On the minimal number of generators of a finite group

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Introduction

- Finding the minimal number of gentators of a finite group H

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Can be reduced to:

- Finding the minimal number of gentators of a finite group H such that $d(H/N) \leq m$ for every non-trivial normal subgroup N, but $d(H) > m$

The case $m = 1$

Theorem

Let H be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup N, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

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Proof.

- $H = P_1 \times \ldots \times P_n$ where P_i is a Sylow p_i-subgroup for $1 \leq i \leq n$ and $p_1, \ldots, p_n$ are distinct primes.

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- $n \geq 2 \implies P_1 \cong H/(1 \times P_2 \ldots \times P_n)$ and thus $d(P_1) = d(H/(1 \times P_2 \ldots \times P_n)) = 1$, contradiction.

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- Since $d(H) = d(H/\Phi(H))$, $\Phi(H) = 1$ hence $H = (\mathbb{Z}_{p_1})^q$.

The case $m = 1$

Proof.

- $q = 2$ since

$$q - 1 = d((\mathbb{Z}_{p_1})^{q-1}) = d(H/(\mathbb{Z}_{p_1} \times 1 \times \ldots \times 1)) = 1.$$

$\square$

The groups L_k

Definition

Given a positive integer k, the group L_k is a subgroup of L^k defined by:

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The group L_k can be described as $diag(L^k)M^k$.

General Case

Theorem

Let m be an integer with $m \geq 1$ and H a finite group such that $d(H/N) \leq m$ for every non-trivial normal subgroup N, but $d(H) > m$. Then there exists a group L which has a unique minimal normal subgroup M and is such that M is either non-abelian or complemented in L and $H \cong L_{f(L,m)}$.