Microeconomics - Convexity

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We will often assume convexity of preferences. Why?

The initial theory formalizied by Walras used additive utilities (basically, $U(x, y) = x + y$).
This aspect neglects the interdependence that may exist between the consumption of different goods on the utility.(Attention it does not mean that goods are substitute, independent or complement).

Edgeworth introduced the generalized utility functions (basically, $U(x, y) = xy$)

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If the preferences are convex (strictly), all the utility functions representing them are quasi-concave (strictly), and the Upper contour sets $U(x) = \{x' | x' \succsim x\}$ are convex (strictly).

In two-goods economies, the easiest way to establish the quasi-concavity may be to show the convexity of the indifference curves.

Example: A Cobb-Douglas utility function on $\mathbb{R}^2_+$:
$U(x, y) = x^\alpha y^{1-\alpha}$

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Assume a function $F : X \to \mathbb{R}$.

The function $F(x)$ is *concave* if and only if $F(\lambda x + (1-\lambda)y) \geq \lambda F(x) + (1-\lambda)F(y),\ \forall \lambda \in [0,1],\ \forall x, y \in X$.

The function $F(x)$ is *strictly concave* if and only if $F(\lambda x + (1-\lambda)y) > \lambda F(x) + (1-\lambda)F(y),\ \forall \lambda \in (0,1),\ \forall x, y \in X,\ x \neq y$.

The function $F(x)$ is *quasi-concave* if and only if $\forall \alpha \in \mathbb{R},\ \{xF(\lambda x + |F(x) \geq \alpha\}$ is a convex set. Equivalently, $F(x)$ is *quasi-concave* if and only if $F(\lambda x + (1-\lambda)y) \geq \min(F(x),\ F(y)),\ \forall \lambda \in [0,1],\ \forall x, y \in X$.

The function $F(x)$ is *semistrictly quasi-concave* if and only if $F(\lambda x + (1-\lambda)y) > F(x),\ \forall \lambda \in (0,1),\ \forall x, y \in X,\ x \neq y,\quad F(x) < F(y)$.

The function $F(x)$ is *strictly quasi-concave* if and only if $F(\lambda x + (1-\lambda)y) > \lambda F(x) + (1-\lambda)F(y) = F(x) = F(y),\ \forall \lambda \in (0,1),\ \forall x, y \in X,\ x \neq y,\ F(x) = F(y)$.

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The function $F(x)$ is *concave* if and only if
$F(x) - F(y) \leq \nabla F(y) \cdot (x - y), \ \forall x, y \in X.$

The function $F(x)$ is *strictly concave* if and only if
$F(x) - F(y) < \nabla F(y) \cdot (x - y), \ \forall x, y \in X, \ x \neq y.$

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The function $F(x)$ is *quasi-concave* if and only if
$F(x) - F(y) \geq 0 \Rightarrow \nabla F(y) \cdot (x - y) \geq 0, \ \forall x, y \in X.$

The function $F(x)$ is *strictly quasi-concave* if and only if
$F(x) - F(y) \geq 0 \Rightarrow \nabla F(y) \cdot (x - y) > 0, \ \forall x, y \in X \ x \neq y.$

The function $F(x)$ is *semistrictly quasi-concave* if and only if
$F(x) - F(y) > 0 \Rightarrow \nabla F(y) \cdot (x - y) > 0, \ \forall x, y \in X \ x \neq y.$

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The definition of *quasiconcavity* does not rule out the possibility of **thick** or **linear** indifference curves.

Strict quasiconcavity implies that the level curves contain no straight lines. This means that **the level curves have no thickness and are neither straight lines nor do they possess even a straight line segment**.

Semistrict quasiconcavity does permit **linear or partially linear level curves** (because there is no condition on two points which the same value by f), but **it does rule out "thick" level curves** as curves as long as the level is less than the maximum value the function can obtain.

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The leading principal minor $M_R(x, F), \;\; R = 1, \dots, n$ of the Hessian $H(x, F)$ is the determinant of the sub-matrix obtained by taking the elements in the first r rows and the first R columns of $H(x, F)$.

The principal minor $m_{ijk\dots}(x, F), \;\; i, j, k \dots \in \{1, n\}$ of the Hessian $H(x, F)$ is the determinant of the sub-matrix obtained by taking the elements in the $ijk\dots$ rows and the $ijk\dots$ columns of $H(x, F)$.

We note $B_R(x, F), \;\; R = 1, \dots, n$ the leading principal minor of the bordered Hessian $B(x, F)$.

We note $b_{ijk\dots}(x, F), \;\; i, j, k \dots \in \{1, n\}$ the principal minor of the bordered Hessian $B(x, F)$.

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The function F is said strongly concave if and only if $(-1)^R M_R(x, F) > 0, \; R = 1, \dots, n$, i.e. $H(x, F)$ is negative definite.

The function F is said weakly concave if and only if $(-1)^r m_{ijk\dots}(x, F) \geq 0, \; r = 1, \dots, n, \; i, j, k \dots \in \{1, n\}$ 0, i.e. $H(x, F)$ is negative semi-definite, where $r = |\{i, j, k \dots\}|$, i.e. the number of rows/columns which are taken.

The matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is such that the leading principal minors are always zero, but the principal minor $m_2(x, F) = 1$ is positive and then $(-1)^r m_{ijk\dots}(x, F) < 0$ since 2 is the only element of $\{i, j, k \dots\}$. Thus the function $f(x, y) = x + y^2$ is not weakly concave.

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The function F is said strongly quasi-concave if and only if $(-1)^{R-1} D_R(x, F) > 0, \;\; R = 2, \dots, n.$

The function F is said weakly quasi-concave if and only if $(-1)^{r-1} b_{1jk\dots}(x, F) \geq 0, \;\; r = 2, \dots, n, \;\; 1, j, k \dots \in \{1, n\}$

It means that at least two rows and two columns are taken into account for the weak definition and that there is always the first row and the first column.

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The condition on the principal minors (the no leading) is equivalent to the fact that the marginal rate of substitution (mrs) are constant or diminishing between goods. However if the diminishing mrs are equivalent to quasi-concavity in the two-good case, it is not the case anymore in the n-goods case.

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Proposition

A function F is concave if and only if it is weakly concave, i.e. the Hessian matrix is negative semi-definite.

Proposition

If F is strongly concave (i.e. the Hessian matrix is negative definite), then it is strictly concave.

Proposition

If the Hessian of F is negative semi-definite and is not identically zero on any segment, then F is strictly concave.

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Proposition

If the Hessian of F is negative semi-definite and is not identically zero on any segment, then F is strictly concave.

To see how this proposition works, study the function:
$F(x_1, x_2) = -x_1^2(1 + \exp(x_2))$.

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Proposition

If F is strongly quasiconcave, then it is strictly quasiconcave. (A counter example for the reverse is $f(x) = -x^4$*).*

Proposition

If a function F is strictly quasiconcave, then it is semistrictly quasiconcave.

Proposition

If a function F is semistrictly quasiconcave then it is quasiconcave.

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Proposition

F is quasiconcave, if and only if it is weakly quasiconcave.

The condition on the principal minors (the no leading) is equivalent to the fact that the marginal rate of substitution (mrs) are constant or diminishing between goods. However if the diminishing mrs are equivalent to quasi-concavity in the two-good case, it is not the case anymore in the n-goods case.

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Example: Show that:
$U = f(x_1, x_2, x_3) = 100x_1 - x_1^2 + 4x_1x_2 + 100x_2 - x_2^2 + x_3$ is not quasi-concave but has diminishing mrs.
Hint: Set $x_3 = h(x_1, x_2, U) = U - 100x_1 + x_1^2 - 4x_1x_2 - 100x_2 + x_2^2$ and derive that: either the indifference functions are not convex and that the mrs between good 1 and good 3 and between good 2 and good 3 are diminishing. The bordered hessian can be used as well for the quasi-concavity.

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Theorem

Let $f(x)$ be a twice differentiable function where $x = (x_1, x_2)$ and assume $x_i \geq 0,\ f_i \geq 0$ for $i = 1, 2$, where $f_i = \dfrac{\partial f(x)}{\partial x_i}$. Then, $f(x)$ is quasi-concave if and only if:

$$f_1^2 f_{22} - 2 f_1 f_2 f_{12} + f_2^2 f_{11} \leq 0$$

(It is equivalent to the fact that marginal rates of substitution are constant or diminishing. A sufficient condition for quasi-concavity is then that $f_{12} \geq 0$ and $f_{11}, f_{22} \leq 0$. Note that the additional condition for concavity is that $f_{11} f_{22} - f_{12} \geq 0$)

This theorem is equivalent to the condition on the bordered hessian since all the principal minors (except the determinant) are non positive in this case.

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Theorem

Let $f(x)$ be a twice differentiable function where $x = (x_1, x_2, \ldots, x_n)$ and assume $x_i \geq 0,\ f_i > 0$ for $i = 1, 2, \ldots, n$, where $f_i = \dfrac{\partial f(x)}{\partial x_i}$. Let x_0 be a row vector with the ith element equal to 1 and all other elements equal to 0 and x_1 be any composite good with the ith element equal to 0. Let: $g(u, v) = f(ux_0 + vx_1), u, v \geq 0$. Then, $f(x)$ is quasi-concave if and only if $g(u, v)$ for all x_0, x_1, i.e. if and only if:

$$g_u^2 g_{vv} - 2g_u g_v g_{uv} + g_v^2 g_{uu} \leq 0$$

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Let the utility function be $U = f(x_1, x_2, \ldots, x_n) = \sum_{j=2}^{n} x_1 x_j$, and set $x_0 = (1, 0, \ldots, 0)$ and $x_1 = (0, x_2^1, x_3^1, \ldots, x_n^1)$.
Check that g is quasi-concave. Check that the sufficient conditions for quasi-concavity are not respected (with the use of leading principal minors of the bordered hessian) but that the necessary conditions are satisfied.

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In the two-good case, the function is quasi-concave if and only if :

$$f_1^2 f_{22} - 2 f_1 f_2 f_{12} + f_2^2 f_{11} \leq 0$$

which can be simplified, if we assume monotonic preferences(i.e. $f_i \geq 0$), in

$$f_{12} \geq 0, f_{11}, f_{22} \leq 0$$

(Inequalities should be strict for strict quasi-concavity).
If the function is concave, it is easier to show that the hessian negative semi-definite.
The easiest way may be to show that the indiferrence curves are convex by setting a function $c(x_1) = x_2$ such that $f(x_1, x_2) = \overline{U}$.

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In the three-good case, there are two equations to establish the strict quasi-concavity and four equations to show the quasi-concavity (since only the principal minors containing the first row and the first column are taken into account). You need to compute the leading principal minors and the next principal minors:

$b_{134}(x, F) = \begin{bmatrix} 0 & f_2 & f_3 \\ f_2 & f_{22} & f_{23} \\ f_3 & f_{32} & f_{33} \end{bmatrix}$ and $b_{124}(x, F) = \begin{bmatrix} 0 & f_1 & f_3 \\ f_1 & f_{11} & f_{13} \\ f_3 & f_{32} & f_{33} \end{bmatrix}$

$(b_{123}(x, F) = D_3(x, F))$. One can check that the second order principal minors (i.e. with 2×2 matrix) are always non-positive (because of the 0).

In the n-good case, if the bordered hessian technique seems to be too long, quasi-concavity may be found with the help of the function $g(u, v) = f(ux_0 + vx_1)$.