Auction Theory for Business

Strategic Market Design and Applications

Strategic Insights

December 18, 2025

Today's Agenda

Introduction

Why Should You Care About Auctions?

Auctions are everywhere in business

- Google Ads ($200+ billion annually, billions of auctions daily)
- Government contracts and procurement
- Spectrum licenses ($100+ billion raised)
- Real estate and property sales
- Online marketplaces (eBay, stock exchanges)
- Art, wine, and collectibles

Key Business Questions

- Which auction format should I use?
- How do I maximize revenue?
- How should I bid when participating?
- How do I prevent manipulation?

Good market design isn't just theory—it's a competitive advantage.

The Stakes Are High

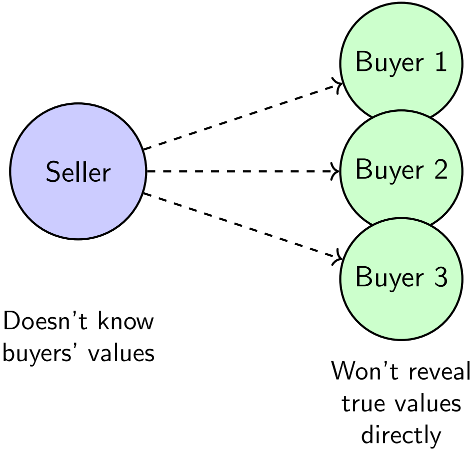
Global Impact: Trillions of dollars transacted via auctions annually

Examples of Design Impact

- **UK spectrum auction (2000):** \$35 billion raised with careful design
- **Switzerland spectrum (2000):** Only \$20 million (poor competition)
- **New Zealand disaster:** Student bid \$1 for spectrum, won due to no reserve price
- **California energy crisis (2000–01):** \$40+ billion in excess costs from poor design

Key Lesson: Small design choices = massive financial impact

The Fundamental Challenge



Solution: Design an auction that reveals true values through competition

Foundations

Game-Theoretic Framework

What Are We Studying? The Auction Game

The Basic Auction Story:

- Multiple people want the same thing
- Only one can have it (or limited quantities)
- How do we decide who gets it and at what price?

Why Game Theory?

- Your best strategy depends on what others do
- Others' strategies depend on what you do
- Everyone is trying to outsmart everyone else
- Information is often incomplete or asymmetric

The Central Questions:

- How should rational bidders behave?
- What auction format maximizes seller revenue?
- How does information structure affect outcomes?

Strategic Form Games

Definition: A game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ consists of:

- **Players:** $N = \{1, 2, \ldots, n\}$
- **Strategy spaces:** S_i for each player i
- **Payoff functions:** $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$

Solution Concepts:

1. **Dominant Strategy:** $s_i^* \in S_i$ such that

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_i \in S_i, \forall s_{-i}$$

2. **Nash Equilibrium:** Profile $(s_1^*, \ldots, s_n^*)$ where

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \forall s_i \in S_i, \forall i$$

Incomplete Information Games

Harsanyi Transformation:

- Nature determines types $\theta = (\theta_1, \ldots, \theta_n)$
- Player i observes only θ_i
- Common prior: $\theta \sim F$ where F is common knowledge
- Strategy: $s_i : \Theta_i \to S_i$

Bayesian Nash Equilibrium:
Strategy profile $(s_1^*, \ldots, s_n^*)$ where each $s_i^* : \Theta_i \to S_i$ satisfies:

$$s_i^*(\theta_i) \in \arg\max_{s_i} \mathbb{E}_{\theta_{-i}|\theta_i}[u_i(s_i, s_{-i}^*(\theta_{-i}), \theta)]$$

Key Insight: Each type best responds to beliefs about others' strategies

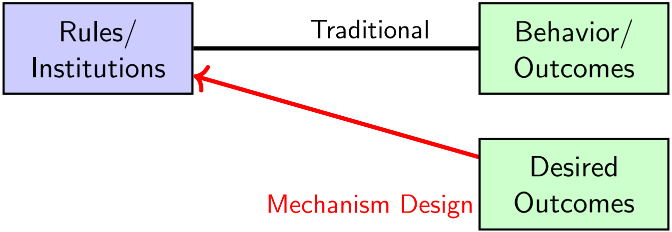
Mechanism Design: The Inverse Problem

Traditional Economics:

- Given rules/institutions → Predict behavior/outcomes
- Example: Given auction format → Find equilibrium

Mechanism Design (Hurwicz 1960s):

- Given desired outcome → Design rules/institutions
- Example: Want revenue maximization → Design optimal auction



Key Question: What mechanisms implement desired outcomes?

Direct Mechanisms

Definition

A **direct mechanism** asks each agent to report their private information (type), then uses these reports to determine:

- Allocation: Who gets what
- Payments: Who pays what

- Agent i has type $\theta_i \in \Theta_i$ (e.g., valuation, cost)
- Mechanism (x, p): $x : \Theta \to X$ (allocation), $p : \Theta \to \mathbb{R}^n$ (payment)

Example (Single Item Auction):

- Reports: $\hat{\theta} = (\hat{v}_1, \ldots, \hat{v}_n)$
- Allocation: $x_i(\hat{\theta}) = \begin{cases} 1 & \text{if } \hat{v}_i = \max_j \hat{v}_j \\ 0 & \text{otherwise} \end{cases}$
- Payment: $p_i(\hat{\theta})$ depends on mechanism

Incentive Compatibility and Individual Rationality

Incentive Compatibility (IC):

- Truthful reporting is optimal
- Dominant Strategy IC (DSIC): Truth optimal regardless of others

$$U_i(\theta_i, \theta_{-i}) \geq U_i(\hat{\theta}_i, \theta_{-i}) \quad \forall \hat{\theta}_i, \forall \theta_{-i}$$

- Bayesian IC (BIC): Truth optimal in expectation

$$\mathbb{E}_{\theta_{-i}}[U_i(\theta_i, \theta_{-i})] \geq \mathbb{E}_{\theta_{-i}}[U_i(\hat{\theta}_i, \theta_{-i})] \quad \forall \hat{\theta}_i$$

Individual Rationality (IR):

- Voluntary participation
- Ex-post IR: $U_i(\theta_i, \theta_{-i}) \geq 0 \quad \forall \theta_{-i}$
- Interim IR: $\mathbb{E}_{\theta_{-i}}[U_i(\theta_i, \theta_{-i})] \geq 0$

In Auctions: Second-price: DSIC and ex-post IR; First-price: BIC and interim IR

The Revelation Principle

Theorem (Revelation Principle)

For any mechanism and any equilibrium of that mechanism, there exists a direct truthful mechanism that achieves the same allocation and utilities.

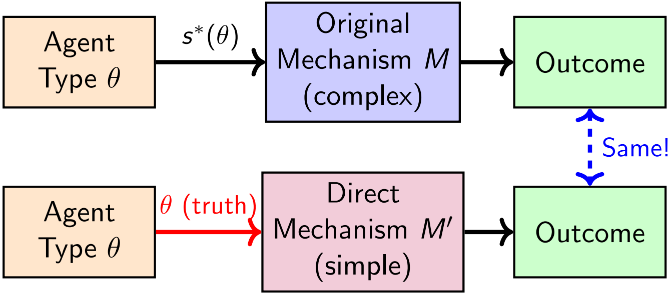
Proof Sketch

1. Start with mechanism M and equilibrium strategies $s^*(\theta)$
2. Construct direct mechanism M': ask for types $\hat{\theta}$, apply $s^*(\hat{\theta})$, run M
3. Truthful reporting optimal: $U_i(\theta_i, M') = U_i(s^*(\theta_i), M)$

Implication: Restrict attention to direct truthful mechanisms!

Caveat: Only works for given equilibrium concept (BNE $\rightarrow$ BIC, Dominant $\rightarrow$ DSIC)

Revelation Principle Illustration



Designer's Advantage: Only need to check IC for truthful reporting!

Order Statistics

Order Statistics: Essential Theory

Definition: For i.i.d. $X_1, \ldots, X_n \sim F$, order statistics are:

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$$

Why Important for Auctions?

- Winner has highest valuation: $v_{(n)} = \max\{v_1, \ldots, v_n\}$
- Second-price auctions: Winner pays $v_{(n-1)}$
- Revenue comparisons require $\mathbb{E}[v_{(n)}]$ and $\mathbb{E}[v_{(n-1)}]$

Key Result

For uniform $[0, 1]$ distribution:

$$\mathbb{E}[X_{(k)}] = \frac{k}{n+1}, \quad \mathrm{Var}(X_{(k)}) = \frac{k(n-k+1)}{(n+1)^2(n+2)}$$

Distribution of Order Statistics

CDF of k-th Order Statistic:

$$G_{(k)}(x) = \sum_{j=k}^{n} \binom{n}{j} F(x)^j [1 - F(x)]^{n-j}$$

PDF of k-th Order Statistic:

$$g_{(k)}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1} [1 - F(x)]^{n-k} f(x)$$

Special Cases:

- Maximum: $G_{(n)}(x) = F(x)^n, \quad g_{(n)}(x) = nF(x)^{n-1} f(x)$
- Minimum: $G_{(1)}(x) = 1 - (1 - F(x))^n$
- Second-highest: $g_{(n-1)}(x) = n(n-1)F(x)^{n-2}[1 - F(x)]f(x)$

Uniform Distribution: Key Results

For $U[0,1]$ distribution:

General Formula:

$$\mathbb{E}[X_{(k)}] = \frac{k}{n+1}$$

Examples:

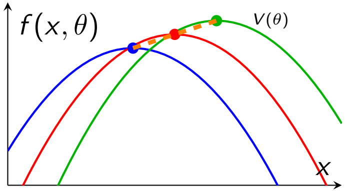
- $n = 2$: $\mathbb{E}[v_{(2)}] = \frac{2}{3}$, $\mathbb{E}[v_{(1)}] = \frac{1}{3}$
- $n = 3$: $\mathbb{E}[v_{(3)}] = \frac{3}{4}$, $\mathbb{E}[v_{(2)}] = \frac{1}{2}$
- $n = 10$: $\mathbb{E}[v_{(10)}] = \frac{10}{11}$, $\mathbb{E}[v_{(9)}] = \frac{9}{11}$

Intuition: Order statistics divide $[0,1]$ into $n+1$ equal segments in expectation

Envelope Theorem

The Envelope Theorem: Intuition

Standard Optimization: $V(\theta) = \max_x f(x, \theta)$
Question: How does optimal value $V(\theta)$ change with parameter θ?
Naive Approach: $\frac{dV}{d\theta} = \frac{\partial f}{\partial x}\frac{dx^*}{d\theta} + \frac{\partial f}{\partial \theta}$
Envelope Theorem: $\frac{dV}{d\theta} = \frac{\partial f}{\partial \theta}\Big|_{x=x^*(\theta)}$
Why? At optimum, $\frac{\partial f}{\partial x} = 0$, so first term vanishes!

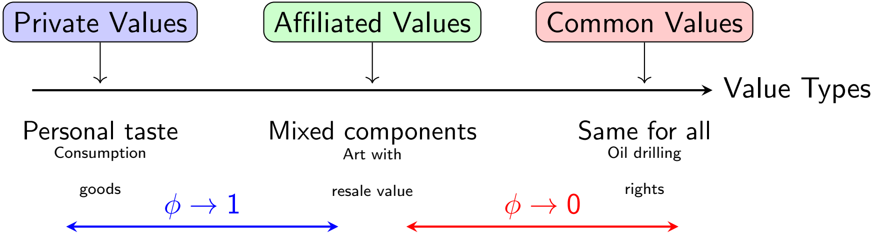


Information Structures

Types of Information in Auctions

1. **Independent Private Values (IPV):**
    - Each bidder knows own valuation exactly: v_i
    - Values drawn i.i.d. from F; learning others' values doesn't change own
    - Example: Personal use items
2. **Common Values (CV):**
    - Single unknown value V same for all bidders
    - Each bidder receives private signal s_i about V
    - Winner's curse: Winning means highest estimate
    - Example: Oil drilling rights
3. **Affiliated Values:**
    - General case: $v_i = v_i(s_1, \ldots, s_n)$
    - Signals correlated (higher signal makes high values more likely)
    - Encompasses both IPV and CV as special cases

Value Types Spectrum



$$v_i = \phi \cdot v_i^{private} + (1 - \phi) \cdot V^{common}$$

Knowledge and Common Knowledge

Why Knowledge Matters in Auctions

The Information Puzzle:

- In auctions, what you know matters
- But what you know that others know also matters
- And what you know that others know that you know...
- This infinite regress is crucial for strategic reasoning!

Common Knowledge in Practice:

- Auction rules must be common knowledge
- Reserve prices often announced publicly
- Some information deliberately kept private
- Strategic information revelation can increase revenue

Knowledge Operators: Formal Framework

Epistemic Logic:

- Knowledge operator K_i: "agent i knows that..."
- $K_i\varphi$ means "agent i knows proposition φ"

S5 Axiom System:

1. **Truth:** $K_i\varphi \to \varphi$
2. **Positive Introspection:** $K_i\varphi \to K_iK_i\varphi$
3. **Negative Introspection:** $\neg K_i\varphi \to K_i\neg K_i\varphi$
4. **Distribution:** $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$

Implication: Agents have perfect knowledge of their own knowledge state

Common Knowledge Definition

Iterative Definition:

$$E^1(\varphi) = \bigwedge_{i \in N} K_i \varphi \quad \text{(everyone knows)}$$

$$E^k(\varphi) = E(E^{k-1}(\varphi)) \quad (k\text{-th level})$$

$$C(\varphi) = \bigwedge_{k=1}^{\infty} E^k(\varphi) \quad \text{(common knowledge)}$$

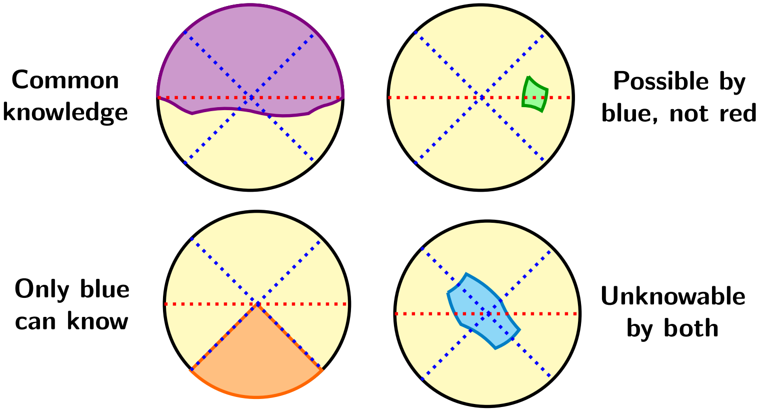
In Auctions, Common Knowledge Includes:

- Number of bidders n; Auction format and rules
- Distribution F (but not realizations); Rationality of all bidders
- Reserve price (if public)

Private Information:

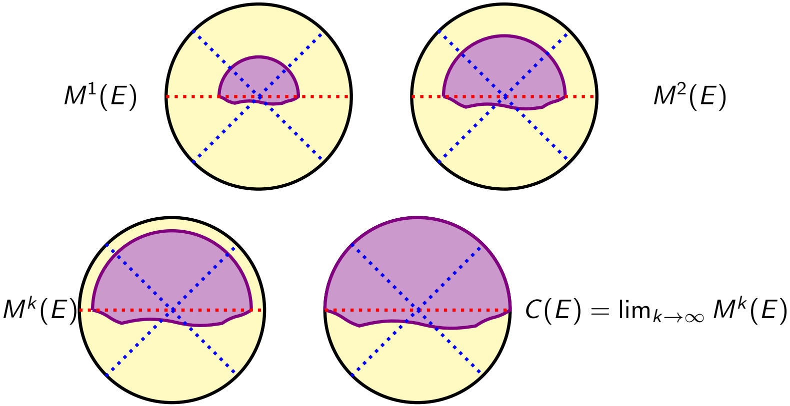
- Own valuation or signal
- Own bidding strategy (in sealed-bid)

Common Knowledge Illustration



Blue dotted lines: sharper agent's partition. Red dotted line: coarser agent's partition.

Mutual Knowledge to Common Knowledge



$M^k(E)$: "Everyone knows that everyone knows that… (k times)… that E occurred."

Single Item

Benchmark Model Setup

Three Key Assumptions:

1. **Independent Private Values (IPV):**
   - Valuations statistically independent
   - Learning others' values doesn't change own value
2. **Risk Neutrality:**
   - Utility $U_i = v_i - p_i$ if win, 0 if lose
   - No risk aversion or risk seeking
3. **Symmetry:**
   - All bidders from same distribution F
   - No ex-ante differences between bidders

Notation:

- n bidders, valuations $v_i \sim F(v)$ on $[\underline{v}, \bar{v}]$
- Density $f(v) = F'(v)$, assume $f > 0$ on support
- Order statistics: $v_{(1)} \geq v_{(2)} \geq \cdots \geq v_{(n)}$
- Seller's valuation: $v_s = 0$ (for simplicity)

Overview: Six Standard Formats

Let v = your valuation, b = your bid, b_1 = highest bid, b_2 = 2nd-highest

1. **English (Ascending Price)**
2. **Dutch (Descending Price)**
3. **Sealed-Bid First-Price:** $\pi = \mathbb{P}(b = b_1) \cdot (v - b)$
4. **Sealed-Bid Second-Price:** $\pi = \mathbb{P}(b = b_1) \cdot (v - b_2)$
5. **All-Pay First Price:** $\pi = \mathbb{P}(b = b_1) \cdot v - b$
6. **All-Pay Second Price:** $\pi = \mathbb{P}(b = b_1) \cdot (v - b_2) - \mathbb{P}(b \neq b_1) \cdot b$

We'll cover: Definition, equilibrium, derivation for each format

English Auction: Definition

Definition

An **English auction** is a sequential mechanism where bidders indicate interest at a current price. Price increases incrementally until only one bidder remains.

Standard Ascending Format:

- Prices increase incrementally
- Bidders signal their interest
- Ends when only one bidder signals
- Only winner pays (pays final price)

Reverse English (Procurement):

- Prices decrease incrementally
- Bidders signal willingness to sell
- Ends when only one bidder signals

English Auction: Equilibrium

Dominant Strategy:

- Stay in auction while price $p < v_i$
- Drop out when price $p \geq v_i$
- Strategy independent of beliefs about others

Equilibrium Outcome:

- Bidder with $v_{(2)}$ drops out at $p = v_{(2)}$
- Bidder with $v_{(1)}$ wins
- Pays price $p = v_{(2)}$
- Winner's surplus: $v_{(1)} - v_{(2)} > 0$

Properties:

- Efficient (highest value wins)
- Simple strategy (no need to know F or n)
- Transparent process
- Information revelation as bidders drop out

English Auction: Proof of Dominance

Claim: Staying active while $p < v_i$ is weakly dominant
Proof: Consider bidder i with value v_i, current price p
Case 1: $p < v_i$

- Stay active: Win if others drop first, payoff $= v_i - p_{final} \geq v_i - p > 0$
- Drop out: Lose, payoff $= 0$
- Staying dominates dropping

Case 2: $p \geq v_i$

- Stay active: If win, pay $p \geq v_i$, payoff ≤ 0
- Drop out: Lose, payoff $= 0$
- Dropping (weakly) dominates staying

Equilibrium Bidding: $b(v_i) = v_i$ (stay until price reaches value)
Winner pays: $p^* = v_{(2)}$

Second-Price Auction: Definition

Definition

A **second-price (Vickrey) auction** is a simultaneous mechanism where the highest bidder wins but pays the second-highest bid.

Sealed-Bid Second-Price Format:

- Bidders submit sealed bids simultaneously
- Highest bid wins the item
- Winner pays the second-highest bid
- Only winner pays

Strategic Property

Bidding your true valuation is a dominant strategy—no incentive to bid higher or lower.

Second-Price: Dominant Strategy Theorem

Theorem (Dominant Strategy)

In a second-price sealed-bid auction, bidding true valuation $b_i = v_i$ is a weakly dominant strategy for all bidders.

Proof: Let $\tilde{b} = \max_{j \neq i} b_j$ (highest other bid)

Case 1: Overbidding $(b > v)$

- If $\tilde{b} > v$: Don't want to win (would pay $> v$)
- If $\tilde{b} < v$: Win by bidding $b = v$ too, same payoff $v - \tilde{b} > 0$
- Bidding v dominates: avoids losses, keeps gains

Case 2: Underbidding $(b < v)$

- If $v > \tilde{b} > b$: Lose, but could win profitably
- If $v > b > \tilde{b}$: Win with same profit as bidding v
- Bidding v dominates: captures all opportunities

□

First-Price Auction: Definition

Definition

A **first-price auction** is a simultaneous mechanism where the highest bidder wins and pays their own bid.

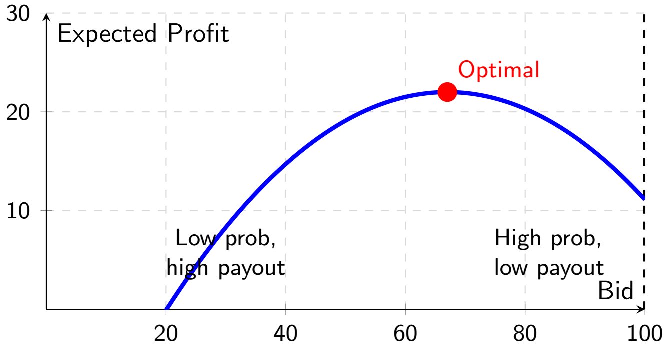
Sealed-Bid First-Price Format:

- Bidders submit sealed bids
- Highest bid wins the item
- Winner pays their own bid
- Only winner pays

Strategic Challenge

Bidders face a trade-off: bidding high increases win probability but decreases profit if winning. This is the **bid-shading problem**.

First-Price: The Trade-off



Key Insight: Higher bid increases win chance but reduces profit

First-Price: Equilibrium Derivation

Setup: If bidder i with value v_i bids as if value w_i:

$$\mathbb{E}[U_i] = (v_i - B(w_i)) \cdot F^{n-1}(w_i)$$

First-Order Condition:

$$\frac{\partial}{\partial w_i} : (v_i - B(w_i))(n-1)f(w_i)F^{n-2}(w_i) - B'(w_i)F^{n-1}(w_i) = 0$$

At symmetric equilibrium $w_i = v_i$:

$$B'(v_i) = (n-1)(v_i - B(v_i))\frac{f(v_i)}{F(v_i)}$$

General Solution (with $B(\underline{v}) = \underline{v}$):

$$B(v_i) = \frac{\int_{\underline{v}}^{v_i} t \, dF^{n-1}(t)}{F^{n-1}(v_i)} = v_i - \frac{\int_{\underline{v}}^{v_i} F^{n-1}(t)dt}{F^{n-1}(v_i)}$$

Note: $B(v_i) < v_i$ (always shade bid below value)

First-Price: Uniform Example

Setup: $v \sim U[0,1]$, so $F(v) = v$, $f(v) = 1$

Differential equation:

$$B'(v_i) = \frac{(n-1)(v_i - B(v_i))}{v_i}$$

Guess linear solution: $B(v_i) = kv_i$

Substituting: $k = \frac{(n-1)(v_i - kv_i)}{v_i} = (n-1)(1-k)$

Solving: $k = \frac{n-1}{n}$

Equilibrium bid function:

$$\boxed{B(v_i) = \frac{n-1}{n} v_i}$$

Examples:

- 2 bidders: bid 50% of value
- 10 bidders: bid 90% of value
- As $n \to \infty$: $B(v) \to v$ (competition eliminates shading)

Dutch Auction: Definition

Definition

A **Dutch auction** is a sequential mechanism where an auctioneer decreases the price until the first bidder signals acceptance.

Standard Descending Format:

- Prices decrease incrementally
- Bidders watch and wait
- First to accept wins at current price
- Only winner pays

Reverse Dutch (Procurement):

- Prices increase incrementally
- First seller to accept wins

Claim: Dutch and First-Price auctions are strategically equivalent

Dutch Decision:

- Choose stopping price b before auction starts
- Accept when clock reaches b
- If someone else accepts first, lose
- Payoff: $(v_i - b) \cdot \mathbb{P}(b > \max_{j \neq i} b_j)$

First-Price Decision:

- Choose bid b to submit in sealed envelope
- If b is highest, win and pay b
- Otherwise lose
- Payoff: $(v_i - b) \cdot \mathbb{P}(b > \max_{j \neq i} b_j)$

Why Equivalent? Same:

- Information at decision time (only own value); Action space (choose a price)
- Payoff structures
- $\Rightarrow$ Identical equilibrium strategies: $b_{Dutch}(v_i) = b_{First}(v_i)$

All-Pay First-Price: Definition

Definition

An **all-pay first-price auction** is a simultaneous mechanism where the highest bidder wins, but **all bidders pay their bids**.

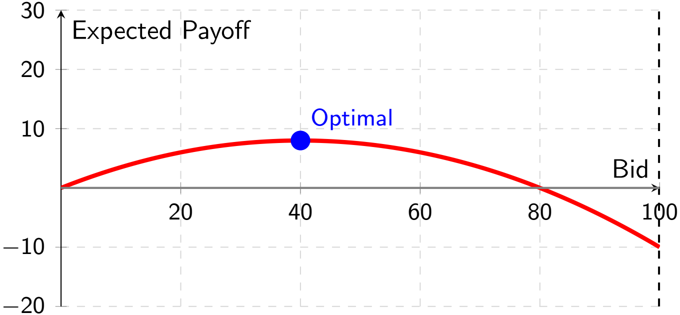
Format:

- Bidders submit sealed bids
- Highest bid wins the item
- **All bidders pay their bids** (winners and losers)
- Winner receives item, losers receive nothing

Applications:

- R&D races (all firms invest, one wins patent)
- Political lobbying (all parties spend, one wins favor)
- Legal contests (all parties pay fees, one wins case)

All-Pay First-Price: The Challenge



Key Insight: Since you pay even if you lose, optimal bids are much lower than standard auctions

All-Pay First-Price: Equilibrium

Symmetric Equilibrium:

$$b(v_i) = \int_{\underline{v}}^{v_i} t dF^{n-1}(t) = v_i F^{n-1}(v_i) - \int_{\underline{v}}^{v_i} F^{n-1}(t)dt$$

Uniform $[0, 1]$ **Case:**

$$\boxed{b(v_i) = \frac{n-1}{n} v_i^n}$$

Comparison to First-Price:

- First-price: $b_{FPA}(v_i) = \frac{n-1}{n} v_i$
- All-pay: $b_{AP}(v_i) = \frac{n-1}{n} v_i^n$
- Note: $b_{AP}(v_i) < b_{FPA}(v_i)$ for all $v_i < 1$

Revenue Equivalence Still Holds:

$$ER_{All-Pay} = \mathbb{E}[v_{(2)}]$$

(All pay, but pay less per person) hy Revenue Equals

All-Pay First-Price: Derivation

Expected Utility: If bidder with value v_i bids as if value w:

$$U(w; v_i) = v_i \cdot F^{n-1}(w) - B(w)$$
$$\frac{\partial U}{\partial w} = v_i(n-1)f(w)F^{n-2}(w) - B'(w) = 0$$

At symmetric equilibrium ($w = v_i$):

$$B'(v_i) = v_i(n-1)f(v_i)F^{n-2}(v_i)$$

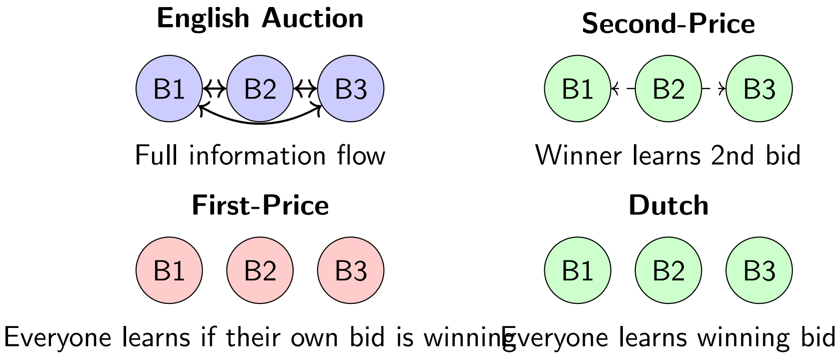
Integrating with $B(\underline{v}) = 0$:

$$B(v_i) = \int_{\underline{v}}^{v_i} t\, dF^{n-1}(t) = v_i F^{n-1}(v_i) - \int_{\underline{v}}^{v_i} F^{n-1}(t)\, dt$$

Uniform Distribution:

$$B(v_i) = v_i^n - \frac{v_i^n}{n} = \frac{n-1}{n} v_i^n$$

Information Flow in Different Formats



Information Revelation Impact:

- More information → Less winner's curse
- Less winner's curse → More aggressive bidding
- More aggressive bidding → Higher revenue

War of Attrition: Definition

Definition

A **war of attrition** (all-pay second-price) is a simultaneous mechanism where the highest bidder wins and pays the second-highest bid, but all losing bidders also pay their own bids.

Format:

- Bidders submit sealed bids
- Highest bid wins the item
- Winner pays the **second-highest bid**
- **All losers pay their own bids**

Applications:

- Standards wars (all invest, winner sets standard)
- Animal contests (all expend energy, winner gets resource)
- Market entry battles

War of Attrition: Equilibrium

Equilibrium Strategy:

$$b(v_i) = -v_i \ln(1 - F^{n-1}(v_i)) + \int_{\underline{v}}^{v_i} \ln(1 - F^{n-1}(t))dt$$

Two-Player Uniform $[0, 1]$ **Case:**

$$\boxed{b(v_i) = -v_i + -\ln(1 - v_i)}$$

Key Properties:

- $\lim_{v_i \to 1} b(v_i) = 1$ (highest type bids their value)
- More aggressive than all-pay first-price
- Revenue equivalence: $ER_{War} = \mathbb{E}[v_{(2)}]$

War of Attrition: Derivation (1)

Expected Utility:

$$U(w; v_i) = F^{n-1}(w) \cdot v_i - \int_{\underline{v}}^{w} B(t) dF^{n-1}(t) - (1 - F^{n-1}(w))B(w)$$

First-Order Condition:

$$\frac{\partial U}{\partial w} = (n-1)f(w)F^{n-2}(w)v_i - B(w)(n-1)f(w)F^{n-2}(w) + (n-1)f(w)F^{n-2}(w)B(w) - B'(w)(1 - F^{n-1}(w)) = 0$$

At $w = v_i$:

$$B'(v_i)(1 - F^{n-1}(v_i)) = (n-1)f(v_i)F^{n-2}(v_i)v_i$$

$$B'(v_i) = \frac{(n-1)f(v_i)F^{n-2}(v_i)}{1 - F^{n-1}(v_i)} v_i$$

War of Attrition: Derivation (2)

Note that:

$$\frac{d}{dv}[-\ln(1 - F^{n-1}(v))] = \frac{(n-1)f(v)F^{n-2}(v)}{1 - F^{n-1}(v)}$$

Therefore:

$$B'(v_i) = -v_i \frac{d}{dv_i} \ln(1 - F^{n-1}(v_i))$$

Integrating by parts with $B(\underline{v}) = 0$:

$$B(v_i) = -v_i \ln(1 - F^{n-1}(v_i)) + \int_{\underline{v}}^{v_i} \ln(1 - F^{n-1}(t))\, dt$$

Two-Player Uniform Case:

$$B(v_i) = -v_i \ln(1 - v_i) + \int_0^{v_i} \ln(1 - t)\, dt = v_i + (1 - v_i) \ln(1 - v_i)$$

Contests

Tullock Contests

Contest Success Function:

$$p_i(x_i, x_{-i}) = \begin{cases} \frac{x_i^r}{\sum_{j=1}^n x_j^r} & \text{if } \sum_j x_j > 0 \\ \frac{1}{n} & \text{otherwise} \end{cases}$$

Parameter r (decisiveness):

- $r \to 0$: Random lottery (effort irrelevant)
- $r = 1$: Linear contest
- $r \to \infty$: All-pay auction (deterministic)

Symmetric Equilibrium:

$$\boxed{x^* = \frac{r(n-1)V}{n^2}}$$

Rent Dissipation: Total effort $= \frac{r(n-1)}{n} \times V$ (fraction of prize value)
Existence: Requires $r \leq 2$ for $n \geq 2$

Tullock: Derivation

Expected Utility: $$U_i = \frac{x_i^r}{x_i^r + \sum_{j \neq i} x_j^r} \cdot V - x_i$$

In symmetric equilibrium ($x_j = x^*$ for all $j \neq i$): $$U_i = \frac{x_i^r}{x_i^r + (n-1)(x^*)^r} \cdot V - x_i$$

FOC: $$1 = \frac{r x_i^{r-1} \cdot (n-1)(x^*)^r}{[x_i^r + (n-1)(x^*)^r]^2} \cdot V$$

At $x_i = x^*$: $$1 = \frac{V \cdot r(n-1)(x^*)^{2r-1}}{[n \cdot x^*]^{2r}}$$

Solving: $$x^* = \frac{r(n-1)V}{n^2}$$

Rank-Order Tournaments

Lazear-Rosen (1981) Model:

- Two workers, output: $q_i = e_i + \varepsilon_i + \theta$
- ε_i: Idiosyncratic shock
- θ: Common shock
- Prizes: $w_1 > w_2$ (winner and loser)

Tournament Rule: Worker i wins if $q_i > q_j$
Equilibrium Effort:

$$e^* = \frac{u(w_1) - u(w_2)}{\sigma_\varepsilon \cdot 2\sqrt{\pi}}$$

Key Result:

- Tournament dominates piece-rate when common shock large
- Piece-rate dominates when idiosyncratic shock large
- Tournament filters common shocks via relative performance

Proof of Tournament Equilibrium (1/2)

Worker i's Optimization Problem:

Given $q_i = e_i + \varepsilon_i + \theta$ and $q_j = e_j + \varepsilon_j + \theta$, worker i wins if:

$$q_i > q_j \quad \Leftrightarrow \quad e_i - e_j > \varepsilon_j - \varepsilon_i$$

Assumptions:

- $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$ i.i.d.
- Cost of effort: $c(e_i) = e_i$ (linear cost)
- Common shock θ cancels in relative comparison

Probability of Winning:

$$\varepsilon_j - \varepsilon_i \sim N(0, 2\sigma_\varepsilon^2)$$

$$P(\text{worker } i \text{ wins}) = \Phi\left(\frac{e_i - e_j}{\sqrt{2}\sigma_\varepsilon}\right)$$

where $\Phi(\cdot)$ is the standard normal CDF.

Proof of Tournament Equilibrium (2/2)

$$\max_{e_i} \; \Phi\left(\frac{e_i - e_j}{\sqrt{2}\sigma_\varepsilon}\right) u(w_1) + \left[1 - \Phi\left(\frac{e_i - e_j}{\sqrt{2}\sigma_\varepsilon}\right)\right] u(w_2) - e_i$$

First-Order Condition: $\phi\left(\frac{e_i - e_j}{\sqrt{2}\sigma_\varepsilon}\right) \cdot \frac{1}{\sqrt{2}\sigma_\varepsilon}[u(w_1) - u(w_2)] = 1$

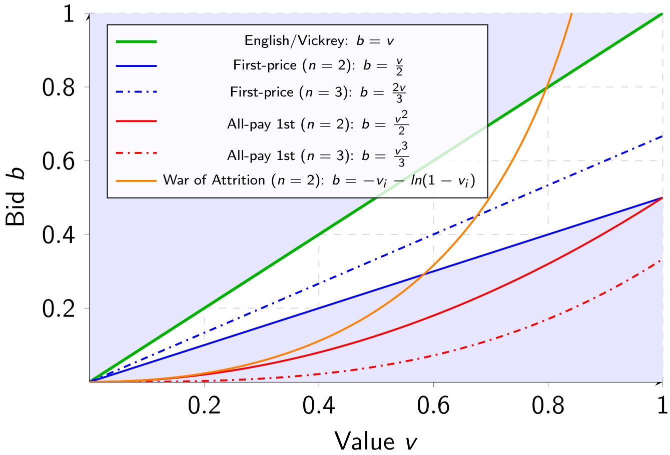
where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the standard normal PDF.

Symmetric Eq.: $e_i = e_j = e^* \implies \phi(0) \cdot \frac{1}{\sqrt{2}\sigma_\varepsilon}[u(w_1) - u(w_2)] = 1$

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}\sigma_\varepsilon}[u(w_1) - u(w_2)] = 1$$

$$\boxed{e^* = \frac{u(w_1) - u(w_2)}{\sigma_\varepsilon \cdot 2\sqrt{\pi}}}$$

Visualization: Bid Functions Across Auction Types



Insight: Mechanism determines effect of competition

Core Results

Strategic and Outcome Equivalences

Strategic and Outcome Equivalence

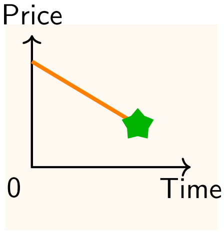
Outcome Equivalence

Ascending Auction ⇠⇢ **Second Price Sealed Bid**

Bidders drop out

Strategic Equivalence

Descending Auction ⇠⇢ **First Price Sealed Bid**

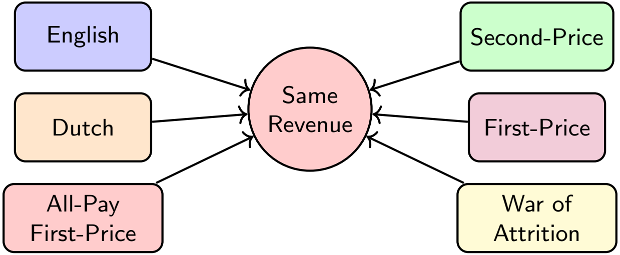


First to accept wins

Revenue Equivalence

Revenue Equivalence: The Surprising Result

All six standard formats generate the same expected revenue!



Conditions Required:

- Independent private values (IPV); risk-neutral bidders
- Symmetric bidders (same distribution F); no collusion

Key Insight: Revenue determined by competition level, not payment rule

Summary: Bid Functions and Revenue (Uniform $[0, 1]$)

Format	**Equilibrium Bid** $B(v)$	**Revenue**
English	v	$\frac{n-1}{n+1}$
Second-Price	v	$\frac{n-1}{n+1}$
First-Price	$\frac{n-1}{n}v$	$\frac{n-1}{n+1}$
Dutch	$\frac{n-1}{n}v$	$\frac{n-1}{n+1}$
All-Pay 1st	$\frac{n-1}{n}v^n$	$\frac{n-1}{n+1}$
War of Attrition	$v + (1 - v)\ln(1 - v)$ (for $n = 2$)	$\frac{n-1}{n+1}$

Key Observation: All yield expected revenue $= \mathbb{E}[v_{(n-1)}] = \frac{n-1}{n+1}$

Revenue Equivalence: The Big Picture

Central Question: Which auction format generates the most revenue?
Surprising Answer: Under standard assumptions, *all* standard auctions yield the same expected revenue!
Standard Assumptions:

- Independent Private Values (IPV)
- Risk-neutral bidders
- Symmetric bidders (same distribution F)
- Efficient allocation (highest value wins)
- Zero payment for lowest type

Implication: Revenue differences come from *violations* of these assumptions

Revenue Equivalence Theorem

Theorem (Revenue Equivalence — Vickrey 1961, Myerson 1981)

Consider any two auction mechanisms satisfying:

1. *Same allocation rule (highest value wins)*
2. *Same expected payment for lowest type ($\underline{v}$ pays 0)*

Then under IPV, risk neutrality, and symmetry, both mechanisms yield identical expected revenue.

All yield expected revenue $= \mathbb{E}[v_{(n-1)}]$

Revenue Equivalence: Proof Sketch (1/2)

Step 1: Expected Utility

Bidder with value v in equilibrium receives: $U(v) = Q(v) \cdot v - M(v)$
where $Q(v)$ = probability of winning, $M(v)$ = expected payment
Envelope Theorem: $\frac{dU}{dv} = Q(v)$
Integrating from $\underline{v}$ to v: $U(v) = U(\underline{v}) + \int_{\underline{v}}^{v} Q(t)\, dt$

Key Insight: Expected utility depends only on $Q(\cdot)$ and $U(\underline{v})$

Revenue Equivalence: Proof Sketch (2/2)

Step 2: Expected Payment

From $U(v) = Q(v) \cdot v - M(v)$: $M(v) = Q(v) \cdot v - U(v) = Q(v) \cdot v - U(\underline{v}) - \int_{\underline{v}}^{v} Q(t)\,dt$

Expected Revenue: $ER = n \cdot \mathbb{E}[M(v)] = n \cdot \int_{\underline{v}}^{\bar{v}} M(v)f(v)dv$

After integration by parts: $ER = \mathbb{E}\left[\sum_i Q_i(v) \cdot \left(v_i - \frac{1-F(v_i)}{f(v_i)}\right)\right] - n \cdot U(\underline{v})$

Conclusion: Revenue depends only on $Q(\cdot)$ and $U(\underline{v})$. Same allocation + same boundary condition $\Rightarrow$ same revenue!

Revenue Equivalence: Verification

First-Price Auction (Uniform $[0, 1]$, n bidders):
Equilibrium bid: $b(v) = \frac{n-1}{n} v$
Expected revenue: $\mathbb{E}[\text{highest bid}] = \mathbb{E}\left[\frac{n-1}{n} v_{(n)}\right] = \frac{n-1}{n} \cdot \frac{n}{n+1} = \frac{n-1}{n+1}$
Second-Price Auction:
Expected revenue: $\mathbb{E}[v_{(n-1)}] = \frac{n-1}{n+1}$ ✓
All-Pay Auction:
Expected total payments: $n \cdot \mathbb{E}\left[\frac{n-1}{n} v^n\right] = (n-1) \cdot \frac{1}{n+1} = \frac{n-1}{n+1}$ ✓

What Revenue Equivalence Does NOT Say

Common Misconceptions:

1. "All auctions are the same" — False
   - Revenue equivalent $\neq$ strategically equivalent
   - Bidding strategies differ dramatically
2. "Revenue always equals $\mathbb{E}[v_{(n-1)}]$" — False
   - Only for efficient mechanisms with $U(\underline{v}) = 0$
   - Reserve prices change this
3. "Real auctions satisfy revenue equivalence" — Rarely
   - Risk aversion breaks it
   - Common values break it
   - Asymmetry breaks it
   - Entry effects break it

Optimal Reserve Prices

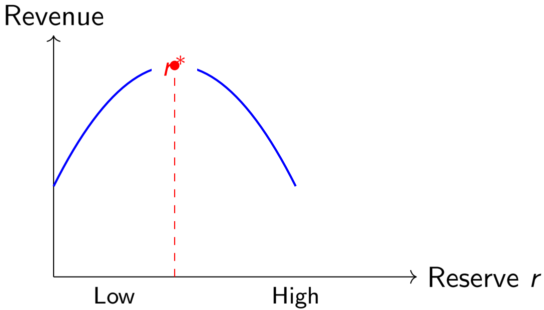
Why Set a Reserve Price?

The Seller's Dilemma:

- Low reserve: More likely to sell, but possibly at low price
- High reserve: Higher conditional price, but risk no sale

Key Insight: Reserve price acts like an additional "phantom bidder"

Trade-off:



Optimal Reserve Price Formula

Theorem (Myerson 1981)

The optimal reserve price r^* satisfies:

$$r^* - \frac{1 - F(r^*)}{f(r^*)} = v_s$$

where v_s is the seller's valuation for the item.

If seller values item at $v_s = 0$:

$$r^* = \frac{1 - F(r^*)}{f(r^*)}$$

Interpretation: Set reserve where "virtual valuation" equals seller's value
Virtual Valuation: $\psi(v) = v - \frac{1-F(v)}{f(v)}$
Reserve screens out bidders with negative virtual valuations

Optimal Reserve: Uniform Example

Setup: $v \sim U[0,1]$, seller value $v_s = 0$

Derivation

$F(v) = v$, $f(v) = 1$

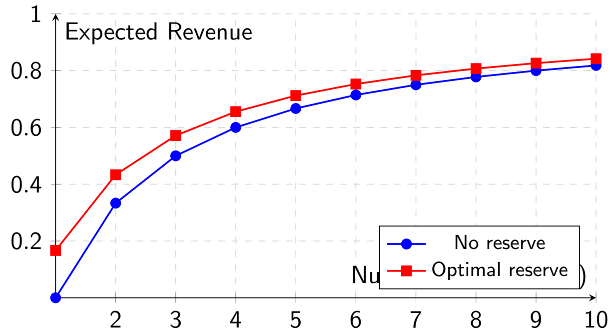
Optimal reserve condition: $r^* = \frac{1-r^*}{1} = 1 - r^*$

Solving: $2r^* = 1 \Rightarrow \boxed{r^* = \dfrac{1}{2}}$

Key Results:

- Optimal reserve is **independent of** n!
- With 2 bidders: $r^* = 0.5$
- With 100 bidders: $r^* = 0.5$
- More bidders $\Rightarrow$ reserve less likely to bind

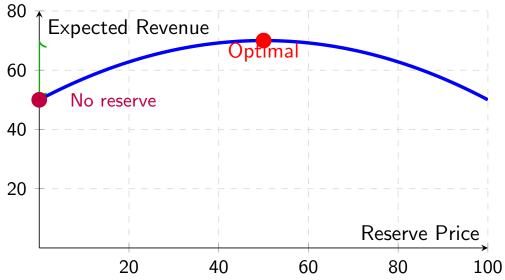
Reserve Price: Revenue Impact



Insight: Reserve price benefit largest with few bidders

Why Reserve Prices Matter

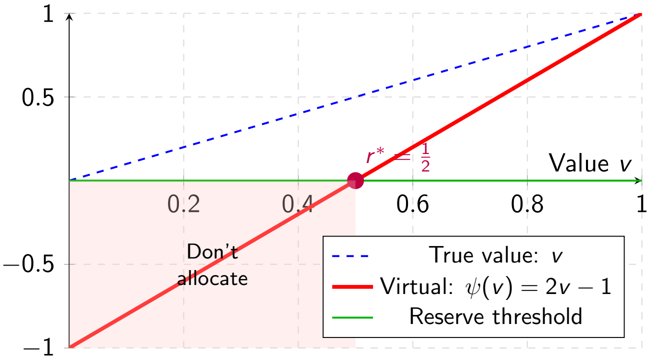
Reserve Price = Minimum acceptable bid



Why They Work:

- Forces bidders to compete more aggressively
- Protects against low-ball bids
- Trade-off: Higher revenue vs. probability of no sale
- New Zealand spectrum disaster: no reserve = $1 winning bid!

Virtual Valuation Illustration



Key: Allocate only when $\psi(v) \geq 0$ (or $\geq v_s$ if seller has value)

Reserve Price: Practical Considerations

When to Use High Reserve:

- Few bidders expected
- Seller has high value for keeping item
- Can re-auction later if no sale

When to Use Low/No Reserve:

- Many bidders expected
- Perishable goods (time-sensitive)
- Reputation concerns (commitment to sell)

Secret vs. Public Reserve:

- Public reserve: Transparent, builds trust
- Secret reserve: Can adjust based on bidding
- Empirically: Public reserves often perform better

Entry and Participation

The Entry Problem

Standard Theory Assumes: Fixed number of bidders n
Reality: Bidders choose whether to participate
Entry Decision:

- Entry cost: $c > 0$ (time, preparation, due diligence)
- Expected profit from participating: $\mathbb{E}[\pi|\text{enter}]$
- Enter if: $\mathbb{E}[\pi|\text{enter}] \geq c$

Implications:

- Entry is endogenous
- Auction format affects entry
- Revenue equivalence may break down

Entry: Free Entry Equilibrium

Free Entry Condition:

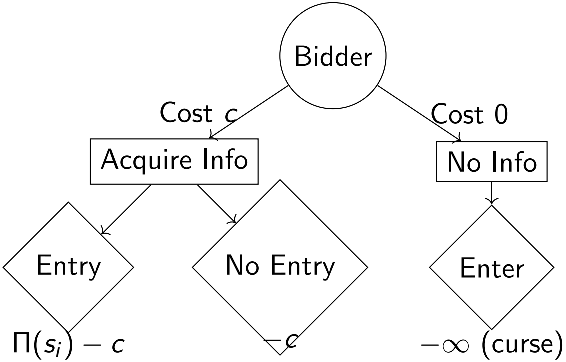
$$\mathbb{E}[\text{profit}|n \text{ entrants}] = c$$

With Symmetric IPV:
For first-price auction with $v \sim U[0,1]$:

$$\mathbb{E}[\pi] = \int_0^1 \frac{v^n}{n} dv = \frac{1}{n(n+1)}$$

Free entry: $\frac{1}{n^*(n^*+1)} = c \Rightarrow n^* \approx \frac{1}{\sqrt{c}}$
Key Result: Higher entry cost $\Rightarrow$ fewer bidders $\Rightarrow$ lower revenue

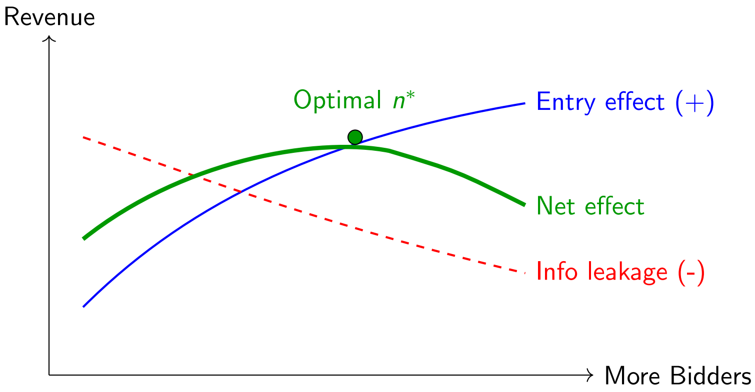
Entry and Information Acquisition



Equilibrium Effects:

- Information acquisition creates entry barrier; Uninformed bidders face winner's curse
- Can lead to thin markets/breakdown; English auctions partially mitigate through revelation

Entry vs. Information Effects



Trade-off:

- More bidders → more competition → higher prices
- More bidders → more information revealed → winner's curse

Auction Format and Entry

	Low Entry Cost	**High Entry Cost**
IPV	Revenue Equivalence holds approximately Format: Flexible	Entry matters more First-price may dominate Format: First-price
CV	English reveals info Reduces winner's curse Format: English	Complex trade-offs Context-dependent Format: Depends

Optimal Auction Design

The Optimal Auction Problem

Seller's Goal: Design mechanism to maximize expected revenue

Constraints:

- Incentive Compatibility (IC): Truth-telling is optimal
- Individual Rationality (IR): Bidders willing to participate

Myerson's Approach:

1. Use Revelation Principle: Focus on direct mechanisms
2. Characterize IC constraints
3. Optimize over feasible mechanisms

Virtual Valuations

Definition (Virtual Valuation)

$$\psi(v) = v - \frac{1 - F(v)}{f(v)}$$

Interpretation:

- v = bidder's actual value
- $\frac{1-F(v)}{f(v)}$ = information rent given to higher types
- $\psi(v)$ = marginal revenue from serving type v

Regularity Condition: $\psi(v)$ increasing in v

Key Result

Expected revenue $= \mathbb{E}[\psi(v) \cdot 1_{\text{win}}]$

To maximize revenue: allocate to bidder with highest $\psi(v_i) \geq 0$

Myerson's Optimal Auction

Theorem (Myerson 1981)

The optimal auction allocates to the bidder with highest virtual valuation, provided it exceeds zero.

Implementation:

1. Compute virtual valuations $\psi_i(v_i)$ for all bidders
2. Allocate to $i^* = \arg\max_i \psi_i(v_i)$ if $\psi_{i^*} \geq 0$
3. Charge payment that makes truth-telling optimal

For Symmetric Regular Distributions:

- Optimal auction = Second-price with optimal reserve r^*
- Reserve satisfies $\psi(r^*) = 0$

Optimal Auction: Uniform Example

Setup: $n = 2$ bidders, $v_i \sim U[0, 1]$
Virtual Valuation: $\psi(v) = v - \frac{1-v}{1} = 2v - 1$
Optimal Reserve: $\psi(r^*) = 0 \Rightarrow 2r^* - 1 = 0 \Rightarrow r^* = \frac{1}{2}$

Revenue Comparison

Without reserve: $ER = \mathbb{E}[v_{(1)}] = \frac{1}{3}$
With optimal reserve $r^* = 0.5$: $ER = \frac{1}{3} + \frac{1}{2} \cdot P(\text{both} < 0.5) \cdot 0.5 = \frac{1}{3} + \frac{1}{8} = \frac{11}{24} \approx 0.458$
Revenue gain: $\frac{11/24 - 1/3}{1/3} = 37.5\%$

Beyond Standard Optimal Auctions

Asymmetric Bidders:

- Different distributions F_i for each bidder
- Optimal auction may favor "weaker" bidders
- Discriminatory reserve prices possible

Risk-Averse Bidders:

- First-price $>$ second-price revenue
- Optimal auction exploits risk aversion

Correlated Values:

- Crémer-McLean mechanism can extract full surplus
- Relies on ability to correlate payments with others' reports
- Fragile to collusion and model misspecification

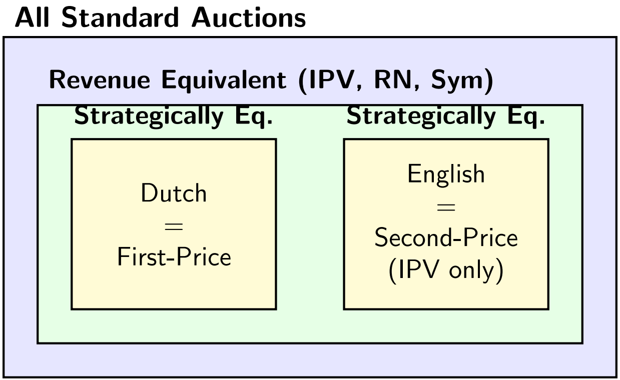
Strategic Equivalences

Strategic vs. Revenue Equivalence

Auction Pair	Strategic Eq.	Revenue Eq.
English vs. Second-Price	IPV only	Yes
Dutch vs. First-Price	Always	Yes
First-Price vs. Second-Price	No	IPV + RN
English vs. Dutch	No	IPV + RN
All-Pay vs. First-Price	No	IPV + RN

Strategic Equivalence: Same strategies in equilibrium
Revenue Equivalence: Same expected revenue (weaker)

The Equivalence Hierarchy



Summary: Core Results

Revenue Equivalence:

- Under IPV + RN + Symmetry: All standard auctions yield same revenue
- Revenue $= \mathbb{E}[v_{(n-1)}]$ with efficient allocation

Optimal Reserve:

- Set where virtual valuation = seller's value
- Independent of number of bidders
- Uniform $[0, 1]$: $r^* = 0.5$

Optimal Auction:

- Allocate to highest virtual valuation ≥ 0
- For symmetric regular: Second-price + optimal reserve

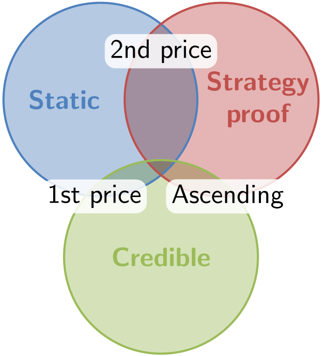
Entry:

- Endogenous participation affects format choice
- Trade-off between competition and information effects

Relaxing Assumptions

The Auction Trilemma: You Can't Have It All

The Three Desirable Properties



- **Static:** No time dimension, all decisions simultaneous
- **Strategy-proof:** Bidding true value is optimal
- **Credible:** Auctioneer cannot manipulate for higher revenue

Relaxing Risk Neutrality

The Assumption: Bidders maximize expected monetary value

Reality: Most bidders are risk-averse

- Utility $U(x)$ is concave: $U''(x) < 0$
- Prefer certain \$50 over 50% chance of \$100

Impact on Standard Formats:

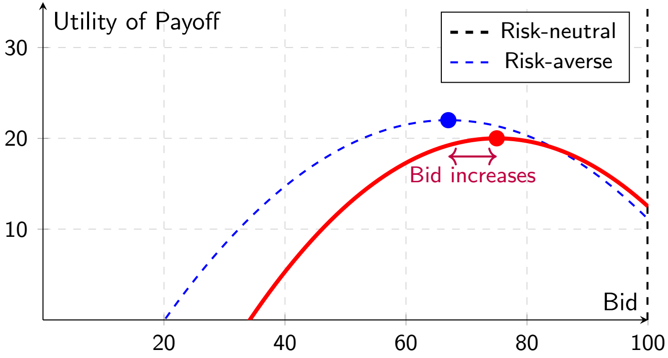
English & Second-Price: No impact!

- Bidding true value still dominant
- Risk preferences irrelevant
- Revenue unchanged

First-Price & Dutch: Higher bids!

- Risk-averse bidders bid closer to true value
- Trade certainty of winning for lower profit
- **Higher revenue for seller**
- Efficiency preserved

Risk Aversion: Why Bid Higher in First-Price?



Key Insight: Risk aversion shifts optimal bid higher

- Values certainty of winning
- Willing to sacrifice profit to increase win probability

Risk Aversion: Revenue Rankings

Revenue Equivalence Breaks Down!

With Risk-Neutral Bidders:

$$\text{First-Price Revenue} = \text{Second-Price Revenue} = \text{English Revenue}$$

With Risk-Averse Bidders:

$$\textbf{First-Price Revenue} > \textbf{Second-Price Revenue} = \textbf{English Revenue}$$

Why:

- Second-price/English: Bidding true value still optimal
- First-price: Risk aversion increases bids
- **Practical implication:** Use first-price when bidders risk-averse

Efficiency:

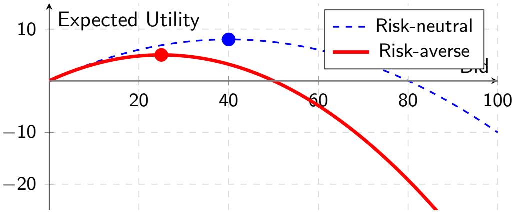
- **Preserved in all formats!**
- Highest-value bidder still wins

All-Pay Auctions with Risk Aversion

The Dangerous Combination: Risk aversion + pay even if you lose

Strategic Effects:

- Risk-averse bidders *extremely* conservative in all-pay auctions
- Two sources of risk:
  1. Risk of losing and paying your bid (standard all-pay risk)
  2. Risk aversion amplifies the pain of losing



Risk aversion + all-pay = very low bidding and participation

Asymmetric Bidders

The Assumption: All bidders from same distribution $F(v)$

Reality: Different bidder types

- Strong bidders: Values from $F_S(v)$
- Weak bidders: Values from $F_W(v)$

Impact:

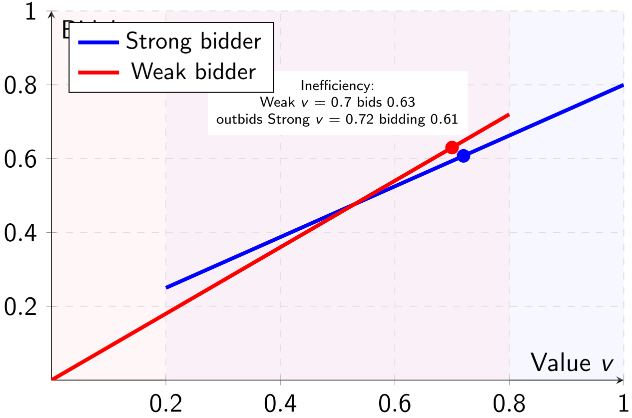
English & Second-Price: No impact!

- Still dominant to bid true value
- Efficient allocation preserved

First-Price & Dutch: Major changes!

- Different equilibrium bid functions $B_i(v)$ per type
- **May lose efficiency**: Weak bidder with higher value may lose
- Revenue effects ambiguous
- Complex equilibrium

Asymmetric Equilibrium Example



Note: Strong: $v \sim U[0.2, 1]$, Weak: $v \sim U[0, 0.8]$

Common Values and Winner's Curse

The Assumption: Each bidder's value independent

Reality: Common Value situations

- True value V same for all
- Each has different signal/estimate s_i
- Example: Oil drilling rights

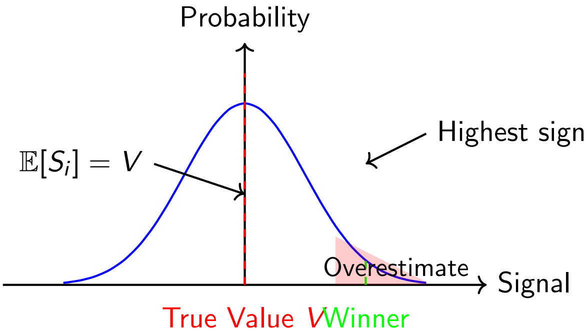
The Winner's Curse:

- Winning means you had highest estimate
- If signals unbiased: $\mathbb{E}[S_i] = V$
- But: $\mathbb{E}[V|S_i = s, \text{Win}] < s$
- Naive bidding leads to losses!

Rational Response:

- Account for adverse selection
- Bid conditional on being "just winning"
- Shade bid more with more competitors

Visualizing the Winner's Curse



Lesson: Winner has most optimistic signal, likely overestimate

Impact on Formats: Common Values

English (Ascending):

- **Advantage:** Information revelation
- Bidders update beliefs as others drop out
- Reduces winner's curse
- More efficient information aggregation

Sealed-Bid:

- **Disadvantage:** No information revelation
- Winner's curse more severe
- Aggressive bid shading
- Lower revenue

Revenue Rankings:

$$\text{English} > \text{First-Price} > \text{Second-Price}$$

Key Result: Revenue equivalence breaks down with common values

Affiliated Values

Affiliation: Random variables are affiliated if:

$$f(x \vee x') \cdot f(x \wedge x') \geq f(x) \cdot f(x')$$

Intuition:

- Higher values of one variable make high values of others more likely
- Stronger than correlation
- Common in auctions

Examples:

- Signals about common value
- Expert evaluations of artwork
- Geological surveys for oil

The Linkage Principle

Milgrom-Weber Result:
More information revelation $\Rightarrow$ Higher expected revenue
Revenue Ranking with Affiliation:

$$R_{English} \geq R_{Second-Price} \geq R_{First-Price}$$

Why:

- English: Progressive information revelation
- Second-price: Winner learns second-highest bid
- First-price: No information revelation

Policy Implication:

- Sellers should reveal information
- Public reserve better than secret reserve
- Provide inspection opportunities

Entry Costs in Auctions

Reality: Participating in auctions is costly

- Due diligence and valuation
- Legal and consulting fees
- Opportunity cost of time
- Bid preparation

Impact:

- Fewer bidders enter than total pool
- Entry decision is strategic
- Affects competition and revenue
- May want to subsidize entry

Key Questions:

1. How many bidders will enter?
2. Should seller charge/subsidize entry?
3. How does entry cost affect format choice?

Endogenous Entry Model (Levin-Smith 1994)

- Values: $v_i \sim F[0,1]$ revealed after entry
- Sequential: Entry $\rightarrow$ Learn value $\rightarrow$ Bid

Entry Decision:

- Enter if: $\mathbb{E}[\text{Surplus}|n \text{ enter}] \geq c$
- Symmetric equilibrium: n^* entrants

Equilibrium Condition:

$$\mathbb{E}_{v,n^*}[\text{Winner's surplus}]/n^* = c$$

Key Result:

$$n^* = \sqrt{\frac{\mathbb{E}[\text{Social surplus}]}{c}}$$

Comparative Statics:

- Higher $c \Rightarrow$ Fewer entrants
- Higher value dispersion $\Rightarrow$ More entry

Optimal Entry Fees

Seller's Problem: Choose entry fee f

- Higher f $\rightarrow$ More revenue per entrant
- Higher f $\rightarrow$ Fewer entrants $\rightarrow$ Less competition

Result: Optimal entry fee can be:

1. **Positive (entry fee):** Extract some rent upfront
2. **Negative (subsidy):** Encourage competition
3. Depends on c (entry cost) and F (value distribution)

Intuition:

- If c small: Competition abundant, charge entry fee
- If c large: Competition scarce, subsidize entry
- Optimal f balances direct revenue vs. competition

Example:

- Google Ads: Free entry (want competition)
- IPOs: Underwriters compete, issuer chooses

Entry Costs: First-Price vs. Second-Price

Surprising Result (Levin-Smith 1994):
With entry costs:

$$\text{Revenue}_{First} > \text{Revenue}_{Second}$$

Why?

- Second-price: Winner pays less, but this is anticipated
- More bidders enter second-price auction
- But: Incremental entrants have low values
- First-price: Fewer entrants, but pay more conditional on winning
- Net effect: First-price can generate higher revenue!

Intuition:

- Revenue equivalence breaks down
- Entry is endogenous, not exogenous
- Format affects who enters, not just how they bid

Implication: When entry costly, first-price may dominate

Numerous Items

Multi-Unit Auction Formats

Setting: K identical units, n bidders

1. **Discriminatory (Pay-as-Bid):**
   - Submit bid schedule for quantities
   - Winners pay own bids
   - Strategic bid shading on all units
2. **Uniform Price:**
   - All winners pay same market-clearing price
   - Demand reduction problem (strategic underbidding)
   - Used in electricity markets, IPOs
3. **Vickrey (Generalized):**
   - Each pays opportunity cost imposed on others
   - Truthful bidding is dominant strategy
   - Complex, rarely used

Demand Reduction Problem

Issue in Uniform Price Auctions: Bidding on marginal unit affects price for ALL units won
Example: 2 units, 2 bidders

- Bidder 1: $v_{11} = 10, v_{12} = 8$
- Bidder 2: $v_{21} = 9, v_{22} = 7$
- Efficient: Each gets one unit

But in equilibrium:

- Bidder 2 may bid: $b_{21} = 9, b_{22} < 7$
- Reduces demand to lower price on first unit
- May result in inefficient allocation

Consequences:

- Inefficient allocation
- Lower revenue than discriminatory auction possible
- Facilitates tacit collusion

The Position Auction Problem: Google Ads Setting

- Multiple ad slots on search results page
- Slots differ in visibility (click-through rate)
- Advertisers compete for slots (per-click bids)
- Payment: pay per click (not per impression)

Ad 1 CTR $= 0.10$

Ad 2 CTR $= 0.07$

Ad 3 CTR $= 0.04$

Organic results

CTR decreases with lower positions

Key Feature: Click-through rate (CTR) decreases with position

Position Auction Model

Setup:

- k positions (slots) with CTRs: $\alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$
- n advertisers with values per click: $v_1, v_2, \ldots, v_n$
- Advertisers submit bids: $b_1, b_2, \ldots, b_n$

Allocation Rule:

- Rank advertisers by bid: $b_{(1)} \geq b_{(2)} \geq \cdots \geq b_{(n)}$
- Assign positions by rank: highest bidder $\rightarrow$ position 1, etc.
- Top k bidders get slots

Payment Rule: Multiple possibilities

1. **First-price:** Pay own bid per click
2. **Generalized Second-Price (GSP):** Pay next-highest bid per click
3. **VCG:** Pay opportunity cost

Generalized Second-Price (GSP) Auction

Format (Used by Google):

- Rank by bid: Highest bidder gets top slot, etc.
- Payment: Pay **next-highest bid** per click
- Bidder in position i pays $b_{(i+1)}$ per click

Example: 3 slots, 3 bidders

Bidder	Value/click	Bid	Position
A	\$10	\$7	1 (pays \$5/click)
B	\$8	\$5	2 (pays \$3/click)
C	\$6	\$3	3 (pays \$0/click)

Expected Payments:

- A: $\alpha_1 \times 5$
- B: $\alpha_2 \times 3$
- C: $\alpha_3 \times 0$

GSP vs. Standard Second-Price

Similarity: Both charge next-highest bid
Key Difference: GSP is **not** truthful!
Why? Winning a higher position affects your payment
Example: 2 positions, CTRs $\alpha_1 = 0.1$, $\alpha_2 = 0.05$
Bidder A with $v_A = 10$:

- Others bid: $b_B = 8$, $b_C = 5$
- If A bids $b_A = 9$: Position 1, pay $8 \times 0.1 = 0.8$
- If A bids $b_A = 7$: Position 2, pay $5 \times 0.05 = 0.25$
- Utility if position 1: $10 \times 0.1 - 0.8 = 0.2$
- Utility if position 2: $10 \times 0.05 - 0.25 = 0.25$
- **Better to bid lower and get position 2!**

Implication: Bidding true value is **not** optimal

Locally Envy-Free Equilibrium

Nash Equilibrium in GSP:

Definition

A bid profile is a **locally envy-free equilibrium** if no bidder wants to move to an adjacent position at current prices.

Conditions: For bidder in position i:

$$\alpha_i(v_i - p_i) \geq \alpha_{i-1}(v_i - p_{i-1}) \quad \text{(don't want higher)}$$
$$\alpha_i(v_i - p_i) \geq \alpha_{i+1}(v_i - p_{i+1}) \quad \text{(don't want lower)}$$

where p_i is price per click for position i.

Interpretation:

- Happy with current position given prices; No incentive to deviate to adjacent slot
- May still want to jump multiple positions; Many equilibria exist in GSP!

VCG for Position Auctions

VCG Applied to Positions:

Allocation: Same as GSP - rank by value; **Intuition:** Pay for displacing others downward

Payment: Bidder i in position j pays:

$$p_i^{VCG} = \sum_{k=j+1}^{n} (v_k - v_{k+1})(\alpha_{k-1} - \alpha_k)$$

Example: 2 positions, values $v_1 = 10, v_2 = 8, v_3 = 5$

- CTRs: $\alpha_1 = 0.10, \alpha_2 = 0.05, \alpha_3 = 0$
- Bidder 1 in position 1:

$$p_1 = (8 - 5)(0.10 - 0.05) + (5 - 0)(0.05 - 0) = 0.15 + 0.25 = 0.40$$

- Bidder 2 in position 2:

$$p_2 = (5 - 0)(0.05 - 0) = 0.25$$

Property: Truthful bidding is dominant strategy!

GSP vs. VCG: Revenue Comparison

Theorem (Edelman-Ostrovsky-Schwarz 2007):
The truthful VCG equilibrium is also a **lowest-revenue** equilibrium of GSP.
Implication: GSP can generate higher revenue than VCG!
Why?

- GSP has multiple equilibria
- Some equilibria have higher prices than VCG
- Advertisers may coordinate on high-price equilibrium

Example: With 2 positions and values $v_1 = 10, v_2 = 8$:

- VCG: Bidder 1 pays based on v_2
- GSP equilibrium: Could have $b_1 = 9, b_2 = 7$
- GSP payment higher than VCG

Google's Choice: Uses GSP, not VCG

- Simpler to explain; Higher revenue in practice
- Price discovery through dynamics

Quality Scores in Practice

Reality: Google doesn't just rank by bid!

Quality Score: Combines multiple factors

- Expected click-through rate (CTR); Ad relevance to query
- Landing page quality; Historical account performance

Effective Ranking: By $b_i \times q_i$ (bid $\times$ quality)

Benefits:

1. **User experience:** Show relevant ads
2. **Platform revenue:** More clicks = more revenue
3. **Efficiency:** High-value ads get prominent positions
4. **Incentives:** Encourage advertisers to improve quality

Trade-off:

- More complex mechanism; Quality scores may be manipulated
- Less transparent to advertisers

Position Auctions: Key Takeaways

GSP (Generalized Second-Price):

- Not truthful - strategic bidding required
- Multiple Nash equilibria
- Can generate higher revenue than VCG
- Simple and practical
- Used by Google, Bing, etc.

VCG for Positions: (Not used in practice)

- Truthful - dominant strategy
- Unique equilibrium
- Lower revenue than GSP equilibria
- More complex payments

Practical Considerations:

- Quality scores improve outcomes; Price discovery through repeated auctions
- Learning and dynamics matter; Simplicity valued over theoretical optimality

Position Auctions: Mathematical Details

$$\text{Surplus}_i = \alpha_j\,(v_i - p_j), \text{where } \alpha_j \text{ is CTR and } p_j \text{ is price/click}$$

$$\begin{aligned}
\text{GSP equilibrium (bidder } i \text{ in pos. } j) : \quad & \alpha_j(v_i - b_{j+1}) \geq \alpha_{j-1}(v_i - b_j), \\
& \alpha_j(v_i - b_{j+1}) \geq \alpha_{j+1}(v_i - b_{j+2}) \\
\Longrightarrow \quad & b_{j+1} \leq v_i - \frac{\alpha_{j-1}}{\alpha_j}\,(v_i - b_j), \\
& b_{j+1} \geq v_i - \frac{\alpha_{j+1}}{\alpha_j}\,(v_i - b_{j+2})
\end{aligned}$$

Remark: Multiple bid vectors can satisfy these inequalities (many GSP equilibria).

Other Position auction applications

1. **Sponsored Search:**
   - Google Ads, Bing Ads
   - \$200+ billion annually
2. **Social Media:**
   - Facebook/Instagram sponsored posts; LinkedIn promoted content
   - Position = visibility in feed
3. **E-commerce:**
   - Amazon sponsored products; eBay promoted listings
   - Position in search results
4. **Video Platforms:**
   - YouTube ads; Pre-roll, mid-roll positions
   - Different values by position

Common Theme: Multiple "slots" with decreasing value

Position Auctions: Summary Table

Feature	GSP	VCG
Allocation	Rank by bid	Rank by value
Payment	Next-highest bid	Opportunity cost
Truthful	No	Yes
Equilibria	Multiple	Unique
Revenue	Higher	Lower
Complexity	Simple	Complex
Used by Google	Yes	No
Efficiency	Yes (in equilibrium)	Yes
Best for:	Revenue & Simplicity	Truth-telling

Industry Standard: GSP with quality scores

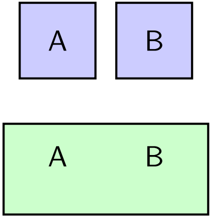
- Balances revenue, efficiency, and user experience
- Proven at massive scale (billions of auctions daily)
- Theory provides insights but practice differs

The Package Bidding Problem

Motivation: Often bidders want combinations of items, not individual items

Examples:

- **Spectrum licenses:** Adjacent frequencies worth more together
- **Airport slots:** Need landing + takeoff slot
- **Supply chains:** Raw materials + manufacturing capacity
- **Transportation:** Multiple routes forming network



Individual values: $v(A) = 5$, $v(B) = 5$

Package value: $v(A, B) = 15$ (synergy!)

Key Issue: $v(A \cup B) > v(A) + v(B)$ (complementarity)

Complementarity vs. Substitutability

Complementarities:
Value of bundle exceeds sum of parts

$$v(S \cup T) > v(S) + v(T)$$

- Left shoe + Right shoe
- Adjacent spectrum licenses
- Hub airport + spoke airports

Substitutes:
Value of bundle less than sum of parts

$$v(S \cup T) < v(S) + v(T)$$

- Two cars (only need one)
- Competing supply contracts
- Alternative shipping routes

Challenge: With m items there are 2^m possible bundles — e.g. 10 items = 1,024; 20 items = 1,048,576. Cannot simply ask for all values.

Vickrey-Clarke-Groves: Efficient Truthful Bidding

Definition

The **VCG mechanism** (direct) asks bidders to report $v_i(S)$ for every bundle and then:

$$S^* = \arg\max_{\{S_i\}} \sum_i v_i(S_i) \quad \text{s.t. } S_i \cap S_j = \varnothing.$$

Each winner pays the opportunity cost imposed on others:

$$p_i = \max_{S_{-i}} \sum_{j \neq i} v_j(S_j) - \sum_{j \neq i} v_j(S_j^*),$$

i.e., the value others would obtain without bidder i.

Payment Interpretation: Pay your **opportunity cost** imposed on others

Key Property: Truthful bidding is dominant strategy!

VCG Example

Setup: 2 items $\{A, B\}$, 2 bidders

Bidder	$v(\{A\})$	$v(\{B\})$	$v(\{A, B\})$
1	5	5	15
2	8	3	10

Step 1: Efficient Allocation

- **Optimal:** Give both to 1, Value $= 15$
- Give A to 2, B to 1, Value $= 8 + 5 = 13$
- Give both to 2 Value $= 10$

Step 2: VCG Payments

- Without 1: Best allocation is both to 2, value $= 10$
- With 1: Others (bidder 2) get nothing, value $= 0$
- **Bidder 1 pays:** $10 - 0 = 10$; **Bidder 2 pays:** 0 (didn't win)

Outcome: Bidder 1 gets $\{A, B\}$, surplus $= 15 - 10 = 5$

VCG, Why Truthful, proof sketch

$$
\begin{aligned}
\text{Bidder } v_i\text{'s utility if reports } \hat{v}_i: \quad & U_i = v_i(S_i^*) - p_i \\
\text{where } S_i^* \text{ is what } i \text{ gets, and} \quad & p_i = \max_{S_{-i}} \sum_{j \neq i} v_j(S_j) - \sum_{j \neq i} v_j(S_j^*) \\
\text{Substituting:} \quad & U_i = v_i(S_i^*) - \max_{S_{-i}} \sum_{j \neq i} v_j(S_j) + \sum_{j \neq i} v_j(S_j^*) \\
& = \underbrace{\left[v_i(S_i^*) + \sum_{j \neq i} v_j(S_j^*) \right]}_{\text{Total value with } i} - \underbrace{\max_{S_{-i}} \sum_{j \neq i} v_j(S_j)}_{\text{Constant (doesn't depend on } \hat{v}_i)} .
\end{aligned}
$$

Key: Maximizing U_i is same as maximizing total value! Truth-telling achieves this maximum. $\square$

VCG: Problems in Practice

Advantages:

- Efficient allocation; Dominant strategy incentive compatible; Individually rational

Problems:

1. **Computational complexity:** NP-hard to find optimal allocation
    - With m items, need to solve over 2^m bundles
    - Approximations may not preserve incentive properties
2. **Low revenue:** Can be arbitrarily low
    - Revenue can be 0 even when values high!
    - Not revenue maximizing
3. **Budget deficit:** $\sum_i p_i < 0$ possible
    - May need to subsidize losing bidders
4. **Vulnerable to collusion:** False-name bids
    - Single bidder submits multiple identities

Practical use: Limited to small-scale applications

Simultaneous Ascending Auction (SAA)

FCC Spectrum Auction Design:

Format:

- Multiple rounds
- Each round: Bidders submit bids on individual items
- Prices increase based on demand
- Bidders can assemble packages dynamically

Activity Rules:

- Must bid on certain # of items each round
- "Use it or lose it" - maintains bidding rights
- Prevents gaming and waiting

Stopping Rule:

- Auction ends when no new bids in a round
- Winners are current high bidders
- Pay own bids (first-price)

SAA: Advantages and Disadvantages

- **Price discovery:** bidders learn values through the process
- **Simple:** bid on individual items (computationally light)
- **Flexible:** can assemble packages dynamically
- **Proven:** raised \$100+ billion for US government

- **Exposure problem:** risk of winning only some
    - e.g. bid on A+B, win A only (worth much less)
- **Not truthful:** strategic bidding required
- **May be inefficient:** due to exposure and strategic play
- **Threshold problem:** small bidders struggle

Example (Exposure)

Want {A,B} valued at 15; items worth 2 each individually. If you win A but not B you get value 2 — bidders often shade bids to avoid this risk.

Combinatorial Clock Auction (UK 4G Auction, 2013)

Phase 1: Clock Phase

- Auctioneer announces prices for each item; Bidders indicate desired quantities at current prices
- Prices increase for over-demanded items; Continue until supply = demand

Phase 2: Supplementary Bids

- Bidders submit additional package bids; Can bid on packages not available in clock phase
- Subject to activity constraints

Phase 3: Winner Determination

- Find allocation maximizing total value; Use all bids from both phases
- Solve optimization problem

Phase 4: Pricing (Core Selecting)

- Set prices in the "core" (coalition-proof); Minimize revenue subject to core constraints

The Core in Combinatorial Auctions

Definition

A set of prices is in the **core** if no coalition of bidders can profitably deviate by forming their own allocation.

Formally: Prices $(p_1, \ldots, p_n)$ are in core if:

$$\sum_{i \in C} (v_i(S_i^*) - p_i) \geq \max_T \sum_{i \in C} v_i(T_i) \quad \forall \text{ coalitions } C$$

Interpretation:

- Winners cannot profitably block allocation and losers cannot outbid winners collectively, stability

Core-Selecting Auctions:

- Choose prices in core that maximize revenue; More revenue than VCG (typically)
- Still efficient allocation; Used in modern spectrum auctions

Combinatorial Auctions: Summary

Property	VCG	SAA	CCA
Efficient	Yes	No	Yes
Truthful	Yes	No	No
Revenue	Low	High	High
Computational	Hard	Easy	Hard
Exposure Risk	No	Yes	No
Price Discovery	No	Yes	Some
Used in Practice	Rare	Often	Growing

Trade-offs:

- Efficiency vs. Simplicity
- Truth-telling vs. Revenue
- Theory vs. Practice

Current Trend: Hybrid designs (CCA) combining advantages

Collusion in Auctions

Definition: Agreement among bidders to coordinate bids

Why It Matters:

- Destroys competition
- Reduces seller revenue dramatically
- Illegal in most jurisdictions
- Difficult to detect and prevent

Famous Examples:

- US school milk contracts (1980s-90s)
- Government procurement worldwide
- Art auctions (dealer rings)
- Spectrum auctions

Key Question: How do bidders sustain collusion?

- Who should win?;How to split gains?;How to enforce agreement?

Ring Formation

Cartel ("Ring") Problem:

1. Decide who bids in main auction
2. Allocate object internally
3. Distribute collusive surplus

Simple Model (Graham-Marshall 1987):

- n ring members with values $v_1, \ldots, v_n$
- Designate bidder with highest value: $v_{(1)}$
- Only $v_{(1)}$ bids seriously in main auction
- Others submit low bids
- Winner pays approximately seller's reserve

Surplus to Split:

$$S = (v_{(1)} - r) - (v_{(1)} - v_{(2)}) = v_{(2)} - r$$

where r is winning price (reserve or competitive bid from outside)

Paradox: Ring's surplus doesn't depend on $v_{(1)}$!

Knockout Auctions

How to Allocate Internally:

- Pre-auction: Ring members hold "knockout" auction
- Winner of knockout bids in main auction
- Losers get side payments

Common Formats:

1. **First-price knockout:**
   - Ring members bid
   - Highest bidder wins right to bid in main auction
   - Pays bid to other ring members (split equally)
2. **Second-price knockout:**
   - Highest bidder wins
   - Pays second-highest bid to ring

Key Insight: Knockout auction induces truthful revelation among ring members!

Evidence: Found in antique auctions (Phillips report), procurement contracts

Collusion Stability

Challenges to Maintaining Cartel:

1. **Incentive to Deviate:**
    - Member might bid seriously to win directly
    - Trade-off: Cheat gain vs. future collusion value
2. **Entry of Outsiders:**
    - New bidders break collusion
    - Need to identify ring members (hard in sealed-bid)
3. **Enforcement:**
    - Illegal agreements unenforceable in court
    - Rely on repeated game punishments
    - Need to detect deviations

Conditions for Stable Collusion:

- Small number of bidders (easier coordination); Frequent interactions (repeated game)
- Observable bids (detect deviations); Symmetric bidders (easier to agree)

Optimal Auctions with Collusion

Seller's Response to Collusion Risk:

Marshall-Marx (2007) Results:

1. **Ascending auctions facilitate collusion**
    - Information revelation helps coordination
    - Easy to detect deviations
2. **Sealed-bid auctions hinder collusion**
    - No information revelation; Hard to detect deviations
    - Members uncertain about others' values
3. **Optimal response:**
    - Use sealed-bid formats; Set high reserve prices
    - Encourage entry of new bidders; Randomize auction procedures

Revenue Ranking with Collusion Risk:

$$\text{Sealed-bid} > \text{English (open ascending)}$$

Reverses standard ranking!

Applications

Google Ads Auction

Generalized Second-Price (GSP):

- Multiple ad slots (positions)
- Bidders submit bids per click
- Allocation: Highest bids get top positions
- Payment: Pay next-highest bid

Key Features:

- Billions of auctions daily
- Quality score adjusts effective bids
- Not fully truthful (strategic bidding remains)
- Revenue: \$200+ billion annually

Design Challenge:

- Balance advertiser incentives with user experience
- Account for click-through rate differences

Spectrum Auctions

FCC Spectrum Auctions:

- Multiple licenses (geography $\times$ frequency)
- Package bidding (combinatorial)
- Complex complementarity and substitution
- Stakes: \$100+ billion raised

Design Evolution:

- 1994: Simultaneous ascending auction
- 2008: Package bidding introduced
- 2017: Incentive auction (buy back + resell)

Lessons:

- Simple ascending works well for many items
- Activity rules prevent gaming
- Information revelation crucial

Treasury Auctions

US Treasury Securities:

- Multiple units (billions of dollars)
- Discriminatory for bills
- Uniform for notes and bonds
- Annual volume: \$12+ trillion (2023)

Why Different Formats?

- Bills: Short-term, simple valuation
- Notes/Bonds: Longer-term, more uncertainty
- Uniform reduces winner's curse for longer maturities

Key Features:

- Non-competitive bids (small investors)
- Competitive bids (large institutions)
- Predictable schedule builds liquidity

Online Marketplaces: eBay vs. Amazon

eBay Model	Amazon Model
English auction	Fixed price
Time-limited bidding	Immediate purchase
Price discovery	Price transparency
Competition visible	Competition hidden
Good for unique items	Good for commodities

Trend: Even eBay emphasizes "Buy It Now" for speed
Lesson: Choose mechanism based on:

- Product uniqueness
- Buyer urgency
- Transaction costs

Reverse Auctions for Procurement

Format: Seller runs auction, buyers bid to supply

Best Practices:

- **Qualify suppliers first:** Don't sacrifice quality for price
- **Clear specifications:** Avoid apples-to-oranges
- **Multiple rounds:** Give suppliers chance to compete
- **Reserve your right:** Don't have to accept lowest bid

Common Mistakes:

- Running with only 2-3 suppliers
- Poorly defined requirements
- Focusing only on price, ignoring total cost
- Burning supplier relationships

When to Avoid:

- Strategic partnerships; Highly customized products
- When you need innovation, not just low price

Building a Marketplace: Two-Sided Markets

Your Platform Connects Buyers and Sellers

Examples: eBay, Airbnb, Uber, AWS

Key Challenges:

- **Chicken-and-egg:** Need both sides
- **Pricing:** Who pays? How much?
- **Quality:** How to maintain standards?

Auction Design Helps:

- **Price discovery:** Market finds right price
- **Matching:** Right buyers find right sellers
- **Efficiency:** Resources to highest value use

Critical Decision: What's your objective?

- Transaction volume? Revenue? Efficiency? Quality?

Marketplace Design: Strategic Choices

Choice	**Option A**	**Option B**
Pricing	Commission	Subscription/fees
Competition	Winner-take-all	Multiple winners
Transparency	Show all bids	Hide info
Speed	Real-time	Batch at intervals
Commitment	Binding immediately	Can cancel

When Should You Use Auction Design?

Good Fit When:

- Selling/buying scarce resources
- Value varies significantly across bidders
- You need price discovery
- You want competition to drive value
- You need transparent, defensible process

Red Flags:

- Few potential bidders (collusion risk)
- High complexity (hard to participate)
- Unclear valuation (creates uncertainty)
- High transaction costs

Decision Framework: 5 Key Questions

Before designing an auction:

1. **What are you selling?**
   - Unique → English auction
   - Commodity → Sealed-bid
2. **Who are your bidders?**
   - Few sophisticated → Watch collusion
   - Many casual → Keep simple
3. **What's your goal?**
   - Max revenue → Set reserve
   - Efficiency → English auction
4. **Time available?**
   - Days/weeks → English
   - Minutes/hours → Sealed-bid
5. **Risk tolerance?**
   - Risk-averse → Higher reserve
   - Risk-neutral → Optimize expected value

Preventing Collusion

Warning Signs:

- Identical bids
- Rotating winners
- Intentionally low bids
- Few active bidders

Countermeasures:

- Use sealed-bid formats; Set meaningful reserves
- Limit information disclosure; Random participation rules
- Prosecute violations; Encourage new entrants

Best Practice:

- Monitor bidding patterns over time; Keep reserve prices confidential
- Use sealed formats when collusion risk high

Implementation Checklist

Before Launching:

1. **Define Success Metrics**
   - Revenue? Efficiency? Fairness? Speed?
2. **Know Your Participants**
   - Sophistication and collusion potential
3. **Set Clear Rules**
   - Reserve prices, activity requirements, payment
4. **Plan for Edge Cases**
   - No bids? Same bids? Manipulation?
5. **Test and Monitor**
   - Start small, gather data, iterate

Red Flags: When Market Design Is Failing

Warning Signs:

- **Low participation:** Few bidders showing up
- **Unusual patterns:** Everyone bids same or very high/low
- **Complaints:** Rules unclear or unfair
- **Gaming:** Evidence of manipulation
- **Instability:** Prices swing wildly
- **Inefficiency:** Wrong winners

What to Do:

- Don't ignore signals
- Gather data and analyze
- Consult participants
- Be willing to redesign
- Test changes carefully

California ignored warnings for months before crisis hit

Double Auctions

Double Auctions: Definition

Definition

A **double auction** is a mechanism where both buyers and sellers submit bids simultaneously. Trades occur when buy bids meet or exceed sell offers.

Key Difference from Standard Auctions:

- Standard: One seller, multiple buyers (or vice versa)
- Double: Multiple buyers AND multiple sellers
- Both sides strategic

Applications:

- Stock exchanges (NYSE, NASDAQ); Commodity markets; Electricity markets
- Carbon emission trading; Decentralized prediction markets

Question: How to match buyers and sellers efficiently?

Double Auction Setup

Players:

- m buyers with private valuations $v_1, \ldots, v_m$
- n sellers with private costs $c_1, \ldots, c_n$
- Values and costs drawn from distributions F_v and F_c

Actions:

- Buyer i submits bid $b_i \leq v_i$
- Seller j submits ask $a_j \geq c_j$

Efficient Allocation:

- Order bids: $b_{(1)} \geq b_{(2)} \geq \cdots \geq b_{(m)}$
- Order asks: $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$
- Trade quantity: $k^* = \max\{k : b_{(k)} \geq a_{(k)}\}$
- Gains from trade: $\sum_{i=1}^{k^*}(v_{(i)} - c_{(i)})$

The k-Double Auction Rule

Trading Rule: If buyer i and seller j are matched:

- Trade occurs if $b_i \geq a_j$
- Trade price: $p = k \cdot b_i + (1 - k) \cdot a_j$ where $k \in [0, 1]$

Special Cases:

- $k = 0$: Buyer-bid auction (buyer pays bid, seller gets bid)
- $k = 1$: Seller-ask auction (buyer pays ask, seller gets ask)
- $k = \frac{1}{2}$: Split-the-difference (average of bid and ask)

Strategic Implications:

- $k = 0$: Buyers bid like first-price auction (shade bids)
- $k = 1$: Sellers bid like first-price auction (inflate asks)
- $k = \frac{1}{2}$: Both sides shade symmetrically
- No k makes truthful bidding dominant for both sides!

General Equilibrium Derivation

$$\begin{aligned} b &\geq a(c); & \\ p &= kb + (1-k)a(c); & \text{becomes} \quad kb + (1-k)b = b; \\ U_B(b|v) &= \Pr(\text{trade}) \cdot (v - p); & \\ U_B(b|v) &= F_c(a^{-1}(b)) \cdot (v - b); & \text{Using p=b} \\ \frac{f_c(v)}{a'(v)} &\cdot [v - b(v)] = F_c(v); & b = b(v); \text{FOC and equilibrium} \\ \frac{f_v(c)}{b'(c)} &\cdot [a(c) - c] = 1 - F_v(c); & \text{symmetrically for seller} \end{aligned}$$

Result: Coupled ODEs determining $b(v)$ and $a(c)$.

Equilibrium Bidding: Uniform Distribution

Setup: One buyer with $v \sim U[0,1]$, one seller with $c \sim U[0,1]$
Equilibrium Strategies (Chatterjee-Samuelson 1983):

$$\text{Buyer bids:} \quad b(v) = kv + \frac{1-k}{2}$$

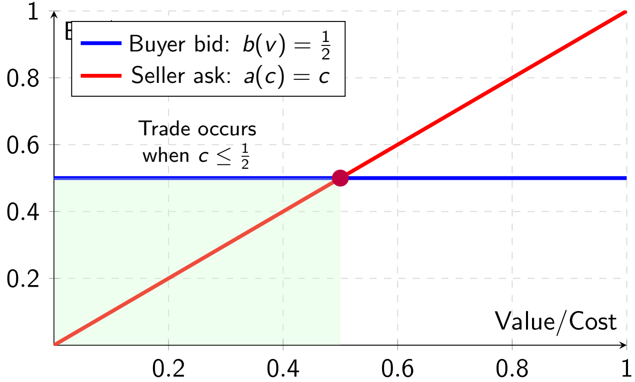
$$\text{Seller asks:} \quad a(c) = (1-k)c + \frac{k}{2}$$

Examples:

k	Buyer Bid $b(v)$	Seller Ask $a(c)$	Trade Price
0	$\frac{1}{2}$	c	$a(c) = c$
$\frac{1}{2}$	$\frac{2v+1}{4}$	$\frac{2c+1}{4}$	$\frac{v+c+1}{4}$
1	v	$\frac{1}{2}$	$b(v) = v$

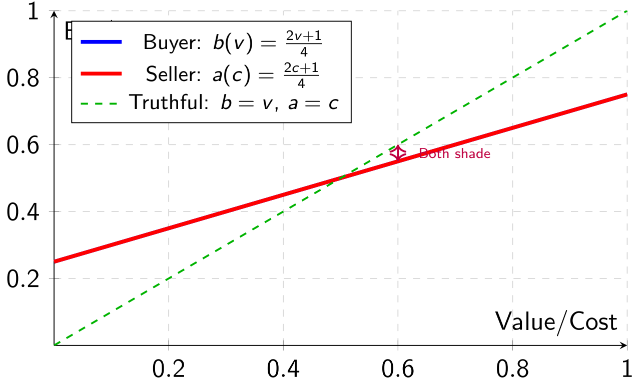
Key Insight: Both sides shade away from truthful reporting

Equilibrium Visualization: $k = 0$ (Buyer-Bid Auction)



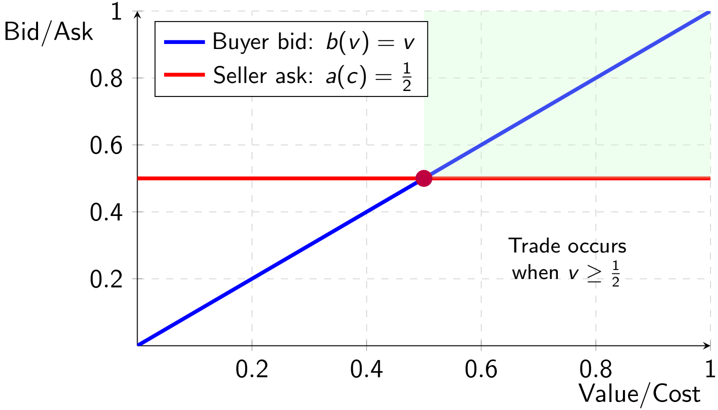
Note: Buyer always bids $\frac{1}{2}$, seller truthfully reveals cost

Equilibrium Visualization: $k = \frac{1}{2}$ (Split-the-Difference)



Note: Symmetric shading - both buyer and seller use same strategy function

Equilibrium Visualization: $k = 1$ (Seller-Ask Auction)



Note: Seller always asks $\frac{1}{2}$, buyer truthfully reveals value

Double Auction: Key Results

Theorem (Chatterjee-Samuelson 1983)

For the k-double auction with one buyer and one seller:

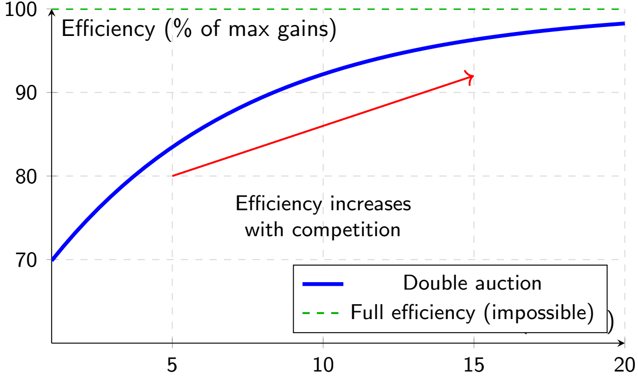
- *No $k \in [0, 1]$ achieves ex-post efficiency; $k = \frac{1}{2}$ maximizes ex-ante gains from trade*
- *Equilibrium involves strategic misrepresentation by both sides*

Myerson-Satterthwaite Impossibility (1983)

No mechanism can simultaneously achieve:

1. Ex-post efficiency (trade whenever $v > c$)
2. Individual rationality (voluntary participation)
3. Balanced budget (no external subsidies)
4. Incentive compatibility (truthful reporting)

Efficiency with Many Traders



Key Insight: With many traders (e.g., 10+ per side), double auctions achieve near-perfect efficiency despite strategic behavior (Gresik-Satterthwaite 1989)

Summary

Summary: Core Results

1. **Equilibrium Strategies:**
   - English/Vickrey: Bid true value (dominant)
   - First-Price/Dutch: Shade bid below value
   - All-Pay: Much more conservative bidding
2. **Revenue Equivalence (IPV, risk-neutral, symmetric):**
   - All standard formats yield same revenue
   - Revenue $= \mathbb{E}[v_{(2)}]$
   - Breaks with risk aversion, asymmetry, or correlation
3. **Optimal Design:**
   - Reserve price improves revenue
   - Virtual valuation determines optimal allocation
   - Information revelation increases revenue with affiliation

Summary: Bid Functions (Uniform Distribution)

Format	**Equilibrium Bid** $B(v)$	**Revenue**
English	v	$\frac{n-1}{n+1}$
Second-Price	v	$\frac{n-1}{n+1}$
First-Price	$\frac{n-1}{n}v$	$\frac{n-1}{n+1}$
Dutch	$\frac{n-1}{n}v$	$\frac{n-1}{n+1}$
All-Pay 1st	$\frac{n-1}{n}v^n$	$\frac{n-1}{n+1}$
War of Attrition	$v + (1-v)\ln(1-v)$ (for $n = 2$)	$\frac{n-1}{n+1}$

Key: Revenue Equivalence holds under benchmark assumptions

Practical Guidelines

DO

- Set a reserve price (almost always)
- Monitor for collusion patterns and use sealed bids when collusion risk high
- Keep rules simple and transparent
- Reveal information with affiliated values

DON'T

- Assume revenue equivalence always holds
- Reveal sensitive competitive information
- Use complex formats without expert help
- Ignore entry barriers and participation costs

Key Takeaways for Business

The Big Ideas:

1. **Design matters more than theory**
   - Real-world context beats textbook
2. **Auctions reveal private information**
   - Better than posted prices when values uncertain
3. **Format choice depends on context**
   - Under ideal conditions, all yield same revenue
   - Choose based on practical considerations
4. **Real markets violate assumptions**
   - Risk aversion, collusion, entry costs matter
5. **Simplicity beats sophistication**
   - Participants must understand mechanism

Additional Resources

Books:

- "Auction Theory" by Vijay Krishna (intermediate)
- "Putting Auction Theory to Work" by Paul Milgrom (advanced)

Industry Examples:

- FCC Spectrum Auctions (www.fcc.gov/auctions)
- Treasury Securities (www.treasurydirect.gov)
- Google Ads Help Center

Expert Consultants:

- For high-stakes auctions (¿$1M)
- Firms: NERA, Compass Lexecon, Analysis Group

Questions?