# Bornö Summerschool
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# Lecture # 4

Gravity waves

- Gravity waves with rotation

Potential vorticity and Rossby adjustment

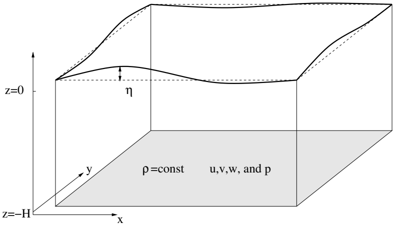
- Potential vorticity
- Geostrophic adjustment

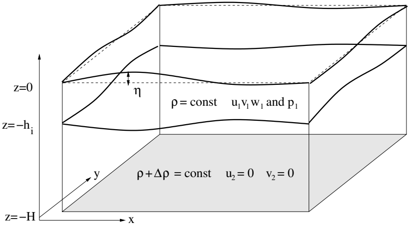
- non-linear "barotropic model" and "baroclinic model"

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} u - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} v + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} h + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- $h$ is total thickness ("barotropic") or layer interface $h_i$ ("baroclinic")
- either $g = 9.81\,\mathrm{m/s^2}$ ("barotropic") or $g \rightarrow g\Delta\rho/\rho_0$ ("baroclinic")



- non-linear "barotropic model" and "baroclinic model"

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + f \underset{\neg}{\boldsymbol{u}} = -g \boldsymbol{\nabla} h \quad , \quad \frac{\partial h}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} h + h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

with $\underset{\neg}{\boldsymbol{u}} = (-v, u)$ which denotes anticlockwise rotation of $\boldsymbol{u}$ by $90^o$

- non-linear "barotropic model" and "baroclinic model"

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + f \ \underset{\neg}{\boldsymbol{u}} = -g \boldsymbol{\nabla} h \ \ , \ \ \frac{\partial h}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} h + h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

with $\underset{\neg}{\boldsymbol{u}} = (-v, u)$ which denotes anticlockwise rotation of $\boldsymbol{u}$ by $90^o$

- split $h = H + h'$ with $H = const \gg |h|$

$$\frac{\partial h'}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} h' + (H + h') \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

$$\frac{\partial g h'}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} g h' + g h' \boldsymbol{\nabla} \cdot \boldsymbol{u} + g H \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

- non-linear "barotropic model" and "baroclinic model"

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + f\,\underset{\neg}{\boldsymbol{u}} = -g\boldsymbol{\nabla}h \;\;,\;\; \frac{\partial h}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}h + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

with $\underset{\neg}{\boldsymbol{u}} = (-v, u)$ which denotes anticlockwise rotation of $\boldsymbol{u}$ by $90^o$

- split $h = H + h'$ with $H = const \gg |h|$

$$\frac{\partial h'}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}h' + (H + h')\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial gh'}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}gh' + gh'\boldsymbol{\nabla}\cdot\boldsymbol{u} + gH\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

- rename to $h = gh'$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + f\,\underset{\neg}{\boldsymbol{u}} = -\boldsymbol{\nabla}h \;\;,\;\; \frac{\partial h}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}h + h\boldsymbol{\nabla}\cdot\boldsymbol{u} + c^2\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

with gravity wave speed (of the waves with $f = 0$) $c = \sqrt{gH}$

- non-linear barotropic or reduced gravity model

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + f \, \underset{\neg}{\underline{\boldsymbol{u}}} = -\boldsymbol{\nabla} h \quad , \quad \frac{\partial h}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} h + h \boldsymbol{\nabla} \cdot \boldsymbol{u} + c^2 \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

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  with gravity wave speed (of the waves with $f = 0$) $c = \sqrt{gH}$
  and $\underset{\neg}{\boldsymbol{u}} = (-v, u)$ which denotes anticlockwise rotation of $\boldsymbol{u}$ by $90^o$

- apply the wave ansatz

$$\boldsymbol{u} = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{\boldsymbol{u}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \quad , \quad h = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{h} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

  with complex amplitudes $\hat{\boldsymbol{u}}(\boldsymbol{k}, t)$, $\hat{h}(\boldsymbol{k}, t)$ and wave number vector $\boldsymbol{k}$
  $\rightarrow$ spectrum of waves instead of a single wave

- non-linear barotropic or reduced gravity model

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + f\,\underset{\neg}{\boldsymbol{u}} = -\boldsymbol{\nabla} h \quad , \quad \frac{\partial h}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} h + h\boldsymbol{\nabla}\cdot\boldsymbol{u} + c^2\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

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  with complex amplitudes $\hat{\boldsymbol{u}}(\boldsymbol{k}, t)$, $\hat{h}(\boldsymbol{k}, t)$ and wave number vector $\boldsymbol{k}$
  $\rightarrow$ spectrum of waves instead of a single wave

- integrals will be real with $\hat{\boldsymbol{u}}(\boldsymbol{k}) = \hat{\boldsymbol{u}}^*(-\boldsymbol{k})$ and $\hat{h}(\boldsymbol{k}) = \hat{h}^*(-\boldsymbol{k})$

- applying $\boldsymbol{u} = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{\boldsymbol{u}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ and $h = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{h} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ to

$$\frac{\partial \boldsymbol{u}}{\partial t} + f\, \underset{\neg}{\boldsymbol{u}} + \boldsymbol{\nabla} h = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} \;\; , \quad \frac{\partial h}{\partial t} + c^2 \boldsymbol{\nabla}\cdot\boldsymbol{u} = -\boldsymbol{u}\cdot\boldsymbol{\nabla} h - h\boldsymbol{\nabla}\cdot\boldsymbol{u}$$

left hand side is linear, right hand side is non-linear

- applying $\boldsymbol{u} = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{\boldsymbol{u}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ and $h = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{h} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ to

  $$\frac{\partial \boldsymbol{u}}{\partial t} + f\ \underset{\neg}{\boldsymbol{u}} + \boldsymbol{\nabla} h = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\ \ ,\ \ \frac{\partial h}{\partial t} + c^2 \boldsymbol{\nabla}\cdot\boldsymbol{u} = -\boldsymbol{u}\cdot\boldsymbol{\nabla} h - h\boldsymbol{\nabla}\cdot\boldsymbol{u}$$

  left hand side is linear, right hand side is non-linear

- ansatz yields on the left hand side

  $$\int_{-\infty}^{\infty} d\boldsymbol{k}\, \frac{\partial \hat{\boldsymbol{u}}}{\partial t} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + f \int_{-\infty}^{\infty} d\boldsymbol{k}\ \ \underset{\neg}{\hat{\boldsymbol{u}}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \int_{-\infty}^{\infty} d\boldsymbol{k}\, i\boldsymbol{k}\hat{h} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \ldots$$

- applying $\boldsymbol{u} = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{\boldsymbol{u}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ and $h = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{h} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ to

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- ansatz yields on the left hand side

$$\int_{-\infty}^{\infty} d\boldsymbol{k}\, \frac{\partial \hat{\boldsymbol{u}}}{\partial t} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + f \int_{-\infty}^{\infty} d\boldsymbol{k}\ \underset{\neg}{\hat{\boldsymbol{u}}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \int_{-\infty}^{\infty} d\boldsymbol{k}\, i\boldsymbol{k}\hat{h} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \ldots$$

- multiplication with $e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}$ and integration over $\boldsymbol{x}$

$$(2\pi)^2 \left( \frac{\partial \hat{\boldsymbol{u}}'}{\partial t} + f\, \underset{\neg}{\hat{\boldsymbol{u}}'} + i\boldsymbol{k}'\hat{h}' \right) = \ldots \quad , \quad (2\pi)^2 \left( \frac{\partial \hat{h}'}{\partial t} + c^2 i\boldsymbol{k}'\cdot\hat{\boldsymbol{u}}' \right) = \ldots$$

with $(2\pi)^2 \delta(\boldsymbol{k}-\boldsymbol{k}') = \int d\boldsymbol{x} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}$ and $\hat{h}' = \hat{h}(\boldsymbol{k}')$, $\hat{\boldsymbol{u}}' = \hat{\boldsymbol{u}}(\boldsymbol{k}')$

- applying $\boldsymbol{u} = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{\boldsymbol{u}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ and $h = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{h} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ to

$$\frac{\partial \boldsymbol{u}}{\partial t} + f\ \underset{\neg}{\boldsymbol{u}} + \boldsymbol{\nabla} h = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\ \ ,\ \ \frac{\partial h}{\partial t} + c^2\boldsymbol{\nabla}\cdot\boldsymbol{u} = -\boldsymbol{u}\cdot\boldsymbol{\nabla} h - h\boldsymbol{\nabla}\cdot\boldsymbol{u}$$

left hand side is linear, right hand side is non-linear

- ansatz yields on the left hand side

$$\int_{-\infty}^{\infty} d\boldsymbol{k}\, \frac{\partial \hat{\boldsymbol{u}}}{\partial t} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + f \int_{-\infty}^{\infty} d\boldsymbol{k}\ \ \underset{\neg}{\hat{\boldsymbol{u}}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \int_{-\infty}^{\infty} d\boldsymbol{k}\, i\boldsymbol{k}\hat{h} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \ldots$$

- multiplication with $e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}$ and integration over $\boldsymbol{x}$

$$(2\pi)^2\left(\frac{\partial \hat{\boldsymbol{u}}'}{\partial t} + f\ \underset{\neg}{\hat{\boldsymbol{u}}'} + i\boldsymbol{k}'\hat{h}'\right) = \ldots\ \ ,\ \ (2\pi)^2\left(\frac{\partial \hat{h}'}{\partial t} + c^2 i\boldsymbol{k}'\cdot\hat{\boldsymbol{u}}'\right) = \ldots$$

with $(2\pi)^2\delta(\boldsymbol{k}-\boldsymbol{k}') = \int d\boldsymbol{x} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}$ and $\hat{h}' = \hat{h}(\boldsymbol{k}')$, $\hat{\boldsymbol{u}}' = \hat{\boldsymbol{u}}(\boldsymbol{k}')$

- rename $\hat{h}' \to \hat{h}$ and $\hat{\boldsymbol{u}}' \to \hat{\boldsymbol{u}}$, $\boldsymbol{k}' \to \boldsymbol{k}$, ...

- applying $\boldsymbol{u} = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{\boldsymbol{u}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ and $h = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{h} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ to

$$\frac{\partial \boldsymbol{u}}{\partial t} + f\, \underset{\neg}{\boldsymbol{u}} + \boldsymbol{\nabla} h = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} \ , \quad \frac{\partial h}{\partial t} + c^2 \boldsymbol{\nabla}\cdot\boldsymbol{u} = -\boldsymbol{u}\cdot\boldsymbol{\nabla} h - h\boldsymbol{\nabla}\cdot\boldsymbol{u}$$

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- applying $\boldsymbol{u} = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{\boldsymbol{u}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ and $h = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{h} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ to

$$\frac{\partial \boldsymbol{u}}{\partial t} + f \underset{\neg}{\boldsymbol{u}} + \nabla h = -\boldsymbol{u}\cdot\nabla\boldsymbol{u} \quad , \quad \frac{\partial h}{\partial t} + c^2 \nabla\cdot\boldsymbol{u} = -\boldsymbol{u}\cdot\nabla h - h\nabla\cdot\boldsymbol{u}$$

left hand side is linear, right hand side is non-linear

- multiplication with $e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}$ and integration over $\boldsymbol{x}$ yields on the right hand side of momentum equation

$$\int e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} d\boldsymbol{x} = \int\int\int d\boldsymbol{x} d\boldsymbol{k}_1 d\boldsymbol{k}_2\, (\hat{\boldsymbol{u}}_1 \cdot i\boldsymbol{k}_2)\hat{\boldsymbol{u}}_2 e^{i(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}')\cdot\boldsymbol{x}}$$
$$= (2\pi)^2 \int\int d\boldsymbol{k}_1 d\boldsymbol{k}_2\, (\hat{\boldsymbol{u}}_1 \cdot i\boldsymbol{k}_2)\hat{\boldsymbol{u}}_2 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}')$$

with $\hat{\boldsymbol{u}}_1 = \hat{\boldsymbol{u}}(\boldsymbol{k}_1)$ and $\hat{\boldsymbol{u}}_2 = \hat{\boldsymbol{u}}(\boldsymbol{k}_2)$, ...

- applying $\boldsymbol{u} = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{\boldsymbol{u}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ and $h = \int_{-\infty}^{\infty} d\boldsymbol{k}\, \hat{h} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ to

$$\frac{\partial \boldsymbol{u}}{\partial t} + f\, \underset{\neg}{\boldsymbol{u}} + \boldsymbol{\nabla} h = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} \quad , \quad \frac{\partial h}{\partial t} + c^2 \boldsymbol{\nabla}\cdot\boldsymbol{u} = -\boldsymbol{u}\cdot\boldsymbol{\nabla} h - h\boldsymbol{\nabla}\cdot\boldsymbol{u}$$

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- multiplication with $e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}$ and integration over $\boldsymbol{x}$ yields on the right hand side of momentum equation

$$\int e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}} (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\, d\boldsymbol{x} = \int\int\int d\boldsymbol{x}\, d\boldsymbol{k}_1 d\boldsymbol{k}_2\, (\hat{\boldsymbol{u}}_1 \cdot i\boldsymbol{k}_2)\hat{\boldsymbol{u}}_2 e^{i(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}')\cdot\boldsymbol{x}}$$
$$= (2\pi)^2 \int\int d\boldsymbol{k}_1 d\boldsymbol{k}_2\, (\hat{\boldsymbol{u}}_1 \cdot i\boldsymbol{k}_2)\hat{\boldsymbol{u}}_2 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}')$$

  with $\hat{\boldsymbol{u}}_1 = \hat{\boldsymbol{u}}(\boldsymbol{k}_1)$ and $\hat{\boldsymbol{u}}_2 = \hat{\boldsymbol{u}}(\boldsymbol{k}_2)$, ...

- and the right hand side of thickness equation

$$\int d\boldsymbol{x} e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}} (\boldsymbol{u}\cdot\boldsymbol{\nabla} h + h\boldsymbol{\nabla}\cdot\boldsymbol{u}) = (2\pi)^2 \int\int d\boldsymbol{k}_1 d\boldsymbol{k}_2$$
$$\times \left(\hat{\boldsymbol{u}}_1 \cdot i\boldsymbol{k}_2 \hat{h}_2 + \hat{h}_2 i\boldsymbol{k}_1 \cdot \hat{\boldsymbol{u}}_1\right) \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}')$$

- taking both sides together yields

$$
\begin{aligned}
\frac{\partial \hat{\boldsymbol{u}}}{\partial t} &= -f\ \underset{\neg}{\hat{\boldsymbol{u}}} - i\boldsymbol{k}\hat{h} + i \int\int d\boldsymbol{k}_1 d\boldsymbol{k}_2\ (\hat{\boldsymbol{u}}_1 \cdot \boldsymbol{k}_2)\hat{\boldsymbol{u}}_2 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}) \\
\frac{\partial \hat{h}}{\partial t} &= -c^2 i\boldsymbol{k} \cdot \hat{\boldsymbol{u}} + i \int\int d\boldsymbol{k}_1 d\boldsymbol{k}_2\ \hat{h}_2 \hat{\boldsymbol{u}}_1 \cdot (\boldsymbol{k}_1 + \boldsymbol{k}_2)\, \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k})
\end{aligned}
$$

- taking both sides together yields

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\frac{\partial \hat{h}}{\partial t} &= -c^2 i\boldsymbol{k}\cdot\hat{\boldsymbol{u}} + i\int\int d\boldsymbol{k}_1 d\boldsymbol{k}_2\ \hat{h}_2\hat{\boldsymbol{u}}_1\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2)\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k})
\end{aligned}
$$

- rewrite as

$$
\frac{\partial \boldsymbol{z}}{\partial t} = i\boldsymbol{A}\cdot\boldsymbol{z} - i\int\int d\boldsymbol{k}_1 d\boldsymbol{k}_2\ \boldsymbol{N}\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k})
$$

  with linear system matrix $\boldsymbol{A}(\boldsymbol{k})$, the state vector $\boldsymbol{z}(\boldsymbol{k},t)$, and the vector function $\boldsymbol{N}$ given by

$$
\boldsymbol{A} = \begin{pmatrix} 0 & -if & -k_x \\ if & 0 & -k_y \\ -c^2k_x & -c^2k_y & 0 \end{pmatrix}\ ,\quad \boldsymbol{z} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix}\ ,\quad \boldsymbol{N} = \ldots
$$

- taking both sides together yields

$$
\begin{aligned}
\frac{\partial \hat{\boldsymbol{u}}}{\partial t} &= -f\ \underset{\neg}{\hat{\boldsymbol{u}}} - i\boldsymbol{k}\hat{h} + i\int\int d\boldsymbol{k}_1 d\boldsymbol{k}_2\ (\hat{\boldsymbol{u}}_1 \cdot \boldsymbol{k}_2)\hat{\boldsymbol{u}}_2 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}) \\
\frac{\partial \hat{h}}{\partial t} &= -c^2 i\boldsymbol{k}\cdot\hat{\boldsymbol{u}} + i\int\int d\boldsymbol{k}_1 d\boldsymbol{k}_2\ \hat{h}_2 \hat{\boldsymbol{u}}_1 \cdot (\boldsymbol{k}_1 + \boldsymbol{k}_2)\,\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k})
\end{aligned}
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with linear system matrix $\boldsymbol{A}(\boldsymbol{k})$, the state vector $\boldsymbol{z}(\boldsymbol{k}, t)$, and the vector function $\boldsymbol{N}$ given by

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$$

- triad wave interaction between $\boldsymbol{z}(\boldsymbol{k}_1)$, $\boldsymbol{z}(\boldsymbol{k}_2)$ and $\boldsymbol{z}(\boldsymbol{k})$ with

$$
\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k} = 0
$$

- taking both sides together yields

$$\frac{\partial \hat{\boldsymbol u}}{\partial t} = -f\,\underset{\neg}{\hat{\boldsymbol u}} - i\boldsymbol k \hat{h} + i \int\int d\boldsymbol k_1 d\boldsymbol k_2\ (\hat{\boldsymbol u}_1 \cdot \boldsymbol k_2)\hat{\boldsymbol u}_2 \delta(\boldsymbol k_1 + \boldsymbol k_2 - \boldsymbol k)$$

$$\frac{\partial \hat{h}}{\partial t} = -c^2 i\boldsymbol k \cdot \hat{\boldsymbol u} + i \int\int d\boldsymbol k_1 d\boldsymbol k_2\ \hat{h}_2 \hat{\boldsymbol u}_1 \cdot (\boldsymbol k_1 + \boldsymbol k_2)\, \delta(\boldsymbol k_1 + \boldsymbol k_2 - \boldsymbol k)$$

- rewrite as

$$\frac{\partial \boldsymbol z}{\partial t} = i\boldsymbol A \cdot \boldsymbol z - i \int\int d\boldsymbol k_1 d\boldsymbol k_2\ \boldsymbol N\, \delta(\boldsymbol k_1 + \boldsymbol k_2 - \boldsymbol k)$$

with linear system matrix $\boldsymbol A(\boldsymbol k)$, the state vector $\boldsymbol z(\boldsymbol k, t)$, and the vector function $\boldsymbol N$ given by

$$\boldsymbol A = \begin{pmatrix} 0 & -if & -k_x \\ if & 0 & -k_y \\ -c^2 k_x & -c^2 k_y & 0 \end{pmatrix} \;,\quad \boldsymbol z = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix} \;,\quad \boldsymbol N = \dots$$

- triad wave interaction between $\boldsymbol z(\boldsymbol k_1)$, $\boldsymbol z(\boldsymbol k_2)$ and $\boldsymbol z(\boldsymbol k)$ with

$$\boldsymbol k_1 + \boldsymbol k_2 - \boldsymbol k = 0$$

- too complicated $\rightarrow$ set $\boldsymbol N = 0$, no non-linear interaction

- linear system matrix $\boldsymbol{A}(\boldsymbol{k})$ and the state vector $\boldsymbol{z}(\boldsymbol{k}, t)$

$$\boldsymbol{A} = \begin{pmatrix} 0 & -if & -k_x \\ if & 0 & -k_y \\ -c^2 k_x & -c^2 k_y & 0 \end{pmatrix} \quad , \quad \boldsymbol{z} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix}$$

- linear system matrix $\boldsymbol{A}(\boldsymbol{k})$ and the state vector $\boldsymbol{z}(\boldsymbol{k}, t)$

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- set $\boldsymbol{z} = \boldsymbol{z}_0 e^{i\omega t}$ which yields in $\partial \boldsymbol{z}/\partial t = i\boldsymbol{A} \cdot \boldsymbol{z}$

$$i\omega \boldsymbol{z}_0 e^{i\omega t} = i\boldsymbol{A} \cdot \boldsymbol{z}_0 e^{i\omega t} \quad \rightarrow \quad \omega \boldsymbol{z}_0 = \boldsymbol{A} \cdot \boldsymbol{z}_0$$

- linear system matrix $\boldsymbol{A}(\boldsymbol{k})$ and the state vector $\boldsymbol{z}(\boldsymbol{k}, t)$

$$\boldsymbol{A} = \begin{pmatrix} 0 & -if & -k_x \\ if & 0 & -k_y \\ -c^2 k_x & -c^2 k_y & 0 \end{pmatrix} \quad , \quad \boldsymbol{z} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix}$$

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$$i\omega \boldsymbol{z}_0 e^{i\omega t} = i\boldsymbol{A} \cdot \boldsymbol{z}_0 e^{i\omega t} \quad \rightarrow \quad \omega \boldsymbol{z}_0 = \boldsymbol{A} \cdot \boldsymbol{z}_0$$

- eigenvalue $\omega$ and eigenvectors $\boldsymbol{z}_0$ to matrix $\boldsymbol{A}$ given by

$$\omega \mathbf{1} \cdot \boldsymbol{z}_0 = \boldsymbol{A} \cdot \boldsymbol{z}_0 \quad \rightarrow \quad (\boldsymbol{A} - \mathbf{1}\omega) \cdot \boldsymbol{z}_0 = 0$$

with unit matrix $\mathbf{1}$

- linear system matrix $\boldsymbol{A}(\boldsymbol{k})$ and the state vector $\boldsymbol{z}(\boldsymbol{k}, t)$

$$\boldsymbol{A} = \begin{pmatrix} 0 & -if & -k_x \\ if & 0 & -k_y \\ -c^2 k_x & -c^2 k_y & 0 \end{pmatrix} \quad , \quad \boldsymbol{z} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix}$$

- set $\boldsymbol{z} = \boldsymbol{z}_0 e^{i\omega t}$ which yields in $\partial \boldsymbol{z}/\partial t = i\boldsymbol{A} \cdot \boldsymbol{z}$

$$i\omega \boldsymbol{z}_0 e^{i\omega t} = i\boldsymbol{A} \cdot \boldsymbol{z}_0 e^{i\omega t} \quad \rightarrow \quad \omega \boldsymbol{z}_0 = \boldsymbol{A} \cdot \boldsymbol{z}_0$$

- eigenvalue $\omega$ and eigenvectors $\boldsymbol{z}_0$ to matrix $\boldsymbol{A}$ given by

$$\omega \mathbf{1} \cdot \boldsymbol{z}_0 = \boldsymbol{A} \cdot \boldsymbol{z}_0 \quad \rightarrow \quad (\boldsymbol{A} - \mathbf{1}\omega) \cdot \boldsymbol{z}_0 = 0$$

  with unit matrix $\mathbf{1}$

- non-trivial solution $\boldsymbol{z}_0 \neq \mathbf{0}$ from characteristic equation

$$|\boldsymbol{A} - \mathbf{1}\omega| = 0$$

- linear system matrix $\boldsymbol{A}(\boldsymbol{k})$ and the state vector $\boldsymbol{z}(\boldsymbol{k}, t)$

$$\boldsymbol{A} = \begin{pmatrix} 0 & -if & -k_x \\ if & 0 & -k_y \\ -c^2 k_x & -c^2 k_y & 0 \end{pmatrix} \quad , \quad \boldsymbol{z} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix}$$

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$$\sum_{s=0,\pm} \boldsymbol{Q}^s e^{i\omega_s t} \frac{\partial a^s}{\partial t} + \sum_{s=0,\pm} \boldsymbol{Q}^s i\omega_s e^{i\omega_s t} a^s = i\boldsymbol{A} \cdot \sum_{s=0,\pm} e^{i\omega_s t} a^s \boldsymbol{Q}^s(\boldsymbol{k})$$

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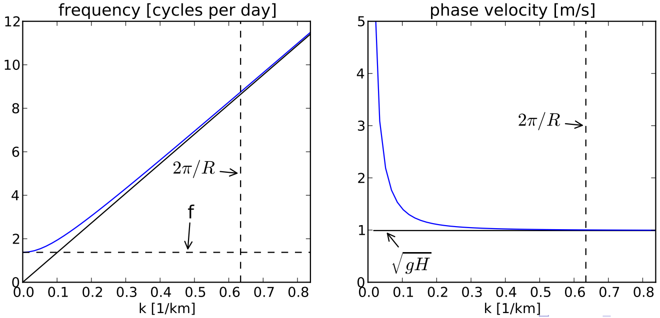
- gravity wave dispersion relation ($f \neq 0$ in blue, $f = 0$ in black)

$$\omega = \pm\sqrt{f^2\left(1 + R^2k^2\right)} \quad , \quad c = \pm\sqrt{f^2\left(1/k^2 + R^2\right)}$$

with Rossby radius $R^2 = gH/f^2$



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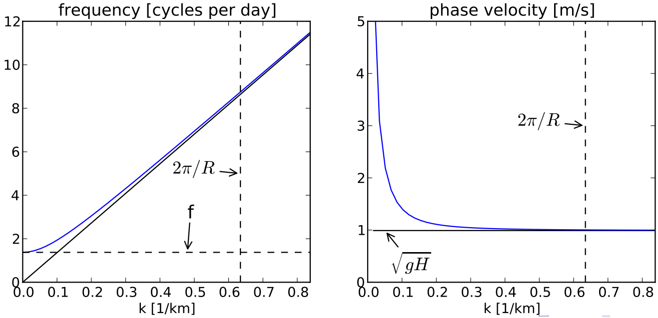

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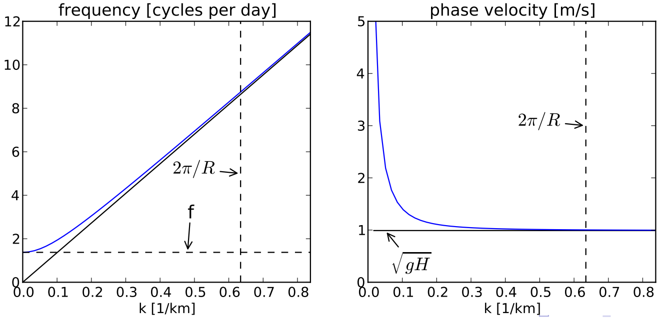

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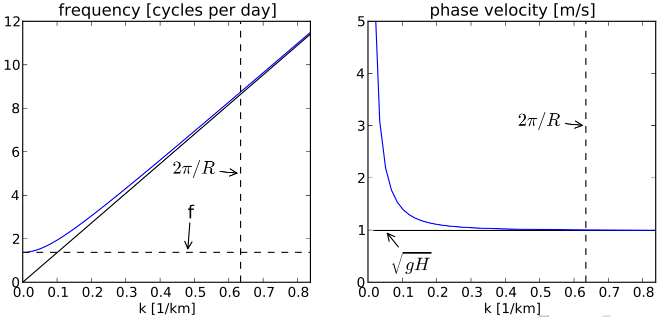
$\rightarrow$ (non-dispersive) gravity waves without rotation (black lines)



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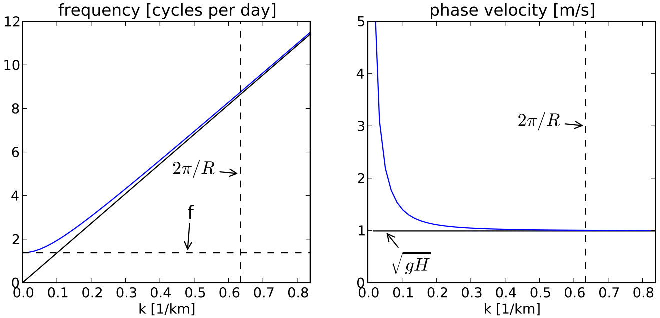

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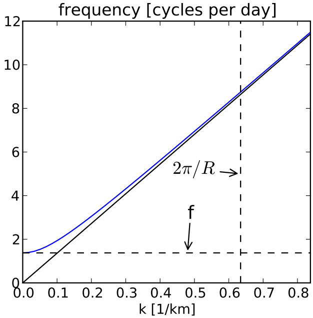


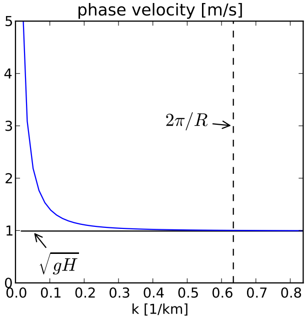

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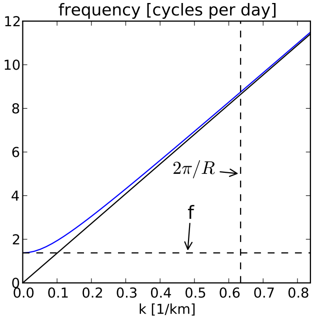
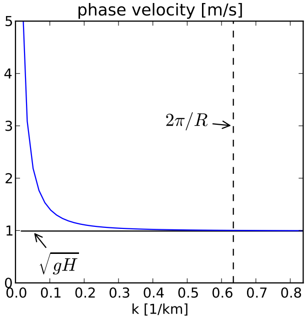
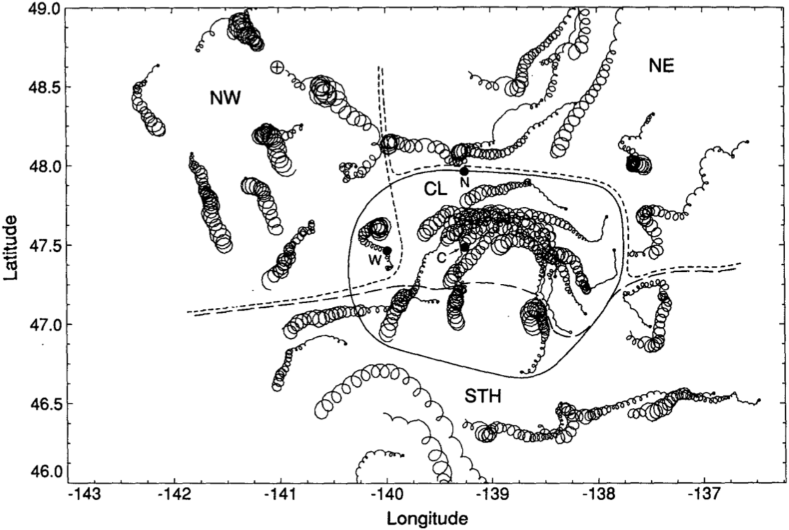
- these are inertial oscillations which also result from

$$\partial u/\partial t - fv = 0 \quad , \quad \partial v/\partial t + fu = 0$$



- trajectories of surface drifter → inertial oscillations




from d'Asaro et al 1995

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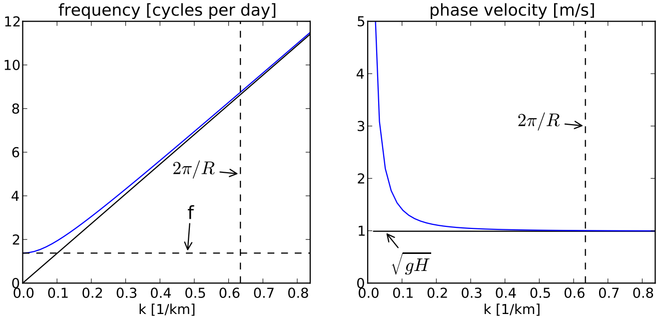

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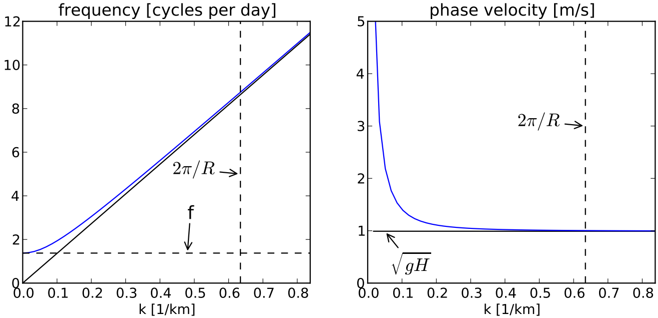
$$\omega = \pm\sqrt{f^2\left(1 + R^2 k^2\right)}$$

- group velocity $\boldsymbol{c}_g = \partial\omega/\partial\boldsymbol{k}$ is given by

$$\boldsymbol{c}_g = \begin{pmatrix} \partial\omega/\partial k_1 \\ \partial\omega/\partial k_2 \end{pmatrix} = \pm\begin{pmatrix} \frac{1}{2}\left(f^2\left(1 + R^2 k^2\right)\right)^{-1/2} f^2 R^2 2k_1 \\ \frac{1}{2}\left(f^2\left(1 + R^2 k^2\right)\right)^{-1/2} f^2 R^2 2k_2 \end{pmatrix} = \frac{gH}{\omega}\boldsymbol{k}$$



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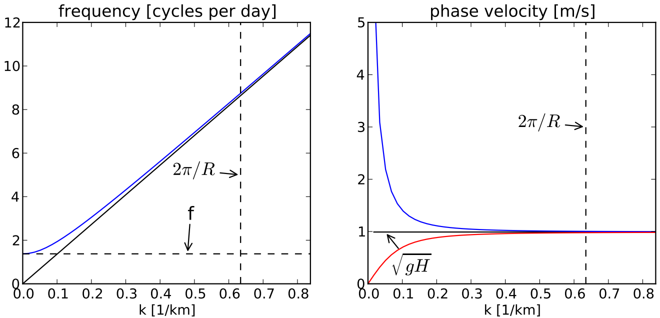

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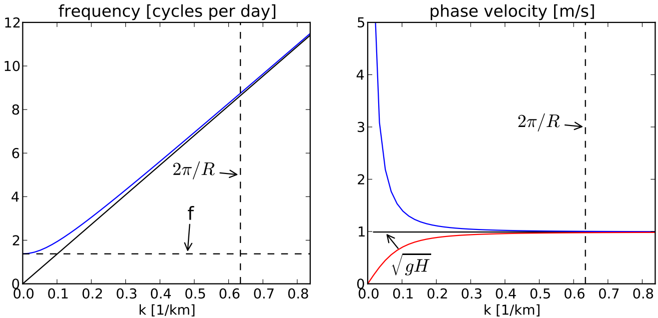
- short wave limit for $\lambda \ll R$

$$\omega \overset{\lambda \ll R}{=} \pm k\sqrt{gH} \quad \rightarrow \quad \boldsymbol{c}_g \overset{\lambda \ll R}{=} \pm\sqrt{gH}\,\boldsymbol{k}/k = c\,\boldsymbol{k}/k$$



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- long wave limit for $\lambda \gg R$

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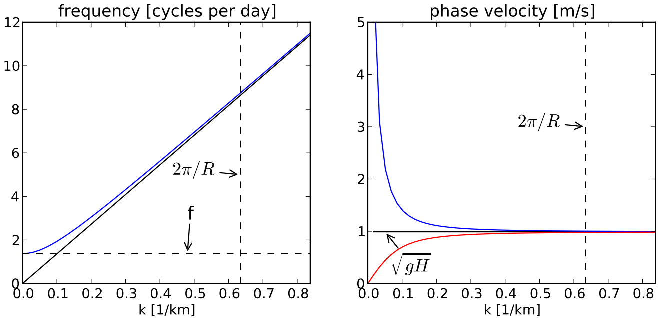
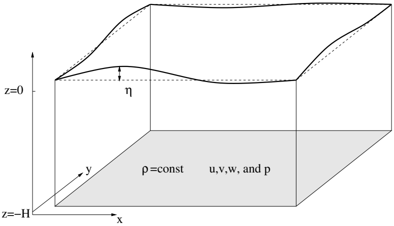


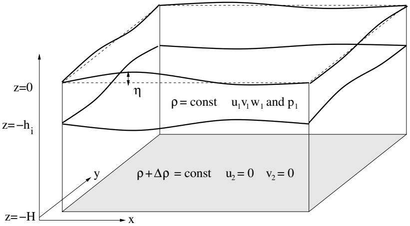
- non-linear "barotropic model" and "baroclinic model"

$$\frac{\partial u}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} u - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} v + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} h + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- $h$ is total thickness ("barotropic") or layer interface $h_i$ ("baroclinic")
- either $g = 9.81\,\mathrm{m/s^2}$ ("barotropic") or $g \rightarrow g\Delta\rho/\rho_0$ ("baroclinic")



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- take curl of mom. equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\frac{\partial}{\partial t}\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}(\boldsymbol{u} \cdot \boldsymbol{\nabla} v) + f\frac{\partial u}{\partial x} = -g\frac{\partial^2 h}{\partial x \partial y}$$

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$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \zeta + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

with relative vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$ and for $f = const$

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- combine vorticity equation with thickness equation
  $\rightarrow$ potential vorticity equation

- combined vorticity and thickness equation $\rightarrow$ potential vorticity

$$\frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u - \frac{f}{h}\frac{Dh}{Dt} = 0$$

with $D/dt = \partial/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$

- combined vorticity and thickness equation $\rightarrow$ potential vorticity

$$\frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u - \frac{f}{h}\frac{Dh}{Dt} = 0$$

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- using

$$\frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}$$

$$= \frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\zeta = -\frac{\zeta}{h}\frac{Dh}{Dt}$$

- combined vorticity and thickness equation → potential vorticity

$$\frac{D\zeta}{Dt} + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u - \frac{f}{h}\frac{Dh}{Dt} = 0$$

with $D/dt = \partial/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$

- using

$$\frac{\partial \boldsymbol{u}}{\partial x} \cdot \boldsymbol{\nabla} v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \boldsymbol{\nabla} u = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}$$
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- it follows that

$$\frac{D\zeta}{Dt} - \frac{\zeta}{h}\frac{Dh}{Dt} - \frac{f}{h}\frac{Dh}{Dt} = \frac{D\zeta}{Dt} - \frac{\zeta + f}{h}\frac{Dh}{Dt} = 0$$
$$\rightarrow \quad \frac{1}{h}\frac{D}{Dt}(\zeta + f) - \frac{\zeta + f}{h^2}\frac{Dh}{Dt} = \frac{D}{Dt}\frac{\zeta + f}{h} = \frac{Dq}{Dt} = 0$$

with potential vorticity $q = (\zeta + f)/h$.

- potential vorticity equation for a single (or 1.5) layer(s)

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

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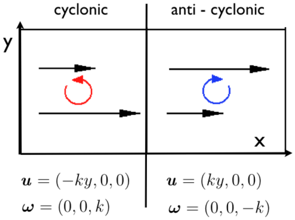
$f$ increases but $q = (f + \zeta)/h$ has to stay constant
$\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ decreases $\rightarrow$ anticyclonic rotation




$u = -ay \ , \ v = 0 \ \rightarrow \ \zeta = a > 0 :$ cyclonic (anticlockwise) rotation
$u = +ay \ , \ v = 0 \ \rightarrow \ \zeta = a < 0 :$ anticyclonic (clockwise) rotation

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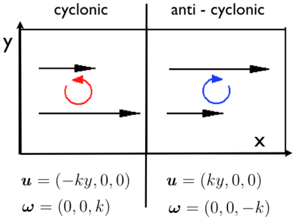
  $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ decreases $\rightarrow$ anticyclonic rotation
- $h = const$, $\zeta$ initially zero, parcel moves southward
  $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ increases $\rightarrow$ more cyclonic rotation

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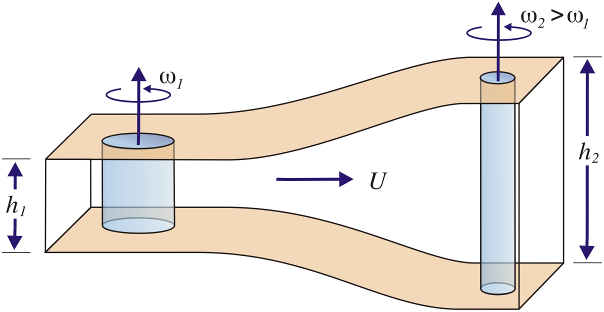
- potential vorticity equation for a single (or 1.5) layer(s)

$$\frac{Dq}{Dt} = 0 \quad , \quad q = \frac{\zeta + f}{h}$$

$q$ is conserved for fluid parcels in single layer

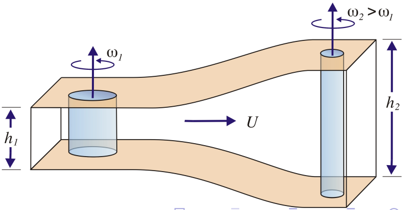
- $f = const$, $\zeta$ initially zero, parcel moves to deeper water $\rightarrow \zeta = \partial v/\partial x - \partial u/\partial y$ increases $\rightarrow$ cyclonic rotation



- quasi-geostrophic potential vorticity equation

$$q = \frac{\zeta + f}{H + h'} \approx \frac{\zeta}{H} + \frac{f/H}{1 + h'/H}$$

for $|\zeta| \ll |f|$ (or $Ro \ll 1$) and $H \gg |h'|$



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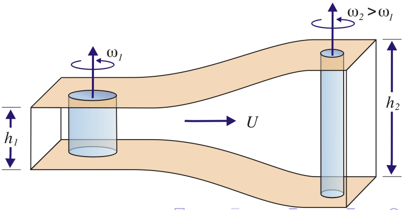

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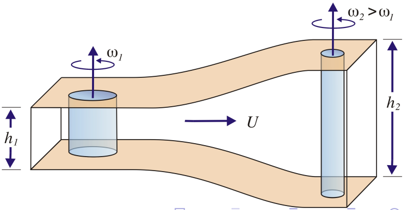

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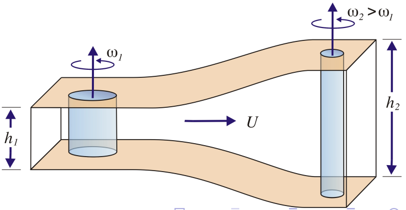
- $\zeta$ is relative vorticity
- $-(f/H)h'$ is stretching vorticity
- $f = f_0 + \beta y$ is planetary vorticity
- $h'$ is streamfunction for the quasi-geostrophic flow



- consider the (linearized) layered model with $f = const$

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x}\ , \quad \frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}\ , \quad \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

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- (linearized, $D/Dt \to \partial/\partial t$, QG) potential vorticity equation

$$\frac{\partial q}{\partial t} = 0\ , \quad q = \zeta - \frac{f}{H}h + f$$

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$$h|_{t=0} = \begin{cases} h_0, & \text{if } x < 0 \\ -h_0, & \text{if } x > 0 \end{cases} \to \ q_0 = q|_{t=0} = \begin{cases} -fh_0/H, & \text{if } x < 0 \\ fh_0/H, & \text{if } x > 0 \end{cases}$$

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- using $q(t) = q_0$ steady state solution ($t \to \infty$) is given by

$$fv_\infty = g\frac{\partial h_\infty}{\partial x}\ , \quad fu_\infty = -g\frac{\partial h_\infty}{\partial y}$$

$$\to\ q_\infty = \frac{g}{f}\frac{\partial^2 h_\infty}{\partial x^2} + \frac{g}{f}\frac{\partial^2 h_\infty}{\partial y^2} - \frac{f}{H}h_\infty = q_0$$

$\to \boldsymbol{\nabla}^2 h_\infty - R^{-2}h_\infty = (f/g)q_0$ with Rossby radius $R = \sqrt{gH}/|f|$

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- initial and steady state solution of $h$ are given by

$$h|_{t=0} = \begin{cases} h_0, & \text{if } x < 0 \\ -h_0, & \text{if } x > 0 \end{cases}, \quad h|_\infty = \begin{cases} h_0(1 - e^{x/R}), & \text{if } x < 0 \\ -h_0(1 - e^{-x/R}), & \text{if } x > 0 \end{cases}$$

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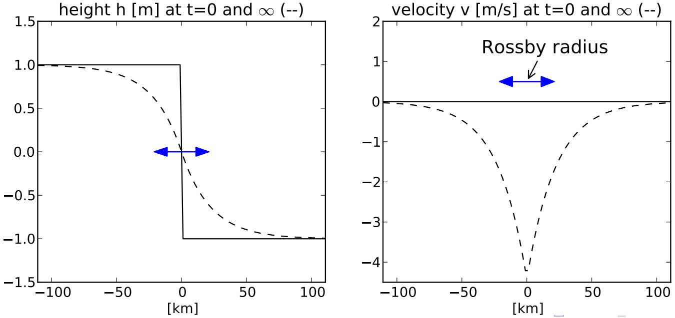

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- velocities from $fv_\infty = g\partial h_\infty/\partial x$ and $fu_\infty = -g\partial h_\infty/\partial y$

$$u_\infty = 0 \quad , \quad v_\infty = (g/f) \begin{cases} -h_0/Re^{x/R}, & \text{if } x < 0 \\ -h_0/Re^{-x/R}, & \text{if } x > 0 \end{cases} = -\frac{gh_0}{fR} e^{-|x|/R}$$