Microeconomics

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Exercise: Continuous preferences

If $u : X \to R$ a continuous function represents $\succeq$, show that $\succeq$ is rational and continuous.

u is defined on X, so all the elements of X are comparable and $\succeq$ is complete.

Suppose x, y and z elements of X such that $u(x) \geq u(y) (\Leftrightarrow x \succeq y)$ and $u(y) \geq u(z)$ $(\Leftrightarrow y \succeq z)$. It implies directly that $\succeq$ is transitive since $u(x) \geq u(z) \;\; \Leftrightarrow x \succeq z$.

If u is continuous, we know by the characterization of the continuity by the limits that: assuming two sequences $(x_n) \to x$ and $(y_n) \to y$ implies $u(x_n) \to u(x)$ and $u(y_n) \to u(y)$. Now suppose that the two sequences are such that $\forall n,\ (x_n) \succeq (y_n)$ $(\Leftrightarrow u(x_n) \geq u(y_n))$.

Here, there are two cases : either the two sequences are such that $\exists N \in \mathbb{N},$ s.t.,$\forall n \geq N, u(x_n) = u(y_n)$ and the result is immediate. Either $\exists N \in \mathbb{N},$ s.t.,$\forall n \geq N, u(x_n) > u(y_n) + \eta$. We know $\forall \epsilon > 0,\ \exists N',\ \forall n > N',\ |u(x_n) - u(x)| < \epsilon,\ |u(y_n) - u(y)| < \epsilon$. Thus it is still true for:

$$\bar{\epsilon} = \frac{\eta}{2}$$

Then we can write that:

$$u(x) \geq u(x_n) - \bar{\epsilon} \geq u(y_n) + \bar{\epsilon} \geq u(y)$$

It means that $u(x) \geq u(y)$ and finally implies by the definition of u that $x \succeq y$, the expected result.

May a continuous preference be represented by a discontinuous utility function?

Yes, an example where we set the function $f : \mathbb{R} \to \mathbb{R}$: $f(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}) \\ x+1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$

Then, for any continuous function u which takes values in $[0, 1]$ (with respect to the Debreu's Theorem) that represents the preferences relation, the $f \circ u$ still represents the preferences (in fact, any composition of a utilty function representing the preferences by a strictly monotonic function will preserves representation of the preferences). Indeed, suppose $x, y \in X$, if $u(x), u(y) < \frac{1}{2}$, it is immediate that $f \circ u$ still represents the preferences. If $u(x),\ u(y) > \frac{1}{2}$, it is straightforward that $u(x) \geq u(y)$ $(\Leftrightarrow x \succeq y) \Leftrightarrow u(x) + 1 \geq u(y) + 1$. The conclusion is the same when x and y are not in the same subset with respect to u.

Show that when the alternatives set is $\mathbb{R}$, the preference represented by the floor function is not continuous

We set two sequences in $\mathbb{R}$ such that with $\epsilon < 0.01$, and $n \in \mathbb{N}^*$:

$$\begin{cases} x_n = 1 - \epsilon^n \to x \equiv 1 \\ y_n = 0.99 + \epsilon^n \to y \equiv 0.99 \end{cases}$$

Since the preferences are represented by the floor function, we get by $u(y) \geq u(x) \Leftrightarrow y \succeq x$ that $\forall n,\ y_n \succeq x_n$ since $\lfloor x_n \rfloor = \lfloor y_n \rfloor = 0$.

However we notice that $\lfloor x \rfloor > \lfloor y \rfloor$. Then $x \succ y$ and finally $\neg(y \succeq x)$. By definition, the preferences are not continuous.