WEAK FACTORIZATION SYSTEMS

Fact: Any map of simplicial sets f, can be factored as

$$f = p \circ j$$

where $p \in \mathrm{InnFib} \rightsquigarrow$ inner fibration.

$j \in \overline{\mathrm{InnHorn}} \rightsquigarrow$ Inner anodyne

This follows from:

Prop: For any A collection of maps in sSet,

$$\bar{S} = {}^{\boxslash}(S^{\boxslash})$$

Proof: $\bar{S} \subseteq {}^{\boxslash}(S^{\boxslash})$ from the fact that ${}^{\boxslash}(-)$ is a weakly saturated class. and $S \subseteq {}^{\boxslash}(S^{\boxslash})$.

$\bar{S} \supseteq {}^{\boxslash}(S^{\boxslash})$: Consider $f \boxslash S^{\boxslash}$, the small object argument tells us we can choose $j \in \bar{S}$, $p \in S^{\boxslash}$ s.t $f = p \circ j$

$$
\begin{tikzcd}
\bullet \arrow[r, "id"] \arrow[d, "f"'] & \bullet \arrow[r, "id"] \arrow[d, "j"] & \bullet \arrow[d] \\
\bullet \arrow[r, dashed, "s"'] \arrow[rr, bend right, "id"'] & \bullet \arrow[r, "p"'] & \bullet
\end{tikzcd}
$$

s exists because

$$
\begin{tikzcd}
\bullet \arrow[r, "j"] \arrow[d, "f"'] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r, "id"'] \arrow[ur, dashed, "s"] & \bullet
\end{tikzcd}
$$

$f \boxslash p \rightsquigarrow p \in S^{\boxslash}$

$f \in {}^{\boxslash}(S^{\boxslash})$

This wizardry has a name: "retract trick"

In our case $S = \text{InnHorn}$

$\bar{S}$ = inner-anodyne

$S^{\boxslash}$ = Inner fibrations

$$\overline{\text{InnHorn}} = {}^{\boxslash}\text{InnFib}$$

&

$$\overline{\text{InnHorn}}^{\boxslash} = \text{InnFib}$$

The pair $(\overline{\text{InnHorn}}, \text{InnFib})$ is a particular case of a weak factorization system

$$\partial\Delta^n = \bigcup_{\{j\} \in [n]} \Delta^{[n] \setminus \{j\}}$$

$$\mathrm{Cell} := \{\partial\Delta^n \subseteq \Delta^n\}_{n \geq 0}$$

$$(\overline{\mathrm{Cell}}, \mathrm{Cell}^{\boxslash})$$

$$\mathrm{TrivFib} := \mathrm{Cell}^{\boxslash}$$

$$A \hookrightarrow X$$

$$X = \operatorname{colim}_k A \cup (\mathrm{Sk}_k(X))$$

$$A \cup (\mathrm{Sk}_0(X)) \hookrightarrow A \cup (\mathrm{Sk}_1(X)) \hookrightarrow \cdots$$

$$\overline{\mathrm{Cell}} \subseteq \mathrm{Mono}$$

$$\boxed{\overline{\mathrm{Cell}} = \mathrm{Mono}}$$

$$A \hookrightarrow X \underset{\text{iso.}}{\rightsquigarrow} A \subseteq X$$

$$\begin{tikzcd}
\coprod_{a \in X_k^{nd} \setminus A_k^{nd}} \partial\Delta^k \arrow[r] \arrow[d, hook] & A \cup (\mathrm{Sk}_{k-1}(X)) \arrow[d] \\
\coprod_{a \in X_k^{nd} \setminus A_k^{nd}} \Delta^k \arrow[r] & A \cup (\mathrm{Sk}_k(X)) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Pushout-product & Pullback-Hom

$f : A \to B$

$g : K \to L$ $\rightsquigarrow$ maps of Simplicial Sets

$h : X \to Y$

Pushout-product

$$\begin{tikzcd}
A \times K \arrow[r, "\mathrm{id} \times g"] \arrow[d, "f \times \mathrm{id}"'] & A \times L \arrow[d] \arrow[ddr, bend left, "f \times \mathrm{id}"] & \\
B \times K \arrow[r] \arrow[drr, bend right, "\mathrm{id} \times g"'] & (A \times L) \cup_{A \times K} (B \times K) \arrow[dr, "f \square g"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & B \times L
\end{tikzcd}$$

Pullback-hom

$$\begin{tikzcd}
\mathrm{Fun}(L,X) \arrow[dr, "h^{\square g}"] \arrow[drr, bend left, "\mathrm{Fun}(g,X)"] \arrow[ddr, bend right, "\mathrm{Fun}(L,h)"'] & & \\
& \mathrm{Fun}(K,X) \times_{\mathrm{Fun}(K,Y)} \mathrm{Fun}(L,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(K,X) \arrow[d, "\mathrm{Fun}(\mathrm{id},h)"] \\
& \mathrm{Fun}(L,Y) \arrow[r, "\mathrm{Fun}(g,\mathrm{id})"'] & \mathrm{Fun}(K,Y)
\end{tikzcd}$$

Remarks:

- $h^{\square g}$ $\rightsquigarrow$ enriched lifting $(h^{\square g})_0$ is the usual lifting set up!
- $\square : \mathrm{Ar}(\mathrm{sSet}) \times \mathrm{Ar}(\mathrm{sSet}) \to \mathrm{Ar}(\mathrm{sSet})$ defines a symmetric monoidal structure & $- \square g \dashv (-)^{\square g}$

ADJUNCTION OF LIFTING PROBLEMS

Prop: $(f \square g) \boxslash h$ iff $f \boxslash (h^{\square g})$

$$
\begin{tikzcd}
(A\times L)\cup(B\times K) \arrow[r, "u"] \arrow[d, "f\square g"'] & X \arrow[d, "h"] \\
B\times L \arrow[r, "w"'] \arrow[ur, dashed, "s"] & Y
\end{tikzcd}
\iff
\begin{tikzcd}
A \arrow[r, "\tilde{u}"] \arrow[d, "f"'] & \mathrm{Fun}(L,X) \arrow[d, "h^{\square g}"] \\
B \arrow[r, "(\tilde{v},\tilde{w})"'] \arrow[ur, dashed, "\tilde{s}"] & \mathrm{Fun}(K,X)\times_{\mathrm{Fun}(B,Y)}\mathrm{Fun}(L,Y)
\end{tikzcd}
$$

Proof uses the adjunction $-\square g \dashv (-)^{\square g}$, but it's checkable with some thought...

Note: If $A=\emptyset$, $K=\emptyset$ or $Y=*$ hold, then:

for instance if $K=\emptyset$, $Y=*$

$$(A\times L \xrightarrow{f\times L} B\times L) \boxslash (X \to *) \text{ iff}$$

$$(A \xrightarrow{f} B) \boxslash (\mathrm{Fun}(L,X) \to *)$$

Prop: S, T any collections of maps

$$\overline{S}\,\square\,\overline{T} \subseteq \overline{S\,\square\,T}$$

$\Big\downarrow$ +

Lemma: $\mathrm{InnHorn}\,\square\,\mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}}$

$(\Lambda^n_k \subseteq \Delta^n)\,\square\,(\partial\Delta^m \subseteq \Delta^m)$ is inner-anodyne $\rightsquigarrow$ $\overline{\mathrm{InnHorn}}\,\square\,\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$

Prop: 1) If $i : A \to B$ is inner-anodyne and $j : K \hookrightarrow L$ is a monomorphism

$$i \,\square\, j : (A \times L) \bigcup_{A \times K} (B \times K) \to B \times L$$

is inner-anodyne

2) If $j : K \hookrightarrow L$ is a monomorphism