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Concavity and Quasiconcavity in Economics*

Modern economic theory is replete with references to notions of concavity and quasiconcavity of functions. We may distinguish two main points of view: the algebraic, involving signs of various determinants and the geometric, involving examination of the function over some line segment. The differing viewpoints are not equivalent though they often tend to be used interchangeably. The present paper attempts to enumerate and clarify some of the concepts most frequently encountered in economic literature. It concludes with some results on the preservation of concavity properties under certain kinds of transformations.

I. Concavity

Because of the dual nature of results involving convexity, only definitions and theorems for concavity will be given, and a similar remark will apply to the next section as well. In this paper all functions will be assumed to be twice continuously differentiable and defined over a convex set.

First, we shall give some geometric definitions relating to concavity.

DEFINITION. The function $F(x)$ is *concave* if and only if $F[\lambda x + (1 - \lambda)y] \geqq \lambda F(x) + (1 - \lambda)F(y)$, for any $\lambda \in [0, 1]$ and any x and y in the domain of F.

DEFINITION. The function $F(x)$ is *strictly concave* if and only if $F[\lambda x + (1 - \lambda)y] > \lambda F(x) + (1 - \lambda)F(y)$, for any $\lambda \in (0, 1)$ and any distinct x and y in the domain of F.

From the above definitions it can be verified that if a function is strictly concave then it is concave. A linear function provides a counterexample to the converse.

A useful geometric characterization of a (strictly) concave function

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is that it always lies (strictly) under its tangent hyperplane through any point (except, of course, at the point of tangency). Alternatively, we may say that for a concave (strictly concave) function F, the linearization $F(y) + \nabla F(y) \cdot (x - y)$ at y never underestimates (always overestimates) $F(x)$ for any $x \neq y$ in the domain of F. This is made rigorous in the following lemmas.

LEMMA 1. *The function $F(x)$ is concave if and only if $F(x) - F(y) \leqq \nabla F(y) \cdot (x - y)$ for any x and y in the domain of F.*

Proof. [4, Theorem 2, pp. 84–85].

LEMMA 2. *The function $F(x)$ is strictly concave if and only if $F(x) - F(y) < \nabla F(y) \cdot (x - y)$ for any distinct x and y in the domain of F.*

Proof. [4, Theorem 2, p. 87].

Some notation is necessary to proceed with the determinantal definitions. Suppose $F(x) = F(x_1, x_2, \ldots, x_n)$, and let $F_i(x)$ denote the ith partial derivative of F evaluated at x. Similarly, let $F_{ij}(x)$ represent the second partial derivative with respect to the ith and jth arguments. The Hessian matrix of the second partial derivatives of the function F evaluated at x is denoted $H(x, F)$. Let $M_r(x, F)$, $r = 1, \ldots, n$, be the leading principal minor of $H(x, F)$ obtained by taking the determinant of those elements in the first r rows and first r columns of $H(x, F)$. For example $M_1(x, F) = |F_{11}(x)|$ and $M_n(x, F) = |H(x, F)|$.

DEFINITION. The function $F(x)$ is *strongly concave* if and only if $(-1)^r M_r(x, F) > 0$ for $r = 1, \ldots, n$.

DEFINITION. The function $F(x)$ is *weakly concave* if and only if $(-1)^r M_r(x, F) \geqq 0$ for $r = 1, \ldots, n$.

The above definitions correspond to the negative definiteness and negative semidefiniteness of the corresponding Hessian matrix. Because of this we have the following proposition.

PROPOSITION 1. *A function F is concave if and only if it is weakly concave.*

Proof. [3, Result 35, pp. 87–88].

PROPOSITION 2. *If F is strongly concave, then it is strictly concave.*

Proof. [3, p. 88].

In shorthand notation we may write strongly concave $\Rightarrow$ strictly concave $\Rightarrow$ concave $\Leftrightarrow$ weakly concave. The function $F(x) = -x^4$ can be used to demonstrate the irreversibility of the first direction of implication. However, since the two notions are often blurred, we should like to examine the relationship between them in greater detail. The following results have been published, and it is hoped that their inclusion here will increase their accessibility to economists.

PROPOSITION 3. *If the Hessian of F is negative semidefinite and is not identically zero on any segment, then F is strictly concave.*

Proof. [3, p. 88].

PROPOSITION 4. *If F is strictly concave, then its Hessian is negative definite except (possibly) on a nowhere dense subset of the domain.*

Proof. [2, Theorem VI, p. 69].

The converse of this proposition is not true, as is shown by the counterexample $F(x_1, x_2) = -(x_1)^2 [1 + \exp(x_2)]$, defined for $x_2 < 0$ (from [2, pp. 70–71]).

PROPOSITION 5. *A function F is strictly concave if and only if its Hessian is negative semidefinite and on any straight line segment L in the domain the set of points in L for which the tangent vector to L is a null eigenvector of the Hessian is nowhere dense in L.*

Proof. [2, Theorem VII, p. 72. Theorem VIII on the same page enables us to substitute "smooth curve" for "straight line segment" in the statement of Proposition 5.]

II. QUASI-CONCAVITY

Again we begin with some geometrically suggestive definitions.

DEFINITION. A function F is *quasiconcave* if and only if for any real number α, the set of x satisfying $F(x) \geqq \alpha$ is convex. Equivalently (see [5, p. 299]), F is *quasiconcave* if and only if for any x and y in the domain of F and any $\lambda \in [0, 1]$ we have $F[\lambda x + (1 - \lambda)y] \geqq \min\,[F(x), F(y)]$.

DEFINITION. The function F is *strictly quasiconcave* if and only if given any distinct x and y in its domain with $F(x) = F(y)$ and for any $\lambda \in (0, 1)$ we have $F[\lambda x + (1 - \lambda)y] > \lambda F(x) + (1 - \lambda)F(y) = F(x) = F(y)$.

DEFINITION. The function F is *semistrictly quasiconcave* if and only if for $F(x) < F(y)$ and $\lambda \in (0, 1)$, we have $F(x) < F[\lambda x + (1 - \lambda)y]$.

For purposes of economic relevance note that the definition of quasiconcavity does not rule out the possibility of thick iso-product (or iso-utility, i.e., indifference) curves. Strict quasiconcavity implies that the level curves contain no straight lines. This means that the level curves have no thickness and are neither straight lines nor do they possess even a straight line segment. The last definition does permit linear or partially linear level curves, but it does rule out "thick" level curves as long as the level is less than the maximum value the function can obtain.

It is possible to give some expression to these notions in terms of gradients, but not as completely as was the case in our discussion on concavity. Specifically we have the following lemmas.

LEMMA 3. *The function F is quasiconcave if and only if $F(x) - F(y) \geqq 0 \Rightarrow \nabla F(y) \cdot (x - y) \geqq 0$ for any x and y in the domain of F.*

Proof. [4, Theorem 4, pp. 134–136].

LEMMA 4. *The function F is strictly quasiconcave if $F(x) - F(y) \geqq 0 \Rightarrow \nabla F(y) \cdot (x - y) > 0$ for any distinct x and y in the domain of F.*

Proof. See proof for Lemma 5.

LEMMA 5. *The function F is semistrictly quasiconcave if $F(x) - F(y) > 0 \Rightarrow \nabla F(y) \cdot (x - y) > 0$ for any distinct x and y in the domain of F.*

Proof. [4, Theorem 5, pp. 143–144]. The same proof works for both lemmas, though Mangasarian's definition of strict quasiconcavity corresponds to our definition of semistrict quasiconcavity. Also, where he appeals to pseudoconcavity, we use the hypothesis of the lemma to be proved. For Lemma 5, the hypothesis is seen to be equivalent to the definition of pseudoconcavity on [4, p. 141]. In Lemma 4 the hypothesis is the defining characteristic of a strictly pseudoconcave function.

Analogous to the role of the Hessian matrix in Part I is that of the bordered Hessian matrix, which is obtained from $H(x, F)$ by bordering it with the first partial derivatives, $F_i(x)$:

$$B(x, F) = \begin{bmatrix} 0 & \nabla F(x) \\ (\nabla F(x))' & H(x, F) \end{bmatrix} = \begin{bmatrix} 0 & F_1(x) & F_2(x) & \cdots & F_n(x) \\ F_1(x) & F_{11}(x) & F_{12}(x) & \cdots & F_{1n}(x) \\ F_2(x) & F_{21}(x) & F_{22}(x) & \cdots & F_{2n}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_n(x) & F_{n1}(x) & F_{n2}(x) & \cdots & F_{nn}(x) \end{bmatrix}$$

Let $D_r(x, F)$, $r = 0, 1,..., n$, be the leading principal minor of the first $r + 1$ rows and columns of $B(x, F)$. Thus, for example, $D_0(x, F) = | 0 |$, $D_1(x, F) = -[F_1(x)]^2$, and $D_n(x, F) = | B(x, F)|$.

DEFINITION. A function F is *strongly quasiconcave* if and only if $(-1)^r D_r(x, F) > 0$ for $r = 1,..., n$.

DEFINITION. A function F is *weakly quasiconcave* if and only if $(-1)^r D_r(x, F) \geqq 0$ for $r = 1,..., n$.

PROPOSITION 6. *If a function is strongly quasiconcave, then it is quasiconcave; and if it is quasiconcave, then it is weakly quasiconcave.*

Proof. [1, Theorem 5, pp. 797–799].

Paraphrasing the second sentence of the proof referred to immediately above, we can state that strong quasiconcavity of a function F at a point x^0 means that x^0 is a strict local maximum of $F(x)$ subject to the constraint $\nabla F(x^0) \cdot x = \nabla F(x^0) \cdot x^0$. Suppose that F is strongly quasiconcave (and, hence, quasiconcave by Proposition 6), but not strictly quasiconcave. Then F is constant along some segment. But $\nabla F(x^0) \cdot (x - x^0) = 0$ for x and x^0 on this segment. Thus, x^0 would not be a strict local maximum subject to the constraint, and the contradiction means that F must be strictly quasiconcave. We have shown the following proposition.

PROPOSITION 7. *If F is strongly quasiconcave, then it is strictly quasiconcave.*

Again we can appeal to the function $F(x) = -x^4$ to exhibit a counterexample to the converse.

PROPOSITION 8. *If a function F is strictly quasiconcave, then it is semistrictly quasiconcave.*

Proof. Suppose $F(x) < F(y)$ and $F(x) \geqslant F[\lambda x + (1 - \lambda)y]$ for some λ, $0 < \lambda < 1$. If the equality holds, then by the strict quasiconcavity of F, $F(x) = F(v) < F[\mu x + (1 - \mu)v]$ for all $\mu \in (0, 1)$, where $v = \lambda x + (1 - \lambda)y$. Pick a μ, and let $w = \mu x + (1 - \mu)v$. Either $F(w) < F(y)$, $F(w) = F(y)$ or $F(w) > F(y)$. If the first case occurs, we can find a point p on the (open) line segment between y and v such that $F(p) = F(w)$. We can express v as a convex combination of p and w, and our above facts and the strict quasiconcavity of F will yield

$$F(v) < F(w) = F(p) < F(v), \text{ a contradiction.}$$

The other two possible cases are handled similarly.

If strict inequality holds, namely $F(x) > F(v)$, then we can find a point p on the line segment $\overline{vy}$ such that $F(x) = F(p)$ and derive a contradiction as before.

PROPOSITION 9. *If a function F is semistrictly quasiconcave then it is quasiconcave.*

Proof. [4, Theorem 3, p. 139]. We note again that in this reference the definition for strictly quasiconcave corresponds to what we have called semistrictly quasiconcave.

Using shorthand notation and "$q - c$" as an abbreviation for quasiconcave, we may summarize the preceding four propositions: strongly $q - c \Rightarrow$ strictly $q - c \Rightarrow$ semistrictly $q - c \Rightarrow q - c \Rightarrow$ weakly $q - c$.

The following two results are of a technical nature and do not seem to have been published. The situation is that of having $g(x) = f(F(x))$, where f is a real-valued function of a single variable, and F is a real-valued function of a vector. We wish to determine the relationship between minors of the Hessian and bordered Hessian of g to the minors of the Hessian and bordered Hessian of F. This may be of general interest, as for example when considering homothetic functions or when examining monotonic transformations of utility functions. For our purposes, the first result will show the invariance of strong or weak quasiconcavity under positively monotonic transformations ($f' > 0$), and the reversal of strong or weak quasiconcavity into strong or weak quasiconvexity under a negatively monotonic transformation ($f' < 0$). (The invariance under a postitively monotonic transformation of the quasiconcavity, strict quasiconcavity, or semistrict quasiconcavity of a function, follows directly from Lemmas 3, 4 and 5, respectively.) The second result shows that the strong or weak concavity of g depends not only on whether F is strongly or weakly concave, but on the signs of f' and f'' as well.

THEOREM 1. *At any point x, $D_r(x, g)$ is equal to $(f')^{r+1} \cdot D_r(x, F)$, for $r = 0, 1, \ldots, n$.*

Proof. By definition, $g_i(x) = f'(F(x)) \cdot F_i(x)$ and $g_{ij}(x) = f'(F(x)) \cdot F_{ij}(x) + f''(F(x)) \cdot F_i(x) \cdot F_j(x)$. Deleting the arguments of the functions, we see

$$D_r(x, g) = \begin{vmatrix} 0 & f'F_1 & f'F_2 & \cdots & f'F_r \\ f'F_1 & f'F_{11} + f''F_1F_1 & f'F_{12} + f''F_1F_2 & \cdots & f'F_{1r} + f''F_1F_r \\ f'F_2 & F'F_{21} + f''F_2F_1 & f'F_{22} + f''F_2F_2 & \cdots & f'F_{2r} + f''F_2F_r \\ \vdots & \vdots & \vdots & & \vdots \\ f'F_r & f'F_{r1} + f''F_rF_1 & f'F_{r2} + f''F_rF_2 & \cdots & f'F_{rr} + f''F_rF_r \end{vmatrix}.$$

Factor out f' from the first row. Then subtract $f''F_j$ times the first row from the $(j+1)$th row. This does not change the value of the determinant and gives us

$$D_r(x, g) = \begin{vmatrix} 0 & F_1 & F_2 & \cdots & F_r \\ f'F_1 & f'F_{11} & f'F_{12} & \cdots & f'F_{1r} \\ f'F_2 & f'F_{21} & f'F_{22} & \cdots & f'F_{2r} \\ \vdots & \vdots & \vdots & & \vdots \\ f'F_r & f'F_{r1} & f'F_{r2} & \cdots & f'F_{rr} \end{vmatrix} \cdot f'.$$

Factor f' out of each of the last r rows to obtain our result

$$D_r(x, g) = (f')^{r+1} \cdot D_r(x, F).$$

THEOREM 2. $M_r(x, g) = (f')^r M_r(x, F) - (f')^{r-1} f'' \cdot D_r(x, F)$.

Proof. The method of this proof is brute force, so it is wise to take a moment to explain some of the symbols which follow. The Greek letter sigma, σ, is an index for the permutations of the integers 1 to r, with the conventional sign association with even and odd permutations.

We shall use the Greek letter rho, ρ, as an index for the set of one-to-one mappings of the set of the $r-1$ integers $\{1, 2, \ldots, j-1, j+1, \ldots, r\}$ onto the set $\{1, \ldots, i-1, i+1, \ldots, r\}$. We may picture such a transformation as

$$\begin{matrix} 1 & 2 & \cdots & j-1 & \begin{pmatrix} j \\ \downarrow \\ i \end{pmatrix} & j+1 & \cdots & r \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \rho(1) & \rho(2) & & \rho(j-1) & & \rho(j+1) & & \rho(r) \end{matrix}$$

with the column in parentheses omitted. For any j and i, there are $(r-1)!$ permutations (of the first r integers) with $\sigma(j) = i$. Each of these has a natural correspondence with one of the ρ, as suggested by the inclusion of the $j \to i$ column above. Thus, for example, we would have

$$\sum_{\rho} \prod_{\substack{t \neq j \\ \rho(t) \neq i}} F_{t\rho(t)} = \sum_{\sigma:\sigma(j)=i} \prod_{t \neq j} F_{t\sigma(t)} .$$

The sign of the mapping ρ is $(-1)^q$, where q is the number of interchanges of adjacent pairs of numbers of the second row of the pictorial representation of ρ in the above paragraph necessary to place these integers in their natural order (with i omitted). Because ρ is one-to-one onto, for every term to the left of i (i.e., to the left of the column in parentheses) that upon rearrangement is to the right of i, there is another term to the right that ends up on the left. Thus, we could reinsert the $j \to i$ column and rearrange around it, which would not affect the sign of ρ since, as we have just seen, that column must be involved in pairs of interchanges. We would arrive at a situation like the following:

$$\begin{array}{ccccccccccc} 1 & 2 & \cdots & i-1 & i & \cdots & j-1 & j & j+1 & \cdots & r, \\ 1 & 2 & \cdots & i-1 & i+1 & \cdots & j & i & j+1 & \cdots & r. \end{array}$$

To compare the signs of the corresponding σ and ρ, we now need to determine the number of additional interchanges necessary to get i in its proper position. This can be seen to be equal to $|i-j|$. Thus for corresponding ρ and σ we know that $sgn\ \rho = (-1)^{|i-j|}$ sgn σ. Continuing the example of the previous paragraph, we may state

$$\sum_{\rho} (\operatorname{sgn} \rho) \prod_{\substack{t \neq j \\ \rho(t) \neq i}} F_{t\rho(t)} = \sum_{\sigma:\sigma(j)=i} (-1)^{|i-j|} (\operatorname{sgn} \sigma) \prod_{t \neq j} F_{t\sigma(t)} . \tag{1}$$

This (finally) ends the preliminaries and we may proceed to the proof.

$$\begin{aligned} M_r(x, g) &= \sum_{\sigma} (\operatorname{sgn} \sigma) \prod_{i=1}^{r} (f' F_{i\sigma(i)} + f'' F_i F_{\sigma(i)}) \\ &= \sum_{\sigma} (\operatorname{sgn} \sigma)(f' F_{1\sigma(1)} + f'' F_1 F_{\sigma(1)}) \\ &\quad \times (f' F_{2\sigma(2)} + f'' F_2 F_{\sigma(2)}) \cdots (f' F_{r\sigma(r)} + f'' F_r F_{\sigma(r)}) \\ &= \sum_{\sigma} (\operatorname{sgn} \sigma) \prod_{i=1}^{r} f' F_{i\sigma(i)} + \sum_{\sigma} (\operatorname{sgn} \sigma) \sum_{i} f'' F_i F_{\sigma(i)} \cdot \prod_{\substack{j=1 \\ j \neq i}}^{r} f' F_{j\sigma(j)} \\ &\quad + \text{other terms.} \end{aligned}$$

The first term is seen to be $(f')^{r} M_r(x, F)$ by definition. Let us look at the "other terms." They all involve a factor of $(f'')^q$, with $q \geqq 2$. Thus, from each term we can factor out $F_i F_{\sigma(i)}\ F_j F_{\sigma(j)}$ for some i and j, $i \neq j$. Now for any permutation σ, there is another permutation $\hat{\sigma}$ such that $\sigma(i) = \hat{\sigma}(j)$, $\sigma(j) = \hat{\sigma}(i)$, and $\sigma(k) = \hat{\sigma}(k)$ for $k \neq i, j$. We also know (sgn σ) $= -$(sgn $\hat{\sigma}$), since we have made one interchange (i.e., $\langle\sigma(1),\ldots, \sigma(r)\rangle$ can be reordered as $\langle\hat{\sigma}(1),\ldots, \hat{\sigma}(r)\rangle$, or vice versa, with $2\,|i-j|-1$ interchanges of adjacent pairs of numbers, as may be verified by the reader). Also $F_i F_{\sigma(i)} F_j F_{\sigma(j)} = F_i F_{\hat{\sigma}(i)} F_j F_{\hat{\sigma}(j)}$, and the rest of the term is unchanged. Since the set of all permutations can be divided in this way, and since they appear with opposite signs, all "other terms" vanish.

It remains to be shown that

$$-\sum_{\sigma} (\operatorname{sgn} \sigma) \sum_{i} F_i F_{\sigma(i)} \prod_{\substack{j=1 \\ j \neq i}}^{r} F_{j\sigma(j)} = D_r(x, F). \tag{2}$$

Expanding $D_r(x, F)$ by cofactors of the first row and then of the first column yields the following expression:

$$D_r(x, F) = \sum_{i=1}^{r} (-1)^i F_i \sum_{j=1}^{r} (-1)^{j+1} F_j \sum_{\rho} (\operatorname{sgn} \rho) \prod_{\substack{t \neq j \\ \rho(t) \neq i}} F_{t\rho(t)} \,. \tag{3}$$

To demonstrate equality of the expressions in (2) and (3), fix i at say i^* and consider terms involving $F_{i^*}F_{i^*}$.
From (2) we get

$$-\sum_{\sigma:\sigma(i^*)=i^*} (\operatorname{sgn} \sigma) F_{i^*}F_{i^*} \prod_{\substack{j=1 \\ j \neq i^*}} F_{j\sigma(j)} \,,$$

and from (3)

$$(-1)^{i^*} F_{i^*}(-1)^{i^*+1} F_{i^*} \sum_{\rho} (\operatorname{sgn} \rho) \prod_{\substack{t \neq i^* \\ \rho(t) \neq i^*}} F_{t\rho(t)} \,,$$

which equals

$$(-1)^{2i^*+1} F_{i^*}F_{i^*} \sum_{\rho} (\operatorname{sgn} \rho) \prod_{\substack{t \neq i^* \\ \rho(t) \neq i^*}} F_{t\rho(t)} \,,$$

and by (1) this equals

$$-\sum_{\sigma:\sigma(i^*)=i^*} (\operatorname{sgn} \sigma) F_{i^*}F_{i^*} \prod_{t \neq i^*} F_{t\sigma(t)} \,,$$

and this is what we had before with only changes in indices.

Now fix j at $j^* \neq i^*$. From (2), terms involving $F_{i^*}F_{j^*}$ can arise in three ways:

$$-\sum_{\substack{\sigma:\sigma(i^*)=j^* \\ \sigma(j^*)=i^*}} (\operatorname{sgn} \sigma) \left[F_{i^*}F_{j^*} \prod_{\substack{j=1 \\ j \neq i^*}}^{r} F_{j\sigma(j)} + F_{j^*}F_{i^*} \prod_{\substack{j=1 \\ j \neq j^*}}^{r} F_{j\sigma(j)}\right];$$

$$-\sum_{\substack{\sigma:\sigma(i^*)=j^* \\ \sigma(j^*)\neq i^*}} (\operatorname{sgn} \sigma) F_{i^*}F_{j^*} \prod_{j \neq i^*} F_{j\sigma(j)} \,;$$

$$-\sum_{\substack{\sigma:\sigma(i^*)\neq j^* \\ \sigma(j^*)=i^*}} (\operatorname{sgn} \sigma) F_{j^*}F_{i^*} \prod_{j \neq j^*} F_{j\sigma(j)} \,.$$

Regrouping terms enables us to write this as

$$-\sum_{\sigma:\sigma(i^*)=j^*} (\operatorname{sgn} \sigma) F_{i^*}F_{j^*} \prod_{j \neq i^*} F_{j\sigma(j)}$$

$$-\sum_{\sigma:\sigma(j^*)=i^*} (\operatorname{sgn} \sigma) F_{i^*}F_{j^*} \prod_{j \neq j^*} F_{j\rho(j)} \,.$$

Meanwhile the expansion of $D_r(x, F)$ in (3) yields

$$(-1)^{i^*+j^*+1} F_{i^*}F_{j^*} \left[\sum_{\rho} (\operatorname{sgn} \rho) \prod_{\substack{t\neq i^* \\ \rho(t)\neq j^*}} F_{t\rho(t)} + \sum_{\rho} (\operatorname{sgn} \rho) \prod_{\substack{t\neq j^* \\ \rho(t)\neq i^*}} F_{t\rho(t)}\right].$$

Noting that $i^* + j^* + 1 + | i^* - j^* |$ equals either $2i^* + 1$ or $2j^* + 1$ and, thus, is always odd, and using (1) suffices to show the equality of the foregoing expressions.

This completes the proof of the theorem.

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