# On the minimal number of generators of a finite group

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# Introduction

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Can be reduced to:

- Finding the minimal number of gentators of a finite group $H$ such that $d(H/N) \leq m$ for every non-trivial normal subgroup $N$, but $d(H) > m$

# The case $m = 1$

**Theorem**

*Let $H$ be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup $N$, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime $p$.*

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## Proof.

- $H = P_1 \times \ldots \times P_n$ where $P_i$ is a Sylow $p_i$-subgroup for $1 \leq i \leq n$ and $p_1, \ldots, p_n$ are distinct primes.

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- By Theorem **??**, $\Phi(H) = 1$ hence $H = (\mathbb{Z}_{p_1})^q$ by Theorem **??**.

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**Proof.**

- $q = 2$ since

$$q - 1 = d((\mathbb{Z}_{p_1})^{q-1}) = d(H/(\mathbb{Z}_{p_1} \times 1 \times \ldots \times 1)) = 1.$$

$\square$