Introduction

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October 5, 2019

Discrete and continous case

$MGF_x(t) := E[e^{tx}] = \sum_x e^{tx} P(x)$ x:discrete

$MGF_x(t) := E[e^{tx}] = \int_x e^{tx} f(x)$ x:discrete

$E(x^n) = \frac{d^n}{dt^n} MGF_x(t)|_{t=0}$

First moment: $E(X) = \frac{d}{dt} MGF_x(t)|_{t=0} = MGF'_x(0)$

Second moment:
$E(X^2) = \frac{d^2}{dt^2} MGF_x(t)|_{t=0} = MGF''_x(0)$

Proof

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

$$e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots + \frac{(tx)^n}{n!}$$

$$E(e^{tx}) = E(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots + \frac{(tx)^n}{n!})$$

$$= E(1) + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \ldots + \frac{t^n}{n!}E(x^n)$$

$$\frac{dE(e^{tx})}{dt} =$$

$$\frac{d}{dt}(E(1) + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \ldots + \frac{t^n}{n!}E(x^n))$$

plug in $t = 0$

$$0 + E(x) + 0 \ldots 0 = E(x)$$

A few facts about MGF

For any MGF, $M(0) = 1$

If you want the k'th moments, derive k times and set the function equal to 0.

The MGF is unique for each distribution

application

$$f_x(x) = \lambda e^{-\lambda x} \text{ if } x > 0$$
$$MGF_x(t) = E[e^{tx}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$
$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx, \text{ require: } t - \lambda < 0$$
$$\lambda | \frac{1}{t-\lambda} e^{t-\lambda} x |_0^\infty$$
$$\lambda(0 - \frac{1}{t-\lambda}) = \frac{\lambda}{\lambda - t}$$