Collective action on an endogenous network

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Motivation

- Consider the following problem faced by an organization: in times of crisis, the different departments of the organization must communicate as much information as possible, and via fast and reliable communication channels.
- **Constraint:** in normal times, a network of departments that is too densely connected is to costly to maintain.
  - Each link is costly: intranet, collaborative platforms, hiring qualified personnel to coordinate communication, etc.
  - Number of departments that are able to share information with each others: size of the biggest component.
  - Speed of transmission - reliability of a communication channel: the length (distance) of a path.
- **Question:** Given a fixed total budget for building a network, what would the most effective network architecture look like?

Context

- Non-cooperative game of endogenous network formation where agents care about the social benefit of their links.

- Games of network formation:
  - Agents value their links for the benefits they can access to via the connections of others.
  - Although links generate positive externalities, agents do not consider the social benefit of their own links.

- Centrality:
  - Agents' actions are complementary to each other.
  - The optimal action is a function of the degree of centrality of the agent in the network.
  - The network structure is exogenous.

Model and objectives

Model

- The game is non-cooperative yet agents value the positive externalities their links generate on others:
  - *agents are beneficiaries of a collective action which success depends on the group's efforts to reach out to each others.*
- An agent's effort is his expenditure in links.
- Agents value their links for the use the group makes out of it.

Objectives

- Characterize the architectures of the strict Nash, cost effective and efficient networks.
- Establish the existence of a Nash equilibrium.
- Two assumptions are explored:
  - how close are agents to each other does not / does matter

- Bala and Goyal (2000)
  A noncooperative model of network formation
  - Links are formed for the informational benefits they give to the player who initiates the connection.
  - The cost of the link is incurred only by the person who initiates it.

- Caria and Fafchamps (2018)
  Can People Form Links to Efficiently Access Information?
  - Experimental design.
  - Do people always connect to the players with the highest reach?
    - No! Agents connect to those who have the most links.

The Model

- Set of players: $N = \{1, \ldots, n\}$, $n \geq 5$.
- At some later point in the game:

  1 player is randomly chosen to be the group representative.
  The group representative is $L \in N$.
  Player L's characteristic determines the level of a group reward.
- The reward is distributed to every agent in N, and it is the same for everybody.

Network formation game

- Players become heterogeneous after they play the network formation game.

- Network formation game
  - agents can form directed links towards other individuals
  - s_i is the set of agents to whom i has a link
  - number of links maintained by i: $\mu(s_i) = |s_i|$
  - each link costs $c > 0$ to the agent who initiates it

$$j \in s_i \quad \Leftrightarrow \quad i \to j \quad \Rightarrow i \text{ pays } c$$

  - all agents' strategies map to a network $\mathrm{g} = (s_1, \dots, s_n)$.

- Characteristic of i, $\kappa^i(\mathrm{g})$: number of agents that i can reach in g.

$$0 \leq \kappa^i(\mathrm{g}) \leq n - 1.$$

Network formation game

- Agent i can reach agent j if and only if i has a directed path to j.
- PATH: a path from i to j is a sequence of links:

$$i \rightarrow i_1 \rightarrow i_2 \rightarrow \ldots i_{k-1} \rightarrow j$$

with $i = i_0$ and $j = i_k$ and all agents along it are **distinct**.

Payoff

- After the network formation, L is randomly picked.
- Group reward: $f(\kappa^L(\mathrm{g}))$, where f is strictly increasing and concave in $\kappa^L(\mathrm{g})$.
  - **Best group representatives:** those who can reach the largest number of individuals.
- Payoff of agent i:

$$f(\kappa^L(\mathrm{g})) - c\mu(s_i)$$

- Expected payoff of agent i:

$$\begin{aligned} u_i(s_i, s_{-i}) &= \frac{1}{n} \sum_{j \in N} f(\kappa^j(\mathrm{g})) - c\mu(s_i) \\ &= v(s_i, s_{-i}) - c\mu(s_i) \end{aligned}$$

Timing

- Agents play the network formation game.

  Decisions are taken simultaneously $g = (s_1, \dots, s_n)$

  Characteristic of agent i is $\kappa^i(g)$

- The group representative L is selected randomly.

  Group reward: $f(\kappa^L(g))$

- Payoffs are realized

Example

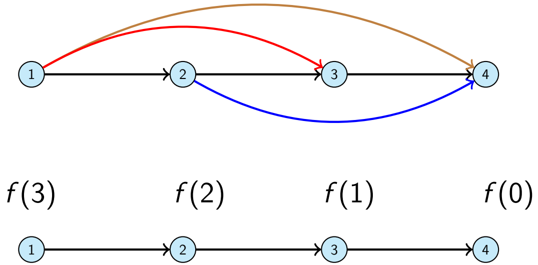

Figure: Two networks that give the same expected reward

Definitions

Strict Nash network

g $= (s_1^*, \ldots, s_n^*)$ is a strict Nash network $\Leftrightarrow (s_1^*, \ldots, s_n^*)$ is a strict NE.

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Strict Nash equilibrium

$g = (s_1^*, \ldots, s_n^*)$ is a strict NE for c iff $\nexists t_i \neq s_i^*$ s.t.:

$$u_i(t_i, s_{-i}^*) \;\geq\; u_i(s_i^*, s_{-i}^*), \forall i \in N.$$

Protocol

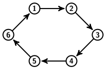
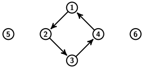
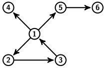
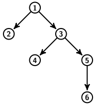
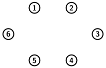
- *Step 1:* Test on all strategy profiles:
  - eliminate all networks $g = (s_1, \ldots, s_n)$ s.t. an agent in g can play a weakly less expensive alternate strategy t_i that improves the reach of everyone.

    $$g' = (s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n) \text{ s.t.}$$
    $$[\kappa^1(g') \ldots \kappa^n(g')] \geq [\kappa^1(g) \ldots \kappa^n(g)]$$

    Use the assumption that:
    1. the expected reward is increasing in any agent's outreach
    2. an agent's payoff is decreasing in the number of links he has.
- *Step 2:* implications of the rest of the assumptions.

Results

- If g is strict Nash, then g is one of the networks below:

	Network characteristics		
	Number of components	Number of links	Topology
The wheel*	1	n	
The isolated wheel*	$n - n_w + 1$	$n > n_w \geq 3$	
The out-tree (cycle)	$n - n_w + 1$	n	
The out-tree (singletons)	n	$n - 1$	
The empty network*	n	0	

The isolated wheel vs the out-tree

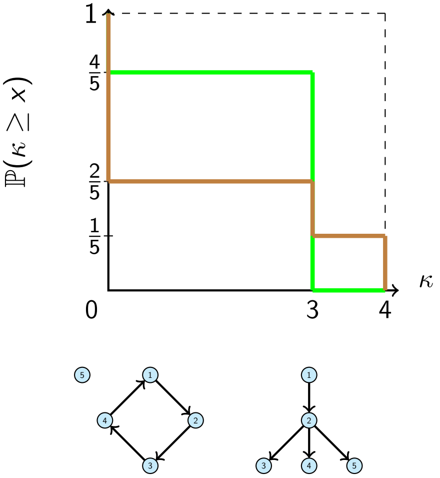


Figure: On the left: the isolated wheel with $n = 5$ and $n_w = 4$ and green distribution; on the right, the out-tree of singletons with n=5, 4 links and brown distribution.

Step 2

- The concavity of f in an agent's outreach enables to rule out the out-tree networks
  $\rightarrow$ *flat* architectures, where the reach of the agents is concentrated around average values, are *preferred* over *hierarchical* structures, that put relatively more weight on extreme values.

- The assumption that v is submodular in any agent's own strategy implies that:
  - g is strict Nash for the value c of the cost of a link when
    1. the link that, if added in g, maximizes the increase in the expected reward is not worth forming,
    2. the link that, if removed in g, minimizes the loss in the expected reward is worth maintaining.

GO

Submodularity assumption

A.1

v is submodular in any agent's own strategy:

$$s_i \subseteq s_i' \;\Rightarrow\; v(s_i \cup \{j\}, s_{-i}) - v(s_i, s_{-i}) \geq v(s_i' \cup \{j\}, s_{-i}) - v(s_i', s_{-i}),$$

for any $s_i, s_i' \in \mathcal{S}_i$, and any $(s_1, \ldots, s_n) \in \mathcal{S}$.

- The return from an additional link that i creates is decreasing in the number of links i already has.

Potential game

- The potential game corresponds to the centralized version of the game.
- The potential function is:

$$P(\mathrm{g}) = v(\mathrm{g}) - c \sum_{i \in N} \mu(s_i)$$

- A network that maximizes the potential function is Nash.
- Best-response dynamics converge to a Nash equilibrium.

Definition and results

Efficient network

An efficient network is a network that maximizes the welfare function, for a given value c of the cost of a link:

$$
\begin{aligned}
\mathrm{g}^{eff} \in \operatorname{argmax} \quad W(\mathrm{g}, c) &= nv(\mathrm{g}) - c \sum_{i \in N} \mu(s_i) \\
&= nP\left(\mathrm{g}, \frac{c}{n}\right)
\end{aligned}
$$

for all $g = (s_1, \ldots, s_n)$.

> **Proposition.** *An efficient network is the wheel network if* $c \leq [f(n-1) - f(0)]$*, and it is the empty network otherwise.*

What happens if distances matter?

- Assume now that the reward depends not only on the outreach of the group representative, but also on how close the group representative is from the rest of the group.
- The distance from the group representative measures his closeness to each of the group members.
- The reward is a function of the shortest distances from the group representative to the rest of the agents.
- The expected payoff of agent i is now:

$$u_i(s_i, s_{-i}) = F(\mathcal{D}(\mathrm{g})) - c\mu(s_i),$$

where $\mathcal{D}(\mathrm{g})$ is the matrix of the shortest distances in g.

Assumption

- Links are maintained to enable agents to reach others **and** to improve their closeness to others.

Definition

Let g and g' be two networks defined on N. Let $\Gamma(x)$ be the distribution of distances in the network x.

If $\Gamma(g)$ is FOSDed by $\Gamma(g')$, then g is dominated by g'.

A.3

If g is dominated by g', then $F(\mathcal{D}(g)) \geq F(\mathcal{D}(g'))$.

- Agents are relatively closer to each others in g than in g′, thus g implies a higher expected reward than g′.

Equilibrium concept

- **Equilibrium concept:** maximum of the potential function:

$$P(\mathrm{g}, c) = F(\mathcal{D}(\mathrm{g})) - c \sum_{i \in N} \mu(s_i)$$

- Recall that a maximum of the potential function for c is:
  - Nash for c,
  - the architecture is efficient when the cost is $\tilde{c} = nc$:

$$\mathrm{argmax}_{\forall \mathrm{g} \in \mathcal{S}} P(\mathrm{g}, c) = \mathrm{argmax}_{\forall \mathrm{g} \in \mathcal{S}} W(\mathrm{g}, \tilde{c})$$

Methodology: brute force approach

- For $n = 5, 6$ and $n \leq K < 2(n-1)$: for each pair (n, K)
  - Get the whole population of networks that have n nodes and K links $\rightarrow \mathcal{N}_{n,K}$.
  - In $\mathcal{N}_{n,k}$, eliminate all networks g such that:

$$\exists \mathrm{g}' \in \mathcal{N}_{n,K} \quad s.t. \quad \mathrm{g}' \text{ is dominated by g.}$$

  - This gives the set $\mathcal{N}^*_{n,K} \subset \mathcal{N}_{n,K}$.

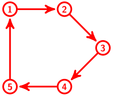
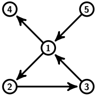
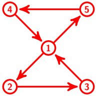
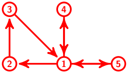
- **Claim:** *If g^* is a maximum of the potential function and if g is on n nodes and K links, then:*

$$\mathrm{g}^* \in \mathcal{N}^*_{n,K}.$$

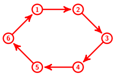
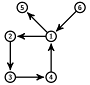
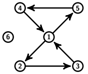
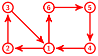
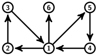
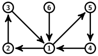
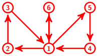
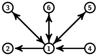
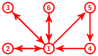
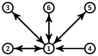
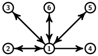
Small groups, intermediate values for the cost

Proposition. *If g is a maximum of the potential function, and if g is on $n \in \{5, 6\}$ nodes and K links, where $n \leq K < 2(n-1)$, then g is one of the following:*

Results n=5

	Topology	
$n = 5$	$\mathrm{g}^f(n, K)$	Other
$K = 5$		
$K = 6$		
$K = 7$		

Results n=6

	Topology		
$n = 6$	$g^f(n, K)$	Other	
$K = 6$			
$K = 7$			
$K = 8$			
$K = 9$			

Flower networks

- **Definition.** *A flower network $g^f(n, K)$ on n nodes and K links is a connected network that partitions the set of agents between:*
  - **a central agent**, j_n,
  - **petals**, which number is $p = n - (K - 1)$. A petal P is a set of agents $\{j_1^P, \ldots, j_l^P\}$ such that:
    - $j_1^P = j_n$ for any petal P,
    - j_n has m links, towards $j_2^1, \ldots, j_2^m$,
    - if $j_k^P \neq j_n$, then j_k^P has 1 link, and this link is directed towards j_{k+1}^P,
    - **the maximum difference in the petals sizes is 1.**

Flower networks, n=6

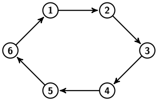


Figure 1: The wheel network on 6 players, 1 petal.

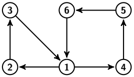


Figure 2: Flower on 6 players and 7 links, 2 petals.

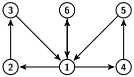


Figure 3: Flower on 6 players and 8 links, 3 petals.

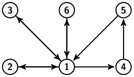


Figure 4: Flower on 6 players and 9 links, 4 petals.

Larger groups, intermediate values of the cost

- For $7 \leq n \leq 9$ and $n \leq K < 2(n-1)$: for each pair (n, K)
  - Take a random sample $S^0_{n,K}$ of 10 million different networks in $\mathcal{N}_{n,K}$
  - Get the sub-sample $S^1_{n,K} \subseteq S^0_{n,K}$ that verifies one of the two constraints:

$$(A) \quad \mathrm{g} \in S^1_{n,K} \quad \Rightarrow \quad \mathrm{g} \text{ is dominated by } \mathrm{g}^f(n, K)$$

or

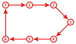
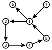
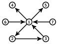
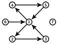
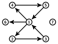
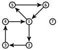
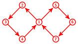
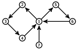
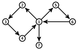
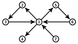
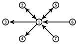
$$(B) \quad \mathrm{g} \in S^1_{n,K} \quad \Rightarrow \quad \mathrm{g} \text{ is not comparable to } \mathrm{g}^f(n, K).$$

No network in $S^1_{n,K}$ is dominated by $\mathbf{g}^f(n, K)$!

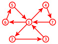
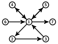
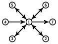
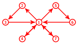
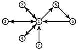
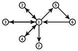
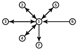
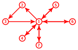
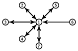
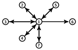
  - Get the sub-sample $S^2_{n,K} \subseteq S^1_{n,K}$ that verifies the last constraint:

$$\mathrm{g} \in S^2_{n,K} \quad \Rightarrow \quad \nexists \mathrm{g}' \in \mathcal{S}^1_{n,K} \;\; s.t. \;\; \mathrm{g}' \text{ is dominated by } \mathrm{g}.$$

Partial results

		Distance distribution Γ							
$n=7$	Topology	**0**	**1**	**2**	**3**	**4**	**5**	**6**	∞
$K=7$		$\frac{1}{7}$	$\frac{14}{49}$	$\frac{21}{49}$	$\frac{28}{49}$	$\frac{35}{49}$	$\frac{42}{49}$	1	1
		$\frac{1}{7}$	$\frac{14}{49}$	$\frac{22}{49}$	$\frac{29}{49}$	$\frac{35}{49}$	$\frac{37}{49}$	1	1
		$\frac{1}{7}$	$\frac{14}{49}$	$\frac{25}{49}$	$\frac{29}{49}$	$\frac{29}{49}$	$\frac{29}{49}$	$\frac{29}{49}$	1
	 	$\frac{1}{7}$	$\frac{14}{49}$	$\frac{24}{49}$	$\frac{30}{49}$	$\frac{32}{49}$	$\frac{32}{49}$	$\frac{32}{49}$	1
		$\frac{1}{7}$	$\frac{14}{49}$	$\frac{23}{49}$	$\frac{31}{49}$	$\frac{35}{49}$	$\frac{37}{49}$	$\frac{37}{49}$	1
$K=8$		$\frac{1}{7}$	$\frac{15}{49}$	$\frac{25}{49}$	$\frac{37}{49}$	$\frac{43}{49}$	$\frac{47}{49}$	1	1
	 	$\frac{1}{7}$	$\frac{15}{49}$	$\frac{26}{49}$	$\frac{36}{49}$	$\frac{41}{49}$	$\frac{43}{49}$	$\frac{43}{49}$	1
		$\frac{1}{7}$	$\frac{15}{49}$	$\frac{28}{49}$	$\frac{36}{49}$	$\frac{38}{49}$	$\frac{38}{49}$	$\frac{38}{49}$	1
		$\frac{1}{7}$	$\frac{15}{49}$	$\frac{29}{49}$	$\frac{29}{49}$	$\frac{29}{49}$	$\frac{29}{49}$	$\frac{29}{49}$	1

Partial results

		Distance distribution Γ							
$n=7$	Topology	**0**	**1**	**2**	**3**	**4**	**5**	**6**	∞
$K=9$		$\frac{1}{7}$	$\frac{16}{49}$	$\frac{31}{49}$	$\frac{43}{49}$	1	1	1	1
		$\frac{1}{7}$	$\frac{16}{49}$	$\frac{32}{49}$	$\frac{38}{49}$	$\frac{38}{49}$	$\frac{38}{49}$	$\frac{38}{49}$	1
		$\frac{1}{7}$	$\frac{16}{49}$	$\frac{33}{49}$	$\frac{33}{49}$	$\frac{33}{49}$	$\frac{33}{49}$	$\frac{33}{49}$	1
$K=10$		$\frac{1}{7}$	$\frac{17}{49}$	$\frac{35}{49}$	$\frac{47}{49}$	1	1	1	1
	 	$\frac{1}{7}$	$\frac{17}{49}$	$\frac{36}{49}$	$\frac{43}{49}$	$\frac{43}{49}$	$\frac{43}{49}$	$\frac{43}{49}$	1
		$\frac{1}{7}$	$\frac{17}{49}$	$\frac{38}{49}$	$\frac{38}{49}$	$\frac{38}{49}$	$\frac{38}{49}$	$\frac{38}{49}$	1
$K=11$		$\frac{1}{7}$	$\frac{18}{49}$	$\frac{41}{49}$	1	1	1	1	1
	 	$\frac{1}{7}$	$\frac{18}{49}$	$\frac{43}{49}$	$\frac{43}{49}$	$\frac{43}{49}$	$\frac{43}{49}$	$\frac{43}{49}$	1

Summary of the results

- Some players form a flower component:
  - the central agent enables every agent in a petal to reach anyone in the other petals + keeps them as close as possible.
- Keeping the number of petals constant, the larger n the larger the set of optimal networks
  - Trade-off between:
    (A) having every player belong to a petal
    $\Rightarrow$ *Bigger flower = more agents can reach each others, however finite distances are relatively longer*
    (B) having a smaller but more densely connected flower
    $\Rightarrow$ *Smaller flower = less agents can reach each others, however finite distances are shorter*
  - As n grows it is increasingly more likely that B is preferable over A.

Concluding remarks

- Starting point of this analysis: Bala and Goyal (2000)
  - *Their assumption:* agents trade off the cost of link formation against their **private** benefit

  - My assumption: agents trade off the cost of link formation against the **social** benefit to the whole group
- Limiting networks:
  - In both cases: wheel, flowers.
  - Difference: in my game, disconnected variants of the wheel and flower networks are equilibrium candidates.