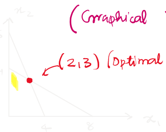
12 January 2024 11:04

Recall that a linear program is $\begin{cases} \max c^T x \\ \text{st.} \quad Ax \le b \\ \quad x \ge 0 \end{cases}$

Eg: $\max \; 7x_1 + 6x_2$

st. $\begin{cases} 2x_1 + 4x_2 \le 16 \\ 3x_1 + 2x_2 \le 12 \\ x_1, x_2 \ge 0 \end{cases}$

(Graphical Method)



(2,3) (Optimal value)

Problems with graphical method:

- Not algorithmic
- Not scalable for dimensions greater than 3.

Simplex Method

An algorithmic approach to solve a linear program.

Let's come back to our original ex.

$\max \; 7x_1 + 6x_2$

st. $\begin{cases} 2x_1 + 4x_2 \le 16 \\ 3x_1 + 2x_2 \le 12 \\ x_1, x_2 \ge 0 \end{cases}$

Step 1: Slack the constraints:

$S_1, S_2 \ge 0$

$2x_1 + 4x_2 \le 16 \rightarrow 2x_1 + 4x_2 + S_1 = 16$

$3x_1 + 2x_2 \le 12 \rightarrow 3x_1 + 2x_2 + S_2 = 12$

Objective f^n: $7x_1 + 6x_2 + 0 \cdot S_1 + 0 \cdot S_2$

Step 2: Set basic and non basic variables:

$S_1 = 16$, $S_2 = 12$ $x_1 = x_2 = 0$ is a basic feasible sol^n (basic $\rightarrow$ non zero value, non basic $\rightarrow$ zeros)

Initially,

basic variables - new ones
non-basic variables - original ones
$\}\rightarrow$ we will modify this using a greedy method.

Greedy method: We bring in x_1 (more weightage in objective f^n)., and push out something at S_1/S_2 out.

Here we bring $x_1 = 4$ in and S_2 will go out.

$\Rightarrow$ New basic variables - x_1, S_1 ; New non basic variables - x_2, S_2.

Now we've $\left.\begin{aligned} S_1 &= 16 - 2x_1 - 4x_2 \\ S_2 &= 12 - 3x_1 - 2x_2 \end{aligned}\right\}$ initially we'd this. basic var. in terms of non basic

Now, we've

basic. $\begin{cases} x_1 = 4 - \dfrac{S_2}{3} - \dfrac{2x_2}{3} \\ S_2 = 8 + \underbrace{\dfrac{2S_2}{3} - \dfrac{8x_2}{3}}_{\text{Non basic}} \end{cases}$

Objective f^n now becomes: $f = 28 + \dfrac{4x_2}{3} - \dfrac{7S_2}{3}$

We will keep updating basic & non basic variables till objective f^{ns} have -ve coefficient.

Simplex Tableu Method

	x_1	x_2	S_1	S_2	b	fraction b/x_i
S_1	2	4	1	0	16	$16/2 = 8$
S_2	3	2	0	1	12	$12/3 = 4$
P	7	6	0	0		

* We push in non basic variable with greater coefficient
* We remove basic variable with smaller fraction.

	x_1	x_2	S_1	S_2	b	fraction b/x_i
S_1	0	$8/3$	1	$-2/3$	8	$8/(8/3) = 3$
x_1	1	$2/3$	0	$1/3$	4	6
P	0	$4/3$	0	$-7/3$		

	x_1	x_2	S_1	S_2	b	fraction b/x_i
x_2	0	1	$3/8$	$-1/4$	3	
x_1	1	0	$-1/4$	$1/2$	6	
P	0	0	$-1/2$	-2		

$\hookrightarrow$ We stop here as coefficients become non-positive.

Convex Optimization:

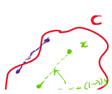
A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if:

$$\boxed{\begin{cases} \forall x, y \in \mathbb{R}^n \\ \forall \lambda \in [0,1] \end{cases} \quad f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)}$$

Ex: $f(x) = x^2$, $f(x_1, x_2) = x_1 x_2$ $(x_1, x_2 \in \mathbb{R})$

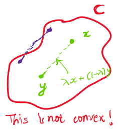
Similarly we can say that a set $C \subseteq \mathbb{R}^n$ is convex if

$$\boxed{\begin{matrix} \forall x, y \in C \\ \forall \lambda \in [0,1] \end{matrix} \quad \lambda x + (1-\lambda) y \in C}$$



Similarly we can say that a set $C \subseteq \mathbb{R}$ is convex if

$$\boxed{\begin{array}{ll} \forall x, y \in C & \lambda x + (1-\lambda) y \in C \\ \forall \lambda \in [0,1] & \end{array}}$$



This is not convex!

Ex: Circular, rectangular region

In covex optimisation

Objective f^n $\quad f(x)$ ← convex f^n

s.t $\quad x \in \mathcal{C}$ ← convex set

Qn: Are the following convex?

- $C = C_1 \cup C_2$: (C_1, C_2 are convex)
- $\{x \in \mathbb{R}^2 : x \geq 0,\ x_1 \cdot x_2 \geq 1\}$
- $f : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$, $f(x) = x_1 x_2$
- $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1^2 + x_2^2 + x_1 x_2$

Ex: (i) $f(x) = e^{ax}$, $a \in \mathbb{R}$ $\quad$ (ii) $f(x) = -\log(x)$, $x > 0$

(iii) $f(x) = x^T x = \|x\|_2^2 = \sum_{i=1}^{n} x_i^2$ (L_2 norm)

(iv) $f(x) = \|x\|_1 = \sum_{i=1}^{n} |x_i|$ (L_1 norm)

Beauty of Convex f^ns:

Def^n: A point x is said to be "globally optimal" if x is feasible and $\nexists$ any other feasible 'y' such that $f(y) < f(x)$
(Ass. minimization problem)

Def^n: A point x is said to be "locally optimal" if x is feasible and $\exists R > 0$ such that, $\forall$ Feasible y with $\|y - x\|_2 \leq R$,
$f(x) \leq f(y)$

Theorem: For a convex optimization problem, all locally optimal points are globally optimal.

Proof: Use contradiction.