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# SIGNALING AND REPUTATION IN REPEATED GAMES, II: STACKELBERG LIMIT PROPERTIES

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Abstract Repeated signaling games are a model of reputation, where past actions influence current expectations of actions via type beliefs. This paper studies continually separating equilibria of infinite horizon repeated signaling games, extending Roddie (2011b).

With a continuum of persistent types, if the signaler is patient he gains Stackelberg leadership, subject to separating from the lowest type. This remains true if the respondent cares directly about the signaler's type in addition to his actions. If the signaler has additively separable utility and is impatient, his actions maximize a discounted form of Stackelberg payoffs. In contrast to the usual theoretical approach to reputation there is no reliance on behavioral types.

## 1. INTRODUCTION

### 1.1. *Reputation: directional and exact approaches*

Reputation is a basic aspect of repeated interaction, and correspondingly a central issue in the theory of repeated games. In most economic analysis, it is the link between what an agent has done and what he is expected to do in future. This link gives rise to "reputational incentives" since actions have consequences for reputation. Firms have an incentive to provide high quality services since so that customers will expect this in future, improving sales. Governments may abstain from expropriating property so that investors do not expect this in future. So reputation is fundamental to important markets.

Game theory has provided two ways to understand reputation. The first is directional: if you have done "better" actions in the past, you will be expected to do "better" actions in the future. The second is "exact", i.e. for a particular action: if you do the action $x$ enough, eventually others will come to expect $x$.[1]

In both approaches reputation and commitment are connected. Call the

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[1]This paper is based on chapter 1 of my PhD thesis, which analyzed a special case of the model. I am grateful to Wolfgang Pesendorfer for advice.

[1]These two notions of reputation emerged early on: Milgrom and Roberts (1982a,b) proposed both as explanations of predatory pricing.

reputation-builder $P1$.[2] Moving from the static simultaneous move game to the Stackelberg game (where $P1$ moves first) makes $P1$ care about the effect of the action on the best response; and directional reputation has the same effect in repeated interaction, so that commitment and reputation move $P1$'s action in the same direction. In the exact approach, the connection is that $P1$ has the ability to commit in the long run to any action $x$.

Both approaches have typically relied on incomplete information, where the actions of an agent convey information about what "type" of agent he is, which then generates expectations of future actions. The exact approach argues that if $P1$ takes an action $x$ repeatedly, he should eventually be expected to play $x$ with probability converging to 1. Demonstrating this conclusion has required the possibility of $P1$ being a *behavioral* type which plays $x$ regardless of strategic considerations, which a *normal* (strategic) player can "pool" with. This approach is represented in a body of literature beginning with Kreps and Wilson (1982), Milgrom and Roberts (1982b), and Fudenberg and Levine (1989), the last paper expressing this central argument.

The conclusion is that if a normal type of $P1$ is sufficiently patient, he gains commitment power, and his average discounted utility will be close to his Stackelberg payoff. By expressing an intuitive argument, with weak assumptions on the stage game, and generating a property that is generalizable across games, the behavioral type approach has dominated theoretical work on reputation.

The dependence on behavioral types is a disadvantage of the approach. These are not natural to rational models, and it is questionable whether they should be necessary for reputation: shouldn't a manager that is known to be astute and responsive to circumstances be able to establish a reputation for his firm?[3] Secondly, commitment type models are not in general tractable. Adapted specifically to demonstrate an argument, the limit case of complete patience can be understood, but not much more. Also the level of patience of the other player must be zero or at least small compared with $P1$.[4] Finally the result is a statement about one quantity, expected discounted utility for the long-run agent; short run and very long run behavior is not determined, and the ability to maintain a reputation may be lost eventually.

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[2] This paper studies one-sided reputation.

[3] While behavioral-type probabilities are allowed to be low, if this is true the discount factor required for payoffs to be near Stackelberg may have to be very close to 1.

[4] Precisely, $1-\delta_{P2}$ must be large compared to $1-\delta_{P1}$.

By contrast, signaling models - representing the directional approach to reputation - do not found reputation on the possibility of non-strategic behavior. Here $P1$, the signaler, has a type which predisposes him to higher or lower actions and typically corresponds to some natural economic variable that is private information. Taking a higher action generates higher beliefs about $P1$'s type, which results in higher expectations about what he will do in the future.

These models are also tractable outside of polar cases, and so have become the main applied approach to reputation. Many models have been two-stage games, with signaling in the first stage; these are solveable in the sense that there exists a unique equilibrium that is for various reasons focal, the dominant separating equilibrium, and this can be simply characterized. Roddie (2011b) shows this tractability extends to arbitrary finite horizons, with continually separating equilibria replacing separating equilibria.[5]

But while the signaling models of reputation, including repeated signaling models, have found directional effects of reputation, they have not quantified these effects in a clear way that is generalizable across settings. They have noted that incomplete information can be advantageous to the signaler, but whether it is and to what extent is left unclear. In the predominant two-stage games, even though they are calculable, the quantitative results found are strongly affected by the simplification of two stages, so that the main message is the direction and not the magnitude of reputational effects.

### 1.2. *Contribution*

This paper shows that a signaling approach to reputation can generate Stackelberg limiting properties. This combines the advantages of a tractable model, with rational types and a directional and calculable structure, with the strong commitment results characteristic of the behavioral type approach. So it strengthens the case for understanding reputational questions in terms of signaling - at least where reputation is for a one-dimensional action, where the directional notion can apply.

To obtain these strong results from a signaling model of reputation we need:

1. Many signaling periods. This allows for a stable level of signaling.
2. A continuum of types (or close to a continuum). This allows an interval of ac-

[5] See this paper for a survey of other repeated signaling models.

tions to be taken in the limit, with the signaler able to latch on to any of them. If the signaler's type is highly persistent, then this is like commitment.

The conclusion is that if the signaler's type is highly persistent, and his discount factor is close to 1, he acts as if he were a Stackelberg leader in the signaling game, subject to the requirement of separating from the lowest type, according to the lowest type's Stackelberg payoff.[6]

Note that since there is continual separation of types, reputation not only survives knowledge that the signaler is strategic, but knowledge of the signaler's exact type. So commitment power does not depend on initial incomplete information at all.[7]

*Reputation under impatience*

In practice agents are not patient, and in a model with discounting we gain from understanding what happens with discount factors significantly different from 1. This is also an important comparative static for reputation: how does reputational ability depend on the discount factor, which can be dependent on the time between interactions or the probability of replacement.

The repeated signaling model is solvable under arbitrary discount factors of both players. Moreover, if the respondent's action enters the signaler's payoff additively, Stackelberg results extend to the case when the signaler's discount factor is less than 1. In this sense the tractability of the model allows it to generate even stronger results than the behavioral type models.

The signaler's payoff is additively separable if it can be written as $u_{P1}(\theta, x, y) = u_X(\theta, x) + u_Y(\theta, y)$, where $\theta$ is type, $x$ is the signaler's action, and $y$ the respondent's action. In this case we find that if type is highly persistent, the signaler acts as if he were a Stackelberg leader in a signaling game with discounted payoff $u_X(\theta, x) + \delta_{P1} \cdot u_Y(\theta, y)$, where $\delta_{P1}$ is his discount factor.

The discount factor of the respondent never affects the results.

[6]Note that not only is signaling responsible for moving the signaler's action up to the Stackelberg action, but there can be residual signaling if the respondent cares directly about the signaler's type.

[7]The main difference is the fact that type has a (possibly small) variability over time in the repeated signaling model. It may be possible to adjust behavioral type models to allow for this, but switching between being a normal and a behavioral type is less plausible than switching between degrees of preference.

### 1.3. *Argument and Results*

The analysis here extends Roddie (2011b), which studies finitely repeated signaling games without explicit consideration of limit cases. This paper studies infinitely repeated signaling games and their limit properties.

There are simultaneous moves in the stage game, and the value of signaling comes from the effect on future stages. The substantive part of the argument is that under appropriate limits, actions taken in the repeated signaling game satisfy the incentive compatibility conditions of the static signaling game where the signaler moves first. And we know (Mailath (1987)) that with a continuum of types, these incentive compatibility conditions uniquely determine the separating equilibria of a signaling game - up to an initial condition, the action of the lowest type, which we can known directly. So we have pinned down limiting strategies.

This argument relies on some technical apparatus. First we show existence of continually separating Markov equilibria in an infinite horizon game. Second, the set of equilibria is shown to be (upper semi-)continuous in the parameters of the game, most importantly the signaler's discount factor and the Markov process on types. Both statements both rely on continuity of the value function iterator in both its argument and in the parameters of the game.

So we know a limit of equilibria of a convergent sequence of games must be an equilibrium of the limit game. We formulate the problem in such a way that this limit game is valid, even with a discount factor of 1, and then apply the substantive methods above to analyze this limit game.

*Equilibria of infinite games and continuity in the parameters*

The value function $V_{t+1}(\hat{\theta},\theta)$ in period $t+1$, a function of current type $\theta$ and type beliefs $\hat{\theta}$, determines the incentive to signal in period $t$. It gives a value of signaling for any given expected action of the respondent in period $t$, and we take the Riley equilibrium of this signaling game. Finally we find the equilibrium action of the respondent by a fixed point argument. This gives us a value function in period $t$.

In a Markov perfect equilibrium, there is no dependence on $t$, and such an equilibrium is a fixed point of value function iteration. To show existence of this fixed point we want to work with a compact space and a continuous value function iterator.

As in Roddie (2011b) we recurse not on $V_t$ but on $v_t^{\sim}$, the value of having signaled $\theta'$

over $\theta_{\min}$: $v_t^{\sim}(\theta',\theta) := V_t(\psi(\theta'),\theta) - V_t(\psi(\theta_{\min}),\theta)$. This is useful because it simplifies the functions and keeps $v_t^{\sim}$ bounded even as $\delta_{P1} \to 1$. We add to Roddie (2011b) by showing that $v_t^{\sim}$ is not only continuous, which we need to analyze the derived signaling games, but has a Lipschitz constant in $\theta$. This combination of properties results in a compact space.

To show continuity of the value function iterator, we decompose it into a number of component parts and show continuity for each individual map. This gives us existence of a fixed point for finite types, and a continuity argument extends this to a continuum of types.

Since we also have continuity of value function iteration in the parameters of the game, the set of equilibria is upper semi-continuous in these parameters. We shall study a limit of patience $\delta_{P1} \to \delta_{P1}^*$, with $\delta_{P1}^* = 1$ an important case, and of the Markov process $\psi$ approaching a fixed point $\psi_{id} : \theta \mapsto [\theta]$ where types stay fixed. The continuity result implies a limit of such equilibria must be an equilibrium under $\delta_{P1} = \delta_{P1}^*$ and $\psi = \psi_{id}$.

*The Stackelberg game*

In the game we study there are simultaneous moves in the stage game. Take a single stage game and alter it so that the signaler moves first. This is the “Stackelberg game”, which is a normal signaling game. If the signaler takes action $x$ and signals type $\theta'$, the respondent best responds to $x$ and $\theta'$.

In the “pure reputation” case, the respondent only cares about expected actions of the signaler. In the repeated game signaling incentives occur because type is used to indicate expected actions in future; in the Stackelberg game there are no signaling incentives, and the unique perfect Bayesian equilibrium involves Stackelberg actions for the signaler. This is an edge case, but also the standard case.

In general if the respondent cares also about type (the “combined case”), there are strict signaling incentives in the Stackelberg game.

*Limit incentive compatibility*

Consider the limit case where $\delta_{P1} = 1$ and types stay the same with probability 1. When a type $\theta$ has just signaled being type $\theta$, then current beliefs are $[\theta]$. The signaler will take some action $\sigma(\theta)$. The respondent's action will be the best response $y(\theta)$ to type $\theta$ and action $\sigma(\theta)$.

Suppose that currently the type belief is $[\theta']$ but the signaler's actual type is $\theta$. In equilibrium $\theta$ will signal his true type immediately, giving some outcome $O$ in that period and $(\sigma(\theta), y(\theta))$ in the next. If instead $\theta$ pretends to be $\theta'$, we have outcome $(\sigma(\theta'), y(\theta'))$ in that period, and $O$ in the next. (Subsequent play is identical.)

Since equilibrium behavior is optimal and $\delta_{P1} = 1$, $\theta$ must weakly prefer $(\sigma(\theta), y(\theta))$ to $(\sigma(\theta'), y(\theta'))$. This means that the strategy $\sigma$ satisfies the incentive compatibility conditions for a separating equilibrium of the Stackelberg game.

With a continuum of types, these IC conditions pin down $\sigma$ when combined with the lowest type's signal $\sigma(\theta_{\min})$ (Roddie (2011a), after Mailath (1987)).[8] We know that the lowest type acts as if there were no signaling incentives, so $\sigma(\theta_{\min})$ is the complete information Nash equilibrium of the stage game. The lowest type would be indifferent to committing to some action $\bar{x}$ above his Stackelberg complete information action. Subsequent types take the Riley equilibrium of the Stackelberg game subject to the restriction of being above $\bar{x}$.[9]

*Results*

In the pure reputation case, all types except $\theta_{\min}$ take (under patience and type-change limits) Stackelberg actions subject to being at least $\bar{x}$. In the combined case the types above the lowest follow the usual differential equation for one-shot signaling with the signaler moving first, with initial value $\bar{x}$.[10] See Figure 1 for graphs illustrating the results.

The lowest type's action gives the difference between equilibria of the Stackelberg signaling game and repeated signaling game. In separating equilibria of both forms of game, the lowest type acts as if there were complete information. In the static signaling game, this means taking his Stackelberg action. In the repeated signaling game, with simultaneous moves, it means takes his complete-information Nash equilibrium action, which may be strictly lower, causing a discontinuity at that point.

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[8]In the combined case, there are strict signaling incentives and the result follows immediately. In the pure reputation case, lack of strict signaling incentives means that we have to show separately that $\sigma(\theta)$ lies weakly above the Stackelberg action for $\theta > \theta_{\min}$. To do this we need to make regularity assumptions on how the Markov process converges to $\psi_{id}$. The result is given in Lemma 5 and proved in the supplementary material.

[9]Where the Riley equilibrium is generalized (Roddie (2011a)) to allow for pooling when there are no signaling incentives.

[10]Without differentiability, the Riley equilibrium of the Stackelberg signaling game, restricted to $x \geq \bar{x}$.

*Impatience*

If patience is less than 1, we can get clean results in the additively separable case when $u_{P1} = u_X(\theta, x) + u_Y(\theta, y)$. Now the signaler's strategy is a function $\sigma(\theta)$ of $\theta$ only, and does not depend on current beliefs.

By mimicking type $\theta'$, the signaler changes payoffs in the current and next periods, but only $u_X$ changes in the current period and only $u_Y$ changes in the next period, where it is discounted by $\delta_{P1}$.

For the equilibrium action $\sigma(\theta)$ to be optimal, $\theta$ must prefer $(\sigma(\theta), y(\theta))$ to $(\sigma(\theta'), y(\theta'))$ according to the utility function $u_X(\theta, x) + \delta_{P1} u_Y(\theta, y)$. Everything said above about the $\delta_{P1} = 1$ case applies here, except we now use this discounted utility function in the Stackelberg game. So under additive separability, the reputational first-mover property is diminished by impatience, in a continuous and calculable way.

## 2. MODEL

There are two players, a signaler (P1) and a respondent (P2) who play over an infinite horizon. In each period both players act simultaneously; actions are observable. The signaler takes actions from $X = [x_{\min}, x_{\max}] \subseteq \mathbb{R}$; the respondent from an interval $Y \subseteq \mathbb{R}$.

### *Types and the Markov process*

At each stage the signaler has a type $\theta_t \in \Theta$, where either $\Theta = \bar{\Theta} := [\theta_{\min}, \theta_{\max}]$ or $\Theta$ is finite with minimal and maximal elements $\{\theta_{\min}, \theta_{\max}\}$. We shall use the measure $\mu$ on $\Theta$: for finite $\Theta$, $\mu$ gives positive measure to each point, and for continuum $\Theta$, $\mu$ is Lebesgue measure, modified to give positive measure to $\theta_{\min}$ and $\theta_{\max}$.

Type varies according to the Markov process $\psi : \Theta \to \triangle\Theta$, with $\psi$ continuous and strictly increasing. We assume that $\psi$ has the following regularity properties:

ASSUMPTION 1

1. *$\mathbb{E}[\psi(\theta)]$ has Lipschitz constant 1, where $\mathbb{E}[m] := \int \theta \, \mathrm{d}m(\theta)$.*
2. *The measure $\mu^* : S \mapsto \int \psi(\theta)(S) \, \mathrm{d}\mu(\theta)$ is absolutely continuous (w.r.t. $\mu$).*
3. *For each $\theta$, the restriction of $\psi(\theta)$ to $\Theta \setminus \{\theta\}$ is absolutely continuous (w.r.t. $\mu$).*

Assumption 1 part 1 asserts that if the current type is increased by $\delta$, the expected type in the next period is increased by at most $\delta$. This plays a role in preserving Lipschitz constants. Part 2 asserts that if the current distribution of types is given by $\mu$, the subsequent distribution of types is absolutely continuous with respect to $\mu$. Part 3 asserts that $\psi(\theta)$ has no point masses, except possibly on $\{\theta_{\min}, \theta, \theta_{\max}\}$.

Strategic incentives in the game come from increasing difference assumptions[11]. To make limiting arguments will need to place Lipschitz conditions on value functions, and this requires bounding some increasing differences:

DEFINITION (Roddie (2011a)) Suppose $f(a,b) \in \mathbb{R}$, where $a$ and $b$ lie in $\mathbb{R}$. $f$ has an increasing differences bound of $\lambda$ if for $a_1 \leq a_2$ and $b_1 \leq b_2$:[12]

$$f(a_2,b_2) + f(a_1,b_1) - f(a_1,b_2) - f(a_2,b_1) \leq \lambda (a_2 - a_1)(b_2 - b_1)$$

*Assumptions for the signaler*

The signaler has discounted utility $\sum_i \delta_{P1}^i u_{P1}(\theta_i, x_i, y_i)$, with $0 < \delta_1 < 1$. The signaler's payoff is supermodular, and he prefers higher responses:

ASSUMPTION 2 $u_{P1}(\theta, x, y)$:

1. *is continuous, with Lipschitz constants $\lambda_\theta$ in $\theta$ and $\lambda_y$ in $y$*
2. *is strictly quasi-concave in $x$*
3. *is strictly increasing in $y$*
4. *has strictly increasing differences in $(\theta, x)$ and weakly increasing differences in $(\theta, y)$ and $(x, y)$*
5. *has increasing difference bounds of $\lambda_{\theta x}$ in $(\theta, x)$, $\lambda_{\theta y}$ in $(\theta, y)$, and $\lambda_{xy}$ in $(x, y)$*

His best response function is:

DEFINITION $\mathrm{BR}_{P1}(\theta, y) = \arg\max_x u_{P1}(\theta, x, y)$

[11]If $A$, $B$ are partially ordered sets, $f : A \times B \to \mathbb{R}$ has *strictly increasing (weakly increasing / constant) differences* if: for $a' > a$ and $b' > b$, $f(a',b') - f(a',b) > (\geq / =)\ f(a,b') - f(a,b)$. Then $f(a,b,c\ldots)$ has weakly increasing differences (etc.) in $(a,b)$ if $f(c,\ldots)(a,b)$ has weakly/strictly increasing differences (etc.) as a function of $(a,b)$. $f$ is weakly *supermodular* if it has weakly increasing differences in all pairs of variables.

[12]If $f$ is twice differentiable, $\frac{\partial^2}{\partial a \partial b} f \leq \lambda$ implies an increasing difference bound of $\lambda$. So if $a, b$ lie in compact sets and $f \in C_2$ then an increasing difference bound must exist.

*Assumptions for the respondent*

We shall place assumptions on the static best response function of the respondent. At any stage the respondent does not know either the current type or the current action of the signaler, so he responds to a distribution over both:

ASSUMPTION 3 *The respondent has a best response function $\rho : \triangle(\Theta \times X) \to Y$, with $\rho$ continuous and weakly increasing.*[13]

The lowest and highest best responses are:

DEFINITION Let $\underline{y} := \rho([\theta_{\min}, x_{\min}])$ and $\overline{y} := \rho([\theta_{\max}, x_{\max}])$. Let $\bar{Y} = \left[\underline{y}, \overline{y}\right]$.

If player 1's strategy in a stage game is a map $\sigma : \Theta \to X$, and player 2's belief about 1's type is $\hat{\theta}$, player 2's belief over $\Theta \times X$ is $\hat{\theta} \circ f^{-1}$, where $f(\theta) := (\theta, \sigma(\theta))$. So the best response to $\sigma$ and $\hat{\theta}$ is:

DEFINITION For $\sigma : \Theta \to X$ measurable, $\rho'(\hat{\theta})(\sigma) := \rho\left(\hat{\theta} \circ f^{-1}\right)$, where $f(\theta) := (\theta, \sigma(\theta))$.

*Joint assumptions*

Both players have Lipschitz constants regulating how their best responses adjust to the actions of each other. These constants multiply to something less than 1, which will imply a unique equilibrium in period $t$ given play in future periods.

ASSUMPTION 4 *There exist $\kappa_{P1} \in [0,\infty)$ and $\kappa_{P2} \in [0,\infty]$ with $\kappa_{P1} \cdot \kappa_{P2} < 1$, such that:*

1. *The difference $u_{P1}\left(\theta, x+dx, y\right) - u_{P1}\left(\theta, x, y\right)$ is decreasing on any line $\left\{\left(x, y\right) = \left(x_0, y_0\right) + \lambda\left(\kappa_{P1}, 1\right)\right\}$, and*
2. *$\rho'(\hat{\theta})(\sigma)$ has Lipschitz constant $\kappa_{P2}$ in $\sigma$.*[14]

To guarantee that there are costly enough signals for separating equilibria to exist, we assume that no change in the respondent's action (within $\bar{Y}$) will compensate for taking $x_{\max}$ over some more moderate action, even for the highest type.

[13] If $Z$ is a metric measure space, give $\triangle Z$ the topology of weak convergence. Then $\triangle Z$ is compact if $Z$ is compact. If $Z \subseteq \mathbb{R}^n$ then $\triangle Z$ is partially ordered by stochastic dominance (**?**).

[14] These are implied by $\left(\kappa_{P1}\left(\frac{\partial}{\partial x}\right)^2 + \frac{\partial}{\partial x \partial y}\right) u_{P1} \leq 0$ and the symmetric condition for $P2$.

ASSUMPTION 5 $u_{P1}(\theta_{\max}, x_{\max}, \overline{y}) \le \max_x u_{P1}(\theta_{\max}, x, \underline{y})$

## 3. BASIC SIGNALING THEORY

A signaling payoff is a function $U(\theta, x, \theta')$, the payoff of type $\theta \in \Theta$ if he takes signal $x \in X$ and is subsequently believed to be $\theta' \in \Theta$.[15] Higher types have a greater incentive to signal a higher type by taking a higher signal:

DEFINITION $U(\theta, x, \theta')$ satisfies *single crossing* if whenever $\theta_1 < \theta_2$, $x_1 < x_2$, and $\theta'_1 \le \theta'_2$, $U(\theta_1, x_2, \theta'_2) \ge U(\theta_1, x_1, \theta'_1)$ implies $U(\theta_2, x_2, \theta'_2) > U(\theta_2, x_1, \theta'_1)$.

This is implied by supermodularity of $\hat{U}$. The basic signaling payoff space satisfies this and some convenient technical conditions:

DEFINITION Let $\Phi$ be the set of $U : \Theta \times X \times \Theta \to \mathbb{R}$ with $U(\theta, x, \theta')$: 1. uniformly continuous in $(\theta, x)$, 2. weakly increasing in $\theta'$, 3. strictly quasi-concave in $x$, 4. with $(U(\theta, x, \theta_1) = U(\theta, x, \theta_2))$ independent of $x$, and 5. satisfying single crossing.

For some payoffs the highest signal is never worth choosing:

DEFINITION Let $\Phi_B$ be the set of payoffs $U \in \Phi$ satisfying $\max_x U(\theta_{\max}, x, \theta_{\min}) \ge U(\theta_{\max}, x_{\max}, \theta_{\max})$.

Given a signaling payoff $U \in \Phi$, a function $f : \Theta \to X$ satisfies individual rationality if for all $\theta$, $U(\theta, f(\theta), \theta) \ge \max_x U(\theta, x, \theta_{\min})$. It satisfies incentive compatibility (IC) if for all $\theta, \theta'$, $U(\theta, f(\theta), \theta) \ge U(\theta, f(\theta'), \theta')$. If both are satisfied, $f$ is *weakly separating*. It is then a separating equilibrium iff it is injective, which it must be if $\hat{U}$ is strictly increasing in $\theta'$.

The set of $U \in \Phi$ admitting at least one weakly separating equilibrium is $\Phi_{\text{Sep}}$, with $\Phi_B \subseteq \Phi_{\text{Sep}}$. On $\Phi_{\text{Sep}}$ there exists a dominant (Riley) separating equilibrium, which uniquely maximizes payoffs for all types of the signaler. This is given by $\mathscr{R} : \Phi_{\text{Sep}} \to \text{Inc}(\Theta, X)$, where $\text{Inc}(\Theta, X)$ are the weakly increasing functions $\Theta \to X$.

## 4. VALUE FUNCTION ITERATION

We shall work with supermodular value functions $V_t(\hat{\theta}, \theta)$ of the signaler. As in Roddie (2011b), we use the reduced value function $v_t^{\sim}(\theta', \theta) := V_t(\psi(\theta'), \theta) -$

[15] This section summarizes material from Roddie (2011a).

$V_t\left(\psi\left(\theta_{\min}\right),\theta\right)$. This is the value to type $\theta$ of having previously signaled $\theta'$ over $\theta_{\min}$. This exists in the the space $\hat{\Pi}$:

DEFINITION Let $B_V := \max_x u_{P1}\left(\theta_{\max}, x, \overline{y}\right) - u_{P1}\left(\theta_{\max}, x_{\max}, \overline{y}\right)$.
Let $\kappa^* := \left(\lambda_{\theta y} + \kappa_{P1} \cdot \lambda_{xy}\right) \cdot \left|\bar{Y}\right|$.
Let $\hat{\Pi}$ be the set of functions $v^{\sim} : \Theta \times \Theta \to \mathbb{R}$

1. weakly increasing, with $v^{\sim}\left(\theta_{\min},\theta\right) = 0$ and $v^{\sim}\left(\theta_{\max},\theta_{\max}\right) \leq B_V$
2. weakly supermodular
3. with a Lipschitz constant $\kappa^*$ in the second argument

The advantage of using $v_t^{\sim}$ rather than $V_t$ here is that we have a compact space $\hat{\Pi}$ on which value function iteration has strong continuity properties. The bound exists since $v_t^{\sim}$ describes a gain in period $t$ only. The Lipschitz constant, which is new here, perserves compactness of $\hat{\Pi}$, while ensuring continuity of $v_t^{\sim}\left(\theta',\theta\right)$ in $\theta$.

### 4.1. *Summary*

Suppose $v_{t+1}^{\sim} \in \hat{\Pi}$. Taking into account discounting and the possibility of type change, $v_{t+1}^{\sim}$ induces in period a value $v_t'\left(\theta',\theta\right)$ of signaling $\theta'$ over $\theta_{\min}$ in period $t$.

Then given any $y$ that is expected in period $t$, the signaler faces signaling payoff $\left(\theta, x, \theta'\right) \mapsto u_{P1}\left(\theta, x, y\right) + v_t'\left(\theta',\theta\right)$ in period $t$. This is in $\Phi_B$, the main condition of single crossing being implied by supermodularity. We then take the Riley equilibrium of this signaling game to get the signaler's strategy $s_Y\left(y,\theta\right)$ in period $t$ - which will depend on $y$.

The signaler's value $W\left(y,\theta\right)$ is then supermodular, the key insight that allows value function iteration. Moreover the benefit of receiving $y$ instead of $\underline{y}$, $w\left(y,\theta\right) := W\left(y,\theta\right) - W\left(\underline{y},\theta\right)$ is not only supermodular and an increasing function, but has Lipschitz constant $\kappa^*$ in $\theta$. This new observation, which depends on further comparative statics of static signaling games, is used to preserve a compact space of value functions to give existence Markov perfect equilibrium as a fixed point.

To complete the analysis, we need to find the respondent's action $y$ as a function $s_{P2}^{\sim}\left(\theta'\right)$ of the previously signaled type $\theta'$. Given $y$, and the signaler's strategy is $\theta \mapsto s_Y\left(y,\theta\right)$, and when beliefs are $\psi\left(\theta'\right)$ the respondent's best response is

$\rho'(\psi(\theta'))(\theta \mapsto s_Y(y,\theta))$. This map $\bar{Y} \to \bar{Y}$ is a contraction, and the $s_{P2}^{\sim}(\theta')$ is the unique fixed point.

Let the signaler's strategy in period $t$ when $\theta'$ was previously signaled (i.e. beliefs are $\psi(\theta')$) be $s_{P1}^{\sim}(\theta',\theta)$. Then we have $s_{P1}^{\sim}(\theta',\theta) = s_Y(s_{P2}^{\sim}(\theta'),\theta)$. We also have value function $v_t^{\sim}(\theta',\theta) = w(s_{P2}^{\sim}(\theta'),\theta) - w(s_{P2}^{\sim}(\theta_{\min}),\theta)$ which is in $\hat{\Pi}$.

### 4.2. *Conclusion*

So we have a value function iterator $F : \hat{\Pi} \to \hat{\Pi}$, with $v_t^{\sim} = F(v_{t+1}^{\sim})$. A formal definition and treatment is carried out in Section B. There $F$ into decomposed into component maps in way that will be useful for showing continuity properties of $F$.

## 5. EXTENDED CONTINUITY OF $F$

### 5.1. *Definition of convergence*

On $\tilde{\Pi}$, use the L1 topology, with $\mu$ as the fundamental measure on $\Theta$. We shall show that $F(v^{\sim})$ is continuous, not only in $v^{\sim}$ but in the parameters $\Theta$, $\psi$ and $\delta_{P1}$. This will be used to give us both existence of stationary Markov equilibria, and upper semi-continuity of these equilibria in these parameters.

The type space $\Theta$ will either be constant (eventually) or a finite set becoming dense in a continuum:

DEFINITION $\Theta_i \nearrow \Theta_\infty$ if $\Theta_i$ has minimal and maximal elements $\theta_{\min}$,$\theta_{\max}$, and $\Theta_i$ is increasing, with $\bigcup_{i<\infty} \Theta_i$ dense in $\Theta_\infty$.

We want a notion of convergence for functions of $\Theta$ - the Markov process, value functions, strategies - even when $\Theta$ is changing. Take $\Theta_i \nearrow \Theta_\infty$. We will have various spaces $S_i$ of functions $f : (\Theta_i)^n \times Z \to \mathbb{R}$. Typically $n = 1$ or 2 and $Z$ is a singleton. $S_i$ is the projection of $S_\infty$ onto $\Theta_i$. If $\bar{f}_i = f_i$ restricted to $\Theta_i$ then we say that $\bar{f}_i$ *extends* $f_i$. We will say that $f_i \nearrow f_\infty$ if for all $\bar{f}_i$ extending $f_i$, $\bar{f}_i \to f_\infty$ in L1.[16]

This gives a definition of convergence for value functions $v_i^{\sim} : \Theta_i^2 \to \mathbb{R}$ and strategies $s_{P1}^{\sim} : \Theta_i^2 \to X$ and $s_{P2}^{\sim} : \Theta_i \to \bar{Y}$. To apply the notion to the Markov process, use distribution functions and consider $\psi$ as the function $(\theta_1,\theta_2) \to \psi(\theta_1)([\theta_{\min},\theta_2])$, which is weakly decreasing in $\theta_1$ and weakly increasing in $\theta_2$.

[16]Then this is true for all such $\bar{f}_i$, provided the functions are monotonic or Lipshitz in each $\Theta$ argument, by Lemma 4.

### 5.2. *The extended continuity result*

Now we can state that $F(v^{\sim})$ is continuous in $v^{\sim}$, $\Theta$, and $\delta_1$:

PROPOSITION 1 *Suppose* $\Theta_i \nearrow \Theta_\infty$, $\psi_i \nearrow \psi_\infty$ *and* $\delta^i_{P1} \to \delta^\infty_{P1}$. *Suppose* $v^{\sim}_i \in \Pi_i$ *with* $v^{\sim}_i \nearrow v^{\sim}_\infty$. *Then* $F\left(v^{\sim}_i\right) \nearrow F\left(v^{\sim}_\infty\right)$.

To show this, we show that each component map in the decomposition of $F$ is continuous in this way. Taking constant $\Theta_i = \Theta$ and $\psi_i = \psi$, we have straight continuity:

COROLLARY 1 *$F$ is continuous.*

## 6. STATIONARY MARKOV EQUILIBRIA

We look for Markov strategies, which are time-independent. So the reduced value function $v^{\sim}$ of such an equilibrium must be time independent, and then $F(v^{\sim}) = v^{\sim}$. Conversely, given such a $v^{\sim}$, if we generate strategies as above, they will be a dynamic Riley equilibrium. So we need to show that such a fixed point of $F$ exists.

### 6.1. *Fixed points*

$\Pi^{\sim}$ is isomorphic to a convex subset of $\mathbb{R}^{|\Theta|^2}$. Since $F$ is continuous and maps into a compact subset, bounded by $B_V$, Corollary 1 implies it must have a fixed point, by Brouwer's fixed point theorem:

COROLLARY 2 *For finite* $\Theta$, *$F$ has a fixed point.*

To extend this to existence of a fixed point for a continuum of types, we use extended continuity of $F(v^{\sim})$ in $v^{\sim}$ and $(\Theta, \psi)$. This extended continuity implies that a convergent sequence of fixed points of finite-type approximations will be an "almost-fixed" point of the continuum:

LEMMA 1 *There exists* $v^{\sim} \in \Pi$ *such that* $F(v^{\sim}) = v^{\sim}$ *a.e..*

To find an actual fixed point, first take an almost fixed point $v^{\sim}_1$. By Assumption 1.3, altering $v^{\sim}_1$ on a measure-zero set $S \in \Theta \times \Theta$ will only affect $F\left(v^{\sim}_1\right)$ on $S$. So we can deal with the points for which $F\left(v^{\sim}_1\right) \neq v^{\sim}_1$ individually, to get a fixed point $v^{\sim}_2$:

LEMMA 2 *If* $v_1^{\sim} \in \Pi$ *satisfies* $F^{\sim}\left(v_1^{\sim}\right) = v_1^{\sim}$ *a.e., then there exists* $v_2^{\sim} \in \Pi$ *with* $v_2^{\sim} = v_1^{\sim}$ *a.e. and* $F\left(v_2^{\sim}\right) = v_2^{\sim}$.

We now have the main result:

PROPOSITION 2 *There exists* $v^{\sim} \in \Pi$ *such that* $F(v^{\sim}) = v^{\sim}$.[17]

PROOF: Immediate from Lemma 1 and Lemma 2. *Q.E.D.*

## 6.2. *Markov equilibria*

Any value function for the Markov Riley equilibrium must be a fixed point of $F$, and conversely any fixed point $v^{\sim}$ of $F$ gives a Markov Riley equilibrium, generating strategies $s_{P1}^{\sim}\left(\theta',\theta\right)$ and $s_{P2}^{\sim}\left(\theta'\right)$ as in 4.1.[18]

## 6.3. *Upper-semicontinuity of Markov equilibria in the parameters*

Finally extended continuity of $F$ shows that the set of Markov equilibria is upper-semicontinuous in $\Theta, \psi$ and $\delta_{P1}$:

PROPOSITION 3 *If* $\Theta_i \nearrow \Theta_\infty$, $\psi_i \nearrow \psi_\infty$ *and* $\delta_{P1}^i \to \delta_{P1}^\infty$, *and* $v_i^{\sim}$ *for* $i < \infty$ *is a convergent sequence of equilibrium (fixed-point) value functions, with corresponding strategies* ${}^i s_{P1}^{\sim}$ *and* ${}^i s_{P2}^{\sim}$, *then there exists an equilibrium value function* $v_\infty^{\sim}$, *with corresponding strategies* ${}^\infty s_{P1}^{\sim}$ *and* ${}^\infty s_{P2}^{\sim}$, *such that* $v_i^{\sim} \nearrow v_\infty^{\sim}$, ${}^i s_{P1}^{\sim} \nearrow {}^\infty s_{P1}^{\sim}$ *and* ${}^i s_{P2}^{\sim} \nearrow {}^\infty s_{P2}^{\sim}$.

Note that $F$ still has a fixed point even in the degenerate case $\psi = \psi_{id}$, and so continually separating equilibria can be defined even when beliefs do not have full support.

The same is true when $\delta_{P1} = \delta_{P1}^* = 1$. Here the game itself is not defined, as value functions would not be finite, but reduced value function iteration takes place in a compact space and still generates a fixed point.

# 7. THE (DISCOUNTED) STACKELBERG GAME

We shall be able to find limit characterizations of Markov Riley equilibria in two cases. The first case is when the signaler becomes completely patient $\delta_{P1} \to \delta_{P1}^* = 1$.

[17] I do not know if $F$ is a contraction, which would give uniqueness of the fixed point.

[18] In the notation of Section B, $s_Y = (\alpha_3 \circ \alpha_2 \circ \alpha_1)(v^{\sim})$, $s_{P2}^{\sim} = (\alpha_5 \circ \alpha_4)(s_Y)$, and $s_{P1}^{\sim}\left(\theta',\theta\right) = s_Y\left(s_{P2}\left(\theta'\right),\theta\right)$.

The second case is when the signaler's utility is additively separable and his discount factor converges to any $\delta^*_{P1}$.

DEFINITION $u_{P1}$ is additively separable if $u_{P1}\left(\theta,x,y\right)=u_X\left(\theta,x\right)+u_Y\left(\theta,y\right)$.

ASSUMPTION 6 *$\delta^*_{P1}=1$ or $u_{P1}$ is additively separable*

Consider the one-shot signaling game formed by taking the stage game of the repeated game and having the signaler move first. Then the reduced form payoff for the signaler is given by the function $\left(\theta,x,\theta'\right)\mapsto u_{P1}\left(\theta,x,\rho\left(\left[\theta',x\right]\right)\right)$. If utility is additively separable and we imagine that the signaler is moving in period 0 and the response is occurring in period 1, then if the signaler's discount factor is $\delta^*_1$, the reduced form payoff for the signaler's action in period 0 and the respondent's response in period 1 is $\left(\theta,x,\theta'\right)\mapsto u_X(\theta,x)+\delta^*_{P1}u_Y\left(\theta,\rho\left(\left[\theta',x\right]\right)\right)$.

The (discounted) Stackelberg game payoff $U_{\text{St}}$ is as follows:

DEFINITION $$U_{\text{St}}\left(\theta,x,\theta'\right):=\begin{cases}u_{P1}\left(\theta,x,\rho\left(\left[\theta',x\right]\right)\right) & \delta^*_{P1}=1\\ u_X(\theta,x)+\delta^*_{P1}u_Y\left(\theta,\rho\left(\left[\theta',x\right]\right)\right) & u_{P1}\text{ additively separable}\end{cases}$$

Note that we can take both $u_X$ and $u_Y$ to be continuous, so $U_{\text{St}}$ is continuous. We assume:

ASSUMPTION 7 $U_{\text{St}}\left(\theta,x,\theta'\right)$ is strictly quasi-concave in $x$.

This implies a unique Stackelberg action maximizing $\hat{U}_{\text{St}}$. More than this, $U_{\text{St}}$ is a regular signaling payoff:

LEMMA 3 $U_{\text{St}}\in\Phi$.

Define $\sigma_{\text{St}}$ as the complete information outcome of the game $U_{\text{St}}$:

DEFINITION $\sigma_{\text{St}}:=\arg\max_x U_{\text{St}}\left(\theta,x,\theta\right)$

## 8. LIMIT INCENTIVE COMPATIBILITY

We shall consider the limit of equilibria as $\delta_{P1}\to\delta^*_{P1}$, and as types become persistent $\psi\to\psi_{id}$, where $\psi_{id}(\theta)\equiv(\theta)$. We deal with a continuum of types, or types becoming a continuum. A limit strategy is:

DEFINITION Let $\sigma_L : \mathrm{Inc}(\Theta, X)$ be a limit of equilibrium strategies ${}^i s^{\sim}_{P1}(\theta,\theta) = s^i_{P1}(\psi_i(\theta),\theta)$ of games $(\delta^i_{P1},\Theta_i,\psi_i)$, where $\delta^i_{P1} \to \delta^*_{P1}$, $\Theta_i \nearrow \bar{\Theta}$, and $\psi_i \nearrow \psi_{id}(\bar{\Theta})$.

By Proposition 3, a limit of equilibria must be an equilibrium of the limit game. So $\sigma_L(\theta) = s^{\sim}_{P1}(\theta,\theta)$, where $s^{\sim}_{P1}$ is an equilibrium strategy of the game with parameters $(\bar{\Theta},\psi_{id},\delta^*_{P1})$.

The key to the limit results is that this limit strategy satisfies the incentive com patability conditions of the (discounted) Stackelberg signaling game:

PROPOSITION 4 *$\sigma_L$ satisfies IC for $U_{\mathrm{St}}$.*

## 9. ACTIONS ABOVE (DISCOUNTED) STACKELBERG IN THE LIMIT

In the combined case, $\rho([\theta',x])$ is strictly increasing in $\theta'$. This implies that $U_{\mathrm{St}}(\theta,x,\theta')$ is strictly increasing in $\theta'$: there are strict signaling incentives in $U_{\mathrm{St}}$. That means that to satisfy IC, $\sigma_L$ must be strictly increasing, which in turn implies that $\sigma_L(\theta) \geq \sigma_{\mathrm{St}}(\theta)$ for $\theta > \theta_{\mathrm{min}}$.

This property is of interest in itself, because it implies a large deviation from the complete information Nash equilibrium $\sigma_{\mathrm{NE}}$ in the limit. But it's main use will be to combine with the IC conditions to give a complete characterization of the limit.

In the pure reputation case, $\rho([\theta',x])$ is constant in $\theta'$, or more generally any case where it is not strictly increasing in $\theta'$, this property does not follow so easily. And in fact some dynamic Riley equilibria of the limit game do not satisfy it.[19] While this is an edge case, it is an important one: the pure reputation case is the natural specification of most reputation models.

This possibility can be ruled out if we place conditions on how the game converges to the limit. Suppose we take a limit of equilibria of non-degenerate Markov processes, where the signaler's strategy can be guaranteed to be strict. Then in any region we can find a point $\theta$ where increasing $x$ by $\delta_x$ to pretend to be a slightly higher type results in expectations of actions increasing by at least $(1-\epsilon)\cdot\delta_x$, if we make regularity assumptions on the Markov process, in particular that it is close to being a convolu-

[19] Suppose $\rho([\theta',x])$ is a function of $x$ only, so that $U_{\mathrm{St}}(\theta,x,\theta') = U_{\mathrm{St}}(\theta,x)$ is constant in $\theta'$. Take any compact set $C$, and let $\sigma'(\theta) \in \arg\max_{x\in C} U_{\mathrm{St}}(\theta,x)$. Then $\sigma'(\theta)$ satisfies IC for $U_{\mathrm{St}}$. If $C$ is finite, then $\sigma'$ will be a step function, and typically each step will be crossed by the curve $\sigma_{\mathrm{St}}$. Then $\sigma'(\theta) < \sigma_{\mathrm{St}}(\theta)$ for some $\theta$, and $\sigma'$ is non-unique, since we have many different $\sigma'$ depending on $C$.

tion with a noise term.[20]

DEFINITION $(\Theta_i, \psi_i) \nearrow (\Theta, \psi_{id})$regularly if $\Theta_i \nearrow \Theta$, and for some measures $\xi_i \neq [0]$ :

1. Each $\Theta_i$ is either $\bar{\Theta}$ or a grid with constant intervals.
2. $\xi_i$ is supported on $[-\epsilon_i, \epsilon_i]$, where $\epsilon_i \to 0$, has mass $\xi_i(\Theta_i) = 1$ and has expectation $\int \theta \, \mathrm{d}\xi_i(\theta) = 0$, and has a single peaked density or mass function, depending on whether $\Theta_i$ is a continuum or finite.
3. For any $\delta > 0$, for sufficiently large $i$, for $\theta$ with $d(\theta, \{\theta_{\min}, \theta_{\max}\}) > \delta$, $\psi_i(\theta) = (1 - \epsilon_i') \cdot [\theta] * \xi_i + \nu_i(\theta)$, where $\nu_i(\theta)$ is weakly increasing in $\theta$ and $\epsilon_i' \leq \delta$.[21]

The last condition is that $\psi_i(\theta)$ is approximately $\theta$ plus an error term with distribution $\xi_i$, and so implies $\psi_i \nearrow \psi_{id}$. It rules out actions below discounted Stackelberg actions, because increasing expectations by $(1-\epsilon) \cdot \delta_x$ is more valuable than the cost of increasing the current action by $\delta_x$ (for small enough $\epsilon$). The argument involves an extended analysis of approximate inequalities.

PROPOSITION 5 *For $\theta > \theta_{min}$, $\sigma_L(\theta) \geq \sigma_{\mathrm{St}}(\theta)$, provided either:*

1. *$\rho([\theta', x])$ is strictly increasing in $\theta'$, or:*
2. *$(\Theta_i, \psi_i) \nearrow (\Theta, \psi_{id})$ regularly,* $\mathrm{BR}_{P1}\left(\theta_{\max}, \underline{y}\right) > x_{\min}$[22], $u_{P1}(\theta, x, y)$ and $\rho([\theta, x])$ are continuously differentiable in $x$ and $y$, and $\left(\frac{\partial}{\partial x}\right) U_{\mathrm{St}} > 0$ below $\sigma_{\mathrm{St}}$ *(strengthening Assumption 7).*

## 10. THE LIMIT RESULT

### 10.1. *Incentive compatibility and uniqueness*

With a continuum of types the incentive compatibility conditions pin down a separating strategy $f$ uniquely, once an initial condition is specified. Suppose the signaling game is $U \in \Phi$ and $x_U^*(\theta) = \arg\max_x U(\theta, x, \theta)$ is the complete information outcome. Then the initial condition $f(\theta_{\min}) = x_U^*(\theta_{\min})$ together with the IC conditions imply that $f$ is the unique separating equilibrium $f = \mathscr{R}(U)$.

[20] They key property is that increasing $\theta$ by $\delta_\theta$ changes $\psi_i(\theta)$ by approximately $\delta_\theta$. Probably the assumptions can be weakened to something nearer this.

[21] $\mu_1 * \mu_2$, where $\mu_i \in \triangle\mathbb{R}$, denotes the convolution of the measures, the distribution of the sum of two independent random variables with distributions $\mu_i$.

[22] The assumption that $\mathrm{BR}_{P1}\left(\theta_{\max}, \underline{y}\right) > x_{\min}$ is used to guarantee strictness outside the limit; otherwise there can be equilibria with $x_{\min}$ taken by all types.

Without this initial condition there exists a unique action $\bar{x}$ weakly above $x^*_U(\theta_{\min})$ which gives type $\theta_{\min}$ the same payoff as $\underline{x} = f(\theta_{\min})$: $U(\theta_{\min}, \bar{x}, \theta_{\min}) = U(\theta_{\min}, \underline{x}, \theta_{\min})$. Given this, the IC conditions pin down $f$ (above $\theta_{\min}$) as the Riley separating equilibrium restricted to $[\bar{x}, x_{\max}]$:

THEOREM 1 *(Roddie (2011a), Proposition 18; after Mailath (1987)) Suppose* $\Theta = \bar{\Theta}$, $U \in \Phi$ *is continuous,* $f$ *satisfies IC, and* $f \geq x^*_U$ *on* $(\theta_{\min}, \theta_{\max}]$. *Then there exists a unique* $\bar{x} \geq x^*_U(\theta_{\min})$ *satisfying* $U(\theta_{\min}, \bar{x}, \theta_{\min}) = U(\theta_{\min}, x^*(\theta_0), \theta_{\min})$. *Then* $U|_{\bar{X}} \in \Phi_{\text{Sep}}(\bar{X})$ *and* $f = \mathscr{R}(U|_{\bar{X}})$ *on* $(\theta_{\min}, \theta_{\max}]$, *where* $\bar{X} := [\bar{x}, x_{\max}]$.

Note the condition $f \geq x^*_U$ on $(\theta_{\min}, \theta_{\max}]$, discussed in Section 9, which is non-trivial when $U(\theta, x, \theta')$ is not strictly increasing in $\theta'$.

## 10.2. *The limit result*

Under complete information in a one-shot game, the signaler of type $\theta$ will take the Nash Equilibrium action $\sigma_{\text{NE}}(\theta)$, the unique fixed point of the contraction $x \mapsto \text{BR}_{P1}(\theta, \rho([\theta, x]))$. Let $\underline{x} := \sigma_{\text{NE}}(\theta_{\min})$. The action of $\theta_{\min}$ is not affected by signaling incentives, so $\sigma_L(\theta_{\min}) = \underline{x}$. Let $\bar{x}$ be the unique action above $\sigma_{\text{St}}$ giving $\theta_{\min}$ the same payoff: $U_{\text{St}}(\theta_{\min}, \bar{x}, \theta_{\min}) = U_{\text{St}}(\theta_{\min}, \underline{x}, \theta_{\min})$.

Proposition 4 and Proposition 5 show that the conditions for Theorem 1 are satisfied, so we can characterize $\sigma_L$:

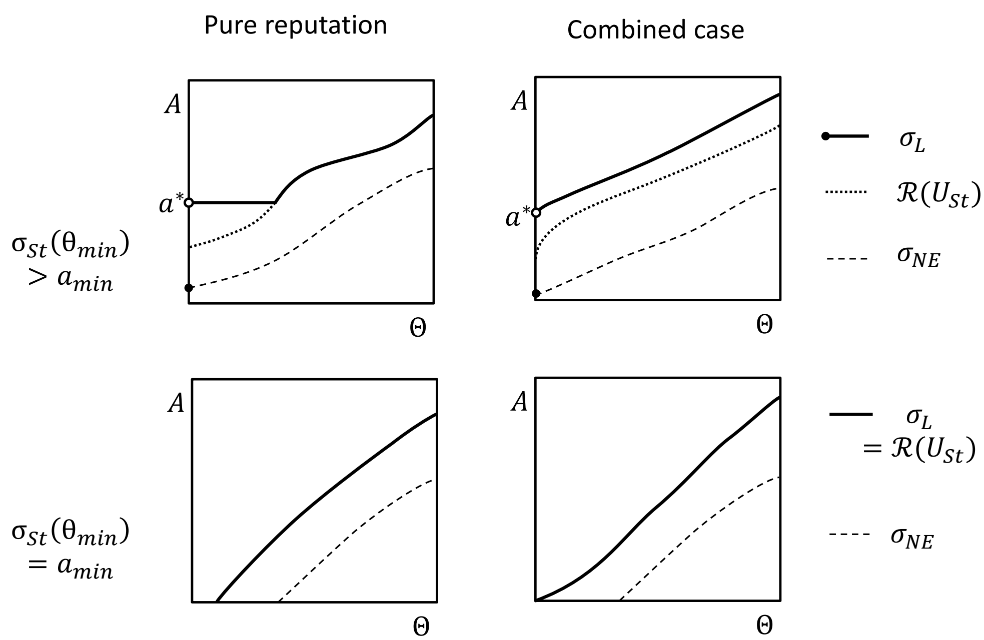
PROPOSITION 6 *Under the assumptions of Proposition 5*:

$$\sigma_L(\theta) := \begin{cases} \underline{x} & \theta = \theta_{\min} \\ \mathscr{R}\left(U_{\text{St}}|_{[\bar{x}, x_{\max}]}\right) & \theta > \theta_{\min} \end{cases}$$

Above $\theta_{\min}$ it is the Riley equilibrium of $U_{\text{St}}$ subject to separating from $\theta_{\min}$ when $\theta_{\min}$ does $\underline{x}$. This is illustrated in Figure 1, for both the pure reputation and combined cases, both when $\sigma_{\text{St}}(\theta_{\min})$ is interior and when it is the corner solution $x_{\min}$.

## 10.3. *Discussion: the limit map $\sigma_L$*

Suppose $\sigma_{\text{St}}(\theta_{\min}) = x_{\min}$. For example in dynamic Cournot competition, the signaler's cost may be so high that no production takes place even as a first-mover. Then the limit map $\sigma_L$ is the Riley equilibrium of $U_{\text{St}}$. If $\delta = 1$ this is just the Stackelberg




Note: $\mathscr{R}(U_{\mathrm{St}}) = \sigma_{\mathrm{St}}$ in the pure reputation case.

FIGURE 1.— The limit map $\sigma_L$

game, and in the reputational case, there are no signaling incentives in $U_{\mathrm{St}}$ and the Riley equilibrium is just Stackelberg actions under complete information.

If $\sigma_{\mathrm{St}}(\theta_{\min}) > x_{\min}$, the action of the lowest type, which is below $\sigma_{\mathrm{St}}$, forces the actions of immediately higher types above $\sigma_{\mathrm{St}}$. In the pure reputation case, the Stackelberg action is taken subject to the lower bound $\bar{x}$. As in Mailath and Samuelson (2001), separation from the lowest type can push a higher type above his Stackelberg action.[23]

In the combined case, with $\sigma_{\mathrm{St}}(\theta_{\min}) > x_{\min}$, separation from the lowest type leads to actions permanently higher than both the separating equilibrium and Stackelberg actions. For smooth payoffs the same differential equation is satisfied but with a higher initial condition $\bar{x}$.

In both cases there is a range of low actions $(\underline{x}, \bar{x})$ which the signaler cannot commit to, which the respondent would not expect to be maintained in future. In the $\psi_{id}$ limit

[23]It may be hard to take an *a priori* view of the preferences of $\theta_{\min}$, in which case the importance of the lower bound is left open. "How low the low type is" determines whether there is a binding $\bar{x}$, and additionally the interval of low types bound by $\bar{x}$ may have low probability.

actions in $(\underline{x}, \bar{x})$ are interpreted as actions of the lowest type, and are never taken.[24] The possibility here is that low actions generate even lower expectations.

Possible examples of $\bar{x}$ binding include seller reputations, where reputations for product quality can be established subject to meeting some minimal standard of functionality, while failing to meet this standard results in a sharp loss of reputation and categorization as a scammer. Or reputations for good social behavior, where even a small degree of criminality is looked upon badly, with a consequent gap between the behavior of criminal and non-criminal types.

## APPENDIX A: KEY

*Variables and sets*

| | | | | | |
|---|---|---|---|---|---|
| $\theta$ | $\in$ | $\Theta$ | $\subseteq$ | $[\theta_{\min}, \theta_{\max}]$ | stage-game type |
| $x$ | $\in$ | $X$ | $=$ | $[x_{\min}, x_{\max}]$ | signaler's (P1's) action |
| $y$ | $\in$ | $Y$ | $=$ | $[y_{\min}, y_{\max}]$ | respondent's (P2's) action |
| $U$ | : | $\Theta \times X \times \Theta \to \mathbb{R}$ | $\in$ | $\Phi_B \subseteq \Phi_{\mathrm{Sep}} \subseteq \Phi$ | static signaling payoff function |
| $v_t^{\sim}$ | : | $\Theta \times \Theta \to \mathbb{R}$ | $\in$ | $\Pi$ | refactored value function |
| $U_{\mathrm{St}}$ | | | $\in$ | $\Phi$ | (discounted) Stackelberg game |

*Main functions*

| | | | |
|---|---|---|---|
| $u_{Pi}$ | : | $\Theta \times A \times R \to \mathbb{R}$ | stage-game payoff |
| $\delta_{Pi}$ | $\in$ | $\mathbb{R}^+$ | discount factor |
| $\psi$ | : | $\Theta \to \Delta\Theta$ | Markov process on type |
| $\mu$ | : | $\sigma(\Theta) \to \mathbb{R}$ | underlying measure on $\Theta$ (counting or modified Lebesgue) |
| $\mathrm{BR}_{P1}$ | : | $\Theta \times R \to X$ | (myopic) best response of 1 |
| $\rho$ | : | $\triangle(\Theta \times X) \to Y$ | best response of 2 |
| $\rho'$ | : | $\triangle\Theta \to (\Theta \to X) \to Y$ | best response of 2 to a type-belief and strategy |
| $\mathscr{R}$ | : | $\Phi_{\mathrm{Sep}} \to (\Theta \to X)$ | Riley equilibrium |
| $F$ | : | $\hat{\Pi} \to \hat{\Pi}$ | value function iteration |

## APPENDIX B: VALUE FUNCTION ITERATION

### B.1. *An induced signaling game*

Suppose $v_{t+1}^{\sim} \in \hat{\Pi}$. The value $v_t'(\theta', \theta)$ of signaling $\theta'$ over $\theta_{\min}$ in period $t$, to type $\theta$, is given by $v_t' = \alpha_1(v_{t+1}^{\sim})$, where:

[24] If beliefs satisfy the recursive D1 refinement. Close to the limit, this property weakens: in-between actions are actually taken by very low types, and reveal being a very low type, with a steep gradient of the map from types to actions.

DEFINITION $\alpha_1(v^{\sim}) : (\theta', \theta) \mapsto \delta_{P1} \cdot \int v^{\sim}(\theta', \theta_{t+1})\, \mathrm{d}[\psi(\theta)](\theta_{t+1})$.

LEMMA $\alpha_1 : \hat{\Pi} \to \hat{\Pi}$.

PROOF: This is shown in Roddie (2011b), except for the Lipschitz constant $\kappa^*$.

$$\begin{aligned}
& v'_t(\theta', \theta_2) - v'_t(\theta', \theta_1) \\
=& \delta_{P1} \cdot \int v^{\sim}_t(\theta', \theta_{t+1})\, \mathrm{d}[\psi(\theta_2)](\theta_{t+1}) - \delta_{P1} \cdot \int v^{\sim}_t(\theta', \theta_{t+1})\, \mathrm{d}[\psi(\theta_1)](\theta_{t+1}) \\
\leq& \delta_{P1} \cdot \int \kappa^* \cdot \theta_{t+1}\, \mathrm{d}[\psi(\theta_2)](\theta_{t+1}) - \delta_{P1} \cdot \int \kappa^* \cdot \theta_{t+1}\, \mathrm{d}[\psi(\theta_1)](\theta_{t+1}) \\
=& \delta_{P1} \cdot \kappa^* \left(\mathbb{E}[\psi(\theta_2)] - \mathbb{E}[\psi(\theta_1)]\right) \\
\leq& \delta_{P1} \cdot \kappa^*
\end{aligned}$$

The first inequality holds since $v^{\sim}_t$ has Lipschitz constant $\kappa^*$ in $\theta_{t+1}$, and the second by Assumption 1.1. *Q.E.D.*

If $y$ is expected from the respondent in period $t$, $v'_t$ gives rise to signaling payoff $\tilde{U}(y) : (\theta, x, \theta') \mapsto u_{P1}(\theta, x, y) + v'_t(\theta', \theta)$ in period $t$. So $\tilde{U} = \alpha_2(v'_t)$, where:

DEFINITION Let $\alpha_2(v'_t) : y \mapsto (\theta, x, \theta') \mapsto u_{P1}(\theta, x, y) + v'_t(\theta', \theta)$

Let $S_3$ be the set of functions $\tilde{U} : \bar{Y} \to \Phi_B$ with $\tilde{U}(y)(\theta, x, \theta')$ weakly increasing in $y$ and weakly supermodular.

FACT *(Roddie (2011b))* $\alpha_2 : \hat{\Pi} \to S_3$

Given $y$, the signaler plays according to the Riley equilibrium of this game:

$$s_Y = \alpha_3(\tilde{U}) : (y, \theta) \mapsto \mathscr{R}(\tilde{U}(y))(\theta)$$

DEFINITION Let $S_4$ be the set of weakly increasing functions $s_Y : \bar{Y} \times \Theta \to X$ with $s_Y(y, \theta)$ $\kappa_{P1}$-Lipschitz in $y$.

FACT *(Roddie (2011b))* $\alpha_3 : S_3 \to S_4$

## B.2. *Finding $s_{P2}$*

Suppose initial beliefs at $t$ are $\psi(\theta')$. Given $y$, the signaler takes strategy $s_Y(y)$. The respondent's best response is $M(\theta')(y) := \rho'(\psi(\theta'))(s_Y(y))$. Then $M = \alpha_4(s_Y)$, where:

DEFINITION Let $\alpha_4(s_Y) : \theta' \mapsto \rho'(\psi(\theta')) \circ s_Y$.

Let $S_5$ be the set of functions $M : \Theta \to \bar{Y} \to \bar{Y}$ with $M(\theta')(y)$ weakly increasing in $\theta'$ and $y$ and $\kappa_{P1} \cdot \kappa_{P2}$-Lipschitz in $y$.

FACT (*Roddie (2011b)*) $\alpha_4 : S_4 \to S_5$

Define $s_{\tilde{P2}}(\theta') := s_{P2}(\psi(\theta'))$: the respondent's action when the signaler previously signaled $\theta'$. In equilibrium $s_{\tilde{P2}}(\theta')$ is the fixed point of the map $M(\theta')$. So $s_{\tilde{P2}} = \alpha_5(M)$, where:

DEFINITION Let $\alpha_5(M) := \text{FixedPoint} \circ M$.
Let $S_6$ be the set of weakly increasing functions $\Theta \to \bar{Y}$.

FACT (*Roddie (2011b)*) $\alpha_5 : S_5 \to S_6$

### B.3. *Preserving a supermodular value function*

The Riley equilibrium of the signaling game $\tilde{U}(y)$ gives rise to a supermodular value function $W(y,\theta) = \tilde{U}(y)(\theta, s_Y(y)(\theta), \theta)$. The difference $w(y,\theta) := W(y,\theta) - W(\underline{y},\theta)$ is also supermodular. We show additionally that $W$, has an increasing difference bound of $\lambda = (\lambda_{\theta y} + \kappa_{P1} \cdot \lambda_{xy})$, and consequently a Lipschitz constant $\kappa^* = \lambda(\bar{y} - \underline{y})$ in $\theta$.

To do this we shall use the following result from Roddie (2011a), which bounds the increasing difference in type and parameter which results from a parametrized supermodular signaling game:

THEOREM 2 (*Roddie (2011a), Proposition 23) Let $\bar{Y} \subseteq \mathbb{R}$ be an interval and $\tilde{U} : Z \to \Phi_{\text{Sep}}$, and suppose for each $z$ there exists some $U \in \Phi(\bar{\Theta})$ extending $\tilde{U}(z)$. Suppose that $\tilde{U}(y)(\theta, x, \theta')$ is weakly supermodular; and suppose it has an increasing difference bound of $\beta_1$ in $(y,\theta)$, $\beta_2$ in $(y,\theta')$ and $\gamma$ in $(x,y)$, and $\delta$ in $(\theta, x)$. Suppose $\mathscr{R}(\tilde{U}(y))(\theta)$ has Lipschitz constant $\alpha$ in $y$. Let $W(y,\theta) := \tilde{U}(y)(\theta, \mathscr{R}(\tilde{U}(y))(\theta), \theta)$. Then $W$ has an increasing difference bound of $\beta_1 + \beta_2 + \alpha \cdot \gamma$.*

DEFINITION Let $\beta_2 = \alpha_2 \circ \beta_3$, where $\beta_3(\tilde{U}) : (y,\theta) \mapsto W(y,\theta) - W(\underline{y},\theta)$, where $W(y,\theta) := \tilde{U}(y)(\theta, \mathscr{R}(\tilde{U}(y))(\theta), \theta)$.
Let $\hat{S}_7$ be the set of weakly increasing, weakly supermodular functions $w : \bar{Y} \times \Theta \to \mathbb{R}$, with $w(\underline{y},\theta) = 0$, with Lipschitz constant $\kappa^*$ in the second argument, and with $w(\bar{y}, \theta_{\max}) \leq B_V$.

LEMMA $\beta_2 : \hat{\Pi} \to \hat{S}_7$

PROOF: It remains from Roddie (2011b) to show that $w(y,\theta)$ has Lipschitz constant $\kappa^*$ in $\theta$.

$\tilde{U}(y)(\theta, x, \theta')$ has an increasing difference bound of $\lambda_{\theta y}$ in $(y,\theta)$, 0 in $(y,\theta')$, $\lambda_{xy}$ in $(x,y)$, and $\lambda_{\theta x}$ in $(\theta, x)$. $s_Y(y,\theta) = \mathscr{R}(\tilde{U}(y))(\theta)$ has Lipschitz constant $\kappa_{P1}$ in $y$. So by Theorem 2, $W(y,\theta) = \tilde{U}(y)(\theta, s_Y(y,\theta), \theta)$ has an increasing difference bound of $\lambda_{\theta y} + \kappa_{P1} \cdot \lambda_{xy}$. By definition $w(y,\theta) = W(y,\theta) - W(\underline{y},\theta)$. So for $\theta_2 \geq \theta_1$:

$$\begin{aligned}
w(y,\theta_2) - w(y,\theta_1) &\leq w(\bar{y},\theta_2) - w(\bar{y},\theta_1) \\
&= W(\bar{y},\theta_2) - W(\bar{y},\theta_1) - W(\underline{y},\theta_2) + W(\underline{y},\theta_1) \\
&\leq (\lambda_{\theta y} + \kappa_{P1} \cdot \lambda_{xy}) \cdot |\bar{Y}| (\theta_2 - \theta_1) \\
&= \kappa^* (\theta_2 - \theta_1)
\end{aligned}$$

$$\begin{tikzcd}
\begin{array}{c} v^{\sim}_{t+1} \in \widehat{\Pi} \\ (\Theta \times \Theta \to \mathbb{R}) \end{array} \arrow[r, "\alpha_1"] \arrow[dd, "F"'] & v'_t \in \widehat{\Pi} \arrow[r, "\alpha_2"] \arrow[d, "\beta_2"] & \begin{array}{c} \widetilde{U} \in S_3 \\ (\bar{Y} \to \Phi_B) \end{array} \arrow[d, "\alpha_3"] \\
 & \begin{array}{c} w \in \hat{S}_7 \\ (\bar{Y} \times \Theta \to \mathbb{R}) \end{array} \arrow[dl] & \begin{array}{c} s_Y \in S_4 \\ (\bar{Y} \times \Theta \to X) \end{array} \arrow[d, "\alpha_4"] \\
v^{\sim}_t \in \widehat{\Pi} & \begin{array}{c} s^{\sim}_{P2} \in S_6 \\ (\Theta \to \bar{Y}) \end{array} \arrow[l, "\alpha_6"] & \begin{array}{c} M \in S_5 \\ (\Theta \to \bar{Y} \to \bar{Y}) \end{array} \arrow[l, "\alpha_5"]
\end{tikzcd}$$

FIGURE 2.— Value function iteration: decomposition

*Q.E.D.*

Since we know $s^{\sim}_{P2}$, we have the value function in period $t$: $v^{\sim}_t(\theta',\theta) = w\left(s^{\sim}_{P2}(\theta'),\theta\right)$. Equivalently $v^{\sim}_t = \alpha_6\left(s^{\sim}_{P2}, w\right)$ where $\alpha_6\left(s^{\sim}_{P2}, w\right) : (\hat{\theta},\theta) \mapsto w\left(s^{\sim}_{P2}(\theta'),\theta\right)$.

LEMMA $\alpha_6 : S_6 \times \tilde{S}_7 \to \tilde{\Pi}$

Figure 2 shows the logic of the text above in a commutative diagram, expressing the value function iterator $F$ as the composition of the individual maps defined above.

## APPENDIX C: PROOFS

### C.1. *Proof of Proposition 1: continuity of $F$ in $v^{\sim}$, $\Theta$, $\psi$ and $\delta_{P1}$*

*Notions of continuity*

See 5.1 for the definition of convergence when $\Theta$ is varying.

NOTATION For each set $S_j$ in the decomposition of $F$, $S^i_j$ denotes the set given type space $\Theta_i$, and for each map $\alpha_j$, $\alpha^i_j$ denotes the map given type space $\Theta_i$, Markov process $\psi_i$, and discount factor $\delta^i_{P1}$.

NOTATION Normalize the measures on $\bar{\Theta}$, $X$, and $\bar{Y}$ to 1.

The following lemma implies that functions on $\Theta_\infty$ agreeing $\Theta_i$ or $\Theta_i \times \Theta_i$ must become close eventually:

LEMMA 4 *Suppose $\Theta_i \nearrow \Theta_\infty$, $B$ is a compact interval, and one of the following holds:*

1. *$f^i_j : \Theta_\infty \to B$ for $j \in \{1,2\}$ are weakly monotonic functions agreeing on $\Theta_i$.*
2. *$f^i_j : \Theta_\infty \times \Theta_\infty \to B$ for $j \in \{1,2\}$ are weakly monotonic functions agreeing on $\Theta_i \times \Theta_i$.*

3. *$f_j^i : \Theta_\infty \times \Theta_\infty \to B$ for $j \in \{1,2\}$ are $\kappa$-Lipschitz in the first argument, weakly monotonic in the second argument, and agree on $\Theta_i \times \Theta_i$.*

*Then $f_i^2 - f_i^1 \to 0$ in L1.*

PROOF: W.l.o.g. replace "weakly monotonic" with "weakly increasing", and take $B = [0,1]$. Each case is trivial for $\Theta_\infty$ finite, so take $\Theta_\infty = \bar{\Theta}$. Since $\max_j \left(f_j^i\right)$ and $\min_j \left(f_j^i\right)$ also have the required properties in each case, assume $f_i^2 \geq f_i^1$. Let $\Theta_i = \{\theta_0 = \theta_{\min}, \theta_1, \dots \theta_n = \theta_{\max}\}$. Suppose $\theta_{i+1} - \theta_i < \delta$.

1. $\int f_2^i - f_1^i \,\mathrm{d}\mu(\theta) \leq \sum_{k=0}^{n-1} \delta \left(f_j^i(\theta_{k+1}) - f_j^i(\theta_k)\right) = \delta \left(f_j^i(\theta_n) - f_j^i(\theta_0)\right) \leq \delta.$

2. Let $U^j(l) = \left\{x \in \Theta \times \Theta : f_j^i(x) \geq l\right\}$. Take $\mu' = \mu \times \mu$. Then $\int \left(f_i^2 - f_i^2\right)(x)\,\mathrm{d}\mu'(x) = \int \mu'\left(U^j(l)\right) \mathrm{d}l \leq \sup_l \mu'\left(U^j(l)\right) = \sup \int \left(1_{U^2(l)} - 1_{U^1(l)}\right)(x)\,\mathrm{d}\mu'(x)$. Let $g^j = 1_{U^j(l)}$. Then the $g^j$ are weakly increasing and agree on $\Theta_i \times \Theta_i$. For $i, j \in \{0,1,\dots n-1\}$, let $S_{i,j} := (\theta_i, \theta_{i+1}) \times \left(\theta_j, \theta_{j+1}\right)$ and let $\chi(i,j) := \left(g^1(x) \neq g^2(x) \text{ for some } x \in S_{ij}\right)$. Then if $\chi(i,j)$ then $g^j\left(\theta_{i+1}, \theta_{j+1}\right) = 1$, so $\neg\chi\left(i^+, j^+\right)$ for $i^+ > i$, $j^+ > j$.
   Let $S := \left\{(i,j) : \chi(i,j)\right\}$, and let $S'$ be the subset with the bottom elements removed: $\left\{(i,j) \in S : \exists j^- < j \text{ with } \left(i, j^-\right) \in S\right\}$. Then there are no two elements $\left(i^-, j\right)$ and $(i,j)$ in $S'$ with $i^- < i$; otherwise there must be an element $\left(i^-, j^-\right) \in S$ with $j^- < j$, which implies $(i,j) \notin S$.
   So $\int \left(1_{U^2(l)} - 1_{U^1(l)}\right)(x)\,\mathrm{d}\mu'(x) \leq \mu'\left(\bigcup_S S_{i,j}\right) \leq \delta + \mu'\left(\bigcup_{S'} S_{i,j}\right) \leq 2 \cdot \delta$.

3. Let $g := f^2 - f^1$. Then $g$ has Lipschitz constant $2 \cdot \kappa$ in the first argument. By part 1, for $\theta_1 \in \Theta_i$, $\int g(\theta_1, \theta_2)\,\mathrm{d}\mu(\theta_2) \leq \delta$. Then the Lipschitz condition implies, for any $\theta_1$, $\int g\left(\theta_1', \theta_2\right) \mathrm{d}\mu(\theta_2) \leq \delta + 2 \cdot \kappa \left|\theta_1' - \theta_1\right|$. So $\int g\left(\theta_1', \theta_2\right) \mathrm{d}\mu(\theta_2) \leq \delta + 2 \cdot \kappa \cdot d\left(\theta_1', \Theta_i\right)$. Integrating over $\theta_1'$, we have: $\int g(x)\,\mathrm{d}\mu'(x) \leq \delta(1+\kappa)$.

*Q.E.D.*

*Some regular continuity results*

In the following we fix $\Theta, \psi$ and $\delta_{P1}$ to get straight continuity results, for those maps where this is useful as an intermediate step to extended continuity.

LEMMA 5 *$\alpha_1$ is continuous.*

PROOF: $\alpha_1$ extends to a linear map from the space of L1 functions $\Theta \times \Theta \to \mathbb{R}$ to itself.

$$\begin{aligned} \|\alpha_1(v^\sim)\| &= \delta_{P1} \cdot \int\int\int \left|v^\sim\left(\theta', \theta_{t+1}\right)\right| \mathrm{d}\left[\psi(\theta_t)\right](\theta_{t+1})\,\mathrm{d}\mu\left(\theta'\right) \mathrm{d}\mu(\theta_t) \\ &= \delta_{P1} \cdot \int\int \left|v^\sim\left(\theta', \theta_{t+1}\right)\right| \mathrm{d}\mu\left(\theta'\right) \mathrm{d}\mu^*(\theta_{t+1}) \end{aligned}$$

Compare to:

$$\|v^\sim\| = \int\int \left|v^\sim\left(\psi(\theta'), \theta_{t+1}\right)\right| \mathrm{d}\mu\left(\theta'\right) \mathrm{d}\mu(\theta_{t+1})$$

Absolute continuity of $\mu^*$ with respect to $\mu$ (Assumption 1) implies that $\alpha_1^{\sim}$ is continuous. *Q.E.D.*

LEMMA 6 *$\alpha_2$ is continuous.*

PROOF: This map has Lipschitz constant 1. *Q.E.D.*

LEMMA 7 *$\alpha_4$ is continuous.*

PROOF: We have Lipschitz constant $\kappa_{P2}$ on $s_Y \mapsto \rho'(\psi(\theta)) \circ s_Y$:

$$\left|\rho'(\psi(\theta)) \circ s_Y^2(y) - \rho'(\psi(\theta)) \circ s_Y^1(y)\right| \leq \kappa_{P2} \cdot \int \left|(s_Y^2(y) - s_Y^1(y))(\theta_t)\right| \mathrm{d}\psi(\theta)(\theta_t)$$

Integrating over $\theta$ and $y$:

$$\left\|\alpha_4(s_Y^2) - \alpha_4(s_Y^1)\right\| \leq \kappa_{P2} \cdot \int\int \left|(s_Y^2(y) - s_Y^1(y))(\theta_t)\right| \mathrm{d}\mu^*(\theta_t)\,\mathrm{d}y$$

So by absolute continuity of $\mu^*$ with respect to $\mu$, $\alpha_4$ is continuous. *Q.E.D.*

LEMMA 8 *$\alpha_5$ is continuous.*

PROOF: Take $M_1, M_2 \in S_5$ and let $s_{P2}^{\sim}(i) = \alpha_5(M_i)$.
Fix $\theta$. Suppose $s_{P2}^{\sim}(2)(\theta') = s_{P2}^{\sim}(1)(\theta') + \delta$, $\delta \geq 0$.
These are the fixed points of $M_i(\theta')$, increasing maps with Lipschitz constant $\kappa_{P1} \cdot \kappa_{P2}$. So:

$$\int \left|M_2(\theta')(y) - M_1(\theta')(y)\right| dy \geq \delta \cdot (1 - \kappa_{P1} \cdot \kappa_{P2})$$

So for each $\theta$, the fixed point map has Lipschitz constant $(1 - \kappa_{P1} \cdot \kappa_{P2})^{-1}$ in the L1 topology, so this is also true for the integrals over $\theta$. *Q.E.D.*

LEMMA 9 *$\alpha_6$ is continuous.*

PROOF: That is, if $w_i \to w_\infty$ and $s_{P2}^{\sim}(i) \to s_{P2}^{\sim}(\infty)$, then if $v_i^{\sim}(\theta', \theta) = w_i\left(s_{P2}^{\sim}(i)(\theta'), \theta\right)$, then $v_i^{\sim} \to v_\infty^{\sim}$.
$w_i(., \theta) \to w_\infty(., \theta)$ in L1 for almost all $\theta$. Sake such a $\theta$. Since $w_\infty(., \theta)$ is continuous and the $w_i(., \theta)$ are monotonically increasing, $w_i(., \theta) \to w_\infty(., \theta)$ uniformly. Then for such $\theta$, in L1:

$$\left(\theta' \to w_i\left(s_{P2}^{\sim}(i)(\theta'), \theta\right)\right) \to \left(\theta' \to w_\infty\left(s_{P2}^{\sim}(\infty)(\theta'), \theta\right)\right)$$

Integrating over $\theta$, using dominated convergence, we have in L1 as required:

$$\left((\theta', \theta) \to w_i\left(s_{P2}^{\sim}(i)(\theta'), \theta\right)\right) \to \left((\theta', \theta) \to w_\infty\left(s_{P2}^{\sim}(\infty)(\theta'), \theta\right)\right)$$

*Q.E.D.*

*The extended continuity result*

PROPOSITION *Suppose $\Theta_i \nearrow \Theta_\infty$, $\psi_i \nearrow \psi_\infty$ and $\delta^i_{P1} \to \delta^\infty_{P1}$. Suppose $v^\sim_i \in \Pi_i$ with $v^\sim_i \nearrow v^\sim_\infty$. Then $F\left(v^\sim_i\right) \nearrow F\left(v^\sim_\infty\right)$.*

This form of continuity holds for each map in the decomposition of $F$, and the continuity property for $F$ follows from those. On each of the sets in Figure 2, use the L1 topology, with $\mu$ as the fundamental measure on $\Theta$.

We know from Roddie (2011a) that the Riley equilibrium is continuous in both the payoff function and type space:

THEOREM 3 *(Roddie (2011a), Proposition 15) Take $\hat{U}_i \in \Phi(\bar{\Theta})$, with $\hat{U}_i \to \hat{U}_\infty$ in L1 and $\hat{U}_i(\theta, x, \theta')$ uniformly continuous in $(a, \theta)$ over $(i, \theta')$. Suppose $\Theta_i \nearrow \Theta_\infty$. Then $\mathscr{R}(\hat{U}_i|_{\Theta_i}) \nearrow \mathscr{R}(\hat{U}_\infty)$ in L1.*

This will give us continuity for $\alpha_3$, the most difficult of the component maps of $F$. Now we proceed with showing extended continuity for each map. In the following, we assume $\left(\Theta_i, \psi_i\right) \nearrow \left(\Theta_\infty, \psi_\infty\right)$ and $\delta^i_{P1} \to \delta^\infty_{P1}$.

LEMMA 10 *Suppose $v^\sim_i \in \Pi^i$, and $v^\sim_i \nearrow v^\sim_\infty$. Let $v'_i = \alpha_1\left(v^\sim_i\right)$. Then $v'_i \nearrow v'_\infty$.*

PROOF: Take $\bar{v}^\sim_i \in \Pi^\infty$ extending $v^\sim_i \in \Pi^i$, with $\bar{v}^\sim_i \to v^\sim_\infty$. Also $\bar{\psi}_i$ extending $\psi_i$ with $\bar{\psi}_i \to \psi_\infty$.

Let $\bar{v}'_i\left(\theta', \theta\right) := \delta^i_{P1} \cdot \int \bar{v}^\sim_i\left(\theta', \theta_{t+1}\right) \mathrm{d}\bar{\psi}_i(\theta)(\theta_{t+1})$. Then $\bar{v}'_i$ extends $v'_i$. Fix $\theta'$ and suppose that $\bar{v}^\sim_i\left(\theta', \cdot\right) \to v^\sim_\infty\left(\theta', \cdot\right)$. This holds for almost all $\theta'$. Then convergence is uniform, since the functions are Lipschitz.

Take $\epsilon > 0$ and $i$ such that $\left\|\bar{v}^\sim_i\left(\theta', \cdot\right) - v^\sim_\infty\left(\theta', \cdot\right)\right\|_\infty < \epsilon$, $\left(\delta^\infty_{P1} - \delta^i_{P1}\right) < \epsilon$. Let $f := v^\sim_\infty\left(\theta', \cdot\right)$.

$$
\begin{aligned}
& \left|\bar{v}'_i\left(\theta', \theta\right) - v'_\infty\left(\theta', \theta\right)\right| \\
= & \left|\delta^i_{P1} \cdot \int \bar{v}^\sim_i\left(\theta', \theta_{t+1}\right) \mathrm{d}\bar{\psi}_i(\theta)(\theta_{t+1}) - \delta^\infty_{P1} \cdot \int f(\theta_{t+1}) \mathrm{d}\bar{\psi}_\infty(\theta)(\theta_{t+1})\right| \\
\leq & \left(\delta^\infty_{P1} - \delta^i_{P1}\right)\left|\int \bar{v}^\sim_i\left(\theta', \theta_{t+1}\right) \mathrm{d}\bar{\psi}_i(\theta)(\theta_{t+1})\right| \delta^\infty_{P1} \cdot \int \left|\bar{v}^\sim_i\left(\theta', \theta_{t+1}\right) - \bar{v}^\sim_\infty\left(\theta', \theta_{t+1}\right)\right| \mathrm{d}\bar{\psi}_i(\theta)(\theta_{t+1}) \\
& + \delta^\infty_{P1} \cdot \left|\int f(\theta_{t+1}) \mathrm{d}\bar{\psi}_i(\theta)(\theta_{t+1}) - \int f(\theta_{t+1}) \mathrm{d}\bar{\psi}_\infty(\theta)(\theta_{t+1})\right| \\
\leq & \epsilon \cdot B_V + \epsilon + \left|\int f(\theta_{t+1}) \mathrm{d}\bar{\psi}_i(\theta)(\theta_{t+1}) - \int f(\theta_{t+1}) \mathrm{d}\bar{\psi}_\infty(\theta)(\theta_{t+1})\right|
\end{aligned}
$$

Integrating over $\theta$:

$$
\begin{aligned}
& \int \left|\bar{v}'_i\left(\theta', \theta\right) - v'_\infty\left(\theta', \theta\right)\right| \mathrm{d}\mu(\theta) \\
\leq & \epsilon \cdot B_V + \epsilon + \left|\int f(\theta_{t+1}) \mathrm{d}\bar{\mu}^*_i(\theta_{t+1}) - \int f(\theta_{t+1}) \mathrm{d}\bar{\mu}^*_\infty(\theta_{t+1})\right| \\
\to & \epsilon \cdot B_V + \epsilon
\end{aligned}
$$

The last term tends to 0 because $f$ is continuous and $\bar{\mu}_i^* \to \bar{\mu}_\infty^*$ in distribution.

So for almost all $\theta'$, $\int \left|\bar{v}_i'(\theta',\theta) - v_\infty'(\theta',\theta)\right| \mathrm{d}\mu(\theta) \to 0$. Integrating with respect to $\theta'$, using dominated convergence, $\bar{v}_i' \to v_\infty'$. By Lemma 4, this is sufficient for $v_i' \nearrow v_\infty'$. *Q.E.D.*

LEMMA 11 *Suppose $v_i' \in \Pi^i$, with $v_i' \nearrow v_\infty'$. Let $\tilde{U}_i = \alpha_2(v_i')$. Then $\tilde{U}_i \nearrow \tilde{U}_\infty$.*

PROOF: Take $\bar{v}_i' \in \Pi^\infty$ extending $v_i' \in \Pi^i$, and $\bar{v}_i' \to v_\infty'$.

Since $\bar{v}_i'$ extends $v_i'$, $\alpha_2(\bar{v}_i')$ extends $\tilde{U}_i = \alpha_2(v_i')$. Suppose $\bar{U}_i$ also extends $\tilde{U}_i$. Then $\bar{U}_i$ and $\alpha_2(\bar{v}_i')$ agree on $\bar{Y} \times \Theta_i \times X \times \Theta_i$. Fixing $y$ and $x$, $\left(\bar{U}_i - \alpha_2(\bar{v}_i')\right)(y)(\cdot,x,\cdot) \to 0$ by Lemma 4. Integrating over $x$ and $y$, $\bar{U}_i - \alpha_2(\bar{v}_i') \to 0$. By continuity of $\alpha_2$, $\alpha_2(\bar{v}_i') \to \alpha_2(v_\infty') = \tilde{U}_\infty$. So $\bar{U}_i \to \tilde{U}_\infty$. *Q.E.D.*

LEMMA 12 *Suppose $\tilde{U}_i \in S_3^i$, with $\tilde{U}_i \nearrow \tilde{U}_\infty$. Let $s_Y^i = \alpha_3(\tilde{U}_i)$. Then $s_Y^i \nearrow s_Y^\infty$.*

PROOF: Take $\bar{U}_i \in S_3^\infty$ extending $\tilde{U}_i \in S_3^i$ with $\bar{U}_i \to \tilde{U}_\infty$. Take $\bar{s}_Y^i(y)$ extending $s_Y^i$.

For almost every $y \in \bar{Y}$, $\bar{U}_i(y) \to \tilde{U}_\infty(y)$. The set of signaling payoffs is also Lipschitz and so uniformly continuous in $(\theta, x)$. Then for such $y$, by Theorem 3, $\bar{s}_Y^i(y) \to s_Y^\infty(y)$. Integrating the difference with respect to $y$, dominated convergence implies $\bar{s}_Y^i \to s_Y^\infty$. *Q.E.D.*

LEMMA 13 *Suppose $s_Y^i \in S_4^i$, with $s_Y^i \nearrow s_Y^\infty$. Suppose $M_i = \alpha_4(s_Y^i)$. Then $M_i \nearrow M_\infty$.*

PROOF: Take $\bar{s}_Y^i \in S_4$ extending $s_Y^i \in S_4^i$, with $\bar{s}_Y^i \to s_Y^\infty$.

Let $\bar{M}_i(\theta')(y) := \rho'(\psi_i(\theta'))(\bar{s}_Y^i(y))$. Then $\bar{M}_i$ extends $M_i$.

Let $(\epsilon) f : x \mapsto f(x+\epsilon)$. Then if $f_i : \bar{\Theta} \to X$ is weakly increasing with $f_i \to f_\infty$ in L1, then $((-\epsilon) f_i)(x) \to [f(x), \infty]$. So $(+\epsilon)(\bar{s}_Y^i(y,\cdot))(\theta) \to [\bar{s}_Y^\infty(y,\theta), \infty]$. So:

$$\begin{aligned}
\bar{M}_i(\theta' + 2\cdot\epsilon)(y) &= \rho'(\psi_i(\theta' + 2\cdot\epsilon))\left(\bar{s}_Y^i(y,\cdot)\right) \\
&\geq \rho'(\psi_\infty(\theta') + \epsilon)\left(\bar{s}_Y^i(y,\cdot)\right) \\
&\geq \rho'(\psi_\infty(\theta'))\left((+\epsilon)\left(\bar{s}_Y^i(y,\cdot)\right)\right) \\
&\to \left[\rho'(\psi_\infty(\theta'))(\bar{s}_Y^\infty(y,\cdot)), \infty\right] \\
&= \left[M_\infty(\theta')(y), \infty\right)
\end{aligned}$$

So $\bar{M}_i(\theta' + 2\cdot\epsilon)(y) \to \left[M_\infty(\theta')(y), \infty\right)$. Symmetrically, $\bar{M}_i(\theta' - 2\cdot\epsilon)(y) \to \left(\infty, M_\infty(\theta')(y)\right]$.

So $\bar{M}_i(\cdot)(y) \to M_\infty(\cdot)(y)$ in L1. Integrating over $y$, $\bar{M}_i \to M_\infty$. By Lemma 4 this is sufficient for $M_i \nearrow M_\infty$. *Q.E.D.*

LEMMA 14 *Suppose $M_i \in S_5^i$, with $M_i \nearrow M_\infty$. Let $s_{P2}^i = \alpha_5(M_i)$. Then $s_{P2}^i \nearrow s_{P2}^\infty$.*

PROOF: Take $\bar{M}_i \in S_5$ extending $M_i \in S_5^i$, with $\bar{M}_i \to M_\infty$. Then $\alpha_5(\bar{M}_i)$ extends $s_{P2}^i$.

Suppose $\bar{s}_{P2}^i \in S_6$ extends $s_{P2}^i$. Then $\alpha_5(\bar{M}_i) - \bar{s}_{P2}^i \to 0$ by Lemma 4. By continuity of $\alpha_5$, $\alpha_5(\bar{M}_i) \to \alpha_5(M_\infty) = s_{P2}^\infty$. So $\bar{s}_{P2}^i \to s_{P2}^\infty$. *Q.E.D.*

LEMMA 15 *Suppose* $v'_i \in S_2^i$, *with* $v'_i \nearrow v'_\infty$. *Let* $w_i = \beta_2\left(v'_i\right)$. *Then* $w_i \nearrow w_\infty$.

PROOF: Let $s_Y^i := \alpha_3 \circ \alpha_2\left(v'_i\right)$. By extended continuity of $\alpha_2$ and $\alpha_3$, $s_Y^i \nearrow s_Y^\infty$. Suppose $\bar{s}_Y^i$ extends $s_Y^i$. Then $\bar{s}_Y^i \to s_Y^i$.

Then $w_i = w_i\left(y,\theta\right) = u_{P1}\left(\theta, s_Y^i\left(y,\theta\right), y\right) - u_{P1}\left(\theta, s_Y^i\left(\underline{y},\theta\right), \underline{y}\right)$.

Then $\bar{w}_i : \left(y,\theta\right) \mapsto u_{P1}\left(\theta, \bar{s}_Y^i\left(y,\theta\right), y\right) - u_{P1}\left(\theta, \bar{s}_Y^i\left(\underline{y},\theta\right), \underline{y}\right)$ extends $w_i$. By continuity of $u_{P1}$, $\bar{w}_i \to w_\infty$.

This is sufficient for $w_i\left(y,\cdot\right) \nearrow w_\infty\left(y,\cdot\right)$ by Lemma 4. Integrating over $y$, we have $w_i \nearrow w_\infty$. *Q.E.D.*

LEMMA 16 *Suppose* $s_{P2}^i \in S_6^i$ *and* $w_i \in S_7^i$, *with* $s_{P2}^i \nearrow s_{P2}^\infty$ *and* $w_i \nearrow w_\infty$. *Let* $v_i^\sim = \alpha_6\left(w_i, s_{P2}^i\right)$. *Then* $v_i^\sim \nearrow v_\infty^\sim$.

PROOF: Take $\bar{s}_{P2}^i \in S_6$ extending $s_{P2}^i \in S_6^i$ and $\bar{w}_i \in S_7$ extending $w_i \in S_7^i$, with $\bar{s}_{P2}^i \to s_{P2}^\infty$ and $\bar{w}_i \to w_\infty$.

Suppose $\bar{v}_i^\sim \in \Pi$ extends $v_i^\sim$. $\alpha_6\left(\bar{w}_i, \bar{s}_{P2}^i\right)$ also extends $v_i^\sim$, so by Lemma 4, $\alpha_6\left(\bar{w}_i, \bar{s}_{P2}^i\right) - \bar{v}_i^\sim \to 0$. By continuity of $\alpha_6$, $\alpha_6\left(\bar{w}_i, \bar{s}_{P2}^i\right) \to \alpha_6\left(w_\infty, s_{P2}^\infty\right) = v_\infty^\sim$. So $\bar{v}_i^\sim \to v_\infty^\sim$. *Q.E.D.*

## C.2. *Proof of Lemma 1*

LEMMA *There exists* $v^\sim \in \hat{\Pi}$ *such that* $F\left(v^\sim\right) = v^\sim$ *a.e..*

PROOF: This is known for finite $\Theta$. Suppose $\Theta = \bar{\Theta}$. Define $\Theta_\infty := \bar{\Theta}$. Then take $\Theta_i$ increasing with $\{\theta_{\min}, \theta_{\max}\} \in \Theta_i$ and $\bigcup_{i<\infty}\Theta_i$ dense in $\Theta_\infty$. Define $\psi_i : \Theta_i \to \triangle\Theta_i$ by $\psi_i\left(\theta\right)\left(\left\{\theta' : \theta' \le \theta^*\right\}\right) = \psi\left(\theta_i\right)\left(\left\{\theta : \theta \le \theta_j\right\}\right)$. Then $\left(\Theta_i, \psi_i\right) \nearrow \left(\Theta_\infty, \psi_\infty\right)$.

By Corollary 2 the map $F_i$ associated with $\Theta_i, \psi_i$ has a fixed point $v_i^\sim \in \hat{\Pi}\left(\Theta_i\right)$. Take $\bar{v}_i^\sim \in \hat{\Pi}$ extending $\bar{v}_i^\sim$.

Then $\bar{v}_i^\sim$ has a subsequence converging to some $v_\infty^\sim$. Discarding other elements, assume $\bar{v}_i^\sim \to v_\infty^\sim$. Then $\bar{v}_i^\sim = F\left(\bar{v}_i^\sim\right) \to F\left(v_\infty^\sim\right)$ by Proposition 1. So $v_\infty^\sim = F\left(v_\infty^\sim\right)$ a.e.. *Q.E.D.*

## C.3. *Proof of Lemma 2*

LEMMA *If* $v_1^\sim \in \Pi$ *satisfies* $F^\sim\left(v_1^\sim\right) = v_1^\sim$ *a.e., then there exists* $v_2^\sim \in \hat{\Pi}$ *with* $v_2^\sim = v_1^\sim$ *a.e. and* $F\left(v_2^\sim\right) = v_2^\sim$.

PROOF: Take $v_1^\sim$ with $F\left(v_1^\sim\right) = v_1^\sim$ a.e..

The map $\beta_w := \left(\beta_2 \circ \alpha_1\right)$ takes value functions $v^\sim$ into $w$. Two $v^\sim \in \hat{\Pi}$ that are equal a.e. give rise to two $w$ that are equal a.e., since the maps involved are continuous, and so equal, since each $w$ is continuous. Let $w^* := \beta_w\left(v_1^*\right)$.

The map $\alpha_{P2} := \left(\alpha_5 \circ \alpha_4 \circ \alpha_3 \circ \alpha_2 \circ \alpha_1\right)$ takes value functions $v^\sim$ into strategies $s_{P2}^\sim$ of the respondent. Two $v^\sim$ that are equal a.e. give rise to two $s_{P2}^\sim$ that are equal a.e.. Let $s_{P2}^\sim := \alpha_{P2}\left(v_1^\sim\right)$. Then $s_{P2}^\sim = \alpha_{P2}\left(F\left(v_1^\sim\right)\right)$ a.e., so they are equal except on a measure 0 set $S$, not including $\theta_{\min}$ and $\theta_{\max}$ (which have positive measure). ($S$ is countable since both functions are weakly increasing.)

We shall adjust $s_{P2}^\sim$ on $S$, (holding $w^*$ constant), to get a fixed point.

Fix $\theta'' \in S$. Let $D\left(\theta''\right) := \left[\sup_{\theta^* \notin S, \theta^* < \theta''} s_{P2}^\sim\left(\theta^*\right), \inf_{\theta^* \notin S, \theta^* > \theta''} s_{P2}^\sim\left(\theta^*\right)\right]$. Provided $y'' \in D$, there exists $s_{P2}^* \in S_6$ with $s_{P2}^* = s_{P2}^\sim$ on $S$, and $s_{P2}^*\left(\theta''\right) = y''$. Then let $v_2^\sim = \alpha_6\left(w^*, s_{P2}^*\right)$, and $s_{P2}^{**} = \alpha_{P2}\left(v_2^\sim\right)$. We know $w^* = \beta_w\left(v_2^{\sim}\right)$, since $v_2^{\sim} = v_1^{\sim}$ a.e.. Let $s_Y := (\alpha_3 \circ \alpha_2 \circ \alpha_1)\left(v_1^{\sim}\right)$ and $s_Y^* := (\alpha_3 \circ \alpha_2 \circ \alpha_1)\left(v_2^{\sim}\right)$. Then $s_Y^*\left(y, \theta\right) = s_Y\left(y, \theta\right)$ for $\theta \notin S$, and $s_Y^*\left(y, \theta'\right)$ is a continuous (increasing) function of $y''$.

Let $M := \alpha_4\left(s_Y\right)$ and $M^* := \alpha_4\left(s_Y^*\right)$. Then by Assumption 1.3, $M = M^*$ outside $S$, and $M\left(y\right)\left(\theta''\right)$ is a continuous (increasing) function of $y''$. Let $s_{P2}^{**} := \alpha_5\left(M^*\right)$. Then $s_{P2}^{**} = s_{P2}^{\sim} = s_{P2}^*$ outside $S$ and $s_{P2}^*\left(\theta''\right)$ is a continuous (increasing) function of $y''$, $f\left(y''\right)$.

Since $s_{P2}^* \in S_6$ and so weakly increasing, we know $f\left(y''\right) \in D$. So $f : D \to D$ is continuous, and so there is a fixed point $y^*$.

For each $\theta'' \in S$, take such a $y^*\left(\theta''\right)$. Then let $s_{P2}^*\left(\theta\right) = \begin{cases} s_{P2}\left(\theta\right) & \theta \in S \\ y^*\left(\theta\right) & \theta \notin S \end{cases}$. Then $s_{P2}^{**} = s_{P2}$. Taking $v_2^* = \alpha_6\left(w^*, s_{P2}^*\right)$ gives the fixed point of $F$. *Q.E.D.*

### C.4. *Proof of Proposition 3*

PROOF: Taking any $v_\infty^{\sim}$ such that $v_i^{\sim} \nearrow v_\infty^{\sim}$ Proposition 1 shows that $F\left(v_\infty^{\sim}\right) = v_\infty^{\sim}$ a.e., 2 shows then there exists $v_\infty^{\sim}$ with $v_i^{\sim} \nearrow v_\infty^{\sim}$ and $F\left(v_\infty^{\sim}\right) = v_\infty^{\sim}$. Using the notation and continuity results of C.1, ${}^i s_{P2}^{\sim} = \alpha_{P2}\left(v_i^{\sim}\right) \nearrow \alpha_{P2}\left(v_\infty^{\sim}\right) = {}^\infty s_{P2}^{\sim}$, where $\alpha_{P2} := (\alpha_5 \circ \alpha_4 \circ \alpha_3 \circ \alpha_2 \circ \alpha_1)$, and similarly for ${}^i s_{P1}^{\sim}$. *Q.E.D.*

### C.5. *Proof of Lemma 3*

LEMMA $U_{\mathrm{St}} \in \Phi$.

PROOF: Since $\rho\left(\left[\theta', x\right]\right)$ is continuous in $\left(\theta', x\right)$, $U_{\mathrm{St}}$ is continuous, so uniformly continuous. It is weakly increasing in $\theta'$. $(U_{\mathrm{St}}(\theta, x, \theta_1) = U_{\mathrm{St}}(\theta, x, \theta_2))$ if and only if $\rho([\theta_1, x]) = \rho([\theta_2, x])$, which is independent of $x$. It satisfies single crossing since $u_{P1}$ satisfies single crossing in $(\theta; x, y)$ and $\rho\left(\left[\theta', x\right]\right)$ is weakly increasing in $\left(\theta', x\right)$. The final condition for $U_{\mathrm{St}} \in \Phi$ is Assumption 7. *Q.E.D.*

### C.6. *Proof of Proposition 4*

PROPOSITION *$\sigma_L$ satisfies IC for $U_{\mathrm{St}}$.*

PROOF: We are in a dynamic Riley equilibrium with specifications $\bar{\Theta}, \psi_{id}, \delta_{P1}^*$.

Define $y := \rho(\left[(\theta, s_{P1}^{\sim}(\theta, \theta)\right])$ and $y' := \rho(\left[(\theta', s_{P1}^{\sim}(\theta', \theta')\right])$. When beliefs are $\left[\theta'\right]$, type $\theta$ takes action $s_{P1}^{\sim}(\theta', \theta)$ rather than $s_{P1}^{\sim}(\theta', \theta')$. So we must have:

$$
\begin{aligned}
& u_{P1}\left(\theta, s_{P1}^{\sim}(\theta', \theta'), y_2\right) + \delta_{P1}^* \cdot u_{P1}\left(\theta, s_{P1}^{\sim}(\theta', \theta), y_2\right) \\
\leq\ & u_{P1}\left(\theta, s_{P1}^{\sim}(\theta', \theta), y_2\right) + \delta_{P1}^* \cdot u_{P1}\left(\theta, s_{P1}^{\sim}(\theta, \theta), y_1\right)
\end{aligned} \tag{1}
$$

First consider the case $\delta_{P1}^* = 1$. Cancelling in the above inequality we get:

$$u_{P1}\left(\theta, s_{P1}^{\sim}(\theta', \theta'), y'\right) \leq u_{P1}\left(\theta, s_{P1}^{\sim}(\theta, \theta), y\right)$$

In other words: $U_{\mathrm{St}}\left(\theta,\sigma_L(\theta'),\theta'\right) \leq U_{\mathrm{St}}(\theta,\sigma_L(\theta),\theta)$. So $\sigma_L$ satisfies IC for $U_{\mathrm{St}}$.

Now consider the additively separable case. Now $s_{P1}^{\sim}(\theta',\theta) = \sigma_L(\theta)$ does not depend on $\theta'$. Equation 1 implies, cancelling terms:

$$u_X\left(\theta,\sigma_L(\theta')\right) + \delta^*_{P1}\cdot u_Y\left(\theta,\rho\left(\left[\theta',\sigma_L(\theta')\right]\right)\right) \leq u_X(\theta,\sigma_L(\theta)) + \delta^*_{P1}\cdot u_Y\left(\theta,\rho\left([\theta,\sigma_L(\theta)]\right)\right)$$

In other words, as required: $U_{\mathrm{St}}(\theta,\sigma_L(\theta'),\theta') \leq U_{\mathrm{St}}(\theta,\sigma_L(\theta),\theta)$. *Q.E.D.*

### C.7. *Proof of Proposition 5*

See supplementary material.

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