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Markus Jochum and Carsten Eden

Institut für Meereskunde, Universität Hamburg

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Lecture # 5

Rossby waves
- Rossby vs. gravity waves

Nonlinear model
- Nonlinear momentum equation
- Nonlinear energy equations
- Spatial discretisation for non-linear model
- Two level time discretisation

- Rossby waves
  - Rossby vs. gravity waves
- Nonlinear model
  - Nonlinear momentum equation
  - Nonlinear energy equations
  - Spatial discretisation for non-linear model
  - Two level time discretisation

- consider the (linearized, $D/Dt \to \partial/\partial t$) layered model

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x} \quad , \quad \frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

with $f = f(y) \to$ non-constant coeff. $\to$ no simple wave solution

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- $\rightarrow$ take divergence of mom. equation: $\partial(1.eqn)/\partial x + \partial(2.eqn)/\partial y$

$$\frac{\partial}{\partial x}(1.\text{eqn}) \;:\; \frac{\partial}{\partial t}\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(fv) = -g\frac{\partial^2 h}{\partial x^2}$$

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\frac{\partial}{\partial y}(2.\text{eqn}) \;:\; \frac{\partial}{\partial t}\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}(fu) &= -g\frac{\partial^2 h}{\partial y^2}
\end{aligned}$$

add both

$$\frac{\partial \xi}{\partial t} - \frac{\partial}{\partial x}(fv) + \frac{\partial}{\partial y}(fu) = -g\boldsymbol{\nabla}^2 h$$

with divergence $\xi = \partial u/\partial x + \partial v/\partial y$ and $\beta = df/dy$

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\end{aligned}$$

add both

$$\begin{aligned}
\frac{\partial \xi}{\partial t} - \frac{\partial}{\partial x}(fv) + \frac{\partial}{\partial y}(fu) &= -g\boldsymbol{\nabla}^2 h \\
\frac{\partial \xi}{\partial t} - f\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \beta u &= -g\boldsymbol{\nabla}^2 h
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with divergence $\xi = \partial u/\partial x + \partial v/\partial y$ and $\beta = df/dy$

- consider the (linearized, $D/Dt \to \partial/\partial t$) layered model

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$$\frac{\partial h}{\partial t} + H\xi = 0 \quad , \quad \frac{\partial \xi}{\partial t} - f\zeta + \beta u = -g\boldsymbol{\nabla}^2 h$$

with divergence $\xi = \partial u/\partial x + \partial v/\partial y$ and relative vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$

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- take curl of mom. equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

$$\frac{\partial}{\partial x}(2.\text{eqn}) \ : \ \frac{\partial}{\partial x}\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(fu) \ = \ -g\frac{\partial^2 h}{\partial x \partial y}$$

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\frac{\partial}{\partial y}(1.\mathrm{eqn}) \; &: \; \frac{\partial}{\partial y}\frac{\partial u}{\partial t} - \frac{\partial}{\partial y}(fv) &= -g\frac{\partial^2 h}{\partial x \partial y}
\end{aligned}$$

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$$\frac{\partial h}{\partial t} + H\xi = 0 \quad , \quad \frac{\partial \xi}{\partial t} - f\zeta + \beta u = -g\nabla^2 h$$

with divergence $\xi = \partial u/\partial x + \partial v/\partial y$ and relative vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$

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\end{aligned}$$

subtract both

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(fu) + \frac{\partial}{\partial y}(fv) = 0$$

with relative vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$ and $\beta = df/dy$

- consider the (linearized, $D/Dt \to \partial/\partial t$) layered model

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- take curl of mom. equation, i.e. $\partial(2.eqn)/\partial x - \partial(1.eqn)/\partial y$

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\frac{\partial}{\partial x}(2.\text{eqn}) \; : \; \frac{\partial}{\partial x}\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(fu) &= -g\frac{\partial^2 h}{\partial x \partial y} \\
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\end{aligned}$$

subtract both

$$\begin{aligned}
\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(fu) + \frac{\partial}{\partial y}(fv) &= 0 \\
\frac{\partial \zeta}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v &= 0
\end{aligned}$$

with relative vorticity $\zeta = \partial v/\partial x - \partial u/\partial y$ and $\beta = df/dy$

- new system of equations with ζ and ξ

$$
\begin{aligned}
\frac{\partial h}{\partial t} &= -H\xi \\
\frac{\partial \zeta}{\partial t} &= -f\xi - \beta v \\
\frac{\partial \xi}{\partial t} &= -g\nabla^2 h + f\zeta - \beta u
\end{aligned}
$$

- new system of equations with ζ and ξ

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\begin{aligned}
\frac{\partial h}{\partial t} &= -H\xi \\
\frac{\partial \zeta}{\partial t} &= -f\xi - \beta v \approx -f\xi - \beta \frac{g}{f}\frac{\partial h}{\partial x} \\
\frac{\partial \xi}{\partial t} &= -g\boldsymbol{\nabla}^2 h + f\zeta - \beta u \approx f\zeta - g\boldsymbol{\nabla}^2 h + \beta \frac{g}{f}\frac{\partial h}{\partial y}
\end{aligned}
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with geostrophic balance $fv \approx g\partial h/\partial x$ and $fu \approx g\partial h/\partial y$

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- neglect $\beta(g/f)\partial h/\partial y$ since $|\partial^2 h/\partial y^2|/|(\beta/f)\partial h/\partial y| \sim (H/L^2)/(H/L\,1/a) = a/L \gg 1$ with Earth radius a

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- rewrite as

$$\frac{\partial \boldsymbol{z}}{\partial t} = i\boldsymbol{A}\cdot\boldsymbol{x}\ ,\ \boldsymbol{A} = \begin{pmatrix} 0 & 0 & iH \\ i\beta\frac{g}{f}\frac{\partial}{\partial x} & 0 & if \\ ig\boldsymbol{\nabla}^2 & -if & 0 \end{pmatrix}\ ,\ \boldsymbol{z} = \begin{pmatrix} h \\ \zeta \\ \xi \end{pmatrix}$$

- new system of equations with ζ and ξ

$$\begin{aligned}
\frac{\partial h}{\partial t} &= -H\xi \\
\frac{\partial \zeta}{\partial t} &= -f\xi - \beta v \approx -f\xi - \beta \frac{g}{f}\frac{\partial h}{\partial x} \\
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- with $\boldsymbol{z} = \boldsymbol{z}_0 \exp i\left(k_1 x + k_2 y - \omega t\right)$

$$0 = \boldsymbol{A}\cdot\boldsymbol{x} + \omega\mathbf{1}\cdot\boldsymbol{x}\ ,\ \boldsymbol{A} = \begin{pmatrix} 0 & 0 & iH \\ -\beta\frac{g}{f}k_1 & 0 & if \\ -igk^2 & -if & 0 \end{pmatrix} \rightarrow |\boldsymbol{A} + \omega\mathbf{1}| = 0$$

- eigenvalue equation $|\boldsymbol{A}+\omega\mathbf{1}|=0$

$$
\begin{aligned}
0 &= \begin{vmatrix} \omega & 0 & iH \\ -\beta\frac{g}{f}k_1 & \omega & if \\ -igk^2 & -if & \omega \end{vmatrix} = \omega \begin{vmatrix} \omega & if \\ -if & \omega \end{vmatrix} + iH \begin{vmatrix} -\beta\frac{g}{f}k_1 & \omega \\ -igk^2 & -if \end{vmatrix} \\
&= \omega(\omega^2-f^2)+iH(i\beta g k_1+\omega igk^2)=\omega(\omega^2-f^2-gHk^2)-\beta gHk_1
\end{aligned}
$$

characteristic equation $\rightarrow$ three roots

- eigenvalue equation $|\boldsymbol{A} + \omega \mathbf{1}| = 0$

$$\begin{aligned} 0 &= \begin{vmatrix} \omega & 0 & iH \\ -\beta \frac{g}{f} k_1 & \omega & if \\ -igk^2 & -if & \omega \end{vmatrix} = \omega \begin{vmatrix} \omega & if \\ -if & \omega \end{vmatrix} + iH \begin{vmatrix} -\beta \frac{g}{f} k_1 & \omega \\ -igk^2 & -if \end{vmatrix} \\ &= \omega(\omega^2 - f^2) + iH(i\beta g k_1 + \omega i g k^2) = \omega(\omega^2 - f^2 - gHk^2) - \beta g H k_1 \end{aligned}$$

characteristic equation $\rightarrow$ three roots

- set $\beta = 0$

$$\omega = 0 \ , \ \omega^2 - f^2 - gHk^2 = 0$$

$\rightarrow$ geostrophy and gravity waves

- eigenvalue equation $|\boldsymbol{A} + \omega \mathbf{1}| = 0$

$$
\begin{aligned}
0 &= \begin{vmatrix} \omega & 0 & iH \\ -\beta \frac{g}{f} k_1 & \omega & if \\ -igk^2 & -if & \omega \end{vmatrix} = \omega \begin{vmatrix} \omega & if \\ -if & \omega \end{vmatrix} + iH \begin{vmatrix} -\beta \frac{g}{f} k_1 & \omega \\ -igk^2 & -if \end{vmatrix} \\
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\end{aligned}
$$

characteristic equation $\rightarrow$ three roots

- set $\beta = 0$

$$\omega = 0 \ , \ \omega^2 - f^2 - gHk^2 = 0$$

$\rightarrow$ geostrophy and gravity waves

- assume $\omega^2 \ll f^2$

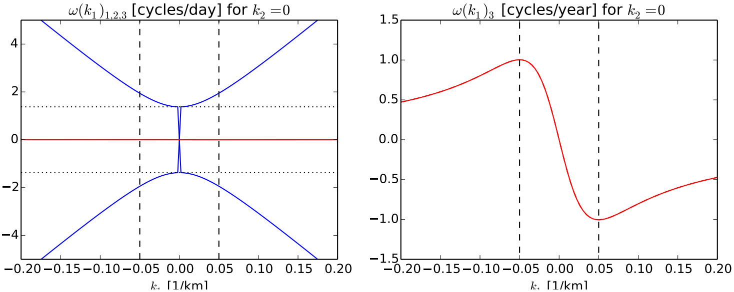
$$\omega(-f^2 - gHk^2) - \beta g H k_1 = 0$$

$\rightarrow$ Rossby waves

- characteristic equation $\rightarrow$ three roots

$$0 = \omega(\omega^2 - f^2 - gHk^2) - \beta gHk_1$$

- set $\beta = 0 \rightarrow$ geostrophy and gravity waves
- assume $\omega^2 \ll f^2 \rightarrow$ Rossby waves
- find roots $\omega_{1,2,3}$ of characteristic equation numerically for $R = \sqrt{gH}/f \approx 20\,\mathrm{km}$ (dashed), $f = 10^{-4}\,\mathrm{s}^{-1}$ (dotted) and $\beta = 2 \times 10^{-11}\,\mathrm{m}^{-1}\mathrm{s}^{-1}$

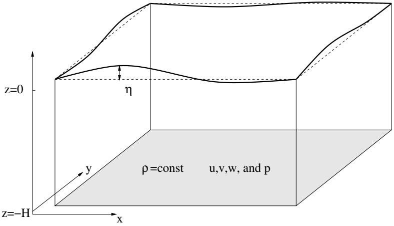


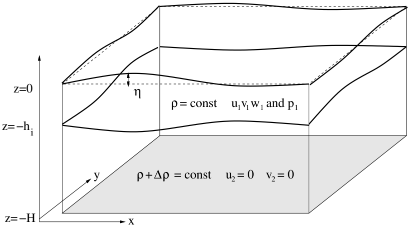
- non-linear barotropic or reduced gravity model

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + f \underset{\neg}{\boldsymbol{u}} = -g \boldsymbol{\nabla} h \quad , \quad \frac{\partial h}{\partial t} + \boldsymbol{\nabla} \cdot (\bar{h} + h)\boldsymbol{u} = 0$$

with mean height $\bar{h} = const$ and total height $H = \bar{h} + h$

and $\underset{\neg}{\boldsymbol{u}} = (-v, u)$ which denotes anticlockwise rotation of $\boldsymbol{u}$ by 90^o





- non-linear "barotropic model" and "baroclinic model"

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + f \underset{\neg}{\boldsymbol{u}} = -g \boldsymbol{\nabla} h \ , \quad \frac{\partial h}{\partial t} + \boldsymbol{\nabla} \cdot H \boldsymbol{u} = 0$$

with $\underset{\neg}{\boldsymbol{u}} = (-v, u)$ which denotes anticlockwise rotation of $\boldsymbol{u}$ by 90^o

and with mean height $\bar{h} = const$ and total height $H = \bar{h} + h$

- non-linear "barotropic model" and "baroclinic model"

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + f \underset{\neg}{\boldsymbol{u}} = -g \boldsymbol{\nabla} h \quad , \quad \frac{\partial h}{\partial t} + \boldsymbol{\nabla} \cdot H \boldsymbol{u} = 0$$

with $\underset{\neg}{\boldsymbol{u}} = (-v, u)$ which denotes anticlockwise rotation of $\boldsymbol{u}$ by 90^o

and with mean height $\bar{h} = const$ and total height $H = \bar{h} + h$

- using the relation $\boldsymbol{\nabla} \boldsymbol{u}^2/2 + \underset{\neg}{\boldsymbol{u}} \underset{\neg}{\boldsymbol{\nabla}} \cdot \boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$ rewrite as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \boldsymbol{u}^2/2 + \underset{\neg}{\boldsymbol{u}} \underset{\neg}{\boldsymbol{\nabla}} \cdot \boldsymbol{u} + f \underset{\neg}{\boldsymbol{u}} = -g \boldsymbol{\nabla} h$$

- non-linear "barotropic model" and "baroclinic model"

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + f \underset{\neg}{\boldsymbol{u}} = -g \boldsymbol{\nabla} h \quad , \quad \frac{\partial h}{\partial t} + \boldsymbol{\nabla} \cdot H \boldsymbol{u} = 0$$

with $\underset{\neg}{\boldsymbol{u}} = (-v, u)$ which denotes anticlockwise rotation of $\boldsymbol{u}$ by 90^o

and with mean height $\bar{h} = const$ and total height $H = \bar{h} + h$

- using the relation $\boldsymbol{\nabla} \boldsymbol{u}^2/2 + \underset{\neg}{\boldsymbol{u}}\, \underset{\neg}{\boldsymbol{\nabla}} \cdot \boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$ rewrite as

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \boldsymbol{u}^2/2 + \underset{\neg}{\boldsymbol{u}}\, \underset{\neg}{\boldsymbol{\nabla}} \cdot \boldsymbol{u} + f \underset{\neg}{\boldsymbol{u}} = -g \boldsymbol{\nabla} h \\ \frac{\partial \boldsymbol{u}}{\partial t} + (f + \underset{\neg}{\boldsymbol{\nabla}} \cdot \boldsymbol{u})\, \underset{\neg}{\boldsymbol{u}} + \boldsymbol{\nabla}(\boldsymbol{u}^2/2 + gh) = 0 \\ \frac{\partial \boldsymbol{u}}{\partial t} + q\, \underset{\neg}{\boldsymbol{U}} + \boldsymbol{\nabla}(K + gh) = 0 \end{aligned}$$

with total transport $\boldsymbol{U} = H\boldsymbol{u}$, relative vorticity $\zeta = \underset{\neg}{\boldsymbol{\nabla}} \cdot \boldsymbol{u}$, potential vorticity $q = (f + \zeta)/H$ and kinetic energy $K = \boldsymbol{u}^2/2$ given by

$$\frac{\partial K}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} K + \boldsymbol{u} \cdot \boldsymbol{\nabla}(gh) = 0$$

since $\boldsymbol{u} \cdot \underset{\neg}{\boldsymbol{u}} H = 0$

- non-linear "barotropic model" and "baroclinic model"

$$\frac{\partial \boldsymbol{u}}{\partial t} + q\ \underset{\neg}{\boldsymbol{U}} + \boldsymbol{\nabla} K = -g\boldsymbol{\nabla} h \quad , \quad \frac{\partial h}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{U} = 0$$

with total transport $\boldsymbol{U} = H\boldsymbol{u}$, relative vorticity $\zeta = \underset{\neg}{\boldsymbol{\nabla}} \cdot \boldsymbol{u}$, potential vorticity $q = (f+\zeta)/H$ and kinetic energy $K = \boldsymbol{u}^2/2$

- total energy $H(K + gH/2)$ is given by adding H times kinetic energy equation+ and gH times thickness equation

$$H\frac{\partial K}{\partial t} + H\boldsymbol{u}\cdot\boldsymbol{\nabla}K + H\boldsymbol{u}\cdot\boldsymbol{\nabla}(gh) = 0 \quad , \quad gH\frac{\partial H}{\partial t} + gH\boldsymbol{\nabla}\cdot\boldsymbol{U} = 0$$

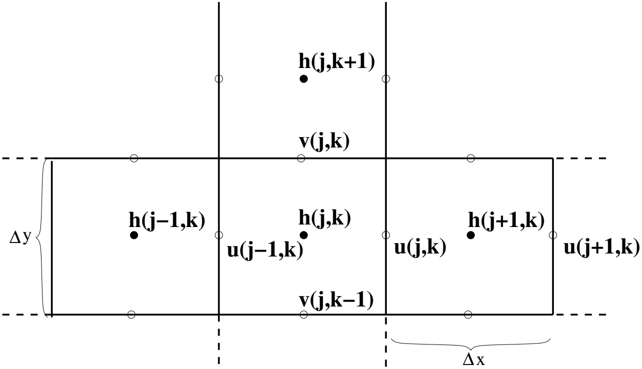
$$H\frac{\partial K}{\partial t} + \boldsymbol{U}\cdot\boldsymbol{\nabla}K + \boldsymbol{U}\cdot\boldsymbol{\nabla}(gH) + gH\frac{\partial H}{\partial t} + gH\boldsymbol{\nabla}\cdot\boldsymbol{U} = 0$$

$$\frac{\partial(HK)}{\partial t} + K\boldsymbol{\nabla}\cdot\boldsymbol{U} + \boldsymbol{U}\cdot\boldsymbol{\nabla}K + gH\frac{\partial H}{\partial t} + \boldsymbol{\nabla}\cdot(gH\boldsymbol{U}) = 0$$

$$\frac{\partial}{\partial t}\left(HK + gH^2/2\right) + \boldsymbol{\nabla}\cdot((K+gH)\boldsymbol{U}) = 0$$

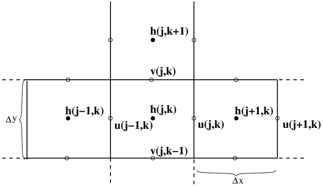
- the linearized 2D shallow water equations with rotation are

$$\frac{\partial u}{\partial t} = fv - \frac{\partial h}{\partial x} \;,\quad \frac{\partial v}{\partial t} = -fu - \frac{\partial h}{\partial y} \;,\quad \frac{\partial h}{\partial t} = -c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$



- discrete linearized shallow water equations with rotation

$$\frac{du_{j,k}}{dt} = f\overline{\overline{v_{j,k}}^{j+}}^{k-} - \delta_x^+ h_{j,k} \ , \ \frac{dv_{j,k}}{dt} = -f\overline{\overline{u_{j,k}}^{j-}}^{k+} - \delta_y^+ h_{j,k}$$



with the finite differencing operators

$$\delta_x^+ h_{j,k} = (h_{j+1,k} - h_{j,k})/\Delta_x \ , \ \delta_x^- h_{j,k} = (h_{j,k} - h_{j-1,k})/\Delta_x$$
$$\delta_y^+ h_{j,k} = (h_{j,k+1} - h_{j,k})/\Delta_y \ , \ \delta_y^- h_{j,k} = (h_{j,k} - h_{j,k-1})/\Delta_y$$

and with the finite averaging operators

$$\overline{h_{j,k}}^{j+} = (h_{j,k} + h_{j+1,k})/2 \ , \ \overline{h_{j,k}}^{j-} = (h_{j,k} + h_{j-1,k})/2$$
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- non-linear "barotropic model" and "baroclinic model"

$$\partial_t \boldsymbol{u} + q \; \underset{\neg}{\boldsymbol{U}} = -\boldsymbol{\nabla}(gh + K) \quad , \quad \partial_t h + \boldsymbol{\nabla} \cdot \boldsymbol{U} = 0$$

  with total transport $\boldsymbol{U} = H\boldsymbol{u}$, relative vorticity $\zeta = \underset{\neg}{\boldsymbol{\nabla}} \cdot \boldsymbol{u}$, potential vorticity $q = (f + \zeta)/H$ and kinetic energy $K = \boldsymbol{u}^2/2$

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- mass flux $\boldsymbol{U} = (U, V)$ is defined as

$$U_{j,k} = u_{j,k}\overline{H}^{i+} \quad , \quad V = v_{j,k}\overline{H}^{j+} \ , \ d_t h = -\delta_x^- U_{j,k} - \delta_y^- V_{j,k}$$

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- gradient force in momentum equation

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- momentum equation is discretized as

$$d_t u_{j,k} = \overline{q_{j,k}\overline{V_{j,k}}^{i+}}^{j-} - \delta_x^+ P_{j,k} \quad , \quad d_t v_{j,k} = -\overline{q_{j,k}\overline{V_{j,k}}^{j+}}^{i-} - \delta_y^+ P_{j,k}$$

- which is the energy conserving scheme by Sadourny (1975)

- consider the linear system again

$$\partial_t \boldsymbol{z} + i\boldsymbol{A}\cdot\boldsymbol{z} = 0$$

with eigenvalues ω_s, left, right eigenvectors $\boldsymbol{Q}^s$ and $\boldsymbol{P}^s$ for $s = 0, \pm$

- interpolate $\boldsymbol{z}$ at time level n, $n-1$ with weights $\alpha = 1.5$, $\beta = -0.5$

$$\boldsymbol{z}^{n+1} - \boldsymbol{z}^n + i\Delta_t\boldsymbol{A}\cdot(\alpha\boldsymbol{z}^n + \beta\boldsymbol{z}^{n-1}) = 0$$

- apply ansatz $\boldsymbol{z} = \boldsymbol{z}_0 e^{-i\bar{\omega}n\Delta_t}$

$$\boldsymbol{z}_0 e^{-i\bar{\omega}\Delta_t} - \boldsymbol{z}_0 + i\Delta_t\boldsymbol{A}\cdot(\alpha\boldsymbol{z}_0 + \beta\boldsymbol{z}_0 e^{i\bar{\omega}\Delta_t}) = 0$$

- expand with eigenvectors $\boldsymbol{z}_0 = \sum_{s=0,\pm} g^s\boldsymbol{Q}^s$ and project on $\boldsymbol{P}^{s_0}$

$$\begin{aligned}
e^{-i\bar{\omega}\Delta_t} - 1 + i\Delta_t\omega_{s_0}(\alpha + \beta e^{i\bar{\omega}\Delta_t}) &= 0\\
e^{-2i\bar{\omega}\Delta_t} - (1 - i\Delta_t\omega_{s_0}\alpha)e^{-i\bar{\omega}\Delta_t} &= -i\Delta_t\omega_{s_0}\beta\\
e^{-i\bar{\omega}\Delta_t} &= (1/2 - i\Delta_t\omega_{s_0}\alpha/2) \pm \sqrt{-i\Delta_t\omega_{s_0}\beta + (1/2 - i\Delta_t\omega_{s_0}\alpha/2)^2}\\
\bar{\omega} &= i\ln\left((1/2 - i\Delta_t\omega_{s_0}\alpha/2) \pm \sqrt{-i\Delta_t\omega_{s_0}\beta + (1/2 - i\Delta_t\omega_{s_0}\alpha/2)^2}\right)/\Delta_t
\end{aligned}$$

- consider the linear system $\partial \boldsymbol{z}/\partial t + i\boldsymbol{A}\cdot\boldsymbol{z} = 0$

$$\boldsymbol{z}^{n+1} - \boldsymbol{z}^{n} + i\Delta_t \boldsymbol{A}\cdot(\alpha \boldsymbol{z}^{n} + \beta \boldsymbol{z}^{n-1}) = 0$$

- interpolate $\boldsymbol{z}$ between time level n and $n-1$ with weights α, β and apply ansatz $\boldsymbol{z} = \boldsymbol{z}_0 e^{-i\bar{\omega} n \Delta_t} \rightarrow \bar{\omega} = i\ln(z)/\Delta_t$ with

$$\begin{aligned} z &= 1/2 - i\Delta_t\omega_{s_0}\alpha/2 \pm \sqrt{-i\Delta_t\omega_{s_0}\beta + (1/2 - i\Delta_t\omega_{s_0}\alpha/2)^2} \\ &= 1/2 - i\Delta_t\omega_{s_0}\alpha/2 \pm \sqrt{-i\Delta_t\omega_{s_0}(\beta+\alpha/2) + 1/4 - (\Delta_t\omega_{s_0}\alpha/2)^2} \end{aligned}$$

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- for small Δ_t we get

$$\begin{aligned} z &= 1/2 - i\Delta_t\omega_{s_0}\alpha/2 \pm (1/2)\sqrt{1 - 4i\Delta_t\omega_{s_0}(\beta+\alpha/2) - 4(\Delta_t\omega_{s_0}\alpha/2)^2} \\ &\approx 1/2 - i\Delta_t\omega_{s_0}\alpha/2 \pm (1/2 - i\Delta_t\omega_{s_0}(\beta+\alpha/2) - (\Delta_t\omega_{s_0}\alpha/2)^2) \end{aligned}$$

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- the negative root becomes with $z = re^{i\phi}$ and $\ln z = \ln r + i\phi$

$$z = i\Delta_t\omega_{s_0}\beta + (\Delta_t\omega_{s_0}\alpha/2)^2$$

and

$$\phi = \tan^{-1}\left(\Delta_t\omega_{s_0}(\alpha/2)^2/\beta\right)$$

which is not converging to analytical frequency $\rightarrow$ computational mode which is strongly damped

- for small Δ_t we get

$$z \approx 1/2 - i\Delta_t\omega_{s_0}\alpha/2 \pm (1/2 - i\Delta_t\omega_{s_0}(\beta + \alpha/2) - (\Delta_t\omega_{s_0}\alpha/2)^2)$$

- the positive root becomes

$$z \approx 1 - (\Delta_t\omega_{s_0}\alpha/2)^2 - i\Delta_t\omega_{s_0}(\alpha + \beta)$$

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- for the real part we find

$$\begin{aligned}\phi &= \tan^{-1}(-\Delta_t\omega_{s_0}(\alpha + \beta)/(1 - (\Delta_t\omega_{s_0}\alpha/2)^2)) \\ &\approx -\Delta_t\omega_{s_0}(\alpha + \beta)/(1 - (\Delta_t\omega_{s_0}\alpha/2)^2) \\ &\approx -\Delta_t\omega_{s_0}(\alpha + \beta)(1 + (\Delta_t\omega_{s_0}\alpha/2)^2) \approx -\Delta_t\omega_{s_0}(\alpha + \beta)\end{aligned}$$

for $\alpha + \beta = 1 \to \lim_{\Delta_t \to 0} \phi = -\Delta_t\omega_{s_0}$ and $\lim_{\Delta_t \to 0} \mathrm{Re}(\bar{\omega}) = \omega$

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  for $\alpha+\beta = 1 \rightarrow \lim_{\Delta_t\rightarrow 0}\phi = -\Delta_t\omega_{s_0}$ and $\lim_{\Delta_t\rightarrow 0}\mathrm{Re}(\bar{\omega}) = \omega$

- imaginary part $\mathrm{Im}(\bar{\omega})$ is larger than zero for $\alpha = 1.5$, $\beta = -0.5$
  but for $\alpha = (1.5+\epsilon)$, $\beta = -(0.5+\epsilon) \rightarrow \mathrm{Im}(\bar{\omega}) < 0$