

NOTE ON THE FIML ESTIMATION OF THE THREE-EQUATION MODEL WITH MIXED POISSON AND QUALITATIVE VARIABLES

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1. NOTATIONS AND PRELIMINARY RESULTS

Consider a two-dimensional random vector $X = (X_1, X_2)'$ such that

$$\mathcal{D}(X) = \mathcal{N}(0, R) \quad (1.1)$$

where

$$R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \quad (\rho^2 < 1). \quad (1.2)$$

Let $\Psi(\rho, a, b)$ be the function defined by

$$\Psi(\rho, a, b) = \frac{1}{2\pi} \int_0^\rho \exp\left(-\frac{1}{2} \frac{a^2 + b^2 - 2tab}{1 - t^2}\right) \frac{dt}{\sqrt{1 - t^2}} \quad (1.3)$$

Hereafter we use the standard notation $\Phi(c) = \int_{-\infty}^c \varphi(x)dx$ and $\varphi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

It has been shown in Lazard Holly and Holly (2002) (see also Huguenin (2004) and Huguenin, Pelgrin and Holly (2013)) that

$$P\{X_1 \geq c_1; X_2 \geq c_2\} = [1 - \Phi(c_1)][1 - \Phi(c_2)] + \Psi(\rho, c_1, c_2) \quad (1.4)$$

and

$$P\{X_1 \geq c_1; X_2 < c_2\} = [1 - \Phi(c_1)]\Phi(c_2) - \Psi(\rho, c_1, c_2) \quad (1.5)$$

$$P\{X_1 < c_1; X_2 \geq c_2\} = \Phi(c_1)[1 - \Phi(c_2)] - \Psi(\rho, c_1, c_2) \quad (1.6)$$

$$P\{X_1 < c_1; X_2 < c_2\} = \Phi(c_1)\Phi(c_2) + \Psi(\rho, c_1, c_2) \quad (1.7)$$

Assume now that instead of (1.1) we have

$$\mathcal{D}(X) = \mathcal{N}(\mu, \Sigma) \quad (1.8)$$

where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

Using the fact that

$$\mathcal{D}\left[\begin{pmatrix} (X_1 - \mu_1)/\sigma_1 \\ (X_2 - \mu_2)/\sigma_2 \end{pmatrix}\right] = \mathcal{N}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right] \quad (1.9)$$

where $\rho = \sigma_{12}/\sigma_1\sigma_2$, we obtain

$$P\{X_1 \geq c_1; X_2 \geq c_2\} = \left[1 - \Phi\left(\frac{c_1 - \mu_1}{\sigma_1}\right)\right] \left[1 - \Phi\left(\frac{c_2 - \mu_2}{\sigma_2}\right)\right] + \Psi\left(\rho, \frac{c_1 - \mu_1}{\sigma_1}, \frac{c_2 - \mu_2}{\sigma_2}\right) \quad (1.10a)$$

$$P\{X_1 \geq c_1; X_2 < c_2\} = \left[1 - \Phi\left(\frac{c_1 - \mu_1}{\sigma_1}\right)\right] \Phi\left(\frac{c_2 - \mu_2}{\sigma_2}\right) - \Psi\left(\rho, \frac{c_1 - \mu_1}{\sigma_1}, \frac{c_2 - \mu_2}{\sigma_2}\right) \quad (1.10b)$$

$$P\{X_1 < c_1; X_2 \geq c_2\} = \Phi\left(\frac{c_1 - \mu_1}{\sigma_1}\right) \left[1 - \Phi\left(\frac{c_2 - \mu_2}{\sigma_2}\right)\right] - \Psi\left(\rho, \frac{c_1 - \mu_1}{\sigma_1}, \frac{c_2 - \mu_2}{\sigma_2}\right) \quad (1.10c)$$

$$P\{X_1 < c_1; X_2 < c_2\} = \Phi\left(\frac{c_1 - \mu_1}{\sigma_1}\right) \Phi\left(\frac{c_2 - \mu_2}{\sigma_2}\right) + \Psi\left(\rho, \frac{c_1 - \mu_1}{\sigma_1}, \frac{c_2 - \mu_2}{\sigma_2}\right) \quad (1.10d)$$

Consider a three-dimensional random vector $X = (X_1, X_2, X_3)'$ such that

$$\mathcal{D}(X) = \mathcal{N}(0, \Sigma) \quad (1.11)$$

where now

$$\Sigma = \begin{pmatrix} 1 & \rho_{12} & \sigma_{13} \\ \rho_{12} & 1 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}$$

which may also be written as

$$\Sigma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13}\sigma_3 \\ \rho_{12} & 1 & \rho_{23}\sigma_3 \\ \rho_{13}\sigma_3 & \rho_{23}\sigma_3 & \sigma_3^2 \end{pmatrix}.$$

Our purpose is to extend formulae (1.10a) through (1.10d) to conditional probabilities with respect to X_3 .

To this end, we observe that

$$\mathcal{D}\left[\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mid X_3\right] = \mathcal{N}\left[\begin{pmatrix} \frac{\rho_{13}}{\sigma_3} X_3 \\ \frac{\rho_{23}}{\sigma_3} X_3 \end{pmatrix}, \begin{pmatrix} 1 - \rho_{13}^2 & \rho_{12} - \rho_{13}\rho_{23} \\ \rho_{12} - \rho_{13}\rho_{23} & 1 - \rho_{23}^2 \end{pmatrix}\right]. \quad (1.12)$$

This implies that the conditional probabilities with respect to X_3 that we are seeking to compute are obtained by substituting in (1.10a) through (1.10d) $\mu_1, \mu_2, \sigma_1, \sigma_2$ and ρ for $\frac{\rho_{13}}{\sigma_3} X_3, \frac{\rho_{23}}{\sigma_3} X_3, \sqrt{1 - \rho_{13}^2}, \sqrt{1 - \rho_{23}^2}$ and $\frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{(1 - \rho_{13}^2)(1 - \rho_{23}^2)}}$ respectively. We thus obtain,

$$P\{X_1 \geq c_1; X_2 \geq c_2 \mid X_3\} = \left[1 - \Phi\left(\frac{c_1 - \frac{\rho_{13}}{\sigma_3} X_3}{\sqrt{1 - \rho_{13}^2}}\right)\right] \left[1 - \Phi\left(\frac{c_2 - \frac{\rho_{23}}{\sigma_3} X_3}{\sqrt{1 - \rho_{23}^2}}\right)\right] \\ + \Psi\left(\frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{(1 - \rho_{13}^2)(1 - \rho_{23}^2)}}, \frac{c_1 - \frac{\rho_{13}}{\sigma_3} X_3}{\sqrt{1 - \rho_{13}^2}}, \frac{c_2 - \frac{\rho_{23}}{\sigma_3} X_3}{\sqrt{1 - \rho_{23}^2}}\right) \quad (1.13a)$$

$$P\{X_1 \geq c_1; X_2 < c_2 \mid X_3\} = \left[1 - \Phi\left(\frac{c_1 - \frac{\rho_{13}}{\sigma_3} X_3}{\sqrt{1 - \rho_{13}^2}}\right)\right] \Phi\left(\frac{c_2 - \frac{\rho_{23}}{\sigma_3} X_3}{\sqrt{1 - \rho_{23}^2}}\right) \\ - \Psi\left(\frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{(1 - \rho_{13}^2)(1 - \rho_{23}^2)}}, \frac{c_1 - \frac{\rho_{13}}{\sigma_3} X_3}{\sqrt{1 - \rho_{13}^2}}, \frac{c_2 - \frac{\rho_{23}}{\sigma_3} X_3}{\sqrt{1 - \rho_{23}^2}}\right) \quad (1.13b)$$

$$P\{X_1 < c_1; X_2 \geq c_2 \mid X_3\} = \Phi\left(\frac{c_1 - \frac{\rho_{13}}{\sigma_3} X_3}{\sqrt{1 - \rho_{13}^2}}\right) \left[1 - \Phi\left(\frac{c_2 - \frac{\rho_{23}}{\sigma_3} X_3}{\sqrt{1 - \rho_{23}^2}}\right)\right] \\ - \Psi\left(\frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{(1 - \rho_{13}^2)(1 - \rho_{23}^2)}}, \frac{c_1 - \frac{\rho_{13}}{\sigma_3} X_3}{\sqrt{1 - \rho_{13}^2}}, \frac{c_2 - \frac{\rho_{23}}{\sigma_3} X_3}{\sqrt{1 - \rho_{23}^2}}\right) \quad (1.13c)$$

$$P\{X_1 < c_1; X_2 < c_2 \mid X_3\} = \Phi\left(\frac{c_1 - \frac{\rho_{13}}{\sigma_3} X_3}{\sqrt{1 - \rho_{13}^2}}\right) \Phi\left(\frac{c_2 - \frac{\rho_{23}}{\sigma_3} X_3}{\sqrt{1 - \rho_{23}^2}}\right) \\ + \Psi\left(\frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{(1 - \rho_{13}^2)(1 - \rho_{23}^2)}}, \frac{c_1 - \frac{\rho_{13}}{\sigma_3} X_3}{\sqrt{1 - \rho_{13}^2}}, \frac{c_2 - \frac{\rho_{23}}{\sigma_3} X_3}{\sqrt{1 - \rho_{23}^2}}\right) \quad (1.13d)$$

Of course, the unconditional probabilities are obtained by taking the expectation with respect to X_3 of the corresponding conditional probabilities.

2. ESTIMATION OF THE MODEL

Our aim is to estimate the simultaneous three-equation model with mixed latent and observed variables.

Define, omitting the index n , $n = 1, \dots, N$, for the individual observations.

$$\begin{cases} y_1^* : \text{propensity to consult} \\ y_2^* : \text{propensity to be prescribed a drug} \\ y_3^* : \text{health status (latent, unobserved)} \end{cases} \quad (2.14)$$

The observed variable y_1 is a count variable for the number of visits:

The distribution of y_1 conditional on a vector of exogenous variable x_1 and an unobserved disturbance term u_1^0 is a Poisson distribution such that

$$\mathcal{D}(y_1 \mid x_1, u_1^0) = \text{Poisson}(\lambda^0), \quad (2.15) \\ \text{where } \lambda^0 = \exp(x_1' \beta_1^0 + u_1^0).$$

Consider the simultaneous-equations model with mixed latent, dichotomous and count variables

$$\begin{cases} y_1 \text{ such that } \mathcal{D}(y_1 \mid x_1, u_1^0) \text{ is the above mentioned Poisson distribution} \\ y_2^* = x_2' \beta_2^0 + \alpha_{21}^0 y_1 + u_2^0 \\ y_3^* = x_3' \beta_3^0 + \alpha_{31}^0 y_1 + \alpha_{32}^0 y_2 + u_3^0 \end{cases} \quad (2.16)$$

We assume that

$$\mathcal{D}(u^0) = \mathcal{N}(0, \Sigma^0), \quad (2.17)$$

where

$$u^0 = \begin{pmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{pmatrix} \quad (2.18)$$

and

$$\Sigma^0 = \begin{pmatrix} \sigma_1^{02} & \rho_{12}^0 \sigma_1^0 & \rho_{13}^0 \sigma_1^0 \\ \rho_{12}^0 \sigma_1^0 & 1 & \rho_{23}^0 \\ \rho_{13}^0 \sigma_1^0 & \rho_{23}^0 & 1 \end{pmatrix} \quad (2.19)$$

Under this assumption, the unconditional distribution of y_1 is the so-called *Poisson log-normal distribution* (hereafter PLN) and not the more familiar negative-binomial distribution. The PLN distribution is obtained as a mixed Poisson distribution if a log-normal distribution, $\mathcal{LN}(\mu^0, \sigma^{02})$ is used as a mixing density.

We observe

$$y_2 = \begin{cases} 1 & \text{if } y_2^* \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad (2.20)$$

and the polychotomous ordered-response variable y_3 such that

$$y_3 = \begin{cases} 1 & \text{if } c_0 \leq y_3^* < c_1 \\ 2 & \text{if } c_1 \leq y_3^* < c_2 \\ 3 & \text{if } c_2 \leq y_3^* < c_3 \\ 4 & \text{if } c_3 \leq y_3^* < c_4 \\ 5 & \text{if } c_4 \leq y_3^* < c_5 \end{cases} \quad (2.21)$$

where the set of constants c 's are such that $c_0 = -\infty$, $c_5 = +\infty$ and $c_0 < c_1 < c_2 < c_3 < c_4 < c_5$. We say that y_3^* belongs to the j -th category if

$$c_{j-1} \leq y_3^* < c_j \quad (j = 1, 2, \dots, J) \text{ with } J = 5.$$

2.1. Intermediary step 1: Estimation of the Poisson log-normal distribution. Define

$d\varphi_{\sigma_1^0}(u_1^0)$ as

$$d\varphi_{\sigma_1^0}(u_1^0) = \frac{1}{\sigma_1^0} \varphi\left(\frac{u_1^0}{\sigma_1^0}\right) du_1^0.$$

In the case where we only seek the marginal distribution of the Poisson log-normal, we have to compute the probabilities:

$$\begin{aligned} P(y_1 = k) &= \int_{-\infty}^{+\infty} P(y_1 = k \mid u_1^0) d\varphi_{\sigma_1^0}(u_1^0) \\ &= \frac{1}{\sqrt{2\pi\sigma_1^0}} \frac{1}{k!} \int_{-\infty}^{+\infty} \exp[-(\mu^0 \exp u_1^0)] (\mu^0 \exp u_1^0)^k \exp\left(-\frac{u_1^{02}}{2\sigma_1^{02}}\right) du_1^0, \end{aligned}$$

where

$$\mu^0 = \exp(x_1' \beta_1^0)$$

We may thus write

$$\begin{aligned} P(y_1 = k) &= \frac{1}{\sqrt{2\pi\sigma_1^0}} \frac{1}{k!} \mu^{0k} \int_{-\infty}^{+\infty} \exp[-(\mu^0 \exp u_1^0)] \exp k u_1^0 \exp\left(-\frac{u_1^{02}}{2\sigma_1^{02}}\right) du_1^0 \\ &= \frac{1}{\sqrt{2\pi\sigma_1^0}} \frac{1}{k!} \mu^{0k} \int_{-\infty}^{+\infty} \exp[-(\mu^0 \exp u_1^0) + k u_1^0] \exp\left(-\frac{u_1^{02}}{2\sigma_1^{02}}\right) du_1^0 \end{aligned} \quad (2.22)$$

The estimation of this model has been considered among others by Greene (1977). He suggested to carry-out the computation of these probabilities using the Gauss-Hermite quadrature. Miranda

(2004) suggested a STATA code for this Gauss-Hermite numerical computation. Specifically, let in (2.22) $v^0 = u_1^0 / \sqrt{2}\sigma_1^0$. With this change of variables (2.22) may be re-expressed as

$$P(y_1 = k) = \frac{1}{\sqrt{\pi}} \frac{1}{k!} \mu^{0k} \int_{-\infty}^{+\infty} \exp \left[- \left(\mu^0 \exp \sqrt{2}\sigma_1^0 v^0 \right) + k\sqrt{2}\sigma_1^0 v^0 \right] \exp(-v^{02}) dv^0 \quad (2.23)$$

As suggested by Greene (1997), the value of the probabilities $P(y_1 = k)$ can be approximated by using Gauss-Hermite quadrature for the integration and compute $P^*(y_1 = k)$ where

$$P^*(y_1 = k) = \frac{1}{\sqrt{\pi}} \frac{1}{k!} \mu^{0k} \sum_{l=1}^L w_l \exp \left[- \left(\mu^0 \exp \sqrt{2}\sigma_1^0 v_l^0 \right) + k\sqrt{2}\sigma_1^0 v_l^0 \right] \quad (2.24)$$

where L is the number of sample points to use for the approximation. The v_l^0 are the roots of the Hermite polynomial $H_L(x)$ ($l = 1, 2, \dots, L$) and the associated weights w_l are given by

$$w_l = \frac{2^{L-1} L! \sqrt{\pi}}{L^2 [H_{L-1}(v_l^0)]^2}$$

2.2. Intermediary step 2: Estimation for first two two-equation model. We now consider as an intermediary step the estimation of the simultaneous equation model made of the first two equation considered above. Explicitly, we consider the following two-equation model

$$\begin{cases} y_1 \text{ such that } \mathcal{D}(y_1 | x_1, u_1^0) \text{ is the above mentioned Poisson distribution} \\ y_2^* = x_2' \beta_2^0 + \alpha_{21}^0 y_1 + u_2^0 \end{cases} \quad (2.25)$$

where we observe

$$y_2 = \begin{cases} 1 & \text{if } y_2^* \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.26)$$

We seek to find the expression of $P(y_1 = k, y_2 = i)$ for $k = 0, 1, \dots$ and $i = 0, 1$.

We have

$$P(y_1 = k, y_2 = i) = \int_{-\infty}^{+\infty} P(y_2 = i | y_1 = k, u_1^0) P(y_1 = k | u_1^0) d\varphi_{\sigma_1^0}(u_1^0) \quad (2.27)$$

We need to derive the expression for the conditional probabilities $P(y_2 = i | y_1 = k, u_1^0)$.

To this end, we use the fact that

$$\mathcal{D}(u_2^0 | u_1^0) = \mathcal{N} \left(\frac{\rho_{12}^0}{\sigma_1^0} u_1^0, 1 - \rho_{12}^{02} \right). \quad (2.28)$$

We have, for $i = 1$,

$$\begin{aligned} P(y_2 = 1 | y_1 = k, u_1^0) &= P(y_2^* \geq 0 | y_1 = k, u_1^0) \\ &= P(x_2' \beta_2^0 + \alpha_{21}^0 y_1 + u_2^0 \geq 0 | y_1 = k, u_1^0) \\ &= P(u_2^0 \geq -x_2' \beta_2^0 - \alpha_{21}^0 y_1 | y_1 = k, u_1^0) \\ &= P \left(\frac{u_2^0 - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \geq -\frac{x_2' \beta_2^0 + \alpha_{21}^0 y_1 + \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \mid y_1 = k, u_1^0 \right) \end{aligned}$$

so that, using (2.28), we obtain

$$\begin{aligned} P(y_2 = 1 \mid y_1 = k, u_1^0) &= 1 - \Phi \left(-\frac{x'_2 \beta_2^0 + \alpha_{21}^0 k + \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \right) \\ &= \Phi \left(\frac{x'_2 \beta_2^0 + \alpha_{21}^0 k + \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \right) \end{aligned} \quad (2.29)$$

We also have, for $i = 0$,

$$\begin{aligned} P(y_2 = 0 \mid y_1 = k, u_1^0) &= 1 - P(y_2 = 1 \mid y_1 = k, u_1^0) \\ &= 1 - \Phi \left(\frac{x'_2 \beta_2^0 + \alpha_{21}^0 k + \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \right) \end{aligned} \quad (2.30)$$

It follows that

$$\begin{aligned} P(y_1 = k, y_2 = 1) &= \int_{-\infty}^{+\infty} P(y_2 = 1 \mid y_1 = k, u_1^0) P(y_1 = k \mid u_1^0) d\varphi_{\sigma_1^0}(u_1^0) \\ &= \frac{1}{\sqrt{2\pi}\sigma_1^0} \frac{1}{k!} \mu^{0k} \int_{-\infty}^{+\infty} \Phi \left(\frac{x'_2 \beta_2^0 + \alpha_{21}^0 k + \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \right) \exp [- (\mu^0 \exp u_1^0) + k u_1^0] \exp \left(-\frac{u_1^{02}}{2\sigma_1^{02}} \right) du_1^0 \end{aligned}$$

where, as before,

$$\mu^0 = \exp(x'_1 \beta_1^0)$$

and

$$\begin{aligned} P(y_1 = k, y_2 = 0) &= \int_{-\infty}^{+\infty} P(y_2 = 0 \mid y_1 = k, u_1^0) P(y_1 = k \mid u_1^0) d\varphi_{\sigma_1^0}(u_1^0) \\ &= \frac{1}{\sqrt{2\pi}\sigma_1^0} \frac{1}{k!} \mu^{0k} \int_{-\infty}^{+\infty} \left[1 - \Phi \left(\frac{x'_2 \beta_2^0 + \alpha_{21}^0 k + \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \right) \right] \exp [- (\mu^0 \exp u_1^0) + k u_1^0] \exp \left(-\frac{u_1^{02}}{2\sigma_1^{02}} \right) du_1^0 \end{aligned}$$

Note that, from the computational point of view, that

$$P(y_1 = k, y_2 = 0) = P(y_1 = k) - P(y_1 = k, y_2 = 1), \quad (2.33)$$

which possibly provides a simpler alternative way to compute $P(y_1 = k, y_2 = 0)$ once $P(y_1 = k)$ and $P(y_1 = k, y_2 = 1)$ have been computed.

The log-likelihood function for a sample with N observations may be written as

$$\mathcal{L}_N = \sum_{n=1}^N \sum_{i=0}^1 \sum_k z_{nki} \log P_{nki} \quad (2.34)$$

where

$$z_{nki} = \begin{cases} 1 & \text{if } y_{n1} = k \text{ and } y_{n2} = i \\ 0 & \text{otherwise} \end{cases} \quad (2.35)$$

and P_{nki} is defined as

$$P_{nki} = P\{z_{nki} = 1\} \quad (2.36)$$

that is to say

$$\begin{aligned} P_{nki} &= P(y_{n1} = k, y_{n2} = i) \\ &= \int_{-\infty}^{+\infty} P(y_{n2} = i \mid y_{n1} = k, u_{n1}) P(y_{n1} = k \mid u_{n1}) d\varphi_{\sigma_1}(u_{n1}) \end{aligned} \quad (2.37)$$

Explicitly, we have

$$P_{nk1} = \frac{1}{\sqrt{2\pi}\sigma_1} \frac{1}{k!} \mu_n^k \int_{-\infty}^{+\infty} \Phi \left(\frac{x'_{n2}\beta_2 + \alpha_{21}k + \frac{\rho_{12}}{\sigma_1} u_{n1}}{\sqrt{1 - \rho_{12}^2}} \right) \exp [- (\mu_n \exp u_{n1}) + k u_{n1}] \exp \left(- \frac{u_{n1}^2}{2\sigma_1^2} \right) du_{n1} \quad (2.38)$$

where

$$\mu_n = \exp(x'_{n1}\beta_1) \quad (2.39)$$

and

$$P_{nk0} = \frac{1}{\sqrt{2\pi}\sigma_1} \frac{1}{k!} \mu_n^k \int_{-\infty}^{+\infty} \left[1 - \Phi \left(\frac{x'_{n2}\beta_2 + \alpha_{21}k + \frac{\rho_{12}}{\sigma_1} u_{n1}}{\sqrt{1 - \rho_{12}^2}} \right) \right] \exp [- (\mu_n \exp u_{n1}) + k u_{n1}] \exp \left(- \frac{u_{n1}^2}{2\sigma_1^2} \right) du_{n1} \quad (2.40)$$

In view of possible computation of the probabilities P_{nk1} and P_{nk0} using the Gauss-Hermite quadrature, we may make the change of variable $v_n = u_{n1}/\sqrt{2}\sigma_1$ in the expressions of these probabilities and obtain

$$P_{nk1} = \frac{1}{\sqrt{\pi}} \frac{1}{k!} \mu_n^k \int_{-\infty}^{+\infty} \Phi \left(\frac{x'_{n2}\beta_2 + \alpha_{21}k + \rho_{12}\sqrt{2}v_n}{\sqrt{1 - \rho_{12}^2}} \right) \exp [- (\mu_n \exp \sqrt{2}\sigma_1 v_n) + k\sqrt{2}\sigma_1 v_n] \exp (-v_n^2) dv_n \quad (2.41)$$

and

$$P_{nk0} = \frac{1}{\sqrt{\pi}} \frac{1}{k!} \mu_n^k \int_{-\infty}^{+\infty} \left[1 - \Phi \left(\frac{x'_{n2}\beta_2 + \alpha_{21}k + \rho_{12}\sqrt{2}v_n}{\sqrt{1 - \rho_{12}^2}} \right) \right] \exp [- (\mu_n \exp \sqrt{2}\sigma_1 v_n) + k\sqrt{2}\sigma_1 v_n] \exp (-v_n^2) dv_n \quad (2.42)$$

2.3. The complete three-equations model. Consider now the complete three-equations model described above. We seek to find the expression of $P(y_1 = k, y_2 = i, y_3 = j)$ for $k = 0, 1, \dots$, $i = 0, 1$ and $j = 1, \dots, 5$.

We have

$$P(y_1 = k, y_2 = i, y_3 = j) = \int_{-\infty}^{+\infty} P(y_2 = i, y_3 = j \mid y_1 = k, u_1^0) P(y_1 = k \mid u_1^0) d\varphi_{\sigma_1^0}(u_1^0) \quad (2.43)$$

where, as before, $d\varphi_{\sigma_1^0}(u_1^0)$ is defined as

$$d\varphi_{\sigma_1^0}(u_1^0) = \frac{1}{\sigma_1^0} \varphi \left(\frac{u_1^0}{\sigma_1^0} \right) du_1^0.$$

2.3.1. Derivation of the probabilities. We need to derive the expression for the conditional probabilities $P(y_2 = i, y_3 = j \mid y_1 = k, u_1^0)$.

We use the fact that

$$\mathcal{D} \left[\begin{pmatrix} u_2^0 \\ u_3^0 \end{pmatrix} \mid u_1^0 \right] = \mathcal{N} \left[\begin{pmatrix} \frac{\rho_{12}^0}{\sigma_1^0} u_1^0 \\ \frac{\rho_{13}^0}{\sigma_1^0} u_1^0 \end{pmatrix}, \begin{pmatrix} 1 - \rho_{12}^{02} & \rho_{23}^0 - \rho_{12}^0 \rho_{13}^0 \\ \rho_{23}^0 - \rho_{12}^0 \rho_{13}^0 & 1 - \rho_{13}^{02} \end{pmatrix} \right]. \quad (2.44)$$

For $i = 0$ we have,

$$\begin{aligned}
P\{y_2 &= 0, y_3 = j \mid y_1 = k, u_1^0\} = P\{y_2^* < 0, c_{j-1} \leq y_3^* < c_j \mid y_1 = k, u_1^0\} \\
&= P\{y_2^* < 0, y_3^* < c_j \mid y_1 = k, u_1^0\} - P\{y_2^* < 0, y_3^* < c_{j-1} \mid y_1 = k, u_1^0\} \\
&= P\{x_2'\beta_2^0 + \alpha_{21}^0 k + u_2^0 < 0, x_3'\beta_3^0 + \alpha_{31}^0 k + u_3^0 < c_j \mid u_1^0\} \\
-P\{x_2'\beta_2^0 + \alpha_{21}^0 k + u_2^0 &< 0, x_3'\beta_3^0 + \alpha_{31}^0 k + u_3^0 < c_{j-1} \mid u_1^0\} \\
&= P\{u_2^0 < -x_2'\beta_2^0 - \alpha_{21}^0 k, u_3^0 < c_j - x_3'\beta_3^0 - \alpha_{31}^0 k \mid u_1^0\} \\
-P\{u_2^0 &< -x_2'\beta_2^0 - \alpha_{21}^0 k, u_3^0 < c_{j-1} - x_3'\beta_3^0 - \alpha_{31}^0 k \mid u_1^0\},
\end{aligned}$$

which, using (1.13d), may be written as

$$\begin{aligned}
P\{y_2 = 0, y_3 = j \mid y_1 = k, u_1^0\} &= \Phi\left(\frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^2}}\right) \left[\Phi\left(\frac{c_j - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^2}}\right) \right. \\
&\quad \left. - \Phi\left(\frac{c_{j-1} - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^2}}\right) \right] \\
&\quad + \Psi\left(\frac{\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^2)(1 - \rho_{13}^2)}}, \frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^2}}, \frac{c_j - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^2}}\right) \\
&\quad - \Psi\left(\frac{\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^2)(1 - \rho_{13}^2)}}, \frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^2}}, \frac{c_{j-1} - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^2}}\right)
\end{aligned} \tag{2.45}$$

The explicit expression for these probabilities when $y_3 = 1$ and $y_3 = 5$ are:

For $y_3 = 1$, we have $c_{j-1} = c_0 = -\infty$ so that

$$\Phi\left(\frac{-\infty - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^2}}\right) = 0.$$

Also, it is easy to verify from equation (1.3) that

$$\Psi(\rho, a, -\infty) = 0,$$

so that $P\{y_2 = 0, y_3 = 1 \mid y_1 = k, u_1^0\}$ simplifies into

$$\begin{aligned}
P\{y_2 = 0, y_3 = 1 \mid y_1 = k, u_1^0\} &= \Phi\left(\frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^2}}\right) \Phi\left(\frac{c_1 - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^2}}\right) \\
&\quad + \Psi\left(\frac{\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^2)(1 - \rho_{13}^2)}}, \frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^2}}, \frac{c_1 - x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^2}}\right)
\end{aligned} \tag{2.46}$$

One can check the validity of this expression by noting that

$$\begin{aligned}
P\{y_2 &= 0, y_3 = 1 \mid y_1 = k, u_1^0\} = P\{y_2^* < 0, y_3^* < c_1 \mid y_1 = k, u_1^0\} \\
&= P\{x_2'\beta_2^0 + \alpha_{21}^0 k + u_2^0 < 0, x_3'\beta_3^0 + \alpha_{31}^0 k + u_3^0 < c_1 \mid u_1^0\} \\
&= P\{u_2^0 < -x_2'\beta_2^0 - \alpha_{21}^0 k, u_3^0 < c_1 - x_3'\beta_3^0 - \alpha_{31}^0 k \mid u_1^0\},
\end{aligned}$$

which using (2.44) and (1.13d) gives exactly the same expression as (2.46).

For $y_3 = 5$, we have $c_j = c_5 = +\infty$ so that

$$\Phi \left(\frac{+\infty - x'_3 \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) = 1.$$

Also, it is easy to verify from equation (1.3) that

$$\Psi(\rho, a, +\infty) = 0,$$

so that $P\{y_2 = 0, y_3 = 5 \mid y_1 = k, u_1^0\}$ simplifies into

$$\begin{aligned} P\{y_2 = 0, y_3 = 5 \mid y_1 = k, u_1^0\} &= \Phi \left(\frac{-x'_2 \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \right) \left[1 - \Phi \left(\frac{c_4 - x'_3 \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) \right] \\ &\quad - \Psi \left(\frac{\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^{02})(1 - \rho_{13}^{02})}}, \frac{-x'_2 \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}, \frac{c_4 - x'_3 \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}}} \right) \end{aligned} \quad (2.47)$$

One can check the validity of this expression by noting that

$$\begin{aligned} P\{y_2 = 0, y_3 = 5 \mid y_1 = k, u_1^0\} &= P\{y_2^* < 0, y_3^* \geq c_4 \mid y_1 = k, u_1^0\} \\ &= P\{x'_2 \beta_2^0 + \alpha_{21}^0 k + u_2^0 < 0, x'_3 \beta_3^0 + \alpha_{31}^0 k + u_3^0 \geq c_4 \mid u_1^0\} \\ &= P\{u_2^0 < -x'_2 \beta_2^0 - \alpha_{21}^0 k, u_3^0 \geq c_4 - x'_3 \beta_3^0 - \alpha_{31}^0 k \mid u_1^0\}, \end{aligned}$$

which using (2.44) and (1.13c) gives exactly the same expression as (2.47).

Similarly, for $i = 1$ we have,

$$\begin{aligned} P\{y_2 = 1, y_3 = j \mid y_1 = k, u_1^0\} &= P\{y_2^* \geq 0, c_{j-1} \leq y_3^* < c_j \mid y_1 = k, u_1^0\} \\ &= P\{y_2^* \geq 0, y_3^* < c_j \mid y_1 = k, u_1^0\} - P\{y_2^* \geq 0, y_3^* < c_{j-1} \mid y_1 = k, u_1^0\} \\ &= P\{x'_2 \beta_2^0 + \alpha_{21}^0 k + u_2^0 \geq 0, x'_3 \beta_3^0 + \alpha_{31}^0 y_1 + \alpha_{32}^0 + u_3^0 < c_j \mid u_1^0\} \\ &\quad - P\{x'_2 \beta_2^0 + \alpha_{21}^0 k + u_2^0 \geq 0, x'_3 \beta_3^0 + \alpha_{31}^0 y_1 + \alpha_{32}^0 + u_3^0 < c_{j-1} \mid u_1^0\} \\ &= P\{u_2^0 \geq -x'_2 \beta_2^0 - \alpha_{21}^0 k, u_3^0 < c_j - x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 \mid u_1^0\} \\ &\quad - P\{u_2^0 \geq -x'_2 \beta_2^0 - \alpha_{21}^0 k, u_3^0 < c_{j-1} - x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 \mid u_1^0\}, \end{aligned}$$

which, using (1.13b), may be written as

$$\begin{aligned} P\{y_2 = 1, y_3 = j \mid y_1 = k, u_1^0\} &= \left[1 - \Phi \left(\frac{-x'_2 \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \right) \right] \times \\ &\quad \times \left[\Phi \left(\frac{c_j - x'_3 \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) - \Phi \left(\frac{c_{j-1} - x'_3 \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) \right] \\ &\quad - \Psi \left(\frac{\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^{02})(1 - \rho_{13}^{02})}}, \frac{-x'_2 \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}, \frac{c_j - x'_3 \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}}} \right) \\ &\quad + \Psi \left(\frac{\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^{02})(1 - \rho_{13}^{02})}}, \frac{-x'_2 \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}, \frac{c_{j-1} - x'_3 \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}}} \right) \end{aligned} \quad (2.48)$$

The explicit expression for these probabilities when $y_3 = 1$ and $y_3 = 5$ are:

For $y_3 = 1$, we have $c_{j-1} = c_0 = -\infty$ so that

$$\Phi \left(\frac{-\infty x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^0}} \right) = 0.$$

so that $P\{y_2 = 1, y_3 = 1 \mid y_1 = k, u_1^0\}$ simplifies into

$$\begin{aligned} P\{y_2 = 1, y_3 = 1 \mid y_1 = k, u_1^0\} &= \left[1 - \Phi \left(\frac{-x'_2 \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^0}} \right) \right] \Phi \left(\frac{c_1 - x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^0}} \right) \\ &\quad - \Psi \left(\frac{\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^0)(1 - \rho_{13}^0)}}, \frac{-x'_2 \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^0}}, \frac{c_1 - x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^0}} \right) \end{aligned} \quad (2.49)$$

One can check the validity of this expression by noting that

$$\begin{aligned} P\{y_2 = 1, y_3 = 1 \mid y_1 = k, u_1^0\} &= P\{y_2^* \geq 0, y_3^* < c_1 \mid y_1 = k, u_1^0\} \\ &= P\{x'_2 \beta_2^0 + \alpha_{21}^0 k + u_2^0 \geq 0, x'_3 \beta_3^0 + \alpha_{31}^0 k + \alpha_{32}^0 + u_3^0 < c_1 \mid u_1^0\} \\ &= P\{u_2^0 \geq -x'_2 \beta_2^0 - \alpha_{21}^0 k, u_3^0 < c_1 - x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 \mid u_1^0\} \end{aligned}$$

which using (2.44) and (1.13b) gives exactly the same expression as (2.49).

For $y_3 = 5$, we have $c_j = c_5 = +\infty$ so that

$$\Phi \left(\frac{+\infty - x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^0}} \right) = 1.$$

so that $P\{y_2 = 1, y_3 = 5 \mid y_1 = k, u_1^0\}$ simplifies into

$$\begin{aligned} P\{y_2 = 1, y_3 = 5 \mid y_1 = k, u_1^0\} &= \left[1 - \Phi \left(\frac{-x'_2 \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^0}} \right) \right] \left[1 - \Phi \left(\frac{c_4 - x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^0}} \right) \right] \\ &\quad + \Psi \left(\frac{\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^0)(1 - \rho_{13}^0)}}, \frac{-x'_2 \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^0}}, \frac{c_4 - x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^0}} \right) \end{aligned} \quad (2.50)$$

One can check the validity of this expression by noting that

$$\begin{aligned} P\{y_2 = 1, y_3 = 5 \mid y_1 = k, u_1^0\} &= P\{y_2^* \geq 0, y_3^* \geq c_4 \mid y_1 = k, u_1^0\} \\ &= P\{x'_2 \beta_2^0 + \alpha_{21}^0 k + u_2^0 \geq 0, x'_3 \beta_3^0 + \alpha_{31}^0 k + \alpha_{32}^0 + u_3^0 \geq c_4 \mid u_1^0\} \\ &= P\{u_2^0 \geq -x'_2 \beta_2^0 - \alpha_{21}^0 k, u_3^0 \geq c_4 - x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 \mid u_1^0\} \end{aligned}$$

which using (2.44) and (1.13a) gives exactly the same expression as (2.50).

Note that

$$\begin{aligned} P\{y_2 = 1, y_3 = j \mid y_1 = k, u_1^0\} &= \left[\Phi \left(\frac{c_j - x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^0}} \right) \right. \\ &\quad \left. - \Phi \left(\frac{c_{j-1} - x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^0}} \right) \right] - P\{y_2 = 0, y_3 = j \mid y_1 = k, u_1^0\}, \end{aligned}$$

which allows to verify that, as one should expect,

$$\begin{aligned} P\{y_3 = j \mid y_1 = k, u_1^0\} &= P\{y_2 = 0, y_3 = j \mid y_1 = k, u_1^0\} + P\{y_2 = 1, y_3 = j \mid y_1 = k, u_1^0\} \\ &= \left[\Phi \left(\frac{c_j - x'_3 \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) - \Phi \left(\frac{c_{j-1} x'_3 \beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) \right] \end{aligned}$$

Equations (2.45) and (2.48) allow us to compute all the probabilities $P(y_1 = k, y_2 = i, y_3 = j)$ for $k = 0, 1, \dots, i = 0, 1$ and $j = 1, \dots, 5$ using (2.43).

2.3.2. The log-likelihood function. The log-likelihood function for a sample with N observation may be written as

$$\mathcal{L}_N = \sum_{n=1}^N \sum_{k=0}^{\infty} \sum_{i=0}^1 \sum_{j=1}^5 z_{nkij} \log P_{nkij} \quad (2.51)$$

where

$$z_{nkij} = \begin{cases} 1 & \text{if } y_{n1} = k, y_{n2} = i \text{ and } y_{n3} = j \quad (k = 0, 1, \dots, i = 0, 1 \text{ and } j = 1, \dots, 5) \\ 0 & \text{otherwise} \end{cases} \quad (2.52)$$

and P_{nkij} is defined as

$$P_{nkij} = P\{z_{nkij} = 1\} \quad (2.53)$$

that is to say

$$\begin{aligned} P_{nkij} &= P(y_{n1} = k, y_{n2} = i, y_{n3} = j) \\ &= \int_{-\infty}^{+\infty} P(y_{n2} = i, y_{n3} = j \mid y_{n1} = k, u_{n1}) P(y_{n1} = k \mid u_{n1}) d\varphi_{\sigma_1}(u_{n1}) \end{aligned} \quad (2.54)$$

where $d\varphi_{\sigma_1^0}(u_1^0)$ is defined as

$$d\varphi_{\sigma_1}(u_{n1}) = \frac{1}{\sigma_1} \varphi \left(\frac{u_{n1}}{\sigma_1} \right) du_{n1}.$$

Explicitly,

$$P_{nkij} = \frac{1}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{+\infty} P(y_{n2} = i, y_{n3} = j \mid y_{n1} = k, u_{n1}) P(y_{n1} = k \mid u_{n1}) \exp \left(-\frac{u_{n1}^2}{2\sigma_1^2} \right) du_{n1}. \quad (2.55)$$

From a numerical point of view, it is convenient to make the change of variable $v_n = u_{n1}/\sqrt{2}\sigma_1$ so that (2.55) may be written as

$$P_{nkij} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} P(y_{n2} = i, y_{n3} = j \mid y_{n1} = k, \sqrt{2}\sigma_1 v_n) P(y_{n1} = k \mid \sqrt{2}\sigma_1 v_n) \exp(-v_n^2) dv_n. \quad (2.56)$$

The probabilities $P(y_{n2} = i, y_{n3} = j \mid y_{n1} = k, \sqrt{2}\sigma_1 v_n)$ can be obtained from (2.45) and (2.48) by making the change in variable $u_1^0 = \sqrt{2}\sigma_1^0 v^0$.

In order to derive the expression for the probability $P(y_{n1} = k \mid \sqrt{2}\sigma_1 v_n)$, it is convenient to define μ_n as

$$\mu_n = \exp(x'_{n1} \beta_1)$$

so that, similarly as in (??) we may write $P(y_{n1} = k \mid \sqrt{2}\sigma_1 v_n)$ as

$$P(y_{n1} = k \mid \sqrt{2}\sigma_1 v_n) = \frac{1}{k!} \mu_n^k \exp \left[- \left(\mu_n \exp \sqrt{2}\sigma_1 v_n \right) + k \sqrt{2}\sigma_1 v_n \right] \quad (2.57)$$

The value of the probabilities P_{nkij} in (2.56) can be approximated by using Gauss–Hermite quadrature for the integration and compute P_{nkij}^* where

$$P_{nkij}^* = \frac{1}{\sqrt{\pi}} \sum_{l=1}^L w_{nl} P(y_{n2} = i, y_{n3} = j \mid y_{n1} = k, \sqrt{2}\sigma_1 v_{nl}) P(y_{n1} = k \mid \sqrt{2}\sigma_1 v_{nl}) \quad (2.58)$$

where, as before, L is the number of sample points to use for the approximation. The v_{nl} are the roots of the Hermite polynomial $H_L(x)$ ($l = 1, 2, \dots, L$) and the associated weights w_l are given by

$$w_{nl} = \frac{2^{L-1} L! \sqrt{\pi}}{L^2 [H_{L-1}(v_{nl})]^2}$$

2.3.3. *The case of a dichotomous y_3 .* Consider the simplest case where y_3 is dichotomous,

$$y_3 = \begin{cases} 1 & \text{if } y_3^* \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad (2.59)$$

For $i = 0$ and $y_3 = 1$ we have,

$$\begin{aligned} P\{y_2 = 0, y_3 = 1 \mid y_1 = k, u_1^0\} &= P\{y_2^* < 0, y_3^* \geq 0 \mid y_1 = k, u_1^0\} \\ &= P\{x_2' \beta_2^0 + \alpha_{21}^0 y_1 + u_2^0 < 0, x_3' \beta_3^0 + \alpha_{31}^0 y_1 + \alpha_{32}^0 y_2 + u_3^0 \geq 0 \mid y_1 = k, u_1^0\} \\ &= P\{x_2' \beta_2^0 + \alpha_{21}^0 k + u_2^0 < 0, x_3' \beta_3^0 + \alpha_{31}^0 k + u_3^0 \geq 0 \mid u_1^0\} \\ &= P\{u_2^0 < -x_2' \beta_2^0 - \alpha_{21}^0 k, u_3^0 \geq -x_3' \beta_3^0 - \alpha_{31}^0 k \mid u_1^0\} \end{aligned} \quad (2.60)$$

which, using (1.13c), may be written as

$$\begin{aligned} P\{y_2 = 0, y_3 = 1 \mid y_1 = k, u_1^0\} &= \Phi \left(\frac{-x_2' \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^2}} \right) \left[1 - \Phi \left(\frac{-x_3' \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^2}} \right) \right] \\ &\quad - \Psi \left(\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^2)(1 - \rho_{13}^2)}}, \frac{-x_2' \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^2}}, \frac{-x_3' \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^2}} \right) \end{aligned} \quad (2.61)$$

Therefore, the unconditional probability $P\{y_1 = k, y_2 = 0, y_3 = 1\}$ is equal to

$$P\{y_1 = k, y_2 = 0, y_3 = 1\} = \int_{-\infty}^{+\infty} P\{y_2 = 0, y_3 = 1 \mid y_1 = k, u_1^0\} P\{y_1 = k \mid u_1^0\} d\varphi_{\sigma_1^0}(u_1^0),$$

and thus,

$$\begin{aligned} P\{y_1 = k, y_2 = 0, y_3 = 1\} &= \frac{1}{\sqrt{2\pi}\sigma_1^0} \frac{1}{k!} \mu^{0k} \int_{-\infty}^{+\infty} \left\{ \Phi \left(\frac{-x_2' \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^2}} \right) \right. \\ &\quad \times \left[1 - \Phi \left(\frac{-x_3' \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^2}} \right) \right] \end{aligned} \quad (2.62)$$

$$\begin{aligned} &\quad \left. - \Psi \left(\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^2)(1 - \rho_{13}^2)}}, \frac{-x_2' \beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^2}}, \frac{-x_3' \beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^2}} \right) \right\} \\ &\quad \times \exp \left[-(\mu^0 \exp u_1^0) + k u_1^0 \right] \exp \left(-\frac{u_1^{02}}{2\sigma_1^2} \right) du_1^0 \end{aligned} \quad (2.63)$$

Similarly, for $i = 0$ and $y_3 = 0$ we have,

$$\begin{aligned}
P\{y_2 &= 0, y_3 = 0 \mid y_1 = k, u_1^0\} = P\{y_2^* < 0, y_3^* < 0 \mid y_1 = k, u_1^0\} \\
&= P\{x_2'\beta_2^0 + \alpha_{21}^0 y_1 + u_2^0 < 0, x_3'\beta_3^0 + \alpha_{31}^0 y_1 + \alpha_{32}^0 y_2 + u_3^0 < 0 \mid y_1 = k, u_1^0\} \\
&= P\{x_2'\beta_2^0 + \alpha_{21}^0 k + u_2^0 < 0, x_3'\beta_3^0 + \alpha_{31}^0 k + u_3^0 \geq 0 \mid u_1^0\} \\
&= P\{u_2^0 < -x_2'\beta_2^0 - \alpha_{21}^0 k, u_3^0 < -x_3'\beta_3^0 - \alpha_{31}^0 k \mid u_1^0\}
\end{aligned} \tag{2.64}$$

which, using (1.13d), may be written as

$$\begin{aligned}
P\{y_2 &= 0, y_3 = 0 \mid y_1 = k, u_1^0\} = \Phi\left(\frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}\right) \left[\Phi\left(\frac{-x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}}\right) \right] \\
&+ \Psi\left(\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^{02})(1 - \rho_{13}^{02})}}, \frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}, \frac{-x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}}\right)
\end{aligned} \tag{2.65}$$

Therefore, the unconditional probability $P\{y_1 = k, y_2 = 0, y_3 = 0\}$ is equal to

$$P\{y_1 = k, y_2 = 0, y_3 = 0\} = \int_{-\infty}^{+\infty} P\{y_2 = 0, y_3 = 0 \mid y_1 = k, u_1^0\} P\{y_1 = k \mid u_1^0\} d\varphi_{\sigma_1^0}(u_1^0),$$

and thus,

$$\begin{aligned}
P\{y_1 &= k, y_2 = 0, y_3 = 0\} = \frac{1}{\sqrt{2\pi}\sigma_1^0} \frac{1}{k!} \mu^{0k} \int_{-\infty}^{+\infty} \left\{ \Phi\left(\frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}\right) \left[\Phi\left(\frac{-x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}}\right) \right] \right. \\
&+ \Psi\left(\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^{02})(1 - \rho_{13}^{02})}}, \frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}, \frac{-x_3'\beta_3^0 - \alpha_{31}^0 k - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}}\right) \left. \right\} \\
&\times \exp\left[-(\mu^0 \exp u_1^0) + k u_1^0\right] \exp\left(-\frac{u_1^{02}}{2\sigma_1^{02}}\right) du_1^0
\end{aligned} \tag{2}$$

For $i = 1$ and $y_3 = 1$ we have,

$$\begin{aligned}
P\{y_2 &= 1, y_3 = 1 \mid y_1 = k, u_1^0\} = P\{y_2^* \geq 0, y_3^* \geq 0 \mid y_1 = k, u_1^0\} \\
&= P\{x_2'\beta_2^0 + \alpha_{21}^0 y_1 + u_2^0 \geq 0, x_3'\beta_3^0 + \alpha_{31}^0 y_1 + \alpha_{32}^0 y_2 + u_3^0 \geq 0 \mid y_1 = k, u_1^0\} \\
&= P\{x_2'\beta_2^0 + \alpha_{21}^0 k + u_2^0 \geq 0, x_3'\beta_3^0 + \alpha_{31}^0 k + \alpha_{32}^0 + u_3^0 \geq 0 \mid u_1^0\} \\
&= P\{u_2^0 \geq -x_2'\beta_2^0 - \alpha_{21}^0 k, u_3^0 \geq -x_3'\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 \mid u_1^0\}
\end{aligned} \tag{2.67}$$

which, using (1.13a), may be written as

$$\begin{aligned}
P\{y_2 &= 1, y_3 = 1 \mid y_1 = k, u_1^0\} = \left[1 - \Phi\left(\frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}\right) \right] \left[1 - \Phi\left(\frac{-x_3'\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}}\right) \right] \\
&+ \Psi\left(\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^{02})(1 - \rho_{13}^{02})}}, \frac{-x_2'\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}, \frac{-x_3'\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}}\right)
\end{aligned} \tag{2}$$

Therefore, the unconditional probability $P\{y_1 = k, y_2 = 1, y_3 = 1\}$ is equal to

$$P\{y_1 = k, y_2 = 1, y_3 = 1\} = \int_{-\infty}^{+\infty} P\{y_2 = 1, y_3 = 1 \mid y_1 = k, u_1^0\} P\{y_1 = k \mid u_1^0\} d\varphi_{\sigma_1^0}(u_1^0),$$

and thus,

$$\begin{aligned} P\{y_1 = k, y_2 = 1, y_3 = 1\} &= \frac{1}{\sqrt{2\pi}\sigma_1^0} \frac{1}{k!} \mu^{0k} \int_{-\infty}^{+\infty} \left\{ \left[1 - \Phi \left(\frac{-x'_2\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \right) \right] \right. \\ &\quad \times \left[1 - \Phi \left(\frac{-x'_3\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) \right] \\ &\quad \left. + \Psi \left(\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^{02})(1 - \rho_{13}^{02})}}, \frac{-x'_2\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}, \frac{-x'_3\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) \right\} \\ &\quad \times \exp \left[-(\mu^0 \exp u_1^0) + k u_1^0 \right] \exp \left(-\frac{u_1^{02}}{2\sigma_1^{02}} \right) du_1^0 \end{aligned} \quad (2.69)$$

Finally, for $i = 1$ and $y_3 = 0$ we have,

$$\begin{aligned} P\{y_2 = 1, y_3 = 0 \mid y_1 = k, u_1^0\} &= P\{y_2^* \geq 0, y_3^* < 0 \mid y_1 = k, u_1^0\} \\ &= P\{x'_2\beta_2^0 + \alpha_{21}^0 y_1 + u_2^0 \geq 0, x'_3\beta_3^0 + \alpha_{31}^0 y_1 + \alpha_{32}^0 + u_3^0 < 0 \mid y_1 = k, u_1^0\} \\ &= P\{x'_2\beta_2^0 + \alpha_{21}^0 k + u_2^0 \geq 0, x'_3\beta_3^0 + \alpha_{31}^0 k + \alpha_{32}^0 + u_3^0 \geq 0 \mid u_1^0\} \\ &= P\{u_2^0 \geq -x'_2\beta_2^0 - \alpha_{21}^0 k, u_3^0 < -x'_3\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 \mid u_1^0\} \end{aligned} \quad (2.70)$$

which, using (1.13b), may be written as

$$\begin{aligned} P\{y_2 = 1, y_3 = 0 \mid y_1 = k, u_1^0\} &= \left[1 - \Phi \left(\frac{-x'_2\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \right) \right] \left[\Phi \left(\frac{-x'_3\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) \right] \\ &\quad - \Psi \left(\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^{02})(1 - \rho_{13}^{02})}}, \frac{-x'_2\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}, \frac{-x'_3\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) \end{aligned} \quad (2.71)$$

Therefore, the unconditional probability $P\{y_1 = k, y_2 = 1, y_3 = 0\}$ is equal to

$$P\{y_1 = k, y_2 = 1, y_3 = 0\} = \int_{-\infty}^{+\infty} P\{y_2 = 1, y_3 = 0 \mid y_1 = k, u_1^0\} P\{y_1 = k \mid u_1^0\} d\varphi_{\sigma_1^0}(u_1^0),$$

and thus,

$$P\{y_1 = k, y_2 = 1, y_3 = 0\} = \frac{1}{\sqrt{2\pi}\sigma_1^0} \frac{1}{k!} \mu^{0k} \int_{-\infty}^{+\infty} \left\{ \left[1 - \Phi \left(\frac{-x'_2\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}} \right) \right] \right. \\ \left. \left[\Phi \left(\frac{-x'_3\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) \right] \right. \quad (2.72)$$

$$\left. - \Psi \left(\frac{\rho_{23}^0 - \rho_{12}^0 \rho_{13}^0}{\sqrt{(1 - \rho_{12}^{02})(1 - \rho_{13}^{02})}}, \frac{-x'_2\beta_2^0 - \alpha_{21}^0 k - \frac{\rho_{12}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{12}^{02}}}, \frac{-x'_3\beta_3^0 - \alpha_{31}^0 k - \alpha_{32}^0 - \frac{\rho_{13}^0}{\sigma_1^0} u_1^0}{\sqrt{1 - \rho_{13}^{02}}} \right) \right\} \\ \times \exp \left[-(\mu^0 \exp u_1^0) + k u_1^0 \right] \exp \left(-\frac{u_1^{02}}{2\sigma_1^{02}} \right) du_1^0 \quad (2.74)$$

The log-likelihood function. The log-likelihood function for a sample with N observation may be written as

$$\mathcal{L}_N = \sum_{n=1}^N \sum_{k=0}^{\infty} \sum_{i=0}^1 \sum_{j=1}^5 z_{nkij} \log P_{nkij} \quad (2.75)$$

where

$$z_{nkij} = \begin{cases} 1 & \text{if } y_{n1} = k, y_{n2} = i \text{ and } y_{n3} = j \quad (k = 0, 1, \dots, i = 0, 1 \text{ and } j = 0, 1) \\ 0 & \text{otherwise} \end{cases} \quad (2.76)$$

and P_{nkij} is defined as

$$P_{nkij} = P\{z_{nkij} = 1\} \quad (2.77)$$

that is to say

$$P_{nkij} = P(y_{n1} = k, y_{n2} = i, y_{n3} = j) \\ = \int_{-\infty}^{+\infty} P(y_{n2} = i, y_{n3} = j \mid y_{n1} = k, u_{n1}) P(y_{n1} = k \mid u_{n1}) d\varphi_{\sigma_1}(u_{n1}) \quad (2.78)$$

where $d\varphi_{\sigma_1^0}(u_1^0)$ is defined as

$$d\varphi_{\sigma_1}(u_{n1}) = \frac{1}{\sigma_1} \varphi \left(\frac{u_{n1}}{\sigma_1} \right) du_{n1}.$$

Explicitly,

$$P_{nkij} = \frac{1}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{+\infty} P(y_{n2} = i, y_{n3} = j \mid y_{n1} = k, u_{n1}) P(y_{n1} = k \mid u_{n1}) \exp \left(-\frac{u_{n1}^2}{2\sigma_1^2} \right) du_{n1}. \quad (2.79)$$

From a numerical point of view, it is convenient to make the change of variable $v_n = u_{n1}/\sqrt{2}\sigma_1$ so that (2.79) may be written as

$$P_{nkij} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} P(y_{n2} = i, y_{n3} = j \mid y_{n1} = k, \sqrt{2}\sigma_1 v_n) P(y_{n1} = k \mid \sqrt{2}\sigma_1 v_n) \exp(-v_n^2) dv_n. \quad (2.80)$$

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