Gabriel Desgranges
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Problem Set: General Equilibrium

I - **Edgeworth Boxes**

Define your own economy with 2 agents $i = A, B$ and 2 goods $l = 1, 2$. Choose 2 utility functions among the following ones:

- Cobb-Douglas $u_i(x_i) = (x_{i1})^{\alpha_i}(x_{i2})^{1-\alpha_i}$ with $\alpha_i = 1/4$, $1/3$, $1/2$, $2/3$ or $3/4$,
- Leontief $u_i(x_i) = \min(x_{i1}, \alpha_i x_{i2})$ with $\alpha_i = 1/10$, $1/2$, 1, 2 or 10,
- linear $u_i(x_i) = x_{i1} + \alpha_i x_{i2}$ with $\alpha_i = 1/10$, $1/2$, 1, 2 or 10,
- CES (Constant Elasticity of Substitution) $u_i(x_i) = \left((x_{i1})^{\alpha_i} + A_i(x_{i2})^{\alpha_i}\right)^{1/\alpha_i}$ with $\alpha_i = 1/2$ or 2 and $A_i = 1$ or 2.

Choose 2 individual endowments among the following ones:

$$\omega_i = (0,1), (1,0), (1,1), (2,1) \text{ or } (1,2).$$

a) Compute the equilibrium (prices and allocation).
b) Compute the PO allocations (denote $\varpi = \omega_A + \omega_B$).
c) Compute the price equilibrium with transfers decentralizing every PO allocation (prices and individual wealths w_i).

II - Exchange economies

a) Consider an economy with I consumers. Every consumer has utility:

$$u(x_i) = \left((x_{i1})^{1/2} + (x_{i2})^{1/2}\right)^2.$$

The endowment of consumer i is ω_i. Global endowment is $\varpi = \sum_i \omega_i$. Compute the equilibrium and the interior PO allocations. Does the equilibrium price depend on ϖ only (or on the whole vector $(\omega_i)_{1\leq i\leq I}$)?

b) Consider an economy with I consumers. Every consumer $i = 1, ..., I-1$ has utility:

$$u(x_i) = (x_{i1})^{1/2}(x_{i2})^{1/2}.$$

The utility of consumer I is:

$$u_I(x_I) = x_{I1} + 2x_{I2}.$$

The endowment of consumer i is ω_i. Global endowment is $\varpi = \sum_{1\leq i\leq I} \omega_i$. Denote $\varpi_{-I} = \sum_{1\leq i\leq I-1} \omega_i$ Answer to the questions a) and b) of Exercise I. Does the equilibrium price depend on ϖ only (or on (ϖ_{-I}, ω_I) or on the whole vector $(\omega_i)_{1\leq i\leq I}$)?

III - Representative agent

a) Consider I agents $i = 1, ..., I$ trading 2 goods $l = 1, 2$ with prices $p_1 = 1$ (numéraire) and p_2. The utility of i is $u_i(x_i) = (x_{i1})^{\alpha_i} (x_{i2})^{1-\alpha_i}$, the endowment of i is ω_i and global endowment is ϖ. Compute the excess demand z_i and the aggregate excess demand Z. Determine the equilibrium price p^*.

b) Consider an economy with one consumer with utility $u(x) = (x_1)^a (x_2)^{1-a}$ and endowment ϖ. Determine the equilibrium price p^{**}. Determine a so that $p^* = p^{**}$.

c) Is the excess demand function in the economy b) equal to Z? Compute $\frac{dp_2^*}{d\omega_{i1}}$ and $\frac{dp_2^{**}}{d\omega_{i1}}$. What is the variation of a needed to have $p^* = p^{**}$ for every ω_{i1}? (that is: compute $\frac{da}{d\omega_{i1}}$ where a is implicitly defined by $p^*(\omega_{i1}) = p^{**}(\omega_{i1}, a)$)

IV - Existence of equilibrium / strict monotonicity of preferences

Consider an economy with two agents $i = A, B$ and two goods $l = 1, 2$ such that:

$$\begin{aligned} u_A(x_A) &= x_{A1} \text{ and } u_B(x_B) = x_{B2}, \\ \omega_A &= (1, 1) \text{ and } \omega_B = (0, 1). \end{aligned}$$

Answer to the questions b) and a) of Exercise I.

V - Production economy

Consider an economy with two consumers $i = A, B$, one firm and two goods $l = 1, 2$. The individual endowments of A and B are $\omega^A = \omega^B = (1/2, 1/2)$. The utility functions are:

$$\begin{aligned} u^A\left(x_1^A, x_2^A\right) &= \ln x_1^A + \ln x_2^A, \\ u^B\left(x_1^B, x_2^B\right) &= \left(x_1^B\right)^{1/4} \left(x_2^B\right)^{3/4}. \end{aligned}$$

The firm produces good 2 using good 1 as input, the production function is $y_2 = \sqrt{y_1}$. The consumer B owns the firm (denote π the firm's profit). Good 2 is the numéraire good (*i.e.* $p_2 = 1$).

a) Determine the demand for good 1 of the consumers and the firm.

b) Show that there is a unique equilibrium price p_1.

c) The production function is now $y_2 = y_1$. Determine the equilibrium (price and allocation).

d) Consider the exchange economy consisting in the two consumers A and B (in other words, eliminate the firm). Determine the equilibrium (price and allocation).

e) Consider again the production economy (A, B and the firm). The production function is now $y_2 = y_1/c$ (where $c > 0$). Determine the values of c such that the the firm is active at equilibrium (*i.e.* $y_1 > 0$) and the values of c such that the firm is not active (*hint:* show that, for some values of c, there is an excess demand of good 1 when the firm is active). Compare the equilibrium of question d) with the equilibrium with the non active firm.

VI - Production economy with CRTS

Consider an economy with two consumers $i = A, B$, one firm (that produces good 2 using good 1 as input) and two goods $l = 1, 2$. The consumer B owns the firm (denote π the firm's profit). Good 2 is the numéraire good (*i.e.* $p_2 = 1$).

a) Consider:

$$
\begin{aligned}
u_A(x_A) &= x_{A1} + \ln x_{A2} \text{ and } u_B(x_B) = \min(x_{B1}, x_{B2}), \\
\omega_A &= (3,0) \text{ and } \omega_B = (0,3).
\end{aligned}
$$

The production function is $y_2 = ay_1$ (with $a > 0$). Compute the equilibrium. Does the consumption sector play a role in the determination of equilibrium prices and allocation?

b) Consider:

$$
\begin{aligned}
u_A(x_A) &= x_{A1} + 4\sqrt{x_{A2}} \text{ and } u_B(x_B) = x_{B1} + 2\sqrt{x_{B2}}, \\
\omega_A &= (4,12) \text{ and } \omega_B = (8,8).
\end{aligned}
$$

The production function is $y_2 = 3y_1$. Compute the equilibrium.

VII - Labor supply

Consider an economy with one consumer, one firm and two commodities: a consumption good (its quantities denoted by x, its price by p) and labour (its quantities denoted by l, its price by p). The endowment of the consumer consists of one unit of labour. The utility function is:

$$u(x,l) = x^{1/2} + a(1-l)^{1/2} \text{ with } a > 0.$$

The technology of the firm is described by the production function: $x_f = \sqrt{l_f}$. The consumer owns the firm.

a) Determine the demand for good and the labor supply of the consumer (denote π the firm's profit); determine the labor demand and the supply of the firm. Determine the equilibrium.

b) Assume that the labor supply can take two values only, namely $l = 0$ or $l = 1$ (the consumer decides either to work or not to work). Determine the demand for good and the labor supply of the consumer (*hint:* compare the two possible decisions of the consumer). Compare the MRS of the two points. Is demand a continuous function of the price? Compute the equilibrium.

VIII - Labor supply 2

Consider an economy with 2 consumers $i = A, B$, one firm and 2 commodities: a consumption good (its quantities denoted by x, its price by p) and labour (its quantities denoted by l, its price by w). The endowment of each consumer consists of one unit of labour. The utility functions are:

$$
\begin{aligned}
u_A(x_A, l_A) &= x_A(1-l_A) \text{ where } x_A \geq 0 \text{ and } 0 \leq l_A \leq 1, \\
u_B(x_B, l_B) &= x_B \text{ where } x_B \geq 0.
\end{aligned}
$$

The technology of the firm is described by the production function: $x_f = \sqrt{l_f}$. Consumer B owns the firm.

a) Compute the supply and demand functions of the consumers and the firm.

b) Prove that there exists a unique equilibrium price.

Solutions

I - **Edgeworth Boxes**

I give here elements that should allow you to solve Problem I. I don't give the exact solutions of all the examples suggested in Problem I.

a) My previous file "Edgeworth Boxes: Examples" partly provides the solution of this question. In particular, it provides individual demands and equilibrium in some examples where different cases needs to be distinguished (like $\frac{p_1}{p_2} > a_i$ or $p_1 p_2 \neq 0$). Notice that the demand of a consumer with a CES utility function $u_i(x_i) = \left((x_{i1})^{\alpha_i} + A_i (x_{i2})^{\alpha_i}\right)^{1/\alpha_i}$ and a wealth $w_i = p.\omega_i$ is:

$$\begin{aligned} x_{i1} &= \frac{w_i}{p_1 + p_2 \left(\frac{A_i p_1}{p_2}\right)^{\frac{1}{1-\alpha_i}}}, \\ x_{i2} &= \frac{w_i}{p_1 \left(\frac{A_i p_1}{p_2}\right)^{\frac{1}{\alpha_i - 1}} + p_2}. \end{aligned}$$

b) I give you 3 ways of computing the PO allocations (they all require that you can apply the Kuhn Tucker Theorem to a maximization problem).

The first way is the more general one. You choose one consumer i_0, $(I-1)$ parameters $\bar{u}_i$ (for every $i \neq i_0$) and you solve:

$$\max u_{i_0}(x_{i_0}),$$

such that:

$$\begin{aligned} \forall i &\neq i_0, u_i(x_i) \geq \bar{u}_i \text{ (utility constraints)}, \\ \forall l, \sum_i x_{il} &= \varpi_l \text{ (resource constraints)}, \\ \forall i, \forall l, x_{il} &\geq 0 \text{ (positivity constraints)}. \end{aligned}$$

As usual, you can omit the positivity constraints (if you end up with a non positive allocation, then you add the positivity constraints and you solve the maximization problem again). The Lagrange multipliers associated with the resource constraints are the shadow prices (see question c below).

The solution of this problem is a PO allocation depending on the $(I-1)$ parameters $\bar{u}_i$. The set of PO allocations is the set of solutions of the problem when $(\bar{u}_i)_{i \neq i_0} \in \mathbb{R}^{I-1}$.

The second way is almost as general as the first one. You choose I non negative parameters λ_i such that $\sum_i \lambda_i = 1$ and you solve:

$$\max \sum_i \lambda_i u_i(x_i),$$

such that:

$$\begin{aligned}\forall l, \sum_i x_{il} &= \varpi_l \text{ (resource constraints)},\\ \forall i, \forall l, x_{il} &\geq 0 \text{ (positivity constraints)}.\end{aligned}$$

Again, you can omit the positive constraints (at least in a first round), and the Lagrange multipliers associated with the resources constraints are the shadow prices (see question c below).

- The reason why this second way is not as general as the first one is that the objective $\sum_i \lambda_i u_i$ of the maximization problem needs to be quasi-concave (in order for the Kuhn Tucker conditions to characterize the solutions of the maximization problem). One always assumes that every u_i is quasi-concave, but this does not imply that $\sum_i \lambda_i u_i$ is quasi-concave. Still, when u_i is concave (this is true for every usual utility function), $\sum_i \lambda_i u_i$ is quasi-concave (it is even concave). Hence, this restriction is a very small one.
- The reason why you could prefer the second way to the first one is that the parameters $(\lambda_1, ..., \lambda_I)$ make more sense than the parameters $(\bar{u}_i)_{i \neq i_0}$ $(1/\lambda_i$ is the marginal utility of income, see Mas-Colell et alii, p 566)

The solution of this problem is a PO allocation depending on the I parameters λ_i. The set of PO allocations is the set of solutions of the problem when $(\lambda_1, ..., \lambda_I)$ is in the simplex.

The third way is the simplest one. Still, this way concerns interior PO allocations only (that is: $x_{il} > 0$ for every i and every l). I advise you to use this way when you are not interested in corner PO solutions (notice that, in many cases, you obtain corner PO allocations when you consider the closure of the set of interior PO allocations, that is: corner PO allocations are on the frontier of the set of PO allocations). If you need to carefully look at corner PO allocations, use the second way above.

Interior PO solutions are the solution of the system of equations:

$$\begin{aligned}\forall i, \forall i', \forall l, \forall l', MRS^i_{l/l'} &= MRS^{i'}_{l/l'},\\ \forall l, \sum_i x_{il} &= \varpi_l,\end{aligned}$$

where $MRS^i_{l/l'}$ is consumer i's marginal rate of substitution between goods l and l'.

c) Restrict attention to the case of an interior PO allocation x and C^2 utility functions. The solution is then very simple: PO implies that the $MRS_i(x_i)$ does not depend on i. Define the relative price by $\frac{p_1}{p_2} = MRS_i(x_i)$. Furthermore, for every i, $w_i = p.x_i$ (give to i what i needs to buy x_i).

It is not necessary that you spend time to solve a more general case. You may want to know that the solution in the general case is to write the FOC of the maximization problem characterizing the PO allocations and to compute the shadow prices (see the "first way" or the "second way" in question b above).

Lastly, I give you some useful insights on some other cases (it may be more useful to understand these insights than to do the big computations in the general case).

- In the case of a corner PO solution and C^2 utility functions, it is possible that $\frac{p_1}{p_2} \neq MRS_i(x_i)$ for a consumer i with $x_{il} = 0$. For example, when $x_{i1} = 0$, $\frac{p_1}{p_2} > MRS_i(x_i)$ is possible (but $\frac{p_1}{p_2} < MRS_i(x_i)$ is not): i finds good 1 so expensive that he wants no good 1 (he would even take a "short position" on good 1, $x_{i1} < 0$, if he had the right to). Still, for i such that $x_i \gg 0$ (that is: $x_{il} > 0$ for every l), $\frac{p_1}{p_2} = MRS_i(x_i)$.

- In the case of utility functions that are not C^2 (example: Leontief preferences), $MRS_i(x_i)$ can be not uniquely defined (the tangent to the indifference curve is not well defined). For example, for $u_i(x_i) = \min(x_{i1}, \alpha_i x_{i2})$, a bundle x_i such that $x_{i1} = \alpha_i x_{i2}$ can be optimal whatever the prices are (for every price, define $w_i = p.x_i$). This is obvious on a figure: any buget line through x_i makes x_i optimal.

II - Exchange economies

a) Demand is $x_i = \left(\frac{w_i}{p_1 + \left(\frac{p_1}{p_2}\right)^2 p_2}, \frac{w_i}{\left(\frac{p_2}{p_1}\right)^2 p_1 + p_2} \right)$ where individual wealth is $w_i = p.\omega_i$. Market clearing for good 1 writes $\sum_i x_{1i} = \varpi_1$. Hence, there is a unique equilibrium price: $\frac{p_1}{p_2} = \sqrt{\frac{\varpi_2}{\varpi_1}}$.

An (interior) PO allocation is a solution of the system:

$$\begin{aligned} \forall i, i', MRS_i &= MRS_{i'}, \\ \sum_i x_i &= \varpi. \end{aligned}$$

Given that $MRS_i = \sqrt{\frac{x_{2i}}{x_{1i}}}$, we have that, at a given PO allocation x, the ratio $\frac{x_{2i}}{x_{1i}}$ does not depend on i. Denote r this ratio ($x_{2i} = r x_{1i}$). $\varpi_2 = \sum_i x_{2i}$ implies then that $\varpi_2 = \sum_i r x_{1i} = r\varpi_1$ and $r = \frac{\varpi_2}{\varpi_1}$. Hence, a PO allocation is a feasible allocation such that $\forall i, x_{2i} = r x_{1i}$. Recall that a feasible allocation is an allocation satisfying the resource constraint $\sum_i x_i = \varpi$. An interior PO allocation is then any allocation such that $\sum_i x_{i1} = \varpi_1$, and, for every i, $x_{2i} = r x_{1i}$.

b) Demand of $i \leq I - 1$ is $x_i = \left(\frac{w_i}{2p_1}, \frac{w_i}{2p_2}\right)$ with $w_i = p.\omega_i$. Furthermore, $x_I = \left(0, \frac{w_I}{p_2}\right)$ if $\frac{p_2}{p_1} < 2$, x_I is any bundle satisfying $p.x_I = w_I$ if $\frac{p_2}{p_1} = 2$, and $x_I = \left(\frac{w_I}{p_1}, 0\right)$ if $\frac{p_2}{p_1} > 2$.

To solve for the equilibrium, distinguish between 3 cases: $\frac{p_2}{p_1} < 2$, $\frac{p_2}{p_1} = 2$, and $\frac{p_2}{p_1} > 2$.

- In the case $\frac{p_2}{p_1} < 2$, market clearing for good 1 writes $\frac{p_1 \varpi_{-I1} + p_2 \varpi_{-I2}}{2p_1} = \varpi_1$, and $\frac{p_2}{p_1} = \frac{2\varpi_1 - \varpi_{-I1}}{\varpi_{-I2}} = \frac{\varpi_1 + \omega_{I1}}{\varpi_2 - \omega_{I2}} > 0$. There is an equilibrium in this case iff $\frac{\varpi_1 + \omega_{I1}}{\varpi_2 - \omega_{I2}} < 2$. This condition rewrites: $\omega_{I1} + 2\omega_{I2} < 2\varpi_2 - \varpi_1$.

- In the case $\frac{p_2}{p_1} > 2$, market clearing for good 2 writes $\frac{p_2}{p_1} = \frac{\varpi_1 - \omega_{I1}}{\varpi_2 + \omega_{I2}} > 0$. There is an equilibrium in this case iff $\frac{\varpi_1 - \omega_{I1}}{\varpi_2 + \omega_{I2}} > 2$. This condition rewrites: $\omega_{I1} + 2\omega_{I2} > \varpi_1 - 2\varpi_2$.

- In the case $\frac{p_2}{p_1} = 2$, market clearing for good 1 ($\frac{p_1 \varpi_{-I1} + p_2 \varpi_{-I2}}{2p_1} + x_{I1} = \varpi_1$) implies that $x_{I1} = \varpi_1 - \left(\frac{\varpi_{-I1}}{2} + \varpi_{-I2}\right)$, market clearing for good 2 ($\frac{p_1 \varpi_{-I1} + p_2 \varpi_{-I2}}{2p_2} + x_{I2} = \varpi_2$) implies that $x_{I2} = \varpi_2 - \left(\frac{\varpi_{-I1}}{4} + \frac{\varpi_{-I2}}{2}\right)$ (x_{I1} and x_{I2} satisfy the budget constraint of I). There is an equilibrium in this case iff $x_{1I} \geq 0$ and $x_{2I} \geq 0$. These 2 conditions rewrites:

$$\begin{aligned} \omega_{I1} + 2\omega_{I2} &\geq 2\varpi_2 - \varpi_1, \\ \omega_{I1} + 2\omega_{I2} &\leq \varpi_1 - 2\varpi_2. \end{aligned}$$

Combining the 3 cases lead to the following conclusion: there is one equilibrium.

- If $2\varpi_2 > \varpi_1$, then this is either $\frac{p_2}{p_1} = \frac{\varpi_1 + \omega_{I1}}{\varpi_2 - \omega_{I2}} < 2$ (when $\omega_{I1} + 2\omega_{I2} < 2\varpi_2 - \varpi_1$) or $\frac{p_2}{p_1} = 2$ (when $\omega_{I1} + 2\omega_{I2} \geq 2\varpi_2 - \varpi_1$).

- If $2\varpi_2 < \varpi_1$, then this is either $\frac{p_2}{p_1} = \frac{\varpi_1 - \omega_{I1}}{\varpi_2 + \omega_{I2}} > 2$ (when $\omega_{I1} + 2\omega_{I2} > \varpi_1 - 2\varpi_2$) or $\frac{p_2}{p_1} = 2$ (when $\omega_{I1} + 2\omega_{I2} \leq \varpi_1 - 2\varpi_2$).

I first compute the interior PO allocations (this is not the answer to the question). The interior PO allocations are the solutions of the system:

$$\begin{aligned} \forall i, i', MRS_i &= MRS_{i'}, \\ \sum_i x_i &= \varpi. \end{aligned}$$

Given that $MRS_I = 1/2$, these are the feasible allocations such that $\forall i \leq I - 1, x_{2i} = x_{1i}/2$ (so that $x_{1I} = \varpi_1 - \sum_{i \leq I-1} x_{i1} \geq 0$ and $x_{2I} = \varpi_2 - \frac{1}{2}\sum_{i \leq I-1} x_{i1} \geq 0$). The set of interior PO allocations is thus parametrized by $I - 1$ parameters $(x_{i1})_{i \leq I-1}$.

But, in the present case, it makes sense to ask whether there are PO solutions with $x_{1I} = 0$ or $x_{2I} = 0$ (you can guess it by looking at I's demand). This is why you are asked to compute all the PO allocations (including the corner ones). To answer the question, you should apply the "second way" defined in Problem I, question b). This is quite tedious but straightforward. I don't do these computations here, I only describe the solution in a (hopefully) intuitive way:

- We already know the interior PO allocations. We now examine every corner allocation to check which ones are PO.

- Consider an allocation with $x_{i1} = 0$ for some $i \leq I - 1$. PO of this allocation requires $x_{i2} = 0$ (because $u_i(0, x_{i2}) = 0$, it is more useful to give good 1 to someone else).

- Consider an allocation with $x_{i2} = 0$ for some $i \leq I - 1$. PO of this allocation requires $x_{i2} = 0$ for the same reason as above.

- Consider an allocation with $x_{I1} = 0$. PO of this allocation requires $MRS_i = MRS_{i'}$ for every $i, i' \leq I - 1$. Denote r the common value of these MRS. We have:
$$\forall i \leq I-1, x_{2i} = rx_{1i}.$$
Given that $\sum_{i \leq I-1} x_{i1} = \varpi_1$, we have:
$$x_{2I} = \varpi_2 - \sum_{i \leq I-1} x_{i2} = \varpi_2 - r\varpi_1.$$
Optimality requires $MRS_I \leq r$ (interpret r as a relative price, the intuition is similar to the one at the end of Problem I: $x_{I1} = 0$ requires that good 1 is less valuable for agent I than for agents $i \leq I - 1$). Recall that $MRS_I = 1/2$. Hence, the following allocations are PO:
$$\begin{aligned} \forall i &\leq I-1, x_{2i} = rx_{1i}, \\ x_{1I} &= 0, \\ x_{2I} &= \varpi_2 - r\varpi_1 \geq 0, \end{aligned}$$
where $1/2 \leq r \leq \varpi_2/\varpi_1$ and $(x_{i1})_{i \leq I-1}$ satisfies $\sum_{i \leq I-1} x_{i1} = \varpi_1$. No allocation of this kind exists if $1/2 > \varpi_2/\varpi_1$.

- Analogously, the following allocations are PO:
$$\begin{aligned} \forall i &\leq I-1, x_{1i} = \frac{x_{2i}}{r}, \\ x_{1I} &= \varpi_1 - \frac{\varpi_2}{r} \geq 0, \\ x_{2I} &= 0, \end{aligned}$$
where $\varpi_2/\varpi_1 \leq r \leq 1/2$ and $(x_{i1})_{i \leq I-1}$ satisfies $\sum_{i \leq I-1} x_{i2} = \varpi_2$. No allocation of this kind exists if $1/2 < \varpi_2/\varpi_1$.

- There is no other PO allocations.

III - Representative agent

a) We have:
$$\begin{aligned} z_i &= \left(\alpha_i \frac{p.\omega_i}{p_1} - \omega_{i1}, (1-\alpha_i)\frac{p.\omega_i}{p_2} - \omega_{i2}\right), \\ Z &= \left(\frac{p.\left(\sum_i \alpha_i \omega_i\right)}{p_1} - \varpi_1, \frac{p.\left(\sum_i (1-\alpha_i)\omega_i\right)}{p_2} - \varpi_2\right), \\ p_2^* &= \frac{\varpi_1 - \sum_i \alpha_i \omega_{i1}}{\sum_i \alpha_i \omega_{i2}}. \end{aligned}$$

b) Excess demand is $\left(a\frac{p.\varpi}{p_1} - \varpi_1, (1-a)\frac{p.\varpi}{p_2} - \varpi_2\right)$, and we have:
$$\begin{aligned} p_2^{**} &= \frac{(1-a)\varpi_1}{a\varpi_2}, \\ a &= \frac{\varpi_1}{\varpi_2 p_2^* + \varpi_1}. \end{aligned}$$

c) The two excess demand function are different. For example, equality between the excess demand for good 1 requires:

$$\forall p, \frac{p.\left(\sum_i \alpha_i \omega_i\right)}{p_1} - \varpi_1 = a\frac{p.\varpi}{p_1} - \varpi_1,$$

that is:

$$\forall p, p.\left(\sum_i \alpha_i \omega_i\right) = p.a\varpi.$$

This in turn implies that the two vectors $\left(\sum_i \alpha_i \omega_i\right)$ and $a\varpi$ are equal, which is not the case (except in some specific cases where $\frac{\sum_i \alpha_i \omega_{il}}{\varpi_l}$ does not depend on good l).

We have (write $\varpi_1 = \sum_i \omega_{i1}$):

$$\begin{aligned}
\frac{dp_2^*}{d\omega_{i1}} &= \frac{1-\alpha_i}{\sum_i \alpha_i \omega_{i2}}, \\
\frac{dp_2^{**}}{d\omega_{i1}} &= \frac{(1-a)}{a\varpi_2}, \\
\frac{da}{d\omega_{i1}} &= \frac{(\varpi_2 p_2^* + \varpi_1) - \varpi_1\left(\varpi_2 \frac{dp_2^*}{d\omega_{i1}} + 1\right)}{(\varpi_2 p_2^* + \varpi_1)^2}, \\
&= \frac{\varpi_2}{(\varpi_2 p_2^* + \varpi_1)^2}\left(p_2^* - \varpi_1 \frac{dp_2^*}{d\omega_{i1}}\right).
\end{aligned}$$

IV - Existence of equilibrium / strict monotonicity of preferences

A cares about good 1 only and B cares about good 2 only. Straightforwardly, the unique PO allocation is $x_A = (1,0)$ and $x_B = (0,2)$.

To show that there is no equilibrium, distinguish between 3 cases:

- If $p_1 p_2 \neq 0$, then $x_A = \left(\frac{p_1+p_2}{p_1}, 0\right)$, $x_B = (0,1)$ and the market for good 2 cannot clear.
- If $p_1 = 0$, then $x_{A1} = +\infty$ and the market for good 1 cannot clear.
- If $p_2 = 0$, then $x_{B2} = +\infty$ and the market for good 2 cannot clear.

V - Production economy

a) x_A solves $\max_{p.x_A = w_A} u_A(x_A)$ with $w_A = p.\omega_A$. Hence, x_A solves $\max_{p.x_A = w_A} \exp\left(\frac{1}{2}u_A(x_A)\right)$, and $x_A = \left(\frac{w_A}{2p_1}, \frac{w_A}{2p_2}\right)$. $x_B = \left(\frac{w_B}{4p_1}, \frac{3w_B}{4p_2}\right)$ with $w_B = p.\omega_B + \pi$ (π is the firm's profit - that is not known yet). $y = (y_1, y_2)$ solves $\max_{y_2 = \sqrt{y_1}} p_2 y_2 - p_1 y_1$. Equivalently, y_1 solves $\max p_2\sqrt{y_1} - p_1 y_1$. Hence, $y_1 = \left(\frac{p_2}{2p_1}\right)^2$, and $y_2 = \left(\frac{p_2}{2p_1}\right)$, and $\pi = \frac{(p_2)^2}{4p_1}$.

b) Market clearing for good 1 writes $x_{1A} + x_{1B} + y_1 = 1$. This is an equation with unknown $\frac{p_2}{p_1}$. This equation rewrites:

$$5\left(\frac{p_2}{p_1}\right)^2 + 6\frac{p_2}{p_1} - 10 = 0.$$

The unique positive solution is $\frac{p_2}{p_1} = \frac{\sqrt{59}-3}{5}$ (notice that market clearing for good 2 writes $x_{2A} + x_{2B} = 1 + y_2$, but this is useless here).

c) The important point in this question is that the firm can be inactive. We proceed as usual: compute first agents' behavior, write then a market clearing equation and solve for the (relative) equilibrium price.

The firm's profit is linear in y_1 (it is $(p_2 - p_1)\, y_1$) so that the solution of profit maximization is not the solution of the "simple" FOC $\frac{d}{dy_1}\left((p_2 - p_1)\, y_1\right) = 0$. The reason for this is that the positivity constraint $y_1 \geq 0$ cannot be omitted in the present case (because this constraint can be binding at the solution: $y_1 = 0$ can be the solution of the maximization problem). To solve the profit maximization, we can either write the Lagrangian $((p_2 - p_1)\, y_1 + \lambda y_1)$ and the associated Kuhn and Tucker FOCs $((p_2 - p_1) + \lambda = 0,\ y_1 \geq 0, \text{ and } \lambda y_1 = 0)$, or we can draw the graph of the profit $(p_2 - p_1)\, y_1$ (as a function of y_1) and see the solution on the figure (it is obvious). Firm's behavior is described as follows:

- In the case $p_2 > p_1$, $y_1 = y_2 = +\infty$ (and $\pi = +\infty$),
- In the case $p_2 < p_1$, $y_1 = y_2 = 0$ (and $\pi = 0$),
- In the case $p_2 = p_1$, every $y_1 \geq 0$ is a solution: we always have $y_2 = y_1$ and $\pi = 0$.

We now solve for the equilibrium. To this purpose, we consider the same 3 cases as above (the important case is the third case):

- In the case $p_2 > p_1$, $y_1 = +\infty$ so that the market for good 1 cannot clear. There is no equilibrium in this case.
- In the case $p_2 < p_1$, $y_1 = 0$ (and $\pi = 0$). Market clearing for good 1 writes $x_{1A} + x_{1B} = 1$. Solving this latter equation for $\frac{p_2}{p_1}$ gives $\frac{p_2}{p_1} = \frac{5}{3} > 1$. Hence, there is no equilibrium in this case.
- In the case $p_2 = p_1$, we compute $x_{A1} = \frac{1}{2}$, $x_{B1} = \frac{1}{4}$ (given that $\pi = 0$, whatever y_1 is). Market clearing for good 1 writes $x_{1A} + x_{1B} + y_1 = 1$. This implies $y_1 = \frac{1}{4} \geq 0$. Hence, there is one equilibrium in this case (the other components of the equilibrium allocation are $x_{2A} = \frac{1}{2}$, $x_{2B} = \frac{3}{4}$, $y_2 = \frac{1}{4}$).

Summing up, there is one equilibrium. At equilibrium, $\frac{p_2}{p_1} = 1$ (the price is determined by the constant marginal productivity of the firm - a standard result in the case of linear technologies). Consumers' demands are then determined by the price. The firm's behavior is chosen so that the markets clear (the firm is indifferent between all the production levels).

d) Demands of A and B are already known (from question a). Market clearing for good 1 writes $x_{1A} + x_{1B} = 1$. The equilibrium price is $\frac{p_2}{p_1} = \frac{5}{3}$. The equilibrium allocation is $x_A = \left(\frac{2}{3}, \frac{2}{5}\right)$, $x_B = \left(\frac{1}{3}, \frac{3}{5}\right)$.

e) Reconsidering the above solution of question c) shows that $\frac{p_2}{p_1} = c$ at an equilibrium with an active firm. At this price, $x_{1A} = \frac{1+c}{4}$ and $x_{1B} = \frac{1+c}{8}$ so that the market clearing for good 1 writes

$$y_1 = \frac{5 - 3c}{8}.$$

Given that $\frac{5-3c}{8} > 0$ iff $c < \frac{5}{3}$, we have:

- If $c < \frac{5}{3}$, then there is one equilibrium. At equilibrium, $\frac{p_2}{p_1} = c$ and the equilibrium allocation is $x_A = \left(\frac{1+c}{4}, \frac{1+c}{4c}\right)$, $x_B = \left(\frac{1+c}{8}, 3\frac{1+c}{8c}\right)$, $y = \left(\frac{5-3c}{8}, \frac{5-3c}{8c}\right)$.
- If $c \geq \frac{5}{3}$, then the above equation shows that the firm cannot be active at equilibrium. Question d) shows that there is one equilibrium with an inactive firm in this case.

To understand this result, recall that an active firm requires $\frac{p_2}{p_1} = c$. Hence, at an equilibrium with an active firm, it must be the case that the marginal productivity $1/c$ of the firm is so high that, when $\frac{p_2}{p_1} = c$, consumers' excess demand for good 1 is negative ($x_{1A} + x_{1B} - 1 < 0$) and consumers' excess demand for good 2 is positive ($x_{2A} + x_{2B} - 1 > 0$), which implies that the firm can use some units of good 1 to produce good 2. Question d) shows that consumers' excess demands are 0 when $\frac{p_2}{p_1} = \frac{5}{3}$. Hence $c < \frac{5}{3}$ is required for the firm to be active.

VI - Production economy with CRTS

This problem is similar to Questions c, d, e in Problem V.

a) As a preliminary remark, notice that, because of the Leontief preferences of B, it could make sense to consider the case $p_1 p_2 = 0$. But, as the preferences of A are strictly monotonic, a zero price is not possible at equilibrium (a zero price implies an infinite demand and the market cannot clear).

We have, with $\pi = 0$ (recall that $\pi = 0$ at equilibrium, whatever the firm's decision is):

$$\begin{aligned} x_A &= \left(2, \frac{p_1}{p_2}\right), \\ x_B &= \left(\frac{3p_2}{p_1 + p_2}, \frac{3p_2}{p_1 + p_2}\right). \end{aligned}$$

To solve for the equilibrium, we distinguish between 2 cases (active/inactive firm).

In the case where the firm is active, we have $\frac{p_1}{p_2} = a$ at equilibrium. Hence, $x_{B1} = \frac{3}{1+a}$ and market clearing for good 1 requires

$$y_1 = \frac{a-2}{1+a}.$$

If $a \geq 2$, then there is one equilibrium in this case. If $a < 2$, then there is no equilibrium in this case.

In the case where the firm is inactive, we have $\frac{p_1}{p_2} \geq a$ at equilibrium. Market clearing for good 1 requires $\frac{p_1}{p_2} = 2$. There is an equilibrium in this case iff $a \leq 2$.

Summing up, there is one equilibrium. Equilibrium price is $\max(2, a)$ (equilibrium allocation is straightforward).

b) We have, with $\pi = 0$ (recall that $\pi = 0$ at equilibrium, whatever the firm's decision is):

$$\begin{aligned}
x_A &= \left(\frac{w_A}{p_1} - 4\frac{p_1}{p_2}, 4\left(\frac{p_1}{p_2}\right)^2\right) \text{ if } \frac{w_A}{p_1} - 4\frac{p_1}{p_2} \geq 0, \\
x_A &= \left(0, \frac{w_A}{p_2}\right) \text{ otherwise,} \\
x_B &= \left(\frac{w_B}{p_1} - \frac{p_1}{p_2}, \left(\frac{p_1}{p_2}\right)^2\right) \text{ if } \frac{w_B}{p_1} - \frac{p_1}{p_2} \geq 0, \\
x_B &= \left(0, \frac{w_B}{p_2}\right) \text{ otherwise,}
\end{aligned}$$

with $w_i = p.\omega_i$.

To solve for the equilibrium, we distinguish between 2 cases (active/inactive firm).

In the case where the firm is active, we have $\frac{p_1}{p_2} = 3$ at equilibrium. Hence, $x_A = (0, 24)$, $x_B = \left(\frac{23}{3}, 9\right)$, and market clearing implies $(y_1, y_2) = \left(\frac{13}{3}, 13\right)$. There is one equilibrium in this case.

In the case where the firm is inactive, we have $\frac{p_1}{p_2} > 3$ at equilibrium. I give two solutions of this case. The first solution is the one that comes to mind first in this kind of problems (at least, to my mind). It consists in 2 subcases:

- Check that there is no equilibrium with $x_{A1} > 0$ and $x_{B1} > 0$. To this purpose, write the market clearing equation for good 2, that is $5\left(\frac{p_1}{p_2}\right)^2 = 20$. This implies $\frac{p_1}{p_2} = 2$, which is a contradiction (we are in the case $\frac{p_1}{p_2} > 3$).
- Check that there is no equilibrium in the 3 other cases ($x_{A1} = 0$, $x_{B1} = 0$), ($x_{A1} > 0$, $x_{B1} = 0$) and ($x_{A1} = 0$, $x_{B1} > 0$)...

The second solution is shorter (but it probably takes more time to think of it). $\frac{p_1}{p_2} > 3$ implies:

$$\frac{w_A}{p_1} - 4\frac{p_1}{p_2} = 4\frac{p_2}{p_1}\left(\frac{p_1}{p_2} + 3 - \left(\frac{p_1}{p_2}\right)^2\right).$$

Consider the polynomial of degree 2 $P[X] = -X^2 + X - 3$. We have $P[3] < 0$ and $P'[3] < 0$, so that $P[X] < X$ for every $X > 3$. Hence,

$$\frac{w_A}{p_1} - 4\frac{p_1}{p_2} = 4\frac{p_2}{p_1}\left(\frac{p_1}{p_2} + 3 - \left(\frac{p_1}{p_2}\right)^2\right) < 0.$$

It follows that $x_{A1} = 0$. Market clearing for good 1 then requires $\frac{w_B}{p_1} - \frac{p_1}{p_2} = 12$, that is

$$\left(\frac{p_1}{p_2}\right)^2 + 4\frac{p_1}{p_2} - 8 = 0$$

The unique positive root of this polynomial is $2\sqrt{3} - 2 < 3$. Hence, there is no equilibrium in this case.

VII - Labor supply
a) Firm's decision is:

$$l_f = \left(\frac{p}{2w}\right)^2, x_f = \frac{p}{2w}, \pi = \frac{p^2}{4w}$$

Consumer's demand solves $\max_{px+w(1-l)=w+\pi} u(x,l)$. Solution is

$$\begin{aligned} x &= \frac{w+\pi}{p+w\left(\frac{ap}{w}\right)^2}, \\ l &= 1-\left(\frac{ap}{w}\right)^2 \frac{w+\pi}{p+w\left(\frac{ap}{w}\right)^2} = \frac{1-\left(\frac{a}{2}\right)^2\left(\frac{p}{w}\right)^3}{1+a^2\frac{p}{w}}, \end{aligned}$$

when $l \geq 0$ in the above expression (that is $\frac{w}{p} \geq \left(\frac{a}{2}\right)^{2/3}$). Otherwise, solution is $x = \frac{p}{4w}$ and $l = 0$.

We first chech that there is no equilibrium with $\frac{w}{p} < \left(\frac{a}{2}\right)^{2/3}$. If $\frac{w}{p} < \left(\frac{a}{2}\right)^{2/3}$, then market clearing for the labor market writes $l = l_f$, that is: $0 = \left(\frac{p}{2w}\right)^2$, which contradicts $\frac{w}{p} < \left(\frac{a}{2}\right)^{2/3}$.

In the case $\frac{w}{p} \geq \left(\frac{a}{2}\right)^{2/3}$, market clearing for the labor market writes $l = l_f$, that is:

$$4\left(\frac{w}{p}\right)^3 - \frac{w}{p} - 2a^2 = 0.$$

This polynomial P of degree 3 (with variable $\frac{w}{p}$) has a unique positive root (there are many ways to show this property. For example, $P'(0) < 0$ and $P(0) < 0$). Hence, there is a unique equilibrium (the equilibrium price is the positive root, equilibrium allocation is computed straightforwardly).

b) If $l = 0$, then the budget constraint implies $x = \frac{\pi}{p}$ and $u(x,0) = \sqrt{\frac{\pi}{p}} + a$. If $l = 1$, then the budget constraint implies that $x = \frac{w+\pi}{p}$ and $u(x,1) = \sqrt{\frac{w+\pi}{p}}$. Hence, consumer's behavior is

$$\begin{aligned} (x,l) &= \left(\frac{\pi}{p}, 0\right) \text{ when } \frac{w}{p} < \left(\sqrt{\frac{\pi}{p}} + a\right)^2 - \frac{\pi}{p}, \\ (x,l) &= \left(\frac{w+\pi}{p}, 1\right) \text{ when } \frac{w}{p} > \left(\sqrt{\frac{\pi}{p}} + a\right)^2 - \frac{\pi}{p}. \end{aligned}$$

The demand x is not continous as a function of the real wage $\frac{w}{p}$.

To solve for the equilibrium, notice that $l = 0$ is not compatible with equilibrium (the labor market cannot clear, see question a) above). Furthermore, an equilibrium with $l = 1$ is such that $l_f = 1$, that is $\frac{w}{p} = \frac{1}{2}$. This implies that $\frac{\pi}{p} = \frac{1}{2}$. Hence, there is an equilibrium as soon as the condition $\frac{w}{p} > \left(\sqrt{\frac{\pi}{p}} + a\right)^2 - \frac{\pi}{p}$ holds. This condition writes: $a < 1 - \sqrt{\frac{1}{2}}$ (the coefficient a involved in the disutility of labor is small, so that the consumer accepts to work).

VIII - Labor supply 2

a) Firm's decision has been computed in Problem VII. Consumers' decisions are:

$$\begin{aligned} x_A &= \frac{w}{2p}, l_A = \frac{1}{2}. \\ x_B &= \frac{w+\pi}{p}, l_B = 1. \end{aligned}$$

To compute quickly (x_A, l_A), consider the budget constraint $px_A = wl_A$ and write $u_A = \frac{w}{p} l_A (1 - l_A)$. Maximizing $l_A (1 - l_A)$ shows $l_A = 1/2$ (this is a simple non constraint optimization problem with one variable only).

b) Market clearing for labor writes $l_A + l_B = l_f$, that is $l_f = \frac{3}{2}$. The expression of l_f implies $\frac{w}{p} = 6^{-1/2}$. There is a unique equilibrium, the allocation is computed straightforwardly.