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STRUCTURAL DYNAMICS

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QUESTION A

a) Establish the equations of motion and the boundary conditions for the system under consideration. Then, calculate the first three eigenfrequencies and design the corresponding eigenmodes of the model with negligible damping. To be compared with the special cases where the beam is semi-articulated or the stiffness of the support springs is negligible.

In Figure 1. the model of the mechanical arrangement with which we will deal is presented. Table 1 presents the data for our system.

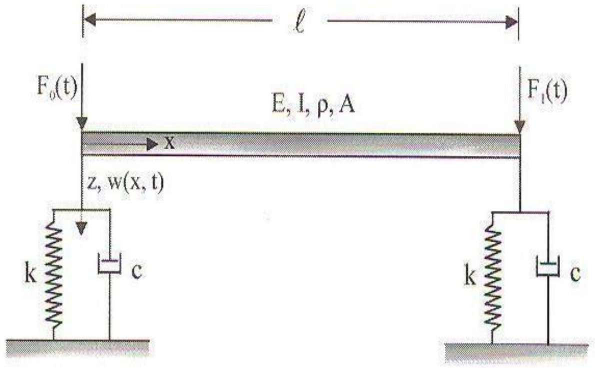


Figure 1. Mechanical layout model, consisting of a beam with known properties

Data			
E	2.1x1011[N/m2]	d	0.015[m]
r	7880 [kg/m3]	fo	50[N]
A	1.767 10^{-4}[m2]	fl	50[N]
k	150[N/m]	Phi	p/4
l	1 [m]	Iyy	2.485 10^{-9}[m4]

Table 1. Mechanical layout data

In Table 1. given the surface A, and the surface moment of inertia Iyy are calculated as follows for a circular cross-section:

$$A = \frac{\pi}{4}\, d^2 \;\; [\, m^2]$$

$$I_{yy} = \int_A z^2 dA$$

where by changing variables we end up with the relation

$$I_{yy} = \frac{\pi}{4} R^4 [\, m^4]$$

First you ask us to set up the equations of motion and the boundary conditions. To achieve this a Free Body Diagram (FBD) is required. The FBD is shown in Figure 2.

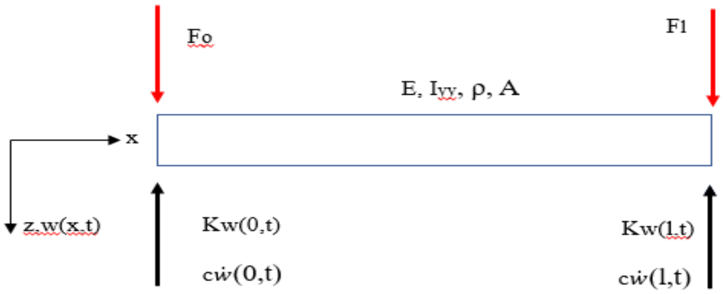


Figure 2. Free body diagram for the beam.

Before proceeding with the construction of the equations of motion, 3 basic assumptions must be considered.

- Ox axis, connects the geometric centers$\Rightarrow \int_A y\, dA = \int_A z\, dA = 0$
- Oy,Oz principal axes, therefore$\int_A yz\, dA = 0$
- Kirchhoff hypothesis : w=w(x,t) and the cross sections of the beam remain perpendicular to Ox$\Rightarrow \gamma_{xz} = \frac{\partial w}{\partial t} + \frac{\partial u}{\partial z} = 0$

Since there is no external load N=EA=0u_0'⇒u_0'=0 Therefore the axial deformation εx = = -zw'' and correspondingly the stress from V.Hooke σφ =E εx = -zEw''$\frac{\partial u}{\partial x}$

The moment in y, with respect to the center of the cross-section, due to normal stresses is calculated from the integral:

My = for constant geometric properties$\int_A z\sigma_x\, dA \;=\; \int_A - z^2$Ew'' dA

My = -Iyy (1) Ew''

Next we isolate an infinite part of the beam, so that we can calculate the transverse load Q(x,t) of our carrier. Applying the equilibrium conditions in Figure 3. we have:

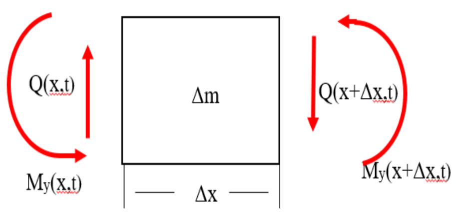


Figure 3. FBD of an infinite part of the carrier

$$\Sigma F_z = \Delta m \ddot{w} \xRightarrow{\Delta m = \rho A \Delta x}$$

$$Q(x+\Delta x) - Q(x,t) = \rho A \ddot{w}$$

By expanding the term Q(x+Δx,t) in Taylor series and keeping only the first order terms we finally have:$Q' = \rho A \ddot{w} \quad (2)$

and$\Sigma M_y = \Delta I_y\, \ddot{\varphi}$where ΔIy = ρIyΔx (mass moment of inertia) and for small changes of w (w'<<) φ= - w' so I have:

$$M(x+\Delta x) - M(x,t) - Q(x+\Delta x)\frac{\Delta x}{2} - Q(x,t)\frac{\Delta x}{2} = \rho I \Delta x(-\ddot{w}')$$

along with (1)$Q = \frac{\partial M}{\partial x} = -(EIw'')'$ (3)

From (2), (3) the equation of motion is derived:$\frac{\partial^2 y}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A\frac{\partial^2 w}{\partial t^2} = 0$ (4)

To find the boundary conditions it is necessary to apply an ideal section to the two ends and to make the FBD of the resulting bodies. Thus, with the following procedure, 4 dynamic boundary conditions arise:

<u>for x=0</u>

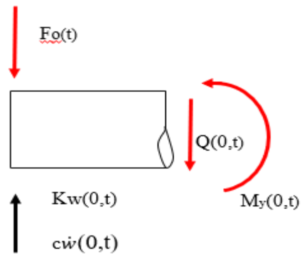


Fig. 4. FBD Section at position x=0

$$\Sigma F_z = \Delta m\ddot{w}(0,t) \xrightarrow{\Delta m \to 0}$$

$$Q(0,t) + F_0(t) - c\dot{w}(0,t) - kw(0,t) = 0$$

$$\overset{(3)}{\Rightarrow} -EIw'''(0,t) + F_0(t) - c\dot{w}(0,t) - kw(0,t) = 0 \quad (5)$$

$$\Sigma M_y = \Delta I_y\ddot{\varphi}(0,t) \xrightarrow{\Delta I \to 0} \quad M(0,t) = 0$$

$$\overset{(1)}{\Rightarrow} w''(0,t) = 0 \quad (6)$$

<u>for x=l</u>

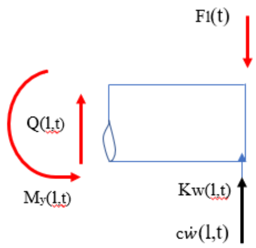


Fig. 5. FBD Section at position x=l

$$\Sigma F_z = \Delta m\ddot{w}(l,t) \xrightarrow{\Delta m \to 0}$$

$$-Q(l,t) + F_l(t) - c\dot{w}(l,t) - kw(l,t) = 0$$

$$\overset{(3)}{\Rightarrow} EIw'''(l,t) + F_l(t) - c\dot{w}(l,t) - kw(l,t) = 0 \quad (7)$$

$$\Sigma M_y = \Delta I_y\ddot{\varphi}(l,t) \xrightarrow{\Delta I \to 0} \quad M(l,t) = 0$$

$$\overset{(1)}{\Rightarrow} w''(l,t) = 0 \quad (8)$$

and initial conditions : w(x,0) = wo(x) ,$\dot{w}(x,0) = \widehat{w_0}(x)$

Therefore, in summary for our system we have:

<u>Equation of Motion</u>

$$\frac{\partial^2 y}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \qquad (4)$$

<u>Boundary Conditions</u>

$$-EIw'''(0,t) + F_0(t) - c\dot{w}(0,t) - kw(0,t) = 0 \qquad (5)$$

$w''(0,t) = 0$ (6)

$$EIw'''(l,t) + F_l(t) - c\dot{w}(l,t) - kw(l,t) = 0 \quad (7)$$

$w''(l,t) = 0$ (8)

Initial conditions

w(x,0) = wo(x) ,$\dot{w}(x,0) = \widehat{w_0}(x)$

To calculate the eigenfrequencies and eigenmodes we apply the Eigenmode Analysis Method. First, the eigenproblem must be solved, for which the terms of external loads and damping are omitted from the equation of motion and the boundary conditions. Therefore, the system for which we will solve the eigenproblem, with fixed properties as the enunciation gives us, comes in the form:

Equation of Motion

$$EIw^{(4)}(x,t) + \rho A\ddot{w}(x,t) = 0 \quad (9)$$

Boundary Conditions

$$-EIw'''(0,t) - kw(0,t) = 0 \quad (10)$$

$$w''(0,t) = 0 \quad (11)$$

$$EIw'''(l,t) - kw(l,t) = 0 \quad (12)$$

$w''(l,t) = 0$ (13)

First, the variables are separated, so we look for a solution of the form W(x,t)=X(x)T(t), by substitution in (9) we get:

$$\frac{EIX^{(4)}}{\rho AX} = -\frac{\ddot{T}}{T} = \alpha = -\omega^2$$

For the first equality to be valid, the two quantities on the right and on the left must be equal to a constant a. We choose this constant equal toω^2.

We arrive at the ordinary differential equation:

$$EIX'''' - \omega^2\rho AX = 0 \quad \Rightarrow \quad X'''' - \beta^4 X = 0 \quad (14)$$

$$\beta^4 = \frac{\rho A}{EI}\omega^2$$

Applying the 4th boundary conditions to equation (14) we get the following system of equations:

$$\begin{cases} -EIX'''(0) = kw(0,t) \\ X''(0) = 0 \\ EIX'''(l) = kw(l,t) \\ X''(l) = 0 \end{cases} \quad (15)$$

The problem that you make up from equation (14) and the system of equations (15) is a regular 4th-order Sturm-Liouville type problem with unknown constants ωnand the functions Xn(x) with n=1,2,....∞.

The solutions are of the form X(x) =α, by substitution in equation (14) we arrive at the equation$e^{\lambda x}$

$\lambda^4 - \beta^4 = 0$⇒λ1,2 = ±iβ and λ3,4 = ±β

using Euler's formulas we arrive at a solution of the form:

$$X_n(x) = a_1 \sin(\beta_n x) + a_2 \cos(\beta_n x) + a_3 \sin h(\beta_n x) + a_4 \cosh(\beta_n x) \quad (16)$$

n=1,2,...∞.

For the analysis that will follow, the calculation of the first three spatial derivatives is required so that we can easily apply the boundary conditions to equation (16) and its derivatives.

$$X'_n(x) = a_1\beta_n\, cos(\beta_n x) - a_2\beta_n \sin(\beta_n x) + a_3\beta_n \sin h(\beta_n x) + a_4\beta_n \cosh(\beta_n x)$$

$$X''_n(x) = -a_1\beta_n^2\, sin(\beta_n x) - a_2\beta_n^2 \sin(\beta_n x) + a_3\beta_n^2 \sin h(\beta_n x) + a_4\beta_n^2 \cosh(\beta_n x)$$

$$X'''_n(x) = -a_1\beta_n^3\, sin(\beta_n x) + a_2\beta_n^3 \sin(\beta_n x) + a_3\beta_n^3 \sin h(\beta_n x) + a_4\beta_n^3 \cosh(\beta_n x)$$

Application 2ndSS $X''(0) = 0$ we have :

$$\beta^2(a_4 - a_2) = 0$$

where β=0 or α2 = α4 .

For β=0 I have ω=0 and which corresponds to a solid body singularity, with$X(x) = ax + b$

For β≠0⇒(17)$a_4 = a_2$

Application of the 1st SS $-EIX'''(0) = kw(0,t)$ we have :

$$EI{\beta_n}^3(\alpha_1 - \alpha_3) = 2k\alpha_2 \quad (18)$$

Application of 3rd SS $EIX'''(l) = kw(l,t)$ we have :

$$EI\beta^3(-\alpha_1 \cos(\beta_n l) + a_2(\sin(\beta_n l) + \sinh(\beta_n l)) + a_3 \cos(\beta_n l)) \\ = k(a_1 \sin(\beta_n l) + a_2(\cos(\beta_n l) + \cosh(\beta_n l)) + a_3 \sinh\,(\beta_n l)) \Rightarrow$$

$\alpha_1(EI\beta^3 \cos(\beta_n l) + k) + + \sin(\beta_n l)\ \ a_2(\cos(\beta_n l) + \cosh(\beta_n l) - EI{\beta_n}^3 \sin(\beta_n l) - EI\beta^3 \sinh(\beta_n l))\, a_3(\sinh(\beta_n l) - EI\beta^3 \cos(\beta_n l)) = 0 \quad (19)$

Application of 4th SS $X''(l) = 0$ we have :

$-\alpha_1 \sin(\beta_n l) - \alpha_2(\cos(\beta_n l)\text{-}\cosh(\beta_n l)) + a_3 \sinh(\beta_n l) = 0 \quad (20)$

Therefore, to calculate the infinite values of the eigenfrequencies and to find values for the constants ai so as to approximate the eigenmodes, it is necessary to solve the system of equations (17),(18),(19),(20) which in matrix form is :

$$\begin{bmatrix} EI{\beta_n}^3 & 2k & -EI{\beta_n}^3 \\ EI\beta^3\cos(\beta_n l) + k\sin(\beta_n l) & B & \sinh(\beta_n l) - EI\beta^3\cos(\beta_n l) \\ -\sin(\beta_n l) & -\cos(\beta_n l) + \cosh(\beta_n l) & \sinh(\beta_n l) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$$

with B=(the table does not fit)$\cos(\beta_n l) + \cosh(\beta_n l) - EI{\beta_n}^3 \sin(\beta_n l) - EI\beta^3 \sinh(\beta_n l)$

For the system to be of the form A(ω^2)$\boldsymbol{a}$=0 non-zero solution (otherwise for a zero solution I have a solid body) the determinant of matrix A must be equal to zero.$|\mathrm{A}(\omega^2)| = 0$

The characteristic equation resulting from the discriminant equation will give us the eigenfrequencies. The equation is a complex transcendental function of ω2 and is therefore solved numerically. Using Matlab we give successive values with a small step to the variable ω, and see where the sign of the determinant changes. Between the 2 values that change sign for ω, we define as frequency ωi the average of the 2 values. Next, for each ω we calculated, we substitute in register A and calculate the coefficients αi as a function of one of them. After choosing some real value for the independent coefficient, we will have the eigenmodes Xi with eigenfrequency ωi . Appendix A1 shows the programming code used to derive the following results.

You ask us for the calculation and visualization of the first 3 eigenfrequencies and eigenmodes. The first eigenfrequency we calculate is that of the solid body for ω=0. For zero eigenfrequency the eigenfunction has the form:

$$X(x) = a_1 x + a_2$$

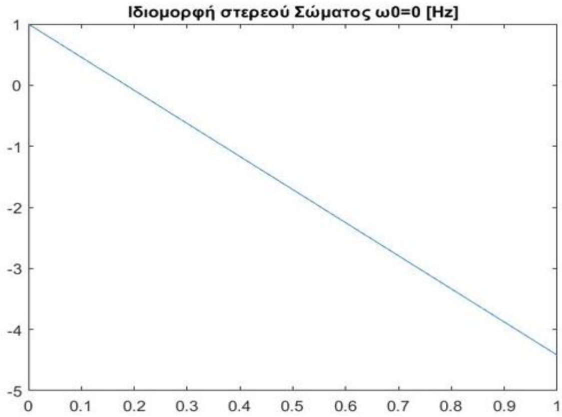
Arbitrarily choosing the value for one of the 2 coefficients we can then apply the orthonormality condition,

$$\int_0^l \rho A X_r X_s dx = \delta_{rs}$$

where δrs =1 for r=s. Let a2 = 1, then:
Solving this integral yields α1 =-5.42 thus

$$X(x) = -5.42x + 1$$

In Figure 6, the solid body singularity corresponding to pure translation and pure rotation for the beam without deformations is plotted.



$X(x) = -5.42x + 1$

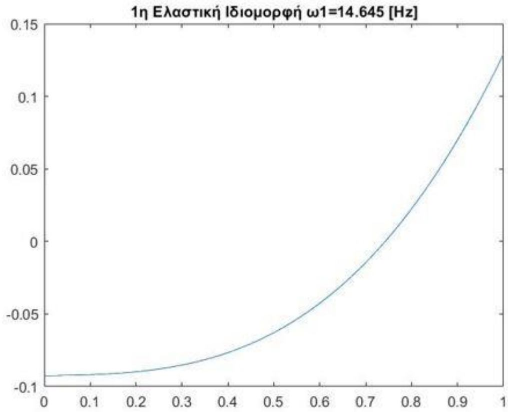
The first 3rd elastic eigenfrequencies and eigenmodes are respectively for independent coefficient α3 = 1:

ω1 =14.645 Hz$X_1(x) = -0.99\sin(0.86x) - 0.0463\,(\cos(0.86x) + \cosh\,(0.88x)) + \sinh\,(0.86x)$

ω2 = 25.415 Hz $X_2(x) = 0.8\sin\big(1.14x\big) - 0.51\,(\cos(1.14x) + \cosh\,(1.14x)) + \sinh\,(1.14x)$

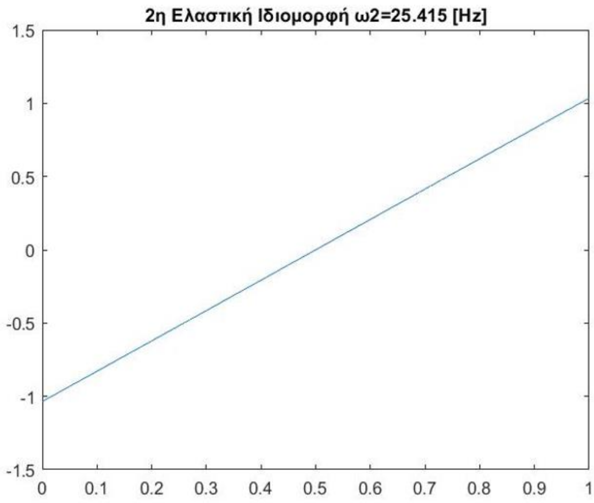
ω3 = 434.11 Hz$X_3(x) = 0.99\sin(4.73x) - 1.01\,(\cos(4.73x) + \cosh\,(4.73x)) + \sinh\,(4.73x)$

In Figures 7-9 are the visualizations of the first 3 elastic eigenmodes and the solid body eigenmodes.



$X_1(x) = -0.9942\sin(0.86x) - 0.0463\,(\cos(0.86x) + \cosh(0.88x)) + \sinh(0.86x)$

Figure 7. 1st Elastic Eigenmode



$$X_2(x) = 0.8\sin(1.14x) - 0.51\,(\cos(1.14x) + \cosh(1.14x)) + \sinh(1.14x)$$

Figure 8. 2nd Elastic Eigenmode

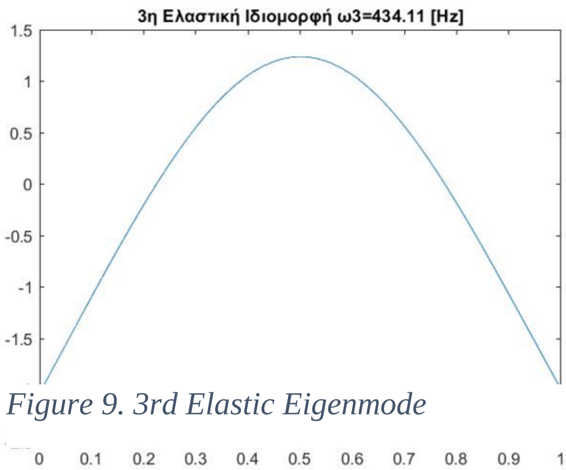


Figure 9. 3rd Elastic Eigenmode

$$X_3(x) = 0.99\sin(4.73x) - 1.01\,(\cos(4.73x) + \cosh(4.73x)) + \sinh(4.73x)$$

Looking at the 3 elastic eigenmodes in Figures 7 to 9 we see that as the value of the eigenfrequency increases, the ranges and complexity of the modes increase.

Still it is evident that the larger singularities could be regarded as a form of superposition of the previous ones. Something that is more noticeable in less complex problems, such as the bi-articulated beam that will be presented later.

If we had chosen orthonormal singularities we could also compare the ranges at each point. We avoided this option because of the high complexity of the integrations that they require.

You ask us to compare the above eigenmodes with those of the bi-jointed beam and the free beam. To verify the functionality of our code and additionally obtain the results for comparison, we will calculate the requests with the same code, changing the spring stiffness value. More specifically for a bi-articulated beam we will consider and for free , the resulting singularities can be checked from the literature for their validity.$k \rightarrow \infty k \rightarrow 0$

(k=<u>$k \rightarrow \infty$1500000)</u>

The first two eigenfrequencies are calculated:
ω1 = 189.7 Hz
ω2 = 743.35 Hz
and are presented in Figure 10. By referring to the book of oscillations (p. 245) the validity of the eigenmodes and therefore of the programming code is verified.

Now comparing it with the elastic singularities we calculated earlier we see that the fundamental singularity of the bi-jointed beam is the same in form, but not in magnitude, as the 3rd. Also, the higher eigenfrequencies in the biarticulate than in our problem are evident.

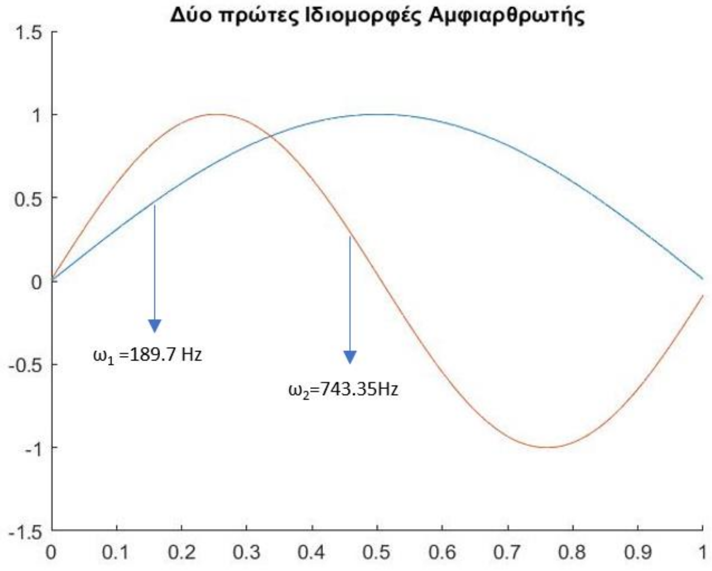


Figure 10. Bifurcated Beam Idiomorphs

$k \rightarrow 0$

We observe in Figure 11, the singularity that results for zero robustness at the bearing. It is obvious, as we expected, that this singularity depicts a simple transfer of the beam, since by setting the stiffness to zero we simulated a free beam model.

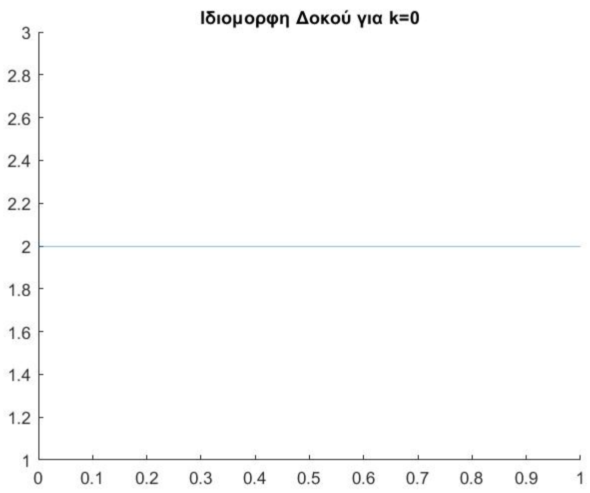


Figure 11. Solid body singularity

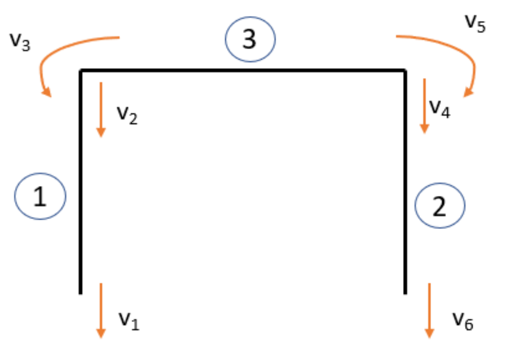
QUESTION B

b) Calculate the first three eigenfrequencies of the system without damping, resulting from replacing the beam with a bending element. Then, to design the corresponding singularities of the system. Repeat the process by discretizing the beam with two bending elements and compare the results.

To calculate the first three eigenfrequencies, first the discretization of our system must be done and the degrees of freedom must be expressed in the correct order in the local and the general coordinate system.

Our system will be converted to a 3-element model, 1 axial element for each spring with zero mass and in the first phase a bending element the beam.

Figure 12 shows the modeling in the general coordinate system with displacement vectors v. According to Figure 13. the relation of general-local coordinates is presented, with the help of Figure 13. the localization matrices will be created by means of which we will then be able to sum the matrices of K and M. The numbers in the circles in the two figures represent the number of each item.



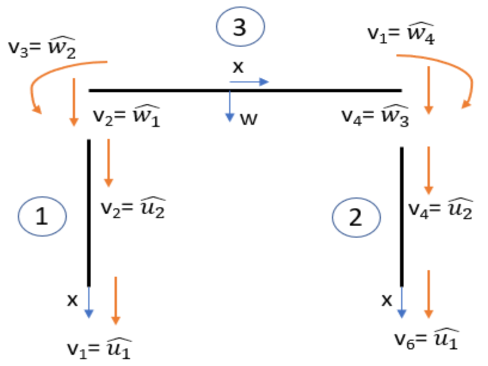


Figure 12. General coordinate system Figure 13. Local-General system correlation

Looking at Figure 13. we extract for each element its locator matrix :

$$l_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, ,l_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} l_3 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Then, in order to arrive at the specific problem that we seek to solve, we must calculate the Kinetic and Dynamic energy, for the calculation of these it is necessary to determine the matrices M and K respectively. In order to be able to calculate the elements of these registers, it is first necessary to define the shape functions which will express the displacements of each element along the entire element.

<u>For the bending elements we have:</u>

We have 4 shape functions since we also have 4 degrees of freedom, through the boundary conditions at the ends of each element the following shape functions arise.

$$\varphi_1(x) = 1 - \left(\frac{x}{l}\right)^2 + 2\left(\frac{x}{l}\right)^3$$

$$\varphi_2(x) = l((\frac{x}{l}) - \left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3)$$

$$\varphi_3(x) = 3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3$$

$$\varphi_4(x) = l(-\left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3)$$

and the draft results from the sum: $w(x,t) = \sum_{n=1}^{4} \varphi_n(x) w_n(t)$

where the displacements at the ends of the bending element .

Then we calculate the kinetic energy of the carrier and arrive at the relation:

$$T = \frac{1}{2}\dot{\boldsymbol{w}}^T M \dot{\boldsymbol{w}}$$

with the mass register elements calculated from the integral

(21)$m_{ij} = \int_0^l \rho A \varphi_i \varphi_j dx$

Accordingly, we calculate the potential energy of the carrier and arrive at the relation:

$$V = \frac{1}{2}\boldsymbol{w}^T K \boldsymbol{w}$$

with the elements of the robustness register calculated from the integral

$$k_{ij} = \int_0^l EI \varphi_i''(x) \varphi_j''(x)\, dx \quad (22)$$

Finally, the matrices K,M in the local coordinate system are calculated for the bending element.

$$K_{e2} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}, ,M_{e2} = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

For the axial elements (springs), the K,M 2x2 matrices are calculated respectively after first calculating the form functions of the axial elements

$\varphi_1(x) = 1 - \frac{x}{l}, \varphi_2(x) = \frac{x}{l}$

and the displacement results: $u(x,t) = \sum_{n=1}^{2} \varphi_n(x) u_n(t)$

calculating the kinetic and potential energy again, the matrices are obtained:

$$K_{e1} = K_{e3} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

$$M_{e1} = M_{e3} = \mathbf{0} \;\; \alpha\varphi o\acute{\upsilon}\; m = 0$$

Having now calculated the matrices in the local coordinate systems, we transform them to the general coordinate system by means of the locating tables. More specifically :

$$\begin{array}{c} [2 \quad 3 \quad 4 \quad 5] \\ K_{e2}l_2 = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \end{array}$$

$$K_2 = \frac{EI}{l^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6l & -12 & 6l & 0 \\ 0 & 6l & 4l^2 & -6l & 2l^2 & 0 \\ 0 & -12 & -6l & 12 & -6l & 0 \\ 0 & 6l & 2l^2 & -6l & 4l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{array}{c} [2 \quad 3 \quad 4 \quad 5] \\ M_{e2}l_2 = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \end{array}$$

$$M_2 = \frac{\rho A l}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 22l & 54 & -13l & 0 \\ 0 & 22l & 4l^2 & 13l & -3l^2 & 0 \\ 0 & 54 & 13l & 156 & -22l & 0 \\ 0 & -13l & -3l^2 & -22l & 4l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Accordingly, for the axial elements, the matrices are calculated in the general coordinate system:

$$K_1 = \begin{bmatrix} k & -k & 0 & 0 & 0 & 0 \\ -k & k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, ,K_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & -k \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & k \end{bmatrix}$$

The total matrices of the system which we will use to solve the eigenproblem are obtained by adding the individual registers. Furthermore, since the springs rest on the ground and have zero displacements at the points of contact with it, we assume that v1 = v6 =0. For this reason, from the total matrices we remove the columns and rows where offsets 1 and 6 are located. Therefore:

K=K1 + K2 + K3

$$K = \frac{EI}{l^3}\begin{bmatrix} 12 + \frac{kl^3}{EI} & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 + \frac{kl^3}{EI} & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

M=M1 + M2 + M3 = M2

$$M = \frac{\rho Al}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

now having the expression of the energies (kinetic and dynamic) we insert them into the Lagrange equation and end up with a system of ordinary differential equations:

$$M\ddot{\boldsymbol{v}} + K\boldsymbol{v} = \boldsymbol{f}(t)$$

with a vector, we don't need to calculate the vector of charges since in this particular problem we are dealing with solving the eigenproblem and finding eigenfrequencies and eigenmodes.$\mathbf{v} = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$

Solving the eigenproblem requires that we find a non-zero solution to the system:

$$|K - \omega^2 M\,|\boldsymbol{v} = 0$$

The calculation of the desired quantities is done using the Matlab software and the first 3 eigenfrequencies and the first 3 eigenvectors of our eigenproblem are obtained respectively:

ω1 =14.6486 Hz$\boldsymbol{v_1} = \begin{bmatrix} 0.8841 \\ 0.02 \\ 0.8441 \\ -0.0202 \end{bmatrix}$

ω2 =25.4166 Hz$\boldsymbol{v_2} = \begin{bmatrix} 1.46 \\ -2.92 \\ -1.46 \\ -2.92 \end{bmatrix}$,$l_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

ω3 = 520.487 Hz $\boldsymbol{v_3} = \begin{bmatrix} 1.896 \\ -11.36 \\ 1.8964 \\ 11.3694 \end{bmatrix}$

where by means of the localization register we map the displacements to the eigenvectors in the local coordinate system for the bending element, so that we can then plot the eigenmodes one by one. For the design of the singularities, now having the ranges of freedom for the first 3 singularities, we apply the sum

$$w(x,t) = \sum_{n=1}^{4} \varphi_n(x) w_n(t)$$

for=$w_n(t)\widehat{w_n}$cos $(\omega_i \mathrm{t} - \theta_i)$

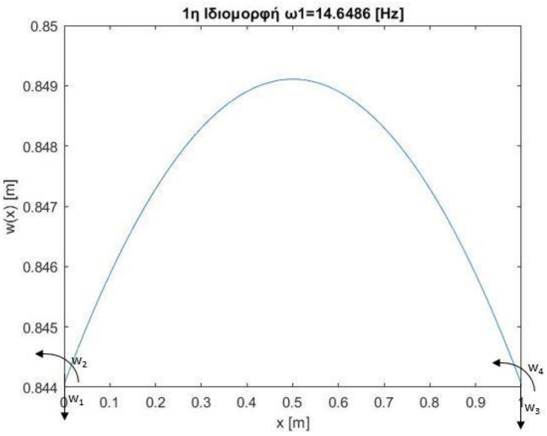
Since we are dealing with visualization of eigenmodes, we are only interested in spatial terms. Therefore with the shape functions we defined above and the

$\widehat{w_n}$ $n = 1,2,3,4$which represent the values in the eigenvectors, the eigenmodes are calculated from the relation:

$$W_n = \sum_{n=1}^{4} \varphi_n(x) \widehat{w_n}$$

thus the following singularities arise.

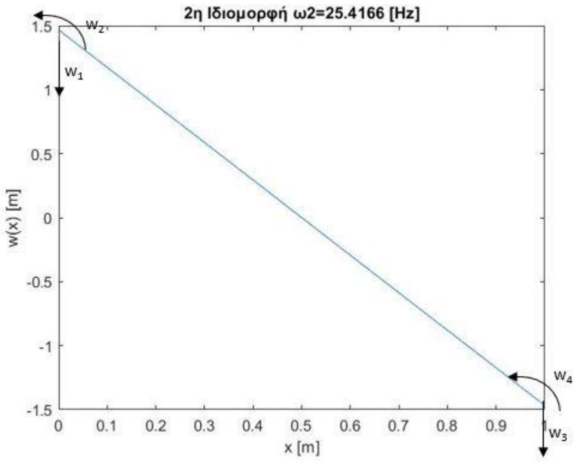
<u>**1st**singular</u>



$$\omega 1 = 14.6486 \text{ Hz} \boldsymbol{v_1} = \begin{bmatrix} 0.8841 \\ 0.02 \\ 0.8441 \\ -0.0202 \end{bmatrix}$$

Figure 14. 1st Eigenmode for a bending element

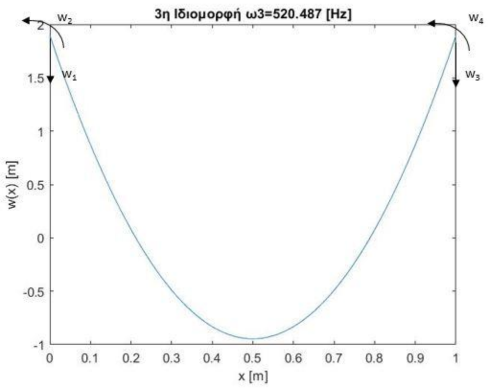
2ndsingular



$$\omega 2 = 25.4166 \text{ Hz} \boldsymbol{v_2} = \begin{bmatrix} 1.46 \\ -2.92 \\ -1.46 \\ -2.92 \end{bmatrix}$$

Figure 15. 2nd Eigenmode for a bending element

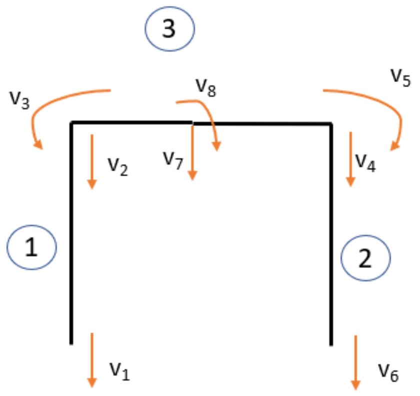
3rdsingular



$$\omega 3 = 520.487 \text{ Hz } \boldsymbol{v_3} = \begin{bmatrix} 1.896 \\ -11.36 \\ 1.8964 \\ 11.3694 \end{bmatrix}$$

Figure 16. 3rd Eigenmode for a bending element

In the second part of question b, you ask us to repeat the process but this time during the discretization we will split the beam into two bending elements. Therefore, our new model is presented in Figure 14. in the general coordinates and correspondingly in Figure 14. the mapping of general and local coordinates is done.



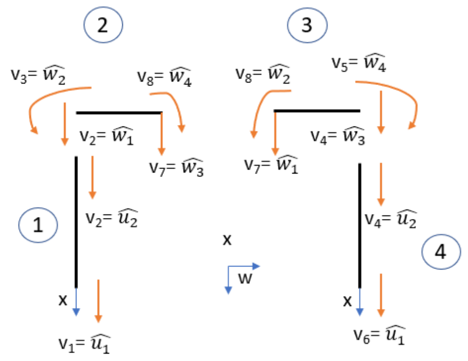


Figure 17. General System coordinates Figure 18. Local-General Coordinates Relationship

What differs in the 2-element analysis for the beam is the locating matrices and the total registers. On the other hand the shape functions used to express the offsets remain the same, as do the calculation equations for the M,K matrices.

The new location matrices are derived from Figure 18. as follows:

$$l_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, , , l_2 = \begin{bmatrix} 2 \\ 3 \\ 7 \\ 8 \end{bmatrix} \; l_3 = \begin{bmatrix} 7 \\ 8 \\ 4 \\ 5 \end{bmatrix} \; l_4 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Then we calculate the local mass and robustness matrices as before the relations:

$$m_{ij} = \int_0^l \rho A \varphi_i \varphi_j dx$$

$$k_{ij} = \int_0^l EI\varphi_i''(x)\varphi_j''(x)\, dx$$

as before. So the resulting local matrices for the bending elements are exactly the same as before, with the only difference that now instead of l, we have l/2 .

$$K_{e2} = K_{e3} = \frac{EI}{(0.5l)^3} \begin{bmatrix} 12 & 60.5l & -12 & 3l \\ 60.5l & 40.25l^2 & -3l & 0.5l^2 \\ -12 & -60.5l & 12 & -3l \\ 6l & 0.5l^2 & -3l & l^2 \end{bmatrix},$$

$$,M_{e2} = M_{e3} = \frac{\rho A 0.5l}{420}\begin{bmatrix} 156 & 11l & 54 & -06.5l \\ 11l & l^2 & 6.5l & -0.75l^2 \\ 54 & 6.5l & 156 & -11l \\ -6.5l & -0.75l^2 & -11l & l^2 \end{bmatrix}$$

Accordingly for the axial elements the matrices are the same as before:

$$K_{e1} = K_{e3} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

$$M_{e1} = M_{e3} = \mathbf{0} \ \ \alpha\varphi o\acute{\upsilon}\, m = 0$$

Using the localization matrices we transform our matrices from the local to the global coordinate system, and we have:

<u>Element 2</u>

[2 3 7 8]

$$K_{e2}l_2 = \frac{EI}{(0.5l)^3}\begin{bmatrix} 12 & 3l & -12 & 3l \\ 3l & l^2 & -3l & 0.5l^2 \\ -12 & -3l & 12 & -3l \\ 3l & 0.5l^2 & -3l & l^2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 7 \\ 8 \end{bmatrix}$$

$$K_2 = \frac{EI}{(0.5l)^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 3l & 0 & 0 & 0 & -12 & 3l \\ 0 & 3l & l^2 & 0 & 0 & 0 & -3l & 0.5l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -3l & 0 & 0 & 0 & 12 & -3l \\ 0 & 6l & 0.5l^2 & 0 & 0 & 0 & -3l & l^2 \end{bmatrix},$$

[2 3 7 8]

$$M_{e2}l_2 = \frac{\rho A 0.5l}{420}\begin{bmatrix} 156 & 11l & 54 & -6.5l \\ 11l & l^2 & 6.5l & -0.75l^2 \\ 54 & 6.5l & 156 & -11l \\ -6.5l & -0.75l^2 & -11l & l^2 \end{bmatrix},\begin{bmatrix} 2 \\ 3 \\ 7 \\ 8 \end{bmatrix}$$

$$M_2 = \frac{\rho A 0.5l}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 11l & 0 & 0 & 0 & 54 & -6.5l \\ 0 & 11l & l^2 & 0 & 0 & 0 & 6.5l & -0.75l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54 & 6.5l & 0 & 0 & 0 & 156 & -11l \\ 0 & -6.5l & -0.75l^2 & 0 & 0 & 0 & -11l & l^2 \end{bmatrix}$$

<u>Item 3</u>

$$[7 \quad 8 \quad 4 \quad 5]$$

$$K_{e3}l_3 = \frac{EI}{(0.5l)^3}\begin{bmatrix} 12 & 3l & -12 & 3l \\ 3l & l^2 & -3l & 0.5l^2 \\ -12 & -3l & 12 & -3l \\ 3l & 0.5l^2 & -3l & l^2 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \\ 4 \\ 5 \end{bmatrix}$$

$$K_3 = \frac{EI}{(0.5l)^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & l^2 & 0 & -12 & 3l \\ 0 & 0 & 0 & -3l & l^2 & 0 & -6l & 0.5l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -12 & -3l & 0 & 12 & 3l \\ 0 & 0 & 0 & 3l & 0.5l^2 & 0 & 3l & l^2 \end{bmatrix},$$

$$[7 \quad 8 \quad 4 \quad 5]$$

$$M_{e3}l_3 = \frac{\rho A 0.5l}{420}\begin{bmatrix} 156 & 11l & 54 & -6.5l \\ 11l & l^2 & 6.5l & -0.75l^2 \\ 54 & 6.5l & 156 & -11l \\ -6.5l & -0.75l^2 & -11l & l^2 \end{bmatrix},\begin{bmatrix} 7 \\ 8 \\ 4 \\ 5 \end{bmatrix}$$

$$M_3 = \frac{\rho A 0.5l}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 156 & -11l & 0 & 54 & 6.5l \\ 0 & 0 & 0 & -11l & l^2 & 0 & -6.5l & -0.75l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 54 & -6.5l & 0 & 156 & 11l \\ 0 & 0 & 0 & 6.5l & -0.75l^2 & 0 & 11l & l^2 \end{bmatrix}$$

Accordingly, for the axial elements, the matrices are calculated in the general coordinate system:

$$K_1 = \begin{bmatrix} k & -k & 0 & 0 & 0 & 0 & 0 & 0 \\ -k & k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

The total matrices of the system which we will use to solve the eigenproblem are obtained by adding the individual registers. Furthermore, since the springs rest on the ground and have zero displacements at the points of contact with it, we assume that v1 = v6 =0. For this reason, from the total matrices we remove the columns and rows where offsets 1 and 6 are located. Therefore:

K=K1 + K2 + K3 + K4

$$K = \frac{EI}{(0.5l)^3} \begin{bmatrix} 12 + \frac{k(0.5l)^3}{EI} & 3l & 0 & 0 & -12 & 3l \\ 3l & l^2 & 0 & 0 & -3l & 0.5l^2 \\ 0 & 0 & \frac{k(0.5l)^3}{EI} + 12 & -3l & -12 & -3l \\ 0 & 0 & -3l & l^2 & -6l & 0.5l^2 \\ -12 & -3l & -12 & -3l & 24 & 0 \\ 6l & \frac{l^2}{2} & -3l & 0.5l^2 & 0 & 2l^2 \end{bmatrix}$$

M=M1 + M2 + M3 = M2

$$M = \frac{\rho Al}{840}\begin{bmatrix} 156 & 11l & 0 & 0 & 54 & -6.5l \\ 11l & l^2 & 0 & 0 & 6.55l & -0.75l^2 \\ 0 & 0 & 156 & -11l & 54 & 6.5l \\ 0 & 0 & -11l & l^2 & -6.5l & -0.75l^2 \\ 54 & 6.5l & 54 & -6.5l & 312 & 0 \\ -6.5l & -0.75l^2 & 6.5l & -0.75l^2 & 0 & 2l^2 \end{bmatrix}$$

now having the expression of the actions (kinetic and dynamic) we insert them into the Lagrange equation, as before and end up with a system of ordinary differential equations:

$$M\ddot{\boldsymbol{v}} + K\boldsymbol{v} = \boldsymbol{f}(t)$$

with a vector, we don't need to calculate the vector of charges since in this particular problem we are dealing with solving the eigenproblem and finding eigenfrequencies and eigenmodes.$\mathbf{v} = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_7 \\ v_8 \end{bmatrix}$

Solving the eigenproblem requires that we find a non-zero solution to the system:

$$|K - \omega^2 M\ |\boldsymbol{v} = 0$$

The calculation of the desired quantities is done using the Matlab software and the first 3 eigenfrequencies and the first 3 eigenvectors of our eigenproblem are obtained respectively:

ω1 =14.6431 Hz$\boldsymbol{v_1} = \begin{bmatrix} 0.8434 \\ 0.0202 \\ 0.8434 \\ -0.0202 \\ 0.8497 \\ -0.000001 \end{bmatrix}$

ω2 =25.4155 Hz$\boldsymbol{v_2} = \begin{bmatrix} 1.4669 \\ -2.9269 \\ 1.4669 \\ -2.9269 \\ -0.000001 \\ -2.9400 \end{bmatrix}$, , $l_2 = \begin{bmatrix} 2 \\ 3 \\ 7 \\ 8 \end{bmatrix}$ $l_3 = \begin{bmatrix} 7 \\ 8 \\ 4 \\ 5 \end{bmatrix}$

ω3 = 435.0829 Hz $\boldsymbol{v_3} = \begin{bmatrix} -1.6999 \\ 7.9067 \\ -1.6999 \\ -\ 7.9067 \\ 1.0371 \\ 0.000001 \end{bmatrix}$

where by means of the localization register we map the displacements to the eigenvectors in the local coordinate system for the bending element, so that we can then plot the eigenmodes one by one. For the design of the singularities, now having the ranges of freedom for the first 3 singularities, we apply the sum

$$w(x,t) = \sum_{n=1}^{4} \varphi_n(x) w_n(t)$$

for=$w_n(t)\widehat{w_n}\cos\left(\omega_i \mathrm{t} - \theta_i\right)$

Since we are dealing with visualization of eigenmodes, we are only interested in spatial terms. Therefore with the shape functions we defined above and the

$\widehat{w_n}\ \ n = 1{,}2{,}3{,}4{,}5{,}6$which represent the values in the eigenvectors, the eigenmodes are calculated from the relation:

$$W_n = \sum_{n=1}^{4} \varphi_n(x)\widehat{w_n}$$

we apply this relationship for each bending element separately. Therefore we apply it 2 times

the first with$\widehat{\boldsymbol{w_{1n}}}$ for n=1,2,5,6

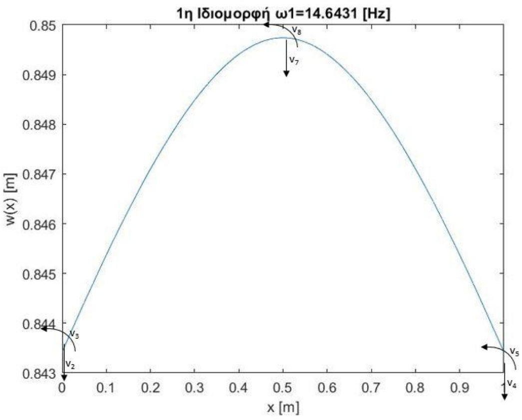
$$W_n = \varphi_1(x)\widehat{v_{2n}} + \varphi_2(x)\widehat{v_{3n}} + \varphi_3(x)\widehat{v_{7n}} + \varphi_4(x)\widehat{v_{8n}}$$

the second for n=5,6,3,4$\widehat{\boldsymbol{w_{2n}}}$

$$W_n = \varphi_1(x)\widehat{v_{7n}} + \varphi_2(x)\widehat{v_{8n}} + \varphi_3(x)\widehat{v_{4n}} + \varphi_4(x)\widehat{v_{5n}}$$

thus the following singularities arise

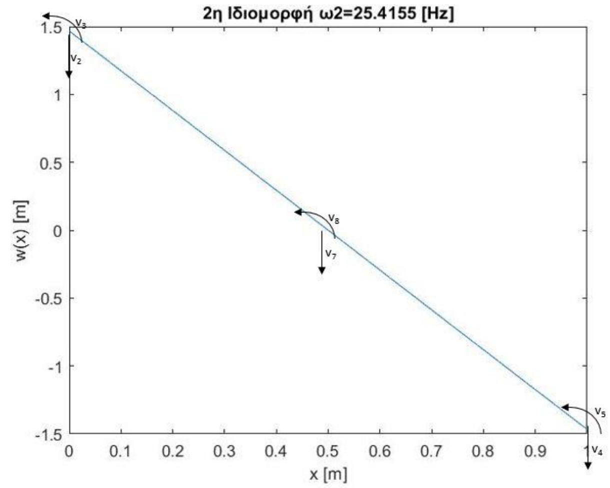
1st eigenmode



ω1 =14.6431 Hz$\boldsymbol{v_1} = \begin{bmatrix} 0.8434 \\ 0.0202 \\ 0.8434 \\ -0.0202 \\ 0.8497 \\ -0.000001 \end{bmatrix}$

Figure 19. 1st EigenMode for 2 bending element

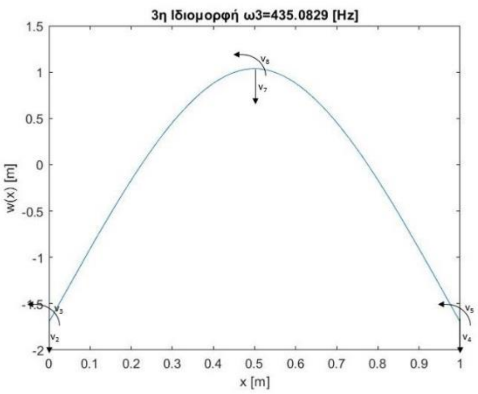
2^{nd} eigenmode



$$\omega 2 = 25.4155\ \text{Hz}\ \boldsymbol{v_2} = \begin{bmatrix} 1.4669 \\ -2.9269 \\ 1.4669 \\ -2.9269 \\ -0.000001 \\ -2.9400 \end{bmatrix}$$

Figure 19. 2nd Eigenmode for 2 bending element

3^{rd} eigenmode



$$\omega 3 = 435.0829\ \text{Hz}\ \boldsymbol{v_3} = \begin{bmatrix} -1.6999 \\ 7.9067 \\ -1.6999 \\ -7.9067 \\ 1.0371 \\ 0.000001 \end{bmatrix}$$

Figure 20. 3rd Eigenmode for 2 bending element

Comparison of the 2 different discretizations

We first observe that the discretization with 2 bending elements offers lower eigenfrequencies deviations from the exact solution than the one bending element. Observing the singularities we see that both cases (1 and 2 flexural) are very close to the singularities of the exact solution. More specifically, with the two bending elements we manage to better approximate the ranges of the eigenmodes. Obviously, as we increase the number of modeling elements, our solution will be continuously improved, until we arrive at the exact solution with zero error, of course, depending on the application and the desired accuracy, we can also work with 2 bending elements as in question (b).

QUESTION C

c)Discretize the beam by applying Lagrange's approximation method and assuming that its response has the form

***w(x,t)* =*q*$_1$(t)+*xq*$_2$(t)+sin(Pi*x/l*)*q*$_3$(t) .**

Compare the eigenfrequencies of the resulting model with the corresponding exact eigenfrequencies. Also, to calculate and design the dependence of the fundamental natural frequency on the robustness of the bearing of the beam. In what range of values is the fundamental natural frequency significantly affected by bearing deformability?

From the approximation of the response given by the pronunciation, we extract the 3 form functions, based on which the approximation will be made:

W1 =1 , W2 = x , W3 = sin(πx/l)

For the application of Lagrange's method, it will help us to initially calculate the first time and the second spatial derivative of the response given to us in the utterance.

$$w(x,t) = q_1(t) + xq_2(t) + \sin\left(\frac{\pi x}{l}\right) q_3(t) \quad (21)$$

$$\dot{w}(x,t) = \dot{q_1}(t) + x\dot{q_2}(t) + \sin\left(\frac{\pi x}{l}\right) \dot{q_3}(t) \quad (22)$$

$$w'(x,t) = q_1(t) + \frac{\pi}{l}\cos\left(\frac{\pi x}{l}\right) q_3(t) \qquad (23)$$

$$w''(x,t) = -\frac{\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right) q_3(t) \qquad (24)$$

and the Lagrange equation:

$$\frac{d}{dx}\left(\frac{\partial T}{\partial \dot{q_J}}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad , i = 1,2,3 \qquad (25)$$

To apply Lagrange's approximation method, we need to calculate kinetic and potential energy. After these calculations are done, we apply the resulting equations to equation (25) through which we will finally arrive at the matrix form of the simplified system, from which the time unknown coefficients will be calculated$q_n(t)$.

Potential energy consists of 3 terms. Two for the two bases and one term from the internal energy of our continuous one-dimensional carrier (beam) and is calculated from the following relationship:

$$V = V_b + \frac{1}{2}k\big(w(0,t)\big)^2 + \frac{1}{2}k\big(w(l,t)\big)^2 \qquad (26)$$

First we calculate the internal energy term:

$V_b = \frac{1}{2}\int_V E\varepsilon_x^2(x,t)dV = \frac{1}{2}\int_0^l \int_A E(-zw''(x,t))^2 dAdx = \frac{1}{2}EI\int_0^l w''(x,t)^2 dx$

By substituting equation (24), we get:

$$V_b = \frac{-EI\pi^4}{2l^3}q_3^2$$

For the actions of the springs we have by substituting the responses at positions 0 and l:

$$V_{\varepsilon\lambda} = kq_1^2 + klq_1q_2 + \frac{1}{2}kl^2q_2^2$$

Substituting the last 2 relations in equation (26) we end up with a relation of the form:

$$V = \frac{1}{2}\boldsymbol{q}^T K\boldsymbol{q} \quad (27)$$

with data for the K register, resulting from

the internal energy of the body:$k_{ij} = \int_0^l EIW_i''(x)W_j''(x)\,dx$

the potential energy of the springs: for α=0 and l$k_{ij}^a = W_i(a)W_j(a)k$

where W are the shape functions.

Thus, from the above, the robustness register K is calculated:

$$K = \begin{bmatrix} 2k & lk & 0 \\ lk & l^2k & 0 \\ 0 & 0 & -\frac{EI\pi^4}{4l^3} \end{bmatrix} \quad (28)$$

Accordingly, the kinetic energy for our system is calculated:

$$dT = \frac{1}{2}dm\dot{w}^2(x,t)$$

By completing the above relation and replacing the equation of the response w(x,t) we end up with an equation of the form:

$$T = \frac{1}{2}\boldsymbol{q}^T M\boldsymbol{q} \quad (29)$$

with the mass register elements calculated from the equation:

$$m_{ij} = \int_0^l \rho AW_i(x)W_j(x)\,dx$$

Thus, from the above, the mass register M is calculated:

$$M = \frac{1}{2}\rho A l \begin{bmatrix} 2 & l & \frac{4}{\pi} \\ l & \frac{2}{3}l^2 & \frac{2l}{\pi} \\ \frac{4}{\pi} & \frac{2l}{\pi} & 1 \end{bmatrix} \quad (30)$$

Substituting equations (27),(28),(29),(30) into the Lagrange equation (25) our system comes in the form:

$$M\ddot{q} + Kq = Q$$

For now we are interested in the calculation of the eigenfrequencies, for this reason we do not calculate the vector Q, since the above will result from the solution of the eigenproblem.

To calculate the first 3 eigenfrequencies, we look for a non-zero solution to the eigenproblem :

$|K - \omega^2 M|\boldsymbol{\alpha}$=**0**

By replacing the values in the elements of the register K,M we arrive at the equation:

$$\left|\begin{bmatrix} 300 & 150 & 0 \\ 150 & 150 & 0 \\ 0 & 0 & -12708 \end{bmatrix} - \omega^2 \begin{bmatrix} 1.392 & 0.6963 & 0.8865 \\ 0.6963 & 0.46 & 0.443 \\ 0.886 & 0.443 & 0.6963 \end{bmatrix}\right| = 0$$

from the development of the above determinant we arrive at the characteristic equation:

$$-0.0213\omega^6 - 2014\omega^4 + 1754000\omega^2 - 2.8\ 10^7 = 0 \quad (31)$$

By numerical solution using Matlab, the first 3 elastic eigenfrequencies are obtained with values:

Exact Solution		*Lagrange solution*	
ω_1	14,645 Hz	ω_1	14,518 Hz
ω_2	25,415 Hz	ω_2	25,559 Hz
ω_3	434.11 Hz	ω_3	308.96 Hz

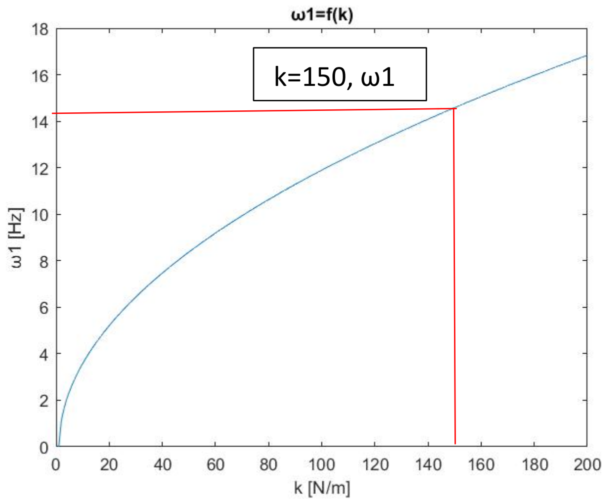
Table 2. Exact and Lagrange Solution Eigenfrequencies

Observing the values of Table 2. we can compare the eigenfrequencies we calculated with those of the exact solution of question (a). We see that the first 2 elastic natural frequencies (since the 1st is generally of the solid body) are almost identical in value, which indicates the validity and correct application of the Lagrange method. On the other hand, of course, we see a large deviation in the values of the 3rd natural frequency. Therefore, the lagrange approximation method by approximating the response to the form given in the utterance is satisfactory for low values of ω, respectively for a better approximation of the high eigenfrequencies we would succeed with a different choice of the form functions (Wi) in our initial hypothesis.

In the 2nd part of this question, you ask us to calculate and plot the dependence of the fundamental natural frequency (ω1) on the robustness of the bearing (k).

Applying the same procedure again and solving the determinant we arrive at the characteristic equation. This time, of course, we keep both ω and k as variables. After obtaining this equation by giving successive values to k, we observe the change of the first eigenfrequency.

Due to the complexity of the final equation, the above process is carried out using the Matlab software and the resulting diagram is presented in Figure 12.



Looking at Figure 21. we see that the maximum slope of the curve is in the range of values 0-30 approximately. In this range of values we also have the sharpest change of the fundamental natural frequency. We also see, as we expected, that for k=0 we have solid body motion and as k→∞ the fundamental natural frequency increases continuously.

Figure 21. Dependence of ω1 on k

QUESTION D

d)Determine the depreciation rate*c*, so that the one-degree-of-freedom system resulting if the rod is assumed to be solid and transported in the vertical direction only, exhibits a damping measureg=0.01. Then, to determine the frequency-response diagram for the acceleration and the maximum stress that develops in the middle of the beam, when it is subjected to excitation in the form of

F0(t) **=*f*⁰cos(Ot)and*fl(t)* =*f*ˆcos(Ωt +*Phi*).** ℓ ℓ

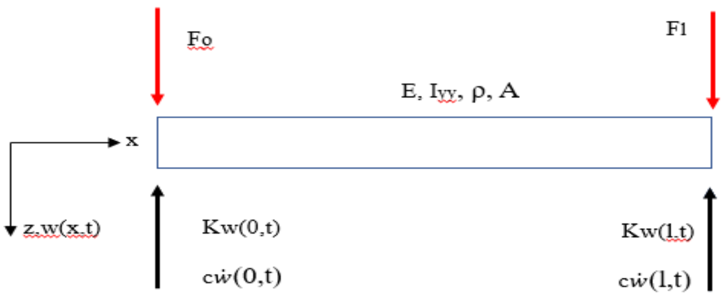


Figure 22. DES beam

Initially, to calculate the damping coefficient, we work with the beam as a solid body with one degree of freedom. We apply Newton's law for w=w(t).

$$F\boldsymbol{e}_{\boldsymbol{z}} = m_b\ddot{w}(t),\ \ m_b = \rho A l \Rightarrow F_0 + F_l - 2kw(t) - 2c\dot{w} = \rho A\ddot{w}(t) \Rightarrow$$

$$\ddot{w}(t) + \frac{2c}{\rho A}\,\dot{w} + \frac{2k}{\rho A}w = \ F_0 + F_l \quad (31)$$

for an oscillatory system with 1 degree of freedom we know:

$$\ddot{w}(t) + 2\zeta\omega_0\dot{w} + {\omega_0}^2 w = 0 \qquad (32)$$

Equating (31), (32) we have for the depreciation coefficient:

$$c = \zeta\omega_0\rho A = \zeta\sqrt{2k\rho A}$$

therefore for ζ=0.01 , damping factor c=0.288 [kg/s]

For a response of the form:$w(x,t) = q_1(t) + xq_2(t) + \sin\left(\frac{\pi x}{l}\right)q_3(t)$

and the modified Lagrange equation:

$$\frac{d}{dx}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_j} = Q_i \quad ,i = 1,2,3 \qquad (25)$$

where the new term$\frac{\partial D}{\partial \dot{q}_j}$represents the energy dissipated by the dampers. As calculated in the previous questions:

with$V = \frac{1}{2}\boldsymbol{q}^T K\boldsymbol{q} \quad K = \begin{bmatrix} 2k & lk & 0 \\ lk & l^2k & 0 \\ 0 & 0 & -\frac{EI\pi^4}{4l^3} \end{bmatrix}$

with$T = \frac{1}{2}\boldsymbol{q}^T M\boldsymbol{q}$ $\quad M = \frac{1}{2}\rho A l\begin{bmatrix} 2 & l & \frac{4}{\pi} \\ l & \frac{2}{3}l^2 & \frac{2l}{\pi} \\ \frac{4}{\pi} & \frac{2l}{\pi} & 1 \end{bmatrix}$

first we enter the energies in the Lagrange equation, it remains to calculate the Rayleigh scattering function D=D(c0)+D(cl)

and of the excitation vector Qn(t).

<u>Scattering Energy</u>

Calculated from the equation:.$D = \frac{1}{2}c\dot{w}^2(0,t) + \frac{1}{2}c\dot{w}^2(l,t)$

By substituting the expression for the response at the boundary, the elements of the register C are obtained in the form:

$$c_{ij}^a = W_i(a)W_j(a)c \quad (34)$$

with shape functions on the boundary:

W1(0)=1 , W2(0)=0 , W1(l)=1 , W2(l)=l , W3(0)=0, W3(l)=0 (35)

From (34), (35) :

$c_{11}^0 = c, c_{11}^l = c$

$c_{12}^0 = 0 c_{12}^l = cl$

$c_{21}^0 = 0 c_{21}^l = lc$

$c_{22}^0 = 0 c_{22}^l = cl^2$

$c_{3i} = 0 \; \gamma\iota\alpha \;\; i = 1,2,3$

We conclude:$D = \frac{1}{2}\boldsymbol{q}^T \begin{bmatrix} 2c & lc & 0 \\ lc & l^2c & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{q}$

<u>Load Qn</u>

It is calculated from the integral where the distributed charge q in our case is expressed with the help of Dirac's delta function as:$Q_n(t) = \int_0^l q(x,t)W_n(x)dx \; q(x,t) = F_o\delta(x) + F_l\delta(x-l)$

therefore for n =1,2,3

$Q_n(t) = \sum_{n=1}^3 \int_0^l \big(F_o\delta(x) + F_l\delta(x-l)\big)W_n(x)dx \; \delta q_n$

$$Q_1(t) = F_0W_1(0) + F_lW_1(l) = F_0 + F_l$$

$$Q_2(t) = F_0W_2(0) + F_lW_2(l) = lF_l$$

$$Q_3(t) = F_0 W_3(0) + F_l W_3(l) = 0$$

Therefore:

$$Q(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} F_0 + \begin{bmatrix} 1 \\ l \\ 0 \end{bmatrix} F_l$$

Substituting the 3 actions T,V,D and the register of the charge in the Lagrange equation, we end up with a system of the form:

$$M\ddot{q} + C\dot{q} + Kq = Q(t) \quad (37)$$

To calculate the response and subsequently the acceleration and the maximum stress, the time coefficients qi(t) need to be calculated for loads of the form : , (38)$F_0 = \widehat{f_o}\cos{(\Omega t)} F_l = \widehat{f_l}\cos{(\Omega t - \varphi)}$

Before proceeding with the process of calculating the time coefficients, it is necessary to transform the trigonometrics in relations (38) so that we can use them in a more functional form. Using the trigonometric identities we have:

$$\cos(\Omega t - \varphi) = A\cos(\Omega t) - Bsin(\Omega t)$$

with$A = Rcos(-\varphi) = \frac{\sqrt{2}}{2}R \qquad B = Rsin(-\varphi) = -\frac{\sqrt{2}}{2}R$

with R=1, finally: cos(Ωt-φ)=$\frac{\sqrt{2}}{2}\cos(\Omega t) + \frac{\sqrt{2}}{2}sin(\Omega t)$

so for the excitation vector we have:

(1) +$\widehat{f_o}\cos{(\Omega t)}\ \widehat{f_l}\cos(\Omega t - \varphi) = \widehat{f_o}\cos(\Omega t) + \widehat{f_l}[\frac{\sqrt{2}}{2}\cos(\Omega t) + \frac{\sqrt{2}}{2}sin(\Omega t)]$

$$= \cos(\Omega t)\left[\widehat{f_o} + \widehat{f_l}\frac{\sqrt{2}}{2}\right] + \frac{\sqrt{2}}{2}\widehat{f_l}sin(\Omega t)$$

(2)$l\widehat{f_l}\cos(\Omega t - \varphi) = l\widehat{f_l}[\frac{\sqrt{2}}{2}\cos(\Omega t) + \frac{\sqrt{2}}{2}sin(\Omega t)]$

presenting the above two equations in matrix form we have:

$$Q(t) = \begin{bmatrix} \widehat{f_o} + \widehat{f_l}\dfrac{\sqrt{2}}{2} \\ l\widehat{f_l}\dfrac{\sqrt{2}}{2} \\ 0 \end{bmatrix}\cos(\Omega t) + \begin{bmatrix} \widehat{f_l}\dfrac{\sqrt{2}}{2} \\ l\widehat{f_l}\dfrac{\sqrt{2}}{2} \\ 0 \end{bmatrix} sin(\Omega t)$$

$$Q(t) = \widehat{\boldsymbol{f_c}}\cos(\Omega t) + \widehat{\boldsymbol{f_s}}\sin{(\Omega t)}$$

Therefore, the differentiated system has the form:

$$M\ddot{q} + C\dot{q} + Kq = Q(t) = \widehat{\boldsymbol{f_c}}\cos(\Omega t) + \widehat{\boldsymbol{f_s}}\sin{(\Omega t)} \quad (39)$$

We are asked for frequency response plots, so the values of qi are calculated for the steady state. After these are calculated, we insert them into the expression of the response and calculate the variables we want.

We assume a solution of the form (40)$q(t) = q_c \cos(\Omega t) + q_s \sin(\Omega t)$

and we substitute in (39) the solution we assumed.

This results in 2 equations where by matching the trigonometric coefficients the system is obtained:

$$\begin{bmatrix} K - \Omega^2 M & \Omega C \\ \Omega C & \Omega^2 M - K \end{bmatrix} \begin{bmatrix} q_c \\ q_s \end{bmatrix} = \begin{bmatrix} f_c \\ f_s \end{bmatrix}$$

By inverting the 4x4 register, qc ,qs are calculated. Then we replace them in equation (40) and thus we end up with the expressions of qi i=1,2,3.

$$\begin{bmatrix} q_c \\ q_s \end{bmatrix} = \begin{bmatrix} K - \Omega^2 M & \Omega C \\ \Omega C & \Omega^2 M - K \end{bmatrix}^{-1} \begin{bmatrix} f_c \\ f_s \end{bmatrix}$$

so we arrive at the answer$w(x,t) = q_1(t) + x q_2(t) + \sin\left(\frac{\pi x}{l}\right) q_3(t)$

We are asked for the frequency response plots for the acceleration and the maximum stress at the middle of the beam, so we first calculate the 2nd time and spatial derivatives of w for x=l/2:

$$\ddot{w}\left(\frac{l}{2}, t\right) = \ddot{q_1} + \frac{l}{2}\ddot{q_2} + \ddot{q_3} \qquad (41)$$

$$w''\left(\frac{l}{2}, t\right) = -\frac{\pi^2}{l^2} q_3 \qquad (42)$$

The 2nd derivatives of q are then calculated:

$q_1(t) = q_{1c}\cos(\Omega t) + q_{1s}\sin(\Omega t)$,$\ddot{q_1}(t) = -\Omega^2 q_{1c}\cos(\Omega t) - \Omega^2 q_{1s}\sin(\Omega t)$ (43)

$q_2(t) = q_{2c}\cos(\Omega t) + q_{2s}\sin(\Omega t)$,$\ddot{q_2}(t) = -\Omega^2 q_{2c}\cos(\Omega t) - \Omega^2 q_{2s}\sin(\Omega t)$(44)

$q_3(t) = q_{3c}\cos(\Omega t) + q_{3s}\sin(\Omega t)$,$\ddot{q_3}(t) = -\Omega^2 q_{3c}\cos(\Omega t) - \Omega^2 q_{3s}\sin(\Omega t)$ (45)

Substituting (43)-(45) into (41),(42) I have:

- $\ddot{w}\left(\frac{l}{2}, t\right) = \cos(\Omega t)\left(-\Omega^2 q_{1c} - \Omega^2 q_{2c} - \Omega^2 q_{3c}\right) - \sin(\Omega t)(-\Omega^2 q_{1s} - \Omega^2 q_{2s} - \Omega^2 q_{3s})$

- $w''\left(\frac{l}{2}, t\right) = -\frac{\pi^2}{l^2}\left(q_{3c}\cos(\Omega t) + q_{3s}\sin(\Omega t)\right)$

In order to be able to visualize the frequency response diagrams we have to keep a trigonometric in the final expression of the response so that the amplitude of the oscillation can be isolated. For this reason we apply once again the identity: Acos(x)-Bsin(x)=Rcos(c+θ) and (we have:

For acceleration

with $A = (-\Omega^2 q_{1c} - \Omega^2 q_{2c} - \Omega^2 q_{3c})$, $B = (-\Omega^2 q_{1s} - \Omega^2 q_{2s} - \Omega^2 q_{3s})$

$$\ddot{w}\left(\frac{l}{2}, t\right) = R_1 \cos(\Omega t + \theta_1), \quad R_1 = \sqrt{A^2 + B^2}, \qquad \theta_1 = \tan^{-1}\frac{B}{A}$$

For the stress

$$w''\left(\frac{l}{2}, t\right) = R_2 \cos(\Omega t + \theta_2)\,, R_2 = \frac{\pi^2}{l^2}\sqrt{{q_{3c}}^2 + {q_{3s}}^2}, \qquad \theta_2 = \tan^{-1}\frac{q_{3s}}{q_{3c}}$$

For the mid-beam acceleration-response frequency plot, we plot R1(Ω). The execution of all the above analysis operations is carried out through the Matlab software, and so finally, by giving values to Ω (with a small step), we end up with the graph shown in Figure 23.

Looking at the diagram though we would expect 3rd resonances since we calculated 3rdeigenfrequencies in question (c), we see 2. This is due to the fact that the 3rd eigenmode is not necessarily excited by the charges. It is also obvious that at the 2 resonance frequencies the acceleration shows extremes, since there we have the maximum oscillation ranges. Finally it is observed that as Ω tends to infinity, the range of acceleration increases continuously.

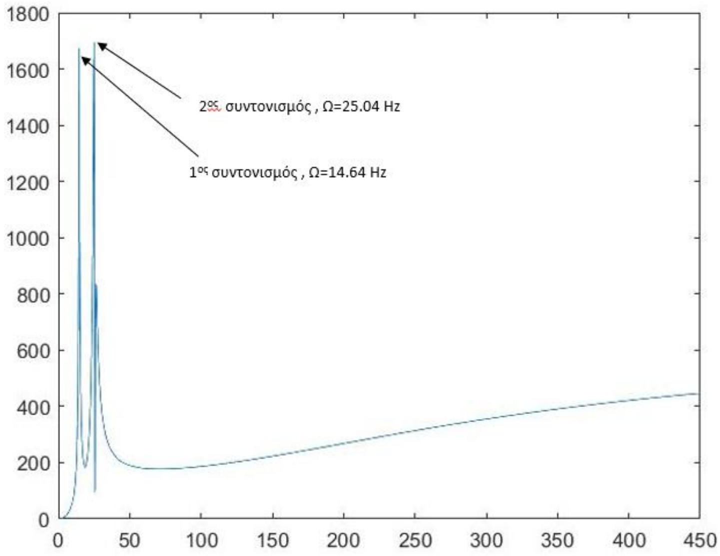


Figure 23. Frequency-response plot for acceleration R1(Ω)

To create the frequency-response diagram for the peak stress we first need to express the peak stress equation:

$$\sigma_{max} = E\varepsilon_x \xRightarrow{\varepsilon_x=-zw''} \sigma_{max} = -Ez_{max}w'' \quad \mu\varepsilon\ z_{max} = \frac{d}{2}$$

Since we know the stress distribution in a bending beam, we also know that the maximum stress is at the boundary of the beam cross-section, hence z_{max} =d/2.

After all :$\sigma_{max}(\frac{l}{2}) = -Ez_{max}R_2\cos(\Omega t + \theta_2)$

The frequency response diagram for the maximum stress is shown in Figure 24. Observing the diagram we see that the large increase in stress occurs mainly during the 2nd resonance, at such a value that for a real structure it would be destructive. We can also distinguish a small increase in stress during the 1st resonance, which of course does not manage to have the effects it has during the 2nd resonance.

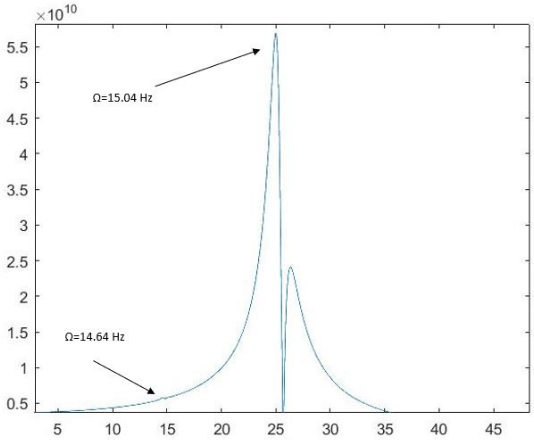


Figure 24. Frequency-response diagram for peak stress R2(Ω)