Temporal Discounting Without Uncertainty: Growth Rates, Preference Reversal and Hyperbolic Discounting

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Abstract

An important question in economics is how people evaluate payoffs in the future. The standard phrasing of the problem is in part psychological. The value we attach to a future payoff is the dollar value of the payoff discounted by a factor whose functional form is determined subjectively and whose (objective) argument is how long we have to wait for the payoff. Here we explore how discounting arises from growth rate optimization, in four specifications of a Certain Intertemporal Payoff Problem (CIPP). Exponential discounting is recovered as optimizing a multiplicative growth rate, and hyperbolic discounting as optimizing an additive growth rate. Preference reversal occurs in two of the specifications we study.

Keywords: Decision theory, Hyperbolic discounting, Ergodicity economics

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1 Introduction

This paper studies temporal discounting in a deterministic setting. In our model a decision maker chooses between two known and different payoffs to be received at known and different times by comparing the growth rates of wealth associated with each option. The model is further specified by assumptions about the wealth dynamics of the decision maker and the time frame of the decision. Depending on the specification, the model produces different forms of discounting – including exponential and hyperbolic.

Preference reversal (PR) is a behavioral phenomenon documented during the past half a century in many studies in economics and psychology (Lichtenstein and Slovic, 1971; Lindman, 1971; Grether and Plott, 1979; Loomes and Sugden, 1983; Tversky, Slovic and Kahneman, 1990; Ainslie, 1992; Laibson, 1997). It takes various forms in different contexts. In its original psychological context (Tversky, 1969; Lichtenstein and Slovic, 1971) it refers to the intransitivity in decision making under uncertainty. It also refers to the phenomenon in which a decision maker changes his mind between two options as time passes.

In our model PR is a consequence of growth rate optimization. Specifications that lead to standard exponential discounting do not generate PR, but others do. The dynamic and time frame specification reflect the situation of a decision maker. Thus a single behavioral model, a single optimand, is consistent with a range of observed behaviors if the situation of the decision maker is formally taken into account.

The main contribution of this paper is the prediction of hyperbolic discounting, and hence PR, from considerations that do not violate the standard von Neumann-Morgenstern axioms. Our framework does not assume any form of irrationality or inconsistency on the part of the decision maker. Changes in the type of discounting come from different dynamics and time frames, *i.e.* properties of the situation of the decision maker, not properties of the decision maker himself. Thus, utility functions or temporal preferences are not part of our model specification – such apparently idiosyncratic properties can be derived from a full specification of the situation.

The paper also contributes to the growing branch of ergodicity economics (Peters and Gell-Mann, 2016; Berman, Peters and Adamou, 2017; Peters and Adamou, 2018). It proposes an alternative framework to expected utility theory or prospect theory, suggesting that agents maximize the growth of their resources averaged over time. This joins recent evidence of a strong effect of changes to the wealth dynamic on choices under uncertainty (Meder et al., 2019).

The paper is organized as follows. Section 2 lays out our model and the basic setup of the problem we are addressing. In Section 3 we present different specifications for the problem in question. We describe how a decision maker will discount payoffs in each specification under our model, giving rise to preference reversal. We conclude in Section 4.

1.1 Related literature

Observations of PR have led to various explanations and theories. One such theory is hyperbolic discounting (Ainslie, 1992; Sozou, 1998; Laibson, 1997), suggesting that the valuation of choices falls hyperbolically in time. This is in contrast to the standard assumption of exponential discounting in economic theory, where no such reversal occurs. Hyperbolic discounting has been established as a plausible explanation for PR. Yet, the dynamically inconsistent preferences it induces have challenged standard economic theory (Laibson, 1997; Starmer, 2000; Thaler, 2016).

Rubinstein (2003) suggested that the same experiments supporting hyperbolic discounting, can also be used to reject it under different axioms. In addition, various behavioral explanations for hyperbolic discounting have been given in the economics literature. One approach considers the information available to decision makers. Sozou (1998); Dasgupta and Maskin (2005) suggested that decision makers learn over time, which can lead to PR. This approach implicitly assumes constructivist rationality similar to that of Smith (2003). In the most basic sense, the methodological approach is to posit the cognitive situation of the agent and to deduce a discounting rule.

The reasoning behind the model of Sozou (1998), for example, is that agents do not know the hazard rate of a random event and learn about it over time through Bayesian updating. Thus, the longer an event does not occur, the more likely it is that it will not occur (decreasing estimated hazard rate). In other words, the agents have some prior on the arrival rate of the event, and the longer they wait without observing the event the more they judge that the arrival rate must be small. Using the Laplace transform of the survival probability, they show that if the survival probability is to be equal to the hyperbolic discount rate, the agents' prior must be an exponential distribution.

On the other hand, Dasgupta and Maskin (2005) assume that the agent knows that an event will occur but not when. The mechanics behind the model are that since an event will occur by some future time, the closer we are to that future time the more certain one of the events will occur very soon. This is because it is initially assumed that the probability of early realizations is the same for both gambles. This, in turn, means that the chance for early realization is more valuable for larger payoffs than smaller payoffs. They later extend their result to say that the density is not the same but that the probability of one event increases relatively more over time which allows for a wider class of hazard functions. This provides a very general class of distributions for which PR occurs.

It is important to note that all the different approaches to hyperbolic discounting in the literature involve some form of uncertainty. As described, this paper discusses temporal discounting, and in particular hyperbolic discounting and preference reversal phenomena, without the need to introduce uncertainty or risk. Our approach can be considered as a micro-foundation of various kinds of discounting.

2 Model

We begin by defining a *Certain Intertemporal Payoff Problem* (CIPP).

Definition 1 *Certain Intertemporal Payoff Problem*

A Certain Intertemporal Payoff Problem (CIPP) is a vector $\{t_0, x(t_0), t_a, \Delta x_a, t_b, \Delta x_b\}$ – a decision maker at time t_0 with wealth $x(t_0)$, chooses between two future cash payoffs, one earlier than the other, whose amounts and payoff times are known with certainty. The two options are:

  a. an earlier payoff of Δx_a at time $t_a > t_0$; and

  b. a later payoff of $\Delta x_b > \Delta x_a$ at time $t_b > t_a$.

A formal decision theory has to provide a criterion for choosing a or b. Here we explore what happens if that criterion is growth rate maximization. A growth rate, g, is defined as the scale parameter of time for an underlying dynamic of wealth. Wealth dynamics can take different forms. Ignoring payoffs Δx_a and Δx_b, a standard assumption is that wealth grows exponentially in time at rate r. We label this dynamic as multiplicative. It corresponds to investing wealth in income-generating assets, where the income is proportional to the amount invested. In this case wealth follows

$$x(t) = x(0) e^{rt}, \tag{2.1}$$

and the scale parameter of time is $g = r$.

Another possibility is additive dynamics, where wealth grows linearly in time, at a rate k. It corresponds to something like labor income, namely to situations where investment income is negligible, and wealth instead changes by a net flow that does not depend on wealth itself. In this case wealth follows

$$x(t) = kt + x(0), \tag{2.2}$$

and the scale parameter of time is $g = k$.

The functional form of the growth rate differs between the dynamics. The growth rate between time $t + \Delta t$ and t under multiplicative dynamics it is $r = \frac{\log x(t+\Delta t) - \log x(t)}{\Delta t}$ and under additive dynamics is $k = \frac{x(t+\Delta t) - x(t)}{\Delta t}$. This can be generalized to other possible wealth dynamics (Peters and Gell-Mann, 2016; Peters and Adamou, 2018).

Now, given a specific wealth dynamic, a CIPP implies two growth rates – g_a, associated with choice a; g_b, associated with choice b. This allows formulating a single axiom:

Axiom 1 *The Optimization of Growth*

Given a wealth dynamic, time t_0, an initial wealth $x(t_0)$, and tuples $(t_a, \Delta x_a)$ and $(t_b, \Delta x_b)$, such that the vector $\{t_0, x(t_0), t_a, \Delta x_a, t_b, \Delta x_b\}$ is a CIPP:

1. *$(t_a, \Delta x_a) \succ (t_b, \Delta x_b)$ if and only if $g_a > g_b$*

2. *$(t_a, \Delta x_a) \sim (t_b, \Delta x_b)$ if and only if $g_a = g_b$*

3. *$(t_a, \Delta x_a) \prec (t_b, \Delta x_b)$ if and only if $g_a < g_b$*

In words, Axiom 1 postulates that a decision maker will prefer choice a if her wealth grows faster under this choice than under choice b, and vice versa. Indifference only occurs if the growth rates are equal. Axiom 1 trivially satisfies the von Neumann-Morgenstern axioms – completeness is satisfied by design, while continuity and independence are irrelevant, since in this setup all the payoffs and times are certain. It also satisfies transitivity.

Proposition 1 *Optimization of Growth is Transitive*

Under the notation of Axiom 1, the Transitivity axiom is satisfied.

2.1 Setup

In a CIPP, $\Delta x_b > \Delta x_a$ because a larger and earlier payoff is trivially always preferred. This setup is illustrated in Fig. 1. We note again that in this setup there is no uncertainty in the payoffs or in the times in which they are realized. Thus, there is no risk.
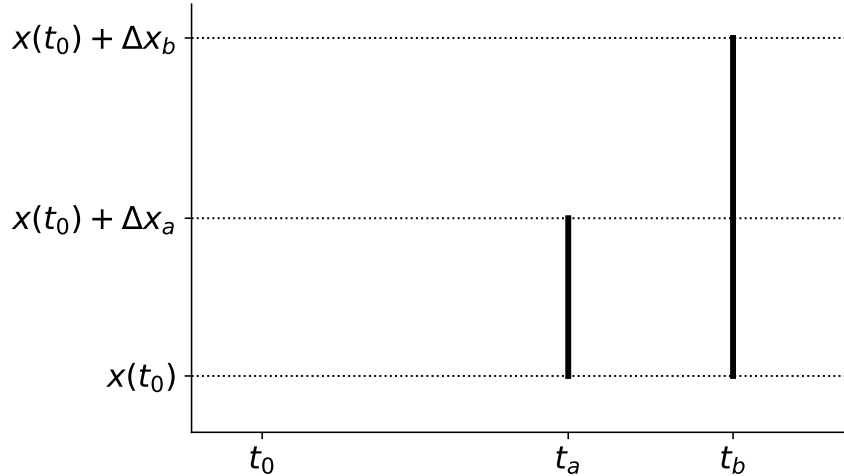

Figure 1: The basic setup of the model. A decision maker faces a choice at time t_0 between option a, which guarantees a payoff of Δx_a at time t_a, and option b, which guarantees a payoff of $\Delta x_b > \Delta x_a$ at time $t_b > t_a$.

This setup corresponds to a standard question that arises in the context of temporal discounting, *e.g.* "would you prefer to receive \$100 tomorrow or \$200 in a month's time?" Despite its apparent simplicity, answering this question requires additional assumptions. Or, put another way, the problem is underspecified. One extra assumption needed concerns the dynamics under which the decision-maker's wealth grows. Often it is assumed that wealth grows exponentially, compounding continuously at a constant riskless rate like funds in a savings account. Another assumption concerns the time frame of the decision, specifically whether a decision-maker accepting the earlier payoff at t_a is free immediately to make his next decision, or whether he must wait until the later time t_b (or, indeed, some other time) before the decision can be repeated. Such assumptions are needed to compute decision-maker's maximand – the growth rate of his wealth – so that the options can be compared quantitatively.

We will describe four different specifications of this basic setup. In each we will calculate the growth rates, g_a and g_b, of wealth associated with options a and b. The decision maker prefers the option whose growth rate is larger.

We will also infer the discount factor (DF) from this analysis. This is the multiplicative factor, δ, by which the later payoff, Δx_b, must be multiplied to equal the earlier payoff, Δx_a, when the payoff amounts and times are such that the decision maker is indifferent between the two options $((t_a, \Delta x_a) \sim (t_b, \Delta x_b))$. In symbols,

$$\delta \equiv \left. \frac{\Delta x_a}{\Delta x_b} \right|_{g_a = g_b}, \tag{2.3}$$

i.e. the ratio of payoffs under the constraint that the growth rates of wealth are equal.

As we show below, this setup predicts decisions equivalent to hyperbolic and exponential discounting under different specifications. Some specifications of the model predict preference reversal.

3 Results

3.1 Specification

We begin by describing four different specifications for our basic setup. Each specifies two aspects necessary to quantify the growth rate of wealth: the time frame of the decision; and the dynamics under which wealth evolves.

The time frame is a key aspect, often left unspecified in similar setups in the literature. Consider the following scenarios:

1. Dana, the real estate developer, loves to work and always wants to keep busy with her building projects, she always gets paid at their completion. Dana has a choice between a project that lasts three months and a project that lasts six months.

2. Every year, Nate the Naval officer must go for either a three month long mission or a six month long mission. He is given the choice at the beginning of every year (both missions finish before the end of the year). He is paid right after his mission is completed.

In the first scenario, the time frame depends on the choice made. We call this the *elastic* time frame because Dana is more flexible to pursue other opportunities if she chooses the shorter project. On the other hand, if she chooses the longer project, it locks her in a for a longer time period, which means it also changes when she will have another choice.

In the second scenario, the important element to note is that no matter which choice is made, it will not affect the timing of future choices. Said otherwise, the time frame is independent of the choice, so we say it is *fixed*.

In our model, we must choose the time period over which the growth rates of wealth in each option are computed. We can choose it to be the time period associated with each payoff, *i.e.* $t_a - t_0$ for option a and $t_b - t_0$ for option b. This specification corresponds to Dana's situation, the elastic time frame specification. Or we can choose it always to be the longer time period, $t_b - t_0$, resembling Nate's dilemma, the fixed time specification.

As described in Section 2, the wealth dynamics can also take different forms, and we will address two specific common cases: additive and multiplicative wealth dynamics. We note that under the multiplicative dynamics it is assumed that the payoff itself is re-invested at the risk-free rate. For additive dynamics there is essentially no re-investment of the payoff – income in this dynamic is independent of wealth.

We will discuss the four specifications, as illustrated in Fig. 2. In each case we will: compute the growth rates g_a and g_b associated with each option; compare them to determine the conditions under which each option is preferred; elicit the form of temporal discounting equivalent to our decision model; and, finally, determine whether PR is predicted.
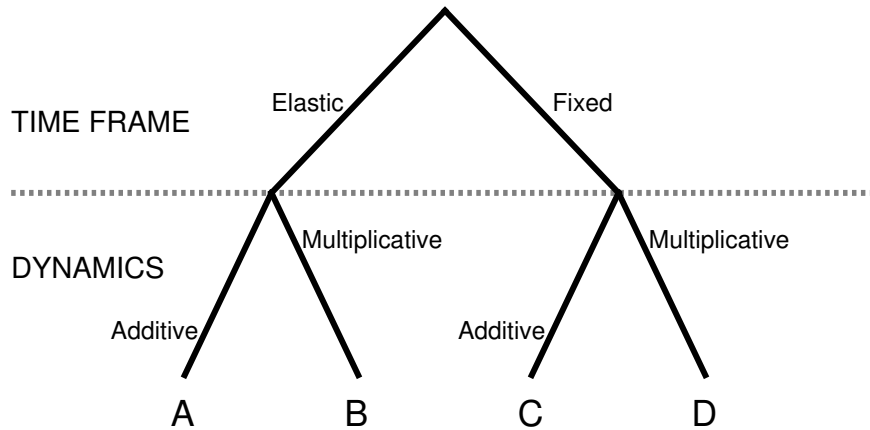

Figure 2: The four model specifications, determined by specifying a time frame and wealth dynamics. The labels A, B, C, and D, are used for the different cases.

3.2 Case A – Elastic time frame with additive dynamics

Specification: the period for computing the growth rate is that between the decision and the chosen payoff; and the wealth dynamics are additive, with growth rate k.

We begin by writing down the final wealth under the two options, evaluated at t_a and t_b respectively:

$$x_a(t_a) = x(t_0) + \Delta x_a + k(t_a - t_0); \tag{3.1}$$

$$x_b(t_b) = x(t_0) + \Delta x_b + k(t_b - t_0). \tag{3.2}$$

The growth rates are:

$$g_a = \frac{x_a(t_a) - x(t_0)}{t_a - t_0} = \frac{\Delta x_a}{t_a - t_0} + k; \tag{3.3}$$

$$g_b = \frac{x_b(t_b) - x(t_0)}{t_b - t_0} = \frac{\Delta x_b}{t_b - t_0} + k. \tag{3.4}$$

It follows that the criterion $g_a > g_b$ is

$$\frac{\Delta x_a}{t_a - t_0} > \frac{\Delta x_b}{t_b - t_0}. \tag{3.5}$$

This criterion suggests that, under this specification, the only thing that matters to the decision maker is the linear payoff rate of each option.

If we treat the payoff amounts, Δx_a and Δx_b, and payoff times, t_a and t_b, as fixed parameters of the problem, then we can elicit the dependence of the decision on the decision time, t_0. When the payoffs are far ahead in the future, the ratio of the growth rates approaches $\lim_{t_0 \to -\infty} \frac{g_a}{g_b} = \frac{\Delta x_a}{\Delta x_b}$, and $g_a < g_b$ since we have assumed $\Delta x_a < \Delta x_b$. When the earlier payoff is imminent, *i.e.* as $t_0 \to t_a$, g_a grows without bound while g_b remains finite and so $g_a > g_b$. In other words, as time passes, our decision model under this specification predicts preference reversal from the later, larger payoff to the earlier, smaller payoff. This is illustrated in Fig. 3.

We can compute the decision time, t_0^{PR}, at which preference reversal occurs by setting $g_a = g_b$ to give

$$t_0^{\mathrm{PR}} = \frac{\Delta x_b t_a - \Delta x_a t_b}{\Delta x_b - \Delta x_a}. \tag{3.6}$$

We can also find the effective discount factor under this specification. When $g_a = g_b$, we have

$$\delta = \frac{\Delta x_a}{\Delta x_b} = \frac{t_a - t_0}{t_b - t_0} = \frac{1}{1 + \frac{t_b - t_a}{t_a - t_0}}, \tag{3.7}$$

where we have made the final manipulation to express δ in hyperbolic form. We see that the discount factor depends on two time periods: that between decision and the earlier payoff, $t_a - t_0$, which we will call the *horizon*; and that between the two payoffs, $t_b - t_a$, which we which we will
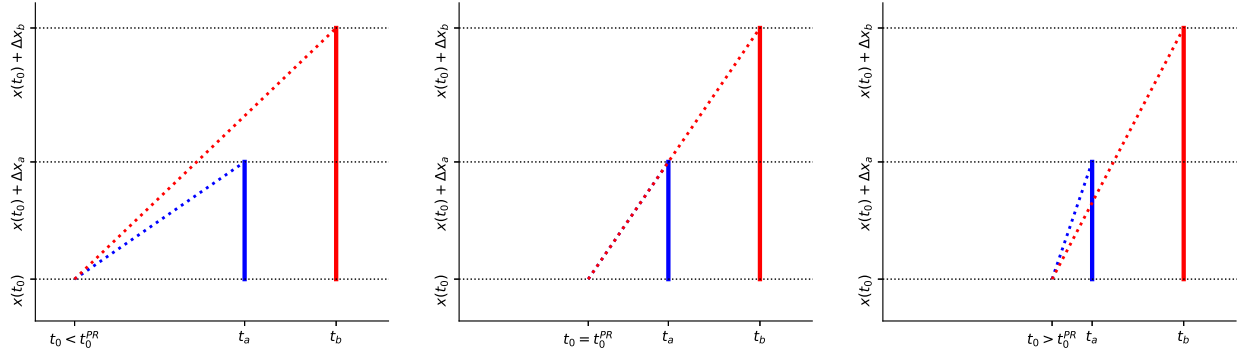
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Figure 3: Preference reversal in case A. From left to right panel, t_0 increases, that is, the time of the payoffs approaches, while all other parameters are unchanged. Initially, option b is preferable, having the higher growth rate. At a later time $t_0 = t_0^{PR}$, given by Eq. (3.6), both options imply equal growth, and preference reversal occurs. At later times option a is preferable.

call the *delay*.[1] If we define $H \equiv t_a - t_0$ and $D \equiv t_b - t_a$, we can write the discount factor as

$$\delta = \frac{1}{1 + D/H}, \tag{3.8}$$

which is expressed in the conventional way as a hyperbolic function of the delay, D. The psychological degree of discounting parameter used in mainstream models is replaced here by $1/H$, the reciprocal of the horizon. As the horizon gets shorter, $1/H$ becomes larger, δ gets smaller, and the later payoff becomes less favorable. No knowledge of the decision-maker's psychology is required in this setup – only the postulate that she prefers her wealth to grow faster rather than slower.

Finally, we note that the background growth rate, k, of the decision-maker's wealth does not appear in the decision criterion. This is because wealth growth under additive dynamics is not affected by exogenous cash flows: the gain $k\Delta t$ over period Δt occurs regardless of other payoffs received. This contrasts with multiplicative dynamics, where payoffs can be subjected to the growth process through re-investment.

3.3   Case B – Elastic time frame with multiplicative dynamics

Specification: the time frame for computing the growth rate is time to the chosen payoff; and the wealth dynamics are multiplicative, with growth rate r.

We follow the same steps as in case A. Wealth evolves to:

$$x_a(t_a) = x(t_0) e^{r(t_a - t_0)} + \Delta x_a; \tag{3.9}$$

$$x_b(t_b) = x(t_0) e^{r(t_b - t_0)} + \Delta x_b. \tag{3.10}$$

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[1]Indeed, the problem is fully specified by these two time periods and the two payoff amounts. The actual times, t_0, t_a, t_b, are not needed to specify the problem because, when computing growth rates, only elapsed times matter. The time origin is arbitrary.

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The corresponding growth rates are:

$$g_a = \frac{1}{t_a - t_0} \log\left(\frac{x_a(t_a)}{x(t_0)}\right) = \frac{1}{t_a - t_0} \log\left(1 + \frac{\Delta x_a}{x(t_0)e^{r(t_a - t_0)}}\right) + r \qquad (3.11)$$

$$g_b = \frac{1}{t_b - t_0} \log\left(\frac{x_b(t_b)}{x(t_0)}\right) = \frac{1}{t_b - t_0} \log\left(1 + \frac{\Delta x_b}{x(t_0)e^{r(t_b - t_0)}}\right) + r. \qquad (3.12)$$

This setting displays preference reversal: $g_a < g_b$ for t_0 sufficiently far away from t_a (long horizon); and $g_a > g_b$ for t_0 sufficiently close to t_a (short horizon). No closed-form expression for the reversal time, t_0^{PR}, is available.

Similarly, the discount factor δ cannot be derived explicitly. However, if we assume small payoffs relative to wealth, *i.e.* $\Delta x_a \ll x(t_0)e^{r(t_a - t_0)}$ and $\Delta x_b \ll x(t_0)e^{r(t_b - t_0)}$, then, setting $g_a = g_b$ and using the first-order approximation $\log(1 + \epsilon) \approx \epsilon$ for $\epsilon \ll 1$, we get

$$\delta = \frac{\Delta x_a}{\Delta x_b} \approx \frac{(t_a - t_0)e^{r(t_a - t_0)}}{(t_b - t_0)e^{r(t_b - t_0)}} = \frac{e^{r(t_a - t_b)}}{1 + \frac{t_b - t_a}{t_a - t_0}}. \qquad (3.13)$$

Using the previous definitions of H and D, we can write this as

$$\delta \approx \frac{e^{-rD}}{1 + D/H}, \qquad (3.14)$$

which is a hybrid of hyperbolic and exponential discounting. We note again that only the elapsed times, H and D, appear in the discount factor. However, that the background wealth growth rate, r, no longer cancels out when dynamics are multiplicative, as does k when they are additive.

Case B also displays another type of preference reversal. Varying initial wealth $x(t_0)$ while keeping all other parameters fixed can lead to a switch from $g_a > g_b$ to $g_a < g_b$, as illustrated in Fig. 4.

The difference $g_a - g_b$, from Eq. (3.11) and Eq. (3.12), is shown as a function of $x(t_0)$ in Fig. 5. This type of preference reversal can be expressed as follows: under certain circumstances, it is growth-optimal for people of lower wealth to choose a small early payoff, whereas it is growth-optimal for wealthier individuals to hold out until the later larger payoff. This predicts the findings of Epper et al. (2018), that "individuals with relatively low time discounting are consistently positioned higher in the wealth distribution".

3.3.1 A simple numerical example of wealth reversal

Suppose the an agent is faced with a choice of receiving \$10 after 1 year or \$25 after two years, to simplify matters, assume the agent has no access to a rate of interest. The claim is that the agent will evaluate the growth rate of each choice. If the agent has wealth \$100, he/she will evaluate that the first bet gives a growth rate, g of $100g = 100 + 10 \Rightarrow g = 1.1$. Similarly for the second choice, which arrives after two years, the agent will evaluate that the growth rate will be $100g^2 = 100 + 25 \Rightarrow g = (1.25)^{\frac{1}{2}} \approx 1.12$. So the agent prefers the later option because $1.12 > 1.1$.
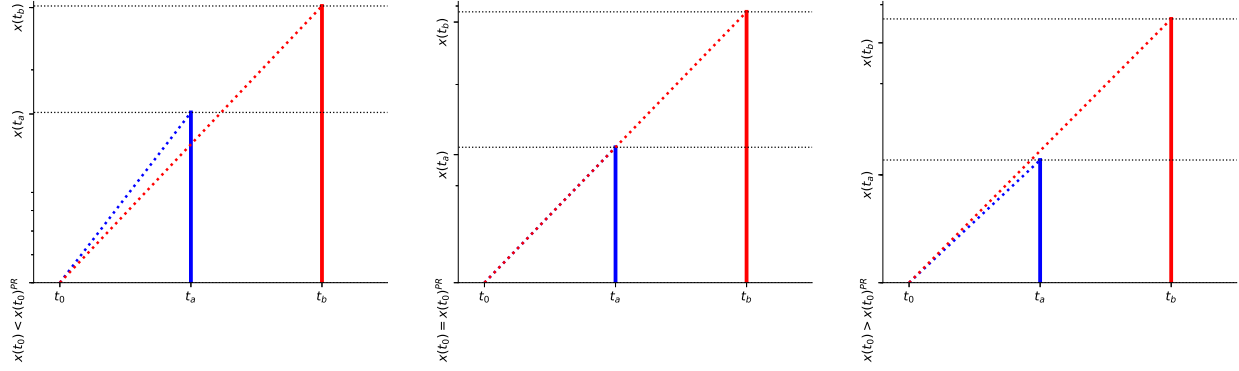
Figure 4: Preference reversal in response to wealth changes, in case B, logarithmic vertical scales. Initial wealth $x(t_0)$ increases from left to right panel (\$500, \$2277, \$5,000), while all other parameters are unchanged ($t_0 = $ today, $t_a = 1$ year from today, $t_b = 2$ years from today, $\Delta x_a = \$1000$, $\Delta x_b = \$2500$, $r = 0.03$ per annum). At low wealth, option a is preferable, having the higher growth rate, according to Eq. (3.11) and Eq. (3.12). At a greater wealth, $x(t_0)^{PR} \approx \$2277$, both options imply equal growth, and preference reversal occurs. At even greater wealth, option b is preferable: the poor behave optimally by choosing the small early payoff.
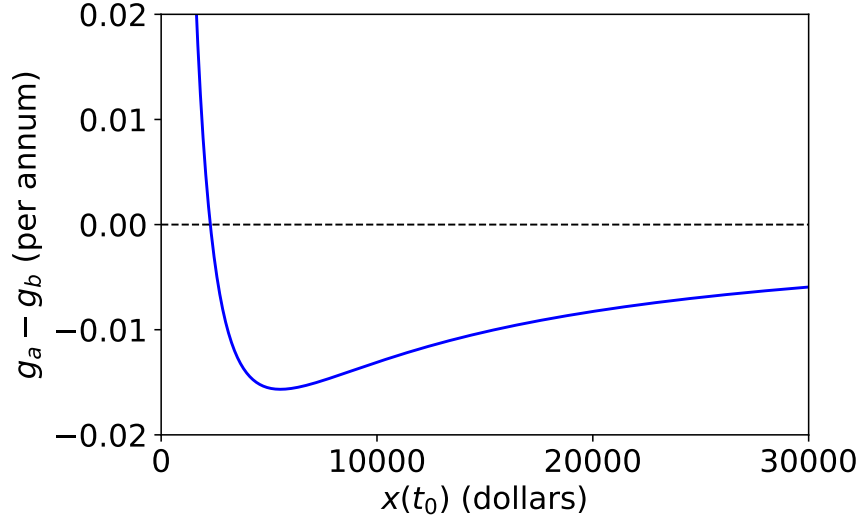

Figure 5: The difference $g_a - g_b$ is positive when the earlier payoff is preferable, and otherwise negative. We see that for small initial wealths $x(t_0)$ the earlier smaller payoff is preferred, whereas for large initial wealth the later larger payoff is preferred (parameters as in Fig. 4).

Now suppose that the agent starts with a wealth of \$10, a similar calculation yields, $g = 2$ for the first option and $g \approx 1.87$ for the second option. Which means that the poorer individual prefers the earlier option.

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3.4   Case C – Fixed time frame with additive dynamics

Now we assume additive dynamics as in case A, but with a fixed time frame so that the outcomes of both choices are compared at t_b. The wealths evolve to:

$$x_a(t_b) = x(t_0) + \Delta x_a + k(t_b - t_0);$$ (3.15)

$$x_b(t_b) = x(t_0) + \Delta x_b + k(t_b - t_0).$$ (3.16)

The growth rates are:

$$g_a = \frac{x_a(t_b) - x(t_0)}{t_b - t_0} = \frac{\Delta x_a}{t_b - t_0} + k;$$ (3.17)

$$g_b = \frac{x_b(t_b) - x(t_0)}{t_b - t_0} = \frac{\Delta x_b}{t_b - t_0} + k.$$ (3.18)

Note that the wealth and its growth rate under option b are the same as in case A, since they were already evaluated at t_b there.

Since we have assumed $\Delta x_b > \Delta x_a$, option b is always preferred to option a. This is a trivial case – if we assume additive wealth dynamics and comparing the growth rates at the same time (or assuming repetition over fixed periods), then the only thing that matters to the decision-maker is payoff size. In this case, the discount factor δ cannot be defined, since the later, larger payoff is always preferred and the indifference condition is never satisfied.

3.5   Case D – Fixed time frame with multiplicative dynamics

Finally, we assume multiplicative dynamics and a fixed time frame. This is the specification that corresponds to the standard assumptions usually considered in temporal discounting – that wealth is continuously compounding at the risk-free rate and that payoffs are re-invested at this rate.

The chief difference from case B is that the earlier payoff, Δx_a, if chosen, is treated as growing exponentially from t_a to t_b. The wealths evolve from t_0 to t_b as follows:

$$x_a(t_b) = x(t_0) e^{r(t_b - t_0)} + \Delta x_a e^{r(t_b - t_a)};$$ (3.19)

$$x_b(t_b) = x(t_0) e^{r(t_b - t_0)} + \Delta x_b.$$ (3.20)

The corresponding growth rates are:

$$g_a = \frac{1}{t_a - t_0} \log\left(\frac{x_a(t_a)}{x(t_0)}\right) = \frac{1}{t_b - t_0} \log\left(1 + \frac{\Delta x_a e^{r(t_b - t_a)}}{x(t_0) e^{r(t_b - t_0)}}\right) + r$$ (3.21)

$$g_b = \frac{1}{t_b - t_0} \log\left(\frac{x_b(t_b)}{x(t_0)}\right) = \frac{1}{t_b - t_0} \log\left(1 + \frac{\Delta x_b}{x(t_0) e^{r(t_b - t_0)}}\right) + r.$$ (3.22)

Note that the evolution of wealth under option b is the same as in case B.

The criterion $g_a > g_b$ is actually very simple, since only the second term in the logarithm is different

and so only this must be compared. Thus, $g_a > g_b$ if

$$\Delta x_a e^{r(t_b - t_a)} > \Delta x_b \,, \tag{3.23}$$

or, in terms of the delay, if

$$\Delta x_a e^{rD} > \Delta x_b \,. \tag{3.24}$$

The discount factor is similarly expressed by setting the growth rates to be equal. Then we get $\Delta x_a e^{rD} = \Delta x_b$ and

$$\delta = \frac{\Delta x_a}{\Delta x_b} = e^{-rD} \,, \tag{3.25}$$

which is the standard exponential discounting result. The interpretation is straightforward: if it is possible to re-invest the earlier payoff such that, by the time of the later payoff, it will exceed the later payoff amount, then option a is preferable to option b (and *vice versa*). Note that, with this specification, the horizon is irrelevant. All that matters is the payoff amount after possible re-investment.

4 Discussion

This paper describes a model in which a decision maker chooses between two certain payoffs realized at different, certain points in time. We assume that a decision is made by comparing the growth rate of wealth associated with each option. We then study temporal discounting in a deterministic setting. In this setting the model is consistent with the standard von Neumann-Morgenstern axioms.

The main finding is that discounting can be interpreted as growth rate optimization. We find that depending on the wealth dynamics assumed by the decision maker, growth rate optimization can be equivalent to hyperbolic discounting. It then predicts preference reversal. It can also be equivalent to a mixed case of hyperbolic and exponential discounting, which also implies preference reversal. Under multiplicative dynamics, we find that growth rate optimization reproduces standard exponential discounting. This reveals the standard form of discounting as one of many possible forms of discounting, each of which is optimal under a different type of wealth growth.

The model predicts another type of reversal under multiplicative dynamics. It can be growth-optimal for people of lower wealth to choose a small early payoff, whereas it is growth-optimal for wealthier individuals to hold out until the later larger payoff. This is inline with the experimental and empirical findings of Epper et al. (2018), that "individuals with relatively low time discounting are consistently positioned higher in the wealth distribution".

We emphasize that no knowledge of the decision-maker's psychology is required in our setup – only the postulate that she prefers her wealth to grow faster rather than slower. This postulate is enough for to predicting a decision maker's discount rates used in mainstream models. Yet, we also note that this paper discusses discounting from a theoretical perspective. An important

complementary step of this research would be comparing the theoretical predictions of the results to empirical and experimental results. In particular, the predicted discount factors and discount rates can be compared to results from controlled experiments. This is planned for future work.

References

Ainslie, George. 1992. Picoeconomics: The Strategic Interaction of Successive Motivational States within the Person. Cambridge University Press.

Berman, Yonatan, Ole Peters, and Alexander Adamou. 2017. "An Empirical Test of the Ergodic Hypothesis: Wealth Distributions in the United States." Available at SSRN.

Dasgupta, Partha, and Eric Maskin. 2005. "Uncertainty and Hyperbolic Discounting." American Economic Review, 95(4): 1290–1299.

Epper, Thomas, Ernst Fehr, Helga Fehr-Duda, Claus T. Kreiner, David D. Lassen, Søren Leth-Petersen, and Gregers N. Rasmussen. 2018. "Time Discounting and Wealth Inequality."

Grether, David M., and Charles R. Plott. 1979. "Economic Theory of Choice and the Preference Reversal Phenomenon." American Economic Review, 69(4): 623–638.

Laibson, David. 1997. "Golden Eggs and Hyperbolic Discounting." Quarterly Journal of Economics, 112(2): 443–478.

Lichtenstein, Sarah, and Paul Slovic. 1971. "Reversals of Preference Between Bids and Choices in Gambling Decisions." Journal of Experimental Psychology, 89(1): 46.

Lindman, Harold R. 1971. "Inconsistent Preferences among Gambles." Journal of Experimental Psychology, 89(2): 390–397.

Loomes, Graham, and Robert Sugden. 1983. "A Rationale for Preference Reversal." American Economic Review, 73(3): 428–432.

Meder, David, Finn Rabe, Tobias Morville, Kristoffer H. Madsen, Magnus T. Koudahl, Ray J. Dolan, Hartwig R. Siebner, and Oliver J. Hulme. 2019. "Ergodicity-Breaking Reveals Time Optimal Economic Behavior in Humans."

Peters, Ole, and Alexander Adamou. 2018. "The Time Interpretation of Expected Utility Theory."

Peters, Ole, and Murray Gell-Mann. 2016. "Evaluating Gambles Using Dynamics." Chaos: An Interdisciplinary Journal of Nonlinear Science, 26(2): 023103.

Rubinstein, Ariel. 2003. ""Economics and psychology"? The Case of Hyperbolic Discounting." International Economic Review, 44(4): 1207–1216.

Smith, Vernon L. 2003. "Constructivist and Ecological Rationality in Economics." American Economic Review, 93(3): 465–508.

Sozou, Peter D. 1998. "On Hyperbolic Discounting and Uncertain Hazard Rates." <u>Proceedings of the Royal Society of London B: Biological Sciences</u>, 265: 2015–2020.

Starmer, Chris. 2000. "Developments in Non-expected Utility Theory: The Hunt for a Descriptive Theory of Choice Under Risk." <u>Journal of Economic Literature</u>, 38(2): 332–382.

Thaler, Richard H. 2016. "Behavioral Economics: Past, Present, and Future." <u>American Economic Review</u>, 106(7): 1577–1600.

Tversky, Amos. 1969. "Intransitivity of Preferences." <u>Psychological Review</u>, 76(1): 31–48.

Tversky, Amos, Paul Slovic, and Daniel Kahneman. 1990. "The Causes of Preference Reversal." <u>American Economic Review</u>, 80(1): 204–217.

A Proofs

A.1 Proof of proposition 1

We assume three tuples $A \equiv (t_a, \Delta x_a)$, $B \equiv (t_b, \Delta x_b)$ and $C \equiv (t_c, \Delta x_c)$, where $t_a < t_b < t_c$. Given time t_0 $(< t_a)$ and an initial wealth $x(t_0)$, the vectors $\{t_0, x(t_0), t_a, \Delta x_a, t_b, \Delta x_b\}$ and $\{t_0, x(t_0), t_b, \Delta x_b, t_c, \Delta x_c\}$ are both CIPPs.

If $A \prec B$ and $B \prec C$ then $g_a < g_b$ and $g_b < g_c$. Also $t_a < t_b < t_c$. Therefore $\{t_0, x(t_0), t_a, \Delta x_a, t_c, \Delta x_c\}$ is a CIPP and $g_a < g_c$, so $A \prec C$.

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