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Department of Mechanical Engineering
MECHANICAL DYNAMICS Laboratory

NUMERICAL METHODS IN OSCILLATIONS OF MECHANICAL SYSTEMS

NUMERICAL METHODS ON DYNAMIC SYSTEMS –

KINEMATIC MECHANICS – PART B

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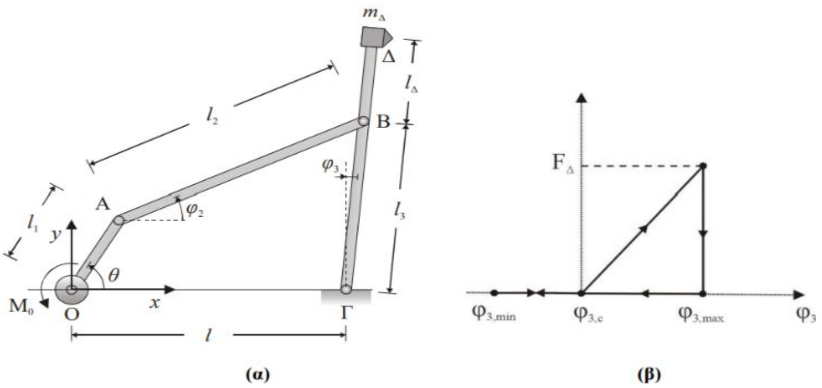
Thessaloniki, January 2022

CONTENTS

PROBLEM DESCRIPTION

The present paper is the 2nd computational Topic in the course 'Numerical Methods in Oscillations of Mechanical Systems'. This work consists in total of 2 distinct parts. In this particular file we study and solve Part B. The problem we are asked to study is the same as Part A with some minor differences, the mechanism under study is presented in Figure 1.



Shape1. Mechanism model, plucker and tip force diagram D as a function of angleφ_3

The results derived in Part A are taken as data for this paper. In the 2nd Part we consider that the mechanism moves with the help of an electric motor now accepting again the load of figure 1b. In addition, the engine needs for the 2nd part of the work to overcome an equivalent viscous resistance torque of the form , where ω is the angular velocity of the driving member.$M_{O,v} = \gamma_O \omega$

Panel1. Prices for Numerical application

$(F\Delta)max$	3 [kN]	l_Δ	15 [cm]
$l1$	15 [cm]	$\varphi_{3,c}$	-10 [°]
$l2$	1[m]	m_1	30[kg]
l	1 [m]	$m_2 = m_3$	60[kg]
$l3$	55 [cm]	m_Δ	90[kg]
m	10 [kg]	γ_O	$4.26[kgm^2/s]$
		R	10 [cm]

The analysis that will follow in all queries is initially carried out using variables, without entering numerical values. During the process of extracting the requested diagrams and results, the programming code will first be compiled in a Matlab environment, where the necessary numerical values will be entered. Table 1 shows a summary of all the numerical values that the pronunciation gives us. Thus, in each question, all the necessary theoretical knowledge and the calculation process are first given, then the code written to extract the results is presented, and the results are presented. Each question is completed with a comment on what is being asked. At the end of this paper in Appendix A, there are all the codes written to solve each question.

EQUATION OF MOTION FOR THE MECHANISM

First we will construct the equation of motion for the mechanism. The equation of motion of the mechanism will be generated using Lagrange's method and the principle of possible works (virtual work principal), so we will be able to use only scalar quantities, which will facilitate the generation of the mechanism equation.

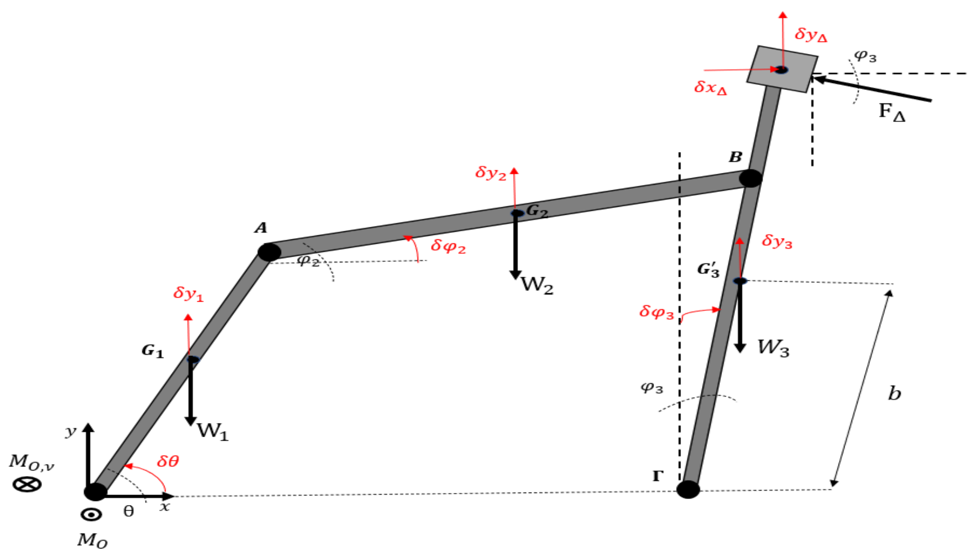
We choose as generalized coordinates the angle θ of the driving member and thus obtains the unique Lagrange equation in the form:$q = \theta$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta + Q'_\theta \quad (\mathbf{1})$$

From the vector summation of the straight segments of the closed polygon OABGO, as we proved in Part A of the paper, 2 more equations result, which will help us to express the relationships of the angles of the mechanism in terms of our unique degree of freedom i. The system results:

$$\begin{cases} l_1 \cos(\theta) + l_2 \cos(\varphi_2) - l_3\, sin(\varphi_3) - l = 0 \\ l_1 \sin(\theta) + l_2 \sin(\varphi_2) - l_3 \cos(\varphi_3) = 0 \end{cases} \quad (\mathbf{2})$$

In equations (1) and (2) the kinetic energy T refers to the total kinetic energy of the mechanism, and the generalized forces include the external, viscous and conservative forces exerted on the Mechanism in the respective directions. Figure 2 shows the mechanism with all the forces and possible displacements that will be introduced in the expression of Lagrange's equations (1).Q_θ



Shape2. Auxiliary mechanism diagram for application of Lagrange

In Part A of the Topic we calculated that the equivalent kinetic energy for the entire mechanism is:$T_{eq} = \frac{1}{2} I_{eq}(\theta)\omega^2$

and using this equation and the expression for the kinetic energy of the mechanism we calculated the equivalent moment of inertia of the mechanism for all positions of the driving member. Since the properties of the system members remain the same and we also do not add any new body to the system, the mass moment of inertia remains constant. So from the above equation we have that since .$T_{O\Lambda} = T(\theta, \dot{\theta})\omega = \dot{\theta}$

Then we calculate the terms of the derivatives found in Lagrange's equation (1) using the equivalent kinetic energy:

$\therefore \boxed{\frac{\partial T_{O\Lambda}}{\partial \dot{\theta}}}$

$$\frac{\partial T_{O\Lambda}}{\partial \dot{\theta}} = I_{eq}(\theta)\omega$$

$\therefore \boxed{\frac{d}{dt}\left(\frac{\partial T_{O\Lambda}}{\partial \dot{\theta}}\right)}$

$$\frac{d}{dt}\left(\frac{\partial T_{O\Lambda}}{\partial \dot{\theta}}\right) = I_{eq}(\theta)\ddot{\theta}$$

$\therefore \boxed{\frac{\partial T_{O\Lambda}}{\partial \theta}}$

$$\frac{\partial T_{O\Lambda}}{\partial \theta} = 0$$

So the equation of motion (1) comes in the form:

$$I_{eq}(\theta)\ddot{\theta} = Q_\theta \quad \mathbf{(3)}$$

At this point we will apply the Principle of Powerful Works (virtual work principal) to calculate the terms of the generalized forces. From the expression of potential work we have:Q_θ

$$\delta W = \sum_{j=1}^{n} Q_j \delta q_j = Q_\theta \delta q_\theta \quad (\mathbf{4})$$

We also have for the conservative (weights), viscous (Moment of Resistance) and applied forces the corresponding possible works. First we calculate for the weights of the members and the concentrated mass D:(F_Δ)

$$\delta W_c = \underline{W_1}\underline{\delta r_{G_1}} + \underline{W_2}\underline{\delta r_{G_2}} + \underline{W_3}\underline{\delta r_{G_3'}}$$

In the above equation, the vectors of possible displacements refer to the position vectors of the centers of gravity for the first 2 members and for the center of gravity of the system member 3 – Body D. The expressions for these vectors are given below where they were defined in full analysis in Part A:$r_{G_3'}$

$$\underline{r}_{G,1} = \frac{l_1}{2}\left(\cos(\theta)\ \underline{e}_x + \sin(\theta)\,\underline{e}_y\right)$$

$$\underline{r}_{G,2} = l_1\left(\cos(\theta)\ \underline{e}_x + \sin(\theta)\,\underline{e}_y\right) + \frac{l_2}{2}\left(\cos(\varphi_2)\underline{e}_x + \sin(\varphi_2)\,\underline{e}_y\right)$$

$$\underline{r}_{G_3'} = \left(\frac{(m_3 + m_\Delta)l}{m} + b\sin(\varphi_3)\right)\underline{e}_x + (b\cos(\varphi_3))\underline{e}_y$$

where $b = \frac{\left(m_3\frac{l_3+l_\Delta}{2} + m_\Delta(l_3+l_\Delta)\right)}{m}$

thus by applying the operator d, we get the equations for the possible displacements:

$$\underline{\delta r_{G_1}} = \frac{l_1}{2}\big(-sin(\theta)\ \underline{e}_x + \cos(\theta)\,\underline{e}_y\big)\delta\theta$$

$$\underline{\delta r_{G_2}} = l_1\big(-sin(\theta)\ \underline{e}_x + \cos(\theta)\,\underline{e}_y\big)\delta\theta + \frac{l_2}{2}\big(-sin(\varphi_2)\underline{e}_x + \cos(\varphi_2)\,\underline{e}_y\big)\delta\varphi_2$$

$$\underline{\delta r_{G_3'}} = b\big(\cos(\varphi_3)\,\underline{e}_x - \sin(\varphi_3)\,\underline{e}_y\big)\delta\varphi_3$$

The expressions for the weights are:$W_1 = m_1 g\underline{e}_y\,, W_2 = m_2 g\ \underline{e}_y\,, W_3 = (m_3 + m_\Delta)g\underline{e}_y$

Substituting all of the above into the expression for the potential work of weights we get:

$$\begin{aligned}\delta W_c = \left(m_1 g\underline{e}_y\right)&\left(\frac{l_1}{2}\big(-sin(\theta)\ \underline{e}_x + \cos(\theta)\,\underline{e}_y\big)\delta\theta\right)\\ &+ \left(m_2 g\ \underline{e}_y\right)\Big(l_1\big(-sin(\theta)\ \underline{e}_x + \cos(\theta)\,\underline{e}_y\big)\delta\theta\\ &+ \frac{l_2}{2}\big(-sin(\varphi_2)\underline{e}_x + \cos(\varphi_2)\,\underline{e}_y\big)\delta\varphi_2\Big)\\ &+ \Big((m_3 + m_\Delta)g\underline{e}_y\Big)\Big(b\big(\cos(\varphi_3)\,\underline{e}_x - \sin(\varphi_3)\,\underline{e}_y\big)\delta\varphi_3\Big)\end{aligned}$$

$$\begin{aligned}\Rightarrow \delta W_c = \left(\frac{1}{2}l_1 m_1 g cos(\theta) + m_2 g\ l_1\cos(\theta)\right)\delta\theta + (m_2 g l_2\cos(\varphi_2))\delta\varphi_2\\ - b(m_3 + m_\Delta)g\sin(\varphi_3)\,\delta\varphi_3 \quad \mathbf{(5)}\end{aligned}$$

The mighty work for herequivalent torque of viscous resistance and the input torque from the electric motor to move the mechanism is:

$$\delta W_v = M_O\delta\theta - M_{O,v}\delta\theta \quad (\mathbf{6})$$

Finally, the potential work for the resistance at end D results:

$$\delta W_\Delta = \underline{F_\Delta}\underline{\delta r_\Delta}$$

with$\underline{F_\Delta} = -F_\Delta\cos(\varphi_3)\,\underline{e}_x + F_\Delta\sin(\varphi_3)\,\underline{e}_y$

$\underline{\delta r_\Delta} = \delta\left(l\underline{e}_x + (l_3 + l_\Delta)\big(\sin(\varphi_3)\,\underline{e}_x + \cos(\varphi_3)\,\underline{e}_y\big)\right)$

$\Rightarrow \underline{\delta r_\Delta} = (l_3 + l_\Delta)\big(cos(\varphi_3)\,\underline{e}_x - \sin(\varphi_3)\,\underline{e}_y\big)\delta\varphi_3$

therefore

$$\delta W_\Delta = \big(-F_\Delta\cos(\varphi_3)\,\underline{e}_x + F_\Delta\sin(\varphi_3)\,\underline{e}_y\big)\Big((l_3 + l_\Delta)\big(cos(\varphi_3)\,\underline{e}_x - \sin(\varphi_3)\,\underline{e}_y\big)\delta\varphi_3\Big)$$

$$\delta W_\Delta = (l_3 + l_\Delta)\ F_\Delta(-\cos^2(\varphi_3) - \sin^2(\varphi_3))\ \delta\varphi_3$$

$$\delta W_\Delta = -(l_3 + l_\Delta)\ F_\Delta\delta\varphi_3\ (\mathbf{7})$$

where for the from Figure 1b. we have, as proved in Part A, that:F_Δ

$$F_\Delta = \begin{cases}-7876.06\varphi_3 - 1374.37 \quad \gamma\iota\alpha\ \varphi_3 \in \{-31.82, -10\} \cap \{\ \underline{\dot{\varphi_3}} \le 0\}\\ 0\ \ \gamma\iota\alpha\{\ \varphi_3\ \notin (-31.82, -10)\} \cup\ \{\underline{\dot{\varphi_3}} \ge 0\}\end{cases}$$

From the system of equations 2 (bond equations) which is a system of the form , by calculating its total differential we have:$\underline{f}(\theta, \varphi_2, \varphi_3) = 0$

$$\underline{f}(\theta,\varphi_2,\varphi_3) = \begin{cases} l_1\cos(\theta) + l_2\cos(\varphi_2) - l_3\, sin(\varphi_3) - l \\ l_1\sin(\theta) + l_2\sin(\varphi_2) - l_3\cos(\varphi_3) \end{cases} = \underline{0}$$

$$df = 0 \Rightarrow \frac{\partial \underline{f}}{\partial\varphi_2}\delta\varphi_2 + \frac{\partial \underline{f}}{\partial\varphi_3}\delta\varphi_3 + \frac{\partial \underline{f}}{\partial\theta}\delta\theta = \underline{0} \ \ (\mathbf{8})$$

Then we calculate one by one the terms of the differential of equation (8):

$\therefore \boxed{\frac{\partial \underline{f}}{\partial\theta}\delta\theta}$

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial\theta} \\ \dfrac{\partial f_2}{\partial\theta} \end{bmatrix}\delta\theta = \begin{bmatrix} -l_1\sin(\theta) \\ l_1\cos(\theta) \end{bmatrix}\delta\theta$$

$\therefore \boxed{\frac{\partial \underline{f}}{\partial\varphi_2}\delta\varphi_2}$

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial\varphi_2} \\ \dfrac{\partial f_2}{\partial\varphi_2} \end{bmatrix}\delta\varphi_2 = \begin{bmatrix} -l_2\sin(\varphi_2) \\ l_2\cos(\varphi_2) \end{bmatrix}\delta\varphi_2$$

$\therefore \boxed{\frac{\partial \underline{f}}{\partial\varphi_3}\delta\varphi_3}$

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial\varphi_3} \\ \dfrac{\partial f_2}{\partial\varphi_3} \end{bmatrix}\delta\varphi_3 = \begin{bmatrix} -l_3\, cos(\varphi_3) \\ l_3\sin(\varphi_3) \end{bmatrix}\delta\varphi_3$$

Substitution of the above terms in equation (8) gives:

$$\begin{bmatrix} -l_2\sin(\varphi_2) \\ l_2\cos(\varphi_2) \end{bmatrix}\delta\varphi_2 + \begin{bmatrix} -l_3\, cos(\varphi_3) \\ l_3\sin(\varphi_3) \end{bmatrix}\delta\varphi_3 + \begin{bmatrix} -l_1\sin(\theta) \\ l_1\cos(\theta) \end{bmatrix}\delta\theta = \underline{0}$$

$$\Rightarrow \begin{bmatrix} l_1\sin(\theta) \\ -l_1\cos(\theta) \end{bmatrix}\delta\theta = \begin{bmatrix} -l_2\sin(\varphi_2) & -l_3\, cos(\varphi_3) \\ l_2\cos(\varphi_2) & l_3\sin(\varphi_3) \end{bmatrix}\begin{bmatrix} \delta\varphi_2 \\ \delta\varphi_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \delta\varphi_2 \\ \delta\varphi_3 \end{bmatrix} = J^{-1}\begin{bmatrix} l_1\sin(\theta) \\ -l_1\cos(\theta) \end{bmatrix}\delta\theta = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\ \delta\theta \ \ (\mathbf{9})$$

$$\mu\varepsilon\ C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = J^{-1}\begin{bmatrix} l_1\sin(\theta) \\ -l_1\cos(\theta) \end{bmatrix}\ (\sigma\upsilon\mu\pi\alpha\gamma\acute{\varepsilon}\sigma\tau\varepsilon\rho\eta\ \mu o\rho\varphi\acute{\eta}\ )$$

where is the Jacobian matrix of the system of constraints, this register is reversed since, as we proved in Part A of the Topic, it does not display any singular point . So from equation (9) we have calculated the possible displacements of the corners as a function of the generalized coordinate.$J = J(\theta,\varphi_2,\varphi_3)(\det(J) \neq 0\ \gamma\iota\alpha\ \kappa\acute{\alpha}\theta\varepsilon\ \theta)\varphi_2,\varphi_3$

Now substituting relations (5), (6), (7) and the equation in (4) we get:$\delta W_c + \delta W_v + \delta W_\Delta = \delta W$

$$Q_\theta \delta q_\theta = \left(\frac{1}{2} l_1 m_1 g cos(\theta) + m_2 g\, l_1 \cos(\theta)\right) \delta\theta + (m_2 g l_2 \cos(\varphi_2))\delta\varphi_2 \\ - b(m_3 + m_\Delta) g \sin(\varphi_3)\, \delta\varphi_3 + M_O \delta\theta - M_{O,v}\delta\theta - (l_3 + l_\Delta)\, F_\Delta \delta\varphi_3$$

$$\Longrightarrow Q_\theta \delta\theta = \left(\frac{1}{2} l_1 m_1 g cos(\theta) + m_2 g\, l_1 \cos(\theta) + M_O - M_{O,v}\right) \delta\theta + (m_2 g l_2 \cos(\varphi_2))\delta\varphi_2 \\ - (b(m_3 + m_\Delta) g \sin(\varphi_3) + (l_3 + l_\Delta)\, F_\Delta)\, \delta\varphi_3$$

We express the possible displacements of the corners of members 2 and 3, with respect to θ from equation (9) and we get:

$$Q_\theta \delta\theta = \left(\frac{1}{2} l_1 m_1 g cos(\theta) + m_2 g\, l_1 \cos(\theta) + M_O - M_{O,v}\right) \delta\theta + (m_2 g l_2 \cos(\varphi_2)) c_1 \delta\theta \\ - (b(m_3 + m_\Delta) g \sin(\varphi_3) + (l_3 + l_\Delta)\, F_\Delta) c_2\, \delta\theta$$

$$\Longrightarrow Q_\theta \delta\theta = \left(\frac{1}{2} l_1 m_1 g cos(\theta) + m_2 g\, l_1 \cos(\theta) + M_O - M_{O,v} + (m_2 g l_2 \cos(\varphi_2)) c_1 \right. \\ \left. - (b(m_3 + m_\Delta) g \sin(\varphi_3) + (l_3 + l_\Delta)\, F_\Delta)\, c_2 \right) \delta\theta$$

So the generalized force in the direction of the generalized coordinate will be:

$$Q_\theta = \left(\frac{1}{2} l_1 m_1 g cos(\theta) + m_2 g\, l_1 \cos(\theta) + M_O - M_{O,v} + (m_2 g l_2 \cos(\varphi_2)) c_1 \right. \\ \left. - (b(m_3 + m_\Delta) g \sin(\varphi_3) + (l_3 + l_\Delta)\, F_\Delta)\, c_2 \right) \quad (\mathbf{10})$$

Substituting the above equation into equation (3) gives:

$$I_{eq}(\theta)\ddot{\theta} = \left(\frac{1}{2} l_1 m_1 g cos(\theta) + m_2 g\, l_1 \cos(\theta) + M_O - M_{O,v} + (m_2 g l_2 \cos(\varphi_2)) c_1 \right. \\ \left. - (b(m_3 + m_\Delta) g \sin(\varphi_3) + (l_3 + l_\Delta)\, F_\Delta)\, c_2 \right)$$

where in a more compact form it becomes:

$$\Longrightarrow \boxed{I_{eq}(\theta)\ddot{\theta} + \gamma_O \dot{\theta} = \Gamma_1 + \Gamma_2 c_1 - \Gamma_3\, c_2 + M_O} \quad (\mathbf{11})$$

with ,$\Gamma_1 = \left(\frac{1}{2} l_1 m_1 g + m_2 g\, l_1\right) cos(\theta) \Gamma_2 = (m_2 g l_2 \cos(\varphi_2))$

$\Gamma_3 = b(m_3 + m_\Delta) g \sin(\varphi_3) + (l_3 + l_\Delta)\, F_\Delta$

Equation (11) constitutes the equation of motion of our mechanism with respect to the unique degree of freedom of i. This equation is a strongly nonlinear ordinary differential equation of the 2nd order, so it does not accept an analytical solution but is solved numerically. For our problem to be fully defined we need to define the initial conditions for our problem. We consider the time that the mechanism starts from rest at the position θ=0. By means of (11) we now express the movement of the entire mechanism with an equation, in terms of a coordinate.$t = 0 \theta(0) = 0\ \kappa\alpha\iota\ \dot{\theta}(0) = 0$

1st QUESTION

In the 1st question we are asked, based on the required average power calculated in Part A and the torque diagrams of the electric motor (which we will choose), to select a suitable electric motor. Then, by applying the central difference method, determine the angular velocity of the driving member in the permanent operating state of the mechanism.

Choice of Electric Motor

In part A we calculated average mechanism power. This power is the load of our mechanism. Next, we will select an electric motor and construct its characteristic curve (Torque-Revolutions) in a common diagram with the load characteristic (Load Torque-Revolutions). The above average power is the load in stable operating conditions (in part A), i.e. operating condition in the permanent state. So for the selection of an electric motor that will satisfy our load, we will compare its characteristic curve with the average torque of the load at rotation speed, so we calculate . It is of great importance that the electric motor we choose can provide the necessary torque to start the rotation of the mechanism.$P_{mean} = 4.31\ [kW]$ $\dot{\theta} = \sigma\tau\alpha\theta.\,240\ rpm$$M_{mean} = \frac{P_{mean}}{\frac{2\pi 240[rpm]}{60}} = 171.5\ [Nm]$

Based on the above, we choose a three-phase motor, short-circuited cage, surface cooling, with internal fan, insulation class B with manufacturer's prices (Book: Consumer Electrical Installations, Petros Dokopoulos):

Rated power$P_N = 15\ kW$

Rated speed$n_N = 1460 \frac{\sigma\tau\rho}{min}$

Asynchronous turns$n_s = 1500\ \frac{\sigma\tau\rho}{min}$

Torque Class: KL 13

Rated Torque :$M_N = 98\ Nm$

Starting Torque :$M_\varepsilon\ = 2.2\ M_N$

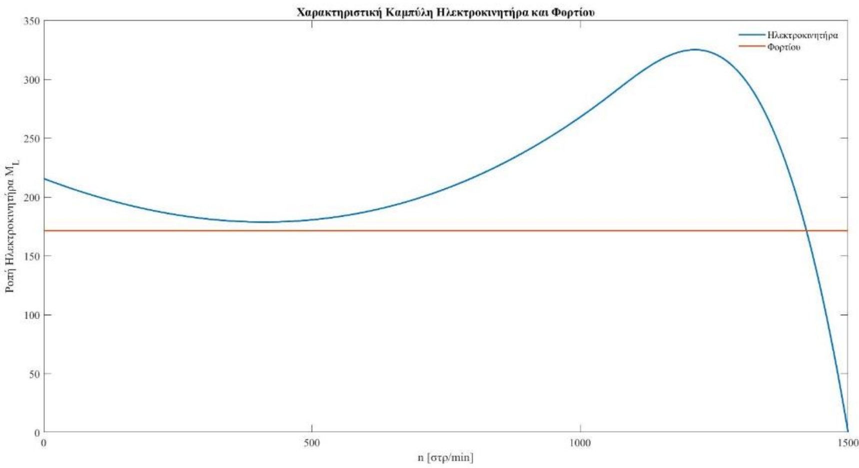
Overturning Moment :$M_\alpha\ = 2.9\ M_N$

Minimum Torque (before stopping):$M_{low} = 25\%\ M_\varepsilon$

Moment of inertia :$I_k =\ J = 0.37\ kgm^2$

Based on the torque values listed above and the value at the operating point that will cover our load, we construct the characteristic curve. The characteristic curve is constructed approximately using the above values, since its exact construction requires measurements on the electric motor and is not offered by the manufacturer.(M_{mean})

The resulting 2 curves are shown in Figure 3. From this figure we see that the electric motor can meet the needs of our load during startup. The point of intersection of the 2 curves is also the operating point for our engine. Also using the values of the characteristic curve for the motor, we can now express the applied torque as a function of the angular velocity of rotation. Next we describe the method of Central differences.$M_O\ = M_O(\dot{\theta})$



Shape3. Characteristic load curves - Electric motor

Central Difference Method

This method is applied to ordinary differential equations of the 2nd order, it is based on the time discretization expressed by the approximation of the time derivatives with algebraic equations that are functions of the previous steps. To apply the method we first discretize the calculation interval of the requested response with step , the step we choose depends on the characteristics of the system we are studying each time, for example, rapidly changing characteristics require a very small step. In the analysis we will use the following symbols:$(t_1, t_2 \dots t_m)$ h $= t_{m+1} - t_m$

$$\theta(\mathrm{t}_m) = \theta_m \quad \kappa\alpha\iota \ f(\theta(t_m), t_m) = f_m$$

Then we develop the terms in Taylor series:θ_{m+1} $\kappa\alpha\iota$ θ_{m-1}

$$\theta_{m+1} = \theta(t_m + h) = \theta(t_m) + h\dot{\theta}(t_m) + \frac{1}{2}h^2\ddot{\theta}(t_m) + O(h^3)$$

$$\theta_{m-1} = \theta(t_m - h) = \theta(t_m) - h\dot{\theta}(t_m) + \frac{1}{2}h^2\ddot{\theta}(t_m) + O(h^3)$$

By subtracting these 2 developments we get:

$$\dot{\theta}_m = \frac{\theta_{m+1} - \theta_{m-1}}{2h} + O(h^2) \quad (\mathbf{12})$$

while with addition:

$$\ddot{\theta}_m = \frac{\theta_{m+1} - 2\theta_m + \theta_{m-1}}{h^2} + O(h^2) \quad (\mathbf{13})$$

Equations (16) and (17) are the central differences with which we discretize the time derivatives in the equation we are studying. Substituting these equations into the equation of motion, solving for the term and using the initial conditions of the problem we can calculate the position values for all time instants. We notice that a problem appears in the initialization of the process (m=0) because the term appears in equations (12) and (13), to calculate this we solve the system of equations (12) and (13) for m=0, in terms of , the termθ_{m+1} $\left(\theta_0 = \theta(0)\,, \dot{\theta}_0 = 75.3\,\frac{rad}{s}\right)$ $\theta_{m+1}\theta_{-1}\theta_1$ $\kappa\alpha\iota$ $\theta_{-1}\ddot{\theta}_0$is calculated from the equation of motion (11) for m=0. After these terms have been calculated, we start the application of the method for m=1. For each time step, after its calculation we calculate the 1st and 2nd time derivatives in the previous step from equations (12) and (13).θ_{m+1}

For our problem now, we apply equations (12) and (13) to the equation of motion (11) after first writing it for the m-th instant and we have:

$$I_{eq}(\theta_m)\left(\frac{\theta_{m+1} - 2\theta_m + \theta_{m-1}}{h^2}\right) + \gamma_O\left(\frac{\theta_{m+1} - \theta_{m-1}}{2h}\right) = \Gamma_1(\theta_m) + \Gamma_2(\theta_m)c_1(\theta_m) - \Gamma_3(\theta_m)\, c_2(\theta_m) + M_O$$

Solving the above equation in terms of we get:θ_{m+1}

$$\theta_{m+1}\left(\frac{I_{eq}(\theta_m)}{h^2} + \frac{\gamma_O}{2h}\right) = \gamma_O\left(\frac{\theta_{m-1}}{2h}\right) + I_{eq}\left(\frac{2\theta_m - \theta_{m-1}}{h^2}\right) + \Gamma_1(\theta_m) + \Gamma_2(\theta_m)c_1(\theta_m) - \Gamma_3(\theta_m)\, c_2(\theta_m) + M_O$$

$$\Rightarrow \theta_{m+1} = \left(\frac{I_{eq}(\theta_m)}{h^2} + \frac{\gamma_O}{2h}\right)^{-1}\left(\gamma_O\left(\frac{\theta_{m-1}}{2h}\right) + I_{eq}\left(\frac{2\theta_m - \theta_{m-1}}{h^2}\right) + \Gamma_1(\theta_m) + \Gamma_2(\theta_m)c_1(\theta_m) - \Gamma_3(\theta_m)\, c_2(\theta_m) + M_O\right) \quad \mathbf{(14)}$$

after$\left(\frac{I_{eq}(\theta_m)}{h^2} + \frac{\gamma_O}{2h}\right) > 0\ \delta\iota\acute{o}\tau\iota\ I_{eq}(\theta_m) > 0\ \kappa\alpha\iota\ h, \gamma_O > 0$

where:,$\Gamma_1 = \left(\frac{1}{2} l_1 m_1 g + m_2 g\, l_1\right) cos(\theta)\Gamma_2 = (m_2 g l_2 \cos(\varphi_2))$

,$\Gamma_3 = b(m_3 + m_\Delta) g \sin(\varphi_3) + (l_3 + l_\Delta)\, F_\Delta \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = J^{-1}\begin{bmatrix} l_1 \sin(\theta) \\ -l_1 \cos(\theta) \end{bmatrix}$

we calculate the terms for m=0 in equations (12) and (13), first of course we calculate the term of the 2nd derivative for m=0, using the initial conditions:$\theta_{-1} \kappa\alpha\iota\ \theta_1 \theta(0) = 0\ \kappa\alpha\iota\ \dot{\theta}(0) = 0$

$$\ddot{\theta}_0 = \frac{(\Gamma_1(\theta_0) + \Gamma_2(\theta_0)c_1(\theta_0) - \Gamma_3(\theta_0)\, c_2(\theta_0) + M_O\ )}{I_{eq}(0)}$$

so we have:

$$\begin{cases} \theta_1 = 2h\dot{\theta}_0 + \theta_{-1} \\ h^2\ddot{\theta}_0 = \theta_1 - 2\theta_0 + \theta_{-1} \end{cases} \Rightarrow \begin{cases} \theta_1 = \theta_{-1} \\ h^2\ddot{\theta}_0 = 2\theta_{-1} \end{cases} \Rightarrow \theta_{-1} = \frac{h^2\ddot{\theta}_0}{2} = \theta_1$$

The terms of the generalized coordinates that appear in the above equations, from Part A of the Topic we know that they are functions of the angle θ, therefore knowing the value for the coordinate we can calculate them from the numerical solution of the system of holonomial ties (2). Specifically, we apply the NR method and solve the system:$\varphi_2, \varphi_3 \theta_m$

$$\underline{f}\left(\underline{\varphi}\,; \theta_m\right) = \begin{cases} l_1 \cos(\theta_m) + l_2 \cos(\varphi_2) - l_3 \sin(\varphi_3) - l = 0 \\ l_1 \sin(\theta_m) + l_2 \sin(\varphi_2) - l_3 \cos(\varphi_3) = 0 \end{cases}$$

applying the calculation scheme:

$$(\,\underline{\varphi}_m^{i+1},\ \theta_m) = \left(\underline{\varphi}^i{}_m + \Delta\underline{\varphi}^i{}_m, \theta_m\right) \quad , \quad \mu\varepsilon\ \underline{\varphi}_m = \begin{bmatrix} \varphi_2(\theta_m) \\ \varphi_3(\theta_m) \end{bmatrix}$$

with$\Delta\underline{\varphi}^{i}{}_{m} = -J^{-1}\left(\underline{\varphi}_m; \theta_m\right)\underline{f_m}\left(\underline{\varphi}_m\, ; \theta_m\right)$

where the subscript m indicates the time step for which we are calculating while the subscript i refers to the iterations in NR space. In addition, the method needs an initial estimate, where we choose it to be the same as the solution in the previous step

$$\left(\underline{\varphi}_m^0, \theta_m\right) = \left(\underline{\varphi}_{m-1}, \theta_m\right)$$

Of course, as we mentioned above, we start applying the method of central differences for m=1. Thus the term that appears in the initial estimate of NR for m=1 needs to be calculated. From the diagrams of Part A, we have that for .$\underline{\varphi}_0 = \begin{bmatrix} \varphi_2(\theta = 0) \\ \varphi_3(\theta = 0) \end{bmatrix} \theta = 0\,,\ \underline{\varphi}_0 \approx \begin{bmatrix} 33.34 \\ -1.5 \end{bmatrix}$

and J o Jacobean$J\left(\underline{\varphi}; \theta\right) = \begin{bmatrix} \frac{\partial f_1}{\partial \varphi_2} & \frac{\partial f_1}{\partial \varphi_3} \\ \frac{\partial f_2}{\partial \varphi_2} & \frac{\partial f_2}{\partial \varphi_3} \end{bmatrix} = \begin{bmatrix} -l_2 \sin(\varphi_2) & -l_3 \cos(\varphi_3) \\ l_2 \cos(\varphi_2) & l_3 \sin(\varphi_3) \end{bmatrix}$

The iterative process for the NR method is completed when the error we define is satisfied:

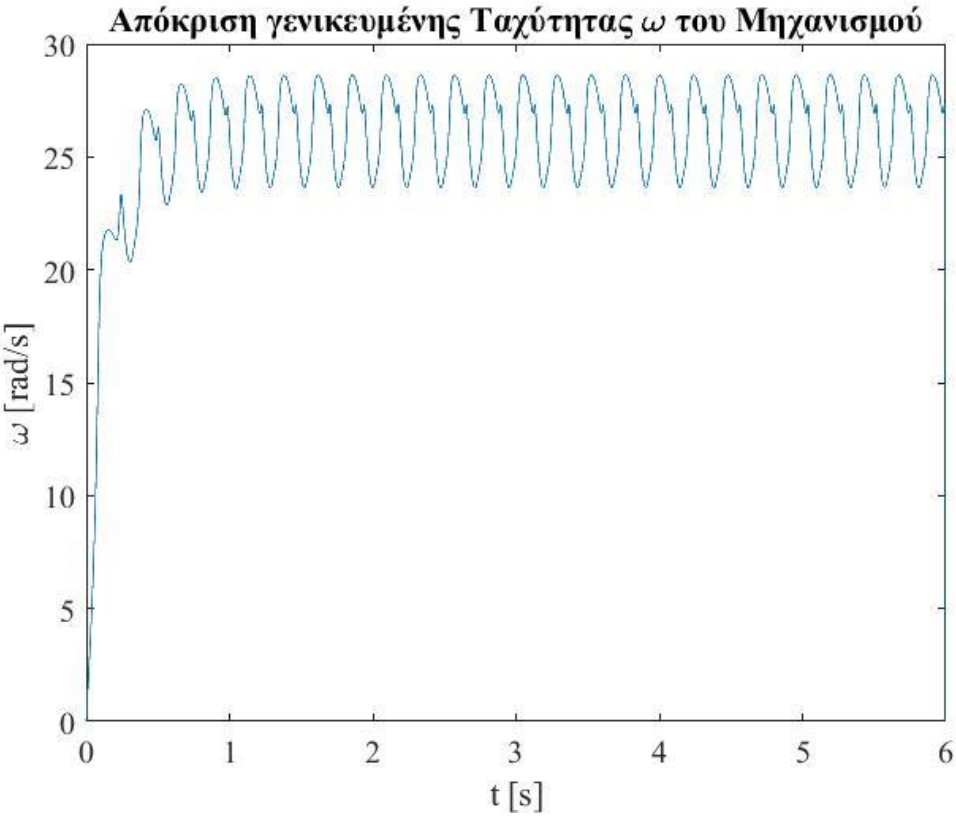
$$\left\| \underline{f}(\ \underline{\varphi}_m^{i+1}, \theta_m)\ \right\| < \varepsilon \text{ , με ε ένας θετικός μικρός αριθμός}$$

as we saw in Part A of the paper, the application of the NR method does not show singular points for any value of the angles.

Therefore, based on all the above, after choosing the numerical values for the basic quantities (h,ε), we can calculate the time histories for the position (ex. 14) and the speed (ex. 12) for the driving member of the mechanism. The file 'SolverMerosB.m' was written to produce the requested diagram, which is provided in Appendix A1. For the calculations we apply all the necessary numerical values of the utterance (Table 1.) and also choose time discretization step after trial and error NR . The step was chosen to be small enough not to lose any of the rapidly changing dynamic features.$h = 10^{-3}\varepsilon = 10^{-7}$

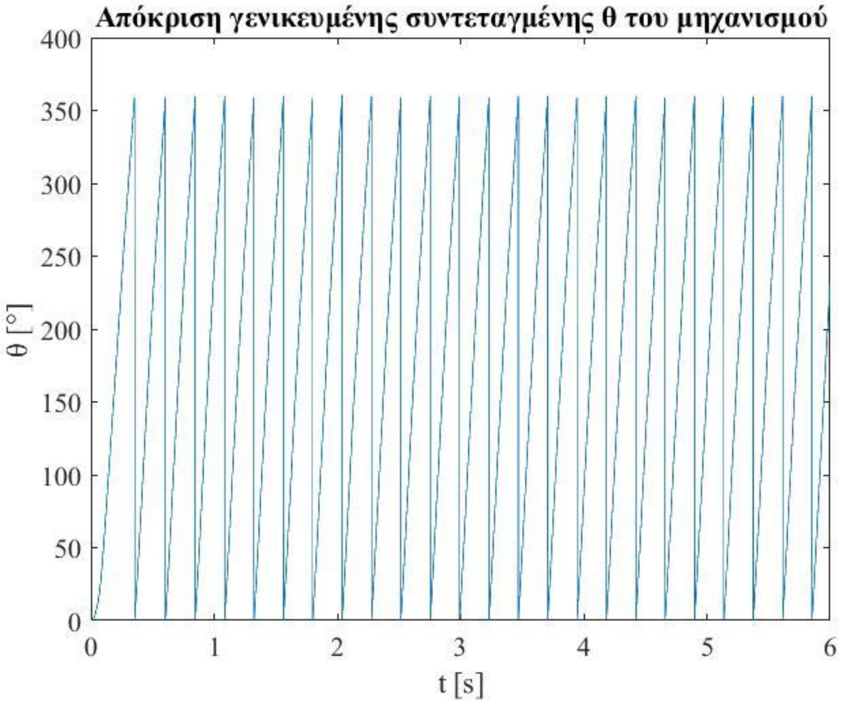
In its structure the SolverMerosB file first constructs the characteristic curve diagrams presented earlier. He then applies the method of central differences, and the Newton-Raphson method to calculate angles where necessary. From this file we call the following functions: 'FUNA.m' bond equations, 'JAC.m' Jacobian bond system, 'NR.m' application of the Newton Raphson method, 'ForceD.m' force calculation in Δ, 'C.m' 'calculation of coefficients, 'Mo.m' calculation of electric motor torque as a function of revolutions with linear interpolation between the values for the corresponding speeds, 'I.m' calculation of moment of inertia of the mechanism at the specific position with linear interpolation to the values of the moment of inertia we calculated in part A. All these files are delivered in Appendix A1.$\varphi_i(\theta)c_i$

Figure 4 shows the generalized velocity response plot and Figure 5 shows the generalized coordinate response plot to verify the functionality of the code.



Shape4. Mechanism Generalized Velocity response diagram$\dot{\theta}$

Observing Figure 4. we easily distinguish the transient and the permanent state. Also for the steady state we observe the variations of the angular velocity around an equilibrium position, which is the operating point for the electric motor we have chosen. From the tests carried out for the selection of the appropriate electric motor, the logical conclusion emerged that the larger (in terms of power and revolutions) the electric motor is, the faster the steady state is achieved, which of course we pay for in stressing the mechanism due to large and rapidly changing characteristics . Of course, choosing a motor with very high characteristics will accelerate the mechanism continuously until it reaches close to its modern revolutions Let's just say that we achieved the steady state in about 1 second. Finally, let us mention that the point around which the angular velocity oscillates in the steady state has a value, while based on the load we chose, we would expect to have values that differ slightly and are due to calculation inaccuracies of the numerical methods and approximations during the construction of the characteristic curve of the electric motor, a precise range of values to construct the characteristic curve would better approximate the nominal values. Figure 5 shows the response of the actuator angle, with reasonable periodic behavior after the steady state is restored.(n_s). $\omega \approx 26\frac{rad}{s} n_N = 240\, rpm = 25.13\frac{rad}{s}$



Shape5. Generalized Coordinate response diagram of the mechanism i

2nd QUESTION

In the 2nd question we are asked, after we introduce a flywheel with a mass moment of inertia with an axis of rotation Oz. To study its effect on the normalization of the angular velocity of the crank. The degree of unevenness of ω should be used as a criterion for comparing the results. It is also asked what its value should be so that δ ≈ 0.01. To redo the calculations for and find the new optima.$I_f I_f \delta = \frac{\omega_{max} - \omega_{min}}{\bar{\omega}}$.$I_f \omega' = \frac{\omega}{4}$ $\kappa\alpha\iota$ $\omega'' = 4\omega I_f$

The flywheel we add has a common axis of rotation with the drive member and performs only rotary motion. Therefore the contribution of this lies in the differentiation of the kinetic energy during the construction of Lagrange's equations and has no influence on the generalized forces since it does not perform any other movement apart from rotation. That is, the resistance to the movement of the mechanism, in the direction of the generalized coordinate, is expressed means of the added moment of inertia to it. The kinetic energy of the flywheel is:

$$T_f = \frac{1}{2} I_f \dot{\theta}^2$$

adding this term to the equivalent kinetic energy brings us the equation of motion (11) in the form:

$$\Rightarrow \left(I_{eq}(\theta) + I_f\right)\ddot{\theta} + \gamma_O \dot{\theta} = \Gamma_1 + \Gamma_2 c_1 - \Gamma_3\, c_2 + M_O$$

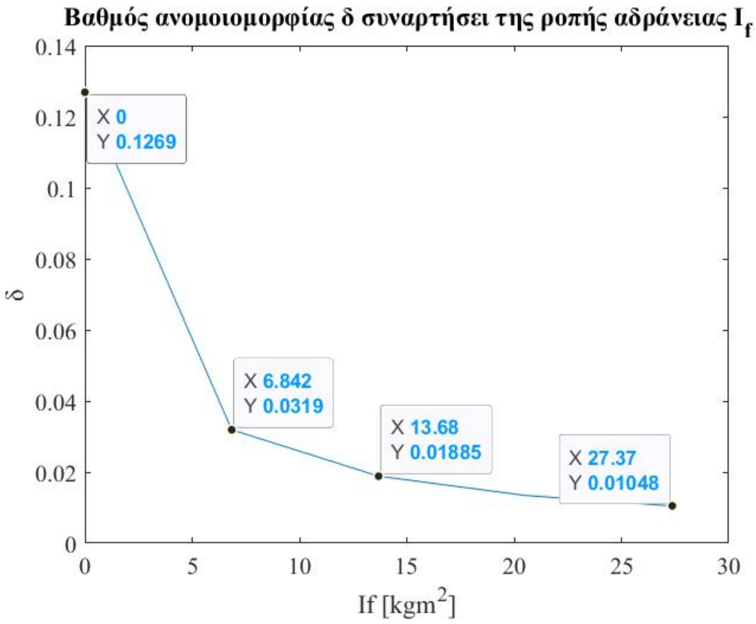
applying the Central Difference method as before, we arrive at the angle calculation equation in the form:θ_{m+1}

$$\Rightarrow \theta_{m+1} = \left(\frac{\left(I_{eq}(\theta_m) + I_f\right)}{h^2} + \frac{\gamma_O}{2h}\right)^{-1} \left(\gamma_O \left(\frac{\theta_{m-1}}{2h}\right) + \left(I_{eq}(\theta_m) + I_f\right)\left(\frac{2\theta_m - \theta_{m-1}}{h^2}\right) + \Gamma_1(\theta_m) + \Gamma_2(\theta_m) c_1(\theta_m) - \Gamma_3(\theta_m)\, c_2(\theta_m) + M_O\right)$$

also the acceleration for m=0 is calculated:

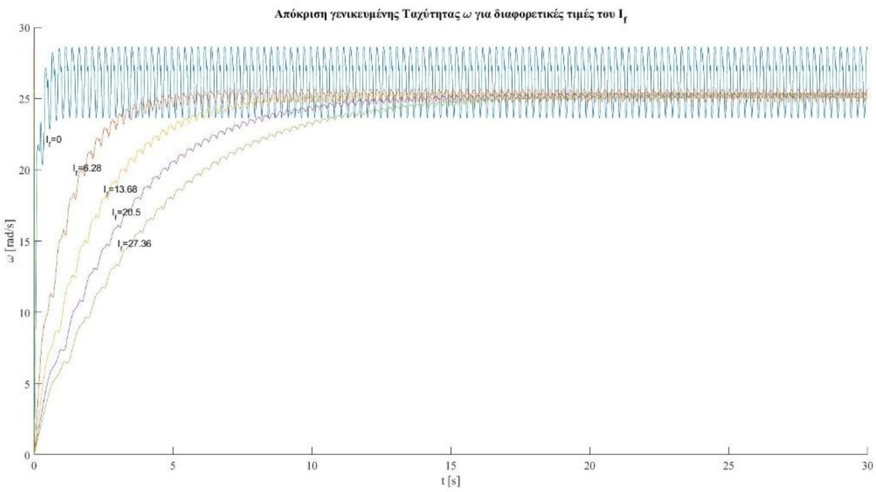
$$\ddot{\theta}_0 = \frac{\left(\Gamma_1(\theta_0) + \Gamma_2(\theta_0) c_1(\theta_0) - \Gamma_3(\theta_0)\, c_2(\theta_0) + M_O\right)}{I_{eq}(0) + I_f}$$

To calculate the values of the non-uniformity coefficient as a function of the moment of inertia, we take into account only the last 1000 values of the vectors constructed by the procedure of query 1. Thus we only include steady state terms where we want to apply the normalization. The factor δ essentially indicates the intensity of the oscillations during the steady state. Figures 6 to 8 show the resulting diagrams.$\delta = f\left(I_f\right)$



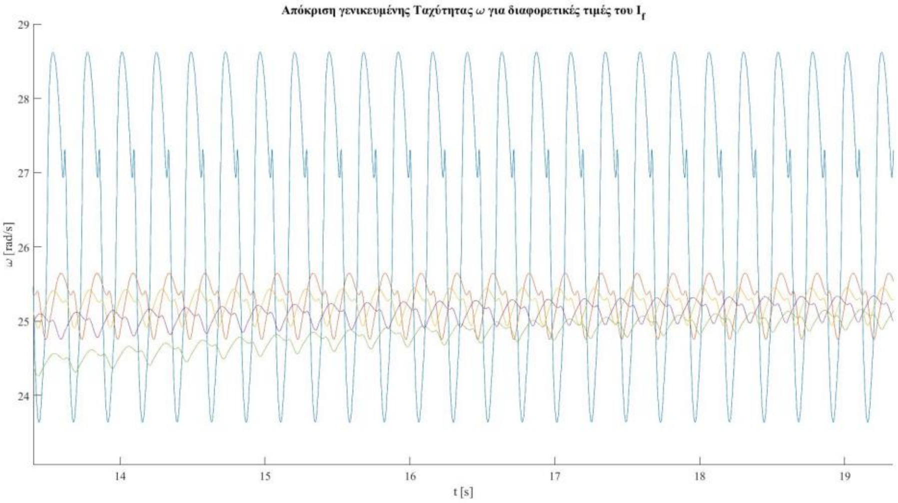
Shape6. Degree of unevenness d as a function of moment of inertiaI_f

Figure 6 shows the variation of the degree of unevenness δ as a function of the mass moment of inertia. We notice that as the moment of inertia of the flywheel increases, d decreases. Therefore, this indicates that the addition of the flywheel improves the behavior in the steady state, it is also evident that in the 1st step for the values of a we have the largest reduction for the coefficient d . In Figure 7, the response diagrams of the angular velocities have been constructed for different values of where and are indicated on the diagram. We see that as it increases, so does the duration of the transition state for the same steps from one value of the moment of inertia to another. The values for are chosen so that the first value gives us the same result as the answer to question 1 and all the others are multiples of the maximum value of the equivalent moment of inertia that we used in Part A .$I_f I_f 0 \rightarrow 6.84 I_f I_f I_f (I_f = 0)(I_f = I_{eq})$



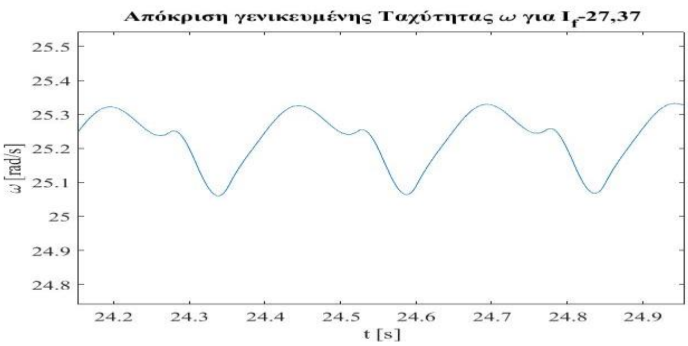
Shape7. Generalized Velocity Response Plots for differentI_f

Finally, Figure 8 shows a detail of Figure 7, in the steady state, where it is easy to see the decrease in the amplitudes of the oscillations in the steady state for the angular velocity. It is important to notice that as its value increases, we approach the operating point of the load for the mechanismI_fn $= 240\ rpm = 25.133\frac{rad}{s}$.



Shape8. Enlargement of the steady state depicted in Figure 7

As we calculated above (Figure 6) we achieve a degree of unevenness for the specific load we apply. The response plot for ω is shown in Figure 9. This result means that to reduce the non-uniformities to 1% of the average value, which as we said in the 1st question for exact motor characteristic is the operating point of the motor, we need to add a moment of inertia with a value approximately six times the maximum equivalent moment of inertia of the mechanism.$\delta = 0.01048 \approx 0.01 I_f = 27.37$



Shape9. Generalized Velocity Response Diagram for δ=0.01048 and$I_f = 27.37$

We now repeat the calculations for$\omega' = \frac{\omega}{4} = 6.283\frac{rad}{s}\ \kappa\alpha\iota\ \omega'' = 4\omega = 100.5\frac{rad}{s}$. In essence, since we keep the same electric motor, what will change is the transmission ratio it gives, that is, the torque of the load that the motor must satisfy. First we check if the specific electric motor can provide the necessary torque for starting. For a load with a constant rotational speed in the steady state we have:$(M_\varepsilon = 215.6\ Nm)$. $P_{mean} = 4.31\ kW \omega'$ή ω''

$\omega' :\ M_{mean} = 685.96\ Nm$

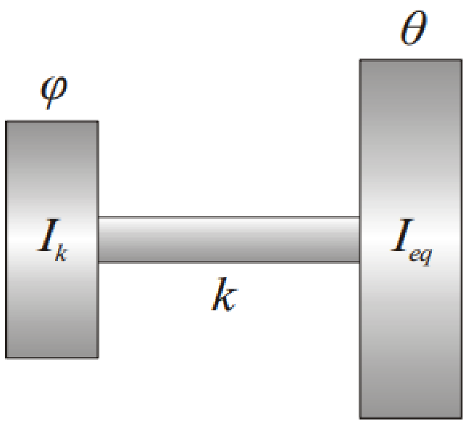
$\omega'' :\ M_{mean} = 42.9\ Nm$

From the above values we see that for speed the electric motor we have chosen cannot provide the necessary torque for starting, so we repeat the calculations only for .$\omega' \omega'' =\ 100.5\frac{rad}{s}$

For the analysis of the 2nd query, the file 'ErotimaB.m' was written which is given in Appendix A2.

3rd QUESTION

In the 3rd question, the construction of the motion equation and its numerical integration with Newmark is requested, considering that the elastic coupling between the electric motor and the mechanism has an equivalent torsional rigidity as can be seen in Figure 10. Initially, select the value , let , for which the natural frequency that corresponds to the elastic singularity of the system is equal to the angular speed of rotation ω. Then it is requested to repeat the calculations for various values of the constant, in the interval . Finally, a comparison of the results with those obtained for a completely undeformed link is requested.$k_t k_t \mathrm{k}_o k_t \left[\frac{k_o}{10}, 10k_o\right]$

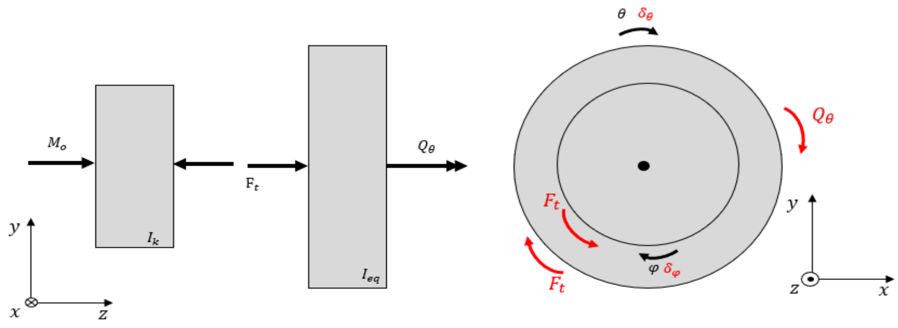


Shape10. Mechanism Model - Electric motor with deformable joint

Formulation of Equations of Motion

To derive the equation of motion we will apply Lagrange. Up to this point in the work we have studied the Mechanism-Electric Motor system as undeformed. Now by introducing a deformable link between them, the spring force of the link is also introduced into the system. Furthermore the electric motor rotates at a different speed than the crank and the torque that the electric motor has to overcome is both the load and its own moment of inertia, so that it ends up in some permanent mode.$\dot{\varphi}$ $\dot{\theta}$

Before we start the construction, we present in Figure 11. a sketch of the Mechanism-Flywheel-Electric Motor system, with the forces acting on it and the ESR of the bodies that make it up.



Shape11. Mechanism-Flywheel-Electric motor system sketch

Having calculated in a previous question the generalized force in the direction of the generalized coordinate θ, we can model (as shown in Figures 10 and 11 the mechanism-flywheel system, as a solid with mass moment of inertia equivalent to) and with external forces in it the generalized torque), with the difference now that the motor torque is exerted in the direction of the generalized coordinate φ. Thus, this entire subsystem is expressed with one degree of freedom and a Torque.$(I'_{eq}(Q_\theta)M_O$

In figure 11. we distinguish the modeling of the Electric Motor-Mechanism-Flywheel system. As generalized coordinates we choose the independent coordinates φ and θ. Furthermore we distinguish the external forces and the force developed in the spring. As we define the equivalent mass moment of inertia of the flywheel and the mechanism, we choose for the moment of inertia of the flywheel, the value we calculated in the previous question as optimal$Q_\theta, M_O I'eq \left(I'_{eq} = I_{eq}(A\ \mu\acute{\varepsilon}\rho o\upsilon\varsigma) + I_f\right) I_f = 27.84\ \ kgm^2$.

To derive the equations of motion we apply Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_{q_i} \quad \textbf{(14)}$$

The kinetic energy results from the sum of the kinetic energies of the 2 bodies as:

$$T = \frac{1}{2} I_k \dot{\varphi}^2 + \frac{1}{2} I_{eq} \dot{\theta}^2,$$

and the spring force: $F_t = k_t(\varphi - \theta)$

Calculating the derivatives of the Lagrange equation for the 2 coordinates we have:

$\therefore \boxed{\frac{\partial T_{O\Lambda}}{\partial \dot{\theta}}}$

$$\frac{\partial T_{O\Lambda}}{\partial \dot{\theta}} = I_{eq}(\theta)\omega$$

$\therefore \boxed{\frac{d}{dt}\left(\frac{\partial T_{O\Lambda}}{\partial \dot{\theta}}\right)}$

$$\frac{d}{dt}\left(\frac{\partial T_{O\Lambda}}{\partial \dot{\theta}}\right) = I_{eq}(\theta)\ddot{\theta}$$

$\therefore \boxed{\frac{\partial T_{O\Lambda}}{\partial \theta}}$

$$\frac{\partial T_{O\Lambda}}{\partial \theta} = 0$$

$\therefore \boxed{\frac{\partial T_{O\Lambda}}{\partial \dot{\varphi}}}$

$$\frac{\partial T_{O\Lambda}}{\partial \dot{\varphi}} = I_k \dot{\varphi}$$

$\therefore \boxed{\frac{d}{dt}\left(\frac{\partial T_{O\Lambda}}{\partial \dot{\varphi}}\right)}$

$$\frac{d}{dt}\left(\frac{\partial T_{O\Lambda}}{\partial \dot{\varphi}}\right) = I_k \ddot{\varphi}$$

$\therefore \boxed{\frac{\partial T_{O\Lambda}}{\partial \varphi}}$

$$\frac{\partial T_{O\Lambda}}{\varphi} = 0$$

From the beginning of the Strong projects we have:

$$Q_\theta \delta\theta + Q_\varphi \delta\varphi = \left(-\gamma_O \dot{\theta} + \Gamma_1 + \Gamma_2 c_1 - \Gamma_3\, c_2\right)\delta\theta + M_O \varphi\theta + F_t \delta(\theta - \varphi)$$

where

$,\Gamma_1 = \left(\frac{1}{2} l_1 m_1 g + m_2 g\, l_1\right) cos(\theta) \Gamma_2 = (m_2 g l_2 \cos(\varphi_2))$

$,\Gamma_3 = b(m_3 + m_\Delta) g \sin(\varphi_3) + (l_3 + l_\Delta)\, F_\Delta \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = J^{-1} \begin{bmatrix} l_1 \sin(\theta) \\ -l_1 \cos(\theta) \end{bmatrix}$

and the expressions of the generalized forces for the 2 coordinates are obtained:

$$\Rightarrow \begin{cases} Q_\theta = -\gamma_O \dot{\theta} + \Gamma_1 + \Gamma_2 c_1 - \Gamma_3\, c_2 + \, k_t(\varphi - \theta) \\ Q_\varphi = M_O - \, k_t(\varphi - \theta) \end{cases}$$

By inserting the forces and the derivatives calculated for the kinetic energy in equation (14) we arrive at the following system which constitutes the equations of motion for our Model:

$$\Rightarrow \begin{cases} I_{eq}(\theta)\ddot{\theta} = \left(-\gamma_O \dot{\theta} + A_1 cos(\theta) \; + A_2 \cos\left(\varphi_2(\theta)\right) c_1 - \left(A_3 \sin\left(\varphi_3(\theta)\right) + A_4\right) c_2\right) + \, k_t(\varphi - \theta) \\ I_k \ddot{\varphi} = M_O - \, k_t(\varphi - \theta) \end{cases}$$

with

$,A_1 = \left(\frac{1}{2} l_1 m_1 g + m_2 g\, l_1\right)$

$A_2 = (m_2 g l_2)$

$A_3 = b(m_3 + m_\Delta) g \quad ,$

$A_4 = (l_3 + l_\Delta)\, F_\Delta$

The above system can be written in compact form:

$$M(\underline{x},t)\underline{\ddot{x}} + \underline{f}(\underline{\dot{x}},\underline{x},t) = \underline{0} \ ,\mu\varepsilon \ \underline{x} = \begin{bmatrix}\theta\\ \varphi\end{bmatrix} \quad (\mathbf{15})$$

with :$M(\underline{x},t) = \begin{bmatrix}I'_{eq} & 0\\ 0 & I_k\end{bmatrix}$

$$\underline{f}(\underline{\dot{x}},\underline{x},t) = \begin{bmatrix}\gamma_O\dot{\theta} - A_1 cos(\theta) - A_2\cos\left(\varphi_2(\theta)\right)c_1 + \left(A_3\sin\left(\varphi_3(\theta)\right) + A_4\right)\ c_2 - k_t(\varphi-\theta)\\ -M_O + \ k_t(\varphi-\theta)\end{bmatrix}$$

The system (15) is a strongly nonlinear ordinary differential system of the 2nd order and therefore cannot be solved analytically but can only be solved numerically. For the above system to be fully defined, it requires 2 initial conditions, so as in the previous question we consider that the mechanism and the electric motor start and accelerate from rest and zero positions for the coordinates φ and θ, so the initial conditions: .$\underline{x}(0) = \underline{x_0} = \begin{bmatrix}\theta_0\\ \varphi_0\end{bmatrix} = \underline{0}\ \kappa\alpha\iota\ \underline{\dot{x}}(0) = \underline{\dot{x_0}} = \begin{bmatrix}\dot{\theta}_0\\ \dot{\varphi}_0\end{bmatrix} = \ \underline{0}$

Select Coefficient kt

We are instructed to initially choose the coefficient equal to where is the value for which the 1st elastic natural frequency equals 240 rpm. We know from the Theory of Oscillations that the singularities of a system of linear differential equations with constant coefficients result from solving the eigenproblem. To solve the eigenproblem we solve the homogeneous problem neglecting the external loads. Thus neglecting the terms resulting from the generalized force in equation (14) we get the following system in matrix form:$k_t = k_o Q_\theta$

$$\begin{bmatrix}I'_{eq} & 0\\ 0 & I_k\end{bmatrix}\underline{\ddot{x}} + \begin{bmatrix}k_t & -k_t\\ -k_t & k_t\end{bmatrix}\underline{x} = \underline{0} \implies \underline{M}\ \underline{\ddot{x}} + \underline{K}\ \underline{x} = \underline{0}$$

With this consideration, in essence, we solve the problem as if we had 2 distinct masses that are joined together by a torsion spring of strength. Because this matrix form is a system of linear differential equations with constant coefficients, the solutions are expressed in exponential form. For these solutions to be solutions of the system, the homogeneous algebraic system must be satisfied: which is an eigenproblem. In order to have a solution, this must. Expanding the determinant gives us:$k_t(K-\omega^2 M)\underline{\hat{x}} = \underline{0}\ |K-\omega^2 M| = 0$

$$\begin{vmatrix}k_t-\omega^2 I'_{eq} & -k_t\\ -k_t & k_t-\omega^2 I_k\end{vmatrix} = 0 \Rightarrow \left(k_t-\omega^2 I'_{eq}\right)(k_t-\omega^2 I_k) - k_t^2 = 0$$

$$\Rightarrow k_t^2 - \omega^2 I_k k_t - \omega^2 I'_{eq}k_t + \omega^4 I'_{eq}I_k - k_t^2 = 0$$

$$\overset{\omega\neq 0}{\Longrightarrow} -I_\kappa k_t - I'_{eq}k_t + \omega^2 I'_{eq}I_k = 0$$

$$\Rightarrow k_t = k_o = \omega^2\frac{I'_{eq}I_k}{I'_{eq}+I_k}\ (\mathbf{16})$$

For and as given by the rated values of the electric motor, the stiffness factor for the joint is calculated using the average value for the equivalent moment of inertia of the Mechanism - Flywheel system. Thus, based on the above, a robustness factor for the torsion spring is calculated$\omega = \frac{2\pi\ 240\ [rpm]}{60} = 25.133\frac{rad}{s} I_k = J_{H/K} = 0.37\ kgm^2 k_t = 43.96$

Newmark application

For the numerical integration we apply the Newmark method. According to this method, we first discretize the integration time interval in m values with step h. Then we write the system (14) in time m+1 using the notations and we have:$(t_1, t_2 \dots . t_m) x(t_{m+1}) = x_{m+1}$

$$M(\underline{x}_{m+1}, t)\underline{\ddot{x}}_{m+1} + \underline{f}(\underline{\dot{x}}_{m+1}, \underline{x}_{m+1}, t_{m+1}) = \underline{0} \ \Rightarrow$$

$$\Rightarrow \begin{cases} I'_{eq}\ddot{\theta}_{m+1} + (\gamma_O\dot{\theta}_{m+1} - A_1 cos(\theta_{m+1}) - A_2\cos(\varphi_2(\theta_{m+1}))\, c_1 + (A_3\sin(\varphi_3(\theta_{m+1})) + A_4)\, c_2) - \ k_t(\varphi_{m+1} - \theta_{m+1}) = 0 \\ I_k\ddot{\varphi}_{m+1} - M_O + k_t(\varphi_{m+1} - \theta_{m+1}) = 0 \end{cases} \quad \mathbf{(16)}$$

According to the Newmark method the 1st and 2nd derivatives are approximated by the equations

$$\underline{\dot{x}}_{m+1} = \underline{\dot{x}}_m + h[(1-\beta)\underline{\ddot{x}}_m + \beta\underline{\ddot{x}}_{m+1}] \quad ,\mu\varepsilon\ \underline{x} = \begin{bmatrix}\theta\\ \varphi\end{bmatrix} \ (\mathbf{17})$$

$$\underline{x}_{m+1} = \underline{x}_m + h\underline{\dot{x}}_m + h^2\left[\left(\frac{1}{2} - a\right)\underline{\ddot{x}}_m + a\underline{\ddot{x}}_{m+1}\right] \quad ,\mu\varepsilon\ \underline{x} = \begin{bmatrix}\theta\\ \varphi\end{bmatrix} \ (\mathbf{18})$$

In the above 2 equations the coefficients α and β are chosen, for which the Newmark method predicts that the acceleration changes linearly between two consecutive times. By choosing these coefficients we solve (18) in terms of and replace it together with (17) in equation (16), so we have:$\alpha = \frac{1}{6}\kappa\alpha\iota\ \beta = \frac{1}{2}\underline{\ddot{x}}_{m+1}$

$$\overset{(18)}{\Longrightarrow} \underline{\ddot{x}}_{m+1} = \frac{6}{h^2}\left(\underline{x}_{m+1} - \underline{x}_m - h\,\underline{\dot{x}}_m - \frac{1}{3}h^2\underline{\ddot{x}}_m\right)$$

$$\overset{(17)}{\Longrightarrow} \underline{\dot{x}}_{m+1} = \underline{\dot{x}}_m + \frac{h}{2}\underline{\ddot{x}}_m + \frac{3}{h}\left[\left(\underline{x}_{m+1} - \underline{x}_m - h\,\underline{\dot{x}}_m - \frac{1}{3}h^2\underline{\ddot{x}}_m\right)\right]$$

and finally

$$\begin{aligned} M(\underline{x}_{m+1}, t)&\left(\frac{6}{h^2}\left(\underline{x}_{m+1} - \underline{x}_m - h\,\underline{\dot{x}}_m - \frac{1}{3}h^2\underline{\ddot{x}}_m\right)\right) \\ &+ \underline{f}\left(\left(\underline{\dot{x}}_m + \frac{h}{2}\underline{\ddot{x}}_m + \frac{3}{h}\left[\left(\underline{x}_{m+1} - \underline{x}_m - h\,\underline{\dot{x}}_m - \frac{1}{3}h^2\underline{\ddot{x}}_m\right)\right]\right), \underline{x}_{m+1}, t_{m+1}\right) \\ &= \underline{0} \ = \underline{g}(\underline{x}_{m+1}) \end{aligned}$$

With initial conditions. In addition, the calculation of the initial acceleration is needed to apply the method. For the calculation we rewrite the system in (16) for the 1st moment m=0 and thus we have:$\underline{x}(0) = \underline{0}\ \kappa\alpha\iota\ \underline{\dot{x}}(0) = \ \underline{0}\ddot{x}_0$

$$\Rightarrow \begin{cases} I'_{eq}\ddot{\theta}_0 + \gamma_O\dot{\theta}_0 + k_t(\theta_0 - \varphi_0) - A_1 cos(\theta_o) \ - A_2\cos\left(\varphi_2(\theta_o)\right) c_1 + \left(A_3\sin\left(\varphi_3(\theta_0)\right) + A_4\right)\ c_2 \ = 0 \\ I_k\ddot{\varphi}_0 - \ k_t(\theta_0 - \varphi_0) = 0 \end{cases}$$

Using the initial conditions we have:

$$\Rightarrow \begin{cases} \ddot{\theta}_0 = \dfrac{A_1 + A_2\cos\left(\varphi_2(\theta_0)\right) c_1 - \left(A_3\sin\left(\varphi_3(\theta_0)\right) + A_4\right)\ c_2}{I'_{eq}(\theta_0)} \\ \ddot{\varphi}_0 = \dfrac{M_O(\dot{\theta}_0)}{I_\kappa} \end{cases}$$

The above system of 2 equations constitutes a system of non-linear algebraic equations and for its solution the application of a numerical method is required. We apply the Newton – Raphson (NR) method for each time value. The NR method was used in the 1st question to calculate the angles. In summary, we state that the calculation algorithm will have the form:$\underline{x}_{m+1} \mathrm{t}_m \varphi_2, \varphi_3$

$$\underline{x}_{m+1}^{[i+1]} = \underline{x}_{m+1}^{[i]} - J^{-1}\left(\underline{x}_{m+1}^{[i]}\right) \underline{g}\left(\underline{x}_{m+1}^{[i]}\right)$$

where J is the Jacobian Matrix of the function, for the derivatives in the Jacobian matrix we have:$\underline{g} J = \begin{bmatrix} \frac{\partial g_1}{\partial \theta_{m+1}} & \frac{\partial g_1}{\partial \varphi_{m+1}} \\ \frac{\partial g_2}{\partial \theta_{m+1}} & \frac{\partial g_2}{\partial \varphi_{m+1}} \end{bmatrix}$

$$\frac{\partial g_1}{\partial \theta_{m+1}} = \frac{6}{h^2} I'_{eq} + \frac{3}{h}\gamma_O + k_t + A_1 sin(\theta_{m+1}) - A_2 \cos\left(\varphi_2(\theta_{m+1})\right) d_1 + \left(A_3 \sin\left(\varphi_3(\theta_{m+1})\right) + A_4\right) d_2$$

$$\frac{\partial g_2}{\partial \theta_{m+1}} = -k_t$$

$$\frac{\partial g_1}{\partial \varphi_{m+1}} = -k_t$$

$$\frac{\partial g_2}{\partial \varphi_{m+1}} = \frac{6I_k}{h^2} + k_t$$

with$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = J^{-1}(\varphi_2, \varphi_3) \begin{bmatrix} l_1 \sin(\theta_{m+1}) \\ -l_1 \cos(\theta_{m+1}) \end{bmatrix} \xrightarrow{\pi\alpha\rho\alpha\gamma\acute{\omega}\gamma\iota\sigma\eta}$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = J^{-1}\left(\varphi_2(\theta_{m+1}), \varphi_3(\theta_{m+1})\right) \begin{bmatrix} l_1 \cos(\theta_{m+1}) \\ l_1 \sin(\theta_{m+1}) \end{bmatrix}$$

As an initial estimate for the application of NR in each time step we take the solution in the previous step. The NR algorithm for each step terminates when where is a small positive number.$t_m \underline{x}_{m+1}^{[0]} = \underline{x}_m \left|\underline{g}\left(\underline{x}_{m+1}^{[i+1]}\right)\right| < \varepsilon_2 \varepsilon_2$

When calculating the step in time, it will be necessary in the NR to find the solution of the system to introduce a 2nd NR so that for each we calculate the angles of the mechanism as a function of θ. The calculation algorithm of the 2nd NR will have the formt$_{m+1} \underline{g}(\underline{x}_{m+1}), \theta_{m+1}^{[i+1]} \varphi_2, \varphi_3$

$$\left(\underline{\varphi}_{m+1}^{[j+1]}, \underline{\theta}_{m+1}^{[i+1]}\right) = \left(\underline{\varphi}_{m+1}^{[j]}, \underline{\theta}_{m+1}^{[i+1]}\right) - \frac{\underline{H}\left(\underline{\varphi}_{m+1}^{[j]}, \underline{\theta}_{m+1}^{[i+1]}\right)}{\underline{H}'\left(\underline{\varphi}_{m+1}^{[j]}, \underline{\theta}_{m+1}^{[i+1]}\right)} \quad , \quad \mu\varepsilon\ \underline{\varphi} = \begin{bmatrix} \varphi_2 \\ \varphi_3 \end{bmatrix}$$

where

$$H(\varphi_2, \varphi_3, \theta) = \begin{cases} l_1 \cos(\theta) + l_2 \cos(\varphi_2) - l_3 sin(\varphi_3) - l = 0 \\ l_1 \sin(\theta) + l_2 \sin(\varphi_2) - l_3 \cos(\varphi_3) = 0 \end{cases}$$

and

$$\underline{H}'(\varphi_2, \varphi_3, \theta) = I\alpha\kappa\omega\beta\iota\alpha\nu\acute{o}\varsigma\ \delta\varepsilon\sigma\mu\acute{\omega}\nu = \begin{bmatrix} -l_2 \sin(\varphi_2) & -l_3 \cos(\varphi_3) \\ l_2 \cos(\varphi_2) & l_3 \sin(\varphi_3) \end{bmatrix}$$

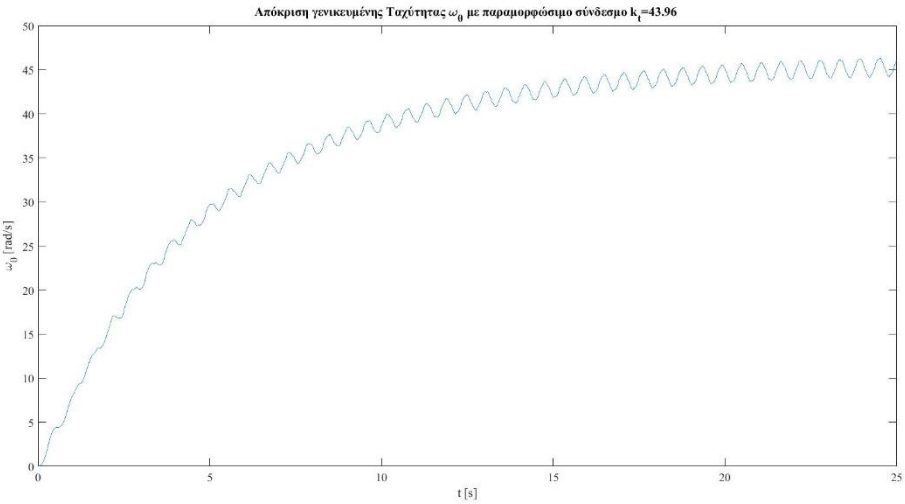
As an initial estimate for the application of the 2nd NR we take the solution in the previous step and for the 1st time step after considering zero initial conditions for the coordinates φ and θ, we get respectively as defined in the 1st part, we also choose an error ε3 such that the 2nd NR to terminate when $.\underline{\varphi}_{m+1}^{[0]} = \underline{\varphi}_m \varphi_2(\theta_0 = 0) = 33.34°\,, \varphi_3(\theta_0 = 0) = -1.5°$ $\left|H\left(\underline{\varphi}_{m+1}^{[j+1]}, \underline{\theta}_{m+1}^{[i+1]}\right)\right| < \varepsilon_3$

The above indicators are expressed as follows

- $m \longrightarrow \chi\rho o\nu\iota\kappa\acute{\eta}\ \sigma\tau\iota\gamma\mu\acute{\eta}\ t_m$
- $i \longrightarrow \varepsilon\pi\alpha\nu\acute{\alpha}\lambda\eta\psi\eta\ i\ 1^{\eta\varsigma}\ N - R\ \gamma\iota\alpha\ \varepsilon\pi\acute{\iota}\lambda\upsilon\sigma\eta\ \tau o\upsilon\ \underline{g}(\underline{x}_{m+1}),$
- $j \longrightarrow \varepsilon\pi\alpha\nu\acute{\alpha}\lambda\eta\psi\eta\ j\ 2^{\eta\varsigma}\ N - R\ \gamma\iota\alpha\ \varepsilon\pi\acute{\iota}\lambda\upsilon\sigma\eta\ \tau o\upsilon\ \sigma\upsilon\sigma\tau\acute{\eta}\mu\alpha\tau o\varsigma\ \delta\varepsilon\sigma\mu\acute{\omega}\nu$

Thus, after calculating the vector for time, we proceed to the next time step and apply the process again.$\underline{x}t_{m+1}$

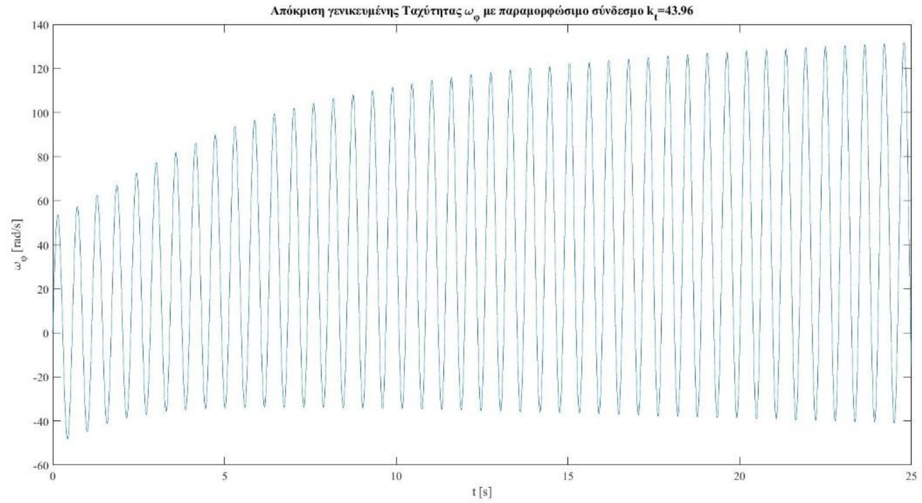
To extract the results and the required graphs, the file 'ErotimaG.m' was written in Matlab. Through this file the initial quantities are defined and the file 'NRNewmark.m' is called in which the discretized derivatives and the NR numerical method are applied to solve the system$\underline{g}(\underline{x}_{m+1})$. The sizes for the step h and the errors we chose for the 2 iterative procedures applied (the 2nd one for the calculation of the internal) were set to the values . These files are delivered in Appendix A3, we should mention that during the calculation process all the files created for the calculations in the 1st question (calculation of electric motor torque, moment of inertia, Jacobian bonds, etc.) are also used. As a reminder we mention that in the following results $.\varepsilon_2, \varepsilon_3 \varphi_2, \varphi_3 \mathrm{h} = 10^{-3}, \varepsilon_2 = 10^{-7} = \varepsilon_3 I_f = 27.84\ kgm^2\ \kappa\alpha\iota\ k_t = k_o = 43.96$
Figures 12 and 13 show the angular velocity responses for θ and φ coordinates respectively.



Shape12. Generalized velocity response for$\dot{\theta} k_t = k_o$

In the figure above, we notice that the mechanism ends up in a permanent state at approximately 20 seconds, also the relatively large oscillations performed by the mechanism around the permanent state can be seen. We now see that the mechanism results in higher rotational speeds which is due to the additional demand in torque introduced to the system by both the deformability of the link to satisfy the dynamics of the torsional spring, and the modeling of the mechanism as a discrete body. Also the load requirement increases due to the modeling of the motor with its moment of inertia. That is, the load that we defined as medium in the previous questions has now increased.

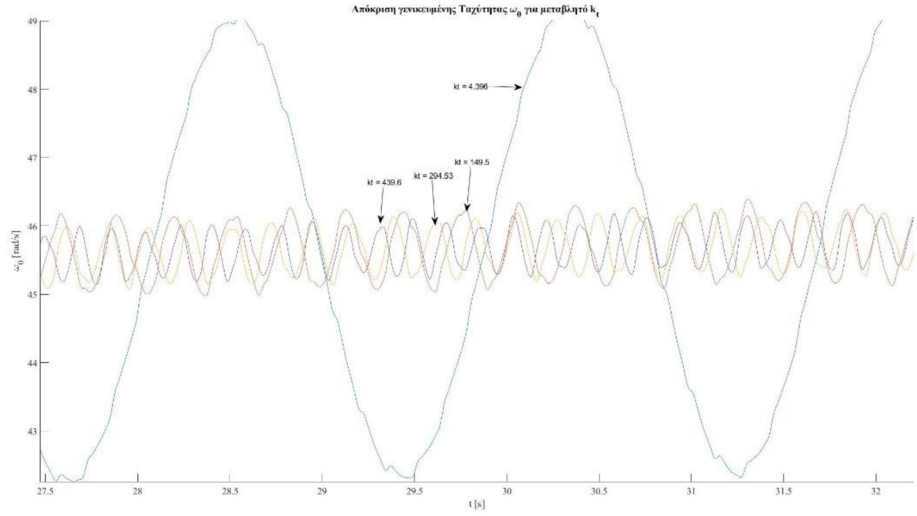
In Figure 13. the speed response of the electric motor is illustrated. In this diagram, the paradox is observed that the electric motor changes direction of rotation continuously to meet the load requirements. Paradoxical in that we expect a motor to provide varying rotational speeds but in the same direction. This behavior is due to the very small moment of inertia of the Electric Motor, to the small robustness of the link, as we will see later, but also to the approximation of its characteristic curve carried out in the 1st question which does not offer the exact torques at the corresponding exact speeds. The electric motor therefore oscillates between 2 positions, such that it can provide the necessary torque to drive the load to the steady state. In general we can see if we identify the 2 graphs in a common diagram, that the rotational speed of the electric motor oscillates around that of the mechanism.



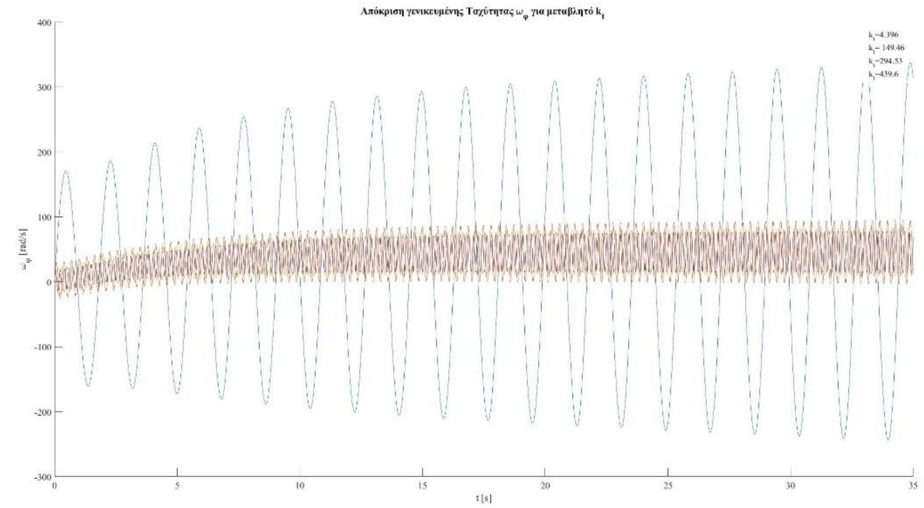
Shape13. Generalized velocity response for$\dot{\varphi} k_t = k_o$

Repeat calculations for variable kt

We repeat the calculations for 4 values of in the interval . For the 1st value of this vector the electric motor we chose cannot provide the required torque to rotate the mechanism.$k_t\left(\frac{k_t}{10}, 10k_t\right)$



Shape14. Generalized Velocity Response for variable link stiffness factor$\dot{\theta}$



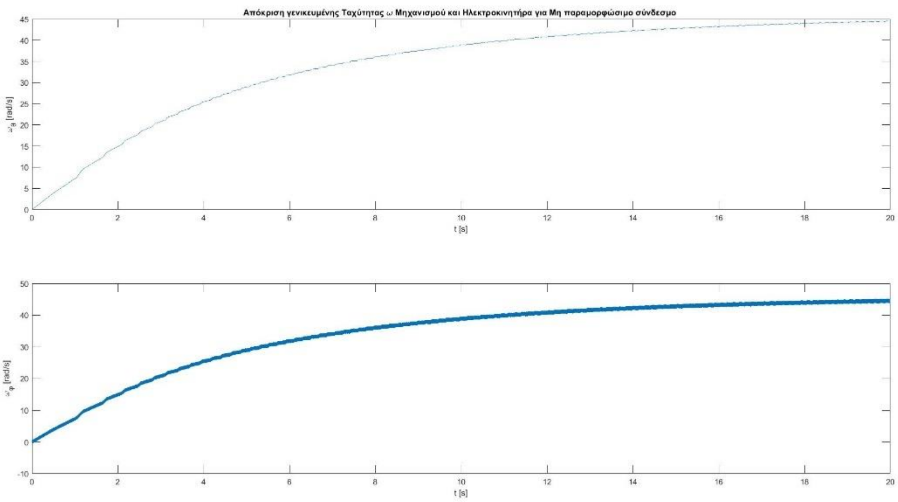
Shape15. Generalized Velocity Response for variable link stiffness factor$\dot{\varphi}$

From both the 2 diagrams we see that as the value of the robustness factor increases, the steady state velocity fluctuations decrease. In addition, we notice that with the discrete increases of the electric motor's angular velocity curve gets closer and closer to that of the mechanism. By increasing it, the 2 bodies can more easily monitor each other's movement and thus the electric motor adjusts its torque to the load requirements. Thinking differently, having kept the load of the whole model constant, therefore constant mechanical energy, by increasing the robustness coefficient we have an increase in potential energy. With the impact of this, the reduction of Kinetic energy and therefore the reduction of rotational speeds.$k_t k_t$

These phenomena in some way testify to the behavior of the mechanism in the next step, where it will be studied as non-deformable by changing the robustness factor.

For the comparison with a non-deformable joint it is sufficient to define a very large (. Thus the joint turns into a finitely rigid connection and we do not need to repeat the construction of the equations of motion and all the calculations from the beginning. By defining we get the diagrams for the generalized velocities in the common diagram, illustrated in Figure 16.$k_t k_t \rightarrow \infty$) $\mathrm{k}_t = 400.000$

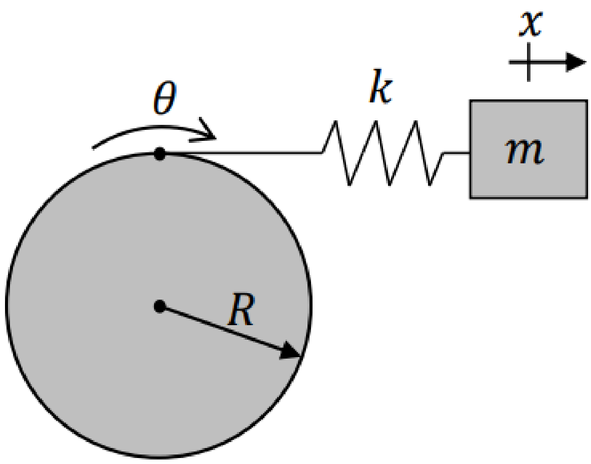
For a non-deformable connection the system rotates as a solid, as we studied in questions 1 and 2, so now we have (θ=φ) at every moment. Accordingly the 2 bodies rotate with the same angular velocity, any small deviations between the velocities are due to calculation errors and the fact that in reality the very large is an approximation of the undeformed. We see in the diagrams of figure 16 that the 2 speeds are characterized by the same values and result in the same speed in the steady state. The electric motor seems to change its characteristics much faster than the mechanism, which is mainly due to the much smaller moment of inertia. One would expect in this case to approach the operating point we studied in question 2, which however does not occur due to the increased load the electric motor has to deal with.k_t



Shape16. Responses of Generalized Coordinate Mechanism - Electric Motor for a undeformed joint

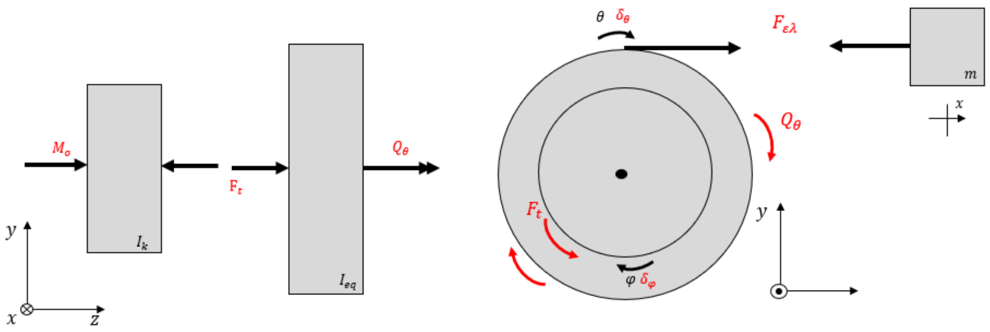
4th QUESTION

In the 4th question it is requested to write the equations of motion for small oscillations around the position θ=0 after first adding a mass m with a spring that represents the deformability of the mechanism as shown in Figure 17. Finally it is requested to determine the eigenfrequencies and eigenforms of the mechanical model which is obtained by Jacobi's method and the iterative force method.$k = \frac{k_t}{3}$



Shape17. Deformability modeling of mechanism members

By adding the mass m as a modeling for the deformability of the mechanism we introduce another degree of freedom into our system. In Figure 18. the DES of the system is illustrated. A generalized force is defined as the force calculated in the 1st question without the torque of the electric motor which is exerted in the direction of the coordinate φ.Q_θ



Shape18. Free Body Diagram of system

The only difference in the formulation of the equations due to the mass m appears in the additional kinetic energy introduced into the system, the spring force exerted on both the mechanism model and the mass m, and the introduction of one more, independent, generalized coordinate x. The amount of kinetic energy added to the system is:$F_{\varepsilon\lambda} = k\big(x - Rsin(\theta)\big)$

$$T_m = \frac{1}{2} m\dot{x}^2$$

and the terms introduced to the generalized forces in the directions of coordinates θ and x (based on the principle of possible works), will be:

$Q_{\theta,m} = RF_{\varepsilon\lambda}andQ_{x,m} = -F_{\varepsilon\lambda}$

Thus we rewrite the system of 3-fold equations of motion by substituting the above into Lagrange's equations for the 3 coordinates:

$$\Rightarrow \begin{cases} I_{eq}(\theta)\ddot{\theta} = \left(-\gamma_O\dot{\theta} + A_1 cos(\theta) \ + A_2\cos\left(\varphi_2(\theta)\right) c_1 - \left(A_3\sin\left(\varphi_3(\theta)\right) + A_4\right) c_2\right) + \ k_t(\varphi-\theta) + Rk\left(x - Rsin(\theta)\right) \\ I_k\ddot{\varphi} = M_O - \ k_t(\varphi-\theta) \\ m\ddot{x} = -k\left(x - Rsin(\theta)\right) \end{cases}$$

with

$,A_1 = \left(\frac{1}{2}l_1 m_1 g + m_2 g\ l_1\right)$

$A_2 = (m_2 g l_2)$

$A_3 = b(m_3 + m_\Delta)g$,

$A_4 = (l_3 + l_\Delta)\ F_\Delta$

$,\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = J^{-1}\begin{bmatrix} l_1\sin(\theta) \\ -l_1\cos(\theta) \end{bmatrix} k = \frac{k_t}{3}$

For oscillations of small amplitude around the position, the trigonometric terms of the equations of motion are transformed to their limit values as follows:$\theta = 0$

$\cos(\theta) \overset{\theta\to 0}{\Longrightarrow} \cos(\theta) = 1, sin(\theta) \overset{\theta\to 0}{\Longrightarrow} \sin(\theta) = \theta$

Thus the following linear system of ordinary differential equations of the 2nd order with respect to the 3 generalized coordinates is obtained:

$$\begin{cases} I_{eq}(\theta)\ddot{\theta} = \left(-\gamma_O\dot{\theta} + A_1 \ + A_2\cos\left(\varphi_2(\theta)\right) c_1 - \left(A_3\sin\left(\varphi_3(\theta)\right) + A_4\right)\ c_2\right) + \ k_t(\varphi-\theta) + Rk(x - R\theta) \\ I_k\ddot{\varphi} = M_O - \ k_t(\varphi-\theta) \\ m\ddot{x} = -k(x - R\theta) \end{cases}$$

$A_1 = \left(\frac{1}{2}l_1 m_1 g + m_2 g\ l_1\right),$

$A_2 = (m_2 g l_2)$

$A_3 = b(m_3 + m_\Delta)g$,

$A_4 = (l_3 + l_\Delta)\ F_\Delta$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = J^{-1}\begin{bmatrix} l_1\sin(\theta) \\ -l_1\cos(\theta) \end{bmatrix}$ όπου $J^{-1}(\varphi_2, \varphi_3) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = l_1\begin{bmatrix} J_{11}\theta - J_{12} \\ J_{21}\theta - J_{22} \end{bmatrix}$

By rearranging the terms of the system we get:

- $I_{eq}(\theta)\ddot{\theta} = -\gamma_O\dot{\theta} + A_1 \ + A_2\cos\left(\varphi_2(\theta)\right) c_1 - \left(A_3\sin\left(\varphi_3(\theta)\right) + A_4\right)\ c_2 + k_t(\varphi-\theta) + Rk(x - R\theta)$

$$\Rightarrow \ I_{eq}(\theta)\ddot{\theta} + \gamma_O\dot{\theta} - \ k_t(\varphi-\theta) - Rk(x - R\theta) = A_1 + A_2 l_1\cos\left(\varphi_2(\theta)\right)(J_{11}\theta - J_{12}) - \left(A_3\sin\left(\varphi_3(\theta)\right) + A_4\right)\ l_1(J_{21}\theta - J_{22})$$

$$\Rightarrow I_{eq}(\theta)\ddot{\theta} + \gamma_O\dot{\theta} + D\theta - k_t\varphi - Rkx = A_1 - D_2 + D_4$$

$whereD = (k_t + R^2 k - D_1 + D_3)$

$D_1 = A_2 l_1\cos\left(\varphi_2(\theta)\right) J_{11}$

$D_2 = J_{12} A_2 l_1 \cos\left(\varphi_2(\theta)\right)$

$D_3 = \left(A_3 \sin\left(\varphi_3(\theta)\right) + A_4\right)\ l_1 J_{21}$

$D_4 = \left(A_3 \sin\left(\varphi_3(\theta)\right) + A_4\right)\ l_1 J_{22}$

$F_\theta = A_1 - D_2 + D_4$

- $I_k \ddot{\varphi} + k_t(\varphi - \theta) = M_O \Rightarrow I_k \ddot{\varphi} + k_t \varphi - k_t \theta = M_O$
- $m\ddot{x} + kx + kR\theta = 0$

After all :

$$\begin{cases} I_{eq}(\theta)\ddot{\theta} + \gamma_O \dot{\theta} + D\theta - k_t \varphi - Rkx = F_\theta \\ I_k \ddot{\varphi} + k_t \varphi - k_t \theta = M_O \\ m\ddot{x} + kx + kR\theta = 0 \end{cases}$$

The above system is written in register form:

$$M = \begin{bmatrix} I_{eq}(\theta) & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & m \end{bmatrix}, C = \begin{bmatrix} \gamma_O & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, K = \begin{bmatrix} D & -k_t & Rk \\ -k_t & k_t & 0 \\ Rk & 0 & k \end{bmatrix}, F = \begin{bmatrix} F_\theta \\ M_o \\ 0 \end{bmatrix}, x = \begin{bmatrix} \theta \\ \varphi \\ x \end{bmatrix}$$

$$\Rightarrow M\underline{\ddot{x}} + C\underline{\dot{x}} + K\underline{x} = \underline{F} \quad (\mathbf{19})$$

Considering that the damping does not significantly affect the response of the system we can neglect it. To solve the resulting equations we apply the eigenform analysis method for which we initially considered$\underline{F} = \underline{0}$and the eigenproblem corresponding to the equation is solved: $M\underline{\ddot{x}} + K\underline{x} = \underline{0}$ $(\mathbf{20})$

By solving the eigenproblem we calculate the eigenmodes and eigenfrequencies of the system, to calculate these the Jacobi method and the iterative force method will be applied. Because an oscillatory response is expected for the system (20) we arrive at the eigenproblem we want, the solution of which will give us the eigenfrequencies and eigenmodes of the system, in the form:

$$(K - \omega^2 M)\underline{\hat{x}} = \underline{0} \Rightarrow (K - \lambda M)\underline{\hat{x}} = \underline{0} \quad (\mathbf{21})$$

We notice that the registers K and M are symmetric, we also know that for discrete systems with oscillatory response we can express the kinetic and potential energy of the system in the form:

$$T = \frac{1}{2}\underline{\dot{x}}^T M \underline{\dot{x}} \; \kappa\alpha\iota \; V = \frac{1}{2}\underline{x}^T K \underline{x}$$

the above quantities are always greater than zero. For our system in particular we have:

$$T = \frac{1}{2}\underline{\dot{x}}^T M \underline{\dot{x}} > 0 \;\; \kappa\alpha\iota \; V = \frac{1}{2}\underline{x}^T K \underline{x} \geq 0 \; (\acute{\iota}\sigma o \; \gamma\iota\alpha \; \kappa\acute{\iota}\nu\eta\sigma\eta \; \sigma\tau\varepsilon\rho\varepsilon o\acute{\upsilon} \; \sigma\acute{\omega}\mu\alpha\tau o\varsigma)$$

Therefore the mass and robustness register of the discrete system (20) are positive semidefinite. From the Cholesky factorization we know that the mass matrix can be written as the product of a lower triangular matrix L and its inverse. Substituting this form into equation (21) we have:

$$M = LL^T \text{ ,}\mu\varepsilon\ L = \begin{bmatrix} \sqrt{I_{eq}(\theta)} & 0 & 0 \\ 0 & \sqrt{I_k} & 0 \\ 0 & 0 & \sqrt{m} \end{bmatrix}$$

$$\overset{(21)}{\Longrightarrow} \lambda LL^T \underline{\hat{x}} = K\underline{\hat{x}} \Rightarrow \lambda L^T \underline{\hat{x}} = L^{-1}K\underline{\hat{x}} \ \ (\boldsymbol{\alpha})$$

we also define the vector$\underline{\hat{y}} = L^T \underline{\hat{x}} \Rightarrow \underline{\hat{x}} = L^{-T} \underline{\hat{y}} \quad (\boldsymbol{\beta})$

$$(\alpha) \overset{(\beta)}{\Longrightarrow} \lambda L^T L^{-T} \underline{\hat{y}} = L^{-1}KL^{-1}\underline{\hat{y}}$$

so since we arrive at the form of the eigenproblem:$L^T L^{-T} = I_{3x3}$

$$A\underline{\hat{y}} = \lambda \underline{\hat{y}} \qquad (\mathbf{22})$$

$and \mu\varepsilon\ A = L^{-1}KL^{-T}A \in \mathbb{R}^{3x3}, \underline{\hat{y}} \in \mathbb{R}^3$

This system consists of linear algebraic equations and appears in the classical form of the eigenproblem.

Jacobi (Planar Rotations)

According to this method we simplify our original matrix A by successive transformations of planar rotations, until it ends up in a diagonal form. The transformation we use is:

$$B = QAQ^{-1} \ \ , \ \ \mu\varepsilon\ \ Q\ o\rho\theta o\gamma\acute{\omega}\nu\iota o\ \mu\eta\tau\rho\acute{\omega}o\ \ QQ^T = I$$

we know that for Q rectangle, the matrices A and B are similar, so B has the same eigenvalues and the eigenvectors of A result from the relation , with z the eigenvectors of B. The above transformation is applied until B ends up on a diagonal form. The register Q will have elements on the main diagonal and in 2 antisymmetric positions off the main diagonal, arrays of the form should generally be formed: e.g.$\underline{y} = Q^{-1}\underline{z}Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\gamma & sin\gamma \\ 0 & -sin\gamma & cos\gamma \end{bmatrix}$

The problem now comes to the definition of the orthogonal transformation matrix Q. To define this we select the largest off-diagonal element and place the trigonometric term sinφ, the rest of the trigonometric terms are placed automatically, so that the above form is formed. That is, even the element with the largest absolute value, then we put . Finally we place the unit in the empty diagonal position (if we had larger registers of three dimensions we would fill the entire diagonal with units). With this process the transformation register is orthogonal. The angle γ, at which we rotate the system is calculated from the relation:$\alpha_{ij}\alpha_{ij} = \sin(\gamma)\ , \alpha_{ji} = -\sin(\gamma)\, , a_{ii} = a_{jj} = \cos(\gamma)$

$$\gamma = \frac{1}{2}\tan^{-1}\left(\frac{2a_{ij}}{a_{ii} - a_{jj}}\right)$$

with the largest off-diagonal element we placed the positive sine on earlier.α_{ij}

Jacobi's method may not lead to exactly a diagonal, but the off-diagonal elements have very small values. For this reason an error is defined in terms of the largest off-diagonal element, at which we consider the register to be diagonal. After arriving at a diagonal matrix repetition, the elements of the main diagonal are the eigenvalues of the initial matrix A due to similarity and the eigenvectors of A are calculated from the relation$B^{[i]}$ $\sigma\tau\eta\nu\ i\underline{\hat{y}}$

$$\underline{\hat{y}} = (Q^{-1})^{[\iota]}(Q^{-1})^{[i-1]}\ldots(Q^{-1})^{[2]}(Q^{-1})^{[1]}\ \underline{z}\ \ (23)$$

after first calculating the values of$\underline{z}$from solving the system$(B-\lambda_B I)\underline{z}=0$

Then we calculate the eigenvectors of our mechanical system from relation (b), .$\underline{\hat{x}} = L^{-T}\underline{\hat{y}}$

From the application of Jacobi's method Using Matlab, we calculate the eigenvalues for the register B and the similarity ratio of A. As we proved in the analysis above, the eigenvalues of A constitute the eigenvalues of the mechanical model:

$$\lambda_1 = 1.245\ ,\lambda_2 = 1.465\ \ ,\lambda_3 = 120{,}28$$

and the corresponding eigenvectors of the mechanical model after applying relation (23) and (b):

$$\underline{\hat{x}}_1 = \begin{bmatrix} 0.15 \\ 0.1653 \\ -0.11 \end{bmatrix}\ \ ,\underline{\hat{x}}_2 = \begin{bmatrix} -0.047 \\ -0.073 \\ -0.272 \end{bmatrix},\underline{\hat{x}}_3 = \begin{bmatrix}\begin{bmatrix} -0.02 \\ 1.63 \\ -2.5\ 10^{-5} \end{bmatrix}\end{bmatrix}$$

To implement the iterative process we set an error for zeroing the diagonal elements and the method terminated after 1 iteration.$\varepsilon = 10^{-4}$

Iterative Force Method

According to the power method, for our matrix which is positive semidefinite , there exists an orthonormal basis of eigenvectors with . By choosing an initial estimate and the relation we can calculate the eigenvalue corresponding to the largest eigenvalue of the register A for k tends to infinity$A \in \mathbb{R}^{3x3}(\lambda\acute{o}\gamma o\ \tau\eta\varsigma\ \sigma\chi\acute{\varepsilon}\sigma\eta\varsigma\ A = L^{-1}KL^{-T})B_n = \{\underline{\hat{y}}_1, \ldots \underline{\hat{y}}_1\}A\underline{\hat{y}} = \lambda\underline{\hat{y}}\underline{z}_0 \in \mathbb{R}^3\underline{z}_k = A^k\underline{z}_0$

$$\lim_{k\to\infty}\underline{z}_k = \lambda_n^k a_n\underline{\hat{y}}$$

and the eigenvalue is calculated from the relation:

$$\frac{\|\underline{z}_{k+1}\|}{\|\underline{z}_k\|} = \lambda_3$$

To find the smallest eigenvalue and the corresponding eigenvector, we replace A with its inverse in the above procedure. Then we solve the system and calculate the eigenvectors of A for the smallest and largest eigenvalue of A. To calculate the intermediate eigenvalue we subtract the contribution of the largest eigenvector (for us and choose the new initial estimate from the equation:$A^{-1}(A-\lambda I)\underline{\hat{y}} = \underline{0}\underline{\hat{y}}_3)$

$$\underline{z}_0^{[2]} = \underline{z}_0 - \alpha_n\underline{\hat{y}}_n\ \ \ \mu\varepsilon\ \ \ \alpha_n\ \tau o\ \varepsilon\sigma\omega\tau\varepsilon\rho\iota\kappa\acute{o}\ \gamma\iota\nu\acute{o}\mu\varepsilon\nu o\ \tau\omega\nu\ \underline{\hat{y}}_n, \underline{z}_0$$

and we apply the above procedure again where the intermediate eigenvalues and eigenvectors are extracted. Finally, the eigenvectors of our mechanical system are derived from relation (b) with their corresponding eigenvalues. The power method completes for k whose norm of the difference of the last 2 iterations is less than an error we define. For error we get the following results.$\underline{\hat{x}} = L^{-T}\underline{\hat{y}}$ $\varepsilon = 10^{-12}$

$$\lambda_1 = 0.8219\,, \lambda_2 = 0.8219\ \ , \lambda_3 = 120{,}285$$

and the corresponding eigenvectors after applying relation (23) and (b):

$$\underline{\hat{x}}_1 = \begin{bmatrix} -0.049 \\ -0.062 \\ -0.29 \end{bmatrix} \ \ , \underline{\hat{x}}_2 = \begin{bmatrix} -0.049 \\ -0.062 \\ -0.29 \end{bmatrix}, \underline{\hat{x}}_3 = \begin{bmatrix} -0.02 \\ 1.63 \\ -2.5\ 10^{-5} \end{bmatrix}$$

Also, the number of iterations needed to calculate each eigenvalue is 131, 131 and 13 respectively for the 1st, 2nd and 3rd eigenvalue.

To derive the results we applied the power method for k=100. In addition, it was observed that for a larger κ (κ=1000) the method led to infinity, possibly due to the computing power we have.

For comparison, the exact eigenvalues calculated by the eigenform analysis method are:$\lambda_1 = 1.217\,, \lambda_2 = 1.493\ \ , \lambda_3 = 120{,}28$

And the eigenvectors

$$\underline{\hat{x}}_1 = \begin{bmatrix} 0.17 \\ 0.1733 \\ -0.1 \end{bmatrix} \ \ , \underline{\hat{x}}_2 = \begin{bmatrix} -0.0579 \\ -0.0586 \\ -0.2996 \end{bmatrix}, \underline{\hat{x}}_3 = \begin{bmatrix} -0.02 \\ 1.63 \\ -2.5\ 10^{-5} \end{bmatrix}$$

We see that for the specific system the Jacobi method gives much better results than the force method. The force method for the specific problem shows a huge error. In terms of speed Jacobi's method is instantaneous (1 iteration), it could also be solved by hand. On the other hand, for the calculation with the force method, a finite number of repetitions is needed, after which of course we do not get a satisfactory result (except for the 3rd eigenvalue).The codes written to implement the above methods are presented in Appendix A4.