On the minimal number of generators of a finite group

Diogo Santos

November 4, 2024

Introduction

- Finding the minimal number of gentators of a finite group H

Introduction

- Finding the minimal number of gentators of a finite group H

Can be reduced to:

- Finding the minimal number of gentators of a finite group H such that $d(H/N) \leq m$ for every non-trivial normal subgroup N, but $d(H) > m$

The case $m = 1$

Theorem

Let H be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup N, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

The case $m = 1$

Theorem

Let H be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup N, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

Proof.

- $H = P_1 \times \ldots \times P_n$ where P_i is a Sylow p_i-subgroup for $1 \leq i \leq n$ and $p_1, \ldots, p_n$ are distinct primes.

The case $m = 1$

Theorem

Let H be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup N, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

Proof.

- $H = P_1 \times \ldots \times P_n$ where P_i is a Sylow p_i-subgroup for $1 \leq i \leq n$ and $p_1, \ldots, p_n$ are distinct primes.
- If $P_1, \ldots, P_n$ are cyclic, we obtain $H \cong \mathbb{Z}_{p_1 \ldots p_n}$ which contradicts $d(H) > 1$.Without loss of generality we can thus assume that P_1 is not cyclic.

The case $m = 1$

Theorem

Let H be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup N, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

Proof.

- $H = P_1 \times \ldots \times P_n$ where P_i is a Sylow p_i-subgroup for $1 \leq i \leq n$ and $p_1, \ldots, p_n$ are distinct primes.
- If $P_1, \ldots, P_n$ are cyclic, we obtain $H \cong \mathbb{Z}_{p_1 \ldots p_n}$ which contradicts $d(H) > 1$.Without loss of generality we can thus assume that P_1 is not cyclic.
- $n \geq 2 \implies P_1 \cong H/(1 \times P_2 \ldots \times P_n)$ and thus $d(P_1) = d(H/(1 \times P_2 \ldots \times P_n)) = 1$, contradiction.

The case $m = 1$

Theorem

Let H be a finite nilpotent group such that $d(H/N) \leq 1$ for every non-trivial normal subgroup N, but $d(H) > 1$. Then $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

Proof.

- $H = P_1 \times \ldots \times P_n$ where P_i is a Sylow p_i-subgroup for $1 \leq i \leq n$ and $p_1, \ldots, p_n$ are distinct primes.
- If $P_1, \ldots, P_n$ are cyclic, we obtain $H \cong \mathbb{Z}_{p_1 \ldots p_n}$ which contradicts $d(H) > 1$.Without loss of generality we can thus assume that P_1 is not cyclic.
- $n \geq 2 \implies P_1 \cong H/(1 \times P_2 \ldots \times P_n)$ and thus $d(P_1) = d(H/(1 \times P_2 \ldots \times P_n)) = 1$, contradiction.
- Since $d(H) = d(H/\Phi(H))$, $\Phi(H) = 1$

The case $m = 1$

Proof.

- $H \cong H/\Phi(H)$ is a $\mathbb{Z}_{p_1}$-vector space and thus $H = (\mathbb{Z}_{p_1})^q$

$\square$

The case $m = 1$

Proof.

- $H \cong H/\Phi(H)$ is a $\mathbb{Z}_{p_1}$-vector space and thus $H = (\mathbb{Z}_{p_1})^q$
- $q = 2$ since

$$q - 1 = d((\mathbb{Z}_{p_1})^{q-1}) = d(H/(\mathbb{Z}_{p_1} \times 1 \times \ldots \times 1)) = 1.$$

$\square$

The groups L_k

Definition

Given a positive integer k, the group L_k is a subgroup of L^k defined by:

$$L_k = \{(l_1, \dots, l_k) \in L^k | l_1 M = \dots = l_k M\}.$$

The groups L_k

Definition

Given a positive integer k, the group L_k is a subgroup of L^k defined by:

$$L_k = \{(l_1, \ldots, l_k) \in L^k | l_1 M = \ldots = l_k M\}.$$

The group L_k can be described as $diag(L^k)M^k$.

Properties of L_k

Properties of L_k

- $soc(L_k) = M^k$

Properties of L_k

- $soc(L_k) = M^k$
- $L_k/M^k \cong L/M$

Properties of L_k

- $soc(L_k) = M^k$
- $L_k/M^k \cong L/M$
- If M is abelian and complemented by C in L, then M^k is complemented by $diag(C^k)$

Properties of L_k

- $soc(L_k) = M^k$
- $L_k/M^k \cong L/M$
- If M is abelian and complemented by C in L, then M^k is complemented by $diag(C^k)$
- The quotient of L_{k+1} by any of its minimal normal subgroups is isomorphic to L_k

General Case

Theorem

Let m be an integer with $m \geq 1$ and H a finite group such that $d(H/N) \leq m$ for every non-trivial normal subgroup N, but $d(H) > m$. Then there exists a group L which has a unique minimal normal subgroup M and is such that M is either non-abelian or complemented in L and $H \cong L_{f(L,m)}$.