Deep Dive of Black and Scholes Framework

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INTRODUCTION

Our objective is to bring value to those eager to learn more about the Black and Scholes framework. The chapters are meant to guide the reader step by step from the basics to the in and out. Indeed, we develop the article as if we were holding a lecture.

The Black-Scholes formula is an expression for the current value of a European call option on a stock which pays no dividends before expiration of the option. The formula is:

$$C = SN(d_1) - Ke^{-rt}N(d_2)$$

where C is the current value of the call, S is the current value of the stock, r is the interest rate (assumed constant), t is the remaining time to expiration of the option, K is the Strike, and $N(d_1)$ and $N(d_2)$ are probability factors with N as the cumulative standard normal distribution function. Going through the d_1 and d_2 computation, we find a volatility factor, σ, which is a constant as well by the Black and Scholes definition:

$$d_1 = \frac{ln(S/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} = d_2 + \sigma\sqrt{t}$$

$$d_2 = \frac{ln(S/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

In the following chapters we will try to give a sense of these formulas through the understanding and demonstration of $N(d_1)$ and $N(d_2)$.

Suggested prerequisite knowledge: Stochastic Processes, Probability Theory, Financial Mathematics and Statistics.

PAYOFF OF THE CALL OPTION

The payoff of a generic European Call Option at maturity T is:

$$C_t = max\{0, S_T - K\} = \begin{cases} S_T - K & \text{if} \quad S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

To enter into this contract the buyer pays a premium and the option will be exercised only if S_T ≥K, otherwise it expires worthless. For our purpose it is useful to split the payoff into two components:

1. The first component is the payment of the exercise price, contingent on the option finishing in the money P(S_T ≥K). This is expressed by the following Unconditional Probability:

$$C_t^1 = \begin{cases} -K & \text{if} \quad S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Expected Payoff} = E[-K|S_T \geq K]$$

and since K is a known quantity, we can take it out of from the expected value:

$$E[-K|S_T \geq K] = -KP(S_T \geq K) = -KN(d_2)$$

However, since K is a future payment, to price the option today, we need to discount K by e^{-rt}:

$$\text{PV(Expected Payoff)} = -Ke^{-rt}N(d_2)$$

as N(d_2) is equal to the probability that the stock price is above the strike price at expiry date.

2. The second component is the expectation of the value of the stock given the exercise price, contingent on the option finishing in the money P(S_T ≥K). There is a big difference with respect to the previous case, if before we had an unconditional probability P($S_T \geq K$) now we have a conditional expectation: how much will be the value of S given the option is ITM * P($S_T \geq K$). S is a random variable and not a constant as of $-K$, therefore we cannot simply take it out from the expectation but rather consider the expected value of S:

$$C_t^2 = \begin{cases} S_T & \text{if} \quad S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Expected Payoff} = E[S_T|S_T \geq K] = SN(d_1)$$

To summarize if we exercise the option at maturity, K represents what we pay and S_T what we get in return. Of course, the exercise depends on P(S_T ≥K). By putting together the two expected payoffs, we get back to the original Black-Scholes formula.

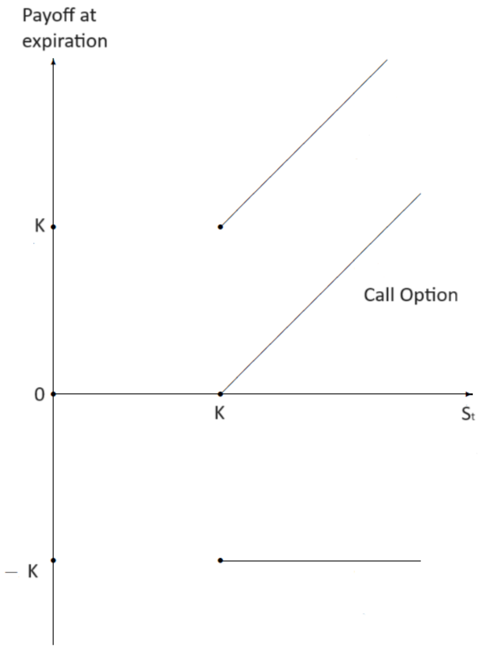

Figure 1: Call payoff and its components

So far we have focused only on the intrinsic value of an option, however a big chunk of the price is coming from time value, composed by time to expiry and volatility. In the next steps we will go through $N(d_1)$ and $N(d_2)$, which are the probability functions that characterize this model. To understand them, first we have to consider the stock price as a Geometric Brownian motion with a drift rate equal to (r - $\frac{\sigma^2}{2}$)t and a diffusion coefficient equal to $\sigma\sqrt{t}$. In this framework prices are log normally distributed [0,+∞) and returns are normally distributed (-∞,+ ∞). Indeed, by risk-neutral measure definition, S_T has the following distribution:

$$S_T \sim S_t \cdot \mathcal{LN}((r - \frac{\sigma^2}{2})t, \sigma^2 t)$$

So:

$$ln(\frac{S_T}{S_t}) \sim \mathcal{N}((r - \frac{\sigma^2}{2})t, \sigma^2 t)$$

with St known and 0<t<T. At this point we substitute the stock price at maturity S_T with the known strike price K to define a threshold for exercise the option.

$$ln(\frac{K}{S_t}) \sim \mathcal{N}((r - \frac{\sigma^2}{2})t, \sigma^2 t)$$

$\mathbf{N(d_2)}$

$N(d_2)$ is the probability that the stock price will be at or above the strike price at expiry. Given that, we must find the rate of growth that it would take to the stock price to be at the strike price when the option expires. We want to determine how many standard deviations away this rate of growth $\ln(\frac{K}{S_t})$ is respect to the expected rate of growth μ. To compute it, we define a random variable z*, which represents the standardization of the normal variable previously mentioned $ln(\frac{K}{S_t})$:

$$z* = \frac{\overset{\text{A}}{X} - \overset{\text{B}}{\mu}}{\sigma}$$

Where, in our specific case, the X represents the normal random variable that we want to transform ($ln(\frac{K}{S_t})$), μ is the drift rate of the Geometric Brownian Motion ($(r - \frac{\sigma^2}{2})t$) and σ is the diffusion coefficient $\sigma\sqrt{t}$:

A Rate of growth that would allow the option to finish ATM:

$$\text{strike price} \longleftarrow K = \overset{\substack{\text{stock price}\\ \uparrow}}{S_t} \cdot \underset{\substack{\downarrow \\ \text{continuous} \\ \text{compound rate}}}{e}^{\,rate\ of\ growth\ from\ stock\ price\ to\ strike\ price}$$

It follows that:

$$\frac{K}{S} = \mathrm{e}^{\text{rate of growth}} \longrightarrow \ln(\frac{K}{S}) = \ln(\mathrm{e}^{\text{rate of growth}}) \longrightarrow \ln(\frac{K}{S}) = \text{rate of growth}$$

B Expected rate of growth = r - $\frac{\sigma^2}{2}$ $\longrightarrow$ Deterministic component of a Generalized Brownian Motion:"drift"

$$P_{t+1} = P_{t_0} \cdot e^{\overset{\substack{\text{drift rate}\\ \uparrow}}{\left(r - \frac{\sigma^2}{2}\right)t} + \sigma w_t \longrightarrow} \text{stochastic component}$$

Once A and B are replaced in the standardization formula we get:

$$z* = \frac{ln(K/S_t) - (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

Thanks to the standardization $z* \sim N(0, 1)$ can be graphically represented as:

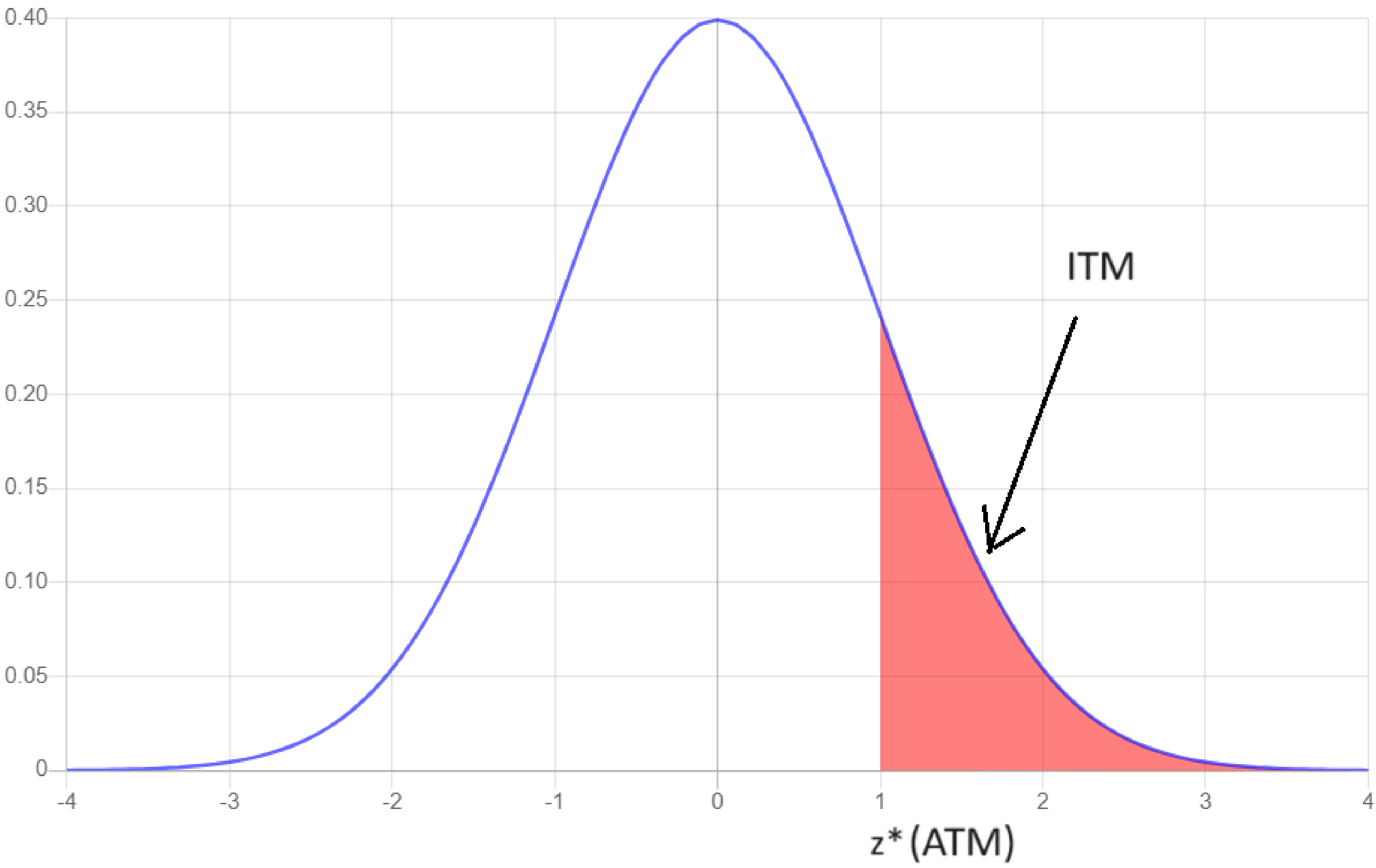

Figure 2: Standard Normal Distribution

z* represents the at the money level, whereas on the right side of z* the call option is in the money, so it is exercisable at expiry. Therefore, the area to the right of the z* is the probability to have $S_t >$ K. This can be mathematically solved, finding the following probability (which represents N(d_2)):

$$P(S_T \geq K)$$

Since we are in the continuous world, we can proceed with the integration:

$$P(X) = \int_{-\infty}^{\infty} f_x \, dx$$

f_x is the probability density function of the normal distribution:

$$PDF = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Therefore we get:

$$P(X) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\, dx$$

Given the standardization $\mu = 0$ and $\sigma = 1$:

$$P(S_T \geq K) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

Since we want to consider just the area above z*, we change the integral interval:

$$\int_{z*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

Now we can proceed with the solution. The integral of a PDF is a cumulative distribution function, which in this case we indicate with N():

$$\int_{z*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz = N(z)|_{z*}^{\infty} \quad = N(\infty) - N(z*) = 1 - N(z*) = N(-z*)$$

$$\text{with } -z* = -\left(\frac{ln(K/S_t) - (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right) = \frac{ln(S_t/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

To summarize, P($S_T \geq K$) = N($\frac{ln(S_t/K)+(r-\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$) which is equal to N($d_2$) by Black and Scholes definition.

N(d_1)

In case of N(d_1) the matter is more complex. N(d_1) is the conditional probability on the future value of the stock price S_T, if and only if S_T is above the strike price K. N(d_1) differs from N(d_2) since it is conditioning on the realization of N(d_2), whereas z* remains the standardization of $ln(\frac{K}{S_t})$:

$$E[S_T|S_T \geq K]$$

From the probability theory in the continuous world, we know that:

$$E[x] = \int_{-\infty}^{\infty} x \cdot f_x \, dx$$

In our case x is equal to the future stock price S_T, and f_x is the same standard normal pdf that we used previously:

$$E[S_T|S_T \geq K] = \int_{-\infty}^{\infty} S_T \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

Since we don't know the value of S_T, because it is a value in the future, we can substitute it with its Geometric Brownian motion: $S_T = S_t e^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{t}z}$, with z $\neq$ z*

$$\int_{-\infty}^{\infty} S_T \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz = \int_{-\infty}^{\infty} S_t e^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{t}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

Since the expected value condition constrains the domain, as in N(d_2), we consider just the area above z*:

$$\int_{z*}^{\infty} S_t e^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{t}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

Now we can solve the definite integral:

$$\int_{z*}^{\infty} S_t e^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{t}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz =$$

$$\int_{z*}^{\infty} \frac{S_t}{\sqrt{2\pi}} e^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{t}z-\frac{z^2}{2}} \, dz =$$

$$\int_{z*}^{\infty} \frac{S_t}{\sqrt{2\pi}} e^{(r-\frac{\sigma^2}{2})t-\frac{1}{2}(z-\sigma\sqrt{t})^2+\frac{1}{2}\sigma^2 t} \, dz =$$

$$\int_{z*}^{\infty} \frac{S_t}{\sqrt{2\pi}} e^{rt-\frac{1}{2}(z-\sigma\sqrt{t})^2} \, dz =$$

$$S_t e^{rt} \int_{z*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sigma\sqrt{t})^2}{2}} \, dz$$

Since the form of the standard normal pdf is $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, we implement the following substitution:

$$x = z - \sigma\sqrt{t}$$

If we derive the above equation we get dx = dz. Moreover, by subtracting $\sigma\sqrt{t}$ from both intervals (the upper limit will remain unchanged since it is equal to $+\infty$) we get:

$$\int_{z^*-\sigma\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\, dx$$

Now we can solve the following CDF:

$$\int_{z^*-\sigma\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\, dx = N(x)|_{z^*-\sigma\sqrt{t}}^{\infty} \quad = N(\infty) - N(z^* - \sigma\sqrt{t}) = 1 - N(z^* - \sigma\sqrt{t}) =$$

$$= N(\sigma\sqrt{t} - z^*) = N(\sigma\sqrt{t} - \frac{ln(K/S_t) - (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}) = N(\frac{ln(S_t/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}) = N(d_1)$$

From the N(d_2) computation we saw that -z* = d_2. Substituting it into N($\sigma\sqrt{t} - z^*$) we get N($\sigma\sqrt{t} + d_2$), which is equal to N(d_1) by Black and Scholes definition.

References

- *Options, Futures, and Other Derivative Securities* - John M. Hull
- *Understanding N(d_1) and N(d_2): Risk-Adjusted Probabilities in the Black-Scholes Model* - Lars Tyge Nielsen
- *Stochastic Processes notes (course MAFINRISK Bocconi University - Probability Theory and Stochastic Processes)* - Professor Francesco Rotondi
- *Monte Carlo for Stochastic Processes notes (course MAFINRISK Bocconi University - Numerical Methods)* - Professor Francesco Rotondi