Fast Fourier Transform

October 16, 2017

1 Introduction

The Fast Fourier Transform (FFT) is a smarter and quicker way to calculate the Discrete Fourier Transform (DFT), which (naturally) is the discrete version of the Fourier Transform (FT). The FT itself is the extension of the Fourier Series, which enables us to write a periodic (but complicated), time-dependent signal $a(t)$ as a superposition of simple wave functions with different frequencies f and amplitudes A. The Fourier Transform $\tilde{a}(f)$ holds the information about f and A. For example:

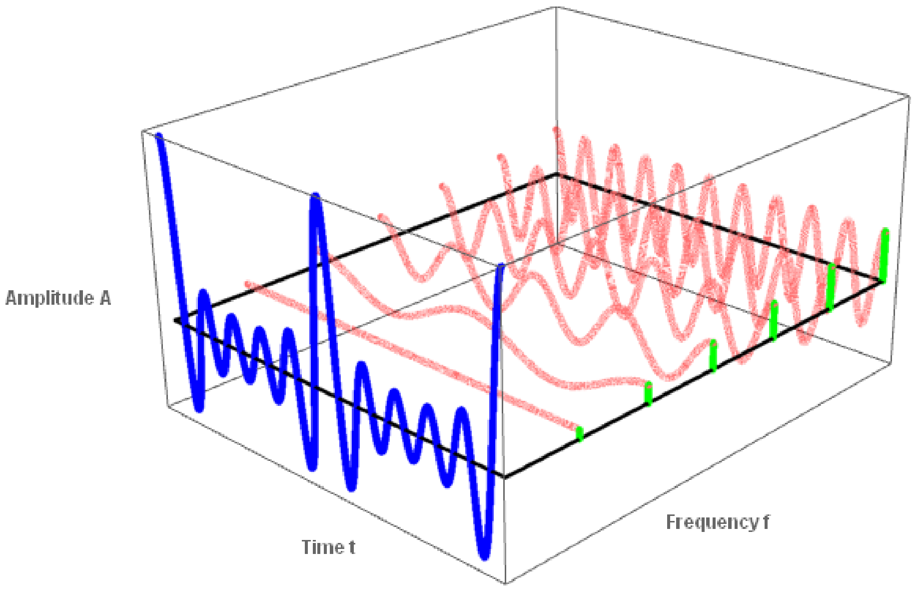

Figure 1: The blue function $a(t)$ can be seen as a superposition of the red wave functions. The green function $\tilde{a}(f)$ is the Fourier Transform of $a(t)$. It shows which frequencies f with which amplitudes A are being used in the red wave functions to construct $a(t)$.

Thus the FT transforms a function from the time-domain t into the frequency-domain f.
The FT can also be calculated for functions $a(t)$ with a period P approaching infinity. Furthermore, the FT can also be a complex function if $a(t)$ has a phase shift. When the phase shift is unequal to zero, the imaginary part of $\tilde{a}(f)$ is also unequal to zero.
The DFT calculates the FT of $a(t)$ for a given array of points $\underline{a}$ on $a(t)$. However, the complexity of the DFT is:

$$N^2$$

with the array-length N of $\underline{a}$.
To achieve a shorter computation time one uses a smarter algorithm to calculate

the FT without loosing any precision: The FFT which has a complexity of:

$$N\,log_2(N) \quad .$$

Thus the FFT is being used instead of the DFT to analyse signals $a(t)$ by decomposing it into several wave functions. The Transformation of a function can also help in solving a differential equation. Linear operations in one domain have corresponding operations in the other. A differentiation in the time-domain becomes a multiplication in the frequency-domain.
The FT is of course not limited to time and frequency. In quantum physics, for example, it is sometimes helpful to switch form the location-domain into the momentum-domain. For less confusion we will stick with time and frequency but keep in mind that this is only an example.

2 Theory

2.1 Discrete Fourier Transform (DFT)

Also the DFT is inferior to the FFT, it is helpful to look at the DFT for a better understanding of the FFT.
Imagine a measurement of a time-dependent function $a(t)$ for a time T which gives an array $\underline{a} = (a_0, a_1, ..., a_{N-1})$ with length N. The measurement frequency m, the number of measurements per time, is:

$$m = \frac{N}{T} \quad .$$

The DFT $\underline{\tilde{a}} = (\tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_{N-1})$ is also an array with length N in the frequency-domain. The f's item of $\underline{\tilde{a}}$ can be calculated with:

$$\tilde{a}_f = \sum_{t=0}^{N-1} a_t \cdot e^{-i\frac{2\pi}{N}\cdot f\cdot t} \quad . \tag{1}$$

The inverse DFT switches back from the frequency-domain into the time-domain and calculates the t's item of $\underline{a}$:

$$a_t = \sum_{f=0}^{N-1} \tilde{a}_f \cdot e^{i\frac{2\pi}{N}\cdot t\cdot f} \quad . \tag{2}$$

The time t between two neighbouring items in $\underline{a}$ is T/N starting with $t = 0\,s$ for a_0.
The frequency f between two neighbouring items in $\underline{\tilde{a}}$ is $1/T$ starting with $f = 0\,Hz$ for $\tilde{a}_0$.

2.1.1 Example

Imagine we are measuring a simple sinusoid function $a(t)$ with frequency $f = 2\,Hz$ and amplitude $A = 1$:

$$a(t) = sin(2\pi \cdot 2\,Hz \cdot t)$$

with $N = 10$ measurement for $T = 1\,s$ measuring time:

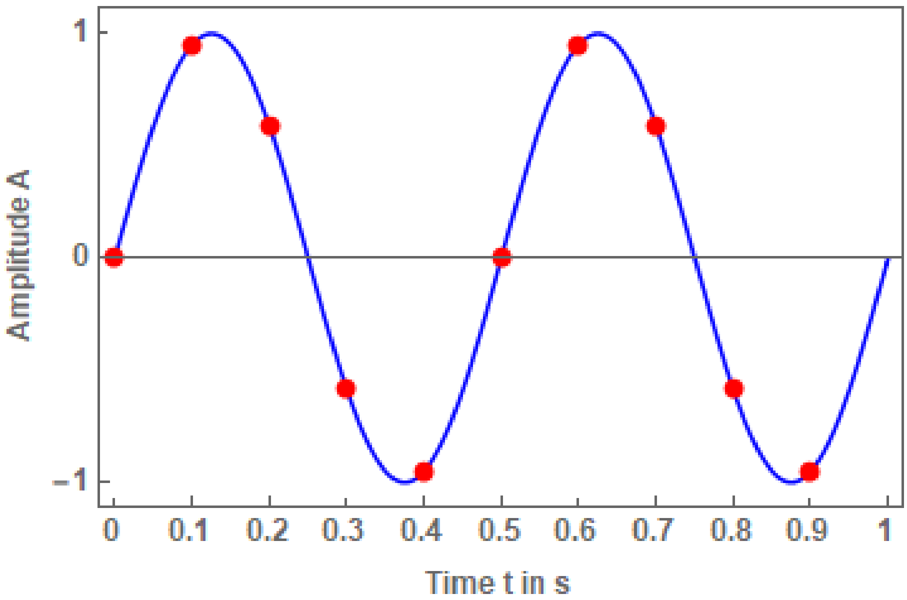

Figure 2: $a(t) = sin(2\pi \cdot 2\,Hz \cdot t)$ in blue with $N = 10$ samples in red.

The measurements gives following array $\underline{a}$:

$$
\begin{aligned}
a_0 &= 0.00 \\
a_1 &= 0.95 \\
a_2 &= 0.59 \\
a_3 &= -0.59 \\
a_4 &= -0.95 \\
a_5 &= 0.00 \\
a_6 &= 0.95 \\
a_7 &= 0.59 \\
a_8 &= -0.59 \\
a_9 &= -0.95
\end{aligned}
$$

Because $a(t)$ is simple function we can predict that the DFT $\underline{\tilde{a}}$ should have one item at $f = 2\,Hz$ with amplitude $A = 1$ and the rest should be zero.

To check the prediction we will calculate $\underline{\tilde{a}}$ with equation (1):

$$\begin{aligned}
\tilde{a}_0 &= \sum_{t=0}^{9} a_t \cdot e^{-i\frac{2\pi}{10}\cdot 0\cdot t} \\
&= a_0 + a_1 + ... + a_{N-1} \\
&= 0.00 + i0.00
\end{aligned}$$

$$\begin{aligned}
\tilde{a}_1 &= \sum_{t=0}^{9} a_t \cdot e^{-i\frac{2\pi}{10}\cdot 1\cdot t} \\
&= a_0 \cdot e^{-i\frac{2\pi}{10}\cdot 1\cdot 0} + a_1 \cdot e^{-i\frac{2\pi}{10}\cdot 1\cdot 1} + ... + a_9 \cdot e^{-i\frac{2\pi}{10}\cdot 1\cdot 9} \\
&= 0.00 + i0.00 \\
&. \\
&. \\
&. \\
\tilde{a}_9 &= \sum_{t=0}^{9} a_t \cdot e^{-i\frac{2\pi}{10}\cdot 9\cdot t} \\
&= a_0 \cdot e^{-i\frac{2\pi}{10}\cdot 9\cdot 0} + a_1 \cdot e^{-i\frac{2\pi}{10}\cdot 9\cdot 1} + ... + a_9 \cdot e^{-i\frac{2\pi}{10}\cdot 9\cdot 9} \\
&= 0.00 + i0.00
\end{aligned}$$

In conclusion:

$$\begin{aligned}
\tilde{a}_0 &= 0.00 + i0.00 \\
\tilde{a}_1 &= 0.00 + i0.00 \\
\tilde{a}_2 &= 0.00 - i5.00 \\
\tilde{a}_3 &= 0.00 + i0.00 \\
\tilde{a}_4 &= 0.00 + i0.00 \\
\tilde{a}_5 &= 0.00 + i0.00 \\
\tilde{a}_6 &= 0.00 + i0.00 \\
\tilde{a}_7 &= 0.00 + i0.00 \\
\tilde{a}_8 &= 0.00 + i5.00 \\
\tilde{a}_9 &= 0.00 + i0.00
\end{aligned}$$

The amplitudes A are the magnitudes of the $\tilde{a}_f$ which are separated by $1\,HZ$:

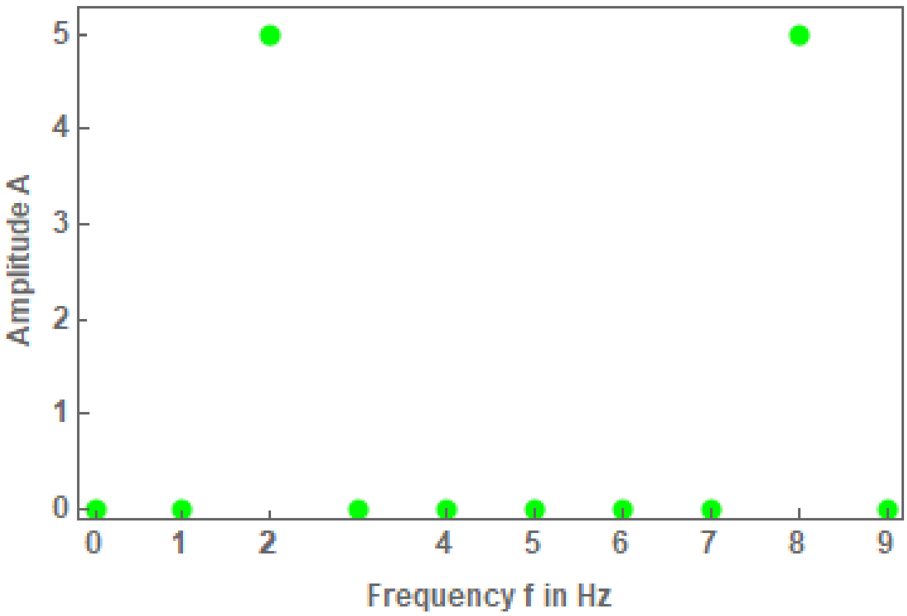

Figure 3: Magnitude of $\underline{\tilde{a}}$ in green.

This result contradicts with the prediction we made above. The reason is that we still have to do two more steps before we arrive at the end result:
1. **The Nyquist Limit $m/2$:** All items of $\underline{\tilde{a}}$ with a frequency above the Nyquist limit are to be ignored and the items under the Nyquist limit are to be doubled. In our example the Nyquist limit is $5\,Hz$, thus we are left with five the points:

$$(0\,Hz|0) \quad , \quad (1\,Hz|0) \quad , \quad (2\,Hz|10) \quad , \quad (3\,Hz|0) \quad , \quad (4\,Hz|0) \qquad .$$

2. **The average:** The last step is to average over the number if measurements N. In our example we made 10 measurements, therefore we arrive at our final result:

$$(0\,Hz|0) \quad , \quad (1\,Hz|0) \quad , \quad (2\,Hz|1) \quad , \quad (3\,Hz|0) \quad , \quad (4\,Hz|0) \qquad .$$

Which is equal to the prediction we made above.

2.2 Fast Fourier Transform (FFT)

The example for the DFT with $N = 10$ showed that we had to do 100 operations before arriving at the end result.
The FFT follows directly from the DFT by rearranging equation (1) and using symmetry relations of the imaginary e-function. First we split the sum into an

even (red) and an odd (blue) part:

$$\begin{aligned}
\tilde{a}_f &= \sum_{t=0}^{N-1} a_t \cdot e^{-i\frac{2\pi}{N}\cdot f\cdot t} \\
&= \sum_{t_1=0}^{N/2-1} a_{2t_1} \cdot e^{-i\frac{2\pi}{N}\cdot f\cdot 2t_1} + \sum_{t_1=0}^{N/2-1} a_{2t_1+1} \cdot e^{-i\frac{2\pi}{N}\cdot f\cdot (2t_1+1)} \\
&= \sum_{t_1=0}^{N/2-1} a_{2t_1} \cdot e^{-i\frac{2\pi}{N/2}\cdot f\cdot t_1} + e^{-i\frac{2\pi}{N} f} \sum_{t_1=0}^{N/2-1} a_{2t_1+1} \cdot e^{-i\frac{2\pi}{N/2}\cdot f\cdot t_1} \\
&= \sum_{t_1=0}^{N/2-1} a_{2t_1} \cdot e^{-i\frac{2\pi}{N/2}\cdot f\cdot t_1} + \quad C_f^1 \quad \sum_{t_1=0}^{N/2-1} a_{2t_1+1} \cdot e^{-i\frac{2\pi}{N/2}\cdot f\cdot t_1} \qquad .
\end{aligned}$$

Now we see that the two sums share the same exponential, which has a symmetry relation between f and $f + N/2$:

$$\begin{aligned}
e^{-i\frac{2\pi}{N/2}\cdot f\cdot t_1} &= e^{-i\frac{2\pi}{N/2}\cdot (f+N/2)\cdot t_1} \\
&= e^{-i\frac{2\pi}{N/2}\cdot f\cdot t_1} \cdot e^{-i\frac{2\pi}{N/2}\cdot N/2\cdot t_1} \\
&= e^{-i\frac{2\pi}{N/2}\cdot f\cdot t_1} \cdot 1 \qquad .
\end{aligned}$$

Therefore the two items $\tilde{a}_f$ and $\tilde{a}_{f+N/2}$ can be calculated at the same time, because the exponentials are the same they only have a different pre-factor.

This procedure of splitting can be repeated for the odd and even sum:

$$\begin{aligned}
&\sum_{t_1=0}^{N/2-1} a_{2t_1} \cdot e^{-i\frac{2\pi}{N/2}\cdot f\cdot t_1} \\
&= \sum_{t_2=0}^{N/4-1} a_{4t_2} \cdot e^{-i\frac{2\pi}{N/2}\cdot f\cdot 2t_2} + \sum_{t_2=0}^{N/4-1} a_{4t_2+2} \cdot e^{-i\frac{2\pi}{N/2}\cdot f\cdot(2t_2+1)} \\
&= \sum_{t_2=0}^{N/4-1} a_{4t_2} \cdot e^{-i\frac{2\pi}{N/4}\cdot f\cdot t_2} + e^{-i\frac{4\pi}{N}f} \sum_{t_2=0}^{N/4-1} a_{4t_2+2} \cdot e^{-i\frac{2\pi}{N/4}\cdot f\cdot t_2} \\
&= \sum_{t_2=0}^{N/4-1} a_{4t_2} \cdot e^{-i\frac{2\pi}{N/4}\cdot f\cdot t_2} + \quad C_f^2 \quad \sum_{t_2=0}^{N/4-1} a_{4t_2+2} \cdot e^{-i\frac{2\pi}{N/4}\cdot f\cdot t_2}
\end{aligned}$$

$$\begin{aligned}
&C_f^1 \sum_{t_1=0}^{N/2-1} a_{2t_1+1} \cdot e^{-i\frac{2\pi}{N/2}\cdot f\cdot t_1} \\
&= C_f^1 \sum_{t_2=0}^{N/4-1} a_{4t_2+1} \cdot e^{-i\frac{2\pi}{N/2}\cdot f\cdot 2t_2} + C_f^1 \sum_{t_2=0}^{N/4-1} a_{4t_2+3} \cdot e^{-i\frac{2\pi}{N/2}\cdot f\cdot(2t_2+1)} \\
&= C_f^1 \sum_{t_2=0}^{N/4-1} a_{4t_2+1} \cdot e^{-i\frac{2\pi}{N/4}\cdot f\cdot t_2} + C_f^1 e^{-i\frac{4\pi}{N}f} \sum_{t_2=0}^{N/4-1} a_{4t_2+3} \cdot e^{-i\frac{2\pi}{N/4}\cdot f\cdot t_2} \\
&= C_f^1 \sum_{t_2=0}^{N/4-1} a_{4t_2+1} \cdot e^{-i\frac{2\pi}{N/4}\cdot f\cdot t_2} + \quad C_f^3 \quad \sum_{t_2=0}^{N/4-1} a_{4t_2+3} \cdot e^{-i\frac{2\pi}{N/4}\cdot f\cdot t_2}
\end{aligned}$$

All four sums share the same exponential, which in this case has a symmetry relation between f and $f + N/4$. Therefore the four items $\tilde{a}_f$, $\tilde{a}_{f+N/4}$, $\tilde{a}_{f+N/2}$ and $\tilde{a}_{f+3N/4}$ can be calculated at the same time.
This procedure can be repeated again and again until the sums only go through one item of $\underline{a}$.

2.2.1 Example

Imagine we are measuring a simple sinusoid function $a(t)$ with frequency $f = 1\,Hz$ and amplitude $A = 1$:

$$a(t) = sin(2\pi \cdot 1\,Hz \cdot t)$$

with $N = 4$ measurement for $T = 1\,s$ measuring time:

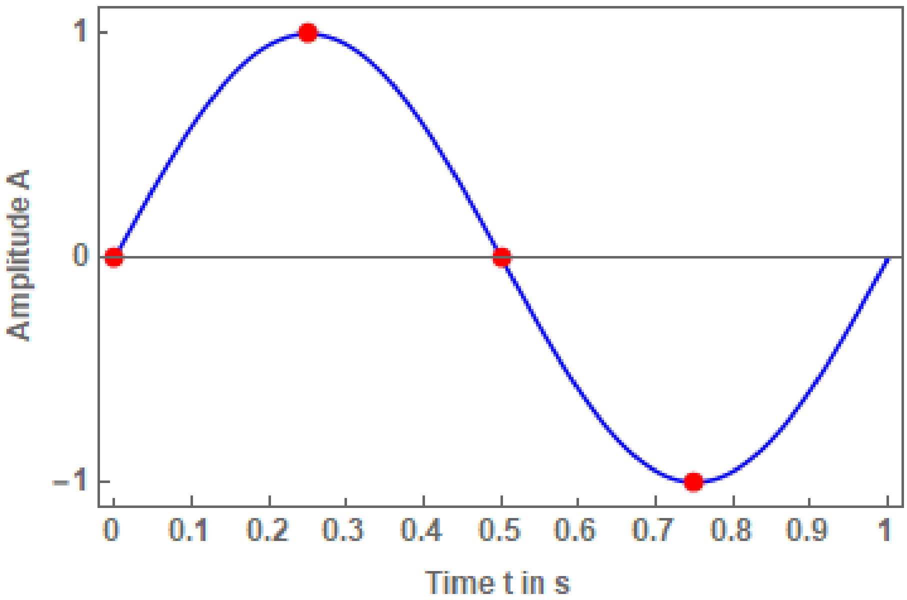

Figure 4: $a(t) = sin(2\pi \cdot 1\,Hz \cdot t)$ in blue with $N = 4$ samples in red.

The measurements gives following array $\underline{a}$:

$$
\begin{aligned}
a_0 &= 0.00 \\
a_1 &= 1.00 \\
a_2 &= 0.00 \\
a_3 &= -1.00
\end{aligned}
$$

Because we made four measurements we can split the sums 2 times:

$$
\begin{aligned}
\tilde{a}_f &= \sum_{t=0}^{3} a_t \cdot e^{-i\frac{2\pi}{4}\cdot f\cdot t} \\
&= \sum_{t_1=0}^{1} a_{2t_1} \cdot e^{-i\frac{2\pi}{2}\cdot f\cdot t_1} + C_f^1 \sum_{t_1=0}^{1} a_{2t_1+1} \cdot e^{-i\frac{2\pi}{2}\cdot f\cdot t_1} \\
&= \sum_{t_2=0}^{0} a_{4t_2} \cdot e^{-i\frac{2\pi}{1}\cdot f\cdot t_2} + C_f^2 \sum_{t_1=0}^{0} a_{4t_2+2} \cdot e^{-i\frac{2\pi}{1}\cdot f\cdot t_2} \\
&\qquad\qquad + C_f^1 \sum_{t_2=0}^{0} a_{4t_2+1} \cdot e^{-i\frac{2\pi}{1}\cdot f\cdot t_2} + C_f^3 \sum_{t_2=0}^{0} a_{4t_2+3} \cdot e^{-i\frac{2\pi}{1}\cdot f\cdot t_2} \\
&= a_0 + C_f^2 a_2 + C_f^1 a_1 + C_f^3 a_3
\end{aligned}
$$

This makes the calculation much faster:

$$\begin{aligned}
\tilde{a}_0 &= 1.00 \cdot C_0^1 - 1.00 \cdot C_0^3 \\
&= 1.00 \cdot e^{-i\frac{2\pi}{4}\cdot 0} - 1.00 \cdot e^{-i\frac{6\pi}{4}\cdot 0} \\
&= 0.00 + i0.00
\end{aligned}$$

$$\begin{aligned}
\tilde{a}_1 &= 1.00 \cdot C_1^1 - 1.00 \cdot C_1^3 \\
&= 1.00 \cdot e^{-i\frac{2\pi}{4}\cdot 1} - 1.00 \cdot e^{-i\frac{6\pi}{4}\cdot 1} \\
&= 0.00 - i2.00
\end{aligned}$$

$$\begin{aligned}
\tilde{a}_2 &= 1.00 \cdot C_2^1 - 1.00 \cdot C_2^3 \\
&= 1.00 \cdot e^{-i\frac{2\pi}{4}\cdot 2} - 1.00 \cdot e^{-i\frac{6\pi}{4}\cdot 2} \\
&= 0.00 + i0.00
\end{aligned}$$

$$\begin{aligned}
\tilde{a}_3 &= 1.00 \cdot C_3^1 - 1.00 \cdot C_3^3 \\
&= 1.00 \cdot e^{-i\frac{2\pi}{4}\cdot 3} - 1.00 \cdot e^{-i\frac{6\pi}{4}\cdot 3} \\
&= 0.00 + i2.00
\end{aligned}$$

After the consideration of the Nyquist limit and the averaging, we get the result, which we would expect:

$$(0\,Hz|0) \quad , \quad (1\,Hz|1) \qquad .$$