Final Project PHY-234

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May 18, 2022

Introduction

In this paper, we analyze an oscillating system subject to a nonlinear restoring force, as modeled by the Duffing Equation:

$$\ddot{x}+\beta\dot{x}+\delta x+\alpha x^3=\gamma\cos\omega t \tag{1}$$

This equation models a non-linear, damped, driven harmonic oscillator. The equation received the name of its main investigator, the German engineer Georg Duffing (Nolte 2022). The parameters β, α, δ, and γ dictate the damping, the non-linearity of the restorative force, the linearity of the restorative force, and the amplitude of the driving force respectively. To analyze this system, we must also scrutinize the Duffing Potential, the potential function of the nonlinear restorative force

$$U(x)=\frac{1}{2}\delta x^2+\frac{1}{4}\alpha x^4 \tag{2}$$

By modeling this type of system, we first want to understand how it differs from systems with a linear restoring force. To achieve this, we will compare analytically and numerically two systems for different values of γ. Both of the following systems have $\beta=0.3$, $\delta=-1$, $\omega=1.2$, and γ ranging from 0.2 to 0.65.

1. A non-linear damped driven oscillator governed by the Duffing equation with $\alpha=1$.
2. A linear damped driven oscillator governed by the Duffing equation with $\alpha=0$.

Then, using visualization tools such as phase space diagrams, and Poincaré sections, we are going to analyze the behavior of a Duffing oscillator for different values of γ.

Outline

To model this system numerically, we used the programming language Python and Jupyter Notebook computational environment, and the odeint function from Scipy to implement Euler's integration method. To graph the time series, phase spaces, and the Poincaré sections, we used Matplotlib Python library.

Assumptions

Each system analyzed is assumed to be a closed system and has one degree of freedom, x. Furthermore, we assume the damping can be modeled by the equation $F_{\text{damp}}=-b\dot{x}$ and that the restorative force of the spring can be modeled by the Duffing Potential.

Choice of values

We are analyzing γ for the following values: 0.2, 0.28, 0.29, 0.37, 0.5, 0.65. The first three values demonstrate period-doubling, 0.37 is a five-period motion, 0.5 gives rise to chaos, whilst 0.65 returns to 2-period motion.

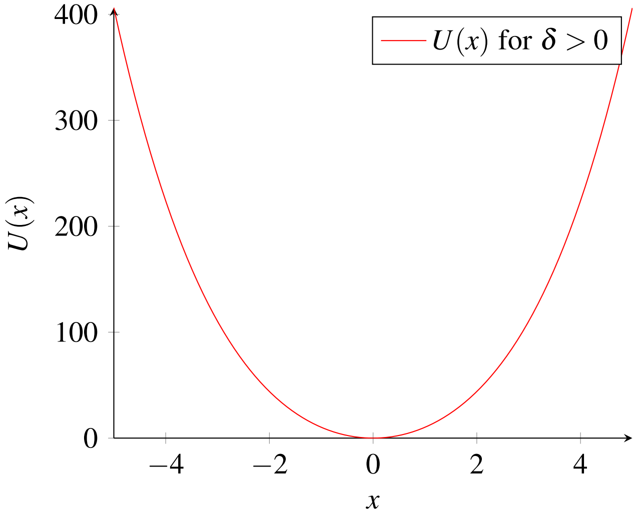
We also analyze the Phase Space for each γ value and for two different values of α: $\alpha=0$ and $\alpha=1$. Note that when $\alpha=0$, the Duffing potential becomes $U(x)=\frac{1}{2}\delta x^2$, which is the elastic potential energy for a linear spring. So, the restorative for in this case is $F=-\delta x$ and the system is can be modeled as a damped, driven linear oscillator.

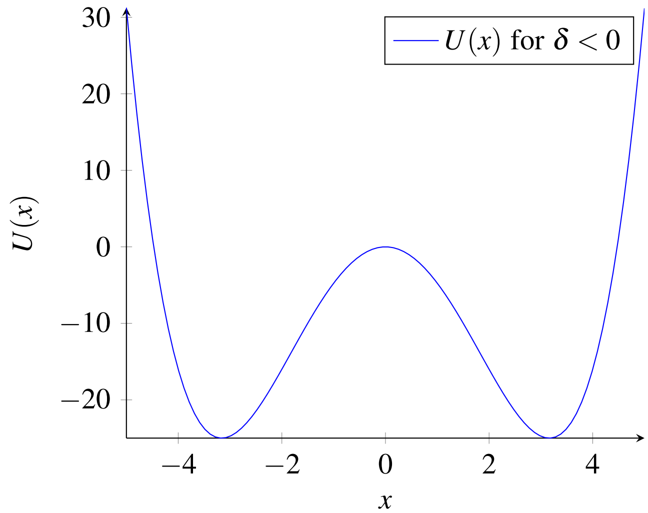
Visual Analysis

We utilized visual analysis tools such as Time Series, Phase Space diagrams, and Poincaré Diagrams to analyze the different γ Duffing oscillators. The use of Poincaré sections is specially convenient for chaotic Duffing oscillators, since we are able to visualize the strange attractor that is hidden behind the intricacy of the Phase Space diagram and the Time Series.

Duffing Potential

To analyze the Duffing Equation and our systems, we first need to have a look at the Duffing Potential. For $\delta > 0$, equation 2 is a parabola with upward concavity, while for $\delta < 0$ $U(x)$ is a double-well potential.





So, when $\delta > 0$, the Duffing potential is similar to a linear spring potential. However, when $\delta < 0$, the double-well shape creates three equilibrium points — the two wells, which are stable, and the bump in the middle, which is unstable.

Findings

Our numerical analysis of the system yielded the following results:

- A system modeled by the Duffing equations with parameters $\alpha = 1$, $\beta = 0.3$, $\delta = -1$, and $\omega = 1.2$ will start with experience period-doubling starting at $\gamma = 0.2$.
- By $\gamma = 0.37$, the motion becomes a 5-period oscillation.
- At $\gamma = 0.5$, the motion of the system is chaotic and the Poincaré section has a strange attractor
- By $\gamma = 0.65$, the motion has returned to a 2-period oscillation.
- At $\gamma = 0.68$, motion becomes chaotic again and remains so at least until $\gamma = 0.725$.
- Attractor for $\gamma = 0.5$ and $0.68 \leq \gamma \leq 0.725$ have a similar shape, but for the second window the attractor is more stretched and has the far-left bit more protruded.

Code Validation

The results obtained through our code agree with textbook example. Namely, we used the same values as the examples cited cited in the Wikipedia page for the Duffing equation (Wikipedia 2021) taken from (D.W. **and** Smith 2007).

Furthermore, we compared our Poincaré sections and Phase Space diagrams to others and both match results obtained in other articles (Kanamaru 2008).

Conclusion

In summary, we numerically analyzed solutions for the Duffing equation with varying γ to better comprehend period-doubling, the route to chaos for Duffing systems, and the strange attractor of the Duffing equation.

References

[1] Jordan D.W. **and** P. Smith. *Nonlinear ordinary differential equations – An introduction for scientists and engineers*. Oxford University Press, 2007.

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