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STRUCTURAL DYNAMICS

NON-LINEAR ELECTRIC ENGINE'S SUPPORT

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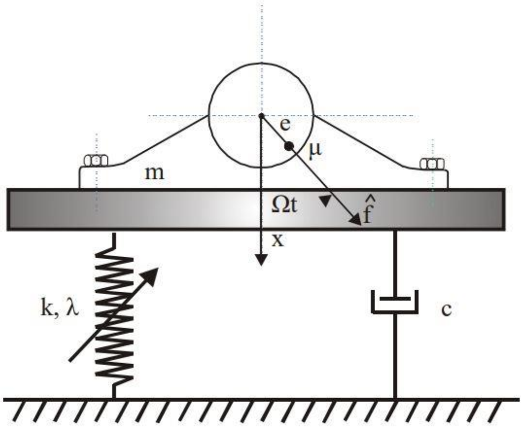
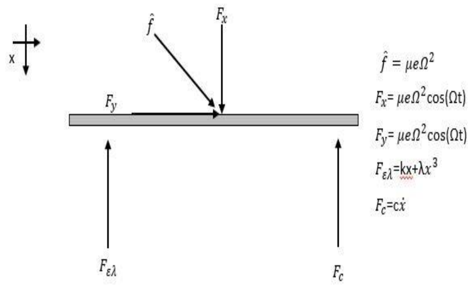
This report studies the dynamical aspects and conducts the desired analysis for a mechanical system. The mechanical system is shown in Figure 1. and includes an electric motor and its mounting system. As noted in Figure 1, the support exhibits Duffing-type stiffness and linear damping. The motor has a mass m and excites the mechanical system with a static imbalance represented by a mass μ << m , at a distance e from the axis of its rotating component. The motor rotates at a constant angular velocity Ω.



Shape1. Mechanical system of an electric motor and its support



Shape2. FBD mechanical layout

Figure 2 shows the free body diagram (FBD) for our layout. The Duffing spring force is equal to:$F_{\varepsilon\lambda} = kx + \lambda x^3$

The force caused by the imbalance is equal to:

$$\hat{f} = \mu e \Omega^2$$

Therefore, after first analyzing the imbalance in its components on the x and y axis, observing Figure 2 we arrive at the following differential equation:

$$\Sigma F_x = m\ddot{x} \;\Rightarrow F - F_{\varepsilon\lambda} - F_c = m\ddot{x} \Rightarrow \mu e \Omega^2 \cos(\Omega t) - kx - \lambda x^3 - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx + \lambda x^3 = \mu e \Omega^2 \cos(\Omega t) \quad (\mathbf{1})$$

1st PROBLEM

In the 1st part we will undimensionalize the differential equation of our mechanical system. Then the frequency-response and phase-response diagrams will be constructed for a state of main resonance and a specific range for the detuning factor σ. In the 2nd part, the response diagrams of the mechanical device will be constructed for various values of the damping measure ζ, and the influence will be commented of the initial conditions during the transitory and permanent state.

Transform variables to dimensionless

We define dimensionless time and dimensionless displacement as follows:

$$\tau = \omega_o t \qquad u(\tau) = \frac{x(t)}{x_c}$$

where : oo the natural frequency of the linear oscillator

xc the distance between the natural length position of the spring and the equilibrium position of the device

deriving the above relationships with respect to time t we have:

$$d\tau = \omega_o dt \;\Rightarrow\; \dot{\tau} = \omega_o \;(\mathbf{2})$$

$$\dot{x} = \frac{dx}{dt} = x_c \frac{du(\tau)}{dt} = x_c \frac{du(\tau)}{d\tau}\frac{d\tau}{dt} \overset{(2)}{\Rightarrow} \dot{x} = x_c \omega_o u'(\tau) \;\; (\mathbf{3})$$

where (u prime) is the derivative of the variable u, with respect to the dimensionless time $\tau . u'(\tau)$

We now also calculate the 2nd time derivative of x :

$$\ddot{x} = x_c {\omega_o}^2 u''(\tau) \;(\mathbf{4})$$

Now we replace relations (3), (4) in (1) and we have:

$$m x_c {\omega_o}^2 u''(\tau) + c x_c \omega_o u'(\tau) + k\, x_c\, u(\tau) + \lambda\,(x_c\, u(\tau))^3 = \mu e \Omega^2 \cos(\Omega t)$$

$$\xRightarrow{:m x_c {\omega_o}^2} u''(\tau) + \frac{c}{m\omega_o} u'(\tau) + \frac{k}{m{\omega_o}^2} u(\tau) + \frac{\lambda x_c^2}{m{\omega_o}^2} u(\tau)^3 = \frac{\mu e \Omega^2}{m x_c {\omega_o}^2}\cos(\Omega t)$$

we have for the linear eigenfrequency of a single-stage oscillator $\omega_o = \sqrt{\frac{k}{m}}$

we also know from the linear theorem for z:$\zeta = \frac{c}{2m\omega_o}$

and for the dimensionless frequency η, is given by the expression:$\eta = \frac{\Omega}{\omega_o}$

Substituting these 3rd variables and the dimensionless time into the last equation we get:$\tau = \omega_o t$,

$$u''(\tau) + 2\zeta u'(\tau) + u(\tau) + \varepsilon u(\tau)^3 = P\eta^2 \cos(\eta\tau)$$

ή $$\ddot{u}(\tau) + 2\zeta \dot{u}(\tau) + u(\tau) + \varepsilon u(\tau)^3 = P\eta^2 \cos(\eta\tau) \quad (\mathbf{5})$$

with parameters :

$\varepsilon = \frac{\lambda x_c^2}{m\omega_o{}^2}$

$P = \frac{\mu e}{m x_c}$

Amplitude-frequency and phase-frequency plots for principal resonance

You then ask us to study the change in amplitude and phase of the oscillation for varying excitation frequency(s). In particular, we study the region of the main resonance in the steady state. We know that for nonlinear oscillators, the amplitude and phase of the oscillation are functions of time. Therefore we need to find a relationship between the variables we want to visualize.$\eta = 1 + \varepsilon\sigma$

The variable σ is a detuning parameter that represents the difference between the frequencies Ω and ω. To allow the range of stimulation to interact in the same order of magnitude as the damping and nonlinearity terms, P=εf and ζ=εd are chosen.

The existence of the non-linear term in the dimensionless equation (5) prevents us from finding exact solutions. Therefore we will look for analytical approximate solutions with the method of multiple time scales. By means of this method our solution is expressed in the form:$u(\tau; \varepsilon) = u_o(\tau_o; \tau_1) + \varepsilon u_1(\tau_o; \tau_1)$.. $(\mathbf{6})$

where ε<<1 and thus for our solution we keep terms up to order ε, the rest are omitted considering that with them we get satisfactory accuracy.

The multi-time scale method essentially considers 2 different time scales for our dynamic problem. This is due to the fact that in our perturbation problem (declining oscillation) we have 2 different dynamical processes running on different time scales. More specifically since we are referring to a decreasing oscillation, we can say that we have the periodic decrease(to) and the exponential decrease(τ1) of the oscillation range. By considering different times we essentially distinguish the 2 different dynamic processes and study them separately. Looking at Figure 3, which shows the response of the free oscillation of a linear oscillator, the definition of these times is understood. The times we consider are:

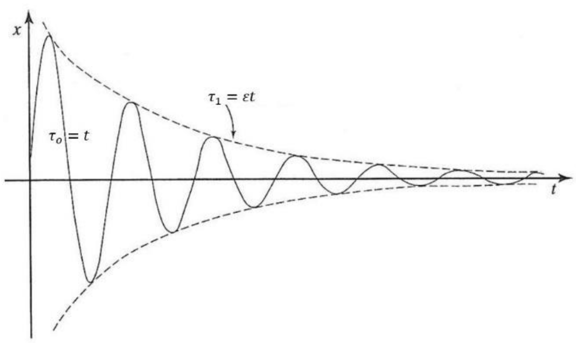


Figure 3. Damped oscillation of a single-stage linear oscillator

Fast time : τ0 = τ

Slow time: τ1 = ετ = ετo

Having now maintained accuracy of solutions up to 1st order, we calculate the 1st and 2nd derivatives of equation (6) using the chain derivative rule and defining its derivative operators we have:$D_o = \frac{\partial}{\partial \tau_o}$, $D_1 = \frac{\partial}{\partial \tau_1}$

$$\dot{u_n} = \frac{du}{d\tau} = \frac{\partial u_n}{\partial \tau_o}\frac{d\tau_o}{d\tau} + \frac{\partial u_n}{\partial \tau_1}\frac{d\tau_1}{d\tau} = \frac{\partial u_n}{\partial \tau_o} + \varepsilon\frac{\partial u_n}{\partial \tau_1} = (D_o + \varepsilon D_1)u_n$$

$$\ddot{u_n} = \frac{d\dot{u_n}(\tau_o, \tau_1)}{d\tau} = (D_o^2 + \varepsilon 2D_oD_1)u_n$$

Applying the above derivatives to our solution will give us:

$$\dot{u}(t;\varepsilon) = (D_o + \varepsilon D_1)[u_o(\tau_o, \tau_1) + \varepsilon u_1(\tau_o, \tau_1)]$$

$$\dot{u}(t;\varepsilon) = (D_o^2 + \varepsilon 2D_oD_1)[u_o(\tau_o, \tau_1) + \varepsilon u_1(\tau_o, \tau_1)]$$

where we have kept the terms up to 1st order, then we replace the last 2 derivatives in equation (5):

$$(D_o^2 + 2\varepsilon D_oD_1)[u_o + \varepsilon u_1] + 2\zeta(D_o + \varepsilon D_1)[u_o + \varepsilon u_1] + [u_o + \varepsilon u_1] + \varepsilon[u_o^3 + \varepsilon u_1^3] = P\eta^2\cos(\eta\tau)$$

After performing the derivations in the last equation, and introducing the assumptions for the range P and the damping measure ζ, we distinguish the terms of zero and 1st order with respect to ε and arrive at the following system:

$$\varepsilon^0 \ : \ \ D_o^2 u_o + u_o = 0$$

$$\varepsilon^1 : 2D_oD_1u_o + D_o^2u_1 + 2\delta D_ou_o + u_1 + u_o^3 = f\eta^2\cos(\eta\tau)$$

The 1st equation of the above system is a linear equation of a single-stage oscillator and for this equation we have the exact homogeneous solution:

$$u_o = \alpha(\tau_1)\cos(\tau + \varphi(\tau_1))$$

Since we consider that the magnitudes of the phase and amplitude during the transient state of the oscillation change slowly with time during a period (T) and that their values during this period are kept approximately constant, then they will depend, approximately, only from the slow time τ1 of the exponential decay of the oscillation. The complex form of the above solution, for our convenience, is written:

$$u_o = \alpha(\tau_1)\cos\big(t + \varphi(\tau_1)\big) = \frac{1}{2}a[\cos(\tau+\varphi) + \sin(\tau+\varphi)] + \frac{1}{2}\alpha[\cos(\tau+\varphi) - \sin(\tau+\varphi)]$$

and using the identity:$cosx = \frac{1}{2}(e^{ix} + e^{-ix})$

$$u_o = \frac{1}{2}a\big[e^{i(\tau+\varphi)} + e^{-i(\tau+\varphi)}\big]$$

$$\Rightarrow u_o = \frac{1}{2}\alpha e^{i\tau}e^{i\varphi} + \frac{1}{2}\alpha e^{-i\tau}e^{i\varphi}$$

$$\Rightarrow u_o = Ae_o + \bar{A}\overline{e_o}$$

$with e_o = e^{i\tau}$, $A = \frac{1}{2}\alpha(\tau_1)e^{i\varphi(\tau_1)}$, $with\ \bar{A},\ cojuagated\ variables$

Now having the linear solution for the 1st term of our uo solution, we substitute in the 2nd equation of the above system, with this process we have now removed the non-linear term and inserted it into the excitation form in the 2nd equation:

$$D_o^2 u_1 + u_1 = (-2D_oD_1)(Ae_o + \bar{A}\bar{e_o}) - 2\delta D_o(Ae_o + \bar{A}\bar{e_o}) - (Ae_o + \bar{A}\bar{e_o})^3 + f\eta^2\cos(\eta\tau_o)\ (\mathbf{7})$$

From the enunciation it is given that for the main resonance we have , along with the definitions for the fast and slow time we gave above, the trigonometric term of the excitation will be converted into:$\eta = 1 + \varepsilon\sigma$

$$\cos(\tau + \tau_1\sigma)\ and\ in\ complex\ form : \cos(\eta\tau_o) = \frac{1}{2}e^{i\tau_o}e^{i\sigma\tau_1} + cc$$

The terms in equation (7) are then calculated separately:

- $D_o u_o = iAe_o - i\bar{A}\overline{e_o} = iAe_o + cc$
- $D_1 u_o = \frac{1}{2}a'e^{i\varphi} + i\frac{1}{2}\alpha\varphi'e^{i\varphi} + cc = A'(\tau_1)e_o + cc$

$$\mu\varepsilon\ A'(\tau_1) = \frac{1}{2}e^{i\varphi(\tau_1)}\left(a'(\tau_1) + \alpha\varphi'(\tau_1)\right)$$

- $D_oD_1u_1 = iA'(\tau_1)e_o + cc$
- $(Ae_o + \bar{A}\overline{e_o})^3 = A^3e_o^3 + 3A^2e_o^2\bar{A}\overline{e_o} + 3Ae_o\overline{A^2}\overline{e_o^2} + \bar{A}^3\overline{e_o^3} = A^3e_o^3 + 3A^2\bar{A}e_o + cc$

We replace the above terms in equation (7) and factorize the terms that have the common exponent eo and thus arrive at the differential equation:

$$D_o^2u_1 + u_1 = -e_o\left[2i(A' + \delta A) + 3A^2\bar{A} - \frac{1}{2}f\eta^2e^{i\sigma\tau_1}\right] - A^3e_o^3 + cc\ \ (\mathbf{8})$$

We observe in equation (8) that the harmonic term eo represents the homogeneous solution of the zero-order differential equation and therefore induces resonance, for this reason we set the equation to zero in the bracket .

$2i(A' + \delta A) + 3A^2\bar{A} - \frac{1}{2}f\eta^2e^{i\sigma\tau_1} = 0$, $\mu\varepsilon\ \alpha\nu\tau\iota\kappa\alpha\tau\acute{\alpha}\sigma\tau\alpha\sigma\eta\ \tau\omega\nu\ \acute{o}\rho\omega\nu\ A, A', \bar{A}$

$$\Rightarrow 2i\left(\frac{1}{2}e^{i\varphi}(a' + i\alpha\varphi') + \delta\frac{1}{2}\alpha e^{i\varphi}\right) + 3\frac{1}{4}\alpha^2e^{i2\varphi}\frac{1}{2}\alpha e^{-i\varphi} - \frac{1}{2}f\eta^2e^{i\sigma\tau_1} = 0$$

$$\Rightarrow i(a' + i\alpha\varphi') + i\delta\alpha + \frac{3}{8}\alpha^3 - \frac{1}{2}f\eta^2e^{i(\sigma\tau_1 - \varphi)} = 0$$

$$\Rightarrow i(a' + i\alpha\varphi') + i\delta\alpha + \frac{3}{8}\alpha^3 - \frac{1}{2}f\eta^2[\cos(\sigma\tau_1 - \varphi) + isin(\sigma\tau_1 + \varphi)] = 0$$

then I separate real from imaginary terms:

$$\begin{cases} a' = -\delta\alpha - \frac{1}{2}f\eta^2\cos(\sigma\tau_1 - \varphi) \\ \alpha\varphi' = \frac{3}{8}\alpha^3 - \frac{1}{2}f\eta^2 sin(\sigma\tau_1 + \varphi) \end{cases}$$

now entering the variable for the frequency we have:$\gamma(\tau_1) = \sigma\tau_1 + \varphi(\tau_1)$

$$\begin{cases} \alpha' = -\,\delta\alpha \,-f_o\cos(\gamma) \\ \alpha\gamma' = \sigma\alpha - \,\beta\alpha^3 - f_o sin(\gamma) \end{cases} \quad (\mathbf{9})$$

where the constant coefficients β,fo are given:

$$\beta = \frac{3}{8}\ , \qquad f_o = \frac{f\eta^2}{2} \approx \frac{f}{2}$$

the approach was made because in the main coordination since we are talking about $\eta = 1 + \varepsilon\sigma \Rightarrow \eta^2 \underset{\varepsilon \ll 1}{\longrightarrow} 1$

The system of 9 equations we came up with is an autonomous system of 2 differential equations of the first order because you do not directly depend on time. We now know that for autonomous differential systems of the form (9) we have a unique solution for given initial conditions as long as the 1st derivatives of α,γ are closed.

More specifically, we will look for the stable solutions of the above system, therefore we refer to the permanent state. To achieve this we set the derivatives to zero and solve the resulting system, which will eventually consist of 2 transcendental equations:α', γ'

$$\begin{cases} f_o\cos(\gamma) = -\,\delta\alpha \\ f_o sin(\gamma) = \sigma\alpha - \,\beta\alpha^3 \end{cases}$$

squaring and adding by terms we get:

$$\beta^2(\alpha^2)^3 - 2\beta\sigma(\alpha^2)^2 + \delta\alpha^2 - f_o^2 = 0 \quad (\mathbf{10})$$

and dividing the 2nd by the 1st equation of the above system we get:

$$\tan(\gamma) = \frac{\delta\alpha}{\beta\alpha^3 - \sigma\alpha} \quad (\mathbf{11})$$

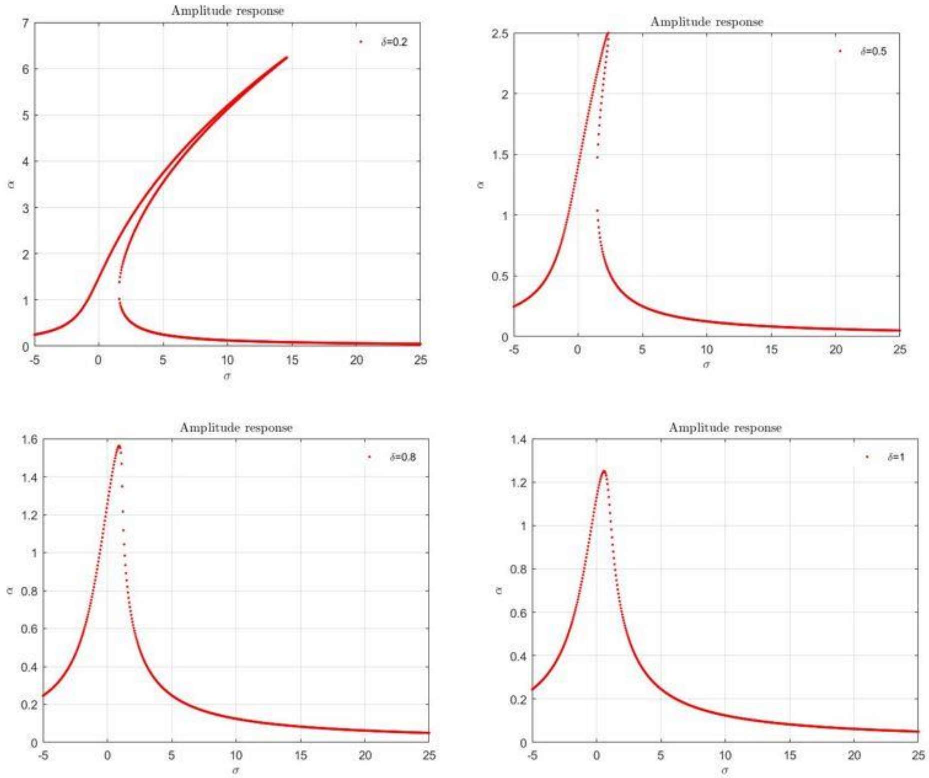
The relation (10) is a polynomial of the 3rd degree with respect to α^2 , the solutions of which will be found numerically by substituting for each value of the detuning coefficient σ. Correspondingly, the relation (11) is an expression of the phase γ as a function of the detuning coefficient σ, the values of γ that verify (11) will also be found numerically. Which specifically we are asked for the steady-state response-frequency and phase-frequency plots for :

$$-5 \leq \sigma \leq 25$$

The visualization of the solutions α=f(σ) and γ=f(σ) is performed using the Matlab programming software. The files **data.m** and **analytical_main.m** are used to obtain the diagrams. After our data for the constants β, f and δ have been entered in the 1st file, we proceed to the 2nd file where the calculations of the points α,γ are made for each σ, with σ moving in the interval we mentioned above with a step of 0.05.

We see that equation (10) is a 3rd degree polynomial in α^2 so we know that it has exactly 3 roots in C. But since we are talking about a response range we want our solutions to be positive and real. The separation of acceptable and unacceptable solutions is done programmatically in the 2nd file I mentioned above. Furthermore, knowing Descartes' rule of signs, we see that it is possible for 1 or 3 solutions to appear for some σ, which depends on the signs of the coefficients of the polynomial.

The process of calculating the solutions will be carried out for 4 different values of δ(0.2 0.5 0.8 1) the values of the constants for numerical application are given by the expression: ε=0.05 f=2.5. Figures 4 and 5 show the response-frequency and phase-frequency diagrams for the four values of δ respectively.

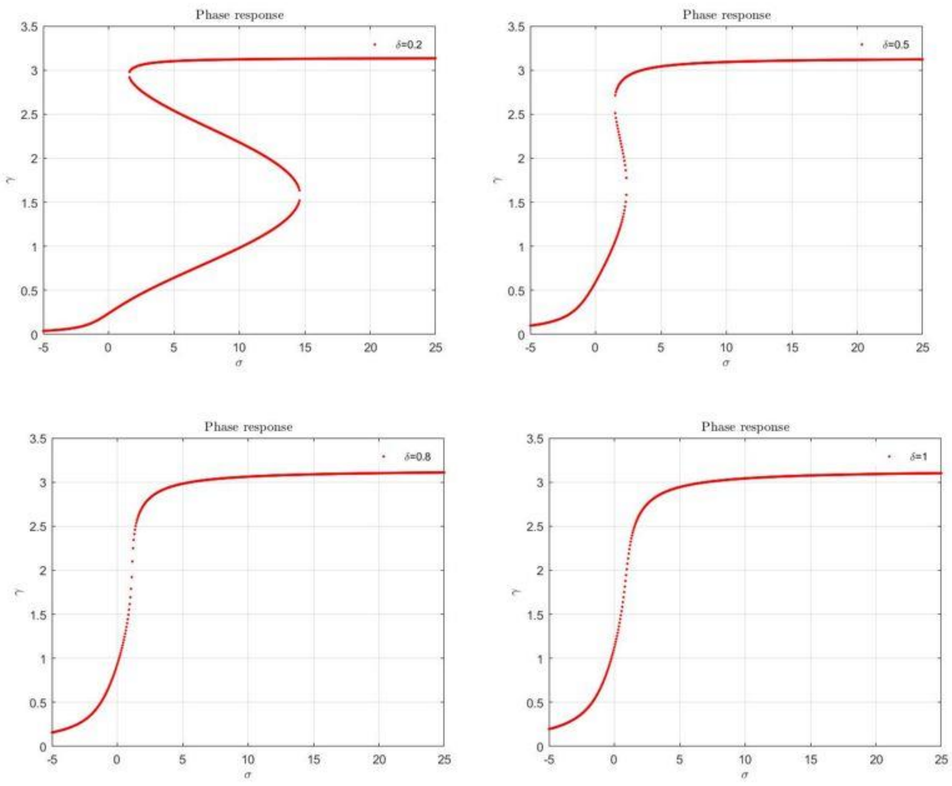


Shape3.Range-Frequency diagrams for principal resonance and for δ=[0.2 0.5 0.8 1]

In Figure 4. we see the different stable solutions ao resulting for different excitation frequencies around the region of the main resonance and the influence of the damping measure (ζ=ε) on them. We notice that for small ζ, the frequency range over which multiple solutions appear is larger. An increase in ζ results in a decrease in the aforementioned area.

In multi-solution regions the steady-state oscillation is mainly determined by the initial conditions. Something that also explains the large range of multiple solutions for small ζ=0.05δ, if we consider that the effect of the initial conditions is damped by the characteristics of the oscillator. Therefore a small ζ measure indicates a slow damping of the initial conditions, resulting in a dominance of the latter in the oscillation response.

In Figure 5. we see the different steady state solutions resulting for different excitation frequencies around the main resonance region and the influence of the damping measure (ζ=ε) on them. The conclusions and results obtained are corresponding to the previous analysis.



Shape4.Phase-Frequency diagrams for principal resonance and for δ=[0.2 0.5 0.8 1]

<u>Oscillation response and influence g</u>

In the 2nd part of the 1st problem we use numerical integration with given initial conditions to make the response-time diagram for the transient and permanent state and to monitor the influence of the damping measure on them.$u(0) = u_o \ \kappa\alpha\iota \ \dot{u}(0) = v_o$

Having now set to zero the term that causes resonance in equation (8) we arrive at the 2nd order differential equation:

$$D_o^2 u_1 + u_1 = -A^3 e_o^3 + cc$$

which considering solution of the form$u_1(\tau) = B\cos\,(3\Omega\tau - 3\gamma_o)$

gives us the partial solution:

$$u_1(\tau) = \frac{a_o^3}{32}\cos\,(3\eta\tau - 3\gamma_o)$$

With the analysis carried out previously we considered that the homogeneous solution of the differential equation of the 1st order terms(ε1) has been integrated into the homogeneous solution of the zeroth order terms.

furthermore we have for the zero-order response:

$$u_o = a\cos\,(\tau + \varphi)$$

$$\gamma\iota\alpha\ \varphi = \sigma\tau_1 - \gamma, \tau_o = \tau, \tau_1 = \varepsilon\tau$$

$$u_o = \alpha \cos(\eta\tau - \varphi)$$

now using the approximate solution we applied at the beginning of the method of multiple time scales: $u(\tau;\varepsilon) = u_o(\tau_o;\tau_1) + \varepsilon u_1(\tau_o;\tau_1) + O(\varepsilon^2)$

we arrive at a response up to 1st order accuracy of the form:

$$u(\tau) = \alpha\cos(\eta\tau - \gamma) \;\; + \varepsilon\frac{a_o^3}{32}\cos\left(3\eta\tau - 3\gamma_o\right) + O(\varepsilon^2)\ (\mathbf{12})$$

To obtain the response diagrams we will apply the numerical integration method to the dimensionless 2nd order differential equation that we constructed at the beginning. To carry out the numerical integration, we develop a code in Matlab using the function, which is based on the Runge-Kutta methods. . The **integration.m** and **solve_duffing.m** files will be used to perform the numerical integration

To use ODE45, we first have to degrade by one order the dimensionless differential equation of the mechanical device and introduce into it the system of 2 differential equations of the 1st order.

The dimensionless equation of motion:$\ddot{u}(\tau) + 2\zeta\dot{u}(\tau) + u(\tau) + \varepsilon u(\tau)^3 = P\eta^2\cos(\eta\tau)$

I put:$\dot{u} = v$

$\dot{v} = -2\zeta\dot{u}(\tau) - u(\tau) - \varepsilon u(\tau)^3 + P\eta^2\cos(\eta\tau)$

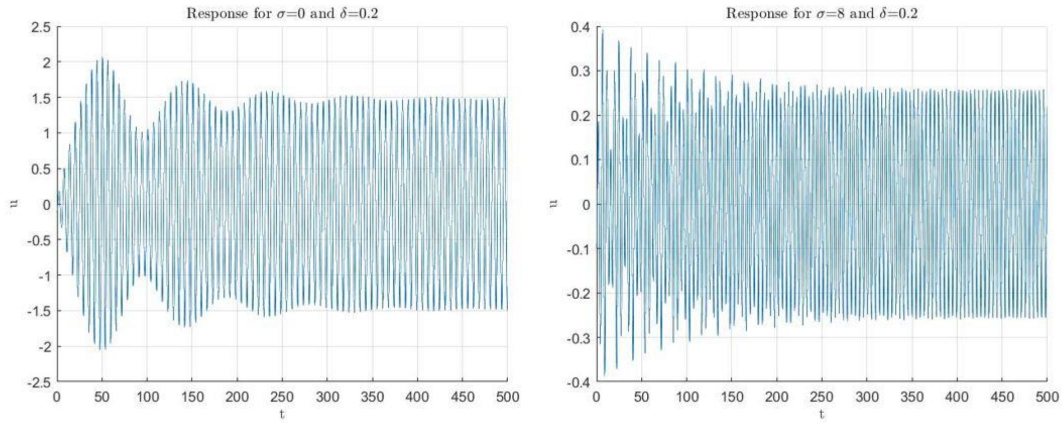
The output of the ODE45 function will be a velocity vector, a position vector, and the discretized time vector. The 2nd file contains the differential equation to which the integration is applied, which in our case is the 2nd of the above. Therefore, by entering what initial conditions, for the discrete time that we define, we will get the diagram of the response.

The numerical integration is carried out for a time from 0 to 500 seconds with a step of 0.01. Initial conditions, ε=0.05, f=2.5. We also maintain the assumptions P=εf and ζ=2δ. We will calculate the response for 2 excitation frequencies σ=0 and σ=8, for 4 values of the damping as in the previous part, δ=0.2, 0.5 and 1.$u(0) = -0.1\ \kappa\alpha\iota\ v = 0.015$

Figures 6,7 and 8 show 2 response plots each. In each figure we keep the damping measure ζ constant and vary the excitation frequency by means of σ. To interpret the diagrams that will follow we must first define the significance of the differences from diagram to diagram.

σ=0 symbolizes that the excitation frequency is equal to the oscillator frequency, so we are in resonance, on the contrary σ=8 thinking about the definition of main resonance we are far from the oscillator frequency. Increasing the delta from application to application means increasing the damping measure, thus faster damping characteristics of the initial conditions. The form of equation (12) ratio of the appearance of frequency in the terms of higher order, which is many times that of the first term, indicates to us that we expect a periodic response in the steady state k and not a harmonic one.$\Omega = \omega_o + \varepsilon\sigma$

In Figures 6 to 8 we can see that the steady state for the same δ but different σ is reached at the same time, which verifies our model. In addition, we can see that for σ=0, i.e. matching the excitation frequency and the natural frequency of the linear oscillator, the oscillation ranges are larger, which indicates the resonance for Ω=ωo . We also notice that for small values of δ, we have the appearance of strong fluctuations in the case of . This phenomenon is due to the fact that our response (12) is expressed by 2 harmonic terms with different frequencies, because the values of the frequencies are of course close and an integral multiple of each other (of the 1st order term three times the zero term) we see the fluctuations they. The intensity of the oscillations is determined by how close the frequencies of the harmonic terms are. $\eta = \frac{\Omega}{\omega_o} = 1 \ (\sigma = 0)$



Shape5. Response diagrams for δ=0.2 and σ=0 and 8

Therefore, during the transitional state, we distinguish 2 periodic processes. We see the periodic fluctuations of the amplitude of the response, which depend mainly on the oscillator and the initial conditions, and in addition we see the response of the oscillation due to the applied excitation.

As the damping coefficient increases, we can distinguish the faster reaching of the permanent state, i.e. the damping coefficient of the initial conditions. For very large values of course, δ=1 and σ=0, we distinguish the complete weakening of the fluctuations and the effect of the initial conditions.

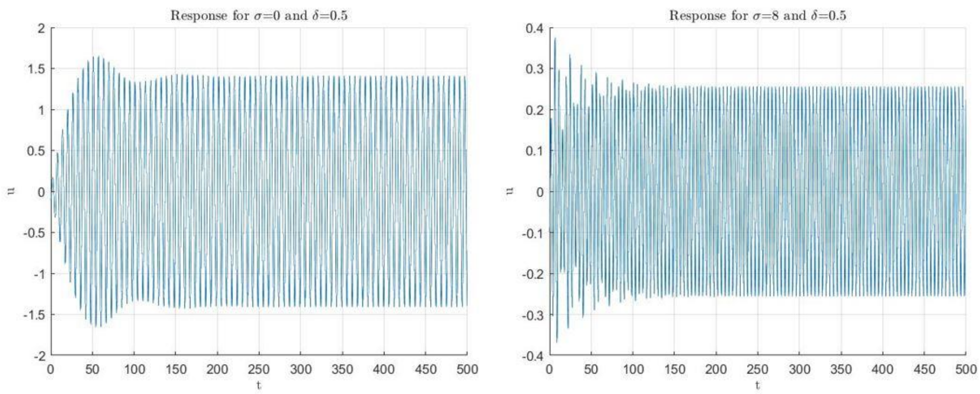
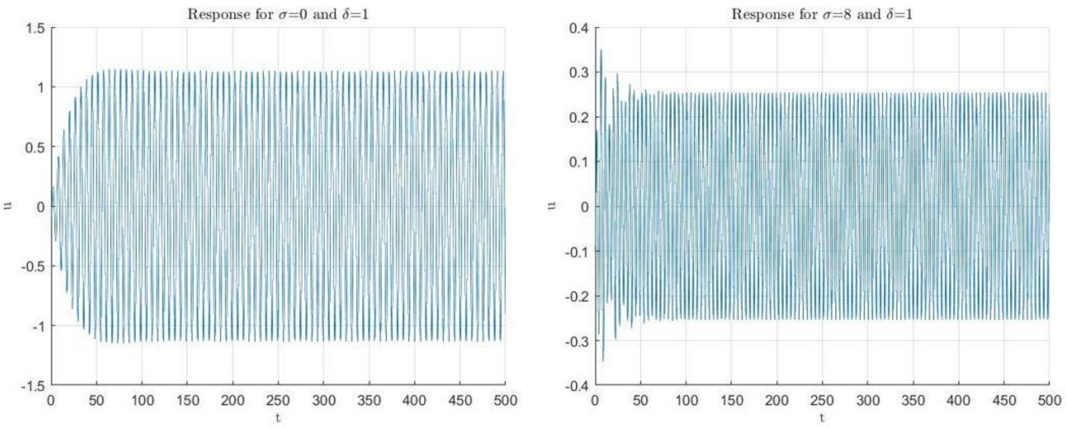


Image6. Response diagrams for δ=0.5 and σ=0 and 8

For now values of the excitation frequency far from the natural frequency of the linear oscillator, we have a periodic response with lower amplitudes. The behavior can be explained by means of the a-s diagrams. As the frequency (σ) increases, we have jump effects, i.e. we have a sharp reduction in the amplitude of the oscillation.

In the response plots with σ=8, the application of the different time scales during the analysis is clearly visible. That is, we can see for the transient state, the rapid oscillation of our system in relation to the slow exponential decrease of the amplitude of the response. Finally for all states in the steady state the amplitudes and the phase are stabilized since our system has now arrived at one of the stable solutions (αo,γo) that we observed in the analysis above.



Shape7. Response plots for δ=1 and σ=0 and 8

At the end of the 1st question, we seek how the initial conditions could be chosen so as to minimize the duration of the transition state.

A qualitative approach to finding these initial conditions would be to construct the phase plane and then choose a solution on some orbital curve that results in a stable solution. With the main goal that this point leads us to the solution as quickly as possible, that is, as close to the stable solution as possible.

Having calculated the response of our oscillator in equation (12) and for

$$u(\tau) = \alpha(\tau_1)\cos(\eta\tau - \gamma(\tau_1)) \;\; + \varepsilon\frac{a_o^3}{32}\cos\left(3\eta\tau - 3\gamma_o\right) + O(\varepsilon^2) \;\; (\mathbf{12})$$

transitory and the permanent state. Our goal is to choose the initial conditions so that our response ends up directly or as close as possible to the steady state response:$(u, \dot{u}) = (u_o, v_o)$

$$u(\tau) = \alpha_o\cos(\eta\tau - \gamma_o) \;\; + \varepsilon\frac{a_o^3}{32}\cos\left(3\eta\tau - 3\gamma_o\right) + O(\varepsilon^2)$$

The selection of the initial conditions for reducing the transition state will be done qualitatively and in a 2nd way. Because the transient state depends mainly on the characteristics of the oscillator, i.e. the homogeneous solution, we consider as a solution of our system the equation:

$$u(\tau) = \alpha(\tau_1)\cos\left(\eta\tau - \gamma(\tau_1)\right) + O(\varepsilon)$$

Looking now at the response plot for δ=1, in Figure 7. we see that the steady state is reached for about time t=50 sec. At this time, for the steady state, we have an oscillation range approximately equal to . for this value of the range we will approximately calculate the phase of the oscillation from relation (11)$\alpha = 1.2 \;\; \tan(\gamma) = \frac{\delta\alpha}{\beta\alpha^3 - \sigma\alpha}$ with and δ=1. In addition, for this calculation you need to define the detuning coefficient σ. In the middle of the range-frequency diagram (Figure 3.) we see that for this range the coefficient.

Thus we calculate:$\beta = \frac{3}{8} \;\; \sigma \approx 0$

$$\tan(\gamma) \approx \frac{1.2}{0.648} = 1.85$$

With these values of α,γ we replace in the answer:

$$u(\tau) = 1.2\cos(\eta\tau - 1.85)$$

where$\eta = 1 + \sigma\varepsilon \approx 1$

$$u(\tau) = 1.2\cos(\tau - 1.85)$$

$$\dot{u}(\tau) = -1.2\sin(\tau - 1.85)$$

We have now expressed the steady state response and for δ=1 damping approximately. For the time now τ=0 we have:

$$u(0) = u_o = 1.2\cos(-1.85)$$

$$\dot{u}(0) = v_o = -1.2\sin(-1.85)$$

Thus we calculate the initial conditions approximately in order to reduce as much as possible the duration of the transition state. For different damping coefficients ζ (d) the same procedure is repeated.

A 3rd way of calculating the initial conditions so that we arrive at the steady state at one of the solution constants we calculated above is as follows. Since we want to reduce the duration of the transient state and not necessarily eliminate it completely, because ε<<1 and because the transient oscillation state is dominated by the homogeneous solution of the oscillator, we neglect the 1st and 2nd order terms with respect to ε we have the system . Then we derive the homogeneous solution with respect to time τ for τ=0:

$$u(0) = u_o = a(0)\cos\big(-\gamma(0)\big)$$

$$\dot{u}(0) = v_o = \varepsilon\alpha'(0)\cos(-\gamma(0)) - \alpha(0)\sin\big(-\gamma(0)\big)\big(\eta - \varepsilon\gamma'(0)\big)$$

$$\begin{cases} u_o = a(0)\cos\big(-\gamma(0)\big) \\ v_o = \varepsilon\alpha'(0)\cos(-\gamma(0)) - \eta\alpha(0)\sin\big(-\gamma(0)\big) - \varepsilon\gamma'(0)\alpha(0)\sin\big(-\gamma(0)\big) \end{cases}$$

In order to go directly to the steady state, i.e. to the stable solution we calculated above (αo,γo) we must from the above equations:

$$\alpha(0) = \alpha_o \; \kappa\alpha\iota \; \gamma(0) = \gamma_o$$

therefore:

$$\begin{cases} u_o = a_o\cos(-\gamma_o) \\ v_o = \varepsilon\alpha'(0)\cos(-\gamma_o) - \eta\alpha_o\sin(-\gamma_o) - \varepsilon\gamma'(0)\alpha_0\sin(-\gamma_o) \end{cases} \quad \mathbf{(13)}$$

in addition to the system of equations (9) we have for the derivatives of the phase and amplitude with respect to the slow time:

$$\begin{cases} a' = -\delta\alpha - f_o\cos(\gamma) \\ \alpha\gamma' = \sigma\alpha - \beta\alpha^3 - f_o sin(\gamma) \end{cases}$$

$$\begin{cases} a'(0) = -\delta\alpha_o - f_o\cos(\gamma_o) \\ \alpha_o\gamma(0)' = \sigma\alpha_o - \beta\alpha_o^3 - f_o sin(\gamma_o) \end{cases}$$

replacing the last 2 in system (13) gives us :

$$\begin{cases} u_o = a_o\cos(-\gamma_o) \\ v_o = \varepsilon[-\delta\alpha_o - f_o\cos(\gamma_o)]\cos(-\gamma_o) - \eta\alpha_o\sin(-\gamma_o) - \varepsilon[\sigma\alpha_o - \beta\alpha_o^3 - f_o sin(\gamma_o)]\sin(-\gamma_o) \end{cases}$$

We therefore end up with 2 equations of which all the quantities on the right are known. Solving these equations gives us the values for the initial conditions so as to reduce the duration of the transient state. In case we wanted a greater reduction in duration (greater accuracy), we could also keep the partial solution while searching for the initial conditions or introduce terms of a higher order in terms of e.

The first 2 trends analyzed could be said to be more mechanical compared to Tuesday which you consider more mathematical. In other words, we see that qualitative analysis can lead to good results, easily and quickly. By means of qualitative methods we get an immediate first assessment of the solutions we are looking for and the goals we want to achieve.

2nd PROBLEM

- **Amplitude-frequency and phase-frequency plots for subharmonic resonance**

The 2nd question asks for the response-frequency and phase-frequency diagrams for the subharmonic resonance condition with . Which means that the excitation frequency is not close to the values of the linear natural frequency of the oscillator. In this case the excitation affects the oscillation only when it has zero-order amplitude. Therefore we start the solution with the same procedure as in question (a) but for . By applying the method of multiple time scales and keeping the zeroth and 1st order terms, we arrive at the system of partial differential equations$\eta = 3 + \varepsilon\hat{\sigma} P = f$

$$\varepsilon^0 \ : \quad D_o^2 u_o + u_o = f\eta^2 \cos(\eta\tau) \quad (\mathbf{13})$$

$$\varepsilon^1 : D_o^2 u_1 + u_1 = -2D_o D_1 u_o - 2\delta D_o u_o - u_o^3 \quad (\mathbf{14})$$

Therefore, the general solution of the zero-order equation will be of the form:

$$u(\tau_o, \tau_1) = u_{o\mu o} + u_\mu$$

with$u_{o\mu o} = A(\tau_1)e_o + cc \quad \kappa\alpha\iota \quad u_\mu = \Lambda e + cc$

where , cc=conjugate complex terms.$A(\tau_1) = \frac{1}{2}\alpha(\tau_1)e^{i\varphi(\tau_1)}, e_o = e^{i\tau}, \Lambda = \frac{f\eta^2}{2(1-\eta^2)}, e = e^{i\eta\tau}$

Replacing all of the above in (14) gives us:

$$\begin{aligned} D_o^2 u_1 + u_1 = &-[2i(A' + \delta A) + 3A^2\bar{A} + 6\Lambda^2 A]e_o - \Lambda(2i\delta\eta + 3\Lambda^2 + 6A\bar{A})e \\ &- \left[A^3 e_o^3 + \Lambda^3 e^3 + 3\Lambda A^2 e e_o^2 + 3\Lambda\overline{A^2} e\overline{e_o^2} + 3\Lambda^2 A e^2 e_o + 3\Lambda^2 A\bar{e}^2 e_o\right] + cc \end{aligned}$$

For subharmonic coordination the terms that cause them in the above equation are zeroed out:

$$2i(A' + \delta A) + 3\left(A^2\bar{A} + 2\Lambda^2 A + \Lambda\bar{A}^2 e^{i\sigma\tau_1}\right) = 0$$

with :$A' = \frac{1}{2}e^{i\varphi}(\alpha' + i\alpha\varphi')$

$\bar{A} = \frac{1}{2}\alpha(\tau_1)e^{-i\varphi(\tau_1)}$

Substitution of the terms in this relationship gives:

$$2i(A' + \delta A) + 3\left(A^2\bar{A} + 2\Lambda^2 A + \Lambda\bar{A}^2 e^{i\sigma\tau_1}\right) = 0$$

$$\begin{aligned} \Rightarrow 2i&\left(\frac{1}{2}e^{i\varphi}(\alpha' + i\alpha\varphi') + \delta\frac{1}{2}\alpha(\tau_1)e^{i\varphi(\tau_1)}\right) \\ &+ 3\left(\frac{1}{8}\alpha^3 e^{i\varphi} + \frac{f^2\eta^4}{4(1-\eta^2)^2}\alpha e^{i\varphi} + \frac{f\eta^2\alpha^2}{8(1-\eta^2)}e^{i(\hat{\sigma}\tau_1 - 2\varphi)}\right) = 0 \end{aligned}$$

$$\overset{:e^{i\varphi}}{\Longrightarrow} -\alpha\varphi' - i(\alpha' + \delta\alpha) + \frac{3\alpha^3}{8} + \frac{3f^2\eta^4}{4(1-\eta^2)^2}a + \frac{3f\eta^2\alpha^2}{8(1-\eta^2)}[\cos(\hat{\sigma}\tau_1 - 3\varphi) + isin(\hat{\sigma}\tau_1 - 3\varphi)] = 0$$

Then we separate the imaginary and the real terms:

(Re) :$-\alpha\varphi' + \frac{3\alpha^3}{8} + \frac{3f^2\eta^4}{4(1-\eta^2)^2}a + \frac{3(f\alpha\eta)^2}{8(1-\eta^2)}\cos(\hat{\sigma}\tau_1 - 3\varphi) = 0$

(Im):$\alpha' + \delta\alpha + \frac{3(f\alpha\eta)^2}{8(1-\eta^2)}sin(\hat{\sigma}\tau_1 - 3\varphi) = 0$

and thus we arrived at the 1st order differential equations of the amplitude and phase of the oscillation in the following autonomous form:

$$\alpha' = -\delta\alpha - \frac{3f\eta^2\alpha^2}{8(1-\eta^2)}sin(\hat{\sigma}\tau_1 - 3\varphi)$$

$$\alpha\varphi' = \frac{3\alpha^3}{8} + \frac{3f^2\eta^4}{4(1-\eta^2)^2}a + \frac{3f\eta^2\alpha^2}{8(1-\eta^2)}\cos(\hat{\sigma}\tau_1 - 3\varphi)$$

now substituting γ=σ1-3φ, the system takes the form:

$$\begin{cases} \alpha' = -\delta\alpha - \dfrac{3f\eta^2\alpha^2}{8(1-\eta^2)}\sin(\gamma) \\ \alpha\gamma' = \alpha\hat{\sigma} - \dfrac{9\alpha^3}{8} - \dfrac{9f^2\eta^4}{4(1-\eta^2)^2}a - \dfrac{9f\eta^2\alpha^2}{8(1-\eta^2)}\cos(\gamma) \end{cases} \quad \mathbf{(15)}$$

and in a more favorable form the system is written:

$$\begin{cases} \alpha' = -\delta\alpha - \dfrac{3\Lambda\alpha^2}{4}\sin(\gamma) \\ \alpha\gamma' = \alpha\hat{\sigma} - \dfrac{9\alpha^3}{8} - 9\Lambda^2\alpha - \dfrac{9\Lambda\alpha^2}{4}\cos(\gamma) \end{cases} \quad \mathbf{(16)}$$

To construct the diagrams now requested, we need to calculate the stable solutions of the autonomous system, to do this we zero out the derivatives$\alpha' = \gamma' = 0$

$$\begin{cases} \dfrac{3\Lambda\alpha_o^2}{4}\sin(\gamma_o) = -\delta\alpha_o \\ \dfrac{9\Lambda\alpha_o^2}{4}\cos(\gamma_o) = \alpha_o\hat{\sigma} - \dfrac{9\alpha_o^3}{8} - 9\Lambda^2\alpha_o \end{cases} \quad \mathbf{(17)}$$

To solve this system we square and add by terms

$$\frac{9\Lambda^2\alpha_o^4}{16} + \frac{81\Lambda^2\alpha_o^4}{16} = \delta^2\alpha_o^2 + \alpha_o^2\hat{\sigma}^2 - \frac{9\alpha_o^4\hat{\sigma}}{4} - 18\alpha_o^2\hat{\sigma}\Lambda^2 + \frac{81\alpha_o^6}{64} + \frac{81\alpha_o^4\Lambda^2}{4} + 81\alpha_o^2\Lambda^4$$

$$\overset{:a_o^2}{\Rightarrow} \frac{81\alpha_o^4}{64} + a_o^2\left(\frac{81\Lambda^2}{4} - \frac{9\hat{\sigma}}{4} - \frac{9\Lambda^2}{16} - \frac{81\Lambda^2}{16}\right) + (81\Lambda^4 + \delta^2 + \hat{\sigma}^2 - 18\hat{\sigma}\Lambda^2) = 0$$

So we ended up with a two-square equation in terms of:α_o^2

$$\alpha_o^4 + a_o^2 p + q = 0$$

$withp = \frac{64}{81}\left(\frac{81}{4}\Lambda^2 - \frac{9\hat{\sigma}}{4} - \frac{90}{16}\Lambda^2\right)\ \kappa\alpha\iota\ q = \frac{64}{81}(81\Lambda^4 - 18\Lambda^2\hat{\sigma} + \delta^2 + \hat{\sigma}^2)$

so the stable solutions for the range arise:

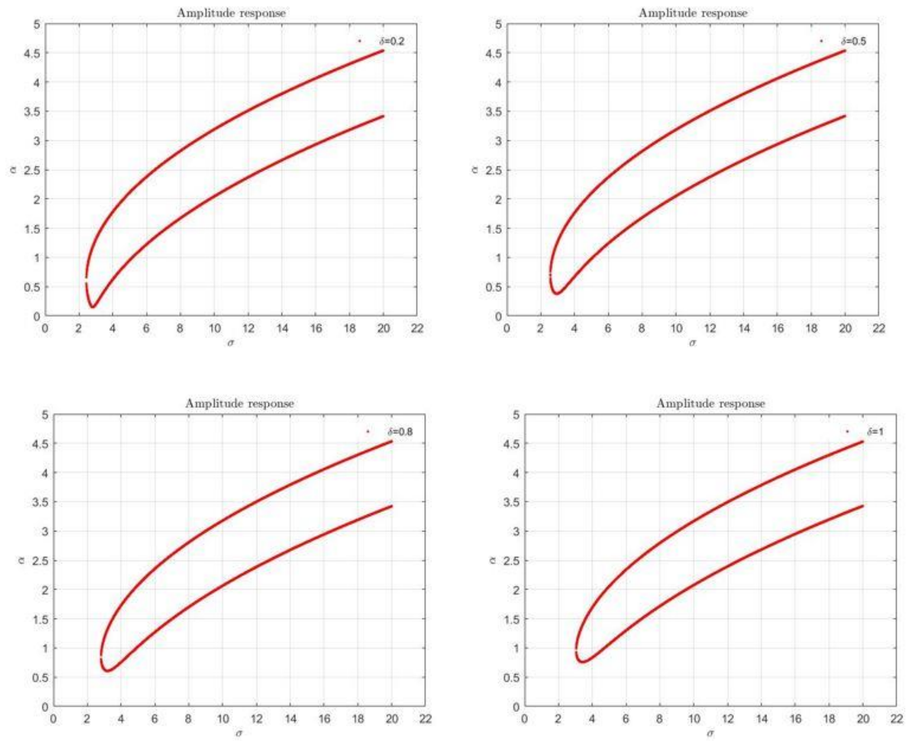
$$\alpha_o = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

From these solutions we keep only the positive real values.

Accordingly, by dividing the equations of system (17) by members, the stable solutions for phase c are obtained:

$$tan(\gamma_o) = -\frac{9\delta}{3\left(\hat{\sigma} - \frac{9\alpha_o^2}{8} - 9\Lambda^2\right)}$$

having calculated the values for the pairs of fixed solutions (ao,o) for a specific value of the excitation frequency (for each), we can construct the requested diagrams. The diagrams were obtained through the Matlab software, specifically using the files provided to us, varying only the calculation relationships of ao,o. Calculations are made for as given in the pronunciation. In total after changing the delta we will get 4 pairs of diagrams, for δ=0.2 0.5 0.8 1. In Figures 9 and 10, the range-frequency and phase-frequency diagrams are presented respectively.$\hat{\sigma}\hat{\sigma}$ $\alpha\pi$ό 0 έ$\omega\varsigma$ 20 $\mu\varepsilon$ βή$\mu\alpha$ 0.005, $P = f = 1$ $\kappa\alpha\iota$ $\varepsilon = 0.05$



Shape8. Range-Frequency diagrams for subharmonic resonance and for δ=[0.2 0.5 0.8 1]

First we observe in Figure 9, in which the amplitude-frequency diagrams for different damping measures of the oscillator are represented, the existence of double solutions as we expected due to the form of the characteristic polynomial. We also observe that by increasing the damping measure, we have a reduction in the range of the region of multiple solutions. From the solution branches in the multi-solution regions we can actually distinguish one of 2 solutions (stable). Therefore the lower branch represents unstable solutions. In other words, we can perceive that as the excitation frequency decreases, we have a jump from stable to unstable solutions. Finally we see that for larger $\delta(\zeta=\varepsilon d)$, the value range of stable solutions ao decreases.

In Figure 10. we distinguish the phase-frequency diagrams for subharmonic resonance, for different values of the damping measure. The conclusions and results obtained are corresponding to the previous analysis for the fixed range solutions.

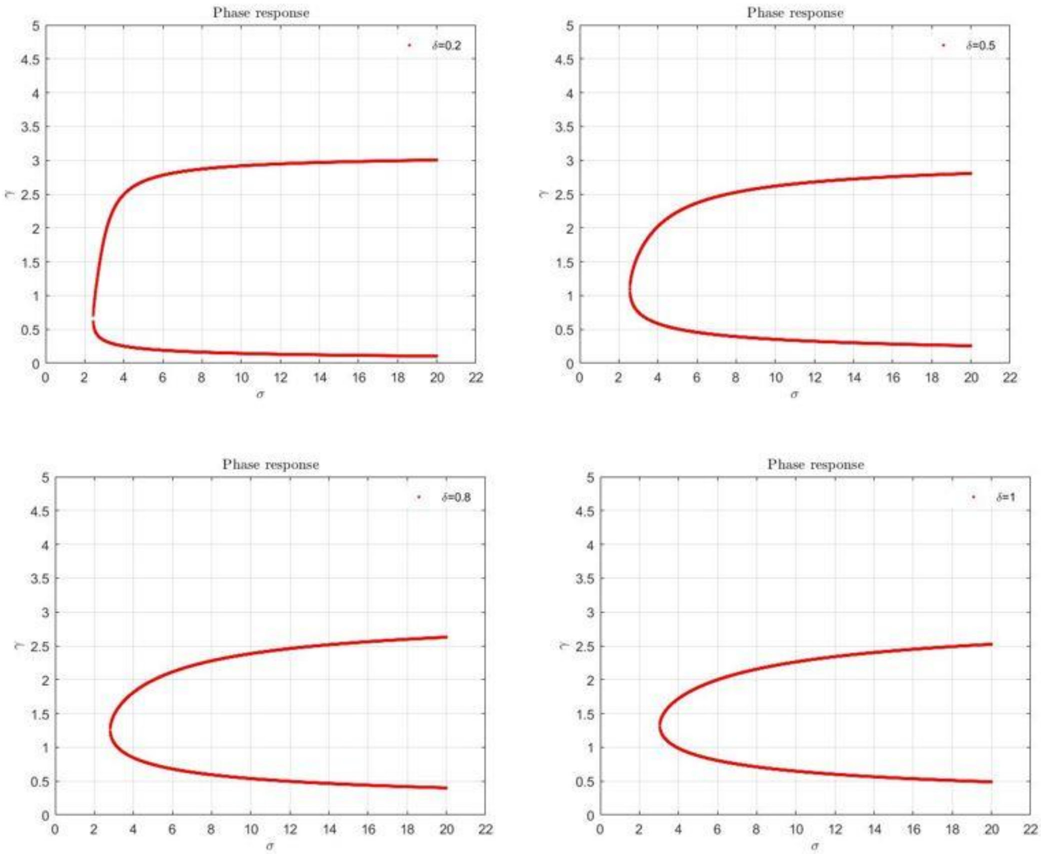


Figure 10. Phase-Frequency diagrams for subharmonic resonance and for δ=[0.2 0.5 0.8 1]

- **Oscillator response plots**

To get the response diagram of the oscillator for the transient and for the permanent state we apply, as in the first question, the method of numerical integration. The procedure you follow to use the method, specifically the Matlab function ODE45, is the same as the one done in the first question.

In summary, after degrading our differential equation to a system of 1st order differential equations, we use the file given to us after first entering our data for the specific problem. The values of the variables entered for numerical application are given by the utterance and are:

$$P = f = 1\,, \varepsilon = 0.05, \hat{\sigma} = 5\ \kappa\alpha\iota\ 15, \text{Αρχικές συνθήκες}\ u(0) = -0.1\ \kappa\alpha\iota\ v = 0.015$$

We will generate the response plots for 3 different values of the damping measure (δ=0.2, 0.5 and 1). Numerical integration will be performed for a time from 0 to 500 seconds with a step of 0.01. Figures 11 to 13 show the responses for various d and in each figure we keep the damping measure constant and vary $\hat{\sigma}$.

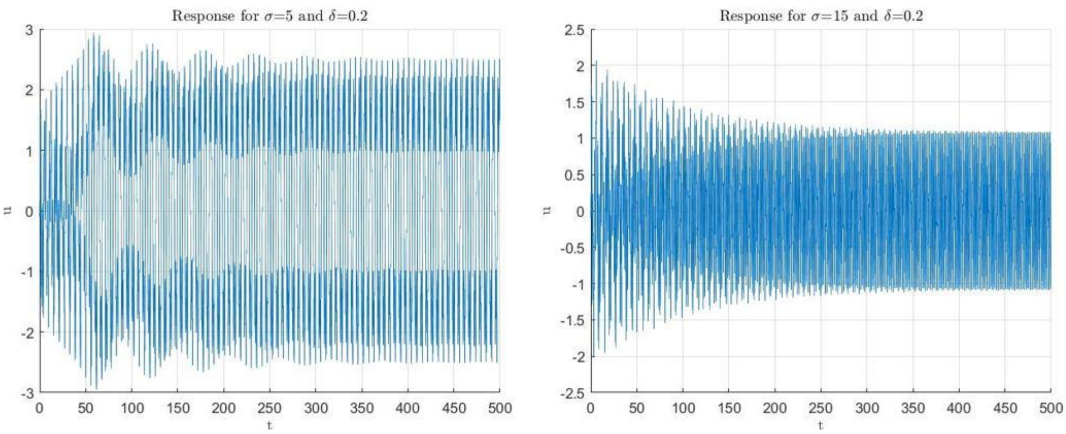


Figure 11. Mechanical response of device for δ=0.2 with frequency σ=5 and 15

First, observing the response diagrams, we compare the influence of the excitation frequency (. We see that for σ=5 we have strong oscillations. This behavior is due to the fact that we have 2 harmonic terms with adjacent frequencies. On the contrary, for large σ, the oscillations weaken and the amplitude response damps exponentially as a function of slow time. Increasing the excitation frequency has a significant impact on small damping measures, the reason for this is the jump effect as we mentioned in the previous question. We notice that with an increase in σ, we have a significant decrease in the amplitude of the oscillation in the permanent state but also avoiding coordinations and fluctuations in the transient state.$\hat{\sigma}$)

Correspondingly with an increase in the damping measure ζ, we see a decrease in the duration of the transition state, a phenomenon we expected because as we know the homogeneous solution that dominates the transition state decays faster. With the same rationale, we also observe smaller fluctuations (for small σ) from the moment they do not have time to develop. In addition we observe that in the steady state for all cases the amplitude of the oscillation remains constant. This proves the existence of the stable solutions, as calculated above

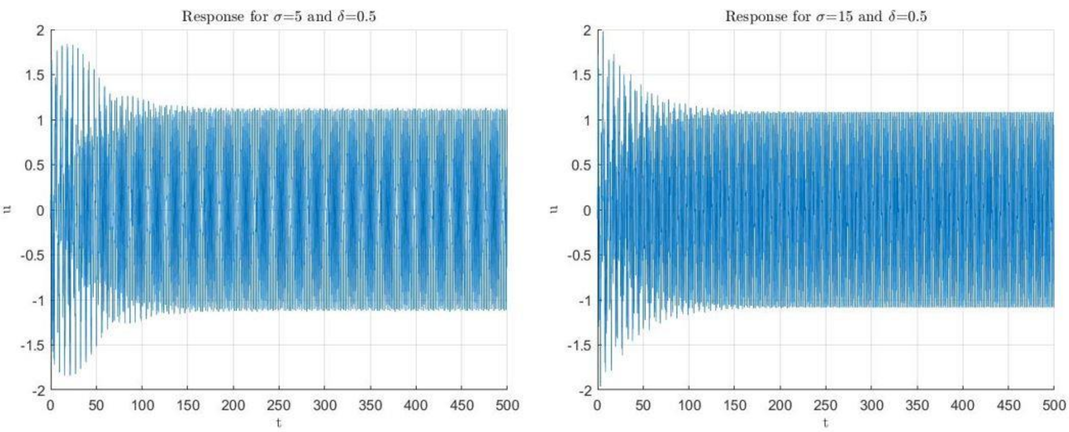


Figure 12. Mechanical response of device for δ=0.5 with frequency σ=5 and 15

The above conclusions are also verified by the algebra of the problem, because for excitation frequencies far from the value (Ω=3ωo , e.g. for σ=15), we zero out different terms to avoid resonance and following the same steps as the previous analysis , we end up with a response of the form:

$$u(\tau) = \alpha_o e^{-\mu\tau_1}\cos(\omega_o\tau + \varphi) + \frac{f\eta^2}{1-\eta^2}\cos\eta\tau$$

where in this relation we distinguish the purely exponential reduction of the amplitude of the oscillation for the homogeneous term that dominates the transition state.

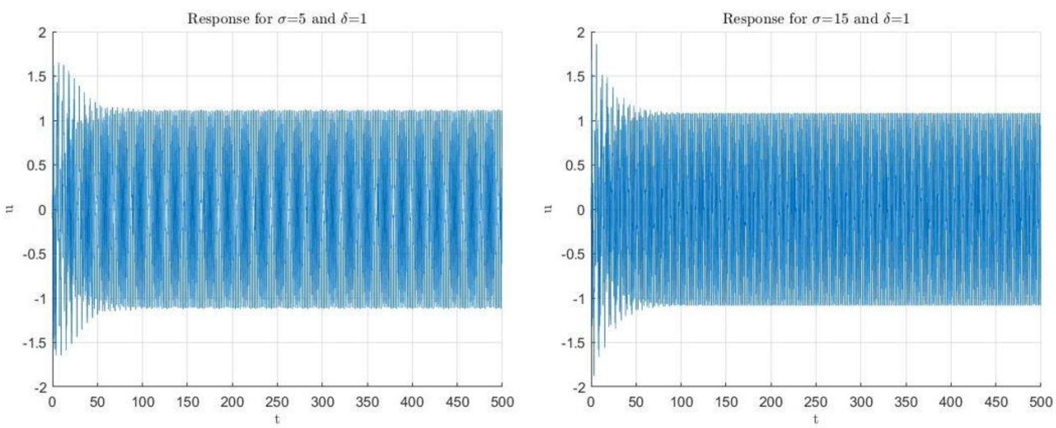


Figure 13. Mechanical response of device for δ=1 with frequency σ=5 and 15

If we followed the linear theory for the design of the support we would not be able to distinguish the subharmonic resonances or the non-linear behavior in general, since the study in the linear models is performed only for main resonances and linearly varying quantities. In addition, in the linear theory the area of occurrence of subharmonic resonances is considered to show the lowest responses. As a result, the latter are destructive to our construction.

Because as we observed in the last diagrams, when we have subharmonic resonance, the transient state is dominated by large fluctuations in the amplitude of the oscillation, which fluctuations are differentiated for different excitation frequencies. Therefore, based on the linear theory, our mechanical device would be designed to cope with and avoid frequencies that cause major resonances(), i.e. in a small frequency range, which in most situations would be far from the frequencies of interest of secondary resonances.

$$\Omega \approx \omega_o$$

Let's think for example, our support consists of a thin material. Material which for high loads could show plastification. As we know plastic deformations are dominated by non-linear characteristics, which the linear model would fail to predict and would lead us to possible failure.

3rd PROBLEM

In this question we study the dynamic response of the system depicted in Figure 14. This system consists of the mechanical system we studied in the previous questions and an obstacle. The obstacle contribution leads our system to have piecewise continuous characteristics. Of course, since the oscillating elements are linear, we can split our motion into 2 parts and work out the equations for each separately.

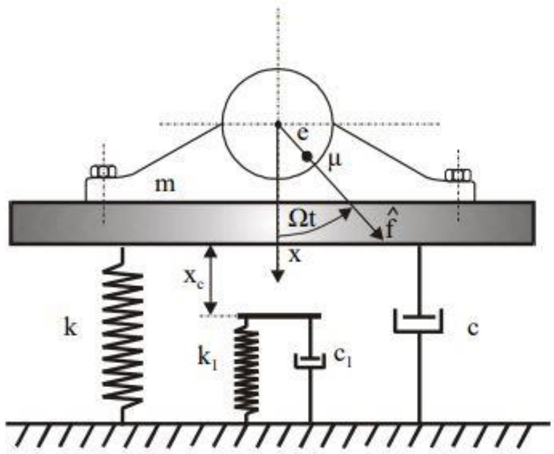


Figure 14. Mechanical system of an electric motor with an

Therefore, 2 equations of motion will be constructed. One will concern for . Where xc is the distance of the support from the obstacle. Therefore we build 2 FBD, for different times. The FBD are shown in Figure 15.

$$x \leq x_c \text{ } and \text{ } second \text{ } x > x_c$$

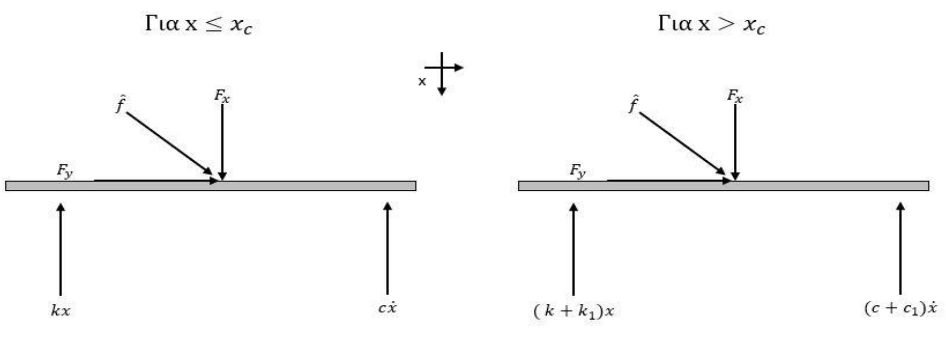


Figure 15. FBD layout mechanics for the 2 different times with fixed characteristics

Thus, from the above diagrams, the equations of motion are derived:

$$m\ddot{x} + c\dot{x} + kx = \hat{f}\cos(\Omega t) \quad \gamma\iota\alpha \text{ } x \leq x_c$$

$$m\ddot{x} + c_2\dot{x} + k_2x + (k - k_2)x_c = \hat{f}\cos(\Omega t) \quad \gamma\iota\alpha \text{ } x > x_c$$

with ,$\hat{f} = \mu e \Omega^2 c_2 = c + c_1$, $k_2 = k + k_1$

Then we dimensionalize the 2 equations of motion above, thus defining the dimensionless time and the dimensionless displacement:$\theta = \Omega t$ $\kappa\alpha\iota$ $\psi(\theta) = \frac{x(t)}{x_c}$

and we calculate the derivatives of x : . Substituting these derivatives and the dimensionless variables into the equations of motion gives us:$\dot{x} = \Omega x_c \dot{\psi}$ $\kappa\alpha\iota$ $\ddot{x} = \Omega^2 x_c \ddot{\psi}$

$$\begin{cases} m\Omega^2 x_c \ddot{\psi} + c\Omega x_c \dot{\psi} + k x_c \psi = \hat{f}\cos(\theta) \quad \gamma\iota\alpha\ \psi \leq 1 \\ m\Omega^2 x_c \ddot{\psi} + c_2 \Omega x_c \dot{\psi} + k_2 x_c \psi + (k - k_2) x_c = \hat{f}\cos(\theta) \quad \gamma\iota\alpha\ \psi > 1 \end{cases}$$

$$\xRightarrow{:m\Omega^2 x_c} \begin{cases} \ddot{\psi} + \frac{c}{m\Omega}\dot{\psi} + \frac{k}{m\Omega^2}\psi = \frac{\mu e}{m x_c}\cos(\theta) \quad \gamma\iota\alpha\ \psi \leq 1 \\ \ddot{\psi} + \frac{c_2}{m\Omega}\dot{\psi} + \frac{k_2}{m\Omega^2}\psi + \frac{(k - k_2)}{m\Omega^2} = \frac{\mu e}{m x_c}\cos(\theta) \quad \gamma\iota\alpha\ \psi > 1 \end{cases}$$

Then we enter the parameters:

$$\overline{\omega_1} = \sqrt{\frac{k}{m}} \ , \ \overline{\omega_2} = \sqrt{\frac{k_2}{m}}$$

$$\zeta_1 = \frac{c_1}{2\sqrt{km}} \quad , \zeta_2 = \frac{c_2}{2\sqrt{k_2 m}}$$

$$\delta_1 = \zeta_1 \omega_1 \quad , \delta_2 = \zeta_2 \omega_2$$

$$\omega_1 = \frac{\overline{\omega_1}}{\Omega} \ , \qquad \omega_2 = \frac{\overline{\omega_2}}{\Omega}$$

By introducing these parameters into the system of motion equations we get the dimensionless linear motion equations for both the 2 situations with continuous characteristics:

$$\begin{cases} \ddot{\psi} + 2\delta_1 \dot{\psi} + \omega_1^2 \psi = P\cos(\theta) \quad \gamma\iota\alpha\ \psi \leq 1 \\ \ddot{\psi} + 2\delta_2 \dot{\psi} + \omega_2^2 \psi = P\cos(\theta) + \omega_2^2 - \omega_1^2 \quad \gamma\iota\alpha\ \psi > 1 \end{cases}$$

$$P = \frac{\mu e}{m x_c}$$

Then the dynamic behavior in the subharmonic response region (n=3) is studied with Poincare sections. Poincare intersections essentially intersect the phase plane trajectories of the dynamical system belonging to the three-dimensional space with a plane (eg ψ=1) and depict these intersection points in the two-dimensional space. When depicting these points, we include only the points with .$(\psi, \dot{\psi}, t)\dot{\psi} > 0$

We apply numerical integration to obtain the Poincare sections. After the numerical integration is done for a time vector with elements that are period subdivisions, we plot the points with a dimensionless time difference of one period. To carry out the numerical integration, we first reduce the order of the differential equations of motion as follows:

We put it like this, our original system turns into a system with 4 equations:$\dot{\psi} = v$

$$\begin{cases} v_1 = \dot{\psi} \\ \dot{v_1} = \frac{\mu e}{m x_c}\cos(\theta) - \frac{c}{m\Omega}\dot{\psi} - \frac{k}{m\Omega^2}\psi \\ v_2 = \dot{\psi} \\ \dot{v_2} = P\cos(\theta) + \omega_2^2 - \omega_1^2 - 2\delta_2\dot{\psi} - \omega_2^2\psi \end{cases}$$

We apply numerical integration to the above system, for zero initial conditions and get the points [ψ,v] we need for the Poincare section. By using Matlab we solve. More specifically, the code consists of the **poincare.m** and **solver.m** files. After we enter our data in the 1st file you perform the numerical integration, calling the functions to be integrated from the second file. Finally the code returns the response, poincare intercept and phase plane plots for each solution included in the poincare points.

We consider the dynamic behavior for dimensionless excitation frequency in the range where the edges are instructed to respond chaotically and periodically respectively. For each value of the dimensionless excitation frequency, the Poincare plots, the phase plane and the time history (response) of x(t) are requested only when it is periodic.$\frac{\Omega}{\omega_1} = 4.35 - 4.46$

The values of the variables we introduce into the problem are given as data:

$\zeta_1 = 0.015, \zeta_2 = 0.101, \omega_2 = 10.05\omega_1 , P = 1, \qquad Initial\ Condition: \psi(0) = 0, v(0) = 0$

We will perform the requested calculations for 4 values of the dimensionless frequency:

$$\frac{\Omega}{\omega_1} = [4.46 , 4.41 , 4.38 , \qquad 4.35]$$

Then the corresponding diagrams are presented

- $\boldsymbol{\frac{\Omega}{\omega_1}}$**= 4.46**

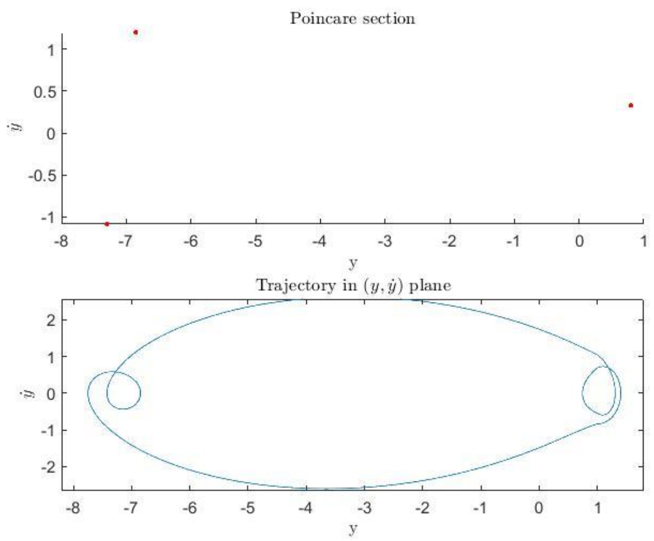


Figure 16. Poincare section and phase plane for$\frac{\Omega}{\omega_1} = 4.46$

Looking at Figure 16 we see the concentration of points on the Poincare section, in 3 points. This indicates the existence of two limit cycles and a closed curve as seen in the phase plane of the same figure. Therefore our response for this constant excitation frequency is periodic with three different periods. The periodic behavior is also verified by the position and velocity response of our oscillator shown in Figure 17.

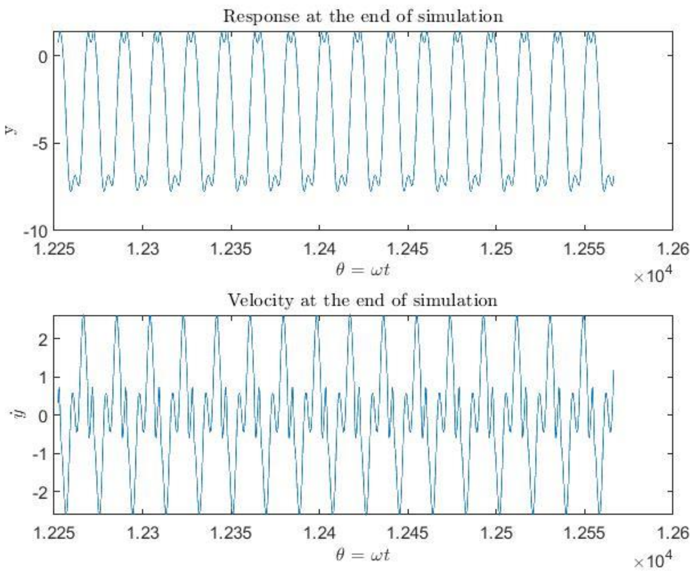


Figure 17. Position and velocity response plots for $\frac{\Omega}{\omega_1} = 4.46$

- $\boldsymbol{\frac{\Omega}{\omega_1}}$ **= 4.41**

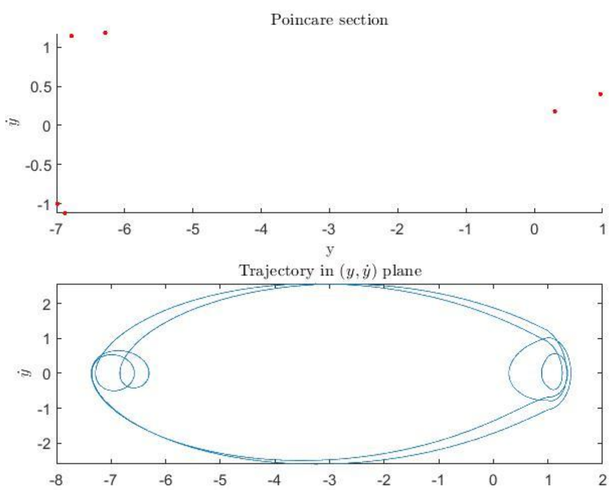


Figure 18. Poincare section and phase plane for $\frac{\Omega}{\omega_1} = 4.41$

As the dimensionless excitation frequency decreases we observe in Figure 18. that we have an increase in the number of points on the Poincare plane. These points include a greater number of periodic solutions than before and correspondingly looking at the phase plane we see the limit cycles and closed curves. Therefore we could say that our solution remains periodic but with multiple periods now. Of course, we cannot be absolutely sure that this solution will end up being periodic (even though it appears to be), which is the main difficulty in nonlinear

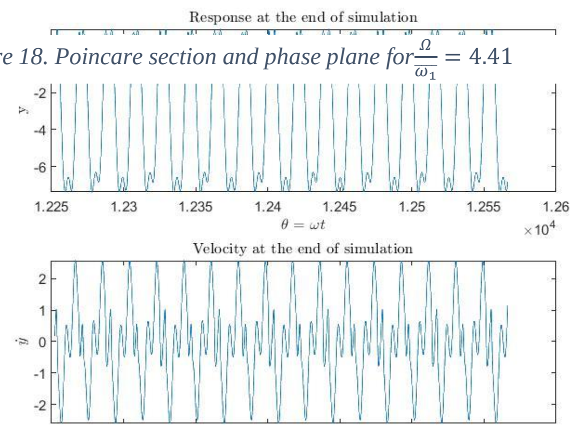


Figure 19. Position and velocity response plots for $\frac{\Omega}{\omega_1} = 4.41$

systems. For more accuracy we could increase the time in which we run our results.

- $\boldsymbol{\frac{\Omega}{\omega_1}}$**= 4.38**

Decreasing the dimensionless frequency further, we can see in Figure 20 the appearance of multiple solutions. We see that the points of contact of the Poincare section with the trajectory are now concentrated in regions and not in points as before. This behavior is aperiodic and we have the appearance of chaotic solutions for our system, with the same uncertainty as before. Figure 21 shows the aperiodic response of the solutions.

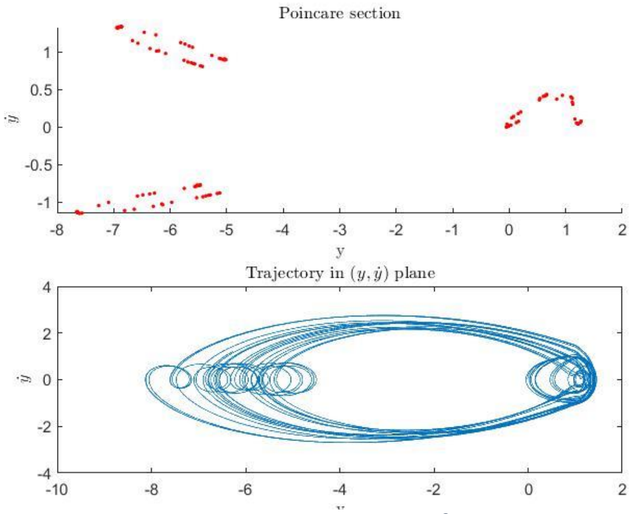


Figure 20. Poincare section and phase plane for$\frac{\Omega}{\omega_1} = 4.38$

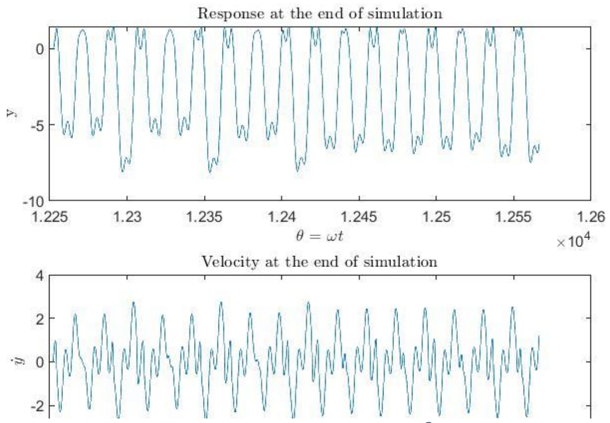


Figure 21. Position and velocity response plots for$\frac{\Omega}{\omega_1} = 4.38$

- $\boldsymbol{\frac{\Omega}{\omega_1}}$**= 4.35**

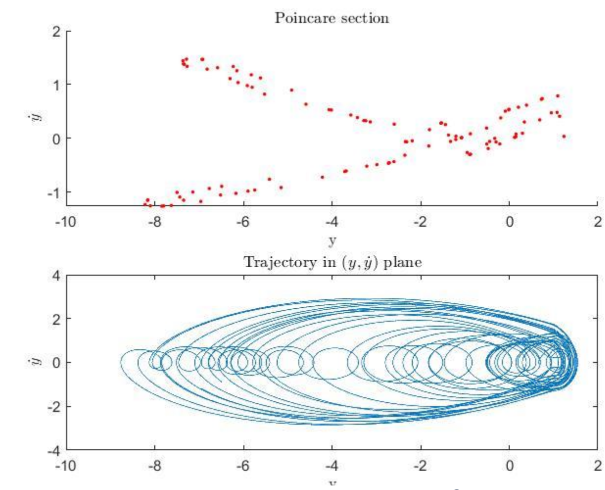


Figure 22. Poincare section and phase plane for $\frac{\Omega}{\omega_1} = 4.35$

Finally in Figure 22 we observe that for values of the dimensionless frequency at 4.35, we have pure chaotic motion. Chaotic motions are characterized by complete aperiodicity, without limit cycles in the phase plane. This aperiodicity can also be seen in Figure 23. of the response.

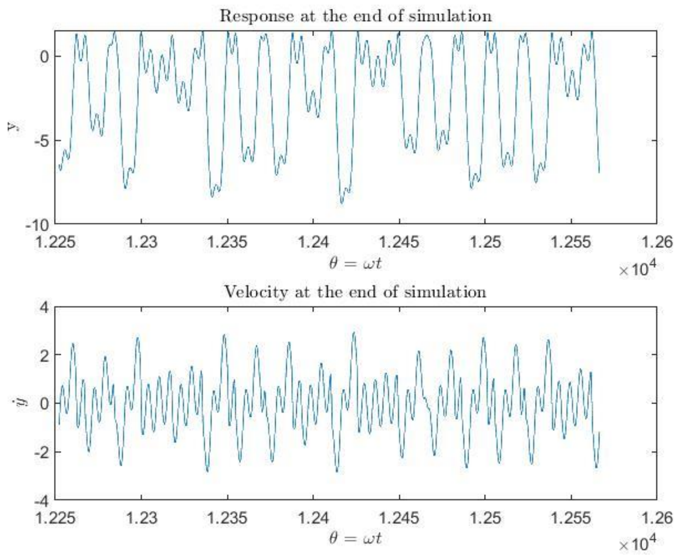


Figure 23. Position and velocity response plots for $\frac{\Omega}{\omega_1} = 4.35$

From the above diagrams we have seen that while our system consists of linear characteristics and linear equations of motion and also our loading is harmonic, it does not always exhibit a periodic response with a period as we would expect in a simple linear system. We saw that by decreasing the excitation frequency, we have increasing points of concentration in the Poincare sections, until we end up with a chaotic solution where our points are now also spread over the whole plane (it would be more noticeable if we ran the simulation for a longer time). That is, we ended up getting from a linear system a behavior similar to the analysis of non-linear systems.

FOURTH QUESTION

In the 4th question, the conditions that lead to a direct determination of periodic response in the steady state of oscillation with 3 changes of the stiffness and damping characteristics in one period will be formulated.

First we define the attribute change times. Let θc1 , θc2 , θc3 , θc4 be the 1st , 2nd and 3rd 4th time characteristic change respectively with , because as we know for harmonic response, the system will give us a periodic response with a period that is an integer multiple of the excitation period. $0 \leq \theta_{ci} \leq 2\pi n\ ,\ i = 1,2,3,4\ and\ n\ integer$

Therefore we have the system of dimensionless equations we defined before, but this time for different intervals each:

$$\begin{cases} \ddot{\psi} + 2\delta_1\dot{\psi} + \omega_1^2\psi = P\cos(\theta) & 0 \leq \theta \leq \theta_{c1} \\ \ddot{\psi} + 2\delta_2\dot{\psi} + \omega_2^2\psi = P\cos(\theta + \theta_{c1} + \gamma_1) + \omega_2^2 - \omega_1^2 & \theta_{c1} \leq \theta \leq \theta_{c2} \\ \ddot{\psi} + 2\delta_1\dot{\psi} + \omega_1^2\psi = P\cos(\theta + \theta_{c2} + \gamma_2) & \theta_{c2} \leq \theta \leq \theta_{c3} \\ \ddot{\psi} + 2\delta_2\dot{\psi} + \omega_2^2\psi = P\cos(\theta + \theta_{c3} + \gamma_3) + \omega_2^2 - \omega_1^2 & \theta_{c3} \leq \theta \leq \theta_{c4} \end{cases}$$

where the parameters of the equations were defined in the previous question, the phases γ are introduced into the excitation so that the response is continuous.

For the 4 different time intervals, based on the linear theory, the responses of the differential equations are calculated:

$$\begin{cases} \psi_1(\theta) = e^{-\delta_1\theta}(A_1\cos(\omega_{d1}\theta) + A_2\sin(\omega_{d1}\theta)) + P_1\cos(\theta - \varphi_1)\ , & 0 \leq \theta \leq \theta_{c1} \\ \psi_2(\theta) = e^{-\delta_2\theta}(A_3\cos(\omega_{d2}\theta) + A_4\sin(\omega_{d2}\theta)) + P_2\cos(\theta + \theta_{c1} + \gamma_1 - \varphi_2) + 1 - P_o & ,\theta_{c1} \leq \theta \leq \theta_{c2} \\ \psi_3(\theta) = e^{-\delta_1\theta}(A_5\cos(\omega_{d1}\theta) + A_6\sin(\omega_{d1}\theta)) + P_1\cos(\theta + \theta_{c2} + \gamma_2 - \varphi_1)\quad , & \theta_{c2} \leq \theta \leq \theta_{c3} \\ \psi_4(\theta) = e^{-\delta_2\theta}(A_7\cos(\omega_{d2}\theta) + A_8\sin(\omega_{d2}\theta)) + P_2\cos(\theta + \theta_{c3} + \gamma_3 - \varphi_2) + 1 - P_o & ,\ \theta_{c3} \leq \theta \leq T \end{cases}$$

with $P_o = \frac{\omega_1^2}{\omega_2^2}$

$$P_i = \frac{P}{\sqrt{\left(\omega_i^2 - 1\right)^2 + 4\delta_i^2}}\ , i = 1,2$$

$$\varphi_i = \tan^{-1}\left(\frac{2\delta_i}{\omega_i^2 - 1}\right)\quad , i = 1,2$$

$$\omega_{di} = \omega_i\sqrt{1 - \zeta_i^2}\quad , i = 1,2$$

and T=2πn since the oscillation period is an integer multiple of the excitation period.

In the above system of differential equations we have 8 unknown constants Ai for i=1 to 8, 3 unknown collision times of the characteristics $\theta_{cj}\ for$ j=1,2,3 and 3 unknown phases γ_i i=1,2,3. Therefore we have a total of 14 unknowns.

In order to have continuity of the solution and to return to the initial state after making 3 changes during a period, the following conditions should be satisfied regarding the position and speed of our oscillator between the equations and also for the initial conditions.

$$\psi_1(0) = \psi_o\ ,\ \ \dot{\psi}_1(0) = \dot{\psi}_o$$

$$\psi_1(\theta_{c1}) = \psi_2(\theta_{c1})\ ,\ \ \dot{\psi}_1(\theta_{c1}) = \ \dot{\psi}_2(\theta_{c1})$$

$$\psi_2(\theta_{c2}) = \psi_3(\theta_{c2})\ ,\ \ \dot{\psi}_2(\theta_{c2}) = \ \dot{\psi}_3(\theta_{c2})$$

$$\psi_3(\theta_{c3}) = \psi_4(\theta_{c3})\ ,\ \ \dot{\psi}_3(\theta_{c3}) = \ \dot{\psi}_4(\theta_{c3})$$

$$\psi_4(\theta_{c4}) = \psi_1(0)\ ,\ \ \dot{\psi}_4(\theta_{c4}) = \dot{\psi}_1(0)$$

Furthermore, we know that the obstacle is weightless, so the change of characteristics always happens in the same position, so:

$$\psi_1(0) = \psi_1(\theta_{c1})$$

$$\psi_2(\theta_{c1}) = \psi_2(\theta_{c2})$$

$$\psi_3(\theta_{c2}) = \psi_3(\theta_{c3})$$

$$\psi_4(\theta_{c3}) = \psi_4(\theta_{c4})$$

Using the above conditions, the 14 unknowns of the system of solutions are calculated. More specifically, using the conditions for the positions, we will obtain relations for the constant coefficients Ai, as a function of θc. Then, using the velocity conditions and the relations for the constant coefficients, we will obtain 3 equations for the times θci. After the times have been calculated we replace and calculate the phases γ, finally we replace in the initial ones and thus we have calculated all the unknowns.