0.1 Additional anomalies

To account for additional anomalies that our model could potentially recover we consider the discount function, δ, derived in each of the cases. For this purpose δ is a function of the problem parameters, including the payments, Δx_a and Δx_b, so that a decision maker would have been indifferent between a and b given the same parameters exactly, but with payment $\delta \Delta x_b$ instead of Δx_b in option b.

- Case A: $\delta = \frac{\Delta x_a}{\Delta x_b}$
- Case B: $\delta = \frac{\Delta x_a e^{rD}}{\Delta x_b}$
- Case C: $\delta = \frac{H+D}{H} \frac{\Delta x_a}{\Delta x_b}$
- Case D: $\delta = \frac{x(t_0)e^{r(H+D)}}{\Delta x_b} \left[\left(1 + \frac{\Delta x_a}{x(t_0)e^{rH}} \right)^{\frac{H+D}{H}} - 1 \right]$

In particular, we are interested in two so-called anomalies that are relevant of the several discussed by **?** – the "sign effect" and the "magnitude effect". The sign effect means that gains are discounted more than losses. In other words, that if Δx_a and Δx_b are positive, δ should be higher than for $-\Delta x_a$ and $-\Delta x_b$. It is clear that for cases A, B and C, such a reflection with respect to zero makes no difference to the value of δ. For Case D it does, as we will soon see.

The magnitude effect means that large payments are discounted at a lower rate than lower payments. This means that if we consider payments Δx_a and Δx_b and $\kappa \Delta x_a$ and $\kappa \Delta x_b$, for some factor $\kappa > 1$, than δ will be lower in the latter case. Again, the magnitude effect is not predicted in cases A, B and C, where a multiplication of Δx_a and Δx_b by a common factor makes no difference to δ. However, it does make a difference to δ in case D.

The sign effect in case D The discount function is

$$\delta \left(D; H; r; x\left(t_0\right); \Delta x_a; \Delta x_b \right) = \frac{x\left(t_0\right) e^{r(H+D)}}{\Delta x_b} \left[\left(1 + \frac{\Delta x_a}{x\left(t_0\right) e^{rH}} \right)^{\frac{H+D}{H}} - 1 \right], \quad (0.1)$$

and the question of whether this recovers the "sign effect", as described by **?**, is whether

$$\delta^{+} \equiv \delta\left(D; H; r; x\left(t_0\right); \Delta x_a; \Delta x_b\right) = \delta\left(D; H; r; x\left(t_0\right); -\Delta x_a; -\Delta x_b\right) \equiv \delta^{-} \qquad (0.2)$$

or not? It is possible that for positive payments $\delta^{+} > \delta^{-}$, *i.e.*, the "sign effect" is recovered.

To show it, let us look at the ratio

$$\frac{\delta^{+}}{\delta^{-}} = -1 \cdot \frac{\left(1 + \frac{\Delta x_a}{x(t_0)e^{rH}}\right)^{\frac{H+D}{H}} - 1}{\left(1 + \frac{-\Delta x_a}{x(t_0)e^{rH}}\right)^{\frac{H+D}{H}} - 1}. \qquad (0.3)$$

It is easy to see that the numerator is positive, assuming positive payments and positive horizon and delay. Assuming $1 + \frac{-\Delta x_a}{x(t_0)e^{rH}}$ is positive, *i.e.* $\Delta x_a < x\left(t_0\right) e^{rH}$, so that the earlier payment is not too high, then the denominator is clearly negative and smaller in size than the numerator. Thus, $\frac{\delta^{+}}{\delta^{-}}$ is positive and larger than 1, and we get the sign effect.

Let us look at a simple numerical example. We assume $H = 1$ day, $D = 1$ day, $x\left(t_0\right) = \$10$ and $r = 10\%/\text{day}$ in case D. With these parameters, a growth-optimal decision maker will be indifferent between an early payment of \$1 and a later payment of \$2.31. However, she will also be indifferent between an early loss of \$1 and a later loss of \$2.11. In other words, the gain is discounted more than a loss.

The magnitude effect in case D The discount function is

$$\delta\left(D; H; r; x\left(t_0\right); \Delta x_a; \Delta x_b\right) = \frac{x\left(t_0\right) e^{r(H+D)}}{\Delta x_b} \left[\left(1 + \frac{\Delta x_a}{x\left(t_0\right) e^{rH}}\right)^{\frac{H+D}{H}} - 1\right], \qquad (0.4)$$

and the question of whether this recovers the "magnitude effect", as described by **?**, is whether

$$\delta \equiv \delta\left(D; H; r; x\left(t_0\right); \Delta x_a; \Delta x_b\right) = \delta\left(D; H; r; x\left(t_0\right); \kappa\Delta x_a; \kappa\Delta x_b\right) \equiv \delta^{\kappa} \qquad (0.5)$$

or not, for $\kappa > 1$? If $\delta > \delta^{\kappa}$, then the "magnitude effect" is recovered.

Let us look at the ratio

$$\frac{\delta}{\delta^{\kappa}} = \kappa \cdot \frac{\left[\left(1 + \frac{\Delta x_a}{x(t_0)e^{rH}}\right)^{\frac{H+D}{H}} - 1\right]}{\left[\left(1 + \frac{\kappa \Delta x_a}{x(t_0)e^{rH}}\right)^{\frac{H+D}{H}} - 1\right]} \stackrel{?}{>} 1\,. \tag{0.6}$$

In the small payment limit, *i.e.* $\frac{\Delta x_a}{x(t_0)e^{rH}} \ll 1$ (and $\frac{\kappa \Delta x_a}{x(t_0)e^{rH}} \ll 1$), we easily get $\frac{\delta}{\delta^{\kappa}} \approx 1$. So, in practical terms, the magnitude effect is not recovered by growth-optimality.