AST 384C: Computational Astrophysics

HW 2 (due by noon central time on Wednesday, March 5)

Complete the following questions by submitting (documented) code and any accompanying answers / plots in a github repository. Email me the repository link once you've committed your solutions. Make sure to clearly document your code; when in doubt, over-explain!

The standard Kepler problem is the bound orbit of a test particle around a body of mass M. The equations are:

$$\dot{\mathbf{r}} = \mathbf{v} \ , \tag{1}$$

$$\dot{\mathbf{v}} = -\frac{GM}{r^3}\mathbf{r} \ , \tag{2}$$

where $\mathbf{r} = (x, y)$ and $\mathbf{v} = (v_x, v_y)$. Initialize the system with the particle at $(0, r_\mathrm{p})$ with velocity $(-v_\mathrm{p}, 0)$, which puts the particle moving counterclockwise with velocity v_p at pericenter r_p. Note:

$$r_\mathrm{p} = a(1-e) \ , \tag{3}$$

$$v_\mathrm{p} = \sqrt{\frac{GM}{a}}\sqrt{\frac{1+e}{1-e}} \tag{4}$$

(here $e \in [0, 1)$ is the eccentricity).
For this homework, assume the orbit is that of a planet at the Earth-Sun separation ($a = 1$ AU) around central mass with $M = 1\,M_\odot$.

- Implement a 4th-order Runge-Kutta solver with **fixed timestep** to solve the Kepler problem.

  First, apply this to a circular orbit ($e = 0$) over a 1 year period. Compare results for a timestep of 1 month (1/12 of a period), 1 week (you can approximate this as 1/48 of a period for this exercise), and 1 day (1/365.25 of a period). Relative to the initial value, how does the error in the final position depend on the timestep? How about the error in the energy?

  Now repeat the exercise for an orbit with ($e = 0.96$) over the same year period. How do your results differ?

- Next, update your fourth-order RK solver to use **adaptive timesteps**. As a reminder, we can do this by evolving from $y(t) \to y(t+\tau)$ with a fixed timestep τ (call this y_τ), then with two steps of size $\tau/2$ (call this $y_{2(\tau/2)}$), and comparing the relative error of the solution at $y(t+\tau)$:

  $$\epsilon_\mathrm{rel} = \mathrm{abs}\left(1 - \frac{y_\tau}{y_{2(\tau/2)}}\right) . \tag{5}$$

  The estimate for timestep that will give us our desired accuracy (ϵ_goal) is

  $$\tau_\mathrm{est} = \tau\left(\frac{\epsilon_\mathrm{goal}}{\epsilon_\mathrm{rel}}\right)^{1/5} \tag{6}$$

  because $\epsilon \propto \tau^5$ for fourth-order Runge-Kutta. If $\epsilon_\mathrm{rel} \leq \epsilon_\mathrm{goal}$, then we adopt the timestep, store the solution at time $t + \tau/2 + \tau/2$ (i.e., use the solution with the smaller of the two timesteps we tried in this iteration), and change to $\tau = \tau_{\text{est}}$ for our next step (increase the timestep). If $\epsilon_{\text{rel}} > \epsilon_{\text{goal}}$, we discard the solution we got for $y(t+\tau)$ and retry the integration at $y(t)$ with $\tau = \tau_{\text{est}}$ (decrease the timestep).

Apply this new time-stepping routine to the $e = 0.96$ case above, starting with an initial timestep of one month and requiring a relative error of $\epsilon_{\text{goal}} = 10^{-5}$. Plot the orbit so that we can see the individual timesteps and separately plot how the timestep evolves over the course of the orbit. How many timesteps did you have to take? How well is energy conserved? Now try for 25 orbits. How does the energy evolve? Would this be a good way to do accurate planetary orbit calculations over the age of the Solar System?