MAIN GOALS

Things we will prove today:

- $\mathrm{Fun}(X, \mathcal{C})$ is an ∞-category if $\mathcal{C}$ is an ∞-category.
- $\mathrm{Comp}_{\mathcal{C}}(f, g)$ is a contractible Kan complex iff $\mathcal{C}$ is an ∞-category.
- $\mathrm{Map}_{\mathcal{C}}(x, y)$ is an ∞-category (unsatisfactory, but we can't do more now...)

but we will prove many more things!

Outline

1. Introduce weakly saturated classes and inner-anodyne maps

2. Introduce :

   2.1 Lifting Calculus

   2.2 Inner Fibrations

3. Digression on Inner Fibrations

4. A tourist's view of :

   4.1 Small Object Argument & Weak Factorization

   4.2 Skeletal Filtrations

   4.3 Trivial Fibrations

5. Pushout-Product & Pullback-Hom (a.k.a enriched lifting)

6. Function Complexes of ∞-categories are ∞-categories

7. Sanity check: $\mathcal{C}^{\Delta^2} \longrightarrow \mathcal{C}^{\Lambda^2_1}$

   does what we want...

Weakly Saturated Classes and Inner-Anodyne Maps

- ∞-categories are defined by an "extension property"

- $\mathcal{C}$ is an ∞-Category if

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & \mathcal{C} \\ L \arrow[ur, dashed] & \end{tikzcd}$$

for every $K \subseteq L \in \{\Lambda^n_j \subseteq \Delta^n : n \geq 2, 0<j<n\}$

Q: Is $\{\Lambda^n_j \subseteq \Delta^n : n \geq 2, 0<j<n\}$ precisely the class of morphisms with this extension property?

not really

Weakly Saturated Classes: A a class of morphisms such that:

1) Contains all isomorphisms $f: X \xrightarrow{\sim} Y$
2) is closed under cobase change

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d] \\ Z \arrow[r] & Y \sqcup_X Z \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

3) is closed under composition
4) is closed under transfinite composition
5) is closed under coproducts
6) is closed under retracts

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

$X \to X' \to X = \mathrm{id}$, $Y \to Y' \to Y = \mathrm{id}$

f is a retract of g

Def: Given a class of maps S, we define its weak saturation to be the smallest saturated class containing S. $S \rightsquigarrow \overline{S}$

Examples:
$\overline{\{\{0,1\} \to \{1\}\}} = \mathrm{Surj}$
$\overline{\{\phi \hookrightarrow \{1\}\}} = \mathrm{Mono}$

Really crucial result!

Prop: Consider a collection $\{C_\alpha\}$ of simplicial sets.
Let $\mathcal{A} := \left\{ i : A \to B \;\middle|\; \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \\ B \arrow[ur, dotted, "\exists g"'] & \end{tikzcd} \;\forall f \in \mathrm{Hom}(A, C_\alpha) \;\; \forall \alpha \right\}$

Then $\mathcal{A}$ is a weakly saturated class

Proof: We must show that this class

i) contains all isomorphisms: Assume that $A \xrightarrow{i} B$ is an isomorphism

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"', shift right] & & C_\alpha \\ & B \arrow[ul, "i^{-1}"', shift right] \arrow[ur, dashed] & \end{tikzcd}$$

2) is closed under cobase change

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C_\alpha \\ B \arrow[ur, dotted, "\exists g"'] & \end{tikzcd} \quad \text{and let} \quad \begin{tikzcd} A \arrow[r, "p"] \arrow[d, "i"'] & C \arrow[d, "i'"] \\ B \arrow[r, "p'"'] & C \sqcup_A B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "i"'] & C \arrow[r, "h"] \arrow[d, "i'"] & C_\alpha \\ B \arrow[r, "p'"'] \arrow[urr, dashed] & C \sqcup_A B \arrow[ur, dashed] & \end{tikzcd}$$

⋮

We could have defined a class of weakly cosaturated maps

We now have a toolset to answer the question!

CLASSES OF "ANODYNE" MORPHISMS

$$\mathrm{InnHorn} := \{\Lambda^n_k \subset \Delta^n : 0 < k < n\}_{n \geq 2}$$
$$\mathrm{LHorn} := \{\Lambda^n_k \subset \Delta^n : 0 \leq k < n\}_{n \geq 1}$$
$$\mathrm{RHorn} := \{\Lambda^n_k \subset \Delta^n : 0 < k \leq n\}_{n \geq 1}$$
$$\mathrm{Horn} := \{\Lambda^n_k \subset \Delta^n : 0 \leq k \leq n\}_{n \geq 1}$$

We will be interested in $\overline{\mathrm{InnHorn}}$ since

Prop: If $\mathcal{C}$ is an ∞-category and $A \subseteq B$ is an inner anodyne inclusion, then any $f: A \to \mathcal{C}$ admits an extension

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \mathcal{C} \\ B \arrow[ur, dashed] & \end{tikzcd}$$

Proof: The class $\left\{ X \hookrightarrow Y \;\middle|\; \begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & \mathcal{C} \\ Y \arrow[ur, dashed] & \end{tikzcd} \text{ for all } \infty\text{-cat. } \mathcal{C} \text{ and } f: X \to \mathcal{C} \right\}$

is weakly saturated as we saw before.

Moreover, it contains the class InnHorn, therefore it must include $\overline{\mathrm{InnHorn}}$. $\square$

$I^n \subseteq \Delta^n$

LIFTING CALCULUS & INNER FIBRATIONS

Move from "extensions" to "lifts"

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \\ B \arrow[ur, dashed] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

- Kan complex
- Weak Kan complex

$\rightsquigarrow$

- Kan Fibration
- Inner Fibration

Let $f: A \to B$ & $g: X \to Y$, a lifting problem for (f, g) is:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "\exists s"] & Y \end{tikzcd}$$

a pair $(u: A \to X,$
$v: B \to Y)$

s.t $g \circ s = v$

$s \circ f = u$

- We call $s: B \longrightarrow X$ a lift for the problem

We write $f \boxslash g$ if $\forall$ $\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$.

Towards Enriched Lifting...

We can redefine $f \boxslash g$ by:

$$\mathrm{Hom}(B, X) \xrightarrow{\text{surjective}} \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, Y)$$

$$s \longmapsto (g \circ s \ , \ s \circ f)$$

Surjectivity means that every lifting problem has a solution

Notation: $f \boxslash g$ ⇝ f has the left lifting property w.r.t g

⇝ g has the right lifting property w.r.t f

$\mathcal{A} \boxslash \mathcal{B}$ if every $f \in \mathcal{A}$, $g \in \mathcal{B}$ have $f \boxslash g$

$\mathcal{A}^{\boxslash} := \{ f \mid a \boxslash f \ \forall a \in \mathcal{A} \}$

⇝ right complement

${}^{\boxslash}\mathcal{A} := \{ g \mid g \boxslash a \ \forall a \in \mathcal{A} \}$

⇝ left complement

Prop: For any class $\mathcal{A}$:

generalization of previous result!

→ ${}^{\boxslash}\mathcal{A}$ the left complement is a weakly saturated class

→ $\mathcal{A}^{\boxslash}$ the right complement is a weakly cosaturated class

INNER FIBRATIONS

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\
\Delta^n \arrow[r, "\sigma"'] \arrow[ur, dashed, "\tilde{\sigma}"] & \mathcal{D}
\end{tikzcd}$$

$\forall$ inner horns

Def: An inner fibration is a map $p : X \to S$ of simplicial sets such that:

$$\mathrm{InnHorn} \boxslash p \quad \rightsquigarrow \quad \forall\, 0 < k < n \quad \forall\, n \geq 2 \quad \begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

$$(\Lambda^n_k \subseteq \Delta^n) \boxslash (X \xrightarrow{p} S)$$

Digression on Inner Fibrations

Harder to motivate inner fibrations, because they have no counterpart in classical category theory!

$f : X \to S$

$$\begin{tikzcd}[column sep=small]
 & \text{f is a trivial fibration} \arrow[d, Rightarrow] & \\
 & \text{f is a Kan fibration} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{f is a left fibration} \arrow[d, Rightarrow] & & \text{f is a right fibration} \arrow[d, Rightarrow] \\
\text{f is a CoCartesian fibration} \arrow[dr, Rightarrow] & & \text{f is a Cartesian fibration} \arrow[dl, Rightarrow] \\
 & \text{f is a Categorical fibration} \arrow[r, Rightarrow] & \text{f is an inner fibration}
\end{tikzcd}$$

Very precise sense:

We can show $p^{-1}(s)$ are Kan Complex

$$\left\{ \begin{array}{l} \mathcal{C} \xrightarrow{F} \mathcal{D} \\ F \text{ (co)fibers } \mathcal{C} \\ \text{in groupoids over} \\ \mathcal{D} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} N_\bullet(F): \mathcal{C} \to \mathcal{D} \\ \text{is a (left)} \\ \text{right fibration} \end{array} \right\}$$

$$\left\{ \; ? \; \right\} \longleftrightarrow \left\{ \begin{array}{l} N_\bullet(F): \mathcal{C} \to \mathcal{D} \\ \text{is an inner} \\ \text{fibration} \end{array} \right\}$$

$\mathrm{InnFib} := \mathrm{InnHorn}^{\boxslash}$

weakly cosaturated (closed under base change, ...)

$\rightsquigarrow \mathrm{InnFib} = \overline{\mathrm{InnFib}}$

${}^{\boxslash}\mathrm{InnFib}$ is weakly cosaturated

$$\Downarrow$$

$${}^{\boxslash}\mathrm{InnFib} = {}^{\boxslash}(\mathrm{InnHorn}^{\boxslash}) \supseteq \mathrm{InnHorn}$$

$$\Downarrow$$

$$\overline{\mathrm{InnHorn}} = {}^{\boxslash}\mathrm{InnFib}$$

in other words:

$$\overline{\mathrm{InnHorn}} \boxslash \mathrm{InnFib}$$

From this abstract reasoning:

- $\mathcal{C} \to *$ is an inner fibration iff $\mathcal{C}$ is an ∞-category

$$\begin{tikzcd} \overset{\text{inner}}{\Lambda^n_k} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & * \end{tikzcd} \rightsquigarrow \text{InnHorn} \boxslash \text{InnFib}$$

- If $p: \mathcal{C} \to \mathcal{D}$ is and p is and inner fibration and $\mathcal{D}$ is an ∞-category then $\mathcal{C}$ is an ∞-category

Why? InnFib is weakly cosaturated, in particular it's closed under compositions:

$$\mathcal{C} \xrightarrow{p} \mathcal{D} \to * \in \text{InnFib}$$

- The fibers of an inner fibration are ∞-categories:

InnFib is closed under base change:

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{D}} \{d\} \arrow[r] \arrow[d, "\text{inner fibration}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "p"] \\ \{*\} \arrow[r, "d"'] & \mathcal{D} \end{tikzcd}$$

$\Rightarrow$ $\mathcal{C} \times_{\mathcal{D}} \{d\}$ is an ∞-category!

More Stuff (exercises): $\mathcal{C}' \overset{i}{\hookrightarrow} \mathcal{C}$

- $\mathcal{C}' \subseteq \mathcal{C}$ of a subcomplex is an inner fibration iff $\mathcal{C}'$ is an ∞-subcategory.
- $N_\bullet(F): N_\bullet\mathcal{C} \to N_\bullet\mathcal{D}$ is an inner fibration for every functor $F: \mathcal{C} \to \mathcal{D}$.