On the minimal number of generators of a finite group

Diogo Santos

September 24, 2024

Introduction

- Finding the minimal number of gentators of a finite group H

Introduction

- Finding the minimal number of gentators of a finite group H

Can be reduced to:

- Finding the minimal number of gentators of a finite group H such that $d(H/N) \leq m$ for every non-trivial normal subgroup N, but $d(H) > m$

The case $m = 1$

Theorem

Let H be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup N, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

The case $m = 1$

Theorem

Let H be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup N, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

Proof.

- $H = P_1 \times \ldots \times P_n$ where P_i is a Sylow p_i-subgroup for $1 \leq i \leq n$ and $p_1, \ldots, p_n$ are distinct primes.

The case $m = 1$

Theorem

Let H be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup N, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

Proof.

- $H = P_1 \times \ldots \times P_n$ where P_i is a Sylow p_i-subgroup for $1 \leq i \leq n$ and $p_1, \ldots, p_n$ are distinct primes.
- If $P_1, \ldots, P_r$ are cyclic, we obtain $H \cong \mathbb{Z}_{p_1 \ldots p_n}$ which contradicts $d(H) > 1$.Without loss of generality we can thus assume that P_1 is not cyclic.

The case $m = 1$

Theorem

Let H be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup N, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

Proof.

- $H = P_1 \times \ldots \times P_n$ where P_i is a Sylow p_i-subgroup for $1 \leq i \leq n$ and $p_1, \ldots, p_n$ are distinct primes.
- If $P_1, \ldots, P_r$ are cyclic, we obtain $H \cong \mathbb{Z}_{p_1 \ldots p_n}$ which contradicts $d(H) > 1$.Without loss of generality we can thus assume that P_1 is not cyclic.
- $n \geq 2 \implies P_1 \cong H/(1 \times P_2 \ldots \times P_n)$ and thus $d(P_1) = d(H/(1 \times P_2 \ldots \times P_n)) = 1$, contradiction.

The case $m = 1$

Theorem

Let H be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup N, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

Proof.

- $H = P_1 \times \ldots \times P_n$ where P_i is a Sylow p_i-subgroup for $1 \leq i \leq n$ and $p_1, \ldots, p_n$ are distinct primes.
- If $P_1, \ldots, P_r$ are cyclic, we obtain $H \cong \mathbb{Z}_{p_1 \ldots p_n}$ which contradicts $d(H) > 1$.Without loss of generality we can thus assume that P_1 is not cyclic.
- $n \geq 2 \implies P_1 \cong H/(1 \times P_2 \ldots \times P_n)$ and thus $d(P_1) = d(H/(1 \times P_2 \ldots \times P_n)) = 1$, contradiction.
- By Theorem **??**, $\Phi(H) = 1$ hence $H = (\mathbb{Z}_{p_1})^q$ by Theorem **??**.

The case $m = 1$

Proof.

- $q = 2$ since

$$q - 1 = d((\mathbb{Z}_{p_1})^{q-1}) = d(H/(\mathbb{Z}_{p_1} \times 1 \times \ldots \times 1)) = 1.$$

$\square$