Dialogue on deriving the normal distribution. Our starting point is to know that the distribution looks like something specific. It is clear we need to use the exponential function, we can try the graphs $e^x, e^{-x}, e^{x^2}, e^{x^-2}$. This last one looks like it so can we use it as a PDF? Well what are the rules for making a PDF? Well the PDF has to be positive for every x, is it true? Yes! Okay what else? $\int_\infty^\infty f(x)dx = 1$. Is this verified? Not sure. Well whatever that integral is, it is some number, k, so maybe we can just multiply the thing by diving by k? Okay so actually the integral gives us $\sqrt{\pi}$, so we know that at the end we will have to divive by that term. However we also want to be able to control the distribution, with two paramers, the mean and variance. So if μ is just a shift then it should be easy, $\int_\infty^\infty f(x-\mu)dx$. We can also multiply the inside of the distribution and this will shrink the height of the distribution so will also shrink the area: $\int_\infty^\infty f(\frac{x}{\sigma})dx = \frac{k}{\sigma}$. So what is missing? $P(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. The denominator, $\frac{1}{\sqrt{2\pi}\sigma}$ might be confusing so to check that the integral adds up this we will integrate the function.

$$\int_\infty^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \int_\infty^\infty e^{-y^2}dx \tag{0.0.1}$$

$$y = \sqrt{\frac{(x-\mu)^2}{2\sigma^2}} \quad \frac{dy}{dx} = \frac{1}{\sqrt{2}\sigma} \rightarrow dx = \sqrt{2}\sigma dy \tag{0.0.2}$$

$$\text{So}: \int_\infty^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \int_\infty^\infty e^{-y^2}\sqrt{2}\sigma dy \tag{0.0.3}$$

$$\sqrt{2}\sigma \int_\infty^\infty e^{-y^2}dy = \sqrt{2}\sigma\sqrt{\pi} \tag{0.0.4}$$