# Preference Reversal and Discounting by Optimizing Growth Rates

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**Abstract**

ABSTRACT

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# 1 Introduction

Preference reversal ($PR$) is a behavioral phenomenon documented during the past half a century in many studies in economics and psychology (Lichtenstein and Slovic, 1971; Lindman, 1971; Grether and Plott, 1979; Loomes and Sugden, 1983; Tversky, Slovic and Kahneman, 1990; Ainslie, 1992; Laibson, 1997). It takes various forms in different contexts. In its original psychological context (Tversky, 1969; Lichtenstein and Slovic, 1971) it refers to the intransitivity in decision making under uncertainty. It also refers to the phenomenon in which a decision maker changes in mind between two options in time.

Observed PR phenomena puzzled economics, leading to various explanations and theories. One particular theory that gives rise to PR is hyperbolic discounting (Ainslie, 1992; Sozou, 1998; Laibson, 1997), suggesting that the valuation of choices falls hyperbolically in time. Practically, this means that decision makers may favor earlier guaranteed reward over higher later reward, in a way that is inconsistent with standard economic theory. While hyperbolic discounting has been established as a plausible explanation for PR, the dynamically inconsistent preferences it induces had challenged standard economic theory (Laibson, 1997; Starmer, 2000; Thaler, 2016), questioning some of the basic axioms of expected utility theory, which predicts exponential discounting, or valuation of choices that falls exponentially in time.

At the same time, it has been suggested that the same experiments used as evidence for supporting the validity of hyperbolic discounting, can also be used to reject it under different axioms (Rubinstein, 2003). In addition, various ways of explaining hyperbolic discounting have been suggested in the economic literature. One approach is to place the conditions as to why it occurs on the information of decision makers. Sozou (1998); Dasgupta and Maskin (2005) suggested that a decision maker is learning over time, which allows for PR. This approach implicitly assumes constructivist rationality similar to that of Smith (2003). In the most basic sense, the methodological approach is to posit the cognitive situation under which the agent is under and to deduce his discounting rule.

The reasoning behind the model of Sozou (1998), for example, is that an event will occur for certain but it is unclear when. The mechanics behind the model are that since an event will occur at some future time, the closer we are to that future time the more certain one of the events will occur very soon. This is because the chance for early realization is more valuable for larger payouts than smaller payouts. This provides a model for how hyperbolic discounting can describe PR under uncertainty.

This paper takes a different approach. Our model consists of a decision maker choosing between two payoffs realized at different points in time by comparing the growth rate of wealth associated with each option. We show that this model gives rise to PR. We find that depending on the wealth dynamics assumed by the decision maker, PR can occur under hyperbolic discounting, but also under other forms of discounting. We also find that this model predicts exponential discounting under multiplicative dynamics. Thus, we conclude that a model that assumes a growth rate maximizing decision maker is consistent with experimental evidence under a variety of specifications.

The main contribution of this paper is therefore to shed new light on the possible explanations for PR and hyperbolic discounting, while demonstrating this can be achieved without being inconsistent with the standard exponential discounting. The importance of these findings lies in the absence of rationality criteria – the same model and the same criteria can produce different types of discounting. We stress that the importance lies in specifying the dynamics of the problem in question.

The paper also contributes to the growing branch of ergodicity economics (Peters and Gell-Mann, 2016; Berman, Peters and Adamou, 2017; Peters and Adamou, 2018), which suggests an alternative for expected utility theory and prospect theory by taking a time perspective to decision theory, without assuming observables are ergodic, as usually done in economic theory.

The paper is organized as follows. Section 2 lays out our model and basic setup of the problem we are addressing. In Section 3 we present different specifications for the problem in question. We describe how a decision maker will discount payoffs in each specification under our model, giving rise to preference reversal. We conclude in Section 4.

## 2 Model

Our model consists of a decision maker facing a choice between two options – $a$ and $b$ – at time $t_0$. Option $a$ guarantees a payoff of $\Delta x_a$ after time period $\Delta t_a$. Option $b$ guarantees a payoff of $\Delta x_b$ after time period $\Delta t_b$. We assume that $\Delta x_b > \Delta x_a$ and that $\Delta t_b > \Delta t_a$. We also assume that at time $t_0$ the wealth (or funds) of the decision maker is $x(t_0)$. $x(t)$ denotes the wealth of the decision maker at time $t$. This setup is illustrated in Fig. 1.

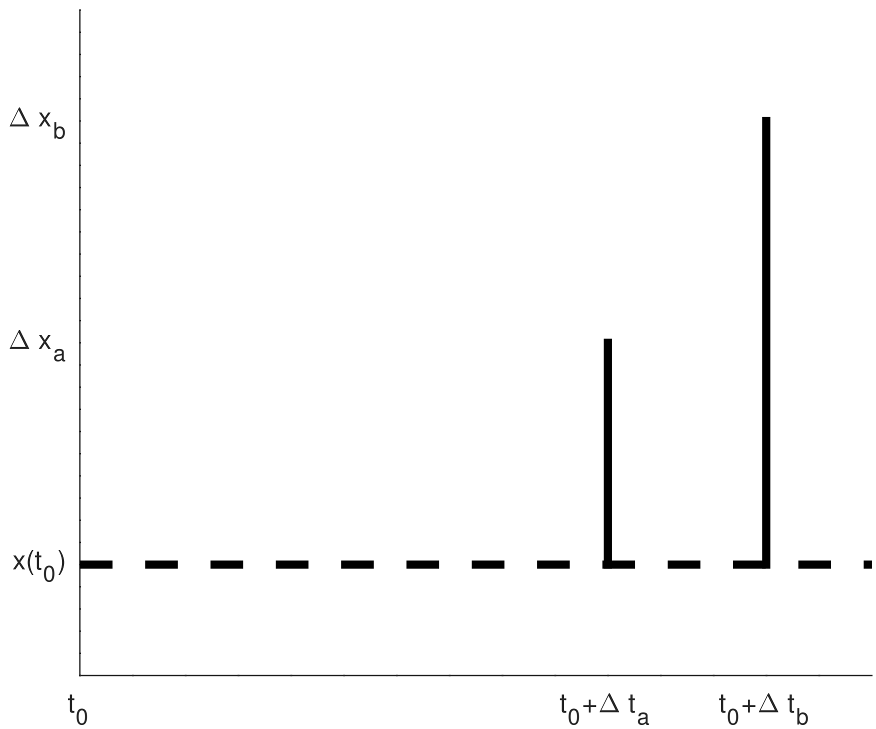


Figure 1: The basic setup of the model. A decision maker faces a choice at time $t_0$ between option $a$, which guarantees a payoff of $\Delta x_a$ after time period $\Delta t_a$, and option $b$, which guarantees a payoff of $\Delta x_b$ after time period $\Delta t_b$.

We note that in this setup there is no uncertainty in the payoffs or in the times in which they are realized. Thus, there is no risk. Because there is no risk, we do not specify a utility function and consider the dollar value of the payoffs.

This setup corresponds to a standard question that arises in the context of intertemporal discounting. Namely, "would you prefer being paid \$1 tomorrow, or \$3 in a month?". Answering this question, despite its simplicity, requires additional assumptions. For example, a standard approach would be to assume that wealth grows following some dynamic, and the standard assumption is exponential growth, continuously compounding at the risk-free rate as is the case for a standard savings account. This would allow a decision maker to quantitatively compare between the possible options.

We will describe four different specifications in which the basic setup can be thought of. In each we will calculate the growth rates of wealth associated with option $a$, $g_a$, and option $b$, $g_b$. The criterion for choosing $a$ or $b$ would be comparing $g_a$ and $g_b$. If $g_a > g_b$, than option $a$ is considered as preferable; if $g_b > g_a$, it is option $b$. This comparison will also produce discounting factors ($DF$). As we show below, this setting predicts both hyperbolic and exponential discounting, and allows preference reversal, assuming the same choice criterion, but under different specifications.

This model differs from many standard models in the literature by assuming that decision makers optimize their wealth growth rate and not the expected utility at some point in time. It has been shown to be equivalent in some cases, under specific dynamics and specific utility function choices (Peters and Adamou, 2018). It is further discussed in Section 4.

# 3 Results

We begin by describing four different specifications for our basic setup. Each specifies two necessary aspects – the time frame, and the wealth dynamics, *i.e.* what happens to the wealth over time.

The time frame is a key aspect, many times left unspecified in similar setups in the literature. One possibility is to assume a fixed time frame – the decision maker faces the choice every $\Delta t_b$. In this case, in order to compare between the two choices, we will evaluate the growth rates between $t_0$ and $\Delta t_b$ in both options. Another possible specification of time frame, is that the decision maker faces a choice after the payoff is exercised, *i.e.* after $\Delta t_a$ if option $a$ was chosen and after $\Delta t_b$ if option $b$ was chosen. In this case, we will evaluate the growth rate for option $a$ at time $\Delta t_a$, and for option $b$ at $\Delta t_b$. The former time frame will be labeled the fixed time frame and the latter the elastic time frame.

The wealth dynamics can also take different forms. A standard assumption would be that wealth grows exponentially in time, assuming a risk-free rate $r$. We label this dynamic as multiplicative. Another option is to assume additive dynamics, *i.e.* that wealth grows linearly in time, assuming a rate $k$. As we will see, this is equivalent to assuming no wealth dynamics, *i.e.* that wealth does not grow in time, apart from the changes that are due to the payouts. The definition of the wealth growth rate differs for the different dynamics. The growth rate between time $t_1$ and $t_2$ under additive dynamics is $\frac{x(t_2)-x(t_1)}{t_2-t_1}$ and under multiplicative dynamics it is $\frac{1}{t_2-t_1}\log\left(\frac{x(t_2)}{x(t_1)}\right)$ (see also (Peters and Adamou, 2018)).

We will discuss the four specifications, as illustrated in Fig. 2. In each case we will write down explicitly the value of $x(t)$ in $\Delta t_a$ or $\Delta t_b$, and use these expressions to calculate the growth rates $g_a$ and $g_b$. Then we will compare the growth rates, to obtain a decision criterion, and determine whether PR is possible and what kind of intertemporal discounting the model predicts.

## 3.1 Case A – Elastic time frame with additive dynamics

We begin by writing down $x(t)$ for the two possible choices in this case, evaluated at $\Delta t_a$ and $\Delta t_b$, and assuming additive dynamics.

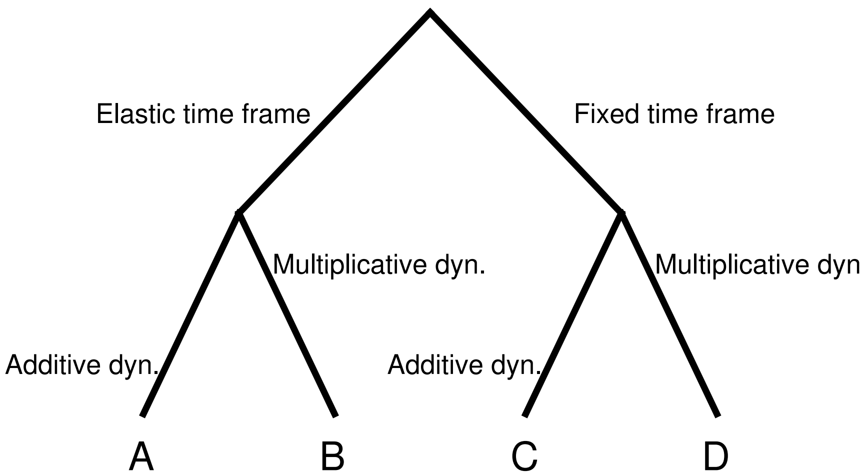


Figure 2: The four model specifications, determined by specifying a time frame and wealth dynamics. The labels A, B, C, and D, are used for the different cases.

$$x(\Delta t_a) = x(t_0) + \Delta x_a + k\Delta t_a \tag{3.1}$$

$$x(\Delta t_b) = x(t_0) + \Delta x_b + k\Delta t_b \,. \tag{3.2}$$

This way the growth rates can be easily obtained:

$$g_a = \tfrac{1}{\Delta t_a}\left(x(\Delta t_a) - x(t_0)\right) = \tfrac{\Delta x_a}{\Delta t_a} + k \tag{3.3}$$

$$g_b = \tfrac{1}{\Delta t_b}\left(x(\Delta t_b) - x(t_0)\right) = \tfrac{\Delta x_b}{\Delta t_b} + k \,. \tag{3.4}$$

It follows that the criterion $g_a > g_b$ is

$$\frac{\Delta x_a}{\Delta t_a} > \frac{\Delta x_b}{\Delta t_b} \,. \tag{3.5}$$

This criterion suggests that under this specification, the only thing that matters to the decision maker is the payout rate of each option with respect to the reference point in time,

$t_0$. The same payouts would translate into different payout rates for each option with a different reference point in time. This, in turn, would allow for preference reversal. This is illustrated in Fig. 3.

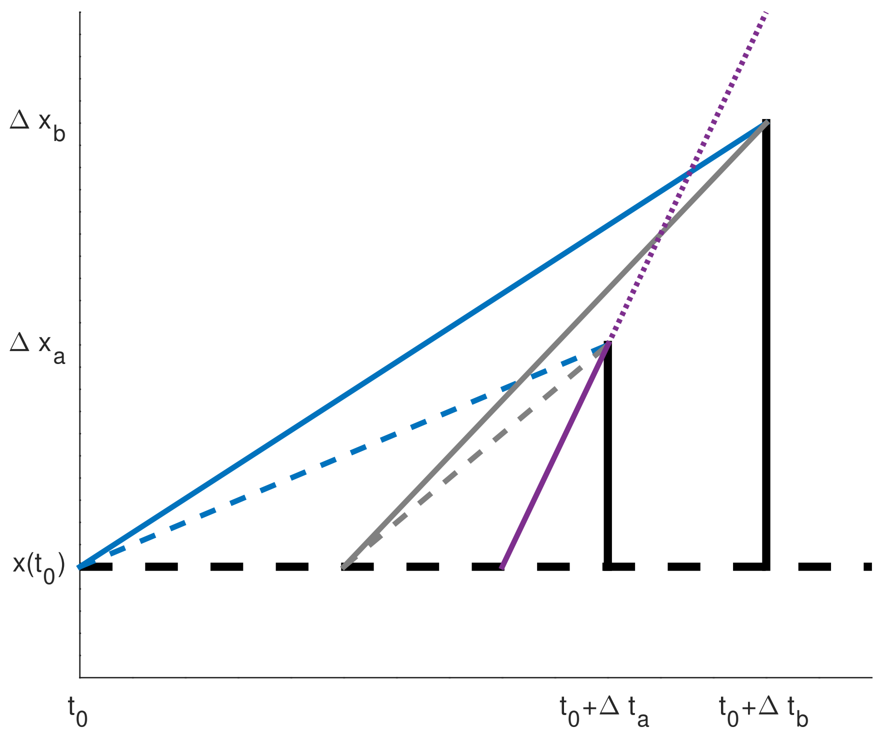


Figure 3: An illustration of preference reversal in case A. Initially, option $b$ is preferable. It is reflected in the slopes of the blue lines. The solid blue line shows the payout rate of option $b$ and the dashed blue line shows the payout rate of option $a$. Even after some time, option $b$ is still preferable, as reflected by the difference in the slopes of the gray lines (solid for option $b$, dashed for $a$). At some point option $a$ becomes preferable, as the payout rate surpasses that of option $b$. This is demonstrated in the purple line.

It shows that the payout rate of option $b$ is higher than that of option $a$, which makes it preferable. Even after some time, this is still the case. However, at some point in time, this changes and the payout rate of option $a$ would be higher. Assuming that initially option $g_b > g_a$, there will always be a point in time in which PR will occur. If time $\tau$ elapsed from $t_0$, than the updated payout rates of options $a$ and $b$ are $\frac{\Delta x_a}{\Delta t_a - \tau}$ and $\frac{\Delta x_b}{\Delta t_b - \tau}$, respectively. The reversal will occur when these are equal, or

$$\tau_{reversal} = \frac{\Delta t_a \Delta x_b - \Delta t_b \Delta x_a}{\Delta x_b - \Delta x_a}\,. \tag{3.6}$$

This case not only demonstrates PR, but also hyperbolic discounting, and the implied discount rate in this case is

$$DF_A = \frac{\Delta x_a}{\Delta x_b}\,. \tag{3.7}$$

We note, in addition, that although additive dynamics were assumed, they are unimportant for the choice criterion. The value of $k$ can be set to zero without loss of generality.

## 3.2 Case B – Elastic time frame with multiplicative dynamics

We follow the same steps as in case A, but assume wealth grows multiplicatively in rate $r$:

$$x\left(\Delta t_a\right) = x\left(t_0\right) e^{r\Delta t_a} + \Delta x_a \tag{3.8}$$

$$x\left(\Delta t_b\right) = x\left(t_0\right) e^{r\Delta t_b} + \Delta x_b\,. \tag{3.9}$$

The corresponding growth rates are now:

$$g_a = \tfrac{1}{\Delta t_a}\log\left(\tfrac{x(\Delta t_a)}{x(t_0)}\right) = \tfrac{1}{\Delta t_a}\log\left(e^{r\Delta t_a} + \tfrac{\Delta x_a}{x(t_0)}\right) \tag{3.10}$$

$$g_b = \tfrac{1}{\Delta t_b}\log\left(\tfrac{x(\Delta t_b)}{x(t_0)}\right) = \tfrac{1}{\Delta t_b}\log\left(e^{r\Delta t_b} + \tfrac{\Delta x_b}{x(t_0)}\right). \tag{3.11}$$

This setting also displays PR, although no closed-form expression for $\tau_{reversal}$ can be derived like in case A. It is clear that PR occurs, since when $\tau$ approaches $\Delta t_a$, $g_a$ will increase indefinitely, while $\log\left(e^{r\Delta t_a}+\frac{\Delta x_a}{x(t_0)}\right)$ remains positive.

### 3.3 Case C – Fixed time frame with additive dynamics

Now we assume additive dynamics again, but with a fixed time frame – the outcomes of both choices are compared at $\Delta t_b$. The growth rate of option $b$ would be the same as in case A, since it was already evaluated at $\Delta t_b$. In this case, option $a$ leads to

$$x\left(\Delta t_b\right)=x\left(t_0\right)+\Delta x_a+k\Delta t_b\,, \tag{3.12}$$

and the corresponding growth rate would be

$$g_a=\frac{1}{\Delta t_b}\left(x\left(\Delta t_b\right)-x\left(t_0\right)\right)=\frac{\Delta x_a}{\Delta t_b}+k\,. \tag{3.13}$$

Recall that $g_b=\frac{\Delta x_b}{\Delta t_b}+k$. So assuming $\Delta x_b>\Delta x_a$, it is clear that option $b$ would always be preferable over option $a$. This is a trivial case – assuming additive dynamics (or no dynamics) the only thing that matters is the payout size, if we compare the options at the same time, or if they repeat at fixed time windows.

### 3.4 Case D – Fixed time frame with multiplicative dynamics

Finally, we assume multiplicative dynamics and a fixed time frame. This is the specification that corresponds to the standard assumptions usually considered in intertemporal discounting – that wealth is continuously compounding at the risk-free rate and that payouts are re-invested at this rate.

The growth rate of option $b$ would be the same as in case B, since it was already evaluated at $\Delta t_b$ ($g_b=\frac{1}{\Delta t_b}\log\left(e^{r\Delta t_b}+\frac{\Delta x_b}{x(t_0)}\right)$). In this case, option $a$ leads to

$$x\left(\Delta t_b\right) = x\left(t_0\right) e^{r\Delta t_b} + \Delta x_a e^{r\left(\Delta t_b - \Delta t_a\right)}\,, \tag{3.14}$$

and the corresponding growth rate would be

$$g_a = \frac{1}{\Delta t_b} \log\left(\frac{x\left(\Delta t_b\right)}{x\left(t_0\right)}\right) = \frac{1}{\Delta t_b} \log\left(e^{r\Delta t_b} + \frac{\Delta x_a e^{r\left(\Delta t_b - \Delta t_a\right)}}{x\left(t_0\right)}\right). \tag{3.15}$$

It follows that the criterion $g_a > g_b$ is

$$\Delta x_a e^{r\left(\Delta t_b - \Delta t_a\right)} > \Delta x_b\,, \tag{3.16}$$

which is the standard exponential discounting result. In other words, if it is possible to re-invest the reward given at time $\Delta t_a$ so that it surpasses the alternative reward given at time $\Delta t_b$, option $a$ is preferable over option $b$, and vice versa.

# 4 Discussion

This paper described a model in which a decision maker chooses between two payoffs realized at different points in time by comparing the growth rate of wealth associated with each option. We show that this simple model gives rise to preference reversal. We find that depending on the wealth dynamics assumed by the decision maker, preference reversal can occur under hyperbolic discounting. We also find that this model predicts exponential discounting under multiplicative dynamics.

A fundamental question about the model is why would someone maximize growth instead of expectation value of utility? This, of course, differs from the traditional framework for analyzing decisions in economics, usually taking it as axiomatic that people maximize utility. Yet, this is only one way of studying optimal choice under different conditions. At the same time it is important to question where does the utility optimizing mechanism come from?

The framework we used here has an answer – evolutionary mechanisms. That is, some utility functions survive better than others. The utility functions that will survive over the long run are the growth utility functions. Though utility functions are traditionally used for individuals, this framework implies that even firms should have utility functions. That is, why should firms discount hyperbolically? If a firm has a fixed staff and cannot undertake numerous projects at the same time and has access to some interest rate, then it should discount hyperbolically. If the evolutionary mechanism is considered, maximizing the expected value can actually be considered as merely a case of bounded rationality. The process of updating beliefs based solely on observation without reference to the environment, may be ecologically irrational.

Whether actual decision makers behave this way is an empirical question, nevertheless this theory relative to the classic case, has more Popperian content inscribed in it. So we consider that this framework offers a rich agenda for research. Yet, the empirical implications of the model do have falsifiable content. That is, after we observe a specific discount factor from someone, we can deduce the environment they are in. For instance, if someone is in a changing environment, we should expect them use the ensemble average. If, on the other hand, the environment is constant then we predict that they will behave more like the time average. An objective measure of opportunity could be one way to construct such a measure.

This is only one way to interpret the model. Another way would be that the preferences which are revealed, also reveal a belief set. However, this case is not as empirically interesting because the belief set is defined by the preferences and does not imply a cognitive awareness of the implied belief. That is, just because a set of beliefs are implied, this does not mean that they can be solicited. Why might we not be able to solicit preferences? For it to be possible to solicit preferences it must assumed that the decision rule used in experiments is the same as the one used in the real world. Well how do we verify that the actual preferences are the same ones as the ones that were solicited? By looking to see consistency between solicitation and the real world.

It is possible to find that agents make use of a specific decision rule only for such experiments. We then should ask why do some people change their decision rules when solicited whilst others do not? Ultimately, this question must be answered without looking at solicited preferences. So there is still a need to look at the environment of the agent. Focusing on beliefs does not in fact ever yield satisfactory answers because the meta question will always exist.

Using solicited preferences does in fact requires going back to the real world in any case, so it can be skipped altogether. The approach in this paper differs by offering the potential to omit the solicitation. Observing data on how decision makers make investment choices in the real world, can describe the discount rate. If we cannot tell which specification of the model should be used, more information may be used. For instance, if someone has access to a good investment manager or if they often have investment opportunities. The advantage of this approach is to bypass the need to look at verbally stated beliefs.

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