Ficha de Exercícios 2

Integrais indefinidos

1. Determine os seguintes integrais indefinidos:

(a) $\displaystyle\int (3x^2 + 5x + 7)\, dx$ (b) $\displaystyle\int \sqrt[3]{x}\, dx$ (c) $\displaystyle\int (x^3+1)^2\, dx$

(d) $\displaystyle\int \frac{\operatorname{arctg} x}{1+x^2}\, dx$ (e) $\displaystyle\int \frac{3x^2}{1+x^3}\, dx$ (f) $\displaystyle\int \frac{1}{x^7}\, dx$

(g) $\displaystyle\int \frac{x+1}{2+4x^2}\, dx$ (h) $\displaystyle\int 4x^3 \cos x^4\, dx$ (i) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\, dx$

(j) $\displaystyle\int \operatorname{sen} x \cos^5 x\, dx$ (k) $\displaystyle\int \operatorname{tg} x\, dx$ (l) $\displaystyle\int \frac{\ln x}{x}\, dx$

(m) $\displaystyle\int e^{\operatorname{tg} x} \sec^2 x\, dx$ (n) $\displaystyle\int x 7^{x^2}\, dx$ (o) $\displaystyle\int \operatorname{sen}(\sqrt{2} x)\, dx$

(p) $\displaystyle\int \frac{x^2+1}{x}\, dx$ (q) $\displaystyle\int \frac{x}{(7+5x^2)^{\frac{3}{2}}}\, dx$ (r) $\displaystyle\int \frac{x^3}{1+x^8}\, dx$

(s) $\displaystyle\int \frac{5x^2}{\sqrt{1-x^6}}\, dx$ (t) $\displaystyle\int \frac{1}{x^2+7}\, dx$ (u) $\displaystyle\int \frac{1}{x^2+2x+5}\, dx$

(v) $\displaystyle\int \frac{x}{1+x^4}\, dx$ (w) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\, dx$ (x) $\displaystyle\int \frac{x^3}{\sqrt{1-x^4}}\, dx$

2. Determine os seguintes integrais indefinidos:

(a) $\displaystyle\int \frac{e^{\operatorname{arcsen} x}}{\sqrt{1-x^2}}\, dx$ (b) $\displaystyle\int \operatorname{tg}^2 x\, dx$ (c) $\displaystyle\int \frac{1}{x}\cos(\ln x)\, dx$

(d) $\displaystyle\int \frac{6}{x \ln^3(4x)}\, dx$ (e) $\displaystyle\int \frac{e^{3x}}{(e^{3x}-2)^6}\, dx$ (f) $\displaystyle\int \operatorname{tg}^3 x\, dx$

(g) $\displaystyle\int \frac{1}{x\sqrt{1-\ln^2 x}}\, dx$ (h) $\displaystyle\int e^x\sqrt{1+e^x}\, dx$ (i) $\displaystyle\int \frac{1}{x \ln x}\, dx$

(j) $\displaystyle\int \frac{1}{\sqrt{x}} e^{\sqrt{x}}\, dx$ (k) $\displaystyle\int \frac{1+\cos x}{x+\operatorname{sen} x}\, dx$ (l) $\displaystyle\int \frac{e^{2x+1}}{e^{2x}+3}\, dx$

(m) $\displaystyle\int x^5 \operatorname{sen}(x^6)\, dx$ (n) $\displaystyle\int \frac{\arccos x - x}{\sqrt{1-x^2}}\, dx$ (o) $\displaystyle\int \frac{\cos(\ln(x^2))}{x}\, dx$

3. Considere a função g definida em $\mathbb{R}^+$ por $g(x) = \frac{(\ln x)^2}{x}$.

(a) Determine a família de todas as primitivas de g.

(b) Indique a primitiva da função g que se anula para $x = e$.

Resolução:

(a) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \frac{(\ln x)^3}{3} + c, \quad c \in \mathbb{R}.$

(b) Para cada $c \in \mathbb{R}$, $G(x) = \frac{(\ln x)^3}{3} + c$ é uma primitiva de g.
Pretendemos então determinar $c \in \mathbb{R}$ tal que $G(e) = 0$.

$$G(e) = 0 \Leftrightarrow \frac{1}{3} + c = 0 \Leftrightarrow c = -\frac{1}{3}$$

Assim, $G(x) = \frac{(\ln x)^3}{3} - \frac{1}{3}$ é a primitiva de g que se anula para $x = e$.

4. Determine a primitiva F para a função $f(x) = \frac{2}{x} + \frac{3}{x^2}$ tal que $F(-1) = 1$.

5. Sabendo que a função f satisfaz a igualdade $\displaystyle\int f(x)\,dx = \operatorname{sen} x - x\cos x - \frac{1}{2}x^2 + c$, com $c \in \mathbb{R}$, determinar $f(\frac{\pi}{4})$.

6. Determine a primitiva da função $f(x) = \dfrac{1}{x^2} + 1$ que se anula no ponto $x = 2$.

7. Determine a primitiva da função f definida por $f(x) = \frac{3\cos(\ln x)}{x}$ que toma o valor 2 em $x = 1$.

8. Determine a função g que verifica as seguintes condições:

$$g'(x) = \frac{1}{(1 + \operatorname{arctg}^2(x))(1 + x^2)} \quad \text{e} \quad \lim_{x \to +\infty} g(x) = 0.$$

9. Determine, usando a técnica de integração por partes, os seguintes integrais indefinidos:

(a) $\displaystyle\int (x + 1)\operatorname{sen} x\,dx$

Resolução: Fazendo

$$\begin{array}{ll} f'(x) = \operatorname{sen} x \quad \text{temos} \quad f(x) = -\cos x \\ g(x) = x + 1 \quad \text{temos} \quad g'(x) = 1 \end{array}$$

Assim,

$$\begin{aligned} \int (x+1)\operatorname{sen} x\,dx &= -(x+1)\cos x + \int \cos x\,dx \\ &= -(x+1)\cos x + \operatorname{sen} x + c, \quad c \in \mathbb{R} \end{aligned}$$

(b) $\displaystyle\int x\cos x\,dx$ (c) $\displaystyle\int x^2\cos x\,dx$ (d) $\displaystyle\int e^{-3x}(2x+3)\,dx$ (e) $\displaystyle\int \ln^2 x\,dx$

(f) $\displaystyle\int \ln x\,dx$ (g) $\displaystyle\int \ln(x^2+1)\,dx$ (h) $\displaystyle\int x\operatorname{arctg} x\,dx$ (i) $\displaystyle\int \cos(\ln x)\,dx$

(j) $\displaystyle\int e^{2x}\operatorname{sen}(x)\,dx$ (k) $\displaystyle\int \operatorname{sen}(\ln x)\,dx$ (l) $\displaystyle\int \operatorname{arcsen} x\,dx$ (m) $\displaystyle\int x\operatorname{arcsen} x^2\,dx$

(n) $\displaystyle\int x^3 e^{x^2}\,dx$ (o) $\displaystyle\int \operatorname{arctg} x\,dx$ (p) $\displaystyle\int \operatorname{arctg}\frac{1}{x}\,dx$ (q) $\displaystyle\int \sqrt{x}\ln x\,dx$

(r) $\displaystyle\int \operatorname{sen} x\cos(3x)\,dx$ (s) $\displaystyle\int \cos^2 x\,dx$ (t) $\displaystyle\int \sec^3 x\,dx$ (u) $\displaystyle\int \frac{x^2}{\sqrt{(1-x^2)^3}}\,dx$

10. Determine, usando a técnica de integração por substituição, os seguintes integrais indefinidos:

(a) $\displaystyle\int x\sqrt{x+1}\,dx$

Resolução:

Consideremos a substituição $x+1=t^2$, com $t\geq 0$. Definindo $\varphi(t)=t^2-1$, $t\geq 0$, temos que φ é invertível, diferenciável e $\varphi'(t)=2t$. Então

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \int (t^2-1)\cdot t\cdot 2t\,dt\\ &= \frac{2t^5}{5}-\frac{2t^3}{3}+c\,.\end{aligned}$$

Atendendo a que $x+1=t^2$, com $t\geq 0$, vem que $t=\sqrt{x+1}$. Assim,

$$\int x\sqrt{x+1}\,dx=\frac{2(x+1)^2\sqrt{x+1}}{5}-\frac{2(x+1)\sqrt{x+1}}{3}+c,\text{ com } c\in\mathbb{R}.$$

(b) $\displaystyle\int \frac{x}{1+\sqrt{x}}\,dx$ (c) $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$ (d) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ (e) $\displaystyle\int \frac{1}{x\sqrt{x^2-5}}\,dx$

(f) $\displaystyle\int x^2\sqrt{1-x}\,dx$ (g) $\displaystyle\int x^2\sqrt{4-x^2}\,dx$ (h) $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx$ (i) $\displaystyle\int \frac{1}{x\sqrt{x^2+4}}\,dx$

(j) $\displaystyle\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$ (k) $\displaystyle\int \frac{x^2}{\sqrt{1-2x-x^2}}\,dx$ (l) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-7}}\,dx$ (m) $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

(n) $\displaystyle\int x(2x+5)^{10}\,dx$ (o) $\displaystyle\int (1+x)^{-2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{3}}\,dx$

(p) $\displaystyle\int e^{\sqrt{x}}\,dx$ (q) $\displaystyle\int \frac{\ln x}{x\cdot\sqrt{1+\ln x}}\,dx$ (r) $\displaystyle\int \frac{1+\operatorname{tg}^2 x}{\sqrt{\operatorname{tg} x-1}}\,dx$

11. Determine os seguintes integrais indefinidos:

(a) $\displaystyle\int \frac{x+2}{(x-1)^2(x^2+4)}\,dx$

Resolução:

A determinação deste integral indefinido passa por decompor em frações simples a fração

$$\frac{x+2}{(x-1)^2(x^2+4)}.$$

Isto é, passa por escrever a dita fração na seguinte forma

$$\frac{x+2}{(x-1)^2(x^2+4)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+4}\quad(*)$$

com $A,\ B,\ C$ e D constantes reais a determinar.

Temos então que

$$\frac{x+2}{(x-1)^2(x^2+4)}=\frac{(A+C)x^3+(-A+B-2C+D)x^2+(4A+C-2D)x-4A+4B+D}{(x-1)^2(x^2+4)}$$

donde resulta a igualdade de polinómios

$$x+2=(A+C)x^3+(-A+B-2C+D)x^2+(4A+C-2D)x-4A+4B+D.$$

Atendendo à condição de igualdade de polinómios resulta que

$$\begin{cases} A + C = 0 \\ -A + B - 2C + D = 0 \\ 4A + C - 2D = 1 \\ -4A + 4B + D = 2 \end{cases} \Leftrightarrow \begin{cases} A = -\frac{1}{25} \\ B = \frac{15}{25} \\ C = \frac{1}{25} \\ D = -\frac{14}{25} \end{cases}$$

Voltando a $(*)$, podemos escrever

$$\frac{x+2}{(x-1)^2(x^2+4)} = \frac{-\frac{1}{25}}{x-1} + \frac{\frac{15}{25}}{(x-1)^2} + \frac{\frac{1}{25}x - \frac{14}{25}}{x^2+4}.$$

Assim

$$\begin{aligned} \int \frac{x+2}{(x-1)^2(x^2+4)}\,dx &= -\tfrac{1}{25}\int \frac{1}{x-1}\,dx + \frac{15}{25}\int (x-1)^{-2}\,dx + \frac{1}{25}\int \frac{x-14}{x^2+4}\,dx \\ &= -\tfrac{1}{25}\ln|x-1| - \tfrac{3}{5(x-1)} + \tfrac{1}{25}\int \frac{x}{x^2+4}\,dx - \frac{14}{25}\int \frac{1}{x^2+4}\,dx \\ &= -\tfrac{1}{25}\ln|x-1| - \tfrac{3}{5(x-1)} + \tfrac{1}{50}\ln(x^2+4) - \tfrac{7}{25}\mathrm{arctg}\,\tfrac{x}{2} + c, \quad c \in \mathbb{R} \end{aligned}$$

(b) $\displaystyle\int \frac{2x-1}{(x-2)(x-3)(x+1)}\,dx$ (c) $\displaystyle\int \frac{1}{(x-1)(x+1)^3}\,dx$ (d) $\displaystyle\int \frac{1}{x^3+8}\,dx$

(e) $\displaystyle\int \frac{x^8}{1+x^2}\,dx$ (f) $\displaystyle\int \frac{1}{x^3(1+x^2)}\,dx$ (g) $\displaystyle\int \frac{8}{x^4+4x^2}\,dx$

(h) $\displaystyle\int \frac{x^5+x^4-8}{x^3-4x}\,dx$ (i) $\displaystyle\int \frac{x^2}{(x-1)^3}\,dx$ (j) $\displaystyle\int \frac{x^3+3x-1}{x^4-4x^2}\,dx$

(k) $\displaystyle\int \frac{x+1}{x^3-1}\,dx$ (l) $\displaystyle\int \frac{x^4}{x^4-1}\,dx$ (m) $\displaystyle\int \frac{1}{x(x^2+1)^2}\,dx$

(n) $\displaystyle\int \frac{x+1}{x^2+4x+5}\,dx$

12. Determine

(a) $\displaystyle\int \mathrm{sen}^2\theta\,d\theta$ (b) $\displaystyle\int \mathrm{sen}^4 x\,dx$ (c) $\displaystyle\int \mathrm{sen}\,x\cos^2 x\,dx$

(d) $\displaystyle\int \mathrm{sen}^3 x\,dx$ (e) $\displaystyle\int \mathrm{sen}^5 x\cos^2 x\,dx$ (f) $\displaystyle\int \cos^3 x\,dx$

(g) $\displaystyle\int \frac{1}{\sqrt{2+x^2}}\,dx$ (h) $\displaystyle\int \frac{\mathrm{sen}\sqrt{x}}{\sqrt{x}}\,dx$ (i) $\displaystyle\int \frac{x}{x^2-5x+6}\,dx$

(j) $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx$ (k) $\displaystyle\int x\sqrt{(1+x^2)^3}\,dx$ (l) $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$

(m) $\displaystyle\int x\ln x\,dx$ (n) $\displaystyle\int \frac{1+e^x}{e^{2x}+4}\,dx$ (o) $\displaystyle\int \frac{x}{\cos^2 x}\,dx$

(p) $\displaystyle\int \frac{\mathrm{sen}\,x}{(1-\cos x)^3}\,dx$ (q) $\displaystyle\int (2x^2+3)\mathrm{arctg}\,x\,dx$ (r) $\displaystyle\int \frac{1}{\sqrt{x^2+2x-3}}\,dx$

(s) $\displaystyle\int \sqrt{1+e^x}\,dx$ (t) $\displaystyle\int \frac{1}{\sqrt{e^x-1}}\,dx$ (u) $\displaystyle\int \cos x\cos(5x)\,dx$

(v) $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$ (w) $\displaystyle\int \mathrm{sen}^5 x\,dx$ (x) $\displaystyle\int \frac{\ln x}{x(\ln^2 x+1)}\,dx$

Exercícios de testes/exames de anos anteriores

13. Determine a primitiva da função $f(x) = \operatorname{tg} x$ cujo gráfico passa pelo ponto de coordenadas $(\pi, 3)$.

(Teste 1, Cálculo I, 2014/2015)

14. Determine a função $f : \mathbb{R} \to \mathbb{R}$ tal que

$$f'(x) = \frac{2e^x}{3 + e^x} \quad \text{e} \quad f(0) = \ln 4.$$

(Exame Final, Cálculo I, 2010/2011)

15. Determine a função $f : \mathbb{R} \to \mathbb{R}$ tal que $f(0) = 1$, $f'(0) = 2$ e $f''(x) = 12x$, para todo o $x \in \mathbb{R}$.

(Miniteste 2, Cálculo I, 2009/2010).

16. Determine os seguintes integrais indefinidos:

(a) $\displaystyle\int \operatorname{sen}(2x) e^{\cos(2x)}\, dx$ *(Miniteste 2, Cálculo I, 2008/2009)*

(b) $\displaystyle\int \frac{1}{\sqrt{x} - \sqrt[4]{x}}\, dx$ *(Miniteste 2, Cálculo I, 2008/2009)*

(c) $\displaystyle\int \frac{1}{x^2\sqrt{x^2 - 9}}\, dx$ *(Exame de Recurso, Cálculo I, 2008/2009)*

(d) $\displaystyle\int x^2 \operatorname{arctg} x\, dx$ *(Exame Época Normal, Cálculo I, 2008/2009)*

(e) $\displaystyle\int \frac{x + 2}{x(x^2 + 4)}\, dx$ *(Miniteste 2, Cálculo I, 2009/2010)*

(f) $\displaystyle\int \frac{x^2}{\sqrt{1 + x^3}}\, dx$ *(Miniteste 2, Cálculo I, 2009/2010)*

(g) $\displaystyle\int \frac{1}{x^2\sqrt{1 + x^2}}\, dx$ *(Exame Época Normal, Cálculo I, 2009/2010)*

(h) $\displaystyle\int \frac{3x - 1}{x^3 + x}\, dx$ *(Exame Final, Cálculo I, 2010/2011)*

(i) $\displaystyle\int \frac{-\cos x}{(1 + \operatorname{sen} x)^2}\, dx$ *(1º Teste, Cálculo I - Agrupamento IV, 2017/2018)*

(j) $\displaystyle\int x \cdot \ln(1 + x^2)\, dx$ *(Exame Final, Cálculo I - Agrupamento IV, 2017/2018)*

(k) $\displaystyle\int \cos x \cdot \ln(\operatorname{sen} x))\, dx$ *(Exame de Recurso, Cálculo I - Agrupamento IV, 2017/2018)*

(l) $\displaystyle\int \frac{1}{x\sqrt{x^2 - 4}}\, dx$ *(Exame de Recurso, Cálculo I - Agrupamento IV, 2017/2018)*

17. Determine a função f tal que

$$f'(x) = \frac{5x - 4}{x(x^2 - 2x + 2)} \quad \text{e} \quad \lim_{x \to +\infty} f(x) = 0.$$

(Exame Época Normal, Cálculo I, 2008/2009)