Bornö Summerschool July 2017

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July 31, 2018

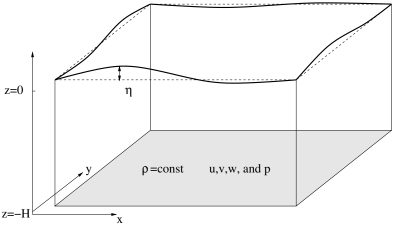
Lecture # 2

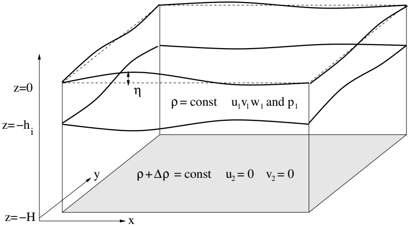
- taken from "Waves in the ocean and atmosphere", Pedlosky (2013), pages 19-27
- general wave kinematics
- exact equations, simplifications
- kinematic, dynamic boundary condition
- irrotational flow, Bernoulli equation
- wave equation, solution

- linear "barotropic" and "baroclinic" layered model for $f = 0$ in 1D

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0$$

- h is total thickness ("barotropic") or layer interface h_i ("baroclinic")
- either $g = 9.81\,\mathrm{m/s^2}$ ("barotropic") or $g \to g\Delta\rho/\rho_0$ ("baroclinic")





Exact conservation laws

- momentum equation

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}p + \boldsymbol{\nabla}\cdot\boldsymbol{\Sigma} - 2\rho\boldsymbol{\Omega}\times\boldsymbol{u} - \rho\boldsymbol{\nabla}\Phi$$

with geopotential ($\Phi = gz$), rotation vector $\boldsymbol{\Omega}$,
and material derivative $D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$

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- continuity equation

$$\frac{D\rho}{Dt} = -\rho\boldsymbol{\nabla} \cdot \boldsymbol{u}$$

- salt conservation equation, ...

$$\rho \frac{DS}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S = \boldsymbol{\nabla} \cdot \kappa_S \boldsymbol{\nabla} S$$

Exact conservation laws

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$$\rho \frac{DS}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S = \boldsymbol{\nabla} \cdot \kappa_S \boldsymbol{\nabla} S$$

- thermodynamics: equation of state with temperature

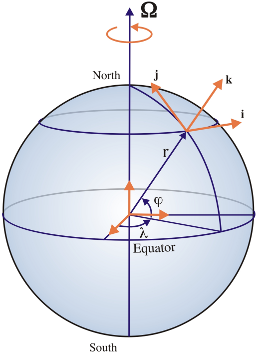
$$\rho = \rho(S, \Theta, p)$$

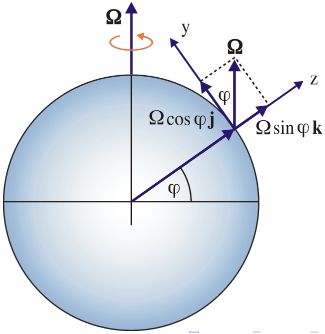
- energy equation $\rightarrow$ conservative temperature equation

$$\rho \frac{D\Theta}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_\Theta + \text{very small source term} \approx \boldsymbol{\nabla} \cdot \kappa_\Theta \boldsymbol{\nabla} \Theta$$

- momentum equation in rotating frame

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \nabla \cdot \boldsymbol{\Sigma} + \boldsymbol{f}^{v} - 2\rho \boldsymbol{\Omega} \times \boldsymbol{u} - \rho \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X})$$





- momentum equation in rotating frame

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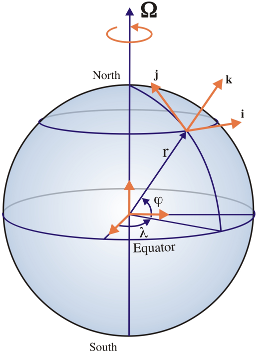
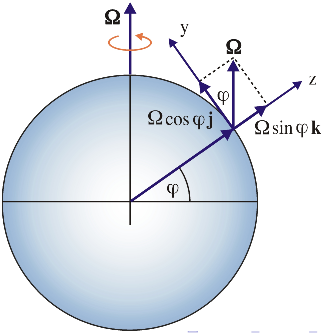
- with $\boldsymbol{\Omega} = (0, \Omega\cos\varphi, \Omega\sin\varphi)$

$$\boldsymbol{\Omega}\times\boldsymbol{u} = \Omega\begin{pmatrix}0\\ \cos\varphi\\ \sin\varphi\end{pmatrix}\times\begin{pmatrix}u\\ v\\ w\end{pmatrix} = \Omega\begin{pmatrix}w\cos\varphi - v\sin\varphi\\ u\sin\varphi\\ -u\cos\varphi\end{pmatrix}$$





- Continuity equation or conservation of mass

$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u} \;\rightarrow\; \frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})$$

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$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u} \;\rightarrow\; \frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})$$

- set $\rho = \rho_0 + \varrho\rho'$ and $p = p_0(z) + p'$ with $\partial p_0/\partial z = -\rho_0 g$ with a large mean value $\rho_0 = 1000\,\mathrm{kg/m^3}$ plus small variations with magnitude $\varrho = 10\,\mathrm{kg/m^3}$ (for water and similar for other fluids)

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- since $\rho_0/\varrho \gg 1$ it follows from scaling that

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) \;\rightarrow\; \frac{\partial \rho'}{\partial t'} = -\frac{\rho_0}{\varrho} \frac{UT}{L} \boldsymbol{\nabla}' \cdot \boldsymbol{u}' - \frac{UT}{L} \boldsymbol{\nabla}' \cdot (\rho' \boldsymbol{u}')$$

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$$\rightarrow\; \boldsymbol{\nabla} \cdot \boldsymbol{u} \approx 0 \;\; : \quad \text{Boussinesq approximation}$$

mass conservation is replaced by volume conservation

- Continuity equation or conservation of mass

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  mass conservation is replaced by volume conservation
- momentum equation simplifies to

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} \;\approx\; -\boldsymbol{\nabla} p' - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} - \varrho\rho' \boldsymbol{\nabla}\phi$$

  pressure p' and density ρ' are now *perturbations*

  (but drop primes from now on)

Conservation laws in Boussinesq approximation

- momentum equation

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} \ - \rho \boldsymbol{\nabla} \Phi$$

with geopotential ($\Phi = gz$)

- continuity equation

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

- salt (or moisture for atmosphere) conservation equation

$$\rho_0 \frac{DS}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_S$$

- "conservative" temperature equation

$$\rho_0 \frac{D\Theta}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_\Theta$$

- equation of state

$$\rho \ = \ \rho(S, \Theta, p_0)$$

- aspect ratio $\delta = H/L \ll 1 \rightarrow$ shallow water

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- from scaling $\nabla \cdot \boldsymbol{u} = 0$ it follows that $W = \delta U$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \to \quad \frac{U}{L}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + \frac{W}{H}\frac{\partial w'}{\partial z'} = 0$$

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- scaling first component of momentum equation

$$\rho_0\left(\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u\right) = -\frac{\partial p}{\partial x} - 2\rho_0\Omega(w\cos\varphi - v\sin\varphi)$$

yields

$$Ro \quad , \quad Ro \quad = \quad \frac{P}{\rho_0 L\Omega U} \quad , \quad \delta \ll 1 \quad , \quad 1$$

with Rossby number $Ro = U/(L\Omega)$

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- Ro compares momentum advection with Coriolis force
  for large-scale flow in ocean and atmosphere $Ro \leq 1$

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with Rossby number $Ro = U/(L\Omega)$

- Ro compares momentum advection with Coriolis force
for large-scale flow in ocean and atmosphere $Ro \leq 1$
- assume dominant geostrophic balance: $P/(\rho_0 L \Omega U) = 1$
but still keep terms of $O(Ro)$ (but not those of $O(\delta)$)!

- aspect ratio $\delta = H/L \ll 1 \quad \rightarrow \quad W = \delta U$
- scaling vertical component of momentum equation

$$\rho_0 \left(\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w \right) = -\frac{\partial p}{\partial z} + 2\rho_0 \Omega u \cos\phi - \rho g$$

yields

$$\delta^2 Ro \ , \ \ \delta^2 Ro \ \ = \ \ 1 \ , \ \ \delta \ , \ \ \frac{\delta \varrho g}{\rho_0 \Omega U} \sim 1$$

all magnitudes are now relative to vertical pressure force

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- all terms except $\partial p / \partial z$ and gravity are $O(\delta)$ or smaller
- since $\delta \ll 1$ neglect all terms except $\partial p / \partial z$ and gravity
  $\rightarrow$ hydrostatic approximation
  $\rightarrow$ primitive equations

Summary hydrostatic approximation

- momentum equation in Boussinesq

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p - 2\rho_0 \boldsymbol{\Omega} \times \boldsymbol{u} - \rho \boldsymbol{\nabla} \Phi$$

with geopotential ($\Phi = gz$) and tidal potential $\Phi_{tide}(\boldsymbol{x}, t)$

- becomes

$$\begin{aligned}
\rho_0 \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + 2\rho_0 \Omega v \sin\phi \\
\rho_0 \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} - 2\rho_0 \Omega u \sin\phi \\
0 &= -\frac{\partial p}{\partial z} - g\rho
\end{aligned}$$

acceleration, advection, etc in 3. momentum equation neglected

- $f = 2\Omega \sin\phi$ is the Coriolis parameter
- other equations are unchanged $\rightarrow$ primitive equations

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- momentum equation in Boussinesq

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with geopotential ($\Phi = gz$)

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$$\begin{aligned} \rho_0 \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \rho_0 f v \\ \rho_0 \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} - \rho_0 f u \\ 0 &= -\frac{\partial p}{\partial z} - g\rho \end{aligned}$$

acceleration, advection, etc in 3. momentum equation neglected

- $f = 2\Omega \sin\phi$ is the Coriolis parameter
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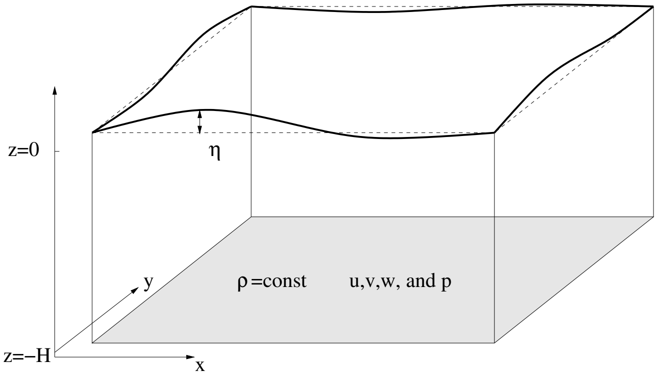
- consider a single layer system in hydrostatic approximation
- assume $\rho = const$ and no vertical shear $\partial u/\partial z = \partial v/\partial z = 0$



- with sea level at $z = \eta$ and the bottom at $z = -H$

- consider a single layer system in hydrostatic approximation

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \nabla u - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \nabla v + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
\cancel{\frac{\partial w}{\partial t}} + \cancel{\boldsymbol{u} \cdot \nabla w} &= -\frac{\partial p}{\partial z} - g\rho \\
\cancel{\frac{1}{\rho}\frac{D\rho}{Dt}} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}
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\end{aligned}
$$

- assume $\rho = const$ and no vertical shear $\partial u/\partial z = \partial v/\partial z = 0$
- now vertically integrate continuity equation from bottom to top

$$
\int_{-H}^{\eta}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz + \int_{-H}^{\eta}\frac{\partial w}{\partial z} dz = 0
$$

with sea level at $z = \eta$ and the bottom at $z = -H$

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\begin{aligned}
\int_{-H}^{\eta} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz + \int_{-H}^{\eta} \frac{\partial w}{\partial z} dz &= 0 \\
(H + \eta) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + w|_{\eta} - w|_{-H} &= 0
\end{aligned}
$$

with sea level at $z = \eta$ and the bottom at $z = -H$

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$$(H+\eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + w|_\eta - w|_{-H} \quad = \quad 0$$

- now use kinematic boundary conditions

$$w_{-H} = 0 \quad , \quad w|_\eta = \frac{\partial \eta}{\partial t} + u|_\eta \frac{\partial \eta}{\partial x} + v|_\eta \frac{\partial \eta}{\partial y}$$

which means no mass flux through upper and lower boundaries

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- this yields

$$(H+\eta)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial x}+v\frac{\partial \eta}{\partial y} \quad = \quad 0$$

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$$\begin{aligned}(H+\eta)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial x}+v\frac{\partial \eta}{\partial y} \quad &= \quad 0\\ h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y} \quad &= \quad 0\end{aligned}$$

which becomes a layer thickness equation for $h = H+\eta$

- assume $\rho = const$ and no vertical shear $\partial u/\partial z = \partial v/\partial z = 0$
- vertically integrate continuity equation from bottom to top

$$(H+\eta)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+w|_{\eta}-w|_{-H} \quad = \quad 0$$

- now use kinematic boundary conditions

$$w_{-H}=0 \quad , \quad w|_{\eta}=\frac{\partial \eta}{\partial t}+u|_{\eta}\frac{\partial \eta}{\partial x}+v|_{\eta}\frac{\partial \eta}{\partial y}$$

which means no mass flux through upper and lower boundaries

- this yields

$$\begin{aligned}
(H+\eta)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial x}+v\frac{\partial \eta}{\partial y} &= 0\\
h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y} &= 0\\
\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(uh)+\frac{\partial}{\partial y}(vh) &= 0
\end{aligned}$$

which becomes a layer thickness equation for $h = H + \eta$

- assume $\rho = const$ and integrate hydrostatic balance from z to top

$$\frac{\partial p}{\partial z} = -g\rho$$

- assume $\rho = const$ and integrate hydrostatic balance from z to top

$$
\begin{aligned}
\frac{\partial p}{\partial z} &= -g\rho \\
\int_z^\eta \frac{\partial p}{\partial z} dz &= p|_\eta - p|_z = -g\rho \int_z^\eta dz = -g\rho(\eta - z)
\end{aligned}
$$

- assume $\rho = const$ and integrate hydrostatic balance from z to top

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p &= p|_{\eta} - g\rho(z - \eta)
\end{aligned}
$$

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\int_z^\eta \frac{\partial p}{\partial z} dz &= p|_\eta - p|_z = -g\rho \int_z^\eta dz = -g\rho(\eta - z) \\
p &= p|_\eta - g\rho(z - \eta) \\
\nabla p &= g\rho\nabla\eta = g\rho\nabla h
\end{aligned}$$

assuming $p|_\eta = p_{air} = const$ and with layer thickness $h = \eta + H$

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$$
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\frac{\partial p}{\partial z} &= -g\rho \\
\int_z^{\eta} \frac{\partial p}{\partial z} dz &= p|_{\eta} - p|_z = -g\rho \int_z^{\eta} dz = -g\rho(\eta - z) \\
p &= p|_{\eta} - g\rho(z - \eta) \\
\nabla p &= g\rho \nabla \eta = g\rho \nabla h
\end{aligned}
$$

assuming $p|_{\eta} = p_{air} = const$ and with layer thickness $h = \eta + H$

- momentum equation becomes

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \nabla u - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \nabla v + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial h}{\partial y}
\end{aligned}
$$

since $h(x, y, t)$ and $\partial u/\partial z = \partial v/\partial z = 0$ equations are now 2-D

- single layer system in hydrostatic approximation

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u - fv &= -g \frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} v + fu &= -g \frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) &= 0
\end{aligned}
$$

- single layer system in hydrostatic approximation

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u - fv &= -g\frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} v + fu &= -g\frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) &= 0
\end{aligned}
$$

- neglecting momentum advection for simplicity
  and assuming $H \gg \eta$ in $h = H + \eta \rightarrow \boldsymbol{\nabla} \cdot (\boldsymbol{u}h) \approx H\boldsymbol{\nabla} \cdot \boldsymbol{u}$

- single layer system in hydrostatic approximation

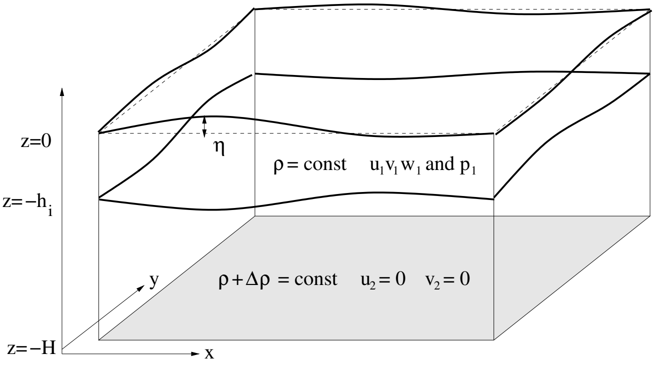
$$
\begin{aligned}
\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u - fv &= -g\frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} v + fu &= -g\frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) &= 0
\end{aligned}
$$

- neglecting momentum advection for simplicity
  and assuming $H \gg \eta$ in $h = H + \eta \rightarrow \boldsymbol{\nabla} \cdot (\boldsymbol{u}h) \approx H\boldsymbol{\nabla} \cdot \boldsymbol{u}$

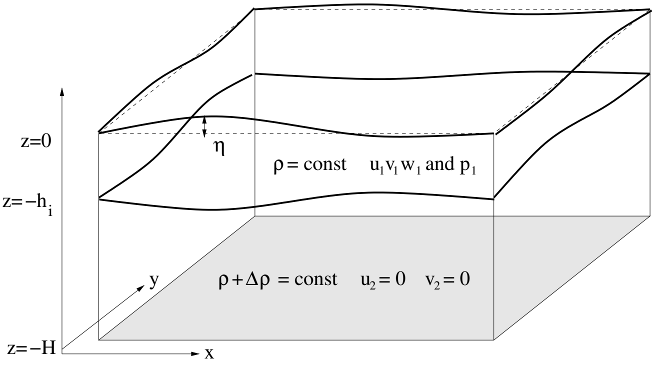
$$
\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -g\frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -g\frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0
\end{aligned}
$$

  simple system which contains almost all relevant dynamics

- two layers with $\rho_1 = \rho = const$ and $\rho_2 = \rho + \Delta\rho = const$
- sea surface at $z = \eta$ and layer interface at $z = -h_i$
- assume again no vertical shear $\partial u_{1,2}/\partial z = \partial v_{1,2}/\partial z = 0$ in layers

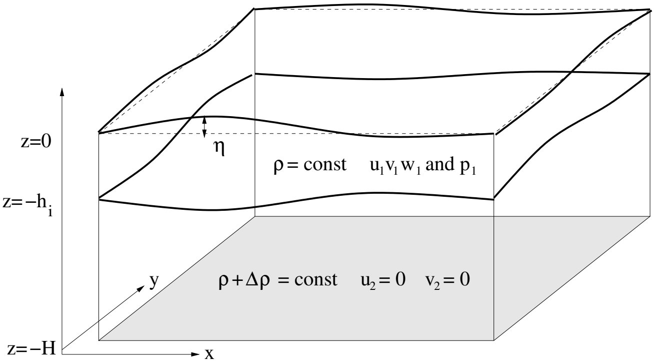


- two layers with $\rho_1 = \rho = const$ and $\rho_2 = \rho + \Delta\rho = const$
- sea surface at $z = \eta$ and layer interface at $z = -h_i$
- assume again no vertical shear $\partial u_{1,2}/\partial z = \partial v_{1,2}/\partial z = 0$ in layers



- pressure gradient in upper layer $g\rho\boldsymbol{\nabla}\eta$

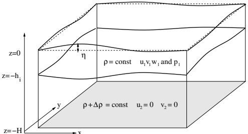
- two layers with $\rho_1 = \rho = const$ and $\rho_2 = \rho + \Delta\rho = const$
- sea surface at $z = \eta$ and layer interface at $z = -h_i$
- assume again no vertical shear $\partial u_{1,2}/\partial z = \partial v_{1,2}/\partial z = 0$ in layers



- pressure gradient in upper layer $g\rho\nabla\eta$
- pressure gradient in lower layer $-g(\rho + \Delta\rho)\nabla h_i + g\rho\nabla(\eta + h_i)$

- upper layer equations

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla} u_1 - f v_1 &= -g \frac{\partial \eta}{\partial x} \\
\frac{\partial v_1}{\partial t} + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla} v_1 + f u_1 &= -g \frac{\partial \eta}{\partial y} \\
\frac{\partial}{\partial t}(\eta + h_i) + \frac{\partial}{\partial x} u_1(\eta + h_i) + \frac{\partial}{\partial y} v_1(\eta + h_i) &= 0
\end{aligned}
$$

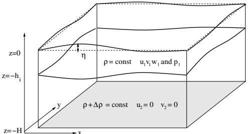


- upper layer equations

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla} u_1 - f v_1 &= -g \frac{\partial \eta}{\partial x} \\
\frac{\partial v_1}{\partial t} + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla} v_1 + f u_1 &= -g \frac{\partial \eta}{\partial y} \\
\frac{\partial}{\partial t}(\eta + h_i) + \frac{\partial}{\partial x} u_1(\eta + h_i) + \frac{\partial}{\partial y} v_1(\eta + h_i) &= 0
\end{aligned}
$$

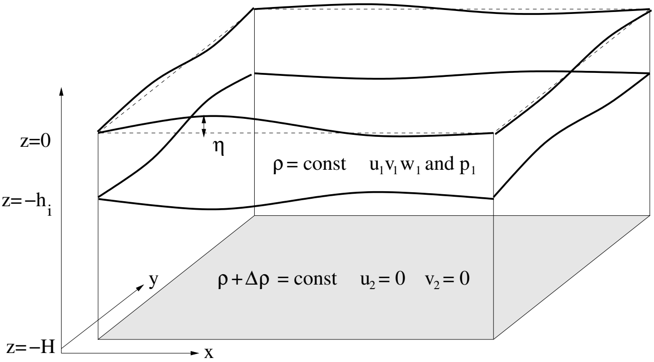
- lower layer equations

$$
\begin{aligned}
\frac{\partial u_2}{\partial t} + \boldsymbol{u}_2 \cdot \boldsymbol{\nabla} u_2 - f v_2 &= g \frac{\Delta \rho}{\rho} \frac{\partial h_i}{\partial x} - g \frac{\partial \eta}{\partial x} \\
\frac{\partial v_2}{\partial t} + \boldsymbol{u}_2 \cdot \boldsymbol{\nabla} v_2 + f u_2 &= g \frac{\Delta \rho}{\rho} \frac{\partial h_i}{\partial y} - g \frac{\partial \eta}{\partial y} \\
\frac{\partial}{\partial t}(H - h_i) + \frac{\partial}{\partial x} u_2(H - h_i) + \frac{\partial}{\partial y} v_2(H - h_i) &= 0
\end{aligned}
$$



- assume that lower layer is motionless

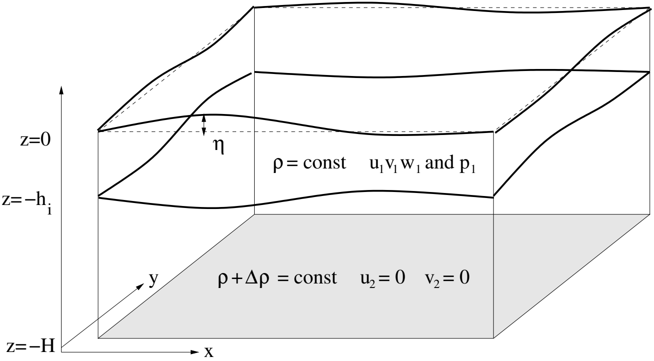
$$0 = g\frac{\Delta\rho}{\rho}\nabla h_i - g\nabla\eta$$



- assume that lower layer is motionless

$$0 = g\frac{\Delta\rho}{\rho}\nabla h_i - g\nabla\eta \quad \rightarrow \quad \frac{\Delta\rho}{\rho}h_i - \eta = const = 0$$

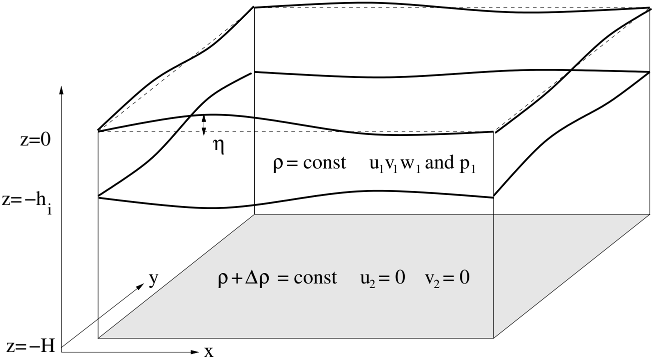
vanishing pressure variations in lower layer



- assume that lower layer is motionless

$$0 = g\frac{\Delta\rho}{\rho}\nabla h_i - g\nabla\eta \quad \rightarrow \quad \frac{\Delta\rho}{\rho}h_i - \eta = const = 0 \quad \rightarrow \quad \eta = \frac{\Delta\rho}{\rho}h_i$$

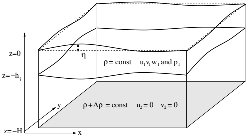
vanishing pressure variations in lower layer



- assume that lower layer is infinitively deep and motionless

$$0 = g\frac{\Delta\rho}{\rho}\boldsymbol{\nabla} h_i - g\boldsymbol{\nabla}\eta \quad \rightarrow \quad \eta = \frac{\Delta\rho}{\rho}h_i$$

vanishing pressure variations in lower layer



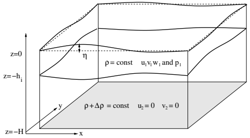
- assume that lower layer is infinitively deep and motionless

$$0 = g\frac{\Delta\rho}{\rho}\nabla h_i - g\nabla\eta \quad \rightarrow \quad \eta = \frac{\Delta\rho}{\rho}h_i$$

vanishing pressure variations in lower layer

- upper layer equations become

$$\begin{aligned}
\frac{\partial u_1}{\partial t} + \boldsymbol{u}_1 \cdot \nabla u_1 - f v_1 &= -g\frac{\partial \eta}{\partial x} \\
\frac{\partial v_1}{\partial t} + \boldsymbol{u}_1 \cdot \nabla v_1 + f u_1 &= -g\frac{\partial \eta}{\partial y} \\
\frac{\partial}{\partial t}(\eta + h_i) + \frac{\partial}{\partial x}u_1(\eta + h_i) + \frac{\partial}{\partial y}v_1(\eta + h_i) &= 0
\end{aligned}$$



- assume that lower layer is infinitively deep and motionless

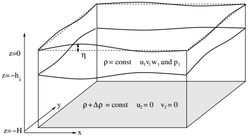
$$0 = g\frac{\Delta\rho}{\rho}\nabla h_i - g\nabla\eta \quad \rightarrow \quad \eta = \frac{\Delta\rho}{\rho}h_i$$

vanishing pressure variations in lower layer

- upper layer equations become

$$\begin{aligned}
\frac{\partial u_1}{\partial t} + \boldsymbol{u}_1\cdot\nabla u_1 - fv_1 &= -g'\frac{\partial h_i}{\partial x}\\
\frac{\partial v_1}{\partial t} + \boldsymbol{u}_1\cdot\nabla v_1 + fu_1 &= -g'\frac{\partial h_i}{\partial y}\\
\frac{\partial}{\partial t}h_i + \frac{\partial}{\partial x}(u_1 h_i) + \frac{\partial}{\partial y}(v_1 h_i) &= 0
\end{aligned}$$

with "reduced gravity" $g' = g\Delta\rho/\rho$

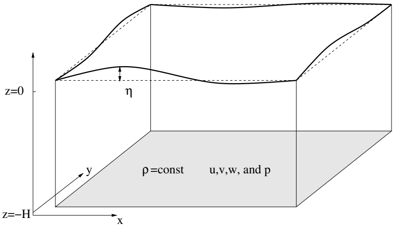


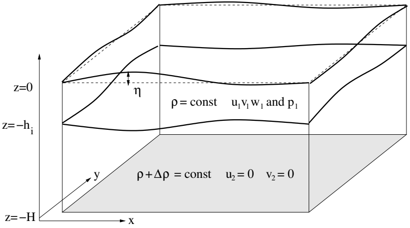
- "barotropic model" and "baroclinic model"

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} v + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) = 0$$

- h is total thickness ("barotropic") or layer interface h_i ("baroclinic")
- either $g = 9.81\,\mathrm{m/s^2}$ ("barotropic") or $g \to g\Delta\rho/\rho_0$ ("baroclinic")





- full three-dimensional equations in hydrostatic approximation (without friction and diffusion)

$$
\begin{aligned}
\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad &, \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
\frac{D\Theta}{Dt} = 0 \quad &, \quad \frac{DS}{Dt} = 0 \\
\frac{\partial p}{\partial z} = -g\rho \quad &, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\end{aligned}
$$

with equation of state $\rho = \rho(S, \Theta, p_0)$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion)

$$
\begin{aligned}
\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad &, \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
\frac{D\Theta}{Dt} = 0 \quad &, \quad \frac{DS}{Dt} = 0 \\
\frac{\partial p}{\partial z} = -g\rho \quad &, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\end{aligned}
$$

  with equation of state $\rho = \rho(S, \Theta, p_0)$
- remember that in Boussinesq approximation $\rho \to \rho' = \rho - \rho_0$ and $p \to p' = p - p_0$ with hydrostatic balance $\partial p_0/\partial z = -g\rho_0$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad , \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$
$$\frac{D\Theta}{Dt} = 0 \quad , \quad \frac{DS}{Dt} = 0$$
$$\frac{\partial p}{\partial z} = -g\rho \quad , \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

with equation of state $\rho = \rho(S, \Theta, p_0)$

- remember that in Boussinesq approximation $\rho \to \rho' = \rho - \rho_0$ and $p \to p' = p - p_0$ with hydrostatic balance $\partial p_0/\partial z = -g\rho_0$
- assume full incompressibility $\partial\rho/\partial p_0 = 0 \to \rho = \rho(S, \Theta)$

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial\Theta}\frac{D\Theta}{Dt} + \frac{\partial\rho}{\partial S}\frac{DS}{Dt} = 0$$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion)

$$
\begin{aligned}
\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad &, \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
\frac{D\Theta}{Dt} = 0 \quad &, \quad \frac{DS}{Dt} = 0 \\
\frac{\partial p}{\partial z} = -g\rho \quad &, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\end{aligned}
$$

with equation of state $\rho = \rho(S, \Theta, p_0)$

- remember that in Boussinesq approximation $\rho \to \rho' = \rho - \rho_0$ and $p \to p' = p - p_0$ with hydrostatic balance $\partial p_0 / \partial z = -g\rho_0$
- assume full incompressibility $\partial \rho / \partial p_0 = 0 \to \rho = \rho(S, \Theta)$

$$
\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial \Theta}\frac{D\Theta}{Dt} + \frac{\partial \rho}{\partial S}\frac{DS}{Dt} = 0
$$

$\to$ replace $D\Theta/Dt = 0$ and $DS/Dt = 0$ with $D\rho/Dt = 0$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion and full incompressibility)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad , \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{D\rho}{Dt} = 0 \quad , \quad \frac{\partial p}{\partial z} = -g\rho \quad , \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

with equation of state $\rho = \rho(S, \Theta)$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion and full incompressibility)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad , \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{D\rho}{Dt} = 0 \quad , \quad \frac{\partial p}{\partial z} = -g\rho \quad , \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

with equation of state $\rho = \rho(S, \Theta)$

- split density and pressure again into background and perturbation

$$\rho = \bar{\rho}(z) + \rho'(\boldsymbol{x}, t) \ , \ p = \bar{p}(z) + p' \text{ with } \frac{\partial \bar{p}}{\partial z} = -g\bar{\rho} \ \rightarrow \ \frac{\partial p'}{\partial z} = -g\rho'$$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion and full incompressibility)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad , \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$
$$\frac{D\rho}{Dt} = 0 \quad , \quad \frac{\partial p}{\partial z} = -g\rho \quad , \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

with equation of state $\rho = \rho(S, \Theta)$

- split density and pressure again into background and perturbation

$$\rho = \bar{\rho}(z) + \rho'(\boldsymbol{x}, t) \; , \; p = \bar{p}(z) + p' \text{ with } \frac{\partial \bar{p}}{\partial z} = -g\bar{\rho} \;\rightarrow\; \frac{\partial p'}{\partial z} = -g\rho'$$

- pressure gradient in momentum equations: $\boldsymbol{\nabla}_h p \rightarrow \boldsymbol{\nabla}_h p'$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion and full incompressibility)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad , \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{D\rho}{Dt} = 0 \quad , \quad \frac{\partial p}{\partial z} = -g\rho \quad , \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

with equation of state $\rho = \rho(S, \Theta)$

- split density and pressure again into background and perturbation

$$\rho = \bar{\rho}(z) + \rho'(\boldsymbol{x}, t) \ , \ p = \bar{p}(z) + p' \text{ with } \frac{\partial \bar{p}}{\partial z} = -g\bar{\rho} \ \rightarrow \ \frac{\partial p'}{\partial z} = -g\rho'$$

- pressure gradient in momentum equations: $\boldsymbol{\nabla}_h p \rightarrow \boldsymbol{\nabla}_h p'$
- substitute $\rho = \bar{\rho} + \rho'$ in "density equation"

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho = 0 \ \rightarrow \ \cancel{\frac{\partial \bar{\rho}}{\partial t}} + \frac{\partial \rho'}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \bar{\rho} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho' = 0$$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion and full incompressibility)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad , \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{D\rho}{Dt} = 0 \quad , \quad \frac{\partial p}{\partial z} = -g\rho \quad , \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

with equation of state $\rho = \rho(S, \Theta)$

- split density and pressure again into background and perturbation

$$\rho = \bar\rho(z) + \rho'(\boldsymbol{x}, t) \ , \ \ p = \bar p(z) + p' \text{ with } \frac{\partial \bar p}{\partial z} = -g\bar\rho \ \rightarrow \ \frac{\partial p'}{\partial z} = -g\rho'$$

- pressure gradient in momentum equations: $\boldsymbol{\nabla}_h p \rightarrow \boldsymbol{\nabla}_h p'$
- substitute $\rho = \bar\rho + \rho'$ in "density equation"

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\rho = 0 \ \rightarrow \ \cancel{\frac{\partial \bar\rho}{\partial t}} + \frac{\partial \rho'}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\bar\rho + \boldsymbol{u}\cdot\boldsymbol{\nabla}\rho' = 0$$

- from $\boldsymbol{u}\cdot\boldsymbol{\nabla}\bar\rho$ only $w\, d\bar\rho/dz$ remains, while $u\partial\bar\rho/\partial x = v\partial\bar\rho/\partial y = 0$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion and full incompressibility)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad , \quad \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{D\rho}{Dt} = 0 \quad , \quad \frac{\partial p}{\partial z} = -g\rho \quad , \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

  with equation of state $\rho = \rho(S, \Theta)$
- split density and pressure again into background and perturbation

$$\rho = \bar{\rho}(z) + \rho'(\boldsymbol{x}, t) \ , \ p = \bar{p}(z) + p' \text{ with } \frac{\partial \bar{p}}{\partial z} = -g\bar{\rho} \ \rightarrow \ \frac{\partial p'}{\partial z} = -g\rho'$$

- pressure gradient in momentum equations: $\boldsymbol{\nabla}_h p \rightarrow \boldsymbol{\nabla}_h p'$
- substitute $\rho = \bar{\rho} + \rho'$ in "density equation"

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho = 0 \ \rightarrow \ \cancel{\frac{\partial \bar{\rho}}{\partial t}} + \frac{\partial \rho'}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \bar{\rho} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho' = 0$$

- from $\boldsymbol{u} \cdot \boldsymbol{\nabla} \bar{\rho}$ only $w\, d\bar{\rho}/dz$ remains, while $u\partial\bar{\rho}/\partial x = v\partial\bar{\rho}/\partial y = 0$
- if perturbation ρ' is small $\rightarrow$ neglect nonlinear term $\boldsymbol{u} \cdot \boldsymbol{\nabla} \rho'$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion and with $\partial\rho/\partial p = 0$)

$$\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} u - fv = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} v + fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$\frac{\partial \rho'}{\partial t} + w\frac{d\bar{\rho}}{dz} = 0 \quad , \quad \frac{\partial p'}{\partial z} = -g\rho' \quad , \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

with the split $\rho = \bar{\rho}(z) + \rho'$ and $p = \bar{p}(z) + p'$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion and with $\partial\rho/\partial p = 0$)

$$\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} u - fv = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} v + fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$\frac{\partial \rho'}{\partial t} + w\frac{d\bar{\rho}}{dz} = 0 \quad , \quad \frac{\partial p'}{\partial z} = -g\rho' \quad , \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

with the split $\rho = \bar{\rho}(z) + \rho'$ and $p = \bar{p}(z) + p'$

- also neglect nonlinear terms in momentum equations neglect $\boldsymbol{u}\cdot\boldsymbol{\nabla} u$ and $\boldsymbol{u}\cdot\boldsymbol{\nabla} v$ $\rightarrow$ linearization $(D/Dt \rightarrow \partial/\partial t)$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion and with $\partial\rho/\partial p = 0$)

$$\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} u - fv = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} v + fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$\frac{\partial \rho'}{\partial t} + w\frac{d\bar{\rho}}{dz} = 0 \quad , \quad \frac{\partial p'}{\partial z} = -g\rho' \quad , \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

  with the split $\rho = \bar{\rho}(z) + \rho'$ and $p = \bar{p}(z) + p'$
- also neglect nonlinear terms in momentum equations neglect $\boldsymbol{u}\cdot\boldsymbol{\nabla} u$ and $\boldsymbol{u}\cdot\boldsymbol{\nabla} v$ $\rightarrow$ linearization $(D/Dt \rightarrow \partial/\partial t)$
- introduce 'stability' or 'Brunt-Väisällä' frequency N with

$$N^2(z) = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}$$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion and with $\partial\rho/\partial p = 0$)

$$\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} u - fv = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} v + fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

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with the split $\rho = \bar\rho(z) + \rho'$ and $p = \bar p(z) + p'$

- also neglect nonlinear terms in momentum equations neglect $\boldsymbol{u}\cdot\boldsymbol{\nabla} u$ and $\boldsymbol{u}\cdot\boldsymbol{\nabla} v$ $\to$ linearization $(D/Dt \to \partial/\partial t)$
- introduce 'stability' or 'Brunt-Väisällä' frequency N with

$$N^2(z) = -\frac{g}{\rho_0}\frac{d\bar\rho}{dz}$$

- for general $\rho(S,\Theta,p)$, stability frequency N is given by

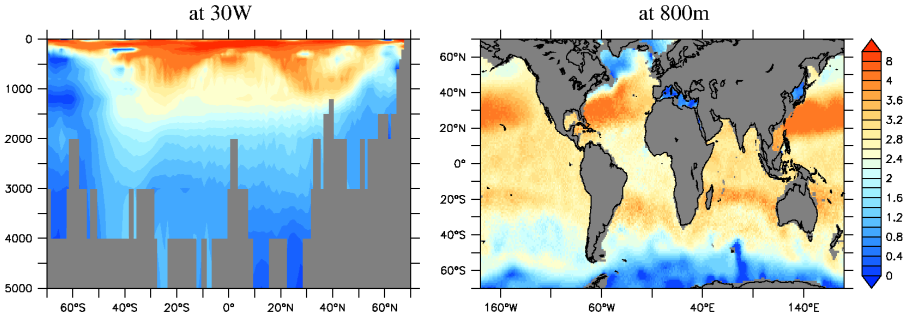
$$N^2 = -\frac{g}{\rho_0}\left(\frac{\partial\rho}{\partial S}\frac{\partial S}{\partial z} + \frac{\partial\rho}{\partial\Theta}\frac{\partial\Theta}{\partial z}\right) = -g\left(\beta\frac{\partial S}{\partial z} - \alpha\frac{\partial\Theta}{\partial z}\right)$$

with thermal and haline expansion coefficients α and β

- "stability" or Brunt-Väisällä" frequency N with

$$N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz} \quad \text{or} \quad N^2 = -g\left(\beta\frac{\partial S}{\partial z} - \alpha\frac{\partial \Theta}{\partial z}\right)$$

  with thermal and haline expansion coefficients α and β
- $N^2 > 0$ for stable stratification $\rightarrow$ lighter above denser water
- stability frequency N in $10^{-3}\,\mathrm{s}^{-1}$ from Levitus climatology



- full three-dimensional equations in hydrostatic approximation (without friction and diffusion, full incompressibility, linearized)

$$
\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \\
\frac{\partial p'}{\partial z} &= -g\rho' \\
\frac{\partial \rho'}{\partial t} - \frac{\rho_0}{g} w N^2(z) &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}
$$

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\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}
$$

- use separation ansatz for the vertical variation given by

$$
\begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ p'(x,y,z,t) \\ w(x,y,z,t) \\ \rho'(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \tilde{u}(x,y,t) & \Phi(z) \\ \tilde{v}(x,y,t) & \Phi(z) \\ \rho_0\tilde{p}(x,y,t) & \Phi(z) \\ \tilde{w}(x,y,t)/N^2(z) & d\Phi(z)/dz \\ \rho_0\tilde{\rho}(x,y,t) & d\Phi(z)/dz \end{pmatrix}
$$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion, full incompressibility, linearized)

$$
\begin{aligned}
\left(\frac{\partial \tilde{u}}{\partial t}\right)\Phi - fv &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \\
\frac{\partial p'}{\partial z} &= -g\rho' \\
\frac{\partial \rho'}{\partial t} - \frac{\rho_0}{g} w N^2(z) &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}
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$$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion, full incompressibility, linearized)

$$
\begin{aligned}
\left(\frac{\partial \tilde{u}}{\partial t}\right)\Phi - f\,\tilde{v}\Phi &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \\
\frac{\partial p'}{\partial z} &= -g\rho' \\
\frac{\partial \rho'}{\partial t} - \frac{\rho_0}{g} w N^2(z) &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
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\begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ p'(x,y,z,t) \\ w(x,y,z,t) \\ \rho'(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \tilde{u}(x,y,t) & \Phi(z) \\ \tilde{v}(x,y,t) & \Phi(z) \\ \rho_0\tilde{p}(x,y,t) & \Phi(z) \\ \tilde{w}(x,y,t)/N^2(z) & d\Phi(z)/dz \\ \rho_0\tilde{\rho}(x,y,t) & d\Phi(z)/dz \end{pmatrix}
$$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion, full incompressibility, linearized)

$$
\begin{aligned}
\left(\frac{\partial \tilde{u}}{\partial t}\right)\Phi - f\tilde{v}\Phi &= -\frac{\rho_0}{\rho_0}\left(\frac{\partial \tilde{p}}{\partial x}\right)\Phi \\
\frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \\
\frac{\partial p'}{\partial z} &= -g\rho' \\
\frac{\partial \rho'}{\partial t} - \frac{\rho_0}{g} w N^2(z) &= 0 \\
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$$
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$$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion, full incompressibility, linearized)

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial t} - f\tilde{v} &= -\frac{\partial \tilde{p}}{\partial x} \\
\frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \\
\frac{\partial p'}{\partial z} &= -g\rho' \\
\frac{\partial \rho'}{\partial t} - \frac{\rho_0}{g} w N^2(z) &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}$$

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\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
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\rho_0 \tilde{p}\frac{d\Phi}{dz} &= -g\rho_0\tilde{\rho}\frac{d\Phi}{dz} \\
\frac{\partial \rho'}{\partial t} - \frac{\rho_0}{g} w N^2(z) &= 0 \\
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$$\begin{aligned}
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\tilde{p} &= -g\tilde{\rho} \\
\frac{\partial \rho'}{\partial t} - \frac{\rho_0}{g} w N^2(z) &= 0 \\
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\tilde{p} &= -g\tilde{\rho} \\
\rho_0 \left(\frac{\partial \tilde{\rho}}{\partial t}\right) \frac{d\Phi}{dz} - \frac{\rho_0}{g} w N^2 &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
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\frac{\partial \tilde{v}}{\partial t} + f\tilde{u} &= -\frac{\partial \tilde{p}}{\partial y} \\
\tilde{p} &= -g\tilde{\rho} \\
\rho_0 \left(\frac{\partial \tilde{\rho}}{\partial t}\right) \frac{d\Phi}{dz} - \frac{\rho_0}{g} N^2 \frac{\tilde{w}}{N^2} \frac{d\Phi}{dz} &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
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\begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ p'(x,y,z,t) \\ w(x,y,z,t) \\ \rho'(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \tilde{u}(x,y,t) & \Phi(z) \\ \tilde{v}(x,y,t) & \Phi(z) \\ \rho_0\tilde{p}(x,y,t) & \Phi(z) \\ \tilde{w}(x,y,t)/N^2(z) & d\Phi(z)/dz \\ \rho_0\tilde{\rho}(x,y,t) & d\Phi(z)/dz \end{pmatrix}
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\tilde{p} &= -g\tilde{\rho} \\
\frac{\partial \tilde{\rho}}{\partial t} - \frac{1}{g}\tilde{w} &= 0 \\
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$$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion, full incompressibility, linearized)

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial t} - f\,\tilde{v} &= -\frac{\partial \tilde{p}}{\partial x} \\
\frac{\partial \tilde{v}}{\partial t} + f\,\tilde{u} &= -\frac{\partial \tilde{p}}{\partial y} \\
\tilde{p} &= -g\,\tilde{\rho} \\
\frac{\partial \tilde{\rho}}{\partial t} - \frac{1}{g}\tilde{w} &= 0 \\
\left(\frac{\partial \tilde{u}}{\partial x}\right)\Phi + \left(\frac{\partial \tilde{v}}{\partial y}\right)\Phi + \frac{\partial w}{\partial z} &= 0
\end{aligned}
$$

- use separation ansatz for the vertical variation given by

$$
\begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ p'(x,y,z,t) \\ w(x,y,z,t) \\ \rho'(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \tilde{u}(x,y,t) & \Phi(z) \\ \tilde{v}(x,y,t) & \Phi(z) \\ \rho_0\tilde{p}(x,y,t) & \Phi(z) \\ \tilde{w}(x,y,t)/N^2(z) & d\Phi(z)/dz \\ \rho_0\tilde{\rho}(x,y,t) & d\Phi(z)/dz \end{pmatrix}
$$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion, full incompressibility, linearized)

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial t} - f\,\tilde{v} &= -\frac{\partial \tilde{p}}{\partial x} \\
\frac{\partial \tilde{v}}{\partial t} + f\,\tilde{u} &= -\frac{\partial \tilde{p}}{\partial y} \\
\tilde{p} &= -g\,\tilde{\rho} \\
\frac{\partial \tilde{\rho}}{\partial t} - \frac{1}{g}\tilde{w} &= 0 \\
\left(\frac{\partial \tilde{u}}{\partial x}\right)\Phi + \left(\frac{\partial \tilde{v}}{\partial y}\right)\Phi + \frac{\partial}{\partial z}\left(\frac{\tilde{w}}{N^2}\frac{d\Phi}{dz}\right) &= 0
\end{aligned}$$

- use separation ansatz for the vertical variation given by

$$\begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ p'(x,y,z,t) \\ w(x,y,z,t) \\ \rho'(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \tilde{u}(x,y,t) & \Phi(z) \\ \tilde{v}(x,y,t) & \Phi(z) \\ \rho_0\tilde{p}(x,y,t) & \Phi(z) \\ \tilde{w}(x,y,t)/N^2(z) & d\Phi(z)/dz \\ \rho_0\tilde{\rho}(x,y,t) & d\Phi(z)/dz \end{pmatrix}$$

- full three-dimensional equations in hydrostatic approximation (without friction and diffusion, full incompressibility, linearized)

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\frac{\partial \tilde{v}}{\partial t} + f\tilde{u} &= -\frac{\partial \tilde{p}}{\partial y} \\
\tilde{p} &= -g\tilde{\rho} \\
\frac{\partial \tilde{\rho}}{\partial t} - \frac{1}{g}\tilde{w} &= 0 \\
\left(\frac{\partial \tilde{u}}{\partial x}\right)\Phi + \left(\frac{\partial \tilde{v}}{\partial y}\right)\Phi + \tilde{w}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right) &= 0
\end{aligned}
$$

- use separation ansatz for the vertical variation given by

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\begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ p'(x,y,z,t) \\ w(x,y,z,t) \\ \rho'(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \tilde{u}(x,y,t) & \Phi(z) \\ \tilde{v}(x,y,t) & \Phi(z) \\ \rho_0\tilde{p}(x,y,t) & \Phi(z) \\ \tilde{w}(x,y,t)/N^2(z) & d\Phi(z)/dz \\ \rho_0\tilde{\rho}(x,y,t) & d\Phi(z)/dz \end{pmatrix}
$$

- rewrite continuity equation

$$\left(\frac{\partial \tilde{u}}{\partial x}\right)\Phi + \left(\frac{\partial \tilde{v}}{\partial y}\right)\Phi + \tilde{w}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right) = 0$$

as

$$\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right)\Phi = -\tilde{w}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right)$$

- rewrite continuity equation

$$\left(\frac{\partial \tilde{u}}{\partial x}\right)\Phi + \left(\frac{\partial \tilde{v}}{\partial y}\right)\Phi + \tilde{w}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right) = 0$$

as

$$\begin{aligned}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right)\Phi &= -\tilde{w}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right) \\ \frac{\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}}{\tilde{w}} &= -\frac{\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right)}{\Phi}\end{aligned}$$

- rewrite continuity equation

$$\left(\frac{\partial \tilde{u}}{\partial x}\right)\Phi + \left(\frac{\partial \tilde{v}}{\partial y}\right)\Phi + \tilde{w}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right) = 0$$

as

$$\begin{aligned}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right)\Phi &= -\tilde{w}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right) \\ \frac{\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}}{\tilde{w}} &= -\frac{\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right)}{\Phi}\end{aligned}$$

- left hand is a function of x, y, t while other side is function of z

- rewrite continuity equation

$$\left(\frac{\partial \tilde{u}}{\partial x}\right)\Phi + \left(\frac{\partial \tilde{v}}{\partial y}\right)\Phi + \tilde{w}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right) = 0$$

as

$$\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right)\Phi = -\tilde{w}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right)$$

$$\frac{\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}}{\tilde{w}} = -\frac{\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right)}{\Phi} = const = 1/(\tilde{g}H)$$

- left hand is a function of x, y, t while other side is function of z
- $\rightarrow$ both sides must be constant, say $1/(\tilde{g}H)$, and thus

- rewrite continuity equation

$$\left(\frac{\partial \tilde{u}}{\partial x}\right)\Phi + \left(\frac{\partial \tilde{v}}{\partial y}\right)\Phi + \tilde{w}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right) = 0$$

as

$$\begin{aligned}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right)\Phi &= -\tilde{w}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right)\\ \frac{\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}}{\tilde{w}} &= -\frac{\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right)}{\Phi} = const = 1/(\tilde{g}H)\end{aligned}$$

- left hand is a function of x, y, t while other side is function of z
- $\rightarrow$ both sides must be constant, say $1/(\tilde{g}H)$, and thus

$$\begin{aligned}\frac{d}{dz}\left(\frac{1}{N^2}\frac{d\Phi}{dz}\right) &= -\Phi/(\tilde{g}H)\\ \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= \tilde{w}/(\tilde{g}H)\end{aligned}$$

- ordinary differential equation in z and partial diff. eq. in x, y, t latter adds to the others $\rightarrow$ system of 2-D partial diff. eqs. in x, y, t

- system of 2-D partial differential equations in x, y, t

$$
\frac{\partial \tilde{u}}{\partial t} - f\tilde{v} = -\frac{\partial \tilde{p}}{\partial x} \quad , \quad \frac{\partial \tilde{v}}{\partial t} + f\tilde{u} = -\frac{\partial \tilde{p}}{\partial y}
$$

$$
\begin{aligned}
\tilde{p} &= -g\tilde{\rho} \\
\frac{\partial \tilde{\rho}}{\partial t} - \frac{1}{g}\tilde{w} &= 0 \\
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= \tilde{w}/(\tilde{g}H)
\end{aligned}
$$

- system of 2-D partial differential equations in x, y, t

$$\frac{\partial \tilde{u}}{\partial t} - f\,\tilde{v} = -\frac{\partial \tilde{p}}{\partial x} \quad , \quad \frac{\partial \tilde{v}}{\partial t} + f\,\tilde{u} = -\frac{\partial \tilde{p}}{\partial y}$$

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- combine last three equations to

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = \tilde{w}/(\tilde{g}H) = g\frac{\partial \tilde{\rho}}{\partial t}/(\tilde{g}H) = -\frac{\partial \tilde{p}}{\partial t}/(\tilde{g}H)$$

- system of 2-D partial differential equations in x, y, t

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial t} - f\tilde{v} = -\frac{\partial \tilde{p}}{\partial x} \quad , \quad \frac{\partial \tilde{v}}{\partial t} + f\tilde{u} &= -\frac{\partial \tilde{p}}{\partial y} \\
\tilde{p} &= -g\tilde{\rho} \\
\frac{\partial \tilde{\rho}}{\partial t} - \frac{1}{g}\tilde{w} &= 0 \\
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= \tilde{w}/(\tilde{g}H)
\end{aligned}
$$

- combine last three equations to

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$$

- remaining equations

$$
\frac{\partial \tilde{u}}{\partial t} - f\tilde{v} = -\frac{\partial \tilde{p}}{\partial x} \, , \, \frac{\partial \tilde{v}}{\partial t} + f\tilde{u} = -\frac{\partial \tilde{p}}{\partial y} \, , \, \frac{\partial \tilde{p}}{\partial t} + \tilde{g}H\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) = 0
$$

- system of 2-D partial differential equations in x, y, t

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} - f\tilde{v} = -\frac{\partial \tilde{p}}{\partial x} \quad &, \quad \frac{\partial \tilde{v}}{\partial t} + f\tilde{u} = -\frac{\partial \tilde{p}}{\partial y} \\ \tilde{p} &= -g\tilde{\rho} \\ \frac{\partial \tilde{\rho}}{\partial t} - \frac{1}{g}\tilde{w} &= 0 \\ \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= \tilde{w}/(\tilde{g}H) \end{aligned}$$

- combine last three equations to

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = \tilde{w}/(\tilde{g}H) = g\frac{\partial \tilde{\rho}}{\partial t}/(\tilde{g}H) = -\frac{\partial \tilde{p}}{\partial t}/(\tilde{g}H)$$

- remaining equations

$$\frac{\partial \tilde{u}}{\partial t} - f\tilde{v} = -\frac{\partial \tilde{p}}{\partial x}, \; \frac{\partial \tilde{v}}{\partial t} + f\tilde{u} = -\frac{\partial \tilde{p}}{\partial y}, \; \frac{\partial \tilde{p}}{\partial t} + \tilde{g}H\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) = 0$$

  are identical to (linearized) layered (barotropic or baroclinic) model for $\tilde{u} \to u$ and $\tilde{v} \to v$ and $\tilde{p} \to \tilde{g}h$

- system of 2-D partial differential equations in x, y, t becomes identical to (linearized) layered model

$$\frac{\partial u}{\partial t} - fv = -\tilde{g}\frac{\partial h}{\partial x}\ , \ \frac{\partial v}{\partial t} + fu = -\tilde{g}\frac{\partial h}{\partial y}\ , \ \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

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- same system as before except that g or g' becomes $\tilde{g}$

- system of 2-D partial differential equations in x, y, t becomes identical to (linearized) layered model

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- same system as before except that g or g' becomes $\tilde{g}$
- $\tilde{g}$ can be determined from ordinary differential equation

$$\frac{d}{dz}\left(\frac{1}{N^2(z)}\frac{d\Phi}{dz}\right) + \Phi/(\tilde{g}H) = 0$$

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$$\frac{d}{dz}\left(\frac{1}{N^2(z)}\frac{d\Phi}{dz}\right) + \Phi/(\tilde{g}H) = 0$$

- solution for $N^2 = const$ is $\Phi = B\cos(n\pi z/H)$ and

$$\tilde{g} = \frac{HN^2}{n^2\pi^2|f|}$$

for $n = 1, 2, ... \rightarrow$ first, second, third, etc baroclinic mode

- system of 2-D partial differential equations in x, y, t becomes identical to (linearized) layered model

$$\frac{\partial u}{\partial t} - fv = -\tilde{g}\frac{\partial h}{\partial x}\ , \ \frac{\partial v}{\partial t} + fu = -\tilde{g}\frac{\partial h}{\partial y}\ , \ \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

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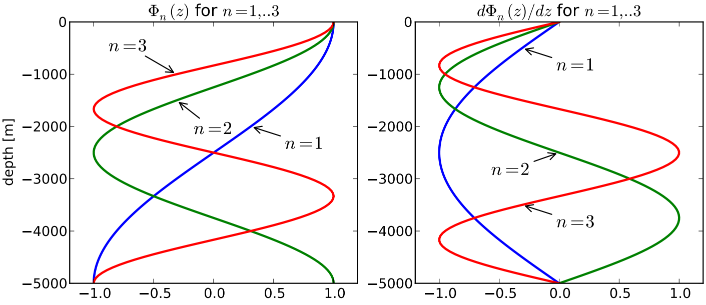
for $n = 1, 2, ... \rightarrow$ first, second, third, etc baroclinic mode
- $n = 0 \rightarrow$ barotropic case is special $\rightarrow$ use barotropic model

- Separation Ansatz with

$$\begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ p'(x,y,z,t) \\ w(x,y,z,t) \\ \rho'(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \tilde{u}(x,y,t) & \Phi(z) \\ \tilde{v}(x,y,t) & \Phi(z) \\ \rho_0 \tilde{p}(x,y,t) & \Phi(z) \\ \tilde{w}(x,y,t)/N^2(z) & d\Phi(z)/dz \\ \rho_0 \tilde{\rho}(x,y,t) & d\Phi(z)/dz \end{pmatrix}$$

- for $N = const \rightarrow \Phi_n = \cos(n\pi z/H)$



with $H = 5000\,\mathrm{m}$ and $N = const$

- gravity wave speed from layered model

$$c = \sqrt{gH}\ ,\ \ g = 9.81\,\mathrm{m/s^2} \text{ or } \ g = 9.81\,\mathrm{m/s^2}\Delta\rho/\rho_0$$

- for $N = const \rightarrow \tilde{g} = HN^2/(n^2\pi^2|f|)$
- define equivalent gravity phase speed $c_n = HN/(n\pi)$
- define equivalent depth $h_n = c_n^2/g = H^2N^2/(n^2\pi^2 g)$

n	h_n [m]	c_n [m/s]	R_n [km]
0	5000	221.5	2215
1	1.03	3.18	31.83
2	0.26	1.59	15.92
3	0.11	1.06	10.61
4	0.06	0.80	7.96
5	0.04	0.64	6.37
6	0.03	0.53	5.31
7	0.02	0.45	4.55
8	0.02	0.40	3.98
9	0.01	0.35	3.54

for $f = 10^{-4}s^{-1}$ and $N = 2\times 10^{-3}\,\mathrm{s}^{-1}$

- consider the (linearized) layered model with $f = 0$
  and also set y dependency to zero $\rightarrow$ $v = 0$

$$\frac{\partial u}{\partial t} - \cancel{fv} = -g\frac{\partial h}{\partial x}\,,\ \cancel{\frac{\partial v}{\partial t}} + \cancel{fu} = \cancel{-g\frac{\partial h}{\partial y}}\,,\ \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}\right) = 0$$

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- combine momentum and thickness equation to wave equation

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial t} = -g\frac{\partial}{\partial x}\frac{\partial h}{\partial x}\;,\; \frac{\partial}{\partial t}\frac{\partial h}{\partial t} + H\frac{\partial}{\partial t}\frac{\partial u}{\partial x} = 0 \;\rightarrow\; \frac{\partial^2 h}{\partial t^2} - gH\frac{\partial^2 h}{\partial x^2} = 0$$

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- try particular solution $h(x,t) = \sin k(x - ct)$

$$\frac{\partial h}{\partial t} \quad = -kc\cos k(x - ct) \quad ,\ \frac{\partial^2 h}{\partial t^2} = -(kc)^2 \sin k(x - ct)$$

- consider the (linearized) layered model with $f = 0$ and also set y dependency to zero $\rightarrow v = 0$

$$\frac{\partial u}{\partial t} - \cancel{fv} = -g\frac{\partial h}{\partial x}\,,\ \cancel{\frac{\partial v}{\partial t}} + \cancel{fu} = \cancel{-g\frac{\partial h}{\partial y}}\,,\ \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}\right) = 0$$

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$$\frac{\partial h}{\partial x} = k\cos k(x-ct)\quad,\quad \frac{\partial^2 h}{\partial x^2} = -k^2\sin k(x-ct)$$

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$$\frac{\partial h}{\partial x} = k\cos k(x - ct) \quad , \quad \frac{\partial^2 h}{\partial x^2} = -k^2 \sin k(x - ct)$$

- this works as long as

$$-(kc)^2 \sin(..) + k^2 gH \sin(..) = 0 \;\rightarrow\; c^2 = gH \;\rightarrow\; c = \pm\sqrt{gH}$$

  which is the dispersion relation for a long gravity wave (for $f = 0$)

- long gravity wave equation (for $f = 0$)

$$\frac{\partial^2 h}{\partial t^2} - gH\frac{\partial^2 h}{\partial x^2} = 0$$

- a particular solution is $h(x,t) = \sin k(x - ct)$

- long gravity wave equation (for $f = 0$)

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- a particular solution is $h(x, t) = \sin k(x - ct)$
- $h = A \sin k(x - ct)$ with constant amplitude A is also solution and also $h = A \sin(k(x - ct) + \phi)$ with constant phase ϕ

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$$h = A \sin k(x - ct) + B \cos k(x - ct)$$

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$$h = A \sin k(x - ct) + B \cos k(x - ct)$$

- or write more compact as

$$h = \mathrm{Re}\left\{ Ce^{ik(x-ct)} \right\}$$

with complex constant C with $\mathrm{Re}\{C\} = C_r$ and $\mathrm{Im}\{C\} = C_i$

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$$h = \mathrm{Re}\left\{Ce^{ik(x-ct)}\right\} = \mathrm{Re}\left\{(C_r + iC_i)\left(\cos k(x - ct) + i\sin k(x - ct)\right)\right\}$$

with complex constant C with $\mathrm{Re}\{C\} = C_r$ and $\mathrm{Im}\{C\} = C_i$
with Euler relation $e^{i\phi} = \cos\phi + i\sin\phi$

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with complex constant C with $\mathrm{Re}\{C\} = C_r$ and $\mathrm{Im}\{C\} = C_i$
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- gravity wave equation (for $f = 0$) $\partial^2 h/\partial t^2 - gH\partial^2 h/\partial x^2 = 0$
- wave solution is given by $h = Ce^{ik(x-ct)}$ with complex amplitude C (Re is often dropped for convenience) as long as $c = \pm\sqrt{gH}$

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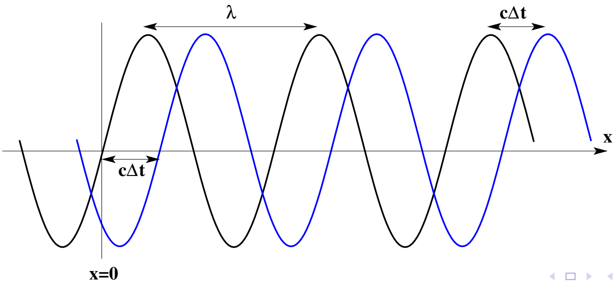
- consider $h = \sin k(x - ct)$ at $t = 0 \rightarrow h = \sin kx$ (black line) $\rightarrow$ wavelength is $\lambda = 2\pi/k$, k is wavenumber



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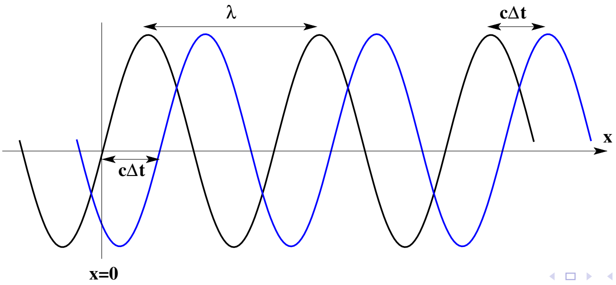

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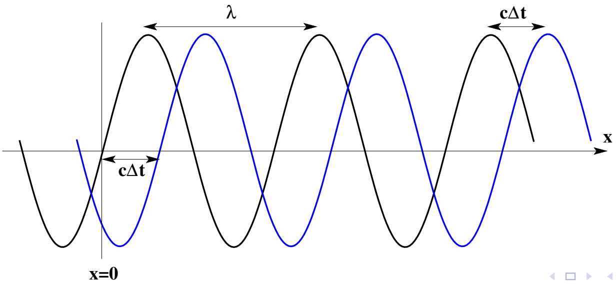
- consider h at $t = 0$ (black line) and at later time $t = \Delta t$ (blue line) phase where $h = 0$ was at $t = 0$ at $x = 0$ but at $t = \Delta t$ at $x = c\Delta t$ $\rightarrow c = dx/dt$ is the velocity at which constant phase propagates $\rightarrow$ phase velocity



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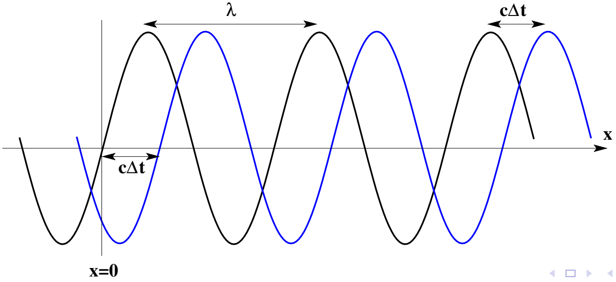
- wavelength $\lambda = 2\pi/k$ with wavenumber k
- phase velocity c with dispersion relation $c = \pm\sqrt{gH}$



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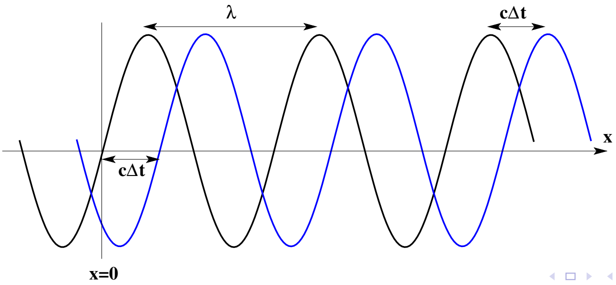
- wavelength $\lambda = 2\pi/k$ with wavenumber k
- phase velocity c with dispersion relation $c = \pm\sqrt{gH}$
- rewrite solution as $h = Ce^{i(kx-\omega t)}$ with frequency $\omega = ck$ and

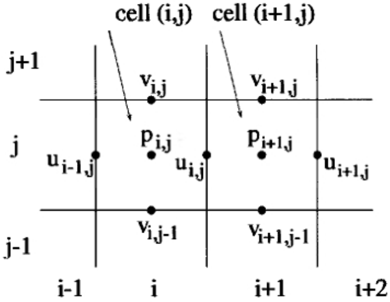
$$\omega = \pm k\sqrt{gH}$$

- $T = 2\pi/\omega$ is the period in which a fixed phase pass a fixed point



- the linearized 1D shallow water equations (for $f = 0$) are

$$\frac{\partial u}{\partial t} = -\frac{\partial h}{\partial x} \ , \quad \frac{\partial h}{\partial t} = -c^2 \frac{\partial u}{\partial x}$$



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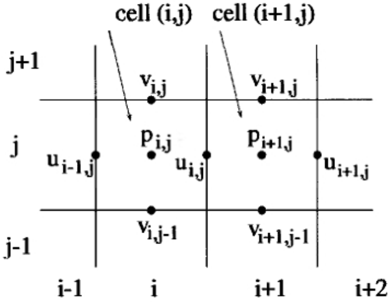

- discrete shallow water equations for staggered grid

$$\frac{du_n}{dt} = -\delta^+ h_n \quad , \quad \frac{dh_n}{dt} = -c^2 \delta^- u_n$$

with the finite differencing operators

$$\delta^+ h_n = (h_{n+1} - h_n)/\Delta \quad , \quad \delta^- h_n = (h_n - h_{n-1})/\Delta$$

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- wave ansatz $u_n = \hat{u}(t)e^{ikx}$, $h_n = \hat{h}(t)e^{ikx}$ with $x = n\Delta$, $n = 0, 1, ..$

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- define $i\hat{k}^+(k, \Delta) = (e^{ik\Delta} - 1)/\Delta$ and $i\hat{k}^-(k, \Delta) = (1 - e^{-ik\Delta})/\Delta$

$$\frac{d}{dt}(\hat{u}e^{ikn\Delta}) = -\hat{h}e^{ikn\Delta} i\hat{k}^+ \quad , \quad \frac{d}{dt}(\hat{h}e^{ikn\Delta}) = -c^2 \hat{u}e^{ikn\Delta} i\hat{k}^-$$

drop factor $e^{ikn\Delta} \rightarrow$ discrete version for staggered grid after Fourier transform

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- discrete version for staggered grid after Fourier transform

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with $i\hat{k}^+ = (e^{ik\Delta} - 1)/\Delta$ and $i\hat{k}^- = (1 - e^{-ik\Delta})/\Delta$

- for $\Delta \to 0$ and $\cos k\Delta \approx 1$ and $\sin k\Delta \approx k\Delta$

$$i\hat{k}^+ = (\cos k\Delta + i \sin k\Delta - 1)/\Delta \approx ik \;\; , \;\; \lim_{\Delta \to 0} \hat{k}^+ = k$$

$$i\hat{k}^- = (1 - \cos k\Delta - i \sin(-k\Delta))/\Delta \approx ik \;\; , \;\; \lim_{\Delta \to 0} \hat{k}^- = k$$

- discrete version converges to analytical Fourier transform

$$\frac{d\hat{u}}{dt} = -ik\hat{h} \;\; , \;\; \frac{d\hat{h}}{dt} = -c^2 ik\hat{u}$$

but is different for finite Δ

- discrete version for staggered grid after Fourier transform

$$\frac{d\hat{u}}{dt} = -i\hat{k}^{+}\hat{h} \quad , \quad \frac{d\hat{h}}{dt} = -c^2 i\hat{k}^{-}\hat{u}$$

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- discrete version for staggered grid after Fourier transform

$$\frac{d\hat{u}}{dt} = -i\hat{k}^+\hat{h} \;\; , \quad \frac{d\hat{h}}{dt} = -c^2 i\hat{k}^-\hat{u}$$

with $i\hat{k}^+ = (e^{ik\Delta} - 1)/\Delta$ and $i\hat{k}^- = (1 - e^{-ik\Delta})/\Delta$

- analytical ansatz in time $\hat{u}(t) = \hat{U}e^{-i\bar{\omega}t}$ and $\hat{h}(t) = \hat{H}e^{-i\bar{\omega}t}$

$$-i\bar{\omega}\hat{U}e^{-i\bar{\omega}t} = -i\hat{k}^+\hat{H}e^{-i\bar{\omega}t} \quad , \quad -i\bar{\omega}\hat{H}e^{-i\bar{\omega}t} = -c^2 i\hat{k}^-\hat{U}e^{-i\bar{\omega}t}$$

$$\bar{\omega}\hat{U} = \hat{k}^+\hat{H} \quad , \quad \bar{\omega}\hat{H} = c^2\hat{k}^-\hat{U}$$

$$\bar{\omega}^2\hat{U} = \hat{k}^+c^2\hat{k}^-\hat{U} \quad \rightarrow \quad \bar{\omega} = \pm c\sqrt{\hat{k}^+\hat{k}^-}$$

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- $\bar{\omega}$ is real since

$$\begin{aligned}
\hat{k}^+\hat{k}^- &= -i(e^{ik\Delta} - 1)/\Delta(-i)(1 - e^{-ik\Delta})/\Delta \\
&= -(e^{ik\Delta} - 1)(1 - e^{-ik\Delta})/\Delta^2 \\
&= -(e^{ik\Delta} - 1 - e^{-ik\Delta}(e^{ik\Delta} - 1))/\Delta^2 = (2 - \cos k\Delta)/\Delta^2
\end{aligned}$$

- compare with analytical dispersion relation $\omega = \pm ck$
- since $\lim_{\Delta\to 0}(\hat{k}^+, \hat{k}^-) = k$ it follows that $\lim_{\Delta\to 0}\bar{\omega} = \omega$
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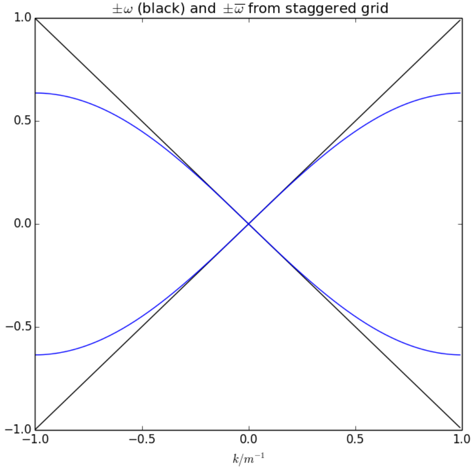
- example for $c = 1\,\mathrm{m/s}$ and $\Delta = \pi\,\mathrm{m}$



- ω (black), $\bar{\omega}$ staggered grid (blue)

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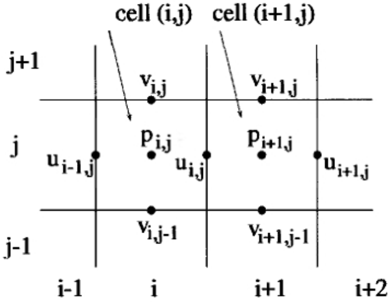

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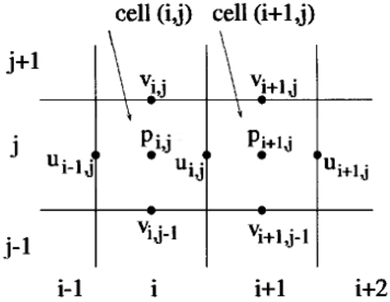

- discrete shallow water equations for unstaggered grid

$$\frac{du_n}{dt} = -(\delta^+ h_n + \delta^- h_n)/2 \quad , \quad \frac{dh_n}{dt} = -c^2(\delta^+ u_n + \delta^- u_n)/2$$

with the finite differencing operators

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- discrete version for unstaggered grid

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- wave ansatz using $\delta^+ h_n = \hat{h} e^{ikn\Delta} i\hat{k}^+$ and $\delta^- u_n = \hat{u} e^{ikn\Delta} i\hat{k}^-$

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$$\frac{d}{dt}(\hat{h} e^{ikn\Delta}) = -c^2 \hat{u} e^{ikn\Delta} i(\hat{k}^+ + \hat{k}^-)/2$$

$$\frac{d\hat{u}}{dt} = -i\hat{h}(\hat{k}^+ + \hat{k}^-)/2 \quad , \quad \frac{d\hat{h}}{dt} = -ic^2 \hat{u}(\hat{k}^+ + \hat{k}^-)/2$$

discrete version for unstaggered grid after Fourier transform

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$$\frac{d\hat{u}}{dt} = -i(\hat{k}^{+} + \hat{k}^{-})\hat{h}/2 \ , \quad \frac{d\hat{h}}{dt} = -c^2 i(\hat{k}^{+} + \hat{k}^{-})\hat{u}/2$$

with $i\hat{k}^{+} = (e^{ik\Delta} - 1)/\Delta$ and $i\hat{k}^{-} = (1 - e^{-ik\Delta})/\Delta$

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- analytical ansatz in time $\hat{u}(t) = \hat{U}e^{-i\bar{\omega}t}$ and $\hat{h}(t) = \hat{H}e^{-i\bar{\omega}t}$

$$-i\bar{\omega}\hat{U}e^{-i\bar{\omega}t} = -i(\hat{k}^+ + \hat{k}^-)\hat{H}e^{-i\bar{\omega}t}/2 \;\rightarrow\; \bar{\omega}\hat{U} = (\hat{k}^+ + \hat{k}^-)\hat{H}/2$$

$$-i\bar{\omega}\hat{H}e^{-i\bar{\omega}t} = -c^2 i(\hat{k}^+ + \hat{k}^-)\hat{U}e^{-i\bar{\omega}t}/2 \;\rightarrow\; \bar{\omega}\hat{H} = c^2(\hat{k}^+ + \hat{k}^-)\hat{U}/2$$

$$\bar{\omega}^2 = c^2(\hat{k}^+ + \hat{k}^-)^2/4 \;\rightarrow\; \bar{\omega} = \pm c/2\sqrt{(\hat{k}^+ + \hat{k}^-)^2}$$

since $(i\hat{k}^+)^* = (e^{-ik\Delta} - 1)/\Delta = -i\hat{k}^- \;\rightarrow\; (\hat{k}^+)^* = \hat{k}^-$
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- compare with analytical dispersion relation $\omega = \pm ck$
and staggered grid relation $\bar{\omega} = \pm c\sqrt{\hat{k}^+\hat{k}^-}$
- since $\lim_{\Delta\to 0}(\hat{k}^+, \hat{k}^-) = k$ it follows that $\lim_{\Delta\to 0}\bar{\omega} = \omega$
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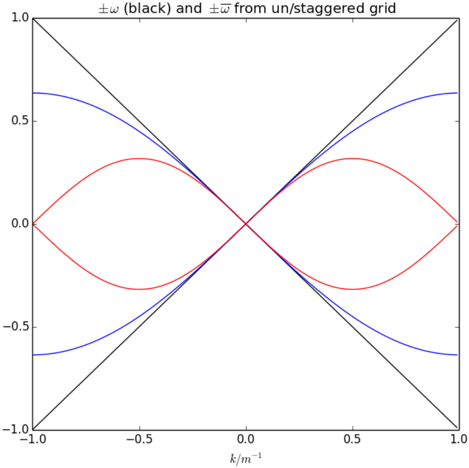
- example for $c = 1\,\mathrm{m/s}$ and $\Delta = \pi\,\mathrm{m}$



- ω (black), $\bar{\omega}$ staggered grid (blue), $\bar{\omega}$ unstaggered grid (red)

- discrete version for staggered grid after Fourier transform

$$\frac{d\hat{u}}{dt} = -i\hat{k}^{+}\hat{h} \quad , \quad \frac{d\hat{h}}{dt} = -ic^2\hat{k}^{-}\hat{u}$$

with $i\hat{k}^{+} = (e^{ik\Delta_x} - 1)/\Delta_x$ and $i\hat{k}^{-} = (1 - e^{-ik\Delta_x})/\Delta_x$

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- time discretisation $d\hat{u}/dt|_{t=m\Delta_t} = (\hat{u}_{m+1} - \hat{u}_m)/\Delta_t$, $t = m\Delta_t$, ...

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- transform in time with $\hat{u}_m = \hat{U}e^{-i\bar{\Omega}m\Delta_t}$ and $\hat{h}_m = \hat{H}e^{-i\bar{\Omega}m\Delta_t}$

$$\hat{U}(e^{-i\bar{\Omega}\Delta_t} - 1) = -i\Delta_t\hat{k}^+\hat{H} \qquad , \qquad \hat{H}(e^{-i\bar{\Omega}\Delta_t} - 1) = -ic^2\Delta_t\hat{k}^-\hat{U}$$

$$\begin{aligned} (e^{-i\bar{\Omega}\Delta_t} - 1)^2 &= -c^2\Delta_t^2\hat{k}^+\hat{k}^- = -\Delta_t^2\bar{\omega}^2 \\ e^{-i\bar{\Omega}\Delta_t} &= 1 \pm i\Delta_t|\bar{\omega}| \\ \bar{\Omega} &= i\ln(1 \pm i\Delta_t|\bar{\omega}|)/\Delta_t \end{aligned}$$

- frequency $\bar{\Omega}$ from time discretisation differs from $\bar{\omega}$ (and ω)

- frequency $\bar{\Omega}$ from forward time discretisation is given by

$$\bar{\Omega} = i \ln(z)/\Delta_t$$

with complex number $z = 1 \pm i\Delta_t|\bar{\omega}|$

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- rewrite z as $z = re^{i\phi}$ with

$$r = \sqrt{1 + \Delta_t^2\bar{\omega}^2} \quad , \quad \phi = \tan^{-1}(\pm\Delta_t|\bar{\omega}|)$$

since $\ln z = \ln r + i\phi$ it follows

$$\bar{\Omega} = (i \ln r - \phi)/\Delta_t \quad , \quad \mathrm{Re}(\bar{\Omega}) = -\phi/\Delta_t \quad , \quad \mathrm{Im}(\bar{\Omega}) = \ln(r)/\Delta_t$$

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- using $\hat{h}_m = \hat{H}e^{-i\bar{\Omega}m\Delta_t} \rightarrow \hat{h}_m = \hat{H}e^{-i\mathrm{Re}(\bar{\Omega})m\Delta_t}e^{\mathrm{Im}(\bar{\Omega})m\Delta_t}$

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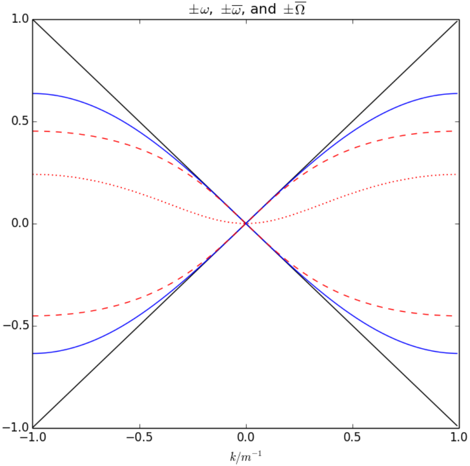
- since $r > 1$ it follows $\mathrm{Im}(\bar{\Omega}) > 0$ and scheme is unstable for large t $\rightarrow$ unconditionally unstable scheme. Do not use it!
- since $\tan^{-1}(x) \approx x$ for small x it follows that $\lim_{\Delta_t \rightarrow 0} \bar{\Omega} = \pm|\bar{\omega}|$

- example for $c = 1\,\mathrm{m/s}$, $\Delta_x = \pi\,\mathrm{m}$, $\Delta_t = 2\,\mathrm{s}$



- ω (black) $\bar{\omega}$ (blue), $\mathrm{Re}(\bar{\Omega})$ (red dotted), $\mathrm{Im}(\bar{\Omega})$ (red dashed)

- discrete version for staggered grid after Fourier transform

$$\frac{d\hat{u}}{dt} = -i\hat{k}^{+}\hat{h} \quad , \quad \frac{d\hat{h}}{dt} = -ic^2\hat{k}^{-}\hat{u}$$

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with $i\hat{k}^{+} = (e^{ik\Delta_x} - 1)/\Delta_x$ and $i\hat{k}^{-} = (1 - e^{-ik\Delta_x})/\Delta_x$

- time discretisation $d\hat{u}/dt|_{t=(m+1)\Delta_t} = (\hat{u}_{m+1} - \hat{u}_m)/\Delta_t$, ...

$$\hat{u}_{m+1} - \hat{u}_m = -i\Delta_t\hat{k}^{+}\hat{h}_{m+1} \quad , \quad \hat{h}_{m+1} - \hat{h}_m = -ic^2\Delta_t\hat{k}^{-}\hat{u}_{m+1}$$

- transform in time with $\hat{u}_m = \hat{U}e^{-i\bar{\Omega}m\Delta_t}$ and $\hat{h}_m = \hat{H}e^{-i\bar{\Omega}m\Delta_t}$

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$$(1 + \Delta_t^2\bar{\omega}^2)e^{-2i\bar{\Omega}\Delta_t} - 2e^{-i\bar{\Omega}\Delta_t} + 1 = 0$$

$$e^{-2i\bar{\Omega}\Delta_t} - 2\alpha e^{-i\bar{\Omega}\Delta_t} + \alpha^2 = -\alpha + \alpha^2 \;\rightarrow\; (e^{-i\bar{\Omega}\Delta_t} - \alpha)^2 = \alpha(\alpha - 1)$$

$$e^{-i\bar{\Omega}\Delta_t} = \alpha \pm \sqrt{\alpha(\alpha - 1)} \;\rightarrow\; \bar{\Omega} = i\ln(\alpha \pm \sqrt{\alpha(\alpha - 1)})/\Delta_t$$

with $\alpha = (1 + \Delta_t^2\bar{\omega}^2)^{-1}$

- frequency $\bar{\Omega}$ from implicit time discretisation is given by

$$\bar{\Omega} = i \ln(\alpha \pm \sqrt{\alpha(\alpha - 1)})/\Delta_t = i \ln(\alpha \pm i\sqrt{\alpha(1 - \alpha)})/\Delta_t$$

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$$r = \sqrt{\alpha^2 + \alpha^2(\alpha - 1)^2} \quad , \quad \phi = \tan^{-1} \pm\sqrt{\alpha(1-\alpha)}/\alpha$$

since $\ln z = \ln r + i\phi$ it follows

$$\bar{\Omega} = (i \ln r - \phi)/\Delta_t \quad , \quad \mathrm{Re}(\bar{\Omega}) = -\phi/\Delta_t \quad , \quad \mathrm{Im}(\bar{\Omega}) = \ln(r)/\Delta_t$$

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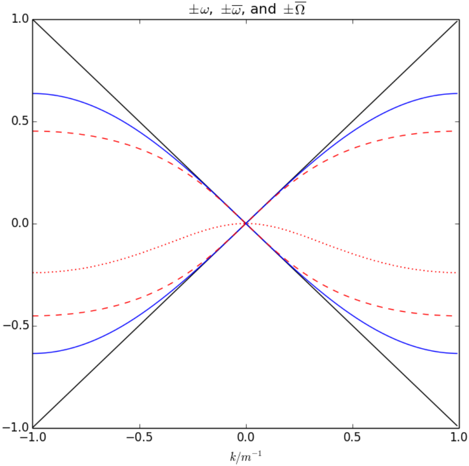
- since $r < 1$ for $0 < \alpha < 1$ it follows that $\mathrm{Im}(\bar{\Omega}) < 0$, scheme is unconditionally stable but strongly damped

- example for $c = 1\,\mathrm{m/s}$, $\Delta_x = \pi\,\mathrm{m}$, $\Delta_t = 2\,\mathrm{s}$



- ω (black) $\bar{\omega}$ (blue), Re($\bar{\Omega}$) (red dotted), Im($\bar{\Omega}$) (red dashed)

- discrete version for staggered grid after Fourier transform

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- use mixed time discretisation

$$\hat{u}_{m+1} - \hat{u}_m = -i\Delta_t\hat{k}^{+}\hat{h}_m \quad , \quad \hat{h}_{m+1} - \hat{h}_m = -ic^2\Delta_t\hat{k}^{-}\hat{u}_{m+1}$$

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$$e^{-2i\bar{\Omega}\Delta_t} + 2\beta e^{-i\bar{\Omega}\Delta_t} + \beta^2 = (e^{-i\bar{\Omega}\Delta_t} + \beta)^2 = \beta^2 - 1$$

$$e^{-i\bar{\Omega}\Delta_t} = \pm\sqrt{\beta^2 - 1} - \beta \; , \; \bar{\Omega} = i\ln(\pm\sqrt{\beta^2 - 1} - \beta)/\Delta_t$$

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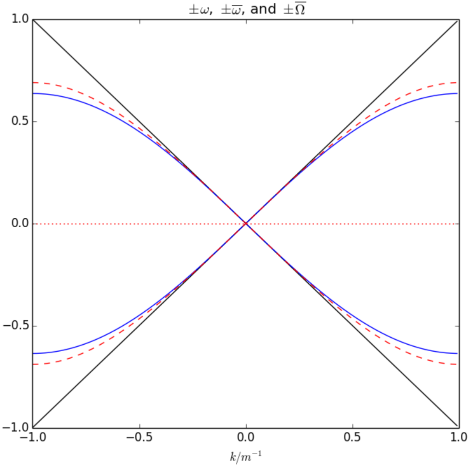
with $\beta = \Delta_t^2\bar{\omega}^2/2 - 1$

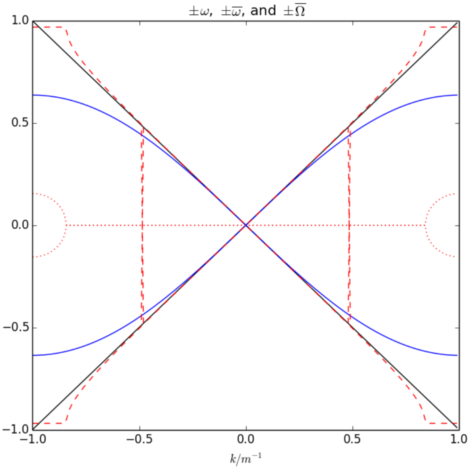
- stable if $\beta^2 > 1 \rightarrow$ conditionally stable scheme

- example for $c = 1\,\mathrm{m/s}$, $\Delta_x = \pi\,\mathrm{m}$, $\Delta_t = 2\,\mathrm{s}$



- ω (black) $\bar{\omega}$ (blue), Re($\bar{\Omega}$) (red dotted), Im($\bar{\Omega}$) (red dashed)

- example for $c = 1\,\mathrm{m/s}$, $\Delta_x = \pi\,\mathrm{m}$, $\Delta_t = (\pi + 0.1)\,\mathrm{s}$



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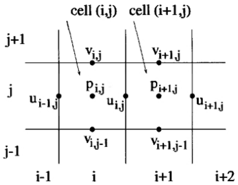
- the linearized 2D shallow water equations (for $f = 0$) are

$$\frac{\partial u}{\partial t} = -\frac{\partial h}{\partial x} \;,\quad \frac{\partial v}{\partial t} = -\frac{\partial h}{\partial y} \;,\quad \frac{\partial h}{\partial t} = -c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$



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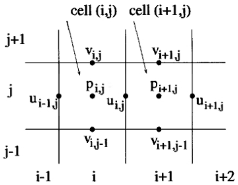

- discrete version for staggered grid

$$\frac{du_{j,k}}{dt} = -\delta_x^+ h_{j,k} \;,\quad \frac{dv_{j,k}}{dt} = -\delta_y^+ h_{j,k} \;,\quad \frac{dh_{j,k}}{dt} = -c^2\left(\delta_x^- u_{j,k} + \delta_y^- v_{j,k}\right)$$

with the finite differencing operators

$$\delta_x^+ h_{j,k} = (h_{j+1,k} - h_{j,k})/\Delta_x \;,\quad \delta_x^- h_{j,k} = (h_{j,k} - h_{j-1,k})/\Delta_x$$
$$\delta_y^+ h_{j,k} = (h_{j,k+1} - h_{j,k})/\Delta_y \;,\quad \delta_y^- h_{j,k} = (h_{j,k} - h_{j,k-1})/\Delta_y$$

- and mixed time discretisation