Lecture 21: Normal Form Game Representation

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Not all games have total utility zero. i.e, Win for one may not mean loss for other

Eg: Arms Race

1 ↓, 2 →	Peace	War
Peace	5, 5	0, 10
War	10, 0	1, 1

Two countries with two possibilies
• be in peace • be in war.

Each entry give utilities for 1 and 2 respectively → see below

$N = \{1, 2, \dots n\}$ Set of players/agents

S_i = strategy set of player $i \in N$

s_i = one strategy of i, $s_i \in S_i$

$(s_1, s_2, \dots s_n)$ = strategy profile. $\in S_1 \times S_2 \dots \times S_n$

$= (s_i, \underline{s}_i)$ where $\underline{s}_i = (s_1, s_2, \dots s_{i-1}, s_{i+1}, \dots s_n)$

$u_i : S_1 \times S_2 \dots \times S_n \longrightarrow \mathbb{R}$ $u_i(s_i, \underline{s}_i) \in \mathbb{R}$ utility of player i

In example shown above $N = \{1, 2\}$, $S_1 = \{\text{Peace}, \text{War}\} = S_2$

$u_1(P, W) = 0$ $u_2(P, W) = 10$

NFG representation: $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

What strategy should 1 and 2 choose?

Dominated Strategy

A strategy s_i' of i is a strictly dominated strategy if $\exists$ another strategy $s_i \in S_i$ such that $\forall \underline{s}_i \in \underline{S}_i = \bigtimes_{j \neq i} S_j$:

$$u_i(s_i, \underline{s}_i) > u_i(s_i', \underline{s}_i)$$

For weak dominated strategy: s_i' is weakly dominated if $\exists s_i \in S_i$

$\forall\ s_{-i} \in S_{-i}$ $\quad u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$

and, $\exists\ \tilde{s}_{-i} \in S_{-i}$ $\quad u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i})$

Example:

	D	E
A	5, 5	0, 5
B	5, 0	1, 1
C	4, 0	1, 1

$u_1(B,D) = u_1(A,D)$

$u_1(B,E) > u_1(A,E)$

$\Rightarrow$ A is weakly dominated

Similarly, C is weakly dominated

as, $u_1(B,D) > u_1(C,D)$

$u_1(B,E) = u_1(C,E)$

Dominant Stratergy

A strategy that dominates every other strategy of the player.

Strictly/weakly dominant: based on type of domination.

In above example: B weakly dominates A and C both.

So, B is weakly dominant strategy for player 1.

Equillibrium: If both players have Strictly dominant / weakly dominant stratergies, the strategy profile of their SDS/WDS is called strictly/weakly dominant stratergy equillibrium (SDSE/WDSE)

Ex: In Arms Race: (War, War) is SDSE

On the other game, (B, E) is WDSE

Remark: Atleast one player having a WDS and others have SDS $\Rightarrow$ WDSE

Professor's Dilemma

Student

	Listen	Sleep

		Listen	Sleep
Professor	Effort	100, 100	−10, 0
	No effort	0, −10	0, 0

Clearly, there is no dominant stratergies for professor & students.

Pure Strategy Nash Equillibrium (Nash 1951)

Some strategy profile from which unilateral (other players actions are fixed, only one player moves) deviations are not benefitial.

A PSNE is a strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ such that

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \quad \forall s_i' \in S_i \quad \forall i \in N$$

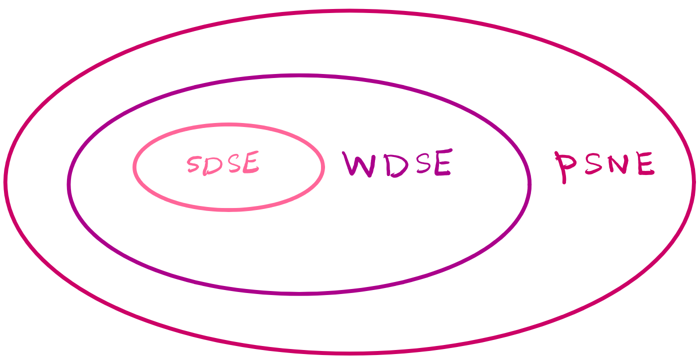
In Professor's Dilemma, (E, L) is a PSNE as:

$$u_1(E, L) > u_1(NE, L)$$
$$u_2(E, L) > u_2(E, S)$$

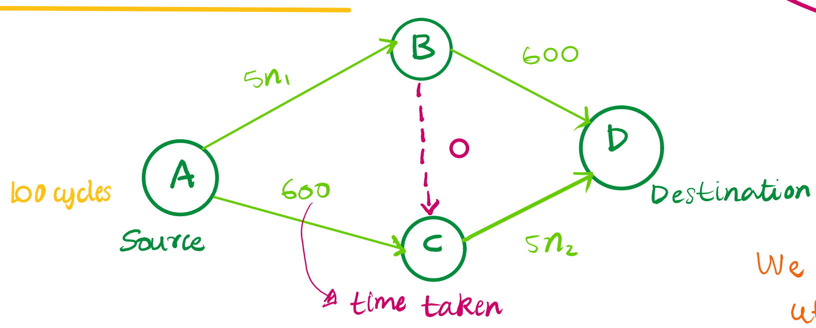
unilateral deviations

Relation b/w WDSE, SDSE, PSNE

- If (s_i^*, s_{-i}^*) is WDSE $\Rightarrow$ (s_i^*, s_{-i}^*) is a PSNE
- SDSE $\Rightarrow$ WDSE $\Rightarrow$ PSNE



Transportation Problem



n_1, n_2: are the no. of vehicles in the path

We need to go from A to D. Our utility is inversely related to time

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Here, equillibrium will be 50 cycles in each path.

Observe that this a PSNE as for a person moving in ABD, moving to ACD won't be beneficial, in this case.

It is not WDSE as it is not satisfied for "all" strategy profiles.

In this equillibrium, total time = 850

What if we introduce a new $B \xrightarrow{0} C$ path which can be reached in 0 time.

New equillibrium will be that everyone (all 100 cycles) choose ABCD

This equillibrium is SDSE as irrespective of others straterges we adopt ABCD, and inequalities are strict.

In this equillibrium, total time = 1000

Braess Paradox (as time increased here)

Recap (Two player zero sum games)

$$\bar{v} = \max_{s_2} \min_{s_1} u(s_1, s_2)$$

$$\underline{v} = \min_{s_2} \max_{s_1} u(s_1, s_2)$$

Lemma: $\bar{v} \geq \underline{v}$

Theorem: A matrix game u has a PSNE (saddle point) iff and only iff $\bar{v} = \underline{v} = u(s_1^*, s_2^*)$, where s_1^* and s_2^* are max-min and min-max straterqies of players 1 and 2

$$s_1^* = \arg\max_{s_1} \min_{s_2} u(s_1, s_2)$$

$$s_2^* = \arg\min_{s_2} \max_{s_1} u(s_1, s_2)$$

[See example in Lec. 20 and verify $\bar{v} = \underline{v}$]

Consider a simpler version of penalty shootout:

Player 1 \ Player 2	L	R
L	-1, 1	1, -1

Player 1

L	-1, 1	1, -1
R	1, -1	-1, 1

Consider all the possible strategy profiles and verify that PSNE does not exist in the above case.

PSNE may not exist always!

So we can define a mixed action for a player.

Ex: $\sigma_2 = \left(\frac{4}{5}, \frac{1}{5}\right)$: player picks L with prob. $\frac{4}{5}$ and R with prob. $\frac{1}{5}$

A mixed stratergy is a propability distribution over the pure stratergies.

We will start looking at expected utilities:

$$u_1(L, \sigma_2) = u_1(L,L) \times \frac{4}{5} + u_1(L,R) \times \frac{1}{5} \qquad \sigma_2 = \left(\frac{4}{5}, \frac{1}{5}\right)$$

$$u_1(\sigma_1, \sigma_2) = u_1(L,L) \times \frac{2}{3} \times \frac{4}{5} + u_1(L,R) \times \frac{2}{3} \times \frac{1}{5} \qquad \sigma_1 = \left(\frac{2}{3}, \frac{1}{3}\right)$$

$$+ u_1(R,L) \times \frac{1}{3} \times \frac{4}{5} + u_1(R,R) \times \frac{1}{3} \times \frac{1}{5}$$

In the example of penalty shootup above:

$$\begin{cases} u_1\left(L, \left(\frac{4}{5}, \frac{1}{5}\right)\right) = -3/5 \\ u_1\left(R, \left(\frac{4}{5}, \frac{1}{5}\right)\right) = 3/5 \end{cases} \quad \text{L is a better choice}$$

$$\begin{cases} u_1\left(L, \left(\frac{1}{5}, \frac{4}{5}\right)\right) = 3/5 \\ u_1\left(R, \left(\frac{4}{5}, \frac{1}{5}\right)\right) = -3/5 \end{cases} \quad \text{R is a better choice}$$

$$u_1(L, (1/2, 1/2)) = 0 = u_1(R, (1/2, 1/2))$$

$$u_1((1/2, 1/2), (1/2, 1/2)) \geq u_1(\sigma_1, (1/2, 1/2))$$

A strategy profile $(\sigma_1^*, \sigma_2^*, \dots \sigma_n^*)$ is a mixed stratergy NE if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \quad \forall \sigma_i' \ \forall i \in N$$