

WEAK FACTORIZATION

SYSTEMS

Fact: Any map of simplicial sets f ,
can be factored as

$$f = p \circ j$$

where $p \in \text{InnFib} \rightsquigarrow$ inner fibration

$j \in \overline{\text{InnFlor}} \rightsquigarrow$ inner anodyne

This follows from:

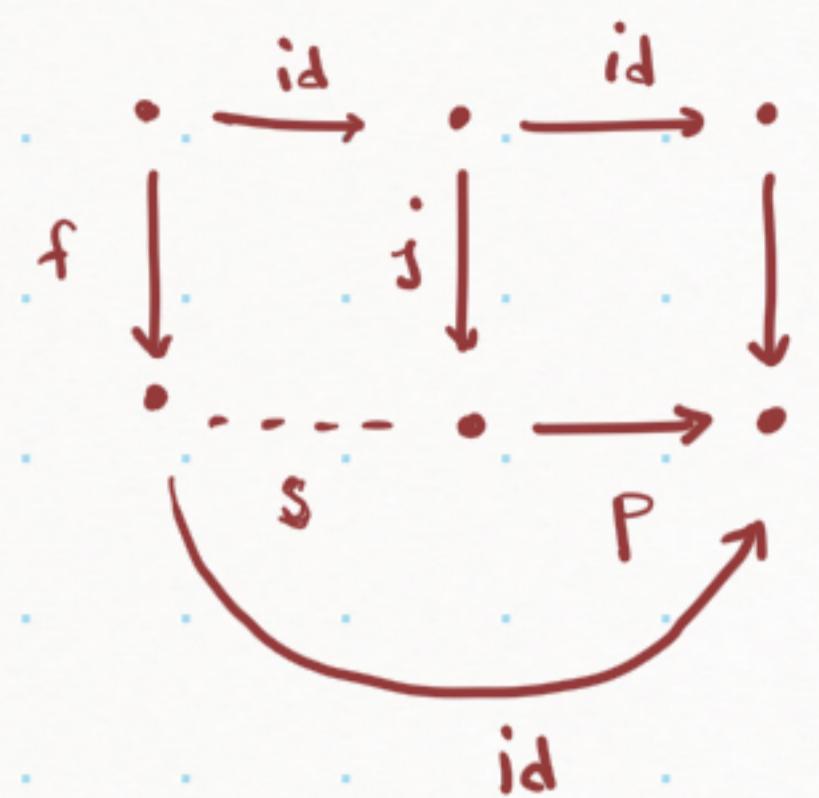
Prop: For any A collection of maps in $s\text{Set}$,

$$\bar{S} = \square(S^\square)$$

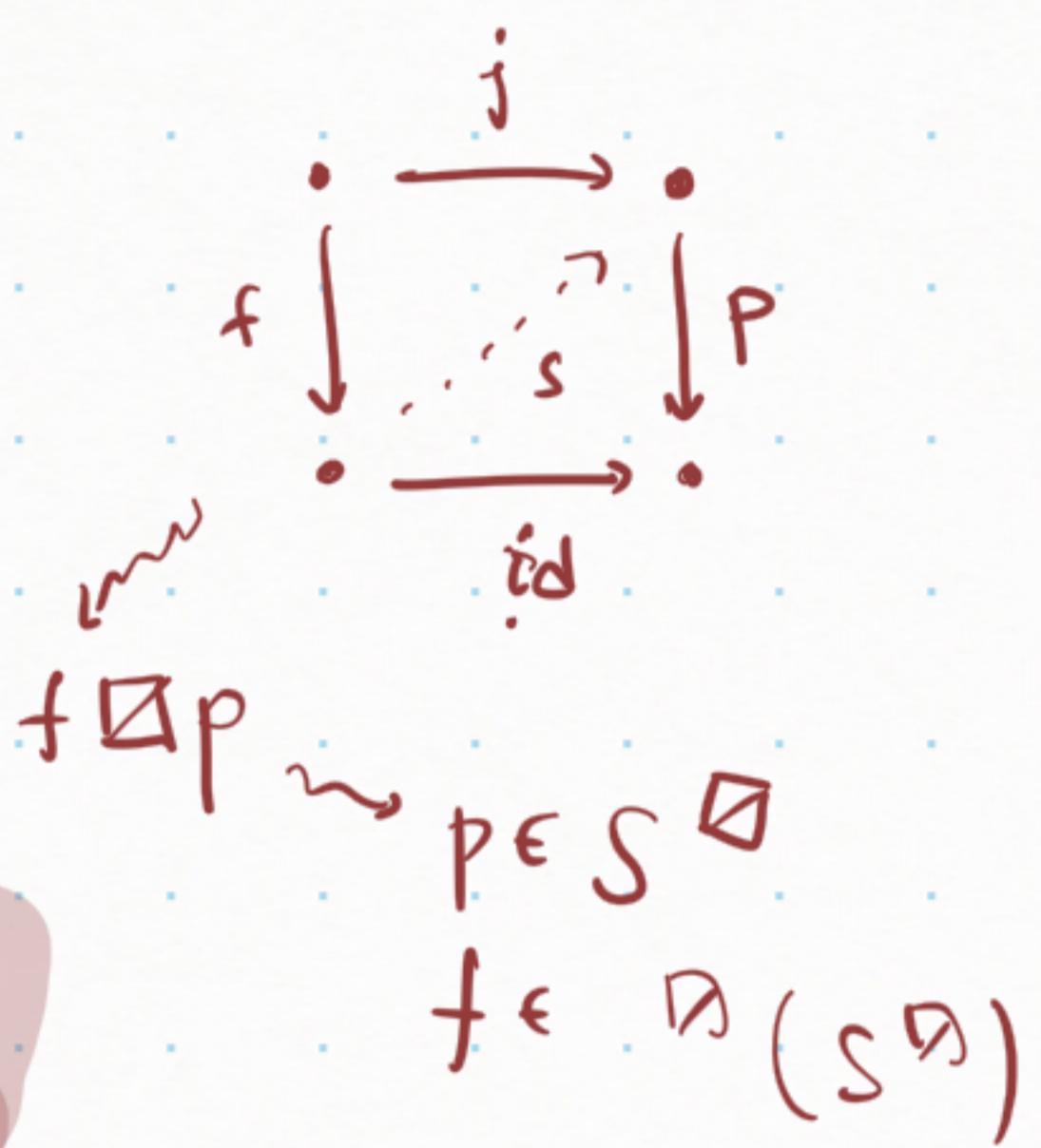
Proof: $\bar{S} \subseteq \square(S^\square)$ from the fact that $\square(-)$
is a weakly saturated class.
and $S \subseteq \square(S^\square)$.

$\bar{S} \supseteq \square(S^\square)$: Consider $f \square S^\square$,

the small object argument tells us we can
choose $j \in \bar{S}, p \in S^\square$ s.t $f = p \circ j$



s exists because



In our case $S = \text{InnHorn}$

$$\overline{\text{InnHorn}} = \square \text{InnFib}$$

&

$$\overline{\text{InnHorn}}^\square = \text{InnFib}$$

The pair $(\overline{\text{InnHorn}}, \text{InnFib})$ is a particular case of a **weak factorization system**.

$\overline{S} = \text{inner-anodyne}$

$S^\square = \text{inner fibrations}$

$$\partial\Delta^n = \bigcup_{\{ij\} \in [n]} \Delta^{[n]}\setminus\{ij\}$$

$$\text{Cell} := \{\partial\Delta^n \subseteq \Delta^n\}_{n \geq 0}$$

$$(\overline{\text{Cell}}, \text{Cell}^\emptyset)$$

$$\text{TrivFib} := \text{Cell}^\square$$

$$A \hookrightarrow X$$

$$X = \underset{k}{\text{colim}} A \cup (S\kappa_k(X))$$

$$A \cup (S\kappa_0(X)) \hookrightarrow A \cup (S\kappa_1(X)) \hookrightarrow \dots$$

$$\overline{\text{Cell}} \subseteq \text{Mors}$$

$$\overline{\text{Cell}} = \text{Mors}$$

$$A \hookrightarrow X \xrightarrow{\sim} A \subseteq X$$

iso.

$$\coprod_{a \in X_k^{\text{nd}} \setminus A_k^{\text{nd}}} \partial\Delta^k \longrightarrow A \cup (S\kappa_{k-1}(X))$$

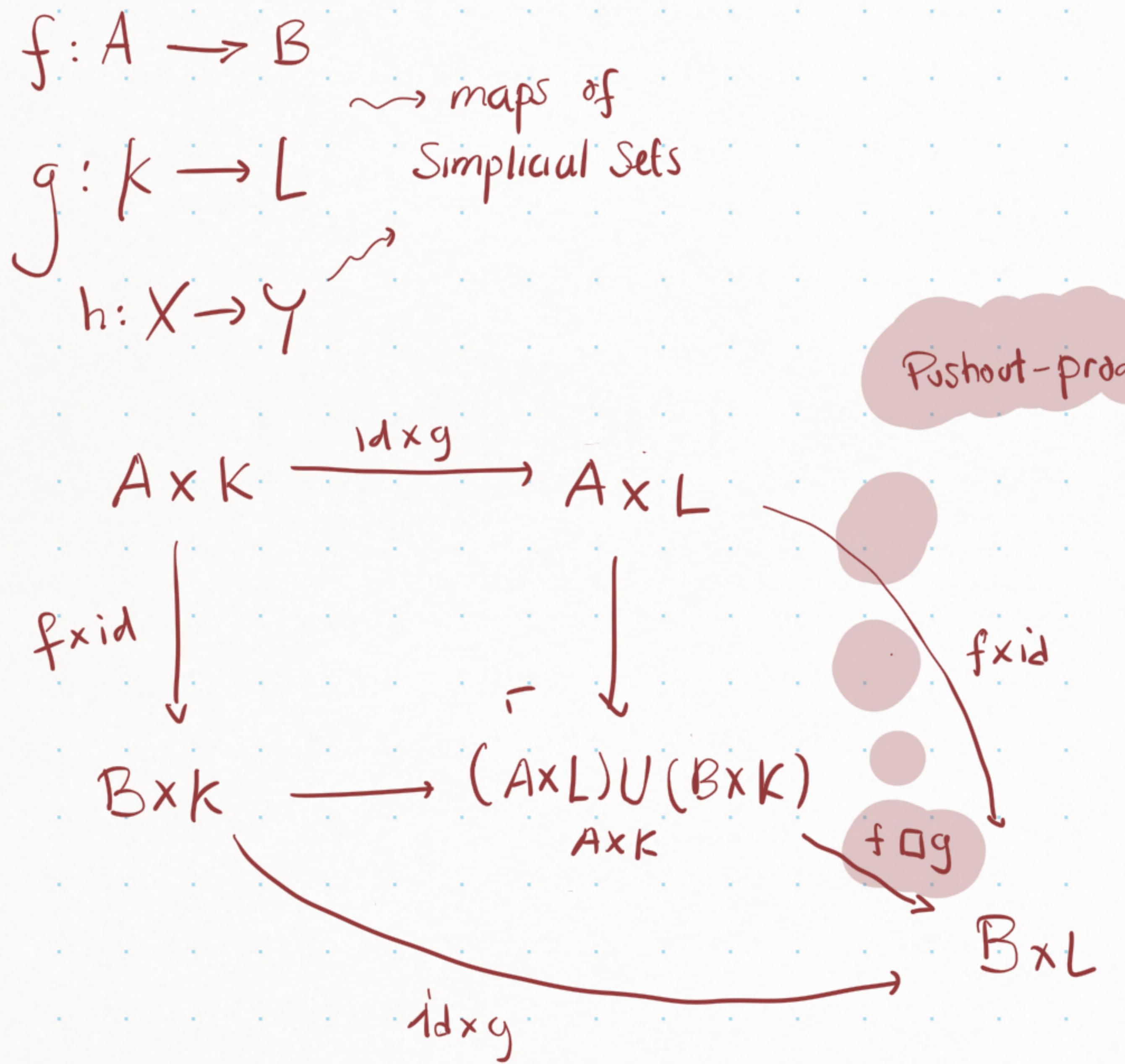
$$\downarrow$$

$$\coprod_{a \in X_k^{\text{nd}} \setminus A_k^{\text{nd}}} \Delta^k$$

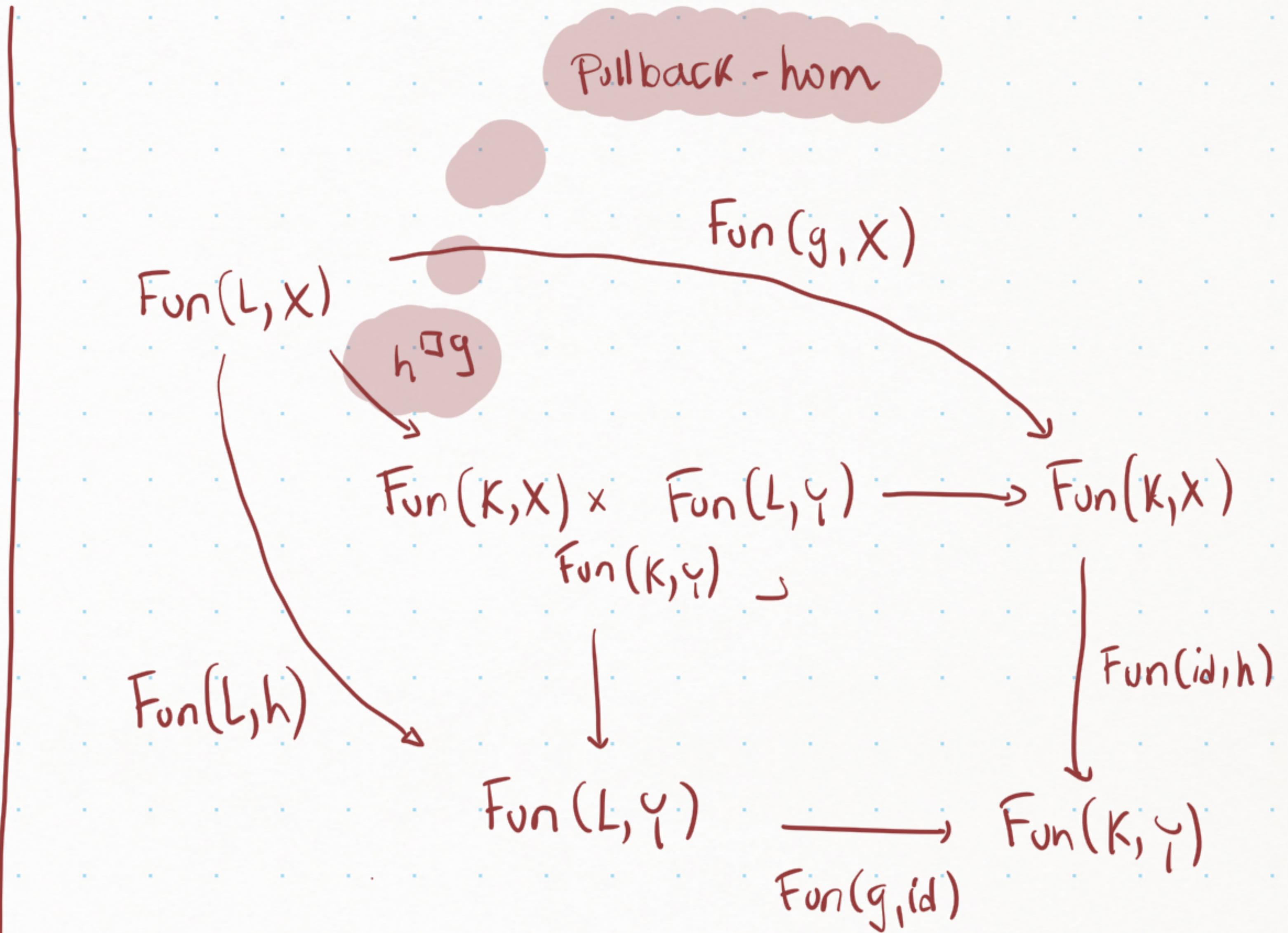
$$\Gamma$$

$$A \cup (S\kappa_k(X))$$

PUSHOUT- PRODUCT & PULLBACK - HOM



Pushout-product



Remarks:

- $h \square g$ \rightsquigarrow enriched lifting ($h \square g$). is the usual lifting set up!
- $\square: \text{Ar}(\text{sSet}) \times \text{Ar}(\text{sSet}) \rightarrow \text{Ar}(\text{sSet})$ defines a symmetric monoidal structure & $- \square g - 1(-)^{\square g}$

ADJUNCTION OF LIFTING PROBLEMS

Prop: $(f \square g) \square h$ iff $f \square (h^{\square g})$

$$\begin{array}{ccc}
 (A \times L) \vee (B \times K) & \xrightarrow{u} & X \\
 f \square g \downarrow \quad \quad \quad \downarrow h & \Rightarrow & f \downarrow \quad \quad \quad \downarrow h^{\square g} \\
 B \times L & \xrightarrow[\omega]{} & Y \\
 & & \\
 A & \xrightarrow{\tilde{u}} & \text{Fun}(L, X) \\
 & & \\
 B & \xrightarrow[(\tilde{v}, \tilde{w})]{} & \text{Fun}(K, X) \times \text{Fun}(L, Y) \\
 & & \text{Fun}(B, Y)
 \end{array}$$

Proof uses the adjunction $- \square g \dashv (-)^{\square g}$, but it's checkable with some thought...

Note: If $A = \emptyset$, $K = \emptyset$ or $Y = *$ hold, then:

for instance if $K = \emptyset$, $Y = *$

$$(A \times L \xrightarrow{f \times L} B \times L) \square (X \rightarrow *) \text{ iff}$$

$$(A \xrightarrow{f} B) \square (\text{Fun}(L, X) \rightarrow *)$$

Prop: S, T any collections of maps

$$\overline{S \sqcup T} \subseteq \overline{S \sqcap T}$$

$$\left. \begin{array}{c} \{ \\ + \\ \end{array} \right\}$$

Lemma: $\overline{\text{InnHorn}} \sqcap \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$

$(\Delta_k^n \subseteq \Delta^n) \sqcap (\partial \Delta^m \subseteq \Delta^m)$ is inner-anodyne

$$\rightsquigarrow \overline{\text{InnHorn}} \sqcap \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$$

Prop: 1) If $i: A \rightarrow B$ is inner-anodyne and

$j: K \hookrightarrow L$ is a monomorphism

$$i \square j: (A \times L) \underset{A \times K}{\cup} (B \times K) \rightarrow B \times L$$

is inner-anodyne

2) If $j: K \hookrightarrow L$ is a monomorphism

