Microeconomics

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Exercise 1.1: Rational preferences

Let $\succsim$ be a rational preference on X (i.e. complete and transitive). Show that:

1. $\succ$ is irreflexive (writing $a \succ a$ is not possible) and transitive.

   a) $a \succ b$ and $b \succ a$ so $b \succsim a$ but $a \succ b$, so $\neg b \succsim a$: contradiction!

   b) If $a \succ b$ and $b \succ c \rightarrow a \succsim$ b and $b \succsim c$ then we have that $a \succsim c$, which indicates that a $\succ$ c or a $\sim$ c.

   By hypothesis, we have that: $\neg c \succsim b$; if $c \succsim a$ (in the case where a $\sim$ c), by transitivity of $\succsim$, we have that $c \succsim b$: contradiction.

2. $\sim$ is reflexive, transitive and symmetric.

   a) Immediate

   b) If $\begin{cases} x \succsim y \\ y \succsim z \end{cases}$ and $\begin{cases} y \succsim z \\ z \succsim y \end{cases} \Rightarrow \begin{cases} x \succsim z \\ z \succsim x \end{cases}$

   c) If $x \sim y \Rightarrow \begin{cases} x \succsim y \\ y \succsim x \end{cases} \Rightarrow y \sim x$

3. if $x \succ y \succsim z$, then $x \succ z$.

$x \succsim y \succsim z \Rightarrow x \succsim z$ by transitivity.

By contradiction, if z $\succ$ x, by transitivity, z $\succ$ y. We have by the hypothesis that z $\sim$ y, and by transitivity of $\sim$ we have that $y \sim x$: which is a contradiction.

Exercise 1.2: Representation of preferences

Let $u : X \rightarrow R$ be utility function which represents the preferences on $\succsim$ on X, such that. $u(x) \geq u(y) \iff x \succsim y, \forall x, y \in X$

Show that for all functions $f : R \rightarrow R$ which are strictly increasing $f \circ u$, also represents $\succsim$. What happens if f is increasing but not strictly?

a) $f(u(x)) \succsim f(u(y)) \iff u(x) \succsim u(y) \iff x \succsim y$

b) In this case, the first equivalence is false: $u(x) \geq u(y) \rightarrow f(u(x)) \geq f(u(y))$ holds always but the inverse relation does not. Example: $f =$constant $, u(z) = z \forall z \in R, x = 0, y = 1$ $f(u(x)) = f(u(y))$ and $u(x) < u(y)$

Exercise 1.3: preferences on a finite set

Let X be a finite set and $\succsim$. Show that their exists a utility function $u : X \rightarrow R$ which represents the preferences.

By induction. Let $M_1 = (x \in X | y \succeq x,\ \forall y \in X) \neq \emptyset$. We let $u(z) = 1 \forall z \in M_1$. If $M_1 = X$ we are done, Otherwise $M_1 \neq X$ and let $X_1 = X \setminus M_1$. We let $u(z) = 2 \forall z \in M_2 = (x \in X | y \succeq x,\ \forall y \in X_1)$ and repeat.

This algorithm takes at most $|X|$ stages and constructs a representative utility function of $\succeq$ with the values of $\mathbb{N}$.

Remark: While X is countable we can represent $\succeq$ by a utility function u: $x \to (0, 1)$.

Exercise 1.4: Weak axiom of revealed preferences

Let $X = \{x, y, z\}$ be an ensemble of alternatives, $G = \{\{x, y\}, \{x, y, z\}\}$ a sub set of X and let C be a function of choice defined on G so that $C(\{x, y\}) = \{x\}$. Show that if C satisfies the weak axiom of revealed preferences, then $C(\{x, y, z\})$ is equal to $\{x\}\{z\}, \{x, z\}$

Reminder that C verifies the weak axiom of revealed preferences if, when x is revealed to be equally preferred to y, y cannot be revealed to be strictly better than x. Said otherwise, there does not exist an $A, B \in G$ so that $x, y \in A \cap B, x \in C(A), y \in C(B)$ and $x \notin C(B)$

Suppose that y $\in C(\{x, y, z\})$ and let $A = \{x, y\}$ and $B = \{x, y, z\}$

Therefore we have that $y, x \in B \cap A$ $y \in C(B)$ $x \in C(A)$

According to the (WA) this implies that $y \in C(A)$, a contradiction. We need only verify that C(B) $= \{x\}$ or $\{x\}$ or $\{x, z\}$ does not contradict the (WA). But this is trivial because $A \cap B = \{x, y\}$ said otherwise, $z \notin A \cap B$ which means that it can't serve as a counterexample to the axiom.

Exercise 1.5: Continuity of preferences

Debreu's Theorem states that $\exists u : X \to \mathbb{R}$, a continuous function such that: $u(x) \geq u(y) \Leftrightarrow x \succeq y$.

Intermediate value Theorem states that:

$$\begin{cases} f : [0,1] \to \mathbb{R} \text{ is continous} \\ f(1) \geq t \geq f(0) \end{cases} \quad \Rightarrow \exists c \in [0,1] \text{ s.t.} f(c) = t$$

$$\text{Applying to } f(c) = \begin{cases} [0,1] \to \mathbb{R} \text{ is continuous} \\ u(cx + (1-c)x) \\ \text{with } t = u(y) \end{cases},$$

We get that there is a c such that $f(c) = t$, and then we deduce the existence of $m = cx + (1-c)z \in [z, x]$ such that $u(m) = u(y)$. We conclude that $m \sim y$.