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Lecture # 3

Gravity waves with rotation

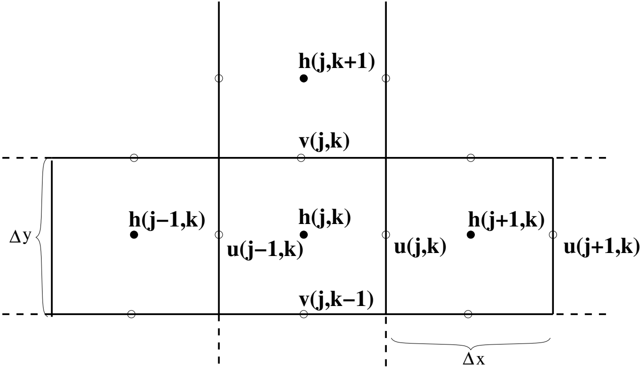
Discrete 2D wave solutions

- Arakawa C grid without rotation
- Arakawa C grid with rotation
- Time discretisation with rotation

- taken from "Waves in the ocean and atmosphere", Pedlosky (2013), chapter 11 (partly) and chapter 13

- the linearized 2D shallow water equations (for $f = 0$) are

$$\frac{\partial u}{\partial t} = -\frac{\partial h}{\partial x} \ , \quad \frac{\partial v}{\partial t} = -\frac{\partial h}{\partial y} \ , \quad \frac{\partial h}{\partial t} = -c^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$



- this arrangement of u, v and h is called Arakawa-C grid
- there are other possibilities: Arakawa-A, B, C, D

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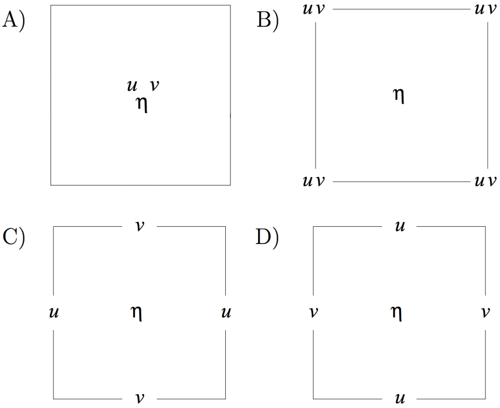

- we are using the C grid arrangement with best properties for waves

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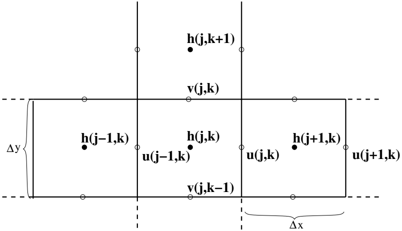

- discrete shallow water equations

$$\frac{du_{j,k}}{dt} = -\delta_x^+ h_{j,k} \ , \quad \frac{dv_{j,k}}{dt} = -\delta_y^+ h_{j,k} \ , \quad \frac{dh_{j,k}}{dt} = -c^2 \left(\delta_x^- u_{j,k} + \delta_y^- v_{j,k} \right)$$

with the finite differencing operators

$$\delta_x^+ h_{j,k} = (h_{j+1,k} - h_{j,k})/\Delta_x \ , \quad \delta_x^- h_{j,k} = (h_{j,k} - h_{j-1,k})/\Delta_x$$
$$\delta_y^+ h_{j,k} = (h_{j,k+1} - h_{j,k})/\Delta_y \ , \quad \delta_y^- h_{j,k} = (h_{j,k} - h_{j,k-1})/\Delta_y$$

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- wave ansatz $h_{j,k} = \hat{h} e^{i(k_x j\Delta_x + k_y k\Delta_y)}$, ... yields

$$\frac{du_{j,k}}{dt} = -\hat{h} e^{i(k_x j\Delta_x + k_y k\Delta_y)} i\hat{k}_x^+ \quad , \quad \frac{dv_{j,k}}{dt} = -\hat{h} e^{i(k_x j\Delta_x + k_y k\Delta_y)} i\hat{k}_y^+$$

$$\frac{dh_{j,k}}{dt} = -c^2 \left(\hat{u} e^{i(k_x j\Delta_x + k_y k\Delta_y)} i\hat{k}_x^- + \hat{v} e^{i(k_x j\Delta_x + k_y k\Delta_y)} i\hat{k}_y^-\right)$$

with $i\hat{k}_x^+ = (e^{ik_x\Delta_x} - 1)/\Delta_x$, $i\hat{k}_y^- = (1 - e^{-ik_y\Delta_y})/\Delta_y$, ...

for which $\lim_{\Delta_x \to} \hat{k}_x^+, \hat{k}_x^- = k_x$ and $\lim_{\Delta_y \to} \hat{k}_y^+, \hat{k}_y^- = k_y$ holds

- wave ansatz $h_{j,k} = \hat{h}e^{i(k_x j\Delta_x + k_y k\Delta_y)}$, ... yields

$$\frac{d\hat{u}}{dt} = -\hat{h}i\hat{k}_x^+ \,, \quad \frac{d\hat{v}}{dt} = -\hat{h}i\hat{k}_y^+ \,, \quad \frac{d\hat{h}}{dt} = -ic^2\left(\hat{u}\hat{k}_x^- + \hat{v}\hat{k}_y^-\right)$$

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- analytical ansatz in time $\hat{u}(t) = \hat{U} e^{-i\bar{\omega} t}$, $\hat{h}(t) = \hat{H} e^{-i\bar{\omega} t}$, ...

$$-i\bar{\omega}\hat{U} = -\hat{H} i \hat{k}_x^+ \ , \quad -i\bar{\omega}\hat{V} = -\hat{H} i \hat{k}_y^+ \ , \quad -i\bar{\omega}\hat{H} = -ic^2 \left(\hat{U} \hat{k}_x^- + \hat{V} \hat{k}_y^- \right)$$

$$\bar{\omega}^2 = c^2 \left(\hat{k}_x^+ \hat{k}_x^- + \hat{k}_y^+ \hat{k}_y^- \right) \quad \rightarrow \quad \bar{\omega} = \pm c \sqrt{\hat{k}_x^+ \hat{k}_x^- + \hat{k}_y^+ \hat{k}_y^-}$$

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$$\bar{\omega}^2 = c^2\left(\hat{k}_x^+\hat{k}_x^- + \hat{k}_y^+\hat{k}_y^-\right) \quad \rightarrow \quad \bar{\omega} = \pm c\sqrt{\hat{k}_x^+\hat{k}_x^- + \hat{k}_y^+\hat{k}_y^-}$$

- since $\hat{k}_x^+ = (\hat{k}_x^-)^*$ and $\hat{k}_y^- = (\hat{k}_y^+)^*$ gravity wave frequency $\bar{\omega}$ is real

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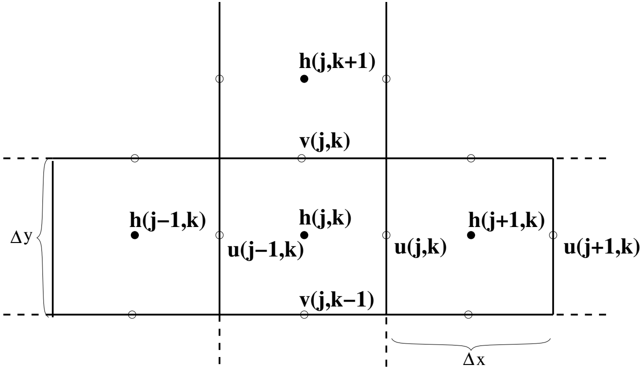
- since $\hat{k}_x^+ = (\hat{k}_x^-)^*$ and $\hat{k}_y^- = (\hat{k}_y^+)^*$ gravity wave frequency $\bar{\omega}$ is real
- compare with analytical dispersion relation $\omega = \pm c \sqrt{k_x^2 + k_y^2}$

  $\rightarrow \lim_{\Delta_x, \Delta_y \rightarrow 0} \bar{\omega} = \omega$ but for finite Δ_x, Δ_y there is grid dispersion

- the linearized 2D shallow water equations with rotation are

$$\frac{\partial u}{\partial t} = fv - \frac{\partial h}{\partial x} \ , \ \frac{\partial v}{\partial t} = -fu - \frac{\partial h}{\partial y} \ , \ \frac{\partial h}{\partial t} = -c^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$



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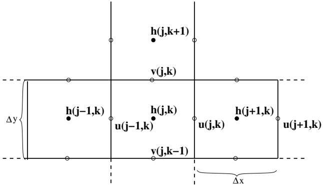

- discrete shallow water equations with rotation

$$\frac{du_{j,k}}{dt} = f\overline{\overline{v_{j,k}}^{j+}}^{k-} - \delta_x^+ h_{j,k} \;,\; \frac{dv_{j,k}}{dt} = -f\overline{\overline{u_{j,k}}^{j-}}^{k+} - \delta_y^+ h_{j,k}$$

with the finite averaging operators

$$\overline{h_{j,k}}^{j+} = (h_{j,k} + h_{j+1,k})/2 \;,\; \overline{h_{j,k}}^{j-} = (h_{j,k} + h_{j-1,k})/2$$
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and the finite difference operators δ_x^+, ... from before

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- with the functions $\hat{1}_x^+(k_x, \Delta_x)$ and $\hat{1}_x^-(k_x, \Delta_x)$

$$\hat{1}_x^+ = \left(1 + e^{ik_x\Delta x}\right)/2 \ , \ \hat{1}_x^- = \left(1 + e^{-ik_x\Delta x}\right)/2$$

with $\lim_{\Delta_x \to 0} \hat{1}_x^+, \hat{1}_x^- = 1$ and similar for $\hat{1}_y^+$ and $\hat{1}_y^-$

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- Fourier transform in x, y yields

$$\frac{d\hat{u}}{dt} = f\hat{1}_x^+\hat{1}_y^-\hat{v} - i\hat{k}_x^+\hat{h} \ , \ \frac{d\hat{v}}{dt} = -f\hat{1}_x^-\hat{1}_y^+\hat{u} - i\hat{k}_y^+\hat{h} \ , \ \frac{d\hat{h}}{dt} = -ic^2\left(\hat{u}\hat{k}_x^- + \hat{v}\hat{k}\right.$$

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- analytical ansatz in time $\hat{u}(t) = \hat{U}e^{-i\bar{\omega}t}$, $\hat{h}(t) = \hat{H}e^{-i\bar{\omega}t}$, ...
  and combining yields

$$\bar{\omega}\left(\bar{\omega}^2 - f^2\hat{1}_x^-\hat{1}_y^+\hat{1}_x^+\hat{1}_y^-\right) = c^2\bar{\omega}(\hat{k}_x^+\hat{k}_x^- + \hat{k}_y^+\hat{k}_y^-)$$

  geostrophic mode $\bar{\omega} = 0$ and gravity wave mode

$$\bar{\omega} = \pm\sqrt{f^2\hat{1}_x^-\hat{1}_y^+\hat{1}_x^+\hat{1}_y^- + c^2(\hat{k}_x^+\hat{k}_x^- + \hat{k}_y^+\hat{k}_y^-)}$$

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- since $(\hat{1}_x^+)^* = \hat{1}_x^-$, $(\hat{k}_x^+)^* = \hat{k}_x^-$, ... gravity wave frequency $\bar{\omega}$ is real
- compare with analytical dispersion relation $\omega = \sqrt{f^2 + c^2(k_x^2 + k_y^2)}$

- shallow water equations with rotation for $h = const$

$$\partial_t u = fv \quad , \quad \partial_t v = -fu$$

- forward time discretisation

$$u^{m+1} - u^m = \Delta_t f v^m \; , \; v^{m+1} - v^m = -\Delta_t f u^m$$

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- semi discrete Fourier transform $u = \hat{U} e^{i(k_x x + k_y y - \omega m \Delta_t)}$, ...

$$\hat{U}(e^{-i\omega\Delta_t} - 1) = \Delta_t f \hat{V} \ , \ \hat{V}(e^{-i\omega\Delta_t} - 1) = -\Delta_t f \hat{U}$$
$$(e^{-i\omega\Delta_t} - 1)^2 = -\Delta_t^2 f^2 \ \rightarrow \ e^{-i\omega\Delta_t} = 1 \pm i\Delta_t f$$
$$\rightarrow \ \omega = i \ln(1 \pm i\Delta_t f)/\Delta_t$$

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$$\begin{aligned} \hat{U}(e^{-i\omega\Delta_t} - 1) = \Delta_t f \hat{V} \ , \ \hat{V}(e^{-i\omega\Delta_t} - 1) &= -\Delta_t f \hat{U} \\ (e^{-i\omega\Delta_t} - 1)^2 = -\Delta_t^2 f^2 \ \rightarrow \ e^{-i\omega\Delta_t} &= 1 \pm i\Delta_t f \\ \rightarrow \ \omega &= i \ln(1 \pm i\Delta_t f)/\Delta_t \end{aligned}$$

- rewrite $z = 1 \pm i\Delta_t f$ as $z = re^{i\phi}$ with $r = \sqrt{1 + \Delta_t^2 f^2}$ and with $\ln z = \ln r + i\phi$ and thus $\mathrm{Im}(\omega) = \ln(r)/\Delta_t$

- shallow water equations with rotation for $h = const$

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- forward time discretisation

$$u^{m+1} - u^m = \Delta_t f v^m \ , \ v^{m+1} - v^m = -\Delta_t f u^m$$

- semi discrete Fourier transform $u = \hat{U} e^{i(k_x x + k_y y - \omega m \Delta_t)}$, ...

$$\hat{U}(e^{-i\omega\Delta_t} - 1) = \Delta_t f \hat{V} \ , \ \hat{V}(e^{-i\omega\Delta_t} - 1) = -\Delta_t f \hat{U}$$
$$(e^{-i\omega\Delta_t} - 1)^2 = -\Delta_t^2 f^2 \ \rightarrow \ e^{-i\omega\Delta_t} = 1 \pm i\Delta_t f$$
$$\rightarrow \ \omega = i \ln(1 \pm i\Delta_t f)/\Delta_t$$

- rewrite $z = 1 \pm i\Delta_t f$ as $z = re^{i\phi}$ with $r = \sqrt{1 + \Delta_t^2 f^2}$ and with $\ln z = \ln r + i\phi$ and thus $\mathrm{Im}(\omega) = \ln(r)/\Delta_t$
- since $r > 1$ it follows $\mathrm{Im}(\omega) > 0$ and scheme is unconditionally unstable

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$$\omega = i \ln(\alpha \pm i\sqrt{\alpha(1-\alpha)})/\Delta_t$$

with $0 < \alpha = 1/(1 + \Delta_t^2 f^2) < 1$

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$$\omega = i \ln \left((1 - \Delta_t^2 f^2/2) \pm i\sqrt{1 - (1 - \Delta_t^2 f^2/2)^2} \right) / \Delta_t$$

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- since $r = 1$ it follows $\mathrm{Im}(\omega) = \ln(r)/\Delta_t = 0$ and scheme is conditionally stable for $\Delta_t < \sqrt{2}/f$