Quantitative Finance Series

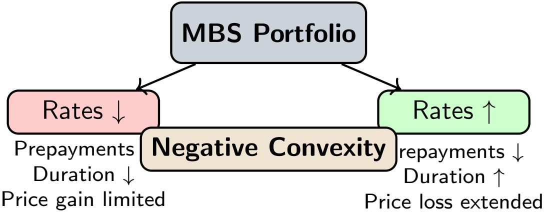
December 19, 2025

Roadmap

Foundations

Why Hedge Mortgage-Backed Securities?

The Fundamental Problem: MBS have complex interest rate sensitivity



Key Risks: Interest rate risk, prepayment risk, basis risk, volatility risk.

The Prepayment Option: Mathematical View

Embedded Option Structure

The MBS holder has **sold** a call option to the borrower: $V_{MBS} = V_{bond} - V_{call}$

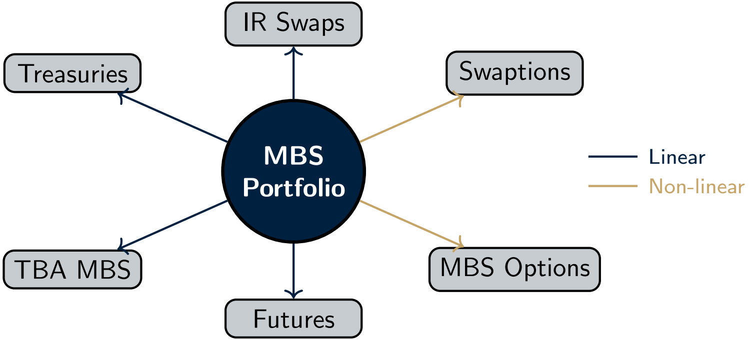
Option Characteristics: Strike = principal balance, American-style exercise, moneyness determined by current rate vs. coupon.

Impact on Price-Yield Relationship:

$$\frac{\partial P}{\partial y} < 0 \quad \text{(always)} \qquad \frac{\partial^2 P}{\partial y^2} < 0 \quad \text{(negative convexity when ITM)}$$

The **gamma** of the embedded short call creates the hedging challenge.

The Hedging Universe: Instrument Overview



Core Risk Metrics

Dollar Value of a Basis Point (DV01)

Definition: DV01

DV01 measures the dollar change in value for a 1 bp change in yield:

$$\boxed{DV01 = -\frac{\partial P}{\partial y} \times 0.0001}$$

Mathematical Derivation: For price $P(y)$: $\Delta P \approx \frac{\partial P}{\partial y}\Delta y$. Setting $\Delta y = 0.0001$:

$$DV01 = -\Delta P = -\frac{\partial P}{\partial y} \times 0.0001$$

For a coupon bond: $DV01 = \frac{1}{10000}\sum_{i=1}^{n} t_i \cdot C_{t_i} e^{-y \cdot t_i}$

Duration: The First-Order Sensitivity

Duration Definitions

Macaulay: $D_{mac} = \frac{\sum_i t_i \cdot PV(C_{t_i})}{P}$ **Modified:** $D_{mod} = \frac{D_{mac}}{1+y/k} = -\frac{1}{P}\frac{\partial P}{\partial y}$

Effective Duration (for MBS):

$$\boxed{D_{eff} = -\frac{P_{-\Delta y} - P_{+\Delta y}}{2 \cdot P_0 \cdot \Delta y}}$$

Key Relationships:

$$\frac{\Delta P}{P} \approx -D_{mod} \cdot \Delta y \qquad DV01 = \frac{D_{mod} \times P}{10000}$$

Convexity: The Second-Order Effect

Convexity Definition

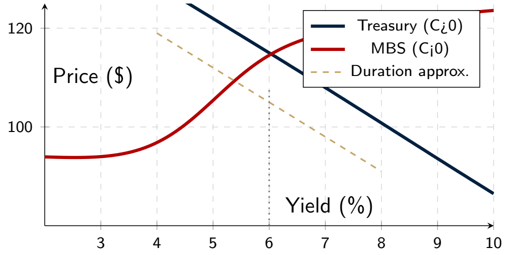
Convexity measures curvature: $\boxed{C = \frac{1}{P} \frac{\partial^2 P}{\partial y^2}}$

Taylor Expansion: $\frac{\Delta P}{P} \approx -D_{mod} \cdot \Delta y + \frac{1}{2} C \cdot (\Delta y)^2$

Effective Convexity: $\boxed{C_{eff} = \frac{P_{-\Delta y} + P_{+\Delta y} - 2P_0}{P_0 \cdot (\Delta y)^2}}$

MBS Convexity: Positive when rates high (OTM), **negative** when rates low (ITM).

Visualizing Duration and Convexity

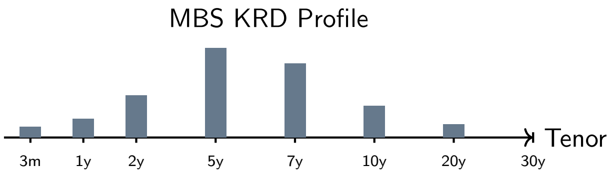


Key Rate Duration (KRD)

Key Rate Duration

KRD = sensitivity to shifts at specific curve points: $KRD_i = -\frac{1}{P}\frac{\partial P}{\partial y_i}$

Decomposition: $D_{eff} = \sum_{i=1}^{n} KRD_i$



Extended Greeks

Vega: Volatility Sensitivity

Option-Adjusted Vega (OAV)

Vega measures price sensitivity to implied volatility: $\boxed{\text{Vega} = \frac{\partial P}{\partial \sigma}}$

For MBS: Since MBS = Bond − Call Option:

$$\text{Vega}_{MBS} = -\text{Vega}_{call} < 0$$

Interpretation: MBS prices *fall* when volatility rises (short option position).

Vega Hedge Ratio:

$$N_{swaption} = -\frac{\text{Vega}_{MBS}}{\text{Vega}_{swaption}}$$

Theta: Time Decay

Time Decay

Theta measures value change with passage of time: $\boxed{\Theta = \frac{\partial P}{\partial t}}$

For Hedged MBS Positions:

$$\Theta_{portfolio} = \Theta_{MBS} + \Theta_{hedge}$$

Key Insight: Swaption hedges have negative theta (time decay cost):

- Long straddle: $\Theta < 0$ (pays for convexity protection)
- Must balance theta cost against convexity benefit

Carry vs. Roll-down: Total return = Coupon + Roll-down + Theta

Cross-Gamma: Rate-Volatility Interaction

Cross-Gamma

Cross-gamma measures how delta changes with volatility: $\boxed{\Gamma_{r,\sigma} = \frac{\partial^2 P}{\partial r\,\partial\sigma} = \frac{\partial\Delta}{\partial\sigma} = \frac{\partial \text{Vega}}{\partial r}}$

Implication for MBS:

- Duration changes when volatility changes
- Vega changes when rates change
- Creates "correlation risk" in hedges

Practical Impact:

$$\Delta P \approx -D \cdot \Delta r + \text{Vega} \cdot \Delta\sigma + \Gamma_{r,\sigma} \cdot \Delta r \cdot \Delta\sigma$$

In volatile markets, cross-gamma can dominate P&L.

Complete Greeks Summary

Greek	Formula	MBS Sign	Hedge With
Delta (Δ)	$\frac{\partial P}{\partial r}$	Negative	Swaps, Treasuries
Gamma (Γ)	$\frac{\partial^2 P}{\partial r^2}$	Negative (low rates)	Swaptions
Vega ($\mathcal{V}$)	$\frac{\partial P}{\partial \sigma}$	Negative	Swaptions
Theta (Θ)	$\frac{\partial P}{\partial t}$	Positive (carry)	–
Cross-Γ	$\frac{\partial^2 P}{\partial r \partial \sigma}$	Variable	Complex structures

Full P&L Decomposition:

$$\Delta P = \Delta \cdot dr + \frac{1}{2}\Gamma \cdot dr^2 + \mathcal{V} \cdot d\sigma + \Theta \cdot dt + \Gamma_{r,\sigma} \cdot dr \cdot d\sigma$$

Prepayment Modeling

CPR, SMM, and PSA Conventions

Prepayment Rate Definitions

CPR (Conditional Prepayment Rate): Annualized prepayment rate

$$CPR = 1 - \left(\frac{\text{Remaining Balance}}{\text{Scheduled Balance}}\right)^{12}$$

SMM (Single Monthly Mortality): Monthly prepayment rate

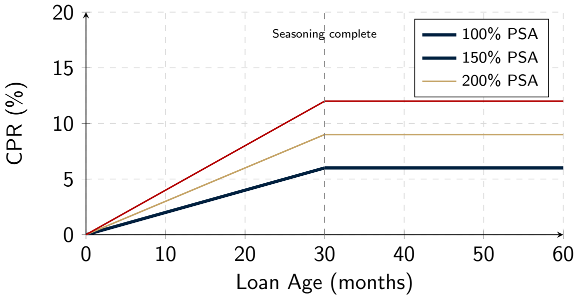
$$\boxed{SMM = 1 - (1 - CPR)^{1/12}}$$

PSA Benchmark:

$$CPR_t = \min\left(\frac{t}{30}, 1\right) \times 6\%$$

where t = loan age in months. 100% PSA ramps from 0% to 6% CPR over 30 months.

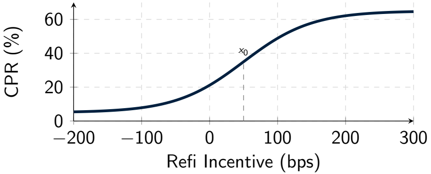
PSA Ramp Visualization



The Prepayment S-Curve

Refinancing Incentive

Incentive $=$ WAC $-$ Current Rate. Prepayment: $CPR(x) = CPR_{min} + \frac{CPR_{max} - CPR_{min}}{1+e^{-\beta(x-x_0)}}$



Burnout Effect: Path Dependency

Burnout Model

Burnout = Reduction in prepayment response after sustained low rates

Cumulative refinancing opportunity:

$$B_t = \int_0^t \max(0, \text{Incentive}_s)\, ds$$

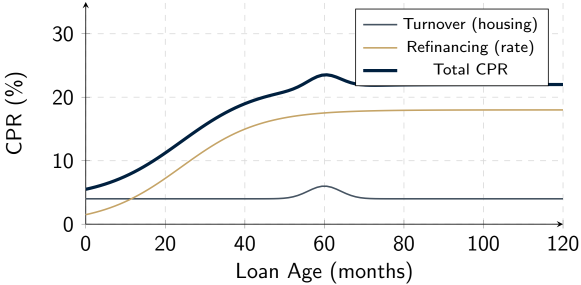
Adjusted CPR:

$$\boxed{CPR_{adj} = CPR_{base} \times e^{-\lambda B_t}}$$

where λ = burnout decay rate.

Intuition: Borrowers most likely to refinance do so first; remaining pool is "burned out."

Prepayment Components



Total CPR = Turnover + Refinancing + Curtailments + Defaults

Hedging Instruments

Treasury Securities as Hedges

Advantages:

- Highest liquidity
- No credit risk
- Multiple maturities
- Simple execution

Disadvantages:

- Convexity mismatch
- Basis risk
- Cannot hedge spread
- Carry cost

Treasury Hedge Ratio

$$HR_{TSY} = \frac{DV01_{MBS}}{DV01_{TSY}} = \frac{D_{MBS} \times P_{MBS}}{D_{TSY} \times P_{TSY}}$$

Interest Rate Swap Hedging

Swap DV01

$$DV01_{swap} = DV01_{fixed} - DV01_{float} \approx DV01_{fixed} \text{ (floating resets to par)}$$

Swap Hedge Ratio

$$\boxed{N_{swap} = -\frac{DV01_{MBS}}{DV01_{swap}} \times P_{MBS}}$$

Example: MBS with DV01 = \$43,050, 5Y swap DV01 = \$45/\$1M

$\Rightarrow$ Hedge notional $= \frac{\$43{,}050}{\$45/\text{M}} = \$95.7\text{M}$

TBA and Dollar Roll

TBA = To-Be-Announced

Forward contract on generic MBS pools. Natural hedge for MBS portfolios.

Dollar Roll: Sell near-month TBA, buy far-month TBA.

$$\text{Roll Value} = P_{near} - P_{far} - \Delta\text{AI}$$

Implied Financing Rate:

$$r_{implied} = \frac{(P_{near} - P_{far}) \times 12/\text{days}}{P_{near}}$$

Advantage: TBA has similar convexity profile to specified pools.

Swaptions: Hedging Negative Convexity

Swaption Greeks

Receiver Swaption: $\Delta < 0$, $\Gamma > 0$, Vega > 0

Straddle Strategy: Buy receiver + payer at same strike

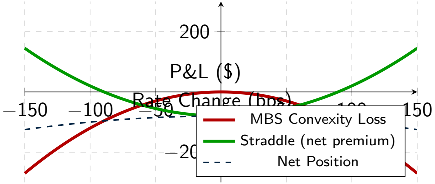
$$\Gamma_{straddle} = \Gamma_{rcvr} + \Gamma_{payer} > 0$$

Hedge Ratios:

$$N_\Gamma = -\frac{C_{MBS} \times P_{MBS}}{\Gamma_{swaption}} \qquad N_\mathcal{V} = -\frac{\text{Vega}_{MBS}}{\text{Vega}_{swaption}}$$

Trade-off: Convexity protection costs theta (time decay).

Swaption Convexity Hedge: Visualization



Hedge Ratios

Optimal Hedge Ratio: Theoretical Foundation

Minimum Variance Hedge Ratio

$$\boxed{h^* = \frac{\text{Cov}(\Delta S, \Delta F)}{\text{Var}(\Delta F)} = \rho \cdot \frac{\sigma_S}{\sigma_F}}$$

Derivation: Portfolio $V = S - h \cdot F$
$\sigma_V^2 = \sigma_S^2 - 2h\text{Cov}(S, F) + h^2\sigma_F^2$
FOC: $\frac{\partial \sigma_V^2}{\partial h} = 0 \Rightarrow h^* = \frac{\text{Cov}(S,F)}{\sigma_F^2}$

Duration-Based Hedge Ratio

Dollar Duration Matching

$$\boxed{h = \frac{D_{MBS} \cdot P_{MBS} \cdot Q_{MBS}}{D_{hedge} \cdot P_{hedge}} = \frac{DV01_{MBS} \cdot Q_{MBS}}{DV01_{hedge}}}$$

Multi-Instrument: $\sum_{j=1}^{n} h_j \cdot DV01_j = DV01_{MBS}$
KRD Matching: $\sum_j h_j \cdot KRD_{ij} = KRD_{i,MBS}$ for each key rate i

Regression-Based Hedge Ratio

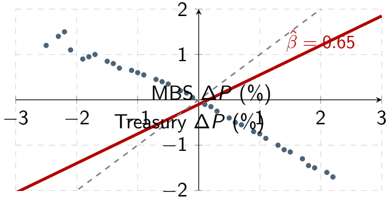
OLS Hedge Ratio

$$\Delta P_{MBS,t} = \alpha + \beta \cdot \Delta P_{hedge,t} + \epsilon_t$$

$$\boxed{\hat{h} = \hat{\beta} = \frac{\text{Cov}(\Delta P_{MBS}, \Delta P_{hedge})}{\text{Var}(\Delta P_{hedge})}}$$

Hedge Effectiveness: $R^2 = \rho^2$
Rolling Window: 60-90 day windows capture time-varying relationships.

Empirical Hedge Ratio



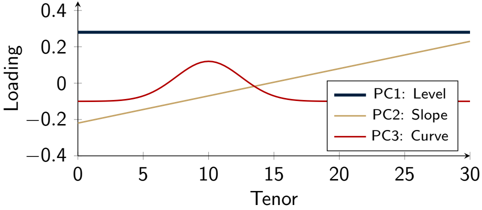
$\hat{\beta} < 1$ reflects MBS negative convexity (smaller gains in rallies).

Principal Component Hedging

PCA Framework

Decompose: $\Delta y_t = \sum_k \beta_k \cdot PC_{k,t}$ where PC1=level, PC2=slope, PC3=curve.

Variance Explained: PC1 ~85%, PC2 ~10%, PC3 ~3%



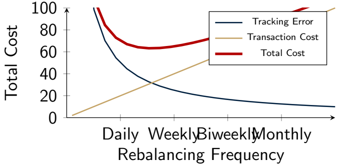
Dynamic Hedging

Rebalancing: Cost vs. Tracking Error

Rebalancing Trade-off

Tracking Error (no rebalancing) vs. **Transaction Costs** (frequent rebalancing)

Optimal Frequency: $\boxed{f^* = \sqrt{\frac{\Gamma^2 \sigma_r^4}{4 \cdot TC}}}$ where TC = transaction cost



Transaction Cost Models

Cost Components

Total Cost = Bid-Ask Spread + Market Impact + Opportunity Cost

Market Impact (Square-Root): $\boxed{MI = \sigma \cdot \sqrt{V/ADV} \cdot \text{sign}(V)}$

Effective Cost: $TC_{eff} = \frac{\text{Spread}}{2} + MI + \text{Delay Cost}$

Typical Costs: Treasuries: 0.5-2 bps, Swaps: 0.25-1 bp, TBA: 1-3 bps, Swaptions: 2-10 bps

Gamma Scalping P&L

Gamma P&L

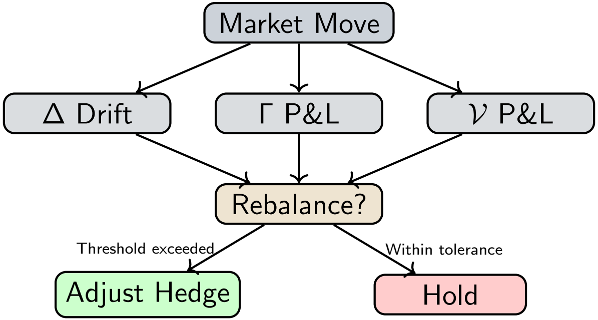
Daily gamma P&L: $\boxed{P\&L_\Gamma = \frac{1}{2}\Gamma \cdot (\Delta r)^2 - \Theta \cdot \Delta t}$

Expected P&L: $\mathbb{E}[P\&L_\Gamma] = \frac{1}{2}\Gamma \cdot \sigma^2 \cdot \Delta t - \Theta \cdot \Delta t$

Break-even Volatility: $\sigma_{BE} = \sqrt{\frac{2\Theta}{\Gamma}}$

Interpretation: If realized vol $>$ implied vol $\Rightarrow$ gamma scalping profitable.

Delta-Gamma-Vega Rebalancing



Threshold Rules: Rebalance when $|\Delta_{actual} - \Delta_{target}| > \epsilon_\Delta$

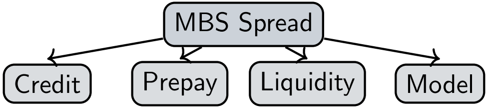
Basis & Spreads

What is Mortgage Basis?

Basis Definition

Basis = MBS yield − Benchmark yield
Primary Basis = MBS − Treasury **Swap Basis** = MBS − Swap

Basis Components:



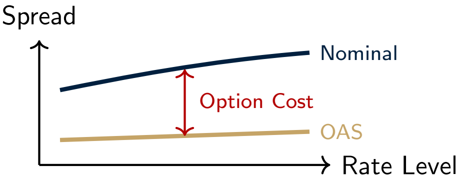
Basis P&L: $\Delta V_{basis} = -D_{OAS} \cdot P \cdot \Delta OAS$

OAS and Basis Relationship

Option-Adjusted Spread

$$P_{MBS} = \sum_{paths} \frac{1}{N} \sum_t \frac{CF_t^{path}}{(1+r_t+OAS)^t}$$

Relationship: Nominal Spread = OAS + Option Cost



Z-Spread vs. OAS

Spread Definitions

Z-Spread: Static spread over spot curve (ignores optionality)

$$P = \sum_{i=1}^{n} \frac{CF_i}{(1 + s_i + Z)^{t_i}}$$

OAS: Spread over forward rates after option adjustment

Relationship:

$$\boxed{Z\text{-Spread} \approx OAS + \text{Option Cost}}$$

When to Use:

- Z-Spread: Non-callable bonds, quick relative value
- OAS: MBS, callable bonds, comparing across structures

LIBOR-OIS and SOFR Basis

Basis Definitions

LIBOR-OIS Spread: Credit/liquidity premium in interbank lending
SOFR Spread Adjustment: Transition from LIBOR to SOFR benchmark

Impact on MBS Hedging:

- Swap hedges reference SOFR (risk-free)
- Historical basis risk during LIBOR transition
- Spread adjustment $\approx$ 26 bps for 3M LIBOR

Hedge Adjustment:

$$HR_{adj} = HR_{base} \times \left(1 + \frac{\Delta \text{Basis}}{\text{Yield}_{hedge}}\right)$$

Cross-Currency Basis

FX Basis

For non-USD investors, total spread includes FX hedging cost:

$$\text{Total Spread} = OAS_{MBS} - \text{FX Basis} - \text{Hedge Cost}$$

FX Basis Components:

- Forward points (interest rate differential)
- Cross-currency basis swap spread
- Counterparty credit adjustment

Example (JPY investor):

$$\text{Yield}_{JPY} = \text{Yield}_{USD} - \text{FX Forward Points} - \text{XCCY Basis}$$

Empirical Performance

Hedge Effectiveness Metrics

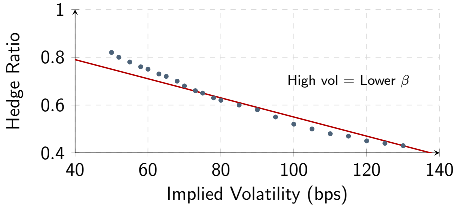
Effectiveness Measures

VRR: $1 - \frac{\text{Var}(\Delta P_{hedged})}{\text{Var}(\Delta P_{unhedged})}$ **TE:** $\sqrt{\text{Var}(\Delta P_{MBS} - h \cdot \Delta P_{hedge})}$

Typical Hedge Effectiveness:

Hedge	VRR	TE (bps/mo)
Treasuries only	70-80%	15-25
Swaps only	75-85%	12-20
Swaps + Swaptions	85-92%	8-15
TBA + Swaps	88-95%	5-12

Regime-Dependent Hedge Ratios



Interpretation: High volatility $\Rightarrow$ MBS underperforms more (negative convexity amplified).

Stress Testing: Rate Shock Scenarios

Scenario	Rate Δ	MBS	Hedged	Residual
Rally 100bp	-100	$+3.5\%$	$+0.8\%$	-2.7%
Rally 50bp	-50	$+2.1\%$	$+0.3\%$	-1.8%
Unchanged	0	0%	0%	0%
Selloff 50bp	$+50$	-2.8%	-0.4%	$+2.4\%$
Selloff 100bp	$+100$	-6.2%	-1.1%	$+5.1\%$

Asymmetry: Hedge performs worse in rally (MBS gains capped) than selloff.

Case Study: 2013 Taper Tantrum

Event: Fed signals tapering of QE (May-Aug 2013)

Market Impact:

- 10Y Treasury: 1.6% $\rightarrow$ 3.0% (+140 bps)
- MBS OAS: 15 bps $\rightarrow$ 45 bps (widened 30 bps)
- Implied Vol: 70 $\rightarrow$ 110 bps

Hedge Performance:

Position	**P&L**	**Attribution**
MBS (unhedged)	−12%	Duration + Convexity + Basis
Duration hedge	+9%	Treasury short
Net (duration only)	−3%	Convexity + Basis

Lesson: Duration hedge insufficient; needed convexity + basis protection.

Case Study: 2020 COVID Crisis

Event: March 2020 liquidity crisis and Fed intervention

Market Dynamics:

- Initial: Rates rally, MBS underperforms (negative convexity)
- Liquidity crisis: Basis widened 100+ bps
- Fed intervention: Purchased $300B+ MBS

Key Observations:

1. Correlation breakdown: MBS-Treasury ρ dropped to 0.3
2. Hedge ratios unstable: Required daily recalibration
3. Liquidity premium: Transaction costs spiked 5-10x

Lesson: Stress periods require liquidity reserves and pre-positioned hedges.

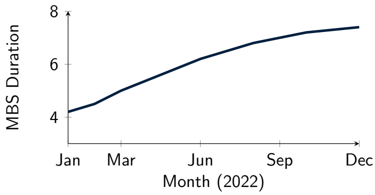
Case Study: 2022 Rate Hiking Cycle

Event: Fed raises rates 425 bps (Mar 2022 - Dec 2022)

MBS Impact:

- Massive duration extension (prepays collapsed)
- Duration increased from 4.5 to 7+ years
- Required continuous hedge rebalancing

Rebalancing Challenge:



Lesson: Hedge ratio must track duration drift; static hedges failed.

Portfolio & Risk Management

Aggregate Portfolio Risk Management

Portfolio DV01

Aggregate DV01: $DV01_{portfolio} = \sum_i w_i \cdot DV01_i$
Aggregate Convexity: $C_{portfolio} = \sum_i w_i \cdot C_i$

Risk Limits Framework:

Metric	Limit	Action if Breached
Net DV01	$\pm$\$50K/bp	Rebalance within 24h
Net Convexity	$\pm$\$500K/bp^2	Add swaptions
OAS Duration	$\pm$1.0 years	Adjust TBA position
Vega	$\pm$\$100K/vol pt	Review option book

Liability-Driven Hedging (ALM)

ALM Framework

For insurers/banks: Hedge **surplus** = Assets − Liabilities

$$\Delta S = \Delta A - \Delta L = -D_A \cdot A \cdot \Delta r + D_L \cdot L \cdot \Delta r$$

Duration Matching Condition:

$$D_A \cdot A = D_L \cdot L \quad \Rightarrow \quad D_A = D_L \cdot \frac{L}{A}$$

Convexity Matching:

$$C_A \cdot A = C_L \cdot L$$

Challenge: MBS negative convexity vs. insurance liabilities' positive convexity.

Regulatory Capital Implications

Basel Treatment

Interest Rate Risk in Banking Book (IRRBB):
Capital charge based on ΔEVE (Economic Value of Equity) under stress scenarios.

Hedge Recognition:

- Derivatives must qualify for hedge accounting
- Documentation requirements (IAS 39 / ASC 815)
- Effectiveness testing: 80-125% range

Capital Relief from Hedging:

$$\text{Capital}_{hedged} = \text{Capital}_{unhedged} \times (1 - \text{HE})$$

where HE = hedge effectiveness ratio.

Monte Carlo for Effective Duration

MC Duration Estimation

Simulate N interest rate paths, compute:

$$D_{eff} \approx -\frac{1}{P_0} \cdot \frac{\bar{P}_{-\Delta y} - \bar{P}_{+\Delta y}}{2\Delta y}$$

where $\bar{P}$ = average price across paths.

Path Generation (Hull-White):

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t$$

Convergence: Standard error $\propto 1/\sqrt{N}$
Typical: $N = 1000 - 5000$ paths for stable Greeks.

Finite Difference Greeks

Bump-and-Reprice

Duration: $D = -\frac{P(y+\Delta y)-P(y-\Delta y)}{2\cdot P(y)\cdot\Delta y}$

Convexity: $C = \frac{P(y+\Delta y)+P(y-\Delta y)-2P(y)}{P(y)\cdot(\Delta y)^2}$

Bump Size Selection:

Greek	Typical Bump	Issue if Too Small/Large
Duration	10-25 bps	Noise / Non-linearity
Convexity	25-50 bps	Instability / Miss curvature
Vega	1-5 vol pts	Noise / Non-linearity

Central vs. Forward Difference: Central is more accurate ($O(h^2)$ vs. $O(h)$).

Model Risk in Hedging

Model Risk Sources

Prepayment Model: Different models $\Rightarrow$ different durations/OAS
Interest Rate Model: Affects option valuation

Prepayment Model Comparison:

Model	Duration	OAS
Model A (aggressive)	3.8 yrs	35 bps
Model B (consensus)	4.2 yrs	42 bps
Model C (conservative)	4.8 yrs	55 bps

Hedge Ratio Range: Can vary 15-20% across models!
Mitigation: Use model ensemble, stress test across models.

Summary

Risk Metrics: Master Formulas

Core Relationships

Duration Family:

$$DV01 = \frac{D_{mod} \times P}{10000} \qquad D_{mod} = -\frac{1}{P}\frac{\partial P}{\partial y}$$

Price Change (Full):

$$\frac{\Delta P}{P} \approx -D \cdot \Delta y + \frac{1}{2} C \cdot (\Delta y)^2 + \frac{\mathcal{V}}{P} \cdot \Delta\sigma$$

Hedge Ratios:

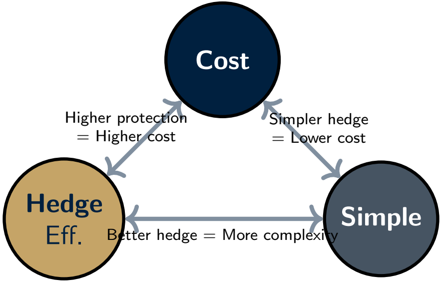
$$h^*_{DV01} = \frac{DV01_{MBS}}{DV01_{hedge}} \qquad h^*_{stat} = \rho \cdot \frac{\sigma_{MBS}}{\sigma_{hedge}}$$

Hedging Strategy Selection

Risk	Instrument	Metric	Cost
Duration	Swaps, Futures	DV01	Low
Curve	Multi-tenor swaps	KRD	Low
Convexity	Swaptions	Gamma	Medium
Volatility	Swaptions	Vega	Medium
Basis	TBA, MBS options	OAS Dur	High

Principles: (1) Match DV01 first, (2) Add convexity for large portfolios, (3) Use TBA for basis, (4) Rebalance dynamically.

Final Thoughts: The Hedging Trade-off



Optimal hedging balances protection, cost, and operational feasibility.

Questions?

References:

- Fabozzi, F. - *The Handbook of Mortgage-Backed Securities*
- Tuckman, B. - *Fixed Income Securities*
- Hayre, L. - *Salomon Smith Barney Guide to MBS*