho, c_1, c_2) &(1.6)\\ P\{X_1 &< c_1; X_2 < c_2\} = \Phi(c_1)\Phi(c_2) + \Psi(\rho, c_1, c_2) &(1.7) \end{aligned}$$

Assume now that instead of (1.1) we have

$$\mathcal{D}(X) = \mathcal{N}(\mu, \Sigma) \tag{1.8}$$

where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

Using the fact that

$$\mathcal{D}\left[\begin{pmatrix} (X_1 - \mu_1)/\sigma_1 \\ (X_2 - \mu_2)/\sigma_2 \end{pmatrix}\right] = \mathcal{N}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right] \tag{1.9}$$

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where $\rho = \sigma_{12}/\sigma_1\sigma_2$, we obtain

$$\begin{aligned}
P\{X_1 &\geqq c_1; X_2 \geqq c_2\} = \left[1 - \Phi\left(\frac{c_1-\mu_1}{\sigma_1}\right)\right]\left[1 - \Phi\left(\frac{c_2-\mu_2}{\sigma_2}\right)\right] \\
&+\Psi\left(\rho, \frac{c_1-\mu_1}{\sigma_1}, \frac{c_2-\mu_2}{\sigma_2}\right) &(1.10a)\\
P\{X_1 &\geqq c_1; X_2 < c_2\} = \left[1 - \Phi\left(\frac{c_1-\mu_1}{\sigma_1}\right)\right]\Phi\left(\frac{c_2-\mu_2}{\sigma_2}\right) - \Psi\left(\rho, \frac{c_1-\mu_1}{\sigma_1}, \frac{c_2-\mu_2}{\sigma_2}\right) &(1.10b)\\
P\{X_1 &< c_1; X_2 \geqq c_2\} = \Phi\left(\frac{c_1-\mu_1}{\sigma_1}\right)\left[1 - \Phi\left(\frac{c_2-\mu_2}{\sigma_2}\right)\right] - \Psi\left(\rho, \frac{c_1-\mu_1}{\sigma_1}, \frac{c_2-\mu_2}{\sigma_2}\right) &(1.10c)\\
P\{X_1 &< c_1; X_2 < c_2\} = \Phi\left(\frac{c_1-\mu_1}{\sigma_1}\right)\Phi\left(\frac{c_2-\mu_2}{\sigma_2}\right) + \Psi\left(\rho, \frac{c_1-\mu_1}{\sigma_1}, \frac{c_2-\mu_2}{\sigma_2}\right) &(1.10d)
\end{aligned}$$

Consider a three-dimensional random vector $X = (X_1, X_2, X_3)'$ such that

$$\mathcal{D}(X) = \mathcal{N}(0, \Sigma) \tag{1.11}$$

where now

$$\Sigma = \begin{pmatrix} 1 & \rho_{12} & \sigma_{13} \\ \rho_{12} & 1 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}$$

which may also be written as

$$\Sigma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13}\sigma_3 \\ \rho_{12} & 1 & \rho_{23}\sigma_3 \\ \rho_{13}\sigma_3 & \rho_{23}\sigma_3 & \sigma_3^2 \end{pmatrix}.$$

Our purpose is to extend formulae (1.10a) through (1.10d) to conditional probabilities with respect to X_3.

To this end, we observe that

$$\mathcal{D}\left[\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mid X_3\right] = \mathcal{N}\left[\begin{pmatrix} \frac{\rho_{13}}{\sigma_3}X_3 \\ \frac{\rho_{23}}{\sigma_3}X_3 \end{pmatrix}, \begin{pmatrix} 1-\rho_{13}^2 & \rho_{12}-\rho_{13}\rho_{23} \\ \rho_{12}-\rho_{13}\rho_{23} & 1-\rho_{23}^2 \end{pmatrix}\right]. \tag{1.12}$$

This implies that the conditional probabilities with respect to X_3 that we are seeking to compute are obtained by substituting in (1.10a) through (1.10d) $\mu_1, \mu_2, \sigma_1, \sigma_2$ and ρ for $\frac{\rho_{13}}{\sigma_3}X_3$, $\frac{\rho_{23}}{\sigma_3}X_3$, $\sqrt{1-\rho_{13}^2}$, $\sqrt{1-\rho_{23}^2}$ and $\rho_{12} - \rho_{13}\rho_{23}$ respectively. We thus obtain,

$$\begin{aligned}
P\{X_1 &\geqq c_1; X_2 \geqq c_2 \mid X_3\} = \left[1-\Phi\left(\frac{c_1-\frac{\rho_{13}}{\sigma_3}X_3}{\sqrt{1-\rho_{13}^2}}\right)\right]\left[1-\Phi\left(\frac{c_2-\frac{\rho_{23}}{\sigma_3}X_3}{\sqrt{1-\rho_{23}^2}}\right)\right] \\
&+\Psi\left(\rho_{12}-\rho_{13}\rho_{23}, \frac{c_1-\frac{\rho_{13}}{\sigma_3}X_3}{\sqrt{1-\rho_{13}^2}}, \frac{c_2-\frac{\rho_{23}}{\sigma_3}X_3}{\sqrt{1-\rho_{23}^2}}\right) \qquad (1.13a) \\
P\{X_1 &\geqq c_1; X_2 < c_2 \mid X_3\} = \left[1-\Phi\left(\frac{c_1-\frac{\rho_{13}}{\sigma_3}X_3}{\sqrt{1-\rho_{13}^2}}\right)\right]\Phi\left(\frac{c_2-\frac{\rho_{23}}{\sigma_3}X_3}{\sqrt{1-\rho_{23}^2}}\right) \\
&-\Psi\left(\rho_{12}-\rho_{13}\rho_{23}, \frac{c_1-\frac{\rho_{13}}{\sigma_3}X_3}{\sqrt{1-\rho_{13}^2}}, \frac{c_2-\frac{\rho_{23}}{\sigma_3}X_3}{\sqrt{1-\rho_{23}^2}}\right) \qquad (1.13b) \\
P\{X_1 &< c_1; X_2 \geqq c_2 \mid X_3\} = \Phi\left(\frac{c_1-\frac{\rho_{13}}{\sigma_3}X_3}{\sqrt{1-\rho_{13}^2}}\right)\left[1-\Phi\left(\frac{c_2-\frac{\rho_{23}}{\sigma_3}X_3}{\sqrt{1-\rho_{23}^2}}\right)\right] \\
&-\Psi\left(\rho_{12}-\rho_{13}\rho_{23}, \frac{c_1-\frac{\rho_{13}}{\sigma_3}X_3}{\sqrt{1-\rho_{13}^2}}, \frac{c_2-\frac{\rho_{23}}{\sigma_3}X_3}{\sqrt{1-\rho_{23}^2}}\right) \qquad (1.13c) \\
P\{X_1 &< c_1; X_2 < c_2 \mid X_3\} = \Phi\left(\frac{c_1-\frac{\rho_{13}}{\sigma_3}X_3}{\sqrt{1-\rho_{13}^2}}\right)\Phi\left(\frac{c_2-\frac{\rho_{23}}{\sigma_3}X_3}{\sqrt{1-\rho_{23}^2}}\right) \\
&+\Psi\left(\rho_{12}-\rho_{13}\rho_{23}, \frac{c_1-\frac{\rho_{13}}{\sigma_3}X_3}{\sqrt{1-\rho_{13}^2}}, \frac{c_2-\frac{\rho_{23}}{\sigma_3}X_3}{\sqrt{1-\rho_{23}^2}}\right) \qquad (1.13d)
\end{aligned}$$

Of course, the unconditional probabilities are obtained by taking the expectation with respect to X_3 of the corresponding conditional probabilities.

2. Estimation of the model

Our aim is to estimate the simultaneous three-equation model with mixed latent and observed variables.

Define, omitting the index n, $n = 1, \ldots, N$, for the individual observations.

$$\begin{cases} y_1^* : \text{ propensity to consult} \\ y_2^* : \text{ propensity to be prescribed a drug} \\ y_3^* : \text{ health status (latent, unobserved)} \end{cases} \qquad (2.14)$$

The observed variable $y_1$is a count variable for the number of visits:

The distribution of y_1 conditional on a vector of exogenous variable x_1 and an unobserved disturbance term u_1^0 is a Poisson distribution such that

$$\begin{aligned} \mathcal{D}(y_1 &\mid x_1, u_1^0) = \text{Poisson } (\lambda^0), \qquad (2.15) \\ \text{where } \lambda^0 &= \exp(x_1'\beta_1^0 + u_1^0). \end{aligned}$$

Consider the simultaneous-equations model with mixed latent, dichotomous and count variables

$$\begin{cases} y_1 \text{ such that } \mathcal{D}(y_1 \mid x_1, u_1^0) \text{ is the above mentioned Poisson distribution} \\ y_2^* = x_2'\beta_2^0 + \alpha_{21}^0 y_1 + u_2^0 \\ y_3^* = x_3'\beta_3^0 + \alpha_{31}^0 y_1 + \alpha_{32}^0 y_2 + u_3^0 \end{cases} \qquad (2.16)$$

We assume that

$$\mathcal{D}\left(u^0\right) = \mathcal{N}(0, \Sigma^0), \qquad (2.17)$$

where

$$u^0 = \begin{pmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{pmatrix} \tag{2.18}$$

and

$$\Sigma^0 = \begin{pmatrix} \sigma_1^{02} & \rho_{12}^0\sigma_1^0 & \rho_{13}^0\sigma_1^0 \\ \rho_{12}^0\sigma_1^0 & 1 & \rho_{23}^0 \\ \rho_{13}^0\sigma_1^0 & \rho_{23}^0 & 1 \end{pmatrix} \tag{2.19}$$

Under this assumption, the unconditional distribution of $y_1$is the so-called *Poisson log-normal distribution* (hereafter PLN) and not the more familiar negative-binomial distribution. The PLN distribution is obtained as a mixed Poisson distribution if a log-normal distribution, $\mathcal{LN}(\mu^0, \sigma^{02})$ is used as a mixing density.

We observe

$$y_2 = \begin{cases} 1 \text{ if } y_2^* \geq 0 \\ 0 \text{ otherwise} \end{cases}, \tag{2.20}$$

and the polychotomous ordered-response variable y_3 such that

$$y_3 = \begin{cases} 1 & \text{if } \alpha_0 \leq y_3^* < \alpha_1 \\ 2 & \text{if } \alpha_1 \leq y_3^* < \alpha_2 \\ 3 & \text{if } \alpha_2 \leq y_3^* < \alpha_3 \\ 4 & \text{if } \alpha_3 \leq y_3^* < \alpha_4 \\ 5 & \text{if } \alpha_4 \leq y_3^* < \alpha_5 \end{cases} \tag{2.21}$$

where the set of constants α's are such that $\alpha_0 = -\infty$, $\alpha_5 = +\infty$ and $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5$. We say that y_3^* belongs to the jth category if

$$\alpha_{j-1} \leq y_3^* < \alpha_j \quad (j = 1, 2, \ldots, J) \text{ with } J = 5.$$

2.1. **The complete three-equations model.** Consider now the complete three-equations model described above. We seek to find the expression of $P(y_1 = k, y_2 = i, y_3 = j)$ for $k = 0, 1, \ldots$, $i = 0, 1$ and $j = 1, \ldots, 5$.

We have

$$P(y_1 = k, y_2 = i, y_3 = j) = \int_{-\infty}^{+\infty} P(y_2 = i, y_3 = j \mid y_1 = k, u_1^0) P(y_1 = k \mid u_1^0) d\varphi_{\sigma_1^0}(u_1^0) \tag{2.22}$$

where $d\varphi_{\sigma_1^0}(u_1^0)$ is defined as

$$d\varphi_{\sigma_1^0}(u_1^0) = \frac{1}{\sigma_1^0}\varphi\left(\frac{u_1^0}{\sigma_1^0}\right) du_1^0.$$

We need to derive the expression for the conditional probabilities $P(y_2 = i, y_3 = j \mid y_1 = k, u_1^0)$.

We use the fact that

$$\mathcal{D}\left[\begin{pmatrix} u_2^0 \\ u_3^0 \end{pmatrix} \mid u_1^0\right] = \mathcal{N}\left[\begin{pmatrix} \frac{\rho_{12}^0}{\sigma_1^0} u_1^0 \\ \frac{\rho_{13}^0}{\sigma_1^0} u_1^0 \end{pmatrix}, \begin{pmatrix} 1 - \rho_{12}^{02} & \rho_{23}^0 - \rho_{12}^0\rho_{13}^0 \\ \rho_{23}^0 - \rho_{12}^0\rho_{13}^0 & 1 - \rho_{13}^{02} \end{pmatrix}\right]. \tag{2.23}$$

For $i = 0$ we have,

$$
\begin{aligned}
P\{y_2 &= 0, y_3 = j \mid y_1 = k, u_1^0\} = P\{y_2^* < 0, \alpha_{j-1} \le y_3^* < \alpha_j \mid y_1 = k, u_1^0\} \\
&= P\{y_2^* < 0, y_3^* < \alpha_j \mid y_1 = k, u_1^0\} - P\{y_2^* < 0, y_3^* < \alpha_{j-1} \mid y_1 = k, u_1^0\} \\
&= P\{x_2'\beta_2^0 + \alpha_{21}^0 k + u_2^0 < 0, x_3'\beta_3^0 + \alpha_{31}^0 k + u_3^0 < \alpha_j \mid u_1^0\} \\
-P\{x_2'\beta_2^0 + \alpha_{21}^0 k + u_2^0 &< 0, x_3'\beta_3^0 + \alpha_{31}^0 k + u_3^0 < \alpha_{j-1} \mid u_1^0\} \\
&= P\{u_2^0 < -x_2'\beta_2^0 - \alpha_{21}^0 k, u_3^0 < \alpha_j - x_3'\beta_3^0 - \alpha_{31}^0 k \mid u_1^0\} \\
-P\{u_2^0 &< -x_2'\beta_2^0 - \alpha_{21}^0 k, u_3^0 < \alpha_{j-1} - x_3'\beta_3^0 - \alpha_{31}^0 k \mid u_1^0\},
\end{aligned}
$$

which, using (1.13d), may be written as

$$
\begin{aligned}
P\{y_2 = 0, y_3 = j \mid y_1 = k, u_1^0\} = \Phi\left(\frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{12}^{02}}}\right) &\left[\Phi\left(\frac{\alpha_j - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right)\right. \\
&\left.-\Phi\left(\frac{\alpha_{j-1} - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right)\right] \\
+\Psi\left(\rho_{23}^0 - \rho_{12}^0\rho_{13}^0, \frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{12}^{02}}}, \frac{\alpha_j - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right)& \\
-\Psi\left(\rho_{23}^0 - \rho_{12}^0\rho_{13}^0, \frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{12}^{02}}}, \frac{\alpha_{j-1} - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right)&
\end{aligned}
\tag{2.24}
$$

Similarly, for $i = 1$ we have,

$$
\begin{aligned}
P\{y_2 &= 1, y_3 = j \mid y_1 = k, u_1^0\} = P\{y_2^* \ge 0, \alpha_{j-1} \le y_3^* < \alpha_j \mid y_1 = k, u_1^0\} \\
&= P\{y_2^* \ge 0, y_3^* < \alpha_j \mid y_1 = k, u_1^0\} - P\{y_2^* \ge 0, y_3^* < \alpha_{j-1} \mid y_1 = k, u_1^0\} \\
&= P\{x_2'\beta_2^0 + \alpha_{21}^0 k + u_2^0 \ge 0, x_3'\beta_3^0 + \alpha_{31}^0 y_1 + \alpha_{32}^0 + u_3^0 < \alpha_j \mid u_1^0\} \\
-P\{x_2'\beta_2^0 + \alpha_{21}^0 k + u_2^0 &\ge 0, x_3'\beta_3^0 + \alpha_{31}^0 y_1 + \alpha_{32}^0 + u_3^0 < \alpha_{j-1} \mid u_1^0\} \\
&= P\{u_2^0 \ge -x_2'\beta_2^0 - \alpha_{21}^0 k, u_3^0 < \alpha_j - x_3'\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 \mid u_1^0\} \\
-P\{u_2^0 &\ge -x_2'\beta_2^0 - \alpha_{21}^0 k, u_2^0 < \alpha_{j-1} - x_3'\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 \mid u_1^0\},
\end{aligned}
$$

which, using (1.13b), may be written as

$$
\begin{aligned}
P\{y_2 = 1, y_3 = j \mid y_1 = k, u_1^0\} = &\left[1 - \Phi\left(\frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{12}^{02}}}\right)\right] \times \\
\times \left[\Phi\left(\frac{\alpha_j - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right) - \Phi\left(\frac{\alpha_{j-1} - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right)\right]& \\
-\Psi\left(\rho_{23}^0 - \rho_{12}^0\rho_{13}^0, \frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{12}^{02}}}, \frac{\alpha_j - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right)& \\
+\Psi\left(\rho_{23}^0 - \rho_{12}^0\rho_{13}^0, \frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{12}^{02}}}, \frac{\alpha_{j-1} - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right)&
\end{aligned}
\tag{2.25}
$$

Note that

$$P\{y_2=1,y_3=j\mid y_1=k,u_1^0\}=\left[\Phi\left(\frac{\alpha_j-x_3'\beta_3^0-\alpha_{31}^0k-\frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right)\right.$$
$$\left.-\Phi\left(\frac{\alpha_{j-1}-x_3'\beta_3^0-\alpha_{31}^0k-\frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right)\right]-P\{y_2=0,y_3=j\mid y_1=k,u_1^0\},$$

which allows to verify that, as one should expect,

$$\begin{aligned}P\{y_3 &= j\mid y_1=k,u_1^0\}=P\{y_2=0,y_3=j\mid y_1=k,u_1^0\}+P\{y_2=1,y_3=j\mid y_1=k,u_1^0\}\\ &= \left[\Phi\left(\frac{\alpha_j-x_3'\beta_3^0-\alpha_{31}^0k-\frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right)-\Phi\left(\frac{\alpha_{j-1}-x_3'\beta_3^0-\alpha_{31}^0k-\frac{\rho_{13}^0}{\sigma_1^0}u_1^0}{\sqrt{1-\rho_{13}^{02}}}\right)\right]\end{aligned}$$

Equations (2.24) and (2.25) allow us to compute all the probabilities $P(y_1=k,y_2=i,y_3=j)$ for $k=0,1,\ldots$, $i=0,1$ and $j=1,\ldots,5$ using (2.22).

The log-likelihood function for a sample with N observation may be written as

$$\mathcal{L}_N=\sum_{n=1}^{N}\sum_{k=0}^{\infty}\sum_{i=0}^{1}\sum_{j=1}^{5}z_{nkij}\log P_{nkij} \tag{2.26}$$

where

$$z_{nkij}=\begin{cases}1 \quad \text{if } y_{n1}=k,\ y_{n2}=i \text{ and } y_{n3}=j \quad (k=0,1,\ldots,i=0,1 \text{ and } j=1,\ldots,5)\\ 0 \text{ otherwise}\end{cases} \tag{2.27}$$

and P_{nij} is defined as

$$P_{nkij}=P\{z_{nkij}=1\} \tag{2.28}$$

References

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