Microeconomics

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Exercise: Lexicographic preferences

The preferences' rationality:

The completeness is immediate since $\geq$ is a total order and if two elements of $\mathbb{R}$ are different, one is greater than the other.

If $x \succeq y$ and $y \succeq z$, then we have to deal with several cases. If $x_1 > y_1 > z_1$, the result is immediate. If $x_1 = y_1 > z_1$ (or $x_1 > y_1 = z_1$), then $x_1 > z_1$ and we get the result. If the three terms are equal, by hypothesis we know that $x_2 \geq y_2 \geq z_2$ by hypothesis and we can conclude that the relation is transitive.

Proof of the non-continuity of the function:

Assume a function $u : \mathbb{R}^2_+ \to \mathbb{R}$ represents the lexicographic preferences.

Then, $u(x + \frac{1}{n}, y - 1) > u(x, y) > u(x - \frac{1}{n}, y + 1)$

The sequences in $\mathbb{R}$ $(x + \frac{1}{n})_n$ and $(x - \frac{1}{n})_n$ are convergent and both have the same limit x.

We notice that $u(x, y - 1) < u(x, y) < u(x, y + 1)$.

We know that a function is continuous if and only if it preserves the limits of sequences. It is immediate that it is not the case here.

Proof of the non-existence of the function:

We want to show that a function does not exist, i.e. the function is not well-defined. In words, a function is a way to associate the elements of two sets. In order to show that a function is not well-defined is to emphasize an incoherence between the function properties and the sets properties. Here we use an associate function q to show that u is not well-defined.

Assume a function $u : \mathbb{R}^2_+ \to \mathbb{R}$ represents the lexicographic preferences.

For any $a \in [0, 1]$, $(a, 1) \succ (a, 0)$ since $(a, 1) \succeq (a, 0)$ but $(a, 0) \succ (a, 1)$ is not true. Thus, if such a function u existed, we would have $u(a, 1) > u(a, 0)$ what implies that the interval $I_a = (u(a, 0), u(a, 1))$ is non-empty.

Assume a function q from $[0, 1]$ into the set of the rational numbers $\mathbb{Q}$ such that for any $b \in [0, 1]$, $q(b) \in I_b = (u(b, 0), u(b, 1))$. Such a function exists since $\mathbb{Q}$ is dense in $\mathbb{R}$.

By definition, we have that $I_a \cap I_b = \emptyset$, so $q(a) \neq q(b)$. Then we get that q is a one-to-one (injective) function, and that:

$$\forall a \neq b, \quad q(a) \neq q(b).$$

In words, there are as many $q(a)$ as a. However, $|\mathbb{Q}| < |\mathbb{R}|$, what contradicts the fact that q is a one-to-one function.