Supervision Assignments in MLRW - Set 3

Dionysis Manousakas

April 2, 2017

1 Matrix formalism in graphs (from Barabasi's book)

Let A be the $N \times N$ adjacency matrix of an undirected unweighted network, without self-loops. Let $\mathbf{1}$ be a column vector of N elements, all equal to 1. In other words $\mathbf{1} = [1, 1, ..., 1]^T$, where the superscript T indicates the transpose operation. Use the matrix formalism (multiplicative constants, multiplication row by column, matrix operations like transpose and trace, etc, but avoid the sum symbol Σ) to write expressions for:

a) The vector k whose elements are the degrees k_i of all nodes $i = 1, 2, ..., N$.

b) The total number of links, L, in the network.

c) The number of triangles T present in the network, where a triangle means three nodes, each connected by links to the other two (Hint: you can use the trace of a matrix).

d) The vector k_{nn} whose element i is the sum of the degrees of node i's neighbors.

e) The vector k_{nnn} whose element i is the sum of the degrees of node i's second neighbors.

Answer :

a) $A\mathbf{1}$

b) $\mathbf{1}^T A\mathbf{1}/2$ or trace$(A^2)/2$

c) trace$(A^3)/6$

d) $A^2\mathbf{1}$

e) $(A^3 - k^2)\mathbf{1}$ (under a mild definition of second neighbors, where a node is considered a second neighbor of herself)

2 Snobbish Network (from Barabasi's book)

Consider a network of N red and N blue nodes. The probability that there is a link between nodes of identical color is p and the probability that there is a link between nodes of different color is q. A network is snobbish if $p > q$, capturing a tendency to connect to nodes of the same color. For $q = 0$ the network has at least two components, containing nodes with the same color.

a) Calculate the average degree of the "blue" subnetwork made of only blue nodes, and the average degree in the full network.

b) Determine the minimal p and q required to have, with high probability, just one component.

c) Show that for large N even very snobbish networks ($p \gg q$) display the small-world property.

Answer :

a) $(N - 1)p$, $(N - 1)p + Nq$

b) $p > 1/(N - 1)$, $q > 1/N$

c) For very large N we will have $p > 1/(N - 1) \approx 0$ and $q > 1/N \approx 0$. Therefore we will have just one component. Within the same coloured subnetworks we will have the small-world property. Hence, starting from any point of the network we will be able to reach a random same coloured node after a small number of transitions and also jump to the differently coloured subnetwork after a small number of transitions.

3 Betweenness (from Newman's book)

Consider an undirected connected tree of n vertices. Suppose that a particular vertex in the tree has degree k, so that its removal would divide the tree into k disjoint regions, and suppose that the sizes of those regions are $n_1....n_k$.

a) Show that the number of shortest paths passing though the vertex is

$$x = n^2 - \sum_{m=1}^{k} n_m^2$$

b) Hence, or otherwise, calculate the betweenness of the ith vertex from the end of a "line graph" of n vertices, i.e. n vertices in a row.

Answer :

a) The shortest paths joining each of the n_i vertices of the ith component with each of the n_j vertices of the jth component will be passing through the vertex, thus

$$x = 2 \sum_{i=1,j=1,i\neq j}^{k} n_i n_j = n^2 - \sum_{m=1}^{k} n_m^2$$

b) The nonzero terms of the sum in the definition of betweenness centrality will be the paths connecting nodes on different sides of i and they will all be 1s. Thus:

$$c_B(i) = (i-1)(n-i)$$