Theory of signaling games

Charles Roddie*

August 28, 2011

Abstract

This paper studies signaling games under weak conditions, generalizing existing results, and finding new continuity and comparative statics properties.

Spence-Mirrlees single crossing is replaced with a more applicable order theoretic condition that is implied by supermodularity. Differentiability of payoffs is also relaxed. Separating equilibria still exist, there is uniqueness with a continuum of types, and the dominant separating equilibrium is selected by the D1 refinement.

Separating and dominant separating equilibria are continuous in the type space as it becomes a continuum, and in the payoff function. Signaling games have strong comparative statics properties, and conditions are found for strategies to be increasing in a parameter, and for value functions to be supermodular. These results allow for the analysis of signaling within larger dynamic games.

Keywords: Signaling, single crossing, supermodular games

Contents

*Nuffield College, Oxford. I am grateful to Meg Meyer for helpful discussions and Gabriel Kriendler for research assistance.

1 Introduction

Signaling models have been used to understand a variety of phenomena in economics, biology, and political science. Tractable signaling models satisfy some form of single crossing condition, which allows higher/better types to separate from lower types by choosing a higher signal.

Most theoretical results have assumed strong forms of single crossing, normally Spence-Mirrlees single crossing (SMSC). But this often fails in applications when there are intrinsic reasons for taking positive signals. This paper shows that the central results hold under a weaker and more applicable form of single crossing: existence of separating equilibria and a dominant separating equilibrium, and selection of this equilibrium by the D1 refinement. The weaker condition also relaxes differentiability, and even without this we find a unique separating equilibrium under a continuum of types.

Secondly, we find new comparative statics properties and continuity properties of separating equilibria, useful for analysis of signaling that takes place within larger games and in particular supermodular games.

These results are developed in a unified mathematical framework, taking a topological approach to deal with a continuum of types.

1.1 Weakening single crossing, with applications

1.1.1 Strong single crossing and (weak) single crossing

Single crossing is key to the structure of signaling games because it gives higher types a greater willingness to choose higher signals than lower types in return for a better response. This is precisely the weak single crossing condition assumed here.

Let $\theta \in \Theta \subseteq \mathbb{R}$ denote the signaler's type, $a \in A \subseteq \mathbb{R}$ the signal chosen, and $b \in B$ either a response or an inferred type, in either case partially ordered.

Definition. A function $u(\theta, a, b)$ satisfies **single crossing** if when $\theta_1 < \theta_2$ and $b_1 \leq b_2$: $u(\theta_1, a_1, b_1) \leq u(\theta_1, a_2, b_2)$ and $a_1 \leq a_2$ imply $u(\theta_2, a_1, b_1) \leq u(\theta_2, a_2, b_2)$, and strictness in either inequality implies $u(\theta_2, a_1, b_1) < u(\theta_2, a_2, b_2)$.[1]

Most existing theory has assumed a strong single crossing condition, where a higher type is always more willing to choose a higher signal even if it results in a worse response or inference.

Definition. A function $u(\theta, a, b)$ satisfies **strong single crossing** if when $\theta_1 < \theta_2$: $u(\theta_1, a_1, b_1) \leq u(\theta_1, a_2, b_2)$ and $a_1 \leq a_2$ imply $u(\theta_2, a_1, b_1) \leq u(\theta_2, a_2, b_2)$, and strictness in either inequality implies $u(\theta_2, a_1, b_1) < u(\theta_2, a_2, b_2)$.[2]

The difference from the stronger form is requirement in the comparison that $b_1 \leq b_2$. Figure 1 illustrates this.[3] Strong single crossing implies that in any perfect Bayesian equilibrium, choice of signal is weakly increasing in type.

There are two cases in which strong and weak forms are equivalent:

Case 1. $u(\theta, a, b)$ is strictly increasing in b and strictly decreasing in a. Then without signaling incentives (under complete information) the minimum signal is always taken. Then $u(\theta_1, a_1, b_1) \leq u(\theta_1, a_2, b_2)$ and $a_1 \leq a_2$ implies $b_1 \leq b_2$, so single crossing implies strong single crossing.

[1]This condition is given in Bagwell and Wolinsky (2002), in the context of a limit pricing model with a binary type and response, and called the "single crossing property". It is close to the notion of single crossing defined in Milgrom and Shannon (1994), in between their single crossing and strict single crossing, when the partial order in their definition is taken to be the usual $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$ and $b_1 \leq b_2$.

[2]This is assumption A4 of Cho and Sobel (1990).

[3]Note that the (probably unavoidable) term "single crossing" has become misleading: there can be more than one crossing of indifference curves, as long as these points are not ordered in $\mathbb{R}^2$.

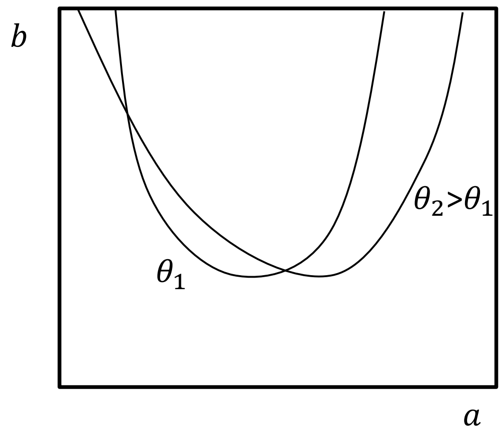

Figure 1: Indifference curves allowed by single crossing but not by strong single crossing.

Case 2. The signaling benefit enters payoffs additively: $u(\theta, a, b) = \alpha(\theta, a) + \beta(b)$.

There are examples within the first case, but it is more common for signals to have some intrinsic value. Outside of these special cases, strong single crossing is typically hard to guarantee in applications.[4]

The common and stronger SMSC condition invokes differentiability and asserts that u_a/u_b is strictly increasing in θ (where A and B are intervals). It implies strong single crossing (Cho and Sobel (1990)).

1.1.2 Single crossing and supermodularity

Recall the standard definitions of increasing differences and supermodularity.

Definition. $f(x, y)$ has weakly increasing differences if for $x' \geq x$ and $y' \geq y$, $f(x', y') + f(x, y) \geq f(x', y) + f(x, y')$. If additionally strictness in the assumptions implies a strict conclusion, f has strictly increasing differences. $f(x, y, z, ...)$ has weakly/strictly increasing differences in (x, y) if for any $(z, \ldots)$, $(x, y) \mapsto f(x, y, z, \ldots)$ has weakly/strictly increasing differences. It is weakly/strictly supermodular if it has weakly/strictly increasing differences in all pairs of variables.

An important advantage of single crossing over strong single crossing is that it is implied by supermodularity, as noted in Bagwell and Wolinsky (2002):

[4] I am not aware of any such applications satisfying strong single crossing, despite the benefit of being able to use existing theoretical results.

Fact. *If $u(\theta, a, b)$ has strictly increasing differences in (θ, a) and weakly increasing differences in (θ, b) then it satisfies single crossing in $(\theta; a, b)$.*

Supermodularity is a common and effective method for construction of signaling games, and more generally multi-stage games with strategic complements or substitutes, since it can describe these and is closed under addition. A supermodular value function $V(\theta, \theta')$ from a second stage, combined with a supermodular payoff $w(\theta, a)$ in the first stage results in a supermodular signaling payoff $w(\theta, a) + \delta \cdot V(\theta, \theta')$.

1.1.3 Examples and applications

Example 1. Consider a signaling model of education. Payoffs for an agent with ability θ from taking education a and being believed to be type θ' are $U(\theta, a, \theta') = -(a - (\theta + 5))^2 + 9\theta \cdot \theta'$. Higher types have a higher reward or lower cost of education; the complete information choice of education is positive, reflecting an intrinsic or job market value[5]; the benefit of signaling higher θ', which opens up higher-level jobs, is higher for a higher type. U is supermodular, so satisfies weak single crossing.

With two types, 1 and 2 the separating equilibria have type 1 taking the complete information choice 6, and type 2 separating by taking education in $[7 - 3\sqrt{2}, 3] \cup [9, 7 + 3\sqrt{2}] \approx [2.8, 3] \cup [9, 11.2]$. The existence of decreasing equilibria shows that U does not satisfy strong single crossing. Existing theoretical results therefore do not apply to this model.

Example 2. In a two-stage Cournot limit pricing model, in the second stage there are profits $\left(P\left(q_I^2 + q_E^2\right) - c_i\right) \cdot q_i^2$, which are supermodular for the incumbent in $\left(-c_I, q_I^2, -q_E^2\right)$ if the inverse demand function P is weakly concave, and similarly for the entrant. The payoff of the maximizing incumbent is then supermodular in $\left(-c_I, -q_E^2\right)$, giving a value function supermodular in $\left(-c_I, -c_I'\right)$, where c_I' is signaled type, since q_E^2 is increasing in c_I'. This gives a signaling payoff supermodular in $\left(-c_I, q_I^1, -c_I'\right)$ in the first stage, implying single crossing. But strong single crossing has been verified to fail here even with a linear demand function.[6]

Maintaining the supermodular structure, we can add additional strategic phenomena between stages, for example a learning by doing or brand loyalty effect of

[5] Spence (1973), the earliest analysis of signaling in economics, assumed that education did not have intrinsic or productive value.

[6] Example on request; I am grateful to Henri Savolainen for calculating this.

market share. Arbitrary type spaces can also be used without difficulty, and uncertainty in the entrant's cost doesn't affect the analysis.

Currently applications often have to reinvent the wheel by deriving existence of separating equilibrium and applying equilibrium refinements from first principles, in simplified settings, since they are unable to use theoretical results that depend on SMSC.

The result is that applications are typically unable to rely on theoretical results, instead duplicating a subset of results in their specific contexts. In order to do this easily, the models are simplified, with two-type models predominant and strategic interactions limited.

1.2 Relaxing technical conditions

1.2.1 Relaxing monotonicity, differentiability, and continuity

Existing theory has assumed differentiability of payoffs (even when SMSC is not used), and often stronger assumptions are used: smooth or C_2 payoffs. If the payoff function is $U\left(\theta, a, \theta'\right)$, a usual further assumption is $\frac{\partial U}{\partial \theta'} > 0$. Here differentiability is removed, and U is allowed to have discontinuities in θ'. Also, U is only required to be weakly increasing in θ', if we correspondingly weaken the notion of separating equilibrium.

There are two types of applications of these these generalizations. Direct applications, where simply specified games violate the stronger conditions. And topological applications, where the improved closure properties of the signaling space allow finding equilibrium of a game as a fixed point.

1.2.2 Direct applications

Discontinuities and non-strictness of monotonicity in θ' can both arise in applications where the response is binary (entry of competitor, agreement to a contract), which leads to a cutoff in θ', with U being discontinuous there and otherwise constant in θ'. This also implies non-differentiability.

Non-differentiability can also result from some effect that starts or stops at a certain point. If the response meets a lower or upper bound: for example in Example 2, the entrant's production will typically be 0 up to a certain signaled cost c^*. This also causes non-monotonicity when the bound is reached. If the response has multiple "causes", there can be non-differentiable points: a firm advertising and selling to two groups of customers, or a government signaling competence to a market that has regular reasons for responding favorably, but an extra effect of illiquidity takes place at low signals.

Multiple equilibria subsequent to the signaling stage can also generate discontinuity or non-differentiability at the point where one type of equilibrium switches to another.

1.2.3 Topological advantages

Taking limits of signaling games is useful for (at least) two reasons. Firstly in the theoretical development here, taking limits is used to extend results, most commonly from the finite type case to the continuum. Secondly in applications fixed point arguments can be used to show existence of equilibrium: for example to show existence of Markov perfect equilibrium in an infinite horizon, we can use a continuous operator on a compact space of value functions. In taking these limits the L1 norm is used here[7], and the theoretical development is proof of its appropriateness.

Differentiability, continuity, and strict monotonicity are not closed in L1: a limit of C_∞ payoff functions $U_i(\theta, a, \theta')$, strictly monotonic in θ', may be discontinuous and only weakly monotonic. This is a problem when using fixed point theorems that require compact spaces. Also when taking a continuum limit, it is important the limit point should be a valid signaling payoff, which closure guarantees. The relaxed conditions here avoid these issues.[8]

Relaxing unnecessary conditions displays the essential structure of signaling games more clearly.

[7] In our context all the L_p norms are identical, for $p < \infty$.

[8] Precisely, useful subspaces are closed. If we restrict to payoffs that are uniformly Lipschitz in θ and a, with a lower bound on the concavity of a (or quasi-concavity, in the manner of Mailath (1987)) and increasing differences in (θ, a), we get a closed space of signaling payoffs.

1.3 Generalized results

Signaling games admit both separating and pooling equilibria, and existing theory and applications have been particularly interested in the separating equilibria. Riley (1979) showed existence of separating equilibrium in the context of a closely related sorting model, assuming SMSC. One equilibrium is dominant in payoffs for the "signaler".

These equilibria exist without differentiability, under just weak single crossing. Some of the separating equilibria not be increasing, as Example 1 shows, but the dominant equilibrium is increasing.

To cut down on equilibria, theoretical and applied work invokes additional restrictions on beliefs. Cho and Sobel (1990) showed that the D1 refinement selects the dominant separating equilibrium. This assumed finite types, strong single crossing and differentiability. Ramey (1996) showed this for a continuum of types under SMSC.[9] However applied work is often unable to use these results; and often has to apply a refinement manually in simplified two-type settings (usually the "intuitive criterion" of Cho and Kreps (1987)).

Here we show that the D1 refinement works under the weaker form single crossing.

The natural specification for most signaling applications is a continuum of types, rather than a finite type space. With a continuum of types, Mailath (1987) showed there is a unique separating equilibrium. The argument was that an incentive compatible strategy must be differentiable and satisfy a differential equation. Here we show the result applies in the absence of differentiability.

1.4 New results and applications

1.4.1 Continuity properties

In most settings a continuum is the natural specification of the type space, and here a topological approach is used to deal with this. Central to the analysis of a continuum of types are continuity properties about separating and dominant separating equilib-

[9] Cho and Sobel (1990) and Ramey (1996) extend to multidimensional signals, the latter with a weaker condition from Engers (1987), but it is hard to have multidimensional conditions that are both tractable and applicable.

ria. (See 3.)

The set of separating equilibria and the dominant separating equilibrium are (upper-semi)continuous in the type space as it becomes dense in a continuum. This extends Mailath (1988), which found the result for separating equilibria, generalizing it from SMSC to single crossing.[10] The result can be used to extend equilibria for finite type spaces to a continuum: both existence of equilibria and results about equilibria.

Both separating and dominant separating equilibria are also continuous with respect to the payoff function. This result can be used for existence of equilibrium, if this is found with a fixed point argument: for example, it is useful if other players move at the same time as the signaler. (These other players may also be signaling.) It is also used to show uniqueness of separating equilibrium for a continuum of types.

The continuity holds when type space and payoffs are varied simultaneously, which is also useful in existence of equilibrium arguments (see Section 5.3).

1.4.2 Comparative statics

What if the signaler is not moving alone in the first stage? What if more than one agent is signaling simultaneously? What if there are stages before or after the signaling and response stages? Regarding the action taken before of simultaneously with the signal as a parameter z, we ask we ask how (dominant separating) strategies and payoffs depend on z.

A condition related to single-crossing is found for the strategy to be increasing in z. This property can give the extended games described above a tractable directional structure.

The condition is satisfied if the payoff function is supermodular, and then we get a value function which is a supermodular function of z and type. This allows us to embed signaling games within supermodular dynamic games: future supermodular stages generate single crossing in the signaling stage, and this preserves a supermodular value function for previous stages. Roddie (2010a) uses this property to study repeated signaling games. Regularity conditions are also given which bound the rate of increase of strategies and the degree of supermodularity.

[10] Manelli (1996) finds that arbitrary equilibria also converge to an equilibrium of the continuum case.

1.5 Contents

Section 2 defines the space of signaling games. The necessary topologies are specified in Section 3.

Section 4 defines separating equilibria and characterizes the (closure of the) set of increasing separating equilibria. This set is (upper semi-)continuous in the type space as it becomes dense in an interval and in the payoff function (Proposition 1). Existence of separating equilibrium is shown (Proposition 3) without requiring differentiability or strong single crossing.

Section 5 studies the payoff-dominant ("Riley") separating equilibrium. This exists uniquely (Proposition 2) and is increasing. Lemma 4 gives further characterization. Section 5.3 extends the continuity results for separating equilibria to the Riley equilibrium (Proposition 3).

Section 6 (Proposition 4) shows uniqueness of separating equilibrium for a continuum of types. This extends Mailath (1987) to non-differentiable settings.

Section 7 studies the comparative statics of the Riley equilibrium. Suppose a signaling payoff is parametrized by z. A condition related to single crossing implies that the Riley equilibrium is increasing in z (Proposition 5). In particular this happens if the signaling payoff is supermodular, and then equilibrium utility is a supermodular function of z and θ (6).

Section 8 shows that the refinement D1 selects the Riley equilibrium, weakening the assumption of strong single crossing in Cho and Sobel (1990).

2 Signaling payoff spaces

Types, signals, beliefs, and payoffs

The type space Θ is either the interval $\bar{\Theta} = [\theta_{\min}, \theta_{\max}]$ or a finite subset with minimal and maximal elements $\theta_{\min}, \theta_{\max}$. The set of signals A is an interval $[a_{\min}, a_{\max}]$. A belief about types is an element $\hat{\theta}$ of $\triangle\Theta$, which is ordered by first-order stochastic dominance.

A signaling game is specified in reduced form by a utility function $u : \Theta \times A \times \triangle\Theta \to \mathbb{R}$, for the signaler, giving payoff $u\left(\theta, a, \hat{\theta}\right)$ to type θ of the signaler when he chooses signal a and is consequently believed to be type $\hat{\theta}$.

To study separating equilibria, we only need to know utility for degenerate posterior beliefs. So we can restrict attention to $U : \Theta \times A \times \Theta \to \mathbb{R}$, where $U(\theta, a, \theta') := u(\theta, a, [\theta'])$, where$[\theta]$ denotes the probability measure placing probability 1 on θ. We will be using these spaces for most of this paper, but will return to define and analyze the fully specified spaces of $u : \Theta \times A \times \triangle\Theta \to \mathbb{R}$ in Section 8, where non-separating equilibria are considered.

Notation. When U is unambiguous, $(a_1, \theta'_1) \preceq_\theta (a_2, \theta'_2)$ will mean $U(\theta, a_1, \theta'_1) \leq U(\theta, a_2, \theta'_2)$, and similarly for $\succeq$, $\sim$, $\prec$ and $\succ$.

The main signaling payoff space

We define a set Φ of signaling game payoffs satisfying the basic conditions; all payoffs in this paper will be in Φ.

Definition. Let Φ be the set of $U : \Theta \times A \times \Theta \to \mathbb{R}$ with $U(\theta, a, \theta')$:

1. uniformly continuous in (θ, a)[11][12]
2. weakly increasing in θ', with $((a, \theta_1) \sim_\theta (a, \theta_2))$ independent of a
3. strictly quasi-concave in a
4. satisfying single crossing

The first condition is a weakening of the usual assumption of differentiability (or more commonly $U \in C_2$ or C_∞). It is weaker than continuity of U, allowing for discontinuities with respect to θ'.

The second condition weakens the usual assumption $\frac{\partial U}{\partial \theta'} > 0$. If U is only weakly increasing, it is possible that some type θ, taking some signal a, does not strictly benefit from being believed to be $\theta_2 < \theta_1$ over θ_1. The condition requires that when this happens does not depend on a. This automatically holds if U is strictly increasing in θ', or if $U = \alpha(\theta, a) + \beta(\theta, \theta')$, a common property of signaling payoffs in a multi-stage game.

[11] This means that for any $\epsilon > 0$, there exists $\delta > 0$ such that for $\|(d\theta, da)\| < \delta$, $|U(\theta + d\theta, a + da, \theta') - U(\theta, a, \theta')| < \epsilon$ for any (θ, a, θ').

[12] Of course, adding any function of θ to U doesn't affect it's real properties.

The third condition is relatively standard. It implies existence of a unique optimal strategy under complete information.

The fourth condition asserts the weak form of single crossing defined in the introduction.

Bounded payoffs

An extra condition makes the highest signal sufficiently costly that taking it would contradict IR even for the highest type. This will ensure there are costly enough signals for a separating equilibrium to exist.

Definition. Let Φ_B be the set of $U \in \Phi$ with $\sup_a U(\theta_{\max}, a, \theta_{\min}) \geq U(\theta_{\max}, a_{\max}, \theta_{\max})$.

Responses

Usually the signaling payoff is derived as $u\left(\theta, a, \hat{\theta}\right) = v\left(\theta, a, \rho\left(\hat{\theta}, a\right)\right)$, where $\rho\left(\hat{\theta}, a\right)$ is a best-response of a respondent. Single crossing in v, combined with monotonicity properties of v and ρ guarantees single crossing for U.

Fact. *Suppose $v(\theta, a, r)$ is continuous, weakly increasing in r and satisfies single crossing. Suppose $\rho\left(\hat{\theta}, a\right)$ is weakly increasing in $\hat{\theta}$ and a and uniformly continuous in a. Then $U\left(\theta, a, \theta'\right) := v\left(\theta, a, \rho\left(\left[\theta'\right], a\right)\right)$ is weakly increasing in θ', satisfies single crossing, and is uniformly continuous in (θ, a). If v is strictly increasing in r, then condition 2 is automatically satisfied. If ρ is a function of θ' only, then quasi-concavity in a follows from quasi-concavity of v in a.*

Separating payoffs and the complete information strategy

If the signaler of type θ takes a strategy $f : \Theta \to A$, and is subsequently known to be type θ, its payoff is:

Definition. Given $U \in \Phi$ and $f : \Theta \to A$, let $U_f(\theta) := U\left(\theta, f(\theta), \theta\right)$.

This will be its payoff in a separating equilibrium described by f. The complete information strategy a_U^* maximizes this:

Definition. For $U \in \Phi$, let $a_U^*(\theta) = \arg\max_a U(\theta, a, \theta)$. Where a^* is used U is implicit.

3 Topologies

We shall be using L1 norms on strategies and signaling payoffs, which requires measures on the initial spaces. On A we use standard Lebesgue measure, normalized to 1. If Θ is finite, we use counting measure. If it is a continuum, we use Lebesgue measure, but altered to give $\theta_{\min}$ and $\theta_{\max}$ positive measure. This ensures that L1 convergence of strategies implies pointwise convergence on $\{\theta_{\min}, \theta_{\max}\}$. Let μ_Θ be this measure on Θ, again normalized to 1.

We also want some notion of convergence in Θ. Let the metric on Θ be as usual, but with $\theta_{\min}$ and $\theta_{\max}$ separated from the other points ($d(\theta_{\min}, \theta) = \theta - \theta_{\min} + 1$ when $\theta \neq \theta_{\min}$ for example).

Signaling payoffs in Φ are measurable functions $U : \Theta \times A \times \Theta \to \mathbb{R}$, and we endow the space Φ with the L1 norm, using the above measures on Θ and A. Strategies are functions $\Theta \to \mathbb{R}$, and we will be particularly interested in the weakly increasing functions, $\mathrm{Inc}(\Theta, A)$. These are measurable. Endow $\mathrm{Inc}(\Theta, A)$ with the L1 norm; it is then a compact space.

Varying Θ

In order to move between finite types and the continuum, both in showing continuity results and extending results to the continuum case, we need a notion of Θ converging to $\bar{\Theta}$.

Definition. $\Theta_i \nearrow \Theta_\infty$ if $\{\theta_{\min}, \theta_{\max}\} \subseteq \Theta_i \in \bar{\Theta}$ is an increasing sequence of sets, with $\bigcup \Theta_i$ dense in Θ_∞.

Suppose $f_i \in \mathrm{Inc}(\Theta_i, A)$ are strategies on spaces $\Theta_i \nearrow \Theta_\infty$. We want some notion of convergence "$f_i \nearrow f_\infty$", despite the fact that they lie in different spaces:

Definition. Suppose $\Theta_i \nearrow \Theta_\infty$ and $f_i \in \mathrm{Inc}(\Theta_i, A)$. Write $f_i \nearrow f_\infty$ if for any $\bar{f}_i \in \mathrm{Inc}(\Theta_\infty, A)$ with $\bar{f}_i|_{\Theta_i} = f_i$, $\bar{f}_i \to f_\infty$.[13]

[13]Lemma B.4 implies that it is sufficient for this to hold for just one sequence (g_i). It is equivalent to convergence under the metric $d(f_i, f_j) := \sup\{d(g_i, f_j) : g_i \in \mathrm{Inc}(\Theta_j, A), g_i = f_i \text{ on } \Theta_i\}$.

4 Separating equilibria

Section 4.1 defines the sets of separating and increasing separating equilibria. For $U \in \Phi_B$, such equilibria exist. The approach to existence is as follows: Section 4.2 characterizes the closure of the set of separating equilibria, allowing Section 4.3 to find continuity properties, including continuity in the type space. Continuity in the type space allows us to extend a construction (4.4) of separating equilibrium for finite types to a continuum.

4.1 Weakly separating strategies

Incentive compatibility is the condition that no type has an incentive to mimic another type. Individual rationality is the condition that no type receives such low utility that he would prefer to take some other action even if the worst possible beliefs result.

Definition. A strategy $f : \Theta \to A$ satisfies **individual rationality** (over $S \subseteq \Theta$) if for all θ ($\in S$) and a, $\left(f(\theta), \theta\right) \succeq_\theta (a, \theta_{\min})$. It satisfies **incentive compatibility** (over S) if for all θ, θ' ($\in S$), $\left(f(\theta), \theta\right) \succeq_\theta \left(f\left(\theta'\right), \theta'\right)$. If both are satisfied, f is **weakly separating** (over S).[14]

Definition. Let $\mathrm{Sep}^{\pm}(U)$ denote the set weakly separating strategies.

We shall be particularly interested in strategies that are weakly increasing:

Definition. Let $\mathrm{Sep}(U) = \mathrm{Sep}^{\pm}(U) \cap \mathrm{Inc}(\Theta, A)$.

If U satisfies strong single crossing, $\mathrm{Sep}(U) = \mathrm{Sep}^{\pm}(U)$.

Relationship to Perfect Bayesian equilibria

Call f separating if it is weakly separating and injective. Provided f is measurable, it is then a perfect Bayesian equilibrium, when combined (for example) with beliefs:

[14] It is possible to remove the IR requirement. This would streamline the theory in some minor ways. But IC+IR corresponds to (separating) perfect Bayesian equilibrium as noted below, the most standard notion of equilibrium, whereas IC alone corresponds to something weaker than Nash equilibrium. IC alone corresponds to Nash equilibrium if for any θ, a', $v\left(\theta, a', r_{\min}\right) \le \inf_{a,\theta'} U\left(\theta, a, \theta'\right)$, where $U\left(\theta, a, \theta'\right) = v\left(\theta, a, \rho\left(\left[\theta'\right], a\right)\right)$.

$$\beta(a) = \begin{cases} f^{-1}(a) & a \in \operatorname{Im}(f) \\ [\theta_{\min}] & a \notin \operatorname{Im}(f) \end{cases}$$

Fact 1. *If $U(\theta, a, \theta')$ is strictly increasing in θ', all weakly separating strategies are separating.*

Proof. IC implies a weakly separating function must be injective. □

If this does not hold, it is possible for a weakly separating strategy to be non-injective. Pooling is allowed to occur if within a pool no type has a strict incentive to appear to be a higher type within the pool. If all types agree on when it is strictly better to signal θ'_2 than θ'_1, then a weakly separating equilibrium with pools is a Perfect Bayesian equilibrium:

Fact 2. *Suppose $u(\theta, a, \hat{\theta})$ is weakly increasing in $\hat{\theta}$, and $\left((a, \theta'_1) \sim_\theta (a, \theta'_2)\right)$ is a function of (θ'_1, θ'_2); i.e. the ranges of θ' that do not affect utility are constant in θ. If $f \in \operatorname{Sep}^{\pm}(U)$ is measurable, then f describes a Perfect Bayesian equilibrium with U_f describing equilibrium utility.*

One such equilibrium sets beliefs to $[\theta_{\min}]$ for actions that are not taken, and a conditional probability of θ given a for actions that are; such a conditional probability is easy to construct if $f \in \operatorname{Sep}(U)$.

4.2 The closure of separating strategies

A weakly separating $f \in \operatorname{Sep}(U)$ may have discontinuities, if U is discontinuous or has regions of constancy in θ'. Then there are weakly increasing functions equal to f except at these discontinuities that are not separating. So we cannot say that $\operatorname{Sep}(U)$ is closed in $\operatorname{Inc}(\Theta, A)$. But when $\operatorname{Sep}(U)$ is adjusted to include these functions, we get a closed set.

Definition. Let $\overline{\operatorname{Sep}}(U) := \left\{g \in \operatorname{Inc}(\Theta, A) : g = f \text{ } a.e. \text{ for some } f \in \operatorname{Sep}(U)\right\}$.

Lemma 1. $\overline{\operatorname{Sep}}(U)$ *is the closure of* $\operatorname{Sep}(U)$ *in* $\operatorname{Inc}(\Theta, A)$.

4.3 Continuity with respect to Θ and U

In the above Θ has been taken as fixed, but now allow it to vary, with the above functions depending implicitly on Θ. The correspondence $\overline{\text{Sep}}$ is (upper semi-)continuous with respect to U and the type space Θ. We will use a general argument to show this simultaneously. All utility functions will be given on the largest type space Θ_∞, and restricted as necessary to smaller ones. Parts 2 and 3 are special cases of part 1.

Proposition 1. *1. (Continuity in U and Θ) Suppose $\Theta_i \nearrow \Theta_\infty$. Suppose $U_i \in \Phi(\Theta_\infty)$, with $U_i \to U_\infty$ (in L1) and $U_i(\theta, a, \theta')$ uniformly continuous in (a, θ) over (i, θ'). Suppose $f_i \in \overline{\text{Sep}}(U_i)$ and $f_i \nearrow f$. Then $f \in \overline{\text{Sep}}(U_\infty)$.*

2. (Continuity in Θ; extends Mailath (1988)) Take $U \in \Phi(\bar{\Theta})$, and let $\theta_{\min} \in \Theta_i \nearrow \bar{\Theta}$. Suppose $f_i \in \text{Inc}(\Theta_i, A)$ with $f_i \nearrow f$. Then $f \in \overline{\text{Sep}}(U)$.

3. (Continuity in U:) Suppose $U_i \in \Phi$, with $U_i \to U_\infty$ in L1 and $U_i(\theta, a, \theta')$ uniformly continuous in (a, θ) over (i, θ'). Suppose $f_i \in \overline{\text{Sep}}(U_i)$ and $f_i \to f$. Then $f \in \overline{\text{Sep}}(U_\infty)$.

4.4 Existence of separating equilibria

In any separating equilibrium, the lowest type must take the complete information signal $a^*_U(\theta_{\min})$. For finite Θ, we can define a separating equilibrium χ inductively: subsequent types maximize utility $U_\chi(\theta)$ subject to separating from lower types.

Lemma 2. *For finite $\Theta = \{\theta_0, \theta_1, \ldots\}$, if $U \in \Phi$, the following definitions are equivalent and uniquely define $\chi \in \text{Sep}(U)$ if a solution exists. If $U \in \Phi_B$ a solution must exist.*

1. *$\chi(\theta_0) = a^*(\theta_0)$. For $i \geq 1$, $\chi(\theta_i)$ maximizes $U(\theta_i, a, \theta_i)$ over $B_i := \{a \geq \chi(\theta_{i-1}) : (\chi(\theta_{i-1}), \theta_{i-1}) \succeq_{\theta_{i-1}} (a, \theta_i)\}$.*
2. *For each θ', $\chi(\theta')$ maximizes $U(\theta', a, \theta')$ over $\{a : a \geq \chi(\theta),\ (\chi(\theta), \theta) \succeq_\theta (a, \theta')\ \text{for}\ \theta < \theta'\}$.*

The construction is standard in the setting of strong single crossing (see Cho and Sobel (1990) or Sobel (2009)) and here we show it works under just single crossing.

Let Φ_{Sep} be the set of signaling payoffs in Φ for which a separating equilibrium exists:

Definition. $\Phi_{\text{Sep}} := \{U \in \Phi : \text{Sep}(U) \neq \varnothing\}$.

This set contains Φ_B, so that $\Phi_B \subseteq \Phi_{\text{Sep}} \subseteq \Phi$. So we have existence of separating equilibrium without differentiability or strong single crossing:

Lemma 3. $\Phi_B \subseteq \Phi_{\text{Sep}}$.

Proof. Lemma 2 shows this for finite Θ. For $\Theta = \bar{\Theta}$, take finite $\Theta_i \nearrow \Theta$ and take a convergent sub-sequence of separating equilibria. Proposition 1.2 shows this is an element of $\overline{\text{Sep}}(U)$, so $\text{Sep}(U)$ must be non-empty. □

5 The dominant separating equilibrium

When a separating equilibrium exists, there exists a dominant separating equilibrium, often known as the Riley equilibrium, which maximizes utility for all types of the signaler.

5.1 Existence and uniqueness

Given a finite collection S of weakly separating strategies f, we can take $g(\theta)$ for each θ to be the $f(\theta)$ that maximizes $U_f(\theta)$ over $f \in S$. This results in a weakly separating g that dominates all the f. With a little care the argument extends to when S is all the separating equilibria. The dominant equilibrium is weakly increasing, and unique in $\text{Sep}(U)$. It is essentially unique in $\text{Sep}^{\pm}(U)$.

Proposition 2. *1. If $U \in \Phi_{\text{Sep}}$, $\exists g \in \text{Sep}(U)$ such that $U_g \geq U_f$ for any $f \in \text{Sep}^{\pm}(U)$.*

2. (a) g is unique in $\text{Sep}(U)$.

(b) g is (i.) unique in $\text{Sep}^{\pm}(U)$ if Θ is finite, and (ii.) unique up to differences on a countable set if $\Theta = \bar{\Theta}$.

Let $\mathscr{R}(U)$ be this equilibrium:

Definition. $\mathscr{R} : \Phi_{\text{Sep}} \to \text{Sep}(\Theta, A)$ gives for each $U \in \Phi_{\text{Sep}}$ the above element g.

5.2 Further characterization

We can give some additional characterizations and properties of $\mathscr{R}(U)$.

Lemma 4. *For $U \in \Phi_{\text{Sep}}$:*

1. *$\mathscr{R}(U)$ is the unique smallest function $f \geq a^*$*

    (a) *which satisfies IC*

    (b) *which is in* $\text{Sep}(U)$

2. *If Θ is finite, fixing $U \in \Phi_{\text{Sep}}$, $\mathscr{R}(U) = \chi$ as defined in Lemma 2.*

5.3 Continuity with respect to Θ and U

In Section 5.3 we obtained continuity results for $\overline{\text{Sep}}$. The Riley equilibrium is (jointly) continuous in the type space and payoff function:

Proposition 3.

1. *Suppose $U_i \in \Phi_{\text{Sep}}(\Theta_\infty)$, $(i \in \mathbb{N} \cup \{\infty\})$ with $U_i \to U_\infty$ (in L1) and $U_i(\theta, a, \theta')$ uniformly continuous in (a, θ) over (i, θ'). Suppose $\Theta_i \nearrow \Theta_\infty$. Then $\mathscr{R}(U_i|_{\Theta_i}) \nearrow \mathscr{R}(U_\infty)$.*

2. *Suppose $U \in \Phi_{\text{Sep}}(\Theta_\infty)$, and $\Theta_i \nearrow \Theta_\infty$. Then $\mathscr{R}(U|_{\Theta_i}) \nearrow \mathscr{R}(U)$.*

3. *Take $U_i \in \Phi_{\text{Sep}}(\Theta)$, with $U_i \to U_\infty$ and $U_i(\theta, a, \theta')$ uniformly continuous in (a, θ) over (i, θ'). Then $\mathscr{R}(U_i) \to \mathscr{R}(U_\infty)$.*

These results are useful for dealing with a continuum of types. Continuity in Θ is useful for extending results from the finite type case to the continuum. It can also be used to describe equilibria when types are finite but close to a continuum.

Continuity in U will be used to show uniqueness of separating equilibrium when $\Theta = \bar{\Theta}$ (Section 6). It can also be used to show existence of equilibria where more than one player is moving in the signaling stage, giving a continuous map from other players' actions to the signaler's strategy.

Simultaneous continuity in Θ and U is stronger than separate continuity, since neither has been shown to be uniform. It can be used in settings with continuous types when more than one player is moving in the signaling stage. It can be used to show that a limit of finite equilibria is an equilibrium for continuous types, giving existence of equilibrium for a continuum of types.[15]

6 Incentive compatibility and uniqueness when Θ is a continuum

Mailath (1987) argued that if the type space is a continuum, there is uniqueness of separating equilibrium, under differentiability and regularity conditions. The main conditions are that $U \in C_2$, with $\frac{\partial U}{\partial \theta'} > 0$ and $\frac{\partial^2 U}{\partial \theta \partial a} > 0$. This uniqueness comes almost entirely from the IC conditions. To satisfy IC, a strategy f must satisfy the differential equation $\frac{df}{d\theta}\frac{\partial U}{\partial a} + \frac{\partial U}{\partial \theta'} = 0$ on $(\theta_{\min}, \theta_{\max}]$. If it also satisfies IR at $\theta_{\min}$, i.e. $f(\theta_{\min}) = a^*(\theta_{\min})$, the solution to the initial value problem is unique. In other words $f = \mathscr{R}(U)$ and is the unique separating equilibrium.

With an arbitrary initial value, f may be discontinuous at $\theta_{\min}$, jumping up to some $\bar{a}$ that is determined by indifference for $\theta_{\min}$. It then is uniquely determined, satisfying the differential equation with initial value $\bar{a}$. In other words it is the Riley equilibrium when the signal space is restricted to $[\bar{a}, a_{\max}]$.

These conclusions do not require differentiability.

Proposition 4. *Suppose $\Theta = \bar{\Theta}$ and $U \in \Phi$ is continuous*[16]. *Suppose $f : \Theta \to A$ satisfies IC, and $f \geq a^*$ on $(\theta_{\min}, \theta_{\max}]$.*

1. *If $f(\theta_{\min}) = a^*(\theta_{\min})$, then $U \in \Phi_{\text{Sep}}$ and $f = \mathscr{R}(U)$.*

2. *(extends Mailath (1987)) For arbitrary $f(\theta_{\min})$ there exists a unique $\bar{a} \geq a^*(\theta_{\min})$ with $\left(f(\theta_{\min}), \theta_{\min}\right) \sim_{\theta_{\min}} (\bar{a}, \theta_{\min})$. Take $\bar{A} := [\bar{a}, a_{\max}]$. Then $U|_{\bar{A}} \in \Phi_{\text{Sep}}\left(\bar{A}\right)$, and $f = \mathscr{R}\left(U|_{\bar{A}}\right)$ on $(\theta_{\min}, \theta_{\max}]$.*

[15] Suppose the actions of the other players are z, and equilibrium of a game with finite types satisfies an equation of the form $F_i(z_i) = z_i$. Suppose $z_i \to z^*$. Then we would like $F_i(z_i) \to F_\infty(z^*)$, which gives $z^* = F_\infty(z^*)$. Since in $F_i(z)$, i affects the type space, and z affects the signaling payoff function, if the Riley equilibrium is involved in F_i, simultaneous continuity in Θ and U is needed.

[16] This is also true for discontinuous U, as the following informal argument shows: take a separating equilibrium f of U; at each discontinuity point θ', expand the type space into an interval, so that the sum of the interval lengths is finite, giving a continuous payoff function U'. We can extend f to a separating f' for U', argue for uniqueness of f' as U' is continuous, and then get unique f. This would require work to formalize as U' does not satisfy single crossing with the required strict inequality, so $U' \notin \Phi$.

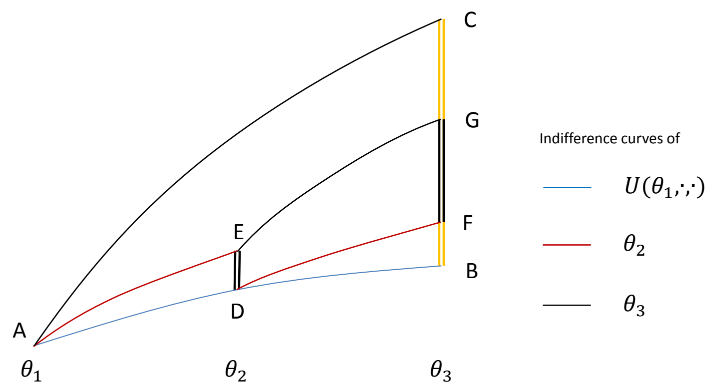

Figure 2: Adding types reduces separating equilibria. Suppose A is the complete information optimum for all types, so $f(\theta_1) = A$. With types $\{\theta_1, \theta_3\}$, the set of incentive compatible values of $f(\theta_3)$ is the interval BC, with AB the Riley equilibrium. With the extra type θ_2 this reduces to FG.

Note that the assumption $f \geq a^*$ must hold if U is strictly increasing in θ':[17]

Fact 3. *If $U(\theta, a, \theta')$ is continuous and strictly increasing in θ', and f is weakly increasing and satisfies IC, then $f \geq a^*$ on $(\theta_{\min}, \theta_{\max}]$.*

The main use of part 1 is the implication of uniqueness of separating equilibrium. In a standard model with an optimizing respondent who makes a cognitive inference, IR is a reasonable assumption so the initial value condition will hold. Part 2 eliminates IR. It can be used in settings where the respondent is known to play a Nash but not Bayesian perfect equilibrium strategy.

6.1 Argument

Adding types imposes additional constraints on separating equilibria, so the set of strategies that are separating over a set S of types is decreasing in S. This is shown graphically in Figure 2 on page 20.

Consider Figure 2 on page 20 with all 3 types active. The Riley equilibrium is ADF. The incentive compatible functions f starting at A all lie between ADF and AEG. AEG is the Riley equilibrium of the altered game U', where $U'(\theta_i, a, \theta') := U(\theta_{i+1}, a, \theta')$. So any incentive compatible f lies between $\mathscr{R}(U)$ and $\mathscr{R}(U')$.

[17]If for example U is independent of θ', then taking any closed set $S \subseteq A$, the strategy $f(\theta) = \arg\max_{a \in S} U(\theta, a)$ satisfies IC. Then $f \geq a^*$ typically does not hold and there is non-uniqueness.

For a continuum of types, suppose f satisfies IC and the initial condition. For any $\delta > 0$, let U_δ be the perturbed payoff $U_\delta : (\theta, a, \theta') \mapsto U(\theta + \delta, a, \theta')$. Then we can bound f between $\mathscr{R}(U)$ and $\mathscr{R}(U_\delta)$. Now $U_\delta \to U$ as $\delta \to 0$, so $\mathscr{R}(U_\delta) \to \mathscr{R}(U)$ by Proposition 3, so we must have $f = \mathscr{R}(U)$. This shows part 2.

To relax the initial condition (part 2), we find that f must jump up to $\bar{a}$, and note that the conditions for part 1 are satisfied when we restrict A to $[\bar{a}, a_{\max}]$.

7 Comparative statics

Here we make statements about how Riley equilibrium strategies and payoffs vary with the payoff function.

7.1 When is $\mathscr{R}(U_1) \leq \mathscr{R}(U_2)$?

Since incentives in typical signaling games are directional, we often want to know the direction of movement of one variable with respect to another. So here we ask when one signaling game leads to a higher equilibrium than another. Just as single crossing causes higher types to take higher actions than lower types in the Riley equilibrium, a single crossing relationship between U_1 and U_2 can imply this.

We can decompose single crossing in terms of the property of "having greater preference for higher a and θ'":

Definition. Suppose A, B are weakly ordered sets. If $v_i : A \times B \to \mathbb{R}$ for $i \in \{1,2\}$, $v_1 \leq_{SC} v_2$ if when $b_1 \leq b_2$, (1) $v_1(a_1, b_1) \leq v_2(a_2, b_2)$ and (2) $a_1 \leq a_2$ imply (3) $v_2(a_1, b_1) \leq v_2(a_2, b_2)$.

Then $u(\theta, a, b)$ satisfies single crossing if for $\theta_1 < \theta_2$, $u(\theta_1, \cdot, \cdot) \leq_{SC} u(\theta_2, \cdot, \cdot)$, and inequality in (1) or (2) implies inequality in (3).

If the preferences of all types are increased in the sense of $\leq_{SC}$, then the Riley equilibrium is weakly increased:

Definition. For $U_1, U_2 \in \Phi$, $U_1 \leq'_{SC} U_2$ iff for any θ, $U_1(\theta, \cdot, \cdot) \leq_{SC} U_2(\theta, \cdot, \cdot)$.

Proposition 5. *If $U_1 \in \Phi$ and $U_2 \in \Phi_{\text{Sep}}$ and $U_1 \leq'_{SC} U_2$ then $U_1 \in \Phi_{\text{Sep}}$ and $\mathscr{R}(U_1) \leq \mathscr{R}(U_2)$.*

7.2 Supermodular comparative statics

As explained in the introduction, supermodular payoffs are a valid input of signaling games and a productive way to model signaling applications. Supermodularity can also be an output of signaling games, and this allows for analysis of signaling within larger supermodular dynamic games. This property is used in Roddie (2010a) to study repeated signaling games.

Suppose $\tilde{U}(z)(\theta, a, \theta')$ is parametrized signaling payoff. The parameter z could be an action played before or simultaneously with the signaler's action. It could also be a state of the world, or capital stock. If $\tilde{U}$ is supermodular, then the resulting payoff $v(z,\theta)$ in the Riley equilibrium is supermodular:

Proposition 6. *Let $Z \subseteq \mathbb{R}$ be an interval and $\tilde{U} : Z \to \Phi_{\text{Sep}}$. Suppose that $\tilde{U}(z)(\theta, a, \theta')$ is weakly supermodular. Let $f(z) := \mathscr{R}\left(\tilde{U}(z)\right)$ and let $v(z,\theta) := \tilde{U}(z)_{f(z)}(\theta)$ be utility of type θ in the Riley equilibrium.*

1. *f is weakly increasing*
2. *v is weakly supermodular.*
3. *If $\tilde{U}(z)(\theta, a, \theta')$ is weakly (strictly) increasing in z, then $v(z,\theta)$ is weakly (strictly) increasing in z.*

7.2.1 Bounds

Here we give conditions bounding the increase in the Riley equilibrium with the parameter z, and bounding the supermodularity of the resulting value function.

The first result uses natural condition for a best response under complete information to be Lipschitz in z, showing that this still applies with signaling incentives. The second result requires a notion of bounds on supermodularity.

Definition. Suppose $f(x, y) \in \mathbb{R}$, where x and y lie in $\mathbb{R}$. f has an increasing differences bound of β if for $x_1 \le x_2$ and $y_1 \le y_2$:

$$f(x_2, y_2) + f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) \le \beta (x_2 - x_1)(y_2 - y_1)$$ [18]

If the increasing differences of $\tilde{U}$ are bounded, then the resulting value function has an increasing difference bound.

[18] If f is twice differentiable, this is equivalent to $\frac{\partial f}{\partial x \partial y} \le \beta$.

Lemma 5. *Let $Z \subseteq \mathbb{R}$ be an interval and $\tilde{U} : Z \to \Phi_{\text{Sep}}$. Again let $f(z) := \mathscr{R}\left(\tilde{U}(z)\right)$ and $v(z,\theta) := \tilde{U}(z)_{f(z)}(\theta)$. Suppose that for $d > 0$, the difference $\tilde{U}(z)\left(\theta, a+d, \theta'\right) - \tilde{U}(z)\left(\theta, a, \theta'\right)$ is decreasing on any line $\{(a,z) = (a_0 + \alpha \cdot \lambda, z_0 + \lambda)\}$.*[19]

1. *$\mathscr{R}\left(\tilde{U}(z)\right)(\theta)$ has Lipschitz constant α in z.*
2. *Suppose there exists some $U \in \Phi\left(\bar{\Theta}\right)$ extending each $\tilde{U}(z)$. Suppose that $\tilde{U}(z)\left(\theta, a, \theta'\right)$ has increasing difference bounds of $\beta_{z\theta}, \beta_{z\theta'}, \beta_{za}, \beta_{\theta a}, \beta_{a\theta'}$ in (z,θ), $\left(z,\theta'\right)$ etc. respectively. Suppose $f(z)(\theta)$ has Lipschitz constant α in z. Then v has an increasing difference bound of $\beta_{\theta z} + \beta_{\theta' z} + \alpha \cdot \beta_{za}$.*

These are regularity properties. The first an be used for uniqueness of equilibrium if equilibrium z is a best response to the signaler's strategy. The second result can be used to give Lipschitz conditions in value function iteration in dynamic signaling games, helping to preserve a compact space.

8 Equilibrium refinement

Equilibrium refinements cut down on the large number of perfect Bayesian equilibria that exist in signaling games. These games admit a multiplicity of pooling equilibria, and for finite types a multiplicity of separating equilibria. Equilibrium refinements work by restricting beliefs off the equilibrium path. They do not fully justify the equilibria they select, but they give a rationale for them by partly modeling out-of-equilibrium behavior.[20] They are an extension of the modeling process, and have become a common tool of applied signaling papers.

Payoffs and equilibrium

So far we have been considering reduced utility functions that depend on a revealed type $\theta' \in \Theta$; this is all that is needed to analyze separating equilibria. In order to consider refinements, we need to allow for pooling equilibria - if only to rule them out -

[19]This is implied by the differential condition $\alpha \cdot \frac{\partial^2 \tilde{U}}{(\partial a)^2} + \frac{\partial \tilde{U}}{\partial a \partial z} \leq 0$.

[20]In the case of D1, this modeling can be seen as specifying infinitesimal probabilities of mistakes by the signaler as a function of errors in beliefs about the subsequent response required to justify suboptimal signals.

and so need specify payoffs on non-degenerate beliefs $\hat{\theta} \in \triangle\Theta$. We also assume that payoffs result, as they usually do in signaling models, from a response $\rho\left(\hat{\theta}, a\right) \in \mathbb{R}$.

Definition. Let $\underline{\Phi}^*$ be the set of $u : \Theta \times A \times \triangle\Theta \to \mathbb{R}$ with $u\left(\theta, a, \hat{\theta}\right) = v\left(\theta, a, \rho\left(\hat{\theta}, a\right)\right)$, where:

1. $\rho\left(\hat{\theta}, a\right) \in \mathbb{R}$ is continuous, weakly increasing in a and strictly increasing in $\hat{\theta}$
2. $v(\theta, a, r)$ is continuous, strictly increasing in r and satisfies strict single crossing
3. $u\left(\theta, a, \hat{\theta}\right)$ is strictly quasi-concave in a
4. $u(\theta_{\max}, a_{\max}, [\theta_{\max}]) \leq u(\theta_{\max}, a, [\theta_{\min}])$ for some a

A fully specified payoff $u \in \underline{\Phi}^*$ gives rise to a signaling payoff $U \in \Phi_B$ with $U\left(\theta, a, \theta'\right)$ continuous[21] and strictly increasing in θ'. The reason for requiring a response r, rather than working directly with beliefs $\hat{\theta}$ is that any two responses are comparable, unlike any two beliefs: so different types have identical preferences over responses (they prefer higher ones), but not over beliefs.

Let $\psi^* \in \triangle\Theta$ be the initial distribution of types. We assume perfect Bayesian equilibrium:

Definition. A perfect Bayesian equilibrium is given by a strategy for the signaler $f : \Theta \to \triangle A$ and a belief function $\beta : A \to \triangle\Theta$ with:

1. $\theta \to f(\theta)(S)$ measurable for any measurable $S \subseteq A$
2. $a \to \beta(a)(T)$ measurable for any measurable $T \subseteq \Theta$
3. $\beta(a)$ giving a conditional distribution of θ given $a = f(\theta)$
4. $f(\theta)$ assigns probability 1 to the set of signals a maximizing $u\left(\theta, a, \beta(a)\right)$

Almost all actions a in the support of $f(\theta)$ give type θ the same equilibrium payoff $u_{f,\beta}(\theta) := u\left(\theta, a, \beta(a)\right)$.

[21] Applying D1 to discontinuous U (for $\Theta = \bar{\Theta}$) may be problematic. In the Riley equilibrium we could have types $< \theta^*$ taking signals below a_1, while θ^* separates from them by taking $a_2 > a_1$; ascribing a belief compatible with D1 to actions in (a_1, a_2) can then be impossible.

The equilibrium refinement D1

The refinement D1 was originally given in Cho and Kreps (1987). The following version can apply to finitely many or a continuum of types:

Definition. An equilibrium f, β satisfies $D1$ if for any $a \in A$ and any $S \subset \Theta$ measurable with $\psi^*(S) > 0$, $\bigcup_{\theta \in S} \{\hat{\theta} : u(\theta, a, \hat{\theta}) > u_{f,\beta}(\theta)\} \subseteq \bigcap_{\theta' \notin S} \{\hat{\theta} : u(\theta', a, \hat{\theta}) \geq u_{f,\beta}(\theta')\}$ implies $\beta(a)(S) = 1$.

Given an action a, if any belief $\hat{\theta}$ associated with a that would give some type outside S a weak incentive to deviate and play a would give all types in S strict incentives to deviate, then beliefs after observing a must lie in S.

Cho and Sobel (1990) showed that under strong single crossing (and differentiability), for a finite set of types, D1 selects the Riley equilibrium uniquely.[22] Ramey (1996) weakened the condition for multidimensional signals from SMSC to the condition given in Engers (1987); in 1 dimension this reduces to SMSC. He found the same result for a continuum of types. Here we restrict to a 1-dimensional signal, but weaken the order theoretic notion of single crossing and allow for non-differentiability.

Proposition 7. *1. Suppose that $u \in \underline{\Phi}^*$ and that ψ^* has full support. Suppose at least one of the following holds: Θ is finite; $\rho([\theta], a)$ is strictly increasing in a; or U satisfies strong single crossing. If f, β satisfy D1 then $f = \mathscr{R}(U)$ (a.e.).*

2. Suppose that $u \in \underline{\Phi}^*$. $f = \mathscr{R}(U)$ satisfies D1 for some β.

The arguments extend those of Cho and Sobel (1990). Differentiability can be set aside easily as it is not important to the argument of Cho and Sobel (1990). Relaxing the order theoretic notion of single crossing requires some alterations. If there is a pool of types P taking signal a, under strong single crossing it is possible to separate from it by taking a slightly higher signal, resulting in beliefs at least $\sup P$. This is not true under our single crossing, when the pool is not above the complete information signals a^*. A larger move, above a^*, must be taken to separate. And downwards separating moves need to be considered in addition to upwards moves, and ruled out.

[22] In fact they allowed for payoffs that do not satisfy the bound condition used here, where separating equilibria may not exist: then D1 uniquely selects an equilibrium which has some pooling by the highest types at the highest action.

The notation for a continuum of types is similar to Ramey (1996), although for a continuum of types the differential methods there are replaced with arguments from uniqueness of separating equilibrium.

A Key: global definitions

Θ	$\subseteq$	$\bar{\Theta} = [\theta_{\min}, \theta_{\max}]$	Space of types (finite or interval)
A	$=$	$[a_{\min}, a_{\max}]$	Space of signals
$\mathrm{Inc}(\Theta, A)$	$\subseteq$	$\{f : \Theta \to \mathbb{R}\}$	Weakly increasing functions
Φ	$\subseteq$	$\{U : \Theta \times A \times \Theta \to \mathbb{R}\}$	Signaling payoffs
Φ_{Sep}	$\subseteq$	Φ	Payoffs for which separating equilibria exist
Φ_B	$\subseteq$	Φ_{Sep}	Payoffs with sufficiently costly signals
$\underline{\Phi}^*$	$\subseteq$	$\{u : \Theta \times A \times \triangle\Theta \to \mathbb{R}\}$	Fully-specified signaling payoffs
a^*	$\in$	$\mathrm{Inc}(\Theta, A)$	Complete information optimal strategy
$\mathrm{Sep}^{\pm}$	:	$\Phi \hookrightarrow A^{\Theta}$	Weakly separating equilibria
Sep	:	$\Phi \hookrightarrow \mathrm{Inc}(\Theta, A)$	Weakly increasing weakly separating eq.
$\overline{\mathrm{Sep}}$			The closure
$\mathscr{R}$	:	$\Phi_{\mathrm{Sep}} \to \mathrm{Inc}(\Theta, A)$	Riley equilibrium

B Proofs

B.1 Map of results

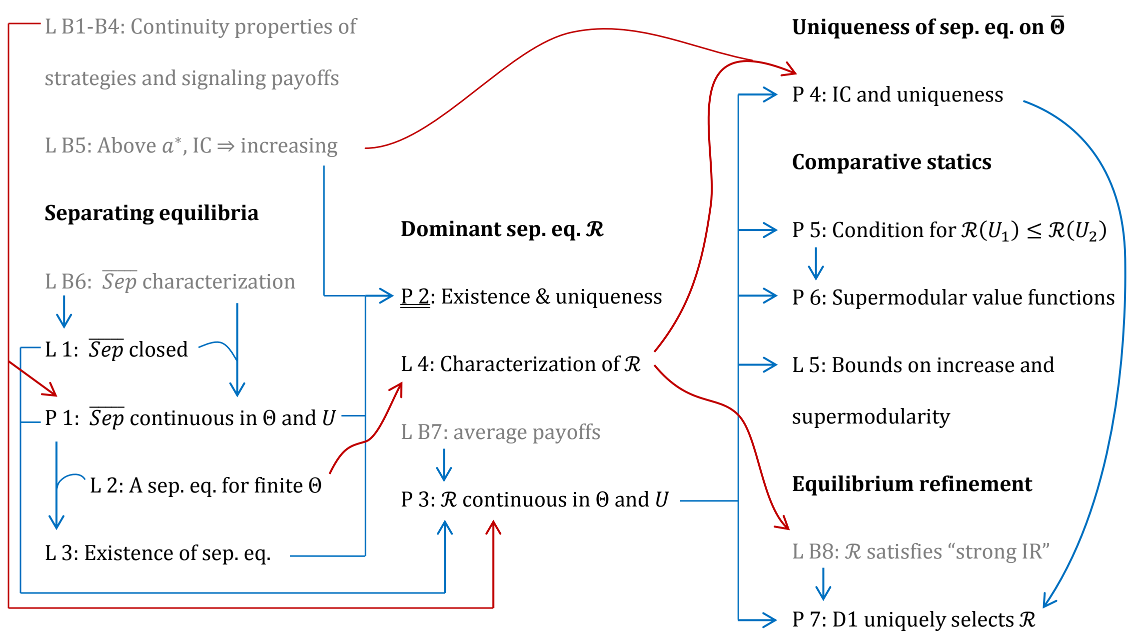

Figure 3: (Proposition 2 is assumed in all subsequent results.)

B.2 Proofs of Facts

These are independent of the main development.

Proof of Fact 2 There is some initial distribution $\hat{\theta}_0$ over Θ. A measurable f results in a distribution over $\Theta \times A$. Let $\beta' : A \to \Delta\Theta$ be a conditional probability.

Let A^* be the set of signals $a \in \mathrm{Im}(f)$ such that $\beta'(a)$ gives weight to types not taking a. The probability of A^* is 0. On this set, using the axiom of choice, let $g(a)$ be some θ with $f(\theta) = a$.

$$\beta(a) = \begin{cases} [\theta_{\min}] & a \notin \mathrm{Im}(f) \\ \beta'(a) & a \in \mathrm{Im}(f) \setminus A^* \\ g(a) & a \in A^* \end{cases}$$

Then β is measurable, and is a conditional probability. It gives a perfect Bayesian equilibrium when combined with f.

Proof of Fact 3 It is immediate that f must be strictly increasing. a^* is continuous since U is. So if $f(\theta^*) < a^*(\theta^*)$ for some $\theta^* > \theta_{\min}$, then $f(\theta^*) < a^*(\theta)$ for some $\theta < \theta^*$. Then $f(\theta) < f(\theta^*) < a^*(\theta)$. Then $\left(f(\theta),\theta\right) \prec_{\theta^*} \left(f(\theta^*),\theta\right) \prec_{\theta} \left(f(\theta^*),\theta^*\right)$, contradicting IC.

B.3 Topological lemmas

Both $f(\theta)$ where $f \in \mathrm{Inc}(\Theta, A)$, and $U\left(\theta, a, \theta'\right)$ where $U \in \Phi$, have common property of being weakly increasing in the final variable, and uniformly continuous in the rest. This gives them similar topological properties.

In the only variable, $f \in \mathrm{Inc}(\Theta, A)$ has continuity points $\mathscr{C}\left(f\right)$. In the final variable, $U \in \Phi$ has continuity points $\mathscr{C}'(U)$:

Definition. For $f \in \mathrm{Inc}(\Theta, A)$, let $\mathscr{C}\left(f\right) := \left\{\theta : f \text{ is continuous at } \theta\right\}$.

For $U \in \Phi$, let $\mathscr{C}'(U) = \left\{\theta' : U(\theta, a, \cdot) \text{ is continuous at } \theta' \text{ for all } (\theta, a)\right\}$.

Equivalently, using uniform continuity of U w.r.t. (θ, a), $\mathscr{C}'(U)$ is the set of continuity of U' in L1, where $U'\left(\theta'\right) : (\theta, a) \mapsto U\left(\theta, a, \theta'\right)$.

Lemma B.1. *Both $\mathscr{C}\left(f\right)$ and $\mathscr{C}'(U)$ are co-countable, have full measure, and are dense.*

There are connections between L1 convergence and pointwise convergence. If $f_i \to f$ in $\mathrm{Inc}(\Theta, A)$, $f_i(\theta) \to f(\theta)$ for $\theta \in \mathscr{C}\left(f\right)$. We can also allow the inner variable to converge, giving $f_i(\theta_i) \to f(\theta)$ for $\theta_i \to \theta$. This property is true for $U \in \Phi$ also.

Lemma B.2.

1. *If $f_i, f \in \mathrm{Inc}(\Theta, A)$, $f_i \to f = f_\infty$, and $\theta_i \to \theta \in \mathscr{C}\left(f\right)$, then $f_i(\theta_i) \to f(\theta)$.*
2. *If $U_i \in \Phi$ with $U_i\left(\theta, a, \theta'\right)$ uniformly continuous in (θ, a) over $\left(i, \theta'\right)$, and $U_i \to U := U_\infty$, and $\theta_i \to \theta$ and $a_i \to a$ and $\theta'_i \to \theta' \in \mathscr{C}'(U)$, then $U_i\left(\theta_i, a_i, \theta'_i\right) \to U\left(\theta, a, \theta'\right)$.*

If strategies and payoffs converge, then separating equilibrium payoffs converge:

Lemma B.3. *If $f_i \to f$ in $\mathrm{Inc}(\Theta, A)$, and $U_i \to U = U_\infty$ with $U_i\left(\theta, a, \theta'\right)$ uniformly continuous in (a, θ) over $\left(i, \theta'\right)$, $(U_i)_{f_i} \to U_f$ in L1.*

When varying Θ, the following result on the closeness of strategies defined on different Θ will be useful:

Lemma B.4. *If $\Theta_i \nearrow \Theta_\infty$, and $f_i, g_i \in \mathrm{Inc}(\Theta_\infty, A)$ agree on Θ_i, then $d\left(f_i, g_i\right) \to 0$. In particular if $g_i = g$ is constant then $f_i \to g$.*

Proof. This is trivial for Θ_∞ finite so take $\Theta_\infty = \bar{\Theta}$. $\{\theta_{\min}, \theta_{\max}\} \in \Theta_i$. Then $d\left(f_i, g_i\right) \le \delta \cdot |A|$, where δ is the width of the largest interval in $\bar{\Theta}/\Theta_i$. □

B.4 Topological lemmas: proofs

Proof of Lemma B.1

$\mathscr{C}\left(f\right)$ is co-countable since discontinuities of an increasing function are countable.

Claim. $\mathscr{C}'(U)$ is co-countable:

Proof. Let $U'\left(\theta'\right):(\theta,a)\mapsto U\left(\theta,a,\theta'\right)$. U' is weakly increasing. So is $J:\theta'\mapsto\int U'\left(\theta'\right)(\theta,a)\,\mathrm{d}(\theta,a)$.

If U has a discontinuity at $\left(\theta_0,a_0,\theta'_0\right)$, the function $\theta'\to U\left(\theta_0,a_0,\theta'\right)$ must be discontinuous at θ'_0, since U is absolutely continuous in (a,θ). The function is weakly increasing, so must jump up at θ'_0, with a discontinuity of some $\delta>0$. Then there exists a ball B of area ϵ around (θ_0,a_0) such that $\left|U'\left(\theta'\right)(\theta,a)-U'\left(\theta'\right)(\theta_0,a_0)\right|<\delta/3$ for any θ'. Then for any $\theta'_-<\theta'_0<\theta'_+$, $U'\left(\theta'_+\right)(\theta,a)>U'\left(\theta'_-\right)(\theta,a)+\delta/3$ on B, so $\int_B U'\left(\theta'_+\right)-\int_B U'\left(\theta'_-\right)\geq\epsilon\cdot\delta/3$, so $J\left(\theta'_+\right)-J\left(\theta'_-\right)>\epsilon\cdot\delta/3$, so J is discontinuous at θ'_0. Since J is weakly increasing, it can have at most countably many discontinuities. □

For finite Θ, $\mathscr{C}\left(f\right)=\mathscr{C}'(U)=\Theta$ and all results are trivial. For $\Theta=\bar{\Theta}$, both $\mathscr{C}\left(f\right)$ and $\mathscr{C}'(U)$ must contain $\{\theta_{\min},\theta_{\max}\}$, since $\{\theta_{\min}\}$ and $\{\theta_{\max}\}$ are open sets. So co-countability implies $\mathscr{C}\left(f\right)$ has full measure and is dense.

Proof of Lemma B.2

Both parts of the lemma follow from the more general statement:

Lemma. *Suppose X and Y are metric measure spaces such that $B(x,\delta)\times B\left(y,\delta\right)$ has positive measure for $x\in X,y\in Y,\delta>0$. Suppose the functions $g_i:X\times Y\to\mathbb{R}$ are measurable, with $g_i\left(x,y\right)$ uniformly continuous in x over $\left(i,y\right)$. Suppose $g_i\to g=g_\infty$. Suppose $x_i\to x$ and $y_i\to y$, with g continuous at $\left(x,y\right)$. Then $g_i\left(x_i,y_i\right)\to g\left(x,y\right)$.*

Proof. Suppose otherwise. Then by uniform continuity of g_i in x, $g_i\left(x,y_i\right)\nrightarrow g\left(x,y\right)$. So, restricting to a sub-sequence, $\left|g_i\left(x,y_i\right)-g\left(x,y\right)\right|\geq 3\epsilon$, for some $\epsilon>0$, for all i.

Take δ such that if $d\left(x^2,x^1\right)<2\delta$, for any i and y^*, $\left|g_i\left(x^2,y^*\right)-g_i\left(x^1,y^*\right)\right|\leq\epsilon$ (by uniform continuity in x), and such that for $d\left(y^*,y\right)<\delta$, $\left|g_i\left(x,y^*\right)-g_i\left(x,y\right)\right|\leq\epsilon$ (by continuity of g_i at $\left(x,y\right)$).

Take the open cube $C=B(x,\delta)\times B\left(y,\delta\right)$. This has measure $M>0$. On C, $\left|g_i-g\right|\geq\epsilon$. Then $\left\|g_i-g\right\|_{L1}\geq M\cdot\epsilon$, contradicting $g_i\to g$. □

Proof of Lemma B.3

Claim. $(U_i)_f$ is uniformly continuous in f over i.

Proof. If $\left\|f-g\right\|_{L1}\leq\delta^2$, $\mu\left(\left\{\theta:\left|f(\theta)-g(\theta)\right|\geq\delta\right\}\right)\leq\delta$. For any $\epsilon>0$, take $0<\delta\leq\epsilon$ such that for $|a_2-a_1|<\delta$, $|U_i(\theta,a_2,\theta)-U_i(\theta,a_1,\theta)|<\epsilon$ for any (i,θ). When $\epsilon=1$, this δ is δ_1. Then for $\left\|f-g\right\|_{L1}\leq\delta^2$, $\int\left|(U_i)_f(\theta)-(U_i)_g(\theta)\right|\mathrm{d}\theta\leq\epsilon+\delta\cdot B\leq\epsilon\cdot(1+B)$, where $B=1+|A|/k$. □

This implies $\left\|(U_i)_{f_i},(U_i)_f\right\|_{L1}\to 0$ as $i\to\infty$. So it is sufficient to show $(U_i)_f\to U_f$.

Claim. $U_i(\theta, f(\theta), \theta) \to U(\theta, f(\theta), \theta)$ for $\theta \in \mathscr{C}'(U)$

Proof. Suppose not. Then $d(U_i(\theta^*, f(\theta^*), \theta^*), U(\theta^*, f(\theta^*), \theta^*)) > 4.\epsilon$ infinitely often for some $\theta^* \in \mathscr{C}'(U)$.

If $U_i > U + 4\epsilon$ at $(\theta^*, f(\theta^*), \theta^*)$ infinitely often, then for $\theta' \in [\theta^*, \theta^* + \delta_2]$ for some $\delta_2 > 0$, $d(U(\theta^*, f(\theta^*), \theta^*), U(\theta^*, f(\theta^*), \theta')) < \epsilon$, and so $d(U_i(\theta^*, f(\theta^*), \theta'), U(\theta^*, f(\theta^*), \theta')) > 3\epsilon$. By uniform continuity in (θ, a), for $d((\theta, a), (\theta^*, f(\theta^*))) < \delta_1$, and $\theta' \in [\theta^*, \theta^* + \delta_2]$ as above, $d(U_i(\theta, a, \theta'), U(\theta, a, \theta')) > \epsilon$ i.o.. This contradicts $U_i \to U$ in L1.

The case of $U > U_i + 4\epsilon$ at $(\theta^*, f(\theta^*), \theta^*)$ i.o. proceeds similarly. □

So restricting to $\mathscr{C}'(U)$, which by Lemma B.1 has full measure, and using dominated convergence (the spaces are bounded a.e.), $\int |U_i(\theta, f(\theta), \theta) - U(\theta, f(\theta), \theta)| \, \mathrm{d}\theta \to 0$ as required.

B.5 IC implies weakly increasing above a^*

Lemma B.5. *If f satisfies IC on S and $f \geq a^*$ on S, then f is weakly increasing on S.*

Proof. Suppose $\theta_1, \theta_2 \in S$ with $\theta_1 < \theta_2$, but $f(\theta_1) > f(\theta_2)$. Then $f(\theta_2) \geq a^*(\theta_2) \geq a^*(\theta_1)$. So $f(\theta_1) > f(\theta_2) \geq a^*(\theta_1)$. So by strict quasi-concavity, $(f(\theta_2), \theta_1) \succ_{\theta_1} (f(\theta_1), \theta_1)$. So by monotonicity $(f(\theta_2), \theta_2) \succ_{\theta_1} (f(\theta_1), \theta_1)$, contradicting IC. □

B.6 Characterization of $\overline{\mathrm{Sep}}(U)$

For a weakly increasing function to be in $\overline{\mathrm{Sep}}(U)$, it is enough that is is weakly separating over a dense subset of Θ:

Lemma B.6. *The following are equivalent:*

1. $f \in \overline{\mathrm{Sep}}(U)$
2. $f \in \mathrm{Inc}(\Theta, A)$ *and is weakly separating over* $\mathscr{C}(f)$
3. $f \in \mathrm{Inc}(\Theta, A)$ *and is weakly separating over a dense subset S of Θ.*

Proof. Despite the possibility of discontinuities in U and f, if f is weakly separating over T, the function U_f must be uniformly continuous on T, as the supremum of the uniformly continuous collection of functions $\{\theta \mapsto U(\theta, f(\theta'), \theta'), \theta' \in T\}$. Combined with uniform continuity of U in the first argument, this implies uniform continuity of $(\theta, \theta') \mapsto U(\theta, f(\theta'), \theta')$ on $T \times T$.

$1 \Rightarrow 2$ Take $g \in \mathrm{Sep}(U)$ with $g = f$ a.e.. If f is continuous at $\theta \notin \{\theta_{\min}, \theta_{\max}\}$, then $f(\theta) = g(\theta)$: otherwise, since g is weakly increasing $f \neq g$ on $[\theta - \epsilon, \theta]$ or on $[\theta, \theta + \epsilon]$ for some $\epsilon > 0$. $f = g$ on $\{\theta_{\min}, \theta_{\max}\}$ because $\{\theta_{\min}\}$ and $\{\theta_{\max}\}$ have positive measure. So $f = g$ on $\mathscr{C}(f)$, and so f is weakly separating over $\mathscr{C}(f)$.

$2 \Rightarrow 3$ $\mathscr{C}(f)$ is dense.

$3 \Rightarrow 1$ U_f is uniformly continuous on S. Let $U^*(\theta) := \lim_{\theta' \in S, \theta' \to \theta} U_f(\theta')$. U^* extends U_f from S to Θ and is continuous.

There exists $g(\theta)$ satisfying: $U_g(\theta) = U^*(\theta)$ and $f(\theta') \leq g(\theta)$ for $\theta' \in S$, $\theta' \leq \theta$ and $g(\theta) \leq f(\theta')$ for $\theta' \in S$, $\theta \leq \theta'$. To achieve this, set $g = f$ on S and outside S there is a solution to the above equations lying between the values of S above and below θ. Such values exist because $\{\theta_{\min}, \theta_{\max}\} \subseteq S$ by the metric on Θ.

$g = f$ on $\mathscr{C}(f)$, so $g = f$ a.e..

$U^*(\theta) = U(\theta, g(\theta), \theta) \geq U(\theta, a, \theta_{\min})$ on S and so, by continuity of U^* and of U in θ, g satisfies IR on Θ.

$U(\theta, g(\theta'), \theta')$ is continuous in θ'as $U(\theta', g(\theta'), \theta')$ is and U is uniformly continuous in the first argument. So since $U(\theta, g(\theta'), \theta') \leq U(\theta, g(\theta), \theta)$ holds on $S \times S$ it must hold on $\Theta \times \Theta$ by continuity in θ and θ'. So g satisfies IC and is weakly separating.

□

B.7 Proof of Lemma 1 ($\overline{\text{Sep}}(U)$ closed)

Since $\overline{\text{Sep}}(U)$ is a subset of the closure of Sep(U), it is sufficient to show:

Claim. $\overline{\text{Sep}}(U)$ is closed.

Proof. Suppose $f'_i \in \overline{\text{Sep}}(U)$, and $f'_i \to f$. Then taking $f_i \in \text{Sep}(U)$ with $f_i = f'_i$ a.e., we have $f_i \to f$. Then $f_i(\theta) \to f(\theta)$ on $\mathscr{C}(f)$ by Lemma B.2. Since $U(\theta, f_i(\theta), \theta) \geq U(\theta, a, \theta_{\min})$ for all a, we must have $U(\theta, f(\theta), \theta) \geq U(\theta, a, \theta_{\min})$ for all a. So f satisfies IR on $\mathscr{C}(f)$. For any $\theta, \theta' \in \mathscr{C}(f)$, $U(\theta, f_i(\theta'), \theta') \leq U(\theta, f_i(\theta), \theta)$, and by taking the same limits, this holds for f. So f satisfies IC on $\mathscr{C}(f)$. By Lemma B.6, $f \in \overline{\text{Sep}}(U)$. □

B.8 Proof of Proposition 1 (continuity of $\overline{\text{Sep}}$)

Parts 2 and 3 follow immediately from part 1:

Proposition. *Suppose $\Theta_i \nearrow \Theta_\infty$. Suppose $U_i \in \Phi(\Theta_\infty)$, with $U_i \to U = U_\infty$ and $U_i(\theta, a, \theta')$ uniformly continuous in (a, θ) over (i, θ'). Suppose $f_i \in \overline{\text{Sep}}(U_i)$ and $f_i \nearrow f = f_\infty$. Then $f \in \overline{\text{Sep}}(U_\infty)$.*

Proof. If Θ_∞ is finite, eventually $\Theta_i = \Theta_\infty$. A limit point of separating equilibria must converge pointwise and be separating by continuity of U in a. So assume $\Theta_\infty = \bar{\Theta}$.

W.l.o.g., assume $f_i \in \text{Sep}(U_i)$. Take $\bar{f}_i \in \text{Inc}(\Theta, A)$ with $f_i = \bar{f}_i|_{\Theta_i}$. Then $\bar{f}_i \to f$.

Let $S := \mathscr{C}'(U) \cap \mathscr{C}(f)$. By Lemma B.1, S is dense. By Lemma B.2, for $\theta \in S$, $\bar{f}_i(\theta_i) \to f(\theta)$ if $\theta_i \to \theta$.

Since $U_i \to U$ in L1, for any (θ, a, θ') at which U_∞ is continuous, there exists a sequence $(\theta_i, a_i, \theta'_i) \to (\theta, a, \theta')$ with $U_i(\theta_i, a_i, \theta'_i) \to U_\infty(\theta, a, \theta')$. Moreover if θ' is interior, the sequences θ_i can be chosen to be monotonic in either direction. By uniform continuity of U_i in (θ, a) we have: $U_i(\theta, a, \theta'_i) \to U_\infty(\theta, a, \theta')$ for some increasing and some decreasing monotonic sequence $\theta'_i \to \theta'$, for interior θ'.

By Lemma B.2, $U_i\left(\theta_i, a_i, \theta_i'\right) \to U_\infty\left(\theta_0, a_0, \theta_0'\right)$ for $\theta_0' \in \mathscr{C}'(U_\infty)$.

For $\theta \in S$, take $\theta_i \to \theta, \theta_i \in \Theta_i$. By IR for U_i, $U_i\left(\theta_i, \bar{f}_i(\theta_i), \theta_i\right) \geq U_i(\theta_i, a, \theta_{\min})$, and by taking limits, $U_\infty\left(\theta, f(\theta), \theta\right) \geq U_\infty(\theta, a, \theta_{\min})$. So f satisfies IR for U_∞.

For $\theta, \theta' \in S$, take $\theta_i, \theta_i' \in \Theta_i$ with $\left(\theta_i, \theta_i'\right) \to \left(\theta, \theta'\right)$. By IC for U_i, $U_i\left(\theta_i, \bar{f}_i(\theta_i), \theta_i\right) \geq U_i\left(\theta_i, \bar{f}_i\left(\theta_i'\right), \theta_i'\right)$, and by taking limits $U_\infty\left(\theta, f(\theta), \theta\right) \geq U_\infty\left(\theta, f\left(\theta'\right), \left[\theta'\right]\right)$, so f satisfies IC for U_∞.

By Lemma B.6, $f \in \overline{\text{Sep}}(U_\infty)$. □

B.9 Proof of Lemma 2 (a finite equilibrium)

Definition 2 implies $\chi(\theta_0) = a^*(\theta_0)$. If 1 uniquely defines a weakly separating equilibrium, that equilibrium must satisfy IC, so must satisfy definition 2, and uniquely since the condition is stricter. So only the claims about definition 1 need to be proved:

Claim. Definition 1 defines a unique χ, weakly increasing, with $\chi \geq a^*$, satisfying IC and IR, when Θ is restricted to $\{\theta_0, \ldots \theta_i\}$, for any i.

Proof. This is true for $i = 0$. Suppose it is true for $i - 1$ (with $i \geq 1$), giving χ' on $\{\theta_0 \ldots \theta_{i-1}\}$.

Let $A_i := \left\{\left(\chi(\theta_{i-1}), \theta_{i-1}\right) \succeq_{\theta_{i-1}} (a, \theta_i)\right\}$. Then $B_i := A_i \cap \left\{a \geq \chi(\theta_{i-1})\right\}$.

For a solution to exist for $\chi(\theta_i)$, B_i must be non-empty. For $U \in \Phi_B$, the set B_i is guaranteed to be non-empty: it contains $a_{\max}$ because $(a_{\max}, \theta_i) \preceq_{\theta_{i-1}} U(a, \theta_{\min})$ for some a by the bound condition, and $(a, \theta_{\min}) \preceq_{\theta_{i-1}} \left(\chi(\theta_{i-1}), \theta_{i-1}\right)$ by IR for θ_{i-1}. The maximum of $U(\theta_i, a, \theta_i)$ on A_i occurs on B_i: if $x \in A_i \setminus B_i$, then $(x, \theta_i) \preceq_{\theta_{i-1}} \left(\chi(\theta_{i-1}), \theta_{i-1}\right) \sim_{\theta_{i-1}} (\min B_i, \theta_i)$, so by single crossing $(x, \theta_i) \prec_{\theta_i} (\min B_i, \theta_i)$. Using the fact that $U(\theta_i, a, \theta_i)$, strictly quasi-concave in a, B_i is a closed finite interval, and definition 1 defines a unique $\chi(\theta_i)$, with $\chi(\theta_i) \geq \chi(\theta_{i-1})$.

So definition 1 gives χ uniquely on $\{\theta_0, \ldots \theta_i\}$, weakly increasing, with $\chi = \chi'$ on $\{\theta_0, \ldots \theta_{i-1}\}$.

χ satisfies IC for (θ, θ_i), $\theta < \theta_i$: by definition $\left(\chi(\theta_{i-1}), \theta_{i-1}\right) \succeq_{\theta_{i-1}} \left(\chi(\theta_i), \theta_i\right)$. Since $\chi(\theta_{i-1}) \leq \chi(\theta_i)$, by single crossing, $\left(\chi(\theta_{i-1}), \theta_{i-1}\right) \succeq_\theta \left(\chi(\theta_i), \theta_i\right)$ for $\theta < \theta_i$. By IC for χ', $\left(\chi(\theta), \theta\right) \succeq_\theta \left(\chi(\theta_{i-1}), \theta_{i-1}\right) \succeq_\theta \left(\chi(\theta_i), \theta_i\right)$.

χ satisfies IC for (θ_i, θ), $\theta < \theta_i$: $\left(\chi(\theta_{i-1}), \theta_{i-1}\right) \sim_{\theta_{i-1}} (\min B_i, \theta_i)$, so by single crossing, $\left(\chi(\theta_{i-1}), \theta_{i-1}\right) \preceq_{\theta_i} (\min B_i, \theta_i) \preceq_{\theta_i} \left(\chi(\theta_i), \theta_i\right)$. By IC for χ', $\left(\chi(\theta_{i-1}), \theta_{i-1}\right) \succeq_{\theta_{i-1}} \left(\chi(\theta), \theta\right)$, and by single crossing, $\left(\chi(\theta_{i-1}), \theta_{i-1}\right) \succeq_{\theta_i} \left(\chi(\theta), \theta\right)$, so $\left(\chi(\theta_i), \theta_i\right) \succeq_{\theta_i} \left(\chi(\theta), \theta\right)$.

So χ satisfies IC. χ also satisfies IR:

Suppose $(a, \theta_{\min}) \succ_{\theta_i} \left(\chi(\theta_i), \theta_i\right)$. Then $a^*(\theta_i) < \chi(\theta_i)$ and $a < \chi(\theta_i)$, or else $(a, \theta_{\min}) \preceq_{\theta_i} (a, \theta_i) \preceq_{\theta_i} \left(\chi(\theta_i), \theta_i\right)$. We know $(a, \theta_{\min}) \succ_{\theta_i} \left(\chi(\theta_i), \theta_i\right) \succeq_{\theta_i} (a_{\max}, \theta_i)$, so by the intermediate value theorem there exists$c \in [a, a_{\max}]$ with $(a, \theta_{\min}) \sim_{\theta_i} (c, \theta_i)$. By single crossing, $(a, \theta_{\min}) \succeq_{\theta_{i-1}} (c, \theta_i)$. By IR for θ_{i-1}, $\left(\chi(\theta_{i-1}), \theta_{i-1}\right) \succeq_{\theta_{i-1}} (a, \theta_{\min})$, so $\left(\chi(\theta_{i-1}), \theta_{i-1}\right) \succeq_{\theta_{i-1}} (c, \theta_i)$. So A_i contains c, and so $\left(\chi(\theta_i), \theta_i\right) \succeq_{\theta_i} (c, \theta_i) \sim_{\theta_i} (a, \theta_{\min})$, contradicting the assumption. So IR is satisfied for θ_i. □

B.10 Proof of Proposition 2 (Riley equilibrium)

Proof of part 1 (existence)

Let $U^{\max}(\theta) := \sup_{f \in \mathrm{Sep}^{\pm}(U)} U_f(\theta)$. Let $A(\theta) := \{a : U(\theta, a, \theta) \geq U^{\max}(\theta)\}$. $A(\theta)$ is a non-empty closed interval containing $a^*(\theta)$ and excluding $a_{\max}$. Let $g(\theta) = \max A(\theta)$. Then $a^* \leq g$, and $U_g \geq U_f$ for $f \in \mathrm{Sep}^{\pm}(U)$.

Claim. $g \in \mathrm{Sep}^{\pm}(U)$.

Proof. For each θ, $g(\theta)$ is a limit point of actions $f(\theta)$ satisfying separating IR for θ, so g satisfies separating IR by continuity of U w.r.t. a.

Now take any θ_1, θ_2. Take a sequence $f_n \in \mathrm{Sep}^{\pm}(U)$ with $U_g(\theta_2) = \lim U_{f_n}(\theta_2)$. We have $U(\theta_1, g(\theta_1), \theta_1) \geq U(\theta_1, f_n(\theta_1), \theta_1) \geq U(\theta_1, f_n(\theta_2), \theta_2)$, the second inequality by IC for f_n, and taking limits gives $U(\theta_1, g(\theta_1), \theta_1) \geq U(\theta_1, g(\theta_2), \theta_2)$. So g satisfies IC. □

Since $g \geq a^*$, g is weakly increasing by Lemma B.5. So $g \in \mathrm{Sep}(U)$.

Proof of part 2 (uniqueness)

As in the proof of part 1, let $U^{\max}(\theta) := \sup_{q \in \mathrm{Sep}^{\pm}(U)} U_q(\theta)$, $A(\theta) := \{a : U(\theta, a, \theta) \geq U^{\max}(\theta)\}$, and $g(\theta) := \max A(\theta)$. Also let $h(\theta) := \min A(\theta)$.

Suppose $f \in \mathrm{Sep}^{\pm}(U)$ and g are both maximal, so that $U_f = U_g$. Let $S := \{\theta : f(\theta) \neq g(\theta)\}$. Then on S, $h(\theta)$ and $g(\theta)$ are distinct and lie strictly on either side of $a^*(\theta)$. On S, $f(\theta) = h(\theta)$, and outside $f(\theta) = g(\theta)$.

Claim. There is no $\theta^* \in S$ with $f(\theta) \leq f(\theta^*)$ for $\theta < \theta^*$.

Proof. Suppose this were true for θ^*.

For some maximal interval $(\theta^{**}, \theta^*]$ or $[\theta^{**}, \theta^*]$, $f(\theta) = f(\theta^*)$.

This can be split into two intervals I_1 and I_2, where $I_1 \nsubseteq S$ and $\emptyset \neq I_2 \subseteq S$, because g is weakly increasing. (If $U(\theta, a, \theta')$ is strictly increasing in θ', $I_1 = \emptyset$ and $I_2 = \{\theta^*\}$.)

For $\theta < \theta^{**}$, $f(\theta) < f(\theta^*)$. IC requires that $(f(\theta), \theta) \succeq_\theta (f(\theta^*), \theta^*)$. So for $\theta < \theta^*$, $f(\theta^*) \notin (h(\theta), g(\theta))$. So $f(\theta^*) \geq g(\theta)$, since $f(\theta^*) > f(\theta)$.

This holds also for I_1. So for θ below I_2, $f(\theta^*) \geq g(\theta)$.

So $(f(\theta), \theta) \succeq_\theta (a^*(\theta'), \theta')$ for $\theta' \in I_2$. So if $f(\theta')$ is altered to $a^*(\theta')$ on I_2, no lower type wants to mimic a type in I_2, nor does a type in I_2 want to mimic any lower type. Moreover because there are no signaling incentives within I_2 (which follows from condition 2 of Φ), no type in I_2 wants to mimic another type in I_2.

Then define $\alpha(\theta)$ to be equal to f below θ^{**}, equal to $a^*(\theta')$ on I_2, and on $[\theta^*, \theta_{\max}]$ equal to a weakly increasing separating equilibrium of $[\theta^*, \theta_{\max}]$ starting at $a^*(\theta^*)$. By Proposition 3, this exists. By considerations above, $\alpha(\theta^*) \in \mathrm{Sep}^{\pm}(U)$, and yet $U_\alpha(\theta^*) > U_f(\theta^*)$, a contradiction. □

a) Suppose $f \in \mathrm{Sep}(U)$. Since f is weakly increasing, $f(\theta) \leq f(\theta^*)$ for $\theta < \theta^*$, and $f = g$ by the above argument.

b) Suppose $f \in \mathrm{Sep}^{\pm}(U)$.

i. Suppose Θ is finite. Let $\theta^* = \min S$. Again $f(\theta) \leq f(\theta^*)$ for $\theta < \theta^*$, so $f = g$.

ii. Suppose Θ is an interval. Take $\theta_1, \theta_3 \in S$ with $\theta_1 < \theta_3$. $\left(g(\theta_1), \theta_1\right) \succeq_{\theta_1} \left(g(\theta_3), \theta_3\right)$ because $g \in \mathrm{Sep}(U)$ and the reverse is true for θ_3, so by continuity of u in θ, we have for some $\theta_2 \in (\theta_1, \theta_3)$: $\left(g(\theta_1), \theta_1\right) \sim_{\theta_2} \left(g(\theta_3), \theta_3\right)$. Since $(h(\theta_1), \theta_1) \sim_{\theta_1} \left(g(\theta_1), \theta_1\right)$, we have by single crossing $(h(\theta_1), \theta_1) \prec_{\theta_2} \left(g(\theta_1), \theta_1\right)$ and similarly $(h(\theta_3), \theta_3) \succ_{\theta_2} \left(g(\theta_3), \theta_3\right)$. Putting the inequalities together, we have $(h(\theta_3), \theta_3) \succ_{\theta_2} (h(\theta_1), \theta_1)$. Therefore we must have $h(\theta_3) > h(\theta_1)$ because otherwise single crossing gives $(h(\theta_3), \theta_3) \succ_{\theta_1} (h(\theta_1), \theta_1)$ and so $(h(\theta_3), \theta_3) \succ_{\theta_1} (h(\theta_1), \theta_1)$, contradicting IC on f. So f must be increasing on S.
Moreover on S, we must have $g(\theta_1) \leq h(\theta_2)$ or else $h(\theta_2) = f(\theta_2) \in \left(h(\theta_1), g(\theta_1)\right)$ and so θ_1 would want to mimic θ_2.
Therefore the non-empty open intervals $\left(h(\theta), g(\theta)\right)$ over $\theta \in S$ are disjoint. Therefore S must be countable.

B.11 Proof of Lemma 4 (characterizations of $\mathscr{R}(U)$)

1.

   (a) Let $f = \mathscr{R}(U)$. Then $f \geq a^*$ and satisfies IC. Suppose, contrary to the claim, that $\alpha \geq a^*$ satisfies IC, with $\alpha(\theta^*) < f(\theta^*)$ for some θ^*. Let $\beta(\theta) := \min\left(\left\{f(\theta), \alpha(\theta)\right\}\right)$.
   $U_\beta(\theta) \geq U_f(\theta)$, so β satisfies IR since f does. Since both f and β satisfy IC, $U\left(\theta, \beta\left(\theta'\right), \theta'\right) \leq \max\left(\left\{U_f(\theta), U_\alpha(\theta)\right\}\right) = U_\beta(\theta)$, so β satisfies IC. So $\beta \in \mathrm{Sep}^{\pm}(U)$.
   But $U_f \ngeq U_\beta$, since $U_\beta(\theta^*) > U_f(\theta^*)$, contradicting $f = \mathscr{R}(U)$.

   (b) This is immediate from 1.a), since elements of $\mathrm{Sep}(U)$ satisfy IC.

2. Suppose χ differs from $\mathscr{R}(U)$ first at θ^*. θ^* must have weakly lower utility under χ, but $\chi(\theta^*)$ uniquely maximizes utility subject to separating from lower types, which $\mathscr{R}(U)(\theta^*)$ must also do. So $\chi(\theta^*) = \mathscr{R}(U)(\theta^*)$.

B.12 Proof of Proposition 3 (continuity of $\mathscr{R}$)

Since $\mathscr{R}(U)$ is dominant, it maximizes any integral of payoffs over types over separating equilibria, and it does so uniquely:

Lemma B.7. *If $I\left(f\right) := \int U_f(\theta)\, \mathrm{d}\alpha(\theta)$, with α a measure on Θ, then $\mathscr{R}(U)$ maximizes I on $\overline{\mathrm{Sep}}(U)$. If α is absolutely continuous with respect to μ_Θ, then the maximum is unique up to a.e.$\left(\mu\right)$ (and so a.e.(α)) equivalence.*

Proof. $I(\mathscr{R}(U)) - I\left(f\right) = \int \left(U_{\mathscr{R}(U)}(\theta) - U_f(\theta)\right) \mathrm{d}\alpha(\theta)$. This is weakly positive since the inner term is for all θ, proving the first part. If the integral is 0, then the function $U_{\mathscr{R}(U)} - U_f$ must be 0a.e.(α). □

Proof of part 2 (continuity in Θ)

Let $i_\Theta : \mathrm{Inc}(\Theta_\infty, A) \to \mathrm{Inc}(\Theta, A)$ be restriction to Θ: $i_\Theta(f) \mapsto f|_\Theta$. Then given a set S of functions $f \in \mathrm{Inc}(\Theta, A)$, $i_\Theta^{-1}(S)$ is the set of functions in $\mathrm{Inc}(\Theta_\infty, A)$ whose restriction to Θ is in S.

If f is a measurable function $\bar{\Theta} \to A$, let $I(f) := \int U_f(\theta)\, d\theta$. By Lemma B.3, I is continuous.

Let $\mathrm{Sep}^*(U) \subseteq \mathrm{Sep}(U)$ be the subset of functions f in $\mathrm{Sep}(U)$ with $f \geq a_U^*$.

Claim. Some member r_i of $i_{\Theta_i}^{-1}\mathscr{R}(U|_{\Theta_i})$ maximizes $I(f)$ over $f \in i_{\Theta_i}^{-1}\mathrm{Sep}^*(U|_{\Theta_i})$.

Proof. For $i = \infty$ this follows from Lemma B.7, part 2. For $i < \infty$:

Let $r_i' := \mathscr{R}(U|_{\Theta_i})$, and let $r_i(\theta) := r_i'(\theta)$ on Θ_i and elsewhere choose r_i to maximize $U(\theta, \cdot, \theta)$ subject to $r_i(\theta) \geq r_i'(\theta^-)$, for all $\theta^- \in \Theta_i, \theta^- \leq \theta$. Then $r_i(\theta) \leq r_i'(\theta^+)$ for $\theta^+ \in \Theta_i, \theta \leq \theta^+$, because $r_i^*(\theta^+) \geq a^*(\theta^+) \geq a^*(\theta)$. So r_i is increasing and so $r_i \in i_{\Theta_i}^{-1}\mathscr{R}(U|_{\Theta_i})$.

Suppose $f \in i_{\Theta_i}^{-1}\mathrm{Sep}^*(U|_{\Theta_i})$. Then $f(\theta) \geq f(\theta^-) \geq r_i^*(\theta^-)$, for $\Theta_i \ni \theta^- \leq \theta$. So $U_{r_i}(\theta) \geq U_f(\theta)$ and so r_i maximizes $\theta \to U_f(\theta)$ on $f \in i_{\Theta_i}^{-1}\mathrm{Sep}^*(U|_{\Theta_i})$ and so maximizes I. □

Let the maximum be $m_i = I(r_i)$. Then $i \to m_i$ is decreasing because $i \to i_{\Theta_i}^{-1}\mathrm{Sep}^*(U|_{\Theta_i})$ is decreasing (w.r.t. set inclusion). In particular $m_i \geq m_\infty$.

Take a convergent sub-sequence of r_i; by Proposition 1.2 this must converge to an element of $r^* \in \overline{\mathrm{Sep}}(U)$. By Lemma B.3, $m_i \to I(r^*)$, so $I(r^*) \geq m_\infty$.

By Lemma B.7, I is uniquely maximized (up to a.e. equality) over $\overline{\mathrm{Sep}}(U)$ with maximum m_∞. So $r^* = \mathscr{R}(U)$ a.e.. So $r_i \to \mathscr{R}(U)$ since all convergent sub-sequences converge to this.

Since $r_i|_{\Theta_i} = \mathscr{R}(U|_{\Theta_i})$, Lemma B.4 implies that $\mathscr{R}(U|_{\Theta_i}) \nearrow \mathscr{R}(U)$.

Proof of part 3 (continuity in U)

Let $r_i := \mathscr{R}(U_i)$, so we need to show $r_i \to r_\infty$.

For finite Θ this is true by using the finite characterization of the Riley equilibrium. This gives pointwise convergence, which is equivalent to L1 convergence for finite Θ. So assume $\Theta = \bar{\Theta}$.

Restrict to a convergent sub-sequence $r_i \to r^*$, suppressing sub-sequence notation. By Proposition 1.1, $r^* \in \overline{\mathrm{Sep}}(U_\infty)$.

Let $I_i(f) := \int (U_i)_f(\theta)\, \mathrm{d}\mu(\theta)$. By Lemma B.3, $I_\infty(r_i) \to I_\infty(r^*)$ and $I_i(r_i) \to I_\infty(r^*)$.

Claim. $I_i(r_i) \to I_\infty(r_\infty)$.

Proof. Define finite approximations Θ_i to $\Theta_\infty := \Theta$, such that each finite Θ_i contains no discontinuities of U_∞. Then on Θ_j, $U_i \to U_\infty$, because $(a \mapsto U_i(\theta, a, \theta')) \to (a \mapsto U_\infty(\theta, a, \theta'))$ uniformly for each (θ, θ'), because $U_i(\theta, a, \theta') \to U_\infty(\theta, a, \theta')$ pointwise with uniform continuity in a. So using the finite result, $\mathscr{R}(U_j|_{\Theta_i}) \to \mathscr{R}(U_\infty|_{\Theta_i})$.

As in the proof of part 2, let $r_j^i \in i_{\Theta_i}^{-1}\mathscr{R}(U_j|_{\Theta_i})$ maximize $I_j(f)$. From the above, $r_j^i \to r_\infty^i$, and so $I_j(r_j^i) \to I_\infty(r_\infty^i)$ by Lemma B.3.

Also $I_j(r_j^i) \searrow I_j(r_j^\infty)$ for each j. That the limit is $I_j(r_j^\infty)$ is implied by part 2. If follows that $I_j(r_j) = I_j(r_j^\infty) \to I_\infty(r_\infty^\infty) = I_\infty(r_\infty)$. □

So $I_\infty(r_\infty) = I_\infty(r^*)$. So $r^* \in \overline{\mathrm{Sep}}(U_\infty)$ maximizes I_∞, so $r^* = \mathcal{R}(U_\infty)$.

Proof of part 1 (continuity in U and Θ)

By Lemma B.4, it is enough to show that there exists $r_i \in \mathrm{Inc}(\Theta_\infty, A)$ with $r_i|_{\Theta_i} = \mathcal{R}\left(U_i|_{\Theta_i}\right)$ and $r_i \to \mathcal{R}(U_\infty)$.

As in the proof of part 3, let $I_j(f) := \int (U_j)_f(\theta)\,\mathrm{d}\mu(\theta)$, and let $r_j^i \in i_{\Theta_i,\Theta_\infty}^{-1}\mathcal{R}\left(U_j|_{\Theta_i}\right)$ maximize $I_j(f)$. By Proposition 1.1, there is an element r^* of $\mathrm{Sep}(U)$ with a sub-sequence of $\left(r_i^i\right)$ converging to r^*. We have: $I_j\left(r_j^i\right) \to I_\infty\left(r_\infty^i\right)$ by part 3, and $I_j\left(r_j^i\right) \searrow I_j\left(r_j^\infty\right)$ by part 2.

Since $I_x\left(r_x^y\right)$ is decreasing in y, for $i \geq j$, $I_i\left(r_i^\infty\right) \leq I_i\left(r_i^i\right) \leq I_i\left(r_i^j\right)$. Taking the limit $i \to \infty$, $I_\infty\left(r_\infty^\infty\right) \leq \lim I_i\left(r_i^i\right) \leq I_\infty\left(r_\infty^j\right)$, where the middle term may be set-valued, and taking limits in j, $I_\infty\left(r_\infty^\infty\right) \leq \lim I_i\left(r_i^i\right) \leq I_\infty\left(r_\infty^\infty\right)$, so $\lim I_i\left(r_i^i\right) = I_\infty\left(r_\infty^\infty\right) = I(\mathcal{R}(U_\infty))$.

By Lemma B.3, for the sub-sequence of $\left(r_i^i\right)$ converging to r^*, $I_i\left(r_i^i\right)$ converges to $I(r^*)$. So $I(r^*) = I(\mathcal{R}(U_\infty))$, and so $r^* = \mathcal{R}(U_\infty)$ and $r_i \to \mathcal{R}(U_\infty)$.

B.13 Proof of Proposition 4 (IC and uniqueness)

Proof of part 1 (uniqueness with initial condition)

f is weakly increasing by Lemma B.5.

Fix $\bar{\delta} > 0$. For $\delta < \bar{\delta}$, let $U_\delta(\theta, a, \theta') := U(\theta + \delta, a, \theta')$ on $[\theta_{\min}, \theta_{\max} - \bar{\delta}]$ (which includes Θ_i).

$f \geq \mathcal{R}(U)$ by Lemma 4.1.

Claim. If $\theta < \theta' < \theta + \delta$, then $f(\theta') \leq \max\{x : U_\delta(\theta, x, \theta') = U_\delta(\theta, f(\theta), \theta)\}$. I.e. $f(\theta')$ is weakly less than what is needed to separate from θ at $f(\theta)$ according to U_δ.

Proof. By IC for f, $U(\theta', f(\theta'), \theta') \geq U(\theta', f(\theta), \theta)$.

Therefore, by single crossing, since f is weakly increasing and $\theta' \leq \theta + \delta$, $U(\theta + \delta, f(\theta'), \theta') \geq U(\theta + \delta, f(\theta), \theta)$

Rewriting, $U_\delta(\theta, f(\theta'), \theta') \geq U_\delta(\theta, f(\theta), \theta)$. So $f(\theta') \leq \max\{x : U_\delta(\theta, x, \theta') = U_\delta(\theta, f(\theta), \theta)\}$. □

Claim. $f \leq r_\delta := \mathcal{R}(U_\delta)$

Proof. 1. It follows from the previous claim that: If $a_{U_\delta}^*(\theta) \leq f(\theta) \leq r_\delta(\theta)$, then $U_\delta(\theta, f(\theta), \theta) \geq U_\delta(\theta, r_\delta(\theta), \theta)$ and so $f(\theta') \leq \max\{x : U_\delta(\theta, x, \theta') = U_\delta(\theta, r_\delta(\theta), \theta)\} \leq r_\delta(\theta')$, for $\theta < \theta' < \theta + \delta$.

2. So if $f \geq a_{U_\delta}^*$ on $[\theta_1, \theta_2]$ and $f \leq r_\delta$ at θ_1, then $f \leq r_\delta$ on $[\theta_1, \theta_2]$ follows, by splitting the interval up into types at most δ apart.

3. Since U and therefore U_δ are continuous, the separating equilibria f and r_δ are continuous.

4. $r_\delta \geq a_{U_\delta}^*$. Suppose $f(\theta^*) > r_\delta(\theta^*)$. From 2., $f \geq a_{U_\delta}^*$ must be violated on $[\theta_{\min}, \theta^*]$. (Since $f(\theta_{\min}) = a^*(\theta_{\min}) \leq a_{U_\delta}^*(\theta_{\min}) = r_\delta(\theta_{\min})$)

5. Take $\underline{\theta} < \theta^*$ minimal such that $f \geq a_{U_\delta}^*$ on $[\underline{\theta}, \theta^*]$. By 4. and continuity, $f(\underline{\theta}) = a_{U_\delta}^*(\underline{\theta})$ and so $f \leq r_\delta$ at $\underline{\theta}$. By 2., $f \leq r_\delta$ on $[\underline{\theta}, \theta^*]$, a contradiction. □

$U_\delta \to U$ as $\delta \to 0$, so by Proposition 3.3, $r_\delta = \mathscr{R}(U_\delta) \to \mathscr{R}(U)$ on $\left[\theta_{\min}, \theta_{\max} - \bar{\delta}\right]$.

Since $\mathscr{R}(U) \le f \le \mathscr{R}(U_\delta)$ for all $\delta > 0$, $f = \mathscr{R}(U)$ on $\left[\theta_{\min}, \theta_{\max} - \bar{\delta}\right]$. This is true for all $\bar{\delta}$, so $f = \mathscr{R}(U)$ on $[\theta_{\min}, \theta_{\max})$, and by continuity of f on $\bar{\Theta}$.

Proof of part 2 (uniqueness without initial condition)

Let $\bar{f}(\theta) := \begin{cases} \bar{a} & \theta = \theta_{\min} \\ f(\theta) & \theta > \theta_{\min} \end{cases}$.

On $\theta > \theta_{\min}$, IC, together with $f \ge a^*$, imply that $f(\theta) \ge \bar{a}$. Lemma B.5 implies that f is weakly increasing on this set. So $\bar{f} \ge a$ and is weakly increasing. By continuity of U and IC for (θ, θ_0), $\bar{f}$ is continuous at $\theta_{\min}$ and so $\bar{f}$ is continuous, since f is continuous on $(\theta_{\min}, \theta_{\max}]$.

$\bar{f}$ satisfies IC on $(\theta_{\min}, \theta_{\max}]$ since f does, and so satisfies IC on $(\theta_{\min}, \theta_{\max}]$ by continuity of U and $\bar{f}$. Restricted to $\bar{A}$, $\bar{f}$ satisfies IR at $\theta_{\min}$: i.e. $a^*_{(U|_{\bar{A}})}(\theta_{\min}) = \bar{f}(\theta_{\min}) = \bar{a}$.

Part 1 then implies that $U|_{\bar{A}} \in \Phi_{\text{Sep}}\left(\bar{A}\right)$ and $\bar{f} = \mathscr{R}\left(U|_{\bar{A}}\right)$.

B.14 Proof of Proposition 5 (a condition for $\mathscr{R}(U_1) \le \mathscr{R}(U_2)$)

First assume $\Theta = \{\theta_0, \ldots \theta_k\}$ is finite. Suppose $\mathscr{R}(U_1)(\theta_i) \le \mathscr{R}(U_2)(\theta_i)$. Then $\mathscr{R}\left(U_j\right)(\theta_{i+1}) = \max\left(s_i^j, a^*_{U_j}(\theta_{i+1})\right)$, where $s_i^j \ge \mathscr{R}\left(U_j\right)(\theta_i)$ and $U_j\left(\theta_i, \mathscr{R}\left(U_j\right)(\theta_i), \theta_i\right) = U_j\left(\theta_i, s_i^j, \theta_{i+1}\right)$. Then:

$$U_1\left(\theta_i, \mathscr{R}(U_1)(\theta_i), \theta_i\right) \ge U_1\left(\theta_i, \mathscr{R}(U_2)(\theta_i), \theta_i\right) \ge U_1\left(\theta_i, s_i^2, \theta_{i+1}\right)$$

The second inequality above holds by $U_1 \le'_{SC} U_2$. So $U_1\left(\theta_i, s_i^1, \theta_{i+1}\right) \ge U_1\left(\theta_i, s_i^2, \theta_{i+1}\right)$, so $s_i^1 \le s_i^2$. Also $a^*_{U_2}(\theta_{i+1}) \ge a^*_{U_1}(\theta_{i+1})$. So $\mathscr{R}(U_1)(\theta_{i+1}) \le \mathscr{R}(U_2)(\theta_{i+2})$.

So by induction, $\mathscr{R}(U_1) \le \mathscr{R}(U_2)$.

Proposition 3.2 extends the result to $\Theta = \bar{\Theta}$.

B.15 Proof of Proposition 6 (supermodular comparative statics)

Proof of part 1 (strategy increasing in parameter)

This is an immediate corollary of Proposition 5.

Proof of part 2 (supermodular value function)

We shall use a result from Topkis (1978):

Theorem. *(Topkis (1978)) If $u\left(a^j, a^{-j}\right)$ is supermodular, then $\max_{a^j} u\left(a^j, a^{-j}\right)$ is supermodular in a^{-j}.*

Claim. v is weakly supermodular.

Proof. First assume Θ is finite. Take adjacent $\theta_1 < \theta_2$.

Let Z_B be the set of z such that θ_2 is weakly bound by separation from θ_1, and Z_U the set of z such that θ_2 is not strictly bound by separation from θ_1:

$$\begin{aligned} Z_B &:= \left\{z : \tilde{U}(z)\left(\theta_1, f(z)(\theta_1), \theta_1\right) = \tilde{U}(z)\left(\theta_1, f(z)(\theta_2), \theta_2\right)\right\} \\ Z_U &:= \left\{z : f(z)(\theta_2) = \arg\max_a \left(\tilde{U}(z)(\theta_2, a, \theta_2)\right)\right\} \end{aligned}$$

On $z \in Z_B$ and $\theta \in \{\theta_1, \theta_2\}$, $v(z,\theta) = \tilde{U}(z)\left(\theta, f(z)(\theta_2), \theta_2\right)$, and $v(z,\theta)$ has weakly increasing differences in (z,θ) on this set since $\tilde{U}(z)(\theta, a, \theta_2)$ has weakly increasing differences in (z,θ) and (z,a) and $f(z)(\theta_2)$ is increasing in x (part 1).

On $z \in X_U$, utility for θ_2 is $\max_a w(z,\theta_2,a)$, where $w(z,\theta,a) := \tilde{U}(z)(\theta,a,\theta)$. w is weakly supermodular, since $\tilde{U}(z)\left(\theta, a, \theta'\right)$ is. So by Topkis (1978), $\max_a w(z,\theta,a)$ is a weakly supermodular function of (z,θ), and more generally so is $\max_{a\geq l} w(z,\theta,a)$ for any l.

Take $z_2 > z_1$ in Z_U and let $l := f(z_1)(\theta_1)$.

$$\begin{aligned} v(z_2,\theta_2) - v(z_1,\theta_2) &= \max_a w(z_2,\theta_2,a) - \max_a w(z_1,\theta_2,a) \\ &= \max_{a\geq l} w(z_2,\theta_2,a) - \max_{a\geq l} w(z_1,\theta_2,a) \\ &\geq \max_{a\geq l} w(z_2,\theta_1,a) - \max_{a\geq l} w(z_1,\theta_1,a) \\ &= \max_{a\geq l} w(z_2,\theta_1,a) - v(z_1,\theta_1) \\ &\geq v(z_2,\theta_1) - v(z_1,\theta_1) \end{aligned}$$

The first inequality holds by weakly increasing differences of $\max_{a\geq l} w(z,\theta,a)$. The second holds because $f(z_2)(\theta_1) \geq l$ by part 1, so $v(z_2,\theta_1) = w\left(z_2,\theta_1, f(z_2)(\theta)\right) \leq \max_{a\geq l} w(z_2,\theta_1,a)$.

So on $\{\theta_1,\theta_2\}$ weakly increasing differences holds over Z_B and Z_U, closed subspaces of Z whose union is Z. Between any two points in $z_B \in Z_B$, $z_U \in Z_U$, there is a point in $Z_B \cap Z_U$, so weakly increasing differences holds over Z. So weakly increasing differences holds over Θ and Z.

Proposition 3.2 and Lemma B.3 extend the result to $\Theta = \bar{\Theta}$. □

Proof of part 3 (value increasing in parameter)

$v(z,\theta_{\min})$ is weakly (strictly) increasing in x: in the Riley equilibrium $\theta_{\min}$ takes an action a maximizing $\tilde{U}(z)(\theta_{\min}, a, \theta_{\min})$, and this maximum must be weakly (strictly) increasing in z since $\tilde{U}(z)\left(\theta, a, \hat{\theta}\right)$ is weakly (strictly) increasing in z.

Weak supermodularity of v (part 2) then implies the result.

B.16 Proof of Lemma 5 (bounds on comparative statics)

Proof of 1 (bound on increase in strategy)

Let $d_z = z_1 - z_0$. Let Θ' be the set of θ for which $\mathscr{R}\left(\tilde{U}(z_0)\right)(\theta) \leq a_{\max} - \alpha \cdot d_z$. Outside this set, the claim holds trivially. Now restrict to Θ'.

Let $\hat{U}\left(\theta, a, \theta'\right) := \tilde{U}(z_1)\left(\theta, a + \alpha \cdot d_z, \theta'\right)$ on $A' := [a_{\min}, a_{\max}] - \alpha \cdot d_z$, and let $A'' := [a_{\min}, a_{\max} - \alpha \cdot d_z]$.

Claim. $\hat{U} \leq'_{SC} \tilde{U}(z_0)$ on A''.

Proof. Let $\Delta = \left(\tilde{U}(z_0) - \hat{U}\right)$. Then $\Delta\left(\theta, a, \theta'\right) = \tilde{U}(z_0)\left(\theta, a + d_z, \theta'\right) - \tilde{U}(z_0 + d_z)\left(\theta, a + \alpha \cdot d_z, \theta'\right)$. Then $\Delta\left(\theta, a_2, \theta'\right) - \Delta\left(\theta, a_1, \theta'\right) \leq 0$ for $a_1 < a_2$, and $\Delta\left(\theta, a, \theta'_2\right) - \Delta\left(\theta, a, \theta'_1\right) = 0$. $\square$

Then $\hat{U} \in \Phi_{\text{Sep}}\left(A''\right)$, and:

$$\begin{aligned}
\mathscr{R}\left(\tilde{U}(z_1)\right) &= \mathscr{R}\left(\hat{U}\right) + \alpha \cdot d_z \\
&\leq \mathscr{R}\left(\hat{U}|_{A''}\right) + \alpha \cdot d_z \\
&\leq \mathscr{R}\left(\tilde{U}(z_0)\right) + \alpha \cdot d_z
\end{aligned}$$

Proof of part 2 (bound on supermodularity)

Let $w(z, \theta, a) := \tilde{U}(z)(\theta, a, \theta)$.

Claim. Suppose $\tilde{U}(z)\left(\theta, a, \theta'\right)$ is continuous. For any l, $w^* : (z, \theta) \mapsto \max_{a \geq l} w(z, \theta, a)$ has an increasing difference bound of $\left(\beta_{z\theta} + \beta_{z\theta'} + \alpha \cdot \beta_{za}\right)$.

Proof. First suppose $\Theta = \bar{\Theta}$.

Let $a^*(z, \theta) = \arg\max_{a \geq l} w(z, \theta, a)$. a^* is continuous, so $a^*(z, \theta)$ is uniformly continuous in θ for each z. Since w has an increasing difference bound of $\delta := \beta_{\theta a} + \beta_{a\theta'}$ in (θ, a), for $\theta < \theta'$:

$$\begin{aligned}
&w\left(z, \theta', a^*\left(z, \theta'\right)\right) + w\left(z, \theta, a^*(z, \theta)\right) - w\left(z, \theta, a^*\left(z, \theta'\right)\right) - w\left(z, \theta', a^*(z, \theta)\right) \\
\leq &\delta \cdot \left(\theta' - \theta\right)\left(a^*\left(z, \theta'\right) - a^*(z, \theta)\right)
\end{aligned}$$

For $\theta_1 < \theta_2$, and $0 \leq i \leq n$, let $\theta^n_i := \frac{i}{n} \cdot \theta_2 + \frac{n-i}{n} \cdot \theta_1$. Then:

$$\begin{aligned}
w^*(z, \theta_2) - w^*(z, \theta_1) &= \sum_{i=0}^{n-1} \left[w^*\left(z, \theta^n_{i+1}\right) - w^*\left(z, \theta^n_i\right)\right] \\
&= \sum_{i=0}^{n-1} \left[w\left(z, \theta^n_{i+1}, a^*\left(z, \theta^n_i\right)\right) - w\left(z, \theta^n_i, a^*\left(z, \theta^n_i\right)\right)\right] + o(1)
\end{aligned}$$

Also w has an increasing difference bound of $\beta_{z\theta} + \beta_{z\theta'}$ in (z, θ). So:

$$
\begin{aligned}
& w^*(z_2,\theta_2)+w^*(z_1,\theta_1)-w^*(z_1,\theta_2)-w^*(z_2,\theta_1)\\
= \quad & \lim\sum_{i=0}^{n-1}\left[w\left(z_2,\theta_{i+1}^n,a^*\left(z_2,\theta_i^n\right)\right)-w\left(z_2,\theta_i^n,a^*\left(z_2,\theta_i^n\right)\right)\right.\\
& \left.-w\left(z_1,\theta_{i+1}^n,a^*\left(z_1,\theta_i^n\right)\right)+w\left(z_1,\theta_i^n,a^*\left(z_1,\theta_i^n\right)\right)\right]\\
\leq \quad & \lim\sum_{i=0}^{n-1}\left[\left(\beta_{z\theta}+\beta_{z\theta'}\right)\left(\theta_{i+1}^n-\theta_i^n\right)(z_2-z_1)+\beta_{za}\cdot\left(a^*\left(z_2,\theta_i^n\right)-a^*\left(z_1,\theta_i^n\right)\right)\left(\theta_{i+1}^n-\theta_i^n\right)\right]\\
= \quad & \left(\beta_{z\theta}+\beta_{z\theta'}\right)(\theta_2-\theta_1)(z_2-z_1)+\beta_{za}\cdot(\theta_2-\theta_1)\cdot\int\left[a^*(z_2,\theta)-a^*(z_1,\theta)\right]\mathrm{d}\theta\\
\leq \quad & \left(\beta_{\theta z}+\beta_{\theta' z}+\alpha\cdot\beta_{za}\right)(\theta_2-\theta_1)(z_2-z_1)
\end{aligned}
$$

This shows the result for $\Theta=\bar{\Theta}$.

For finite Θ the assumption that there is a member of $\Phi\left(\bar{\Theta}\right)$ extending each $\tilde{U}(z)$ means the result still applies. □

Claim. The result holds when Θ is finite and $\tilde{U}(z)\left(\theta,a,\theta'\right)$ is strictly increasing in θ'.

Proof. We can assume $\tilde{U}(z)\left(\theta,a,\theta'\right)$ is continuous in z by subtracting the function $\tilde{U}(z)(\theta_{\min},a_{\min},\theta_{\min})$ of z: this is continuous by the assumed increasing difference bounds. Then $\tilde{U}(z)\left(\theta,a,\theta'\right)$ is continuous.

Take two adjacent $\theta_1<\theta_2$. As in the proof of Proposition 6.2, define:

$$
\begin{aligned}
Z_B &:= \left\{z:\tilde{U}(z)\left(\theta_i,f(z)(\theta_i),\theta_i\right)=\tilde{U}(z)\left(\theta_i,f(z)(\theta_{i+1}),\theta_{i+1}\right)\right\}\\
Z_U &:= \left\{z:f(z)(\theta_{i+1})=\arg\max_a\left(\tilde{U}(z)(\theta_{i+1},a,\theta_{i+1})\right)\right\}
\end{aligned}
$$

1. On $z\in Z_B$ and $\theta\in\{\theta_1,\theta_2\}$, $v(z,\theta)=\tilde{U}(z)\left(\theta,f(z)(\theta_2),\theta_2\right)$.

For $z_1,z_2\in Z_B$, $z_1<z_2$, take $a_i:=f(z_i)(\theta_2)$, so that $0\leq a_2-a_1\leq\alpha\cdot(z_2-z_1)$:

$$
\begin{aligned}
& v(z_2,\theta_2)+v(z_1,\theta_1)-v(z_2,\theta_1)-v(z_1,\theta_2)\\
= \quad & \tilde{U}(z_2)(\theta_2,a_2,\theta_2)+\tilde{U}(z_1)(\theta_1,a_1,\theta_2)-\tilde{U}(z_2)(\theta_1,a_2,\theta_2)-\tilde{U}(z_1)(\theta_2,a_1,\theta_2)\\
= \quad & \left[\tilde{U}(z_2)(\theta_2,a_1,\theta_2)+\tilde{U}(z_1)(\theta_1,a_1,\theta_2)-\tilde{U}(z_2)(\theta_1,a_1,\theta_2)-\tilde{U}(z_1)(\theta_2,a_1,\theta_2)\right]\\
& +\left[\tilde{U}(z_2)(\theta_2,a_2,\theta_2)+\tilde{U}(z_2)(\theta_1,a_1,\theta_2)-\tilde{U}(z_2)(\theta_1,a_2,\theta_2)-\tilde{U}(z_2)(\theta_2,a_1,\theta_2)\right]\\
\leq \quad & \beta_{z\theta}\cdot(z_2-z_1)(\theta_2-\theta_1)+\beta_{za}\cdot(a_2-a_1)(\theta_2-\theta_1)\\
\leq \quad & \left(\beta_{z\theta}+\alpha\cdot\beta_{za}\right)(z_2-z_1)(\theta_2-\theta_1)
\end{aligned}
$$

2. Now take $z_1,z_2\in Z_U$, $z_1<z_2$.

Let $l:=f(z_2)(\theta_1)\leq f(z_1)(\theta_2)$. Define w^* and a^* accordingly. Then $v(z_2,\theta_1)=w^*(z_2,\theta_1)$ and $v(z_1,\theta_1)\leq w^*(z_1,\theta_1)$, so:

$$\begin{aligned}
v(z_2,\theta_1) - v(z_1,\theta_1) &\geq w^*(z_2,\theta_1) - w^*(z_1,\theta_1) \\
v(z_2,\theta_2) - v(z_1,\theta_2) &= w^*(z_2,\theta_2) - w^*(z_1,\theta_2)
\end{aligned}$$

So:

$$\begin{aligned}
& v(z_2,\theta_2) + v(z_1,\theta_1) - v(z_1,\theta_2) - v(z_2,\theta_1) \\
\leq\ & w^*(z_2,\theta_2) + w^*(z_1,\theta_1) - w^*(z_1,\theta_2) - w^*(z_2,\theta_1) \\
\leq\ & \left(\beta_{z\theta} + \beta_{z\theta'} + \alpha \cdot \beta_{za}\right)(\theta_2 - \theta_1)(z_2 - z_1)
\end{aligned}$$

Since $\tilde{U}(z)\left(\theta,a,\theta'\right)$ is strictly increasing in θ', $f(z)(\theta_2) > f(z)(\theta_1)$. Since f is continuous in z, $\delta := \max\left(f(z)(\theta_2) - f(z)(\theta_1)\right)$ is attained, so $\delta > 0$. Since f is uniformly continuous in z, for $z_2 - z_1 < \epsilon$, $f(z_2)(\theta_i) - f(z_2)(\theta_i) < \delta$. Then for $z_2 - z_1 < \epsilon$, $f(z_2)(\theta_1) < f(z_1)(\theta_1) + \delta \leq f(z_1)(\theta_2)$.

So the property holds when $z_2 - z_1 < \epsilon$.

3. Therefore v has an increasing difference bound of $\beta_{z\theta} + \beta_{z\theta'} + \alpha \cdot \beta_{za}$ on $\{\theta_1,\theta_2\} \times [z, z+\epsilon]$ for any interval $[z, z+\epsilon] \in Z$. So it has the increasing difference bound on $\{\theta_1,\theta_2\} \times Z$. So it has it on $\Theta \times Z$. □

Claim. The result holds when Θ is finite.

Proof. Define $U_i'(z)\left(\theta,a,\theta'\right) := \tilde{U}(z)\left(\theta,a,\theta'\right) + \epsilon_i\theta'$, where $\epsilon_i \to 0$. The result holds for U_i' by the previous claim, and taking limits, for $\tilde{U}$.[23] □

Claim. The result holds for continuum Θ.

Proof. Take finite approximations $\Theta_i \nearrow \Theta = \bar{\Theta}$ and define: $U_i'(z)\left(\theta,a,\theta'\right) := \int \tilde{U}(z)\left(\theta,a,\theta'+\zeta\right) \mathrm{d}\mu_i(\zeta)$, where μ_i has a truncated $N(0,\epsilon_i)$ distribution (say) and where $\epsilon_i \to 0$. $U_i'(z)\left(\theta,a,\theta'\right)$ preserves the increasing difference bounds and converges in L1 to $\tilde{U}(z)$ for each z. Then Proposition 3.1 gives the result for $\tilde{U}$. □

B.17 Proof of Proposition 7

B.17.1 Proof of 1

The strategy f and initial distribution ψ^* result in a measure $\mu_{\Theta\times A}$ over types and signals, which projected to A is $\mu_A(S) := \mu_{\Theta\times A}(\Theta \times S)$.

Let $\mathrm{Sp}(a) = \bigcap\left\{T : \mu_{\Theta\times A}\left(T^C \times O\right) = 0 \text{ for some open set } O \text{ containing } a\right\}$. I.e. $\mathrm{Sp}(a)$ is the set of types taking action a.

[23] To ensure $U_i'(z) \in \Phi_{\mathrm{Sep}}$, A may have to be extended and $U_i'\left(\theta, a_{\max}+\epsilon, \theta'\right)$ set to $U_i'\left(\theta, a_{\max}, \theta'\right) + v(\theta,\epsilon)$, where v is strictly supermodular and strictly decreasing in ϵ. This will guarantee that $U_i'(z) \in \Phi_{\mathrm{Sep}}$.

1. No pooling Suppose $\psi^*(S) > 0$, and $\mathrm{Sp}(a_1) \cap S \neq \varnothing$. I.e. a_1 is in the support of actions taken by types in a set S of positive measure. (This is equivalent to: for any open set O containing a_1, $\mu_{\Theta\times A}(S \times O) > 0$.)

Let $r_1 := \limsup_{a \to a_1} \rho(\beta(a), a)$ be the highest response to an action close to a_1.[24] Then $u_{f,\beta}(\theta^*) = v(\theta^*, a_1, r_1)$ for some type θ^* in S.

Let a_h be the higher solution of $v(\theta^*, a, r_1) = v(\theta^*, a_1, r_1)$, so that $v(\theta^*, a, r_1)$ is strictly decreasing after a_h. For some ϵ, for all $a_2 \in [a_h, a_h + \epsilon)$, there exists $r(a_2)$ such $(a_1, r_1) \sim_{\theta^*} (a_2, r(a_2))$. $r(a_h) = r_1$, and r is strictly increasing.

All lower types strictly prefer (a_1, r_1) to $(a_2, r(a_2))$. So all lower types prefer their equilibrium actions to (a_1, r_1) (weakly) to $(a_2, r(a_2))$ (strictly). So for any $\delta > 0$ there exists some some r' such that types above $\theta^* - \delta$ strictly prefer (a_2, r') to (a_1, r_1), and lower types reverse this preference. In the continuous case, r' is found by making type $\theta^* - \delta$ indifferent.

So by D1, $\beta(a_2) \geq [\theta^* - \delta]$ for all δ, so $\beta(a_2) \geq [\theta^*]$. So $\rho(\beta(a_2), a_2) \geq \rho([\theta^*], a_2) \geq \rho([\theta^*], a_1)$. It follows that $\rho([\theta^*], a_1) \leq r_1$ or else $\rho(\beta(a_2), a_2) \geq \rho([\theta^*], a_1) > r_1$ for all $a_2 > a_h$ and type θ^* would gain by taking a_2 close to a_h.

Since this is true for some $\theta^* \in S$ for all positive measure S with $\mathrm{Sp}(a_1) \cap S \neq \varnothing$, taking $S_\epsilon = [x - \epsilon, x]$, where $x = \sup(\mathrm{Sp}(a_1))$, and taking a limit $\epsilon \to 0$ gives $\rho([x - \epsilon], a_1) \leq r_1$, so by continuity $\rho(\sup(\mathrm{Sp}(a_1)), a_1) \leq r_1$.

We now have the result for finite types: in a pool with highest type θ^*, take $S = \{\theta^*\}$; then $\rho([\theta^*], a_1) > \rho(\beta(a_1), a_1)$ because $[\theta^*] > \beta(a_1)$. In general:

Suppose $\beta(a)$ is non-degenerate for actions $a \in A_P$ with positive measure $\mu_A(A_P) > 0$. Then for almost all (μ_A) $a_1 \in A_P$, $\beta(a_1) < \sup(\mathrm{Sp}(a_1))$. So $\rho(\beta(a_1), a_1) < \rho(\sup(\mathrm{Sp}(a_1)), a_1) \leq r_1$. This means taking a_1 is not rational for any type, since taking a nearby signal is superior. So a_1 is not taken by any type, contradicting that such a_1 have positive measure.

Therefore $\beta(a)$ is degenerate for almost all (μ_A) a.

2. Riley equilibrium Here we need to apply one of the extra conditions.

a) Finite types By induction. Suppose the equilibrium has $f(\theta) = \mathscr{R}(U)(\theta)$ up to θ_i. Let $\bar{a} := \mathscr{R}(U)(\theta_{i+1})$, and $\bar{r} := \rho([\theta_{i+1}], \bar{a})$. Since the equilibrium is separating, by the definition of $\mathscr{R}$, type θ_{i+1} weakly prefers $(\bar{a}, \bar{r})$ to his equilibrium action(s). Since $\mathscr{R}(U)$ is separating, $\theta_1, \ldots \theta_i$ weakly prefer their equilibrium actions to $(\bar{a}, \bar{r})$.

Taking a^+ just above $\bar{a}$, type θ_{i+1} is indifferent between $(\bar{a}, \bar{r})$ and (a^+, r^+), where $r^+ > \bar{r}$. Then θ_{i+1} weakly prefers (a^+, r^+) to his equilibrium action(s), and by single crossing, lower types strictly prefer their equilibrium actions to (a^+, r^+). So by D1, $\beta(a^+) \geq [\theta_{i+1}]$.

So θ_{i+1} by deviating can achieve a payoff of at least $v(\theta_{i+1}, a^+, \rho([\theta_{i+1}], a^+))$, and by continuity in a, must achieve payoff at least the maximum separating payoff $v(\theta_{i+1}, \bar{a}, \bar{r})$, and since the Riley equilibrium is unique and the equilibrium is separating, we must have $f(\theta_{i+1}) = \mathscr{R}(U)(\theta_{i+1})$.

[24] Note that $a = a_1$ is allowed inside the limit.

b) Continuum of types, U satisfying strong single crossing If U satisfies strong single crossing, the separating equilibrium is increasing, and so Proposition 4.1 implies $f = \mathscr{R}(U)$ a.e..

c) Continuum of types, $\rho([\theta], a)$ strictly increasing in a Equilibrium payoff of type θ is $u^E(\theta)$. $f(\theta)$ places positive measure on $A(\theta) := \{a : \beta(a) = [\theta]\}$ for almost all θ, on a set Θ^-. Choose $g(\theta) \in A(\theta)$ on Θ^- such that $U^E(\theta) = U(\theta, g(\theta), \theta)$. Then g is separating over Θ^-.

i. Suppose $g(\theta) < a^*(\theta)$ on $I \cap \Theta^-$, where I is an interval of positive measure. Take $\theta^* \in I \cap \Theta^-$ and in the interior of I. Let $x = g(\theta^*)$, $r_x = \rho([\theta^*], x)$.

Take $y \geq a^*(\theta^*)$ satisfying $U^E(\theta^*) = U(\theta^*, y, \theta^*)$. Then $y > g(\theta)$ for $\theta < \theta^*$; otherwise type θ would strictly prefer $(g(\theta^*), \theta^*)$, contradicting IC. Moreover $y > g(\theta) + \epsilon$ for $\theta < \theta^*$. Let $r_y = \rho([\theta^*], y)$. By the assumption on ρ, $r_y > r_x$.

Take $y^- < y$ close to y. Then $a^*(\theta^*) < y$, $y^- > y - \epsilon$. Let r^- satisfy $U^E(\theta^*) = v(\theta^*, y^-, r^-)$. Taking y^- close enough to y, $r^- > r_x$. Then type θ^* is indifferent to (y^-, r^-), while lower types are lose by choosing (x, r_x) (by IC) and lose further (by single crossing) by choosing (y^-, r^-). So $\beta(y^-) \geq [\theta^*]$. Then type θ^* strictly gains by choosing y^-, a contradiction.

ii. Suppose, on the other hand $g(\theta) \geq a^*(\theta)$ on a set that is dense in Θ. g is separating on this set. Then there exists $h(\theta)$ with $h \geq a^*$ and $g = h$ on a dense set, with h separating. By Proposition 4.1, $g = \mathscr{R}(U)$. f gives the same payoff as g, so $f = \mathscr{R}(U)$ a.e. by Proposition 2.2.

B.17.2 A strengthening of IR satisfied by the Riley equilibrium

We know that in a Riley equilibrium, no type strictly prefers any $(a, \theta_{\min})$ to equilibrium. This is the IR condition. We can generalize it. Consider the set $\mathrm{NIC}(\theta')$ of actions a which if taken by θ' would make some $\theta < \theta'$ want to mimic θ', when θ takes the Riley equilibrium action:

Definition. Let $\mathrm{NIC}(\theta, \theta') := \{a : (a, \theta') \succ_\theta (\mathscr{R}(U)(\theta), \theta)\}$, and $\mathrm{NIC}(\theta') := \bigcup_{\theta < \theta'} \mathrm{NIC}(\theta, \theta')$.

With two types, $\Theta = \{\theta_0, \theta_1\}$, to separate from θ_0 an action a outside $\mathrm{NIC}(\theta_1)$ must be taken. Provided this happens, no type strictly prefers (a, θ_1) to equilibrium utility. In general, fixing θ', to separate from types $\theta < \theta'$ an action a outside $\mathrm{NIC}(\theta')$ must be taken. Then no type strictly prefers (a, θ') to equilibrium utility.

Lemma B.8. *For any θ', if $a \notin \mathrm{NIC}(\theta')$, then for any θ^*, $(a, \theta') \preceq_{\theta^*} (\mathscr{R}(U)(\theta^*), \theta^*)$.*

Proof. This is immediate for $\theta^* < \theta'$. For $\theta^* \geq \theta'$:

Let $S := \mathrm{NIC}(\theta')$. This is open, so $A \setminus S$ is closed.

<u>i.</u> For $\theta'' \geq \theta'$, on $A \setminus S$, $U(\theta'', a, \theta'')$ is maximized at some $x \geq \sup S$. To see this, suppose it is maximized at $x < \sup S$, and this is strictly better than any $x > \sup S$. Take $y \in S$, $y > x$. If $(x, \theta'') \succeq_{\theta''} (y, \theta'')$, then $(x, \theta'') \succ_\theta (y, \theta'')$ for $\theta < \theta''$ and so $x \in S$. So for $y \in S$, $y > x$, $(x, \theta'') \prec_{\theta''} (y, \theta'')$, and taking limits as $y \to \sup S$, we have $(x, \theta'') \preceq_{\theta''} (\sup S, \theta'')$, contradicting the assumption.

<u>ii.</u> Let $\hat{\Theta} := \{\theta \in \Theta : \theta \geq \theta'\}$ and $\hat{A} := [\sup S, a_{\max}]$.

Take $f = \begin{cases} \mathscr{R}(U) & \theta \in \Theta \cap [\theta_{\min}, \theta') \\ \mathscr{R}\left(U|_{\hat{\Theta},\hat{A}}\right) & \theta \in \hat{\Theta} \end{cases}$.

f is weakly increasing. f satisfies IC on $\tilde{\Theta} := \Theta \cap [\theta_{\min}, \theta']$ and $\hat{\Theta}$. If $\theta_1 \in \tilde{\Theta}$ and $\theta_2 \in \hat{\Theta}$, $(\theta_1, f(\theta_1)) \succeq_{\theta_1} (\theta', f(\theta'))$ and $(\theta', f(\theta')) \succeq_{\theta'} (\theta_2, f(\theta_2))$, so by single crossing $(\theta', f(\theta')) \succeq_{\theta_1} (\theta_2, f(\theta_2))$ and so $(\theta_1, f(\theta_1)) \succeq_{\theta_1} (\theta_2, f(\theta_2))$. Similarly $(\theta_2, f(\theta_2)) \succeq_{\theta_2} (\theta_1, f(\theta_1))$. So f satisfies IC.

Since f is also above a^*, by Lemma 4.1, $f \geq \mathscr{R}(U)$; but f maximizes payoffs subject to weaker conditions, so $f = \mathscr{R}(U)$.

iii. IR for $\mathscr{R}\left(U|_{\hat{\Theta},\hat{A}}\right)$ implies $(f(\theta^*), \theta^*) \succeq_{\theta^*} (a, \theta')$ for $a \geq \sup S$, and so by part i., for all $a \notin S$. □

B.17.3 Proof of part 2 ($f = \mathscr{R}(U)$ satisfies $D1$ for some β)

First we define β. Let S be the set of (θ, a) for which $\{r : v(\theta, a, r) \geq U_f(\theta)\}$ is non-empty, and S_A be the set of a such that $(\theta, a) \in S$ for some θ. S is closed by continuity of v and U_f. On S let $\bar{r}(a, \theta) := \min\{r : v(\theta, a, r) \geq U_f(\theta)\}$. This is continuous in θ. On S_A, let $M(a) := (\arg\min)_\theta \bar{r}(a, \theta)$. By continuity of $\bar{r}$, this is a closed set. Set $\beta(a) = [\max M(a)]$ on S_A, and arbitrarily outside.

This satisfies D1 by construction. It remains to show that β satisfies Bayes' rule and f is rational given β. Let the equilibrium separating response be $\rho'(\theta) := \rho([\theta], f(\theta))$.

Claim. $\beta(f(\theta)) = [\theta]$ (Bayes' rule)

Proof. By single crossing, and since f is strictly increasing, if $\theta_0 < \theta_1 < \theta_2$, θ_0 strictly prefers $(f(\theta_1), \rho'(\theta_1))$ to $(f(\theta_2), \rho'(\theta_2))$, and so strictly prefers $U_f(\theta_0)$ to $(f(\theta_2), \rho'(\theta_2))$, so $\bar{r}(f(\theta_2), \theta_0) > \rho'(\theta_2)$ and the same holds for $\theta_2 > \theta_0$.

Therefore for Θ a continuum, $M(f(\theta)) = \{\theta\}$. For finite Θ, the above gives $\{\theta_i\} \subseteq M(f(\theta_i)) \subseteq \{\theta_{i-1}, \theta_i, \theta_{i+1}\}$. But we know from the finite characterization of $\mathscr{R}(U)$ that θ_{i+1} strictly prefers his equilibrium action to that of θ_i. So $\{\theta_i\} \subseteq M(f(\theta_i)) \subseteq \{\theta_{i-1}, \theta_i\}$. In either case, $\beta(f(\theta)) = [\max M(f(\theta))] = [\theta]$. □

Let $o(\hat{\theta}, a) := (a, \rho(\hat{\theta}, a))$ be the outcome of beliefs $\hat{\theta}$ and signal a.

Claim. If $a \in \mathrm{NIC}(\theta')$, then $\beta(a)$ is supported on $[\theta_{\min}, \theta' - \delta]$ for some $\delta > 0$.

Proof. Suppose $\theta < \theta'$ and $a \in \mathrm{NIC}(\theta, \theta')$. Let $\alpha(\theta, \theta') := \sup(\mathrm{NIC}(\theta, \theta'))$. Then $([\theta], f(\theta)) \sim_\theta ([\theta'], \alpha(\theta, \theta'))$. Then $(a, r) \sim_\theta o([\theta'], \alpha(\theta, \theta'))$ for some $r < \rho([\theta'], a) \leq \rho([\theta'], \alpha(\theta, \theta'))$. By single crossing, $(a, r) \prec_{\theta''} o([\theta'], \alpha(\theta, \theta')) \preceq_{\theta''} ([\theta'], f(\theta'))$ for $\theta'' \geq \theta'$. By continuity of $v(\theta, a, r)$ in θ, this must also hold for $\theta'' \geq \theta' - \delta$ for some $\delta > 0$. So by D1, $\beta(a)$ is supported on $[\theta_{\min}, \theta' - \delta]$ for some $\delta > 0$. □

Claim. No type has a profitable deviation

Proof. Suppose there is a profitable deviation: $([\theta], f(\theta)) \prec_\theta (\beta(a), a)$. Define θ' by: $\rho([\theta'], a) = \rho(\beta(a), a)$. (There exists a solution by continuity of ρ; it is unique by monotonicity of ρ in $\hat{\theta}$.)

Then $([\theta], f(\theta)) \prec_\theta ([\theta'], a)$. So by Lemma B.8, $a \in \mathrm{NIC}(\theta')$. Then by the previous claim $\beta(a) \leq [\theta' - \delta]$ for some δ. This contradicts $\rho([\theta'], a) = \rho(\beta(a), a)$. □

References

K. Bagwell & A. Wolinsky (2002): “Game Theory and Industrial Organization”, Handbook of Game Theory, Volume 3

I.-K. Cho & D. Kreps (1987): "Signaling games and stable equilibria", Quarterly Journal of Economics

I.-K. Cho & J. Sobel (1990): "Strategic stability and uniqueness in signaling games", Journal of Economic Theory

M. Engers (1987): “Signaling with many signals”, Econometrica

G. Mailath (1987): “Incentive compatibility in signaling games with a continuum of types”, Econometrica

G. Mailath (1988): “On the Behavior of Separating Equilibria of Signaling Games with a Finite Set of Types as the Set of Types Becomes Dense in an Interval”, Journal of Economic Theory

G. Mailath, E.-L. von Thadden, “Incentive Compatibility and Differentiability: New Results and Classic Applications”, mimeo

A. Manelli (1996): “On the convergence of equilibrium strategies of approximating signaling games”, Economic Theory

P. Milgrom & C. Shannon (1994): “Monotone Comparative Statics”, Econometrica

G. Ramey (1996): "D1 Signalling Equilibria with Multiple Signals and a Continuum of Types", Journal of Economic Theory

J. Riley (1979): “Informational Equilibrium”, Econometrica

C. Roddie (2011): “Signaling and Reputation in Repeated Games, Part I: Finite Games” (mimeo) (link)

C. Roddie (2011): “Signaling and Reputation in Repeated Games, Part II: Stackelberg Limit Properties (mimeo) (link)

J. Sobel (2009): “Signaling games”, Encyclopedia of Complexity and Systems Science

M. Spence (1973) “Job market signaling”, Quarterly Journal of Economics

M. Topkis (1978): “Minimizing a Submodular Function on a Lattice”, Operations Research