Lecture 22: Simultaneous Move Games

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Recall the penalty shoot-out game:

		L	R
Player 1	L	-1, 1	1, -1
	R	1, -1	-1, 1

← Player 2

We have seen that we need mixed stratergy here as there is no PSNE

NB: Utility of player 1 is the same for both L and R if player 2 chooses $\sigma_2 = (1/2, 1/2)$. Similarly, player 2 is also indifferent b/w L and R if player 1 chooses $(1/2, 1/2)$

We've:

$$u_1(\sigma_1, \sigma_2) = \sum_{s_2 \in S_2} \sum_{s_1 \in S_1} \sigma_1(s_1)\, \sigma_2(s_2)\, u_1(s_1, s_2)$$

Eg:

$$u_1\left(\left(\tfrac{2}{3}, \tfrac{1}{3}\right), \left(\tfrac{4}{5}, \tfrac{1}{5}\right)\right) = \frac{2}{3} \times \frac{4}{5}(-1) + \frac{2}{3} \times \frac{1}{5}(1) + \frac{1}{3} \times \frac{4}{5}(1) + \frac{1}{3} \times \frac{1}{5}(-1)$$

$$= \frac{-8+2+4-1}{15} = \underline{\underline{\frac{-1}{5}}}$$

For more than two players:

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_n \in S_n} \cdots \sum_{s_1 \in S_1} \sigma_1(s_1) \cdots \sigma_n(s_n)\, u_i(s_1, s_2, \ldots s_n)$$

Expected utility of player i.

Def^n: A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is MSNE if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \quad \forall \sigma_i' \quad \forall i \in N$$

How to find MSNE's ?

"Support" of a probability distribution:

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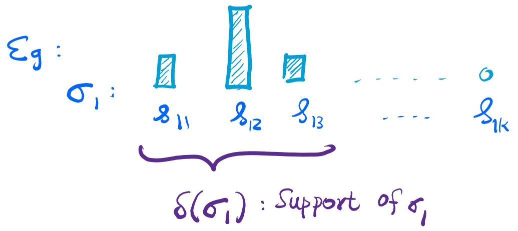
Subset of state space (here, pure strategies) where positive probability mass is placed by σ.



Theorem: A mixed strategy profile $(\sigma_1^*, \sigma_2^*, \ldots \sigma_n^*)$ is an MSNE if and only if:

i) $u_i(s_i, \sigma_{-i}^*)$ is the same for all $s_i \in \delta(\sigma_i^*)$ ← support of σ_i^*

NB: $u_i(s_i, \sigma_{-i}^*) = \sum_{\substack{s_j \in S_j \\ j \neq i}} \prod_{j \neq i} \sigma_j(s_j)\, u_i(s_i, s_{-i})$

ii) $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$ where $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$

Eg:

Player 1 \ Player 2	L	R
L	-1, 1	1, -1
R	1, -1	-1, 1

Possible supports: $\{L\}, \{R\}, \{L, R\}$

i) $(\underset{\delta_1}{\{L\}}, \underset{\delta_2}{\{L\}})$ Not possible as $u_1(L,L) < u_1(R,L)$: violates cond^n (ii)

ii) $(\{R\}, \{R\})$ Not possible : similar to (i)

iii) $(\{L\}, \{L,R\})$ Not possible as $u_2(L,L) \neq u_2(L,R)$: violates cond (i)

iv) $(\{L,R\}, \{L,R\})$: Let $\sigma_1 = (\overset{L}{p}, \overset{R}{1-p})$ and $\sigma_2 (\overset{L}{q}, \overset{R}{1-q})$

Condition (i) for player 1:

$$u_1(L, (q, 1-q)) = u_1(R, (q, (1-q)))$$

$$\Rightarrow q(-1) + (1-q)(1) = q(1) + (1-q)(-1)$$

$\Rightarrow \quad q = 1/2$

Condition (i) for player 2:

$$u_2((p,(1-p)),L) = u_2((p,(1-p)),R)$$

$$\Rightarrow p = 1/2$$

Exercise: Find MSNE for professor's Dilemma game in Lecture 21.

So far:

"Given a game, what is the rational outcome?"

How about the question:

"Given an outcome, how the game should be designed such that in the equilibrium of that game the desired outcome will be obtained?"

Mechanism Design / Social Choice

Voting:

$N = \{1, 2, \ldots n\}$

$A = \{a_1, a_2, \ldots a_m\}$ (Set of alternatives)

Every agent has strict preferences over A: $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = P_i$ → of i^{th} player → any permutation of alternatives.

Voting / Social Choice function:

$$f(P_1, P_2, \ldots P_n) \in A$$

eg: $$f\begin{array}{c} \begin{array}{ccc} P_1 & P_2 & \ldots P_n \end{array} \\ \left(\begin{array}{ccc} a_2 & a_1 & \vdots \\ a_1 & a_2 & \vdots \\ a_4 & a_3 & \\ a_3 & a_4 & \end{array}\right) \end{array} = a_4$$

Common Voting rules:

Each voter votes for one candidate (most preferred one i.e. $P_i[0]$).

The candiates with most votes win. (Plurality rule)

Eg:

	1	2	3	4	5
3	a	a	b	c	d
2	b	b	c	b	b
1	d	c	d	d	c
0	c	d	a	a	a

four candidates: $\{a, b, c, d\}$

According to plurality:

Winner is 'a'

Borda's Voting rule:

For each position in preference order, a weight is assigned:

$$w_1 = m-1$$
$$w_2 = m-2$$
$$\vdots$$
$$w_{m-1} = 1$$
$$w_m = 0$$

$a = 6 \quad b = 11 \quad c = 7 \quad d = 6$

for Borda's voting rule.

Single Transferrable Vote (STV)

Runs in multiple rounds. In each round one candidate with minimum plurality score is eliminated.

Eg:

1	2	3	4	5
~~a~~	~~a~~	~~b~~	c	~~d~~
~~b~~	~~b~~	c	~~b~~	~~b~~
~~d~~	c	~~d~~	~~d~~	c
c	~~d~~	~~a~~	~~a~~	~~a~~

Round 1: b is eliminated

Round 2: d is eliminated

Round 3: a is eliminated

Winner: 'c'

Condorcet Consistency

A condorcet winner is a candidate who beats every other candidate in pairwise elections:

Eg

a	b	c
b	c	a
c	a	b

$(a,b) \xrightarrow{\text{winner}} a$

$(b,c) \xrightarrow{\text{winner}} b$

$(c,a) \xrightarrow{\text{winner}} c$

So condorcet winner may not exist!

A condorcet consistent voting rule always output a condorcet winner if it exists.

Eg: Is plurality condorcet consistent?

a	b	c
b	a	a
c	c	b
30%	30%	40%

Pairwise elections:

(a,b) : 70% with a

(a,c) : 60% with a

So, a is condorcet winner.

But c is plurality winner (40%)

Copeland rule:

Copeland scores for a candidate (a) = No. of wins it has in pairwise elections.

Candidate with highest copeland score wins.

Copeland rule is condorcet consistent.