Math and Physics Cheat Sheet

Diogo Almeida

November 2023

Contents

1 Introduction

The present notes accompany my study of math and indicate interesting theorems and results that might come in handy in the future. [?] [?] [?] [?]

2 Calculus and Algebra

2.1 Pre-College algebra

$$a^3+b^3=(a+b)(a^2-ab+b^2) \tag{1}$$

2.2 Sums

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1) \tag{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}(n+1)(2n+1) \tag{3}$$

$$e^x = \lim_{N\to\infty}\sum_{n=0}^{N}\frac{x^n}{n!} \tag{4}$$

$$\sum_{n=1}^{N} u_n = \sum_{n=0}^{N-1}(an+b) = \frac{N}{2}(u_1+u_N) \tag{5}$$

$$\sum_{n=1}^{N} u_n = \sum_{n=0}^{N-1}(ar^n) = u_1\frac{1-r^n}{1-r} \tag{6}$$

$$\sum_{n=0}^{N-1}(an+b)r^n = \frac{a-[a+(N-1)d]r^N}{1-r} + \frac{rd(1-r^{N-1})}{(1-r)^2} \tag{7}$$

$$\sum_{n=1} Nn^3 = \left(\sum_{n=1}^{N} n\right)^2 \tag{8}$$

2.3 Trigonometry

$$\sin^2(x)+\cos^2(x)=1 \tag{9}$$

$$\tan^2(x)+1=\sec^2(x) \tag{10}$$

$$\cot^2(x)+1=\csc^2(x) \tag{11}$$

$$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y) \tag{12}$$

$$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y) \tag{13}$$

$$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \tag{14}$$

$$\sin(2x)=2\cos(x)\sin(x) \tag{15}$$

$$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x) \tag{16}$$

$$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)} \tag{17}$$

$$\cos^2(x)=\frac{1+cos(2x)}{2} \tag{18}$$

$$\sin^2(x)=\frac{1-cos(2x)}{2} \tag{19}$$

$$\cos(x)\cos(y)=\frac{1}{2}\left(\cos(x-y)+\cos(x+y)\right) \tag{20}$$

$$\sin(x)\sin(y)=\frac{1}{2}\left(\cos(x-y)-\cos(x+y)\right) \tag{21}$$

$$\sin(x)\cos(y)=\frac{1}{2}\left(\sin(x-y)+\sin(x+y)\right) \tag{22}$$

$$\cos(x)+\cos(y)=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \tag{23}$$

$$\cos(x)-\cos(y)=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \tag{24}$$

$$\sin(x)-\sin(y)=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \tag{25}$$

$$2i\sin(x)=e^{ix}-e^{-ix} \tag{26}$$

$$2\cos(x)=e^{ix}+e^{-ix} \tag{27}$$

$$2i\sin(nx)=e^{nix}-e^{-nix} \tag{28}$$

$$2\cos(nx)=e^{nix}+e^{-nix} \tag{29}$$

2.4 Hyperbolic Functions

$$\sinh(x)=\frac{1}{2}(e^x-e^{-x}) \tag{30}$$

$$\cosh(x)=\frac{1}{2}(e^x+e^{-x}) \tag{31}$$

$$\text{sech}(x)=\frac{1}{\cosh(x)} \tag{32}$$

$$\text{csch}(x)=\frac{1}{\sinh(x)} \tag{33}$$

$$\tanh(x)=\frac{\sinh(x)}{\cosh(x)} \tag{34}$$

$$\coth(x)=\frac{\cosh(x)}{\sinh(x)} \tag{35}$$

$$\cosh^2(x)-\sinh^2(x)=1 \tag{36}$$

$$\cosh(x+y)=\cosh(x)\cosh(y)+\sinh(x)\sinh(y) \tag{37}$$

$$\sinh(x+y)=\cosh(x)\sinh(y)+\sinh(x)\cosh(y) \tag{38}$$

$$\sinh^{-1}(x)=\ln\left(\sqrt{1+x^2}+x\right) \tag{39}$$

$$\cosh^{-1}(x)=\ln\left(\sqrt{x^2-1}+x\right) \tag{40}$$

2.5 Complex Numbers

$$e^{i\theta}=\cos(\theta)+i\sin(\theta) \tag{41}$$

$$|z|^2=z\overline{z} \tag{42}$$

$$\overline{z+w}=\overline{w}+\overline{z} \tag{43}$$

$$\overline{zw}=\overline{w}\cdot\overline{z} \tag{44}$$

$$\overline{\left(\frac{z}{w}\right)}=\frac{\overline{w}}{\overline{z}} \tag{45}$$

$$e^{in\theta}=\left(e^{i\theta}\right)^n \tag{46}$$

$$\cos(i\theta)=\cosh(\theta) \tag{47}$$

$$\sin(i\theta)=i\sinh(\theta) \tag{48}$$

2.6 Differentiation

$$\partial_x x^a = a x^{a-1} \tag{49}$$

$$\partial_x e^x = e^x \tag{50}$$

$$\partial_x a^x = \ln(a) a^x \tag{51}$$

$$\partial_x \ln(x) = \frac{1}{x} \tag{52}$$

$$\partial_x \log_a(x) = \frac{1}{x \ln(a)} \tag{53}$$

$$\partial_x \sin(x) = \cos(x) \tag{54}$$

$$\partial_x \cos(x) = -\sin(x) \tag{55}$$

$$\partial_x \tan(x) = \sec^2(x) \tag{56}$$

$$\partial_x \cot(x) = -\csc^2(x) \tag{57}$$

$$\partial_x \sec(x) = \sec(x)\tan(x) \tag{58}$$

$$\partial_x \csc(x) = -\csc(x)\cot(x) \tag{59}$$

$$\frac{\mathrm{d}\sin^{-1}(x)}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}} \tag{60}$$

$$\frac{\mathrm{d}\cos^{-1}(x)}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}} \tag{61}$$

$$\frac{\mathrm{d}\tan^{-1}(x)}{\mathrm{d}x} = \frac{1}{1+x^2} \tag{62}$$

$$\frac{\mathrm{d}\sec^{-1}(x)}{\mathrm{d}x} = \frac{1}{x\sqrt{x^2-1}} \tag{63}$$

$$\frac{\mathrm{d}\csc^{-1}(x)}{\mathrm{d}x} = -\frac{1}{x\sqrt{x^2-1}} \tag{64}$$

$$\frac{\mathrm{d}\cot^{-1}(x)}{\mathrm{d}x} = -\frac{1}{x^2+1} \tag{65}$$

$$\frac{\mathrm{d}\sinh(x)}{\mathrm{d}x} = \cosh(x) \tag{66}$$

$$\frac{\mathrm{d}\cosh(x)}{\mathrm{d}x} = \sinh(x) \tag{67}$$

$$\frac{\mathrm{d}\tanh(x)}{\mathrm{d}x} = \mathrm{sech}^2(x) \tag{68}$$

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \tag{69}$$

$$\frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} \tag{70}$$

$$\frac{\mathrm{d}fg}{\mathrm{d}x} = \frac{\mathrm{d}g}{\mathrm{d}x} f + \frac{\mathrm{d}f}{\mathrm{d}x} g \tag{71}$$

$$\frac{\mathrm{d}\frac{f}{g}}{\mathrm{d}x} = \frac{\frac{\mathrm{d}f}{\mathrm{d}x} g - \frac{\mathrm{d}g}{\mathrm{d}x} f}{g^2} \tag{72}$$

$$f(x) = f(a) + \sum_{n=1}^{\infty} \frac{\mathrm{d}^n f}{\mathrm{d}x^n}\Big|_{x=a} \frac{(x-a)^n}{n!} \tag{73}$$

The following can be generalized to more variables:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \tag{74}$$

For a function z, such that $z = f(x,y)$, if $z = C$

$$\frac{\partial x}{\partial y} = \left(\frac{\partial y}{\partial x}\right)^{-1} \tag{75}$$

2.6.1 Lagrange multipliers

For a function $f(\vec{x})$ with restrains $g(\vec{x}) = c$, then to find the stationary points, solve:

$$\begin{gathered} \nabla f = \lambda \nabla g \\ g(\vec{x}) = c \end{gathered} \tag{76}$$

2.6.2 Feynman's integration trick

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(x,t)\mathrm{d}t = \int_a^b \frac{\partial}{\partial x} f(x,t)\mathrm{d}t \tag{77}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_v^u f(x,t)\mathrm{d}t = f(x,u)\frac{\mathrm{d}u}{\mathrm{d}x} - f(x,v)\frac{\mathrm{d}v}{\mathrm{d}x} + \int_v^u \frac{\partial}{\partial x} f(x,t)\mathrm{d}t \tag{78}$$

2.6.3 Implicit partial differentiation

DON'T TRY TO USE IMPLICIT INTEGRATION WITH PARTIAL DERIVATIVES THE WAY YOU DO WITH TOTAL DERIVATIVES. IT IS A HUGE MISTAKE. USE DIFFERENTIALS INSTEAD. IT IS ALSO ALWAYS HELPFUL TO THINK IN TERMS OF OPERATORS.

2.6.4 Finding maxima, minima and saddle points

Consider the matrix:

$$M_{ij} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f \tag{79}$$

Then, if the eigenvalues of M are positive, we have a minimum. If the eigenvalues of M are negative, we have a maximum. If the eigenvalues of M have mixed signs, then we have a saddle point.

2.7 Integration

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \tag{80}$$

$$\int e^x \mathrm{d}x = e^x + C \tag{81}$$

$$\int x^a \mathrm{d}x = \frac{x^{a+1}}{a} + C \tag{82}$$

$$\int a^x \mathrm{d}x = \frac{a^x}{\ln a} + C \tag{83}$$

$$\int \frac{1}{x}\mathrm{d}x = \ln|x| + C \tag{84}$$

$$\int \ln(x)\mathrm{d}x = x\ln(x) - x + C \tag{85}$$

$$\int \log_a(x)\mathrm{d}x = \frac{x\ln(x)-x}{\ln(a)} + C \tag{86}$$

$$\int \sin(x)\mathrm{d}x = -\cos(x) + C \tag{87}$$

$$\int \cos(x)\mathrm{d}x = \sin(x)+C \tag{88}$$

For more trig integrals refer to the differentiation section.

$$\int \tan(x)\mathrm{d}x = -\ln|\cos(x)|+C \tag{89}$$

$$\int \sec(x)\mathrm{d}x = \ln|\tan(x)+\sec(x)|+C \tag{90}$$

$$\int \csc(x)\mathrm{d}x = -\ln|\cot(x)+\csc(x)|+C \tag{91}$$

$$\int \cot(x)\mathrm{d}x = \ln|\sin(x)|+C \tag{92}$$

$$\int \cos^{-1}\mathrm{d}x = x\cos^{-1}(x)-\sqrt{1-x^2}+C \tag{93}$$

$$\int \sin^{-1}\mathrm{d}x = x\sin^{-1}(x)+\sqrt{1-x^2}+C \tag{94}$$

Using Feynman's integration trick it is easy to show that:

$$\int_0^\infty \frac{\sin(x)}{x}\mathrm{d}x = \frac{\pi}{2} \tag{95}$$

$$\int_{-\infty}^{+\infty} e^{-x^2} = \sqrt{\pi} \tag{96}$$

$$\int f\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int f\mathrm{d}u \tag{97}$$

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u \tag{98}$$

2.7.1 The Gamma Function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t \text{ for } \alpha > 0 \tag{99}$$

2.7.2 Average Value of a Function

$$\langle f\rangle = \frac{1}{A}\int_\Omega f\mathrm{d}A \tag{100}$$

(Can be extendend to more dimensions)

2.7.3 Jacobians and Change of Variables

$$J = \frac{\partial(x,y,z,\dots)}{\partial(u,v,w,\dots)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} & \dots \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \tag{101}$$

$$\mathrm{d}A_{x,y} = |J|\mathrm{d}A_{u,v} \tag{102}$$

$$\frac{\partial(x_1,x_2,\dots)}{\partial(y_1,y_2,\dots)} = \frac{\partial(x_1,x_2,\dots)}{\partial(z_1,z_2,\dots)}\frac{\partial(z_1,z_2,\dots)}{\partial(y_1,y_2,\dots)} \tag{103}$$

$$\frac{\partial(x_1,x_2,\dots)}{\partial(y_1,y_2,\dots)} = \left(\frac{\partial(y_1,y_2,\dots)}{\partial(x_1,x_2,\dots)}\right)^{-1} \tag{104}$$

2.8 Differentials

The formulas in this subsection use Einstein Summation Convention

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i \tag{105}$$

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta \tag{106}$$

$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z = \rho^2\sin(\phi)\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta \tag{107}$$

A differential $A\mathrm{d}x + B\mathrm{d}y$ is exact, iff

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \tag{108}$$

2.9 Operators

The Laplacian in 2 dimensions:

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial\theta^2} \tag{109}$$

2.10 Vectors

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} \tag{110}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c} \tag{111}$$

$$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a} \tag{112}$$

2.11 Tensors

Derivitive, with respect to time, of a rank 1 tensor:

$$\frac{\mathrm{d}\overline{T}^\alpha}{\mathrm{d}t} = \frac{\mathrm{d}T^\beta}{\mathrm{d}t}\frac{\partial\overline{x}^\alpha}{\partial x^\beta}+T^\beta\frac{\partial}{\partial x^\gamma}\frac{\partial\overline{x}^\alpha}{\partial x^\beta}\frac{\mathrm{d}x^\gamma}{\mathrm{d}t} \tag{113}$$

A $\binom{M}{N}$ tensor is a linear mapping from M one-forms and N vectors to a real number.

$$T^\alpha_\beta = \mathbf{T}(\tilde{\omega}^\alpha,\vec{e}_\beta) \tag{114}$$

$$\frac{\partial\overline{x}^\alpha}{\partial x^\beta} = \Lambda^\alpha_\beta \tag{115}$$

$$\frac{\partial x^\beta}{\partial\overline{x}^\alpha} = (\Lambda^{-1})^\beta_\alpha \tag{116}$$

$$\overline{A}^\alpha = \Lambda^\alpha_\beta A^\beta \tag{117}$$

$$\overline{\vec{e}}_\alpha = (\Lambda^{-1})^\beta_\alpha\vec{e}_\beta \tag{118}$$

$$\overline{V}_\alpha = (\Lambda^{-1})^\beta_\alpha V_\beta \tag{119}$$

$$\overline{\tilde{\omega}}^\alpha = \Lambda^\alpha_\beta\tilde{\omega}^\beta \tag{120}$$

$$V_\alpha = g_{\alpha\beta}V^\beta \tag{121}$$

$$V^\alpha = g^{\alpha\beta}V_\beta \tag{122}$$

$$\phi_{,\gamma} \rightarrow \{\frac{\partial\phi}{\partial x^\gamma}\} \tag{123}$$

2.12 Fourier Series

2.12.1 The Dirichlet conditions

- The function must be periodic;
- it must be single-valued and continuous, except possibly at a finite number of finite discontinuities over the period;
- it must have only a finite number of maxima and minima within one period;
- the integral over one period of $|f(x)|$ must converge.

If this conditions are met, then the Fourier series of f converge where f is continuous.

2.12.2 The Fourier coefficients

$$f(x) = \frac{a_o}{2} + \sum_{r=1}^{\infty} a_r \cos\left(\frac{2\pi r x}{T}\right) + b_r \sin\left(\frac{2\pi r x}{T}\right) \tag{124}$$

$$a_r = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\cos\left(\frac{2\pi r x}{T}\right)\mathrm{d}x \tag{125}$$

$$b_r = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\sin\left(\frac{2\pi r x}{T}\right)\mathrm{d}x \tag{126}$$

Note: When there are discontinuities in the period, the Fourier series for the function does not converge on that point, but halfway.

2.12.3 Integration and Differentiation of Fourier Series

One can integrate the series of a function to get its integral and the same for the derivative. In the derivative case, $f'(x)$ must satisfy the Dirichlet conditions. If $f(x)$ is a continuous function of x for all x and $f(x)$ is also periodic then the Fourier series that results from differentiating term by term converges to $f'(x)$, provided that $f'(x)$ itself satisfies the Dirichlet conditions.

2.12.4 Complex Fourier Series

One can write the Fourier Series for Complex functions in the following way:

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{\frac{2\pi}{T} irx} = \sum_{r=-\infty}^{\infty} c_r \exp\left(\frac{2\pi}{T} irx\right) \tag{127}$$

where

$$c_r = \frac{1}{T}\int_{x_0}^{x_o+T} f(x) e^{-\frac{2\pi}{T} irx}\mathrm{d}x \tag{128}$$

The relationship between a_r, b_r and c_r is given by:

$$\begin{aligned} c_r &= \frac{1}{2}(a_r - ib_r) \\ c_{-r} &= \frac{1}{2}(a_r + ib_r) \end{aligned} \tag{129}$$

There is one more regard worth mentioning. If f is real, then $c_{-r} = \overline{c_r}$

2.12.5 Parseval's theorem

$$\frac{1}{T}\int_{x_0}^{x_o+T} |f(x)|^2 \mathrm{d}x = \sum_{r=-\infty}^{\infty} |c_r|^2 \tag{130}$$

2.13 Fourier Transform

The Fourier transform is a type of integral transformation given by the following:

$$\tilde{f}(\omega) = \mathcal{F}[f(x)](\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-i\omega x}\mathrm{d}x \tag{131}$$

And its inverse:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega x}\mathrm{d}\omega \tag{132}$$

2.13.1 Delta Functions

$$\delta(t) = 0 \forall t \neq 0 \tag{133}$$

$$\int_{\Omega} f(t)\delta(t-a) = f(a) \text{ if } a \in \Omega \tag{134}$$

$$\delta(t) = \delta(-t) \tag{135}$$

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{136}$$

$$t\delta(t) = 0 \tag{137}$$

$$\int_{-\infty}^{\infty} f(t)\frac{\mathrm{d}\delta(t-a)}{\mathrm{d}t}\mathrm{d}t = -\frac{\mathrm{d}f}{\mathrm{d}t}|_{t=a} \tag{138}$$

2.13.2 Properties of the Fourier Transform

Here are some of the properties of the Fourier Transform:

$$\mathcal{F}[f'(x)](\omega) = i\omega\tilde{f}(\omega) \tag{139}$$

$$\mathcal{F}\left[\int f(t)\mathrm{d}t\right](\omega) = \frac{1}{i\omega}\tilde{f}(\omega) + 2\pi c\delta(\omega) \tag{140}$$

$$\mathcal{F}[f(at)](\omega) = \frac{1}{a}\tilde{f}(\frac{\omega}{a}) \tag{141}$$

$$\mathcal{F}[f(t+a)](\omega) = e^{ia\omega}\tilde{f}(\omega) \tag{142}$$

$$\mathcal{F}[e^{\alpha t} f(t)] = \omega \tilde{+} i\alpha \tag{143}$$

2.14 Convolutions

$$(f*g)(z) = \int_{-\infty}^{\infty} f(x)g(z-x)\mathrm{d}x \tag{144}$$

This operation is commutative, associative and distributive.

$$\mathcal{F}[f(x)g(x)](\omega) = \frac{1}{\sqrt{2\pi}}(\tilde{f}*\tilde{g})(\omega) \tag{145}$$

$$\mathcal{F}[(f*g)(x)](\omega) = \sqrt{2\pi}\tilde{f}(\omega)\tilde{g}(\omega) \tag{146}$$

2.15 Correlation Functions

$$(f\otimes g)(z) = \int_{-\infty}^{\infty} \overline{f(x)} g(x+z)\mathrm{d}x \tag{147}$$

This operation is associative and distributive, but not commutative. In fact,

$$(f\otimes g)(z) = \overline{(g\otimes f)}(-z) \tag{148}$$

$$\mathcal{F}[(f\otimes g)(x)](\omega) = \sqrt{2\pi}\overline{\tilde{f}(\omega)}\tilde{g}(\omega) \tag{149}$$

$$\mathcal{F}[\overline{f(x)}g(x)] = \frac{1}{\sqrt{2\pi}}((\tilde{f}\otimes\tilde{g})(\omega)) \tag{150}$$

2.16 Higher Dimensions of Fourier Series

$$\tilde{f}(\mathbf{w}) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{w}\cdot\mathbf{x}}\mathrm{d}^n\mathbf{x} \tag{151}$$

$$f(\mathbf{x}) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} \tilde{f}(\mathbf{w}) e^{i\mathbf{w}\cdot\mathbf{x}}\mathrm{d}^n\mathbf{w} \tag{152}$$

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\mathbf{w}\cdot\mathbf{x}}\mathrm{d}^n\mathbf{w} \tag{153}$$

2.17 Laplace Transform

$$\overline{f}(s) = \mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t \tag{154}$$

2.17.1 Standard Laplace Transforms

$$\mathcal{L}[c](s) = \frac{c}{s} \tag{155}$$

$$\mathcal{L}[ct^n](s) = \frac{cn!}{s^{n+1}} \tag{156}$$

$$\mathcal{L}[\sin(bt)](s) = \frac{b}{s^2+b^2} \tag{157}$$

$$\mathcal{L}[\cos(bt)](s) = \frac{s}{s^2+b^2} \tag{158}$$

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a} \tag{159}$$

$$\mathcal{L}[t^n e^{at})](s) = \frac{n!}{(s-a)^{n+1}} \tag{160}$$

$$\mathcal{L}[\sinh(at)](s) = \frac{a}{s^2-b^2} \tag{161}$$

$$\mathcal{L}[\cosh(at)](s) = \frac{s}{s^2-b^2} \tag{162}$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \tag{163}$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \tag{164}$$

$$\mathcal{L}[t^{\frac{1}{2}}](s) = \frac{1}{2}\left(\frac{\pi}{s^3}\right)^{\frac{1}{2}} \tag{165}$$

$$\mathcal{L}[t^{-\frac{1}{2}}](s) = \left(\frac{\pi}{s}\right)^{\frac{1}{2}} \tag{166}$$

$$\mathcal{L}[\delta(t-a)](s) = e^{-sa} \tag{167}$$

$$\mathcal{L}[H(t-a)](s) = e^{-sa}/s \tag{168}$$

The region of convergence is $\mathbb{R}^+$ for all, except for (159), (160), (161), (162), (163) and (164). For (161) and (162) it is $|a|$, for the remaining, it is a.

2.17.2 Derivatives and Integrals

$$\mathcal{L}[\frac{\mathrm{d}^n f}{\mathrm{d}t^n}](s) = s^n\overline{f} - s^{n-1}f(0) - s^{n-2}\frac{\mathrm{d}f}{\mathrm{d}t}(0) - \cdots - \frac{\mathrm{d}^{n-1}f}{\mathrm{d}t^{n-1}}(0) \tag{169}$$

$$\mathcal{L}\left[\int_0^t f(u)\mathrm{d}u\right](s) = \frac{1}{s}\mathcal{L}[f(u)](s) \tag{170}$$

2.17.3 Properties of Laplace Transform

$$\mathcal{L}[e^{at}f(t)](s) = \overline{f}(s-a) \tag{171}$$

$$\mathcal{L}[f(at)](s) = \frac{1}{a}\overline{f}(\frac{s}{a}) \tag{172}$$

$$\mathcal{L}[t^n f(t)](s) = (-1)^n\frac{\mathrm{d}^n\overline{f}}{\mathrm{d}s^n} \tag{173}$$

$$\mathcal{L}\left[\frac{f(t)}{t}\right](s) = \int_s^\infty \overline{f}(u)\mathrm{d}u \tag{174}$$

3 Differential Equations

There are many ways one can solve differential equations in practise.

3.1 Separation of Variables

Simple to do. Easy to understand.

3.2 Exact Equations

$$A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y = 0 \tag{175}$$

Just do what you would do if this was the differential of a multi-variable function.

3.3 Linear Equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x) \tag{176}$$

To solve this, multiply both sides by the integrating factor $\mu(x)$

$$\mu(x) = \exp\left[\int P(x)\mathrm{d}x\right] \tag{177}$$

Then you can simplify the LHS to a single integral $(\frac{\mathrm{d}y\mu(x)}{\mathrm{d}x})$.

3.4 Homogeneous Equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(\frac{y}{x}) \tag{178}$$

Make the substitution $y = vx$ and solve by the method of separable equations.

3.5 Constant Coefficient Equations

$$\sum_{n=0}^N a_n\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = F(x) \tag{179}$$

First solve the equation (find the roots)

$$\sum_{n=0}^N a_n z^n = 0 \tag{180}$$

The complementary solution is given by

$$y_c = \sum_{i=1}^m\left[e^{\lambda_i x}\sum_{p=0}^{k_i-1} c_{ip}x^p\right] \tag{181}$$

Where m is the number of roots, λ_i are the different roots and k_i is the multiplicity of the ith root.

To find the particular solution, try:

- If $F(x) = ae^{rx}$, the $y_p = be^{rx}$.
- If $F(x) = a_1\sin(rx) + a_2\cos(rx)$, the $y_p = b_1\sin(rx) + b_2\cos(rx)$.
- If F is a polynomial, then try a polynomial of the same degree.

3.6 Non Constant Coeficient Equations

$$a_n(\alpha x+\beta)^n\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \cdots + a_1(\alpha x+\beta)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x) \tag{182}$$

Just make the substitution $\alpha x + \beta = e^t$. The equation will simplify to a solvable form.

3.7 Exact Equations

Imagine we have an equation of the type:

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \cdots + a_0(x)y = f(x) \tag{183}$$

Then if

$$a_0(x) - D_x a_1(x) + D_x^2 a_2(x) - \cdots + (-1)^n D_x^n a_n(x) = 0 \tag{184}$$

The equation is called exact, which means that we can factor out a derivative, that is, the equation in (183) can be written as

$$D_x[b_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + b_0(x)y] = f(x) \tag{185}$$

The integrate both sides to remove the derivative. Repeat the process if possible.

3.8 Partically known complemetary function

If we know $u(x)$ is a solution to a differential equation of the type of (183), then we may make the substitution $y = u(x)v(x)$ and the equation might prove to be solvable.

3.9 Variation of parameters

4 Physics

4.1 Center of mass

$$\bar{x} = \frac{1}{M}\int_\Omega x\mathrm{d}M \tag{186}$$

Or, more generally:

$$\bar{\vec{r}} = \frac{1}{M}\int_\Omega \vec{r}\mathrm{d}M \tag{187}$$

4.2 Moment of Inertia

$$I = \int_\Omega l^2 \mathrm{d}M \tag{188}$$

4.3 Newton's Law

$$\vec{F} = m\vec{a} \tag{189}$$

$$||\vec{F_g}|| = \frac{GM_1M_2}{r^2} \tag{190}$$

$$\vec{F_g} = -\frac{GM_1M_2\vec{r}}{||\vec{r}||^3} \tag{191}$$

4.4 Quantum

4.4.1 Time invariant one dimensional Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V\psi = E\psi \tag{192}$$

4.5 Maxwell Equations

$$\nabla \cdot E = \rho \tag{193}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{194}$$

$$\nabla \cdot B = 0 \tag{195}$$

$$c^2 \nabla \times B = j + \frac{\partial E}{\partial t} \tag{196}$$

5 Linear Algrebra

5.1 Generalized Eigenvectors

Given two matrices S and M (M is positive definite), we define generalized eigenvectors and generalized eigenvalues:

$$Sx = \lambda Mx \tag{197}$$

The eigenvectors are the ones that solve the following problem:

$$Hy = \lambda y \text{ where } H = M^{-\frac{1}{2}}SM^{-\frac{1}{2}} \text{ and } y = M^{\frac{1}{2}}x \tag{198}$$

6 Probabilities and Statistics

Any probability function must satisfy the following conditions (199) and (200):

$$P(\Omega) = 1 \tag{199}$$

$$P(A \cup B) = P(A) + P(B) \text{ ,for } A \cap B = \{\} \tag{200}$$

Also, here are De Morgan's laws

$$(A \cup B)^C = A^C \cap B^C \tag{201}$$

$$(A \cap B)^C = A^C \cup B^C \tag{202}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{203}$$

Here is the law of total probability:
For disjoint events $C_1, ..., C_n$, such that $C_1 \cup \cdots \cup C_n = \Omega$

$$P(A) = P(A|C_1)P(C_1) + \cdots + P(A|C_n)P(C_n) \tag{204}$$

For disjoint events $C_1, ..., C_n$, such that $C_1 \cup \cdots \cup C_n = \Omega$

$$P(C_i|A) = \frac{P(A|C_i)P(C_i)}{P(A|C_1)P(C_1) + \cdots + P(A|C_n)P(C_n)} \tag{205}$$

The event A is independent of B if:

$$P(A|B) = P(A) \tag{206}$$

6.1 Random variables

The *probability mass function* of a discrete random variable X is the function:

$$p_X(x) = P(X = x) \tag{207}$$

The *distribution function* of a random variable X (not necessarily discrete) is the function F_X:

$$F_X(x) = P(X \leq x) \tag{208}$$

Moreover, for *discrete random variables*:

$$F_X(a) = \sum_{a_i \leq a} p_X(a_i) \tag{209}$$

For *continuous random variables*, we have the probability density function f_X:

$$P(a \leq X \leq b) = \int_a^b f_X(x)\mathrm{d}x \tag{210}$$

One can also write the *probability distribution function* F_X:

$$F_X(x) = \int_{-\infty}^x f_X(x)\mathrm{d}x \tag{211}$$

From this one can deduce the following proposition:

$$\frac{\mathrm{d}F_X(x)}{\mathrm{d}x} = f_X(x) \tag{212}$$

6.1.1 Discrete probability distributions

The Bernoulli distribution

$$X \hookrightarrow \mathrm{Ber}(p)$$

For $p \in [0,1]$

$$\begin{aligned} p_X(1) &= p \\ p_X(0) &= 1-p \end{aligned} \tag{213}$$

The Binomial distribution

$$X \hookrightarrow \mathrm{Bin}(n,p)$$

For $k \in \{1,2,3,\dots,n\}$
For $n \in \mathbb{N}$
For $p \in [0,1]$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{214}$$

The Geometric distribution

$$X \hookrightarrow \mathrm{Geo}(p)$$

For $p \in [0,1]$
For $k \in \mathbb{N}$

$$p_X(k) = (1-p)^{k-1} p \tag{215}$$

The Poisson distribution

$$X \hookrightarrow \mathrm{Pois}(\mu)$$

For $\mu > 0$
For $k \in \mathbb{N}_0$

$$f_X(k) = \frac{\mu^k}{k!} e^{-\mu} \tag{216}$$

6.1.2 Continuous random variables distributions

There are also many continuous random variable distribution. Remember that, for every distribution, we have that

$$\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1 \tag{217}$$

$$f_X(x) \geq 0 \ \forall x \in \mathbb{R} \tag{218}$$

The Uniform distribution

$$X \hookrightarrow \mathrm{U}(\alpha,\beta)$$

For $\alpha < \beta$
For $x \in [\alpha,\beta]$

$$f_X(x) = \frac{1}{\beta-\alpha} \tag{219}$$

The Exponential distribution

$$X \hookrightarrow \mathrm{Exp}(\lambda)$$

For $\lambda > 0$
For $x \geq 0$

$$f_X(x) = \lambda e^{-\lambda x} \tag{220}$$

The Pareto distribution

$$X \hookrightarrow \mathrm{Par}(\alpha)$$

For $\alpha > 0$
For $x \geq 1$

$$f_X(x) = \frac{\alpha}{x^{\alpha+1}} \tag{221}$$

The following distribution is one of the most important in probability and statistics.

The Normal distribution

$$X \hookrightarrow \mathrm{N}(\mu,\sigma^2)$$

For $\sigma^2 > 0$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \tag{222}$$

The Log-Normal distribution

$$X \hookrightarrow \mathrm{LN}(\mu,\sigma^2)$$

For $\sigma^2 > 0$
For $x \geq 0$

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(\ln(x)-\mu)^2}{\sigma^2}} \tag{223}$$

There is one special case of the normal distribution with $\mu = 1$ and $\sigma^2 = 1$, which is called the *standard normal distribution*:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \tag{224}$$

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\mathrm{d}x \tag{225}$$

The Gamma distribution

$$X \hookrightarrow \mathrm{Gam}(\alpha,\lambda)$$

For $\alpha > 0$
For $\lambda > 0$
For $x \geq 0$

$$f_X(x) = \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \tag{226}$$

6.2 Quantiles

Let X be a *continuous random variable* and let p be a number between 0 and 1. The pth quantile or 100pth percentile of the distribution of X is the smallest number q such that

$$F(q) = P(X \leq q) = p$$

. The median of a distribution is its 50th percentile.

6.3 Mean and variance

For a *discrete random variable*

$$\langle X \rangle = \sum_i a_i p_X(a_i) \tag{227}$$

For a *continuous random variable*

$$\langle X \rangle = \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x \tag{228}$$

Important remark: The mean may not exist

- The mean of a geometric distribution is $\frac{1}{p}$
- The mean of an exponential distribution is $\frac{1}{\lambda}$
- The mean of a normal distribution is μ
- The mean of a poisson distribution is μ

$$\langle g(X)\rangle = \sum_i g(a_i)p_X(a_i) \tag{229}$$

$$\langle g(X)\rangle = \int_{-\infty}^{\infty} g(x)f_X(x)\mathrm{d}x \tag{230}$$

$$\mathrm{Var}(X) = \left\langle (X-\langle X\rangle)^2\right\rangle = \left\langle X^2\right\rangle - \langle X\rangle^2 \tag{231}$$

The variance of a normal distribution is σ^2
The variance of a poisson distribution is μ

$$\langle rX+t\rangle = r\langle X\rangle + t \tag{232}$$

$$\mathrm{Var}(rX+t) = r^2\,\mathrm{Var}(X) \tag{233}$$

For two indepenent random variables,

$$\langle XY\rangle = \langle X\rangle\langle Y\rangle \tag{234}$$

6.4 Jensen's Inequality

$$g(\langle X\rangle) \le \langle g(X)\rangle \tag{235}$$

6.5 Change of Variable Theorem

Supppose X and Y are random variables such that

$$P(X\le x) = P(Y \le h(x)) \tag{236}$$

Then,

$$f_X(x) = f_Y(h(x))\frac{\mathrm{d}h}{\mathrm{d}x} \tag{237}$$

6.6 Moment Generating Function

The h-th moment of a function is given by

$$m_k = \left\langle X^k\right\rangle \tag{238}$$

Its moment generation function is given by

$$M_X(t) = \left\langle e^{tX}\right\rangle \tag{239}$$

On top of that, the k-th derivitive at 0 of the moment generating function is equal to the t-th moment

$$\frac{\mathrm{d}^k M_X}{\mathrm{d}t^k}|_{t=0} = m_k \tag{240}$$

There is also an interesting result regarding the moment generatin function: If two distributions have the same moment generating function, then they have the same distribution.

Note: This does not necessarly mean that two distributions with the same h-th moments, then they are the same!

6.7 Joint distributions

$$p_{XY}(x,y) = P(X=x, Y=y) \tag{241}$$

$$F_{XY}(x,y) = P(X\le x, Y\le y) \tag{242}$$

For a continuous random variable, we have the following:

$$P(a_1\le X\le a_2, b_1\le X\le b_2) = \int_{a_1}^{a_2}\int_{b_1}^{b_2} f_{XY}(x,y)\mathrm{d}x\mathrm{d}y \tag{243}$$

$$f_{XY}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F(x,y) \tag{244}$$

$$F(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f_{XY}(x,y)\mathrm{d}x\mathrm{d}y \tag{245}$$

We can also go from joint probability to marginal probability (from f_{XY} to f_X or f_Y).

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\mathrm{d}y \tag{246}$$

$$f_X(x) = \sum_i p_{XY}(x,b_i) \tag{247}$$

6.8 Independence of random variables

Two random variables are independence iff

$$F_{XY}(a,b) = F_X(a)F_Y(b) \tag{248}$$

One result worth stating is that of the propagation of independence:
Let $X_1, X_2\ldots, X_n$ be independent random variables. For each i, let $h_i:\mathbb{R}\longrightarrow\mathbb{R}$ be a function and define the random variable $Y_i = h_i(X_i)$.
Then $Y_1, Y_2, \ldots, Y_n$ are also independent

6.9 Covariance and Correlation

For random variables X and Y we have:

$$\langle g(X,Y)\rangle = \sum_i\sum_j g(a_i,b_j)p_{XY}(a_i,b_i) \tag{249}$$

$$\langle g(X,Y)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)p_{XY}(x,y)\mathrm{d}x\mathrm{d}y \tag{250}$$

Also, the average operator is linear, as we have already seen.

$$\langle rX+sY+t\rangle = r\langle X\rangle + s\langle Y\rangle + t \tag{251}$$

6.9.1 Covariance

$$\mathrm{Cov}(X,Y) = \langle XY\rangle - \langle X\rangle\langle Y\rangle \tag{252}$$

Note: If two random variables are independent, then they are not correlated, therefore, Cov$(X,Y)=0$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y) \tag{253}$$

$$\mathrm{Cov}(rX+s, tY+u) = rt\mathrm{Cov}(X,Y) \tag{254}$$

The correlation coefficient can be written as

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} \tag{255}$$

The correlation coefficient is independent of units.

6.10 Computations with random variables

For more reference on sum, difference, product and quotient of random variables go check out chapter 11 of [**?**]

6.11 Law of Large Numbers

$$\overline{X}_n = \frac{X_1+\cdots+X_n}{n} \tag{256}$$

$$\left\langle \overline{X}_n\right\rangle = \langle X_i\rangle \tag{257}$$

$$\mathrm{Var}(\overline{X}_n) = \frac{\mathrm{Var}(X_i)}{n} \tag{258}$$

$$P(|Y-\langle Y\rangle| \ge a) \le \frac{1}{a^2}\mathrm{Var}(Y) \tag{259}$$

Here is the law of large numbers:

$$\lim_{n\to\infty} P(|\overline{X}_n - \langle X\rangle| > \epsilon) = 0 \tag{260}$$

6.12 Central Limit Theorem

Let $X_1,\ldots,X_n$ be independent, identical random variables, with mean μ and variance σ^2

$$Y_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i - \mu \tag{261}$$

The distribution of Y_n converges to the distribution of $N(0,\sigma^2)$

6.13 Stochastic Processes

A stochastic process is a collection of random variables indexed by time. They can be discrete or continuous.

- Discrete: $X_1, X_2, X_3, \ldots$
- Continuous: $\{X_t\}_{t\geq 0}$

There are 3 questions one is interested in studying about stochastic processes.

1. What are the dependencies in the sequence of values (can I predict the future);
2. What is the long term behaviour? (law of large numbers, central limit theorem);
3. What are the boundary events (how often will a stock price drop more that 10% for more that 5 days in a row).

6.13.1 Simple Random Walk

A simple random walk is a stochastic process. Let $Y_1,\ldots$ be random variables such that

$$Y_i = \begin{cases} 1(\text{prob } \frac{1}{2}) \\ -1(\text{prob } \frac{1}{2}) \end{cases}$$

For each t, $X_t = \sum_{i=1}^{t} Y_i$. Then $X_1,\ldots$ is a simple random walk. Properties of a simple random walk:

- $\mathbb{E}[X_k] = 0$
- If $t_0 \leq \cdots \leq t_k$, then $X_{t_{i+1}} - X_{t_i}$ are mutually independent.
- For all $h \geq 1, k \geq 0$, the distribution of $X_{t+h} - X_k$ is the same as the distribution of X_h

6.13.2 Markov Chain

Stochastic process whose effect of the past on the future is summarized only by the current state. A simple random walk is a markov chain.

$$P(X_{t+1} = s|X_0,\ldots,X_t) = P(X_{t+1} = s|X_t) \tag{262}$$

If all X_i values are in S, a finite set, then the information about a markov chain can be summarized into a matrix, the transition probability matrix:

$$P_{ij} = P(X_{t+1} = j|X_t = i) \tag{263}$$

One can multiply the transition probability matrix to get the transition probability matrix of 2 steps, 3 steps, and so on ...

The eigenvector of the eigenvalue 1 of the transition probability matrix is called the stationamry distribution. It is what the distribution of the random process converges to as $t \to \infty$. This eigenvector is garrenteed to exist.

$$P\psi = \psi \tag{264}$$

6.13.3 Martingales

A stochastic process is a martingale iff

$$\mathbb{E}[X_{t+1}|X_0,\ldots,X_t] = X_t \tag{265}$$

There is an interesting theorem that further shows that martingales model far games:

Given a stochastic process, a non-negative integer random variable τ, is called a stopping time, if for all $k \in \mathbb{N}_0, \tau \leq k$ depends only on $X_1,\ldots,X_k$.

Now suppose $X_1,\ldots$ is a martingale and τ is a stopping time. Furthermore, there is a constant T such that $\tau \leq T$. Then $\mathbb{E}[X_\tau] = X_0$

6.14 Regression Analysis

$$Y = X\beta + \epsilon \tag{266}$$

Where Y is a vector random variable and so is ϵ. The actual data one gets for y is just a realization of the random vector Y. The ϵ are the residuals.

6.14.1 Ordinary Least Squares

There are a number of assumptions so that ordinary least squares works right. Those are:

- X has full column rank
- No collinearity between the predictors;
- The residuals are not auto correlated;
- The residuals must be normally distributed;
- The mean of the residuals must be 0;
- The residuals must be independent;
- The residuals must be of constant variance (Make a predicted-value-residuals plot to check this);

Then, the $\hat{\beta}$ for OLS is given by:

$$\hat{\beta} = (X^T X)^{-1} X^T y \tag{267}$$

Note: β is a random variable, while $\hat{\beta}$ is just a realization of that random variable.

One can then use β to estimate the expected values and so get the realization of the residuals.

6.14.2 Generalized Least Squares

This version of least sqaures loosens the requirement that residuals cannot be correlated.

- X has full column rank;
- Expected value of the residuals is 0;
- The residuals can be correlated;

We now define

$$\text{Cov}(\epsilon) = \sigma^2\Sigma \tag{268}$$

The $\hat{\beta}$ is then given by

$$\hat{\beta} = [X^T\Sigma^{-1}X]^{-1}X^T\Sigma^{-1}Y \tag{269}$$

6.14.3 More information on this topic

For a boatload more of this, visiti this site [?]

6.14.4 Brownian Motion

Here are the defining properties of Brownian Motion:

- $\forall 0 \leq s < t : B(t) - B(s) \;\; N(0, t-s)$
- If $[s_i, t_i]$ are not overlapping, then $B(t_i) - B(s_i)$ are independent random variables
- $B(0) = 0$

Interesting facts about Brownian motion

1. Crosses the t-axis infinitely often
2. Does not deviate much from $t = y^2 \Leftrightarrow y = \pm\sqrt{t}$
3. Is nowhere differentiable.

For all $t > 0$ and $a > 0$

$$\mathbb{P}(M(t) > a) = 2\mathbb{P}(B(t) > a) \tag{270}$$

where $M(t) = \max_{s: s \leq t} B(s)$

6.15 Itô's Calculus

7 Logic

$$\neg(\forall p : q) \iff \exists p : \neg q \tag{271}$$

$$\neg(p \implies q) \iff p \land \neg q \tag{272}$$

8 Computing

8.1 Classical Logic Gates

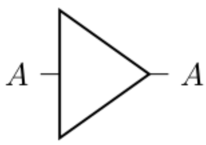

Figure 1: Identity gate

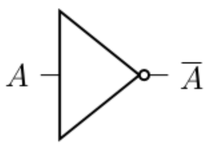

Figure 2: Identity with NOT gate

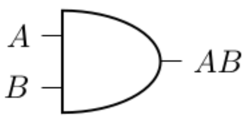

Figure 3: AND gate

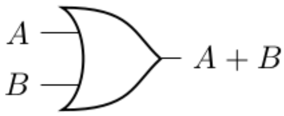

Figure 4: OR gate

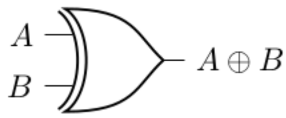

Figure 5: XOR gate

8.2 Quantum Computing

8.2.1 Basics

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{273}$$

$$\langle\psi|\psi\rangle = 1 \tag{274}$$

For quantum gates, we have the following:

$$U^H U = I \tag{275}$$

Identity gate:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{276}$$

Pauli X gate:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{277}$$

Pauli Y gate:

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{278}$$

Pauli Z gate:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{279}$$

Phase S gate:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \tag{280}$$

T gate:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} \tag{281}$$

Hadamard gate:

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{282}$$

8.2.2 Multiple Quantum Gates

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{283}$$

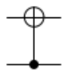

Figure 6: CNOT gate