STYLIZED FACTS OF DAILY RETURN SERIES AND THE HIDDEN (SEMI) MARKOV MODEL

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Presentation for the Course:
Statistics in Stochastic Processes

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Preliminary

Stock market

- Indexes (EURO STOXX 600)
- Sectors (banks ,chemicals ...)
- Opening price p_t
- Time series
- Returns (Daily/weekly/monthly)
  $r_t = log(p_{t+1}) - log(p_t)$ Prices/ Returns

Purpose of our Presentation

- Modelling of the distributional and temporal properties of a daily return series r_t .

  EURO STOXX 600 banks

- Fit a hidden Markov models HMM (Ryden).

  TOBIAS RYDEN,TIMO TERASVIRTA, STEFAN ASBRINK *STYLIZED FACTS OF DAILY RETURN SERIES AND THE HIDDEN MARKOV MODEL*. JOURNAL OF APPLIED ECONOMETRICS , 1998.

- Fit a hidden semi-Markov models HSMM (Bulla Bulla) .

  Jan Bulla, Ingo Bulla *Stylized facts of financial time series and hidden semi-Markov models*. Elsevier B.V. , 2006.

- Comparing the models based on the capacity of reproducing the stylized facts for daily return series listed by Granger and Ding .

Stylized facts of financial time series

Autocorrelation function (Acf)

$$\rho_k = corr(r_t, r_{t-k}) = \frac{Cov(r_t, r_{t-k})}{\sqrt{Var(r_t)}\sqrt{Var(r_{t-k})}}, k \in (1, 2, \dots)$$

Granger and Ding (1995)

Stylized facts of financial time series

- Temporal and properties
- Distributional properties

Temporal Properties

- TP1: Returns r_t are not autocorrelated (except for, possibly, at lag one). TP1

Stylized facts of financial time series

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- TP4: The autocorrelations of sign $(r_t)$are negligibly small . TP4

Stylized facts of financial time series

Distributional Properties

- DP1: $|r_t|$ and $\mathrm{sign}(r_t)$ are independent.

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  An exponentially distributed stochastic variable x_t has the following properties:
- PED1: $E[x_t] = \sqrt{Var(x_t)}$ see DP2
- PED2: $\frac{E[x_t - E[X_t]]^3}{(Var(x_t))^{3/2}} = 2$
- PED3: $\frac{E[x_t - E[X_t]]^4}{(Var(x_t))^2} = 9$

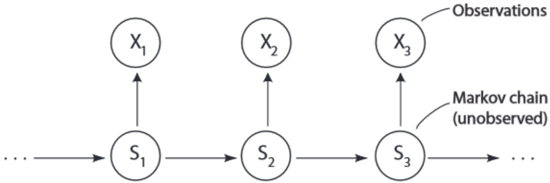
Semi-Markov Process

- S_t : Semi markov discrete time process
- Initial probabilities :
  $\pi_j := P(S_0 = j)$with $\sum_j \pi_j = 1$
- Transition probabilities for the state i : for each $j \neq i$ T.M.
  $P(S_{t+1} = j | S_{t+1} \neq i, S_t = i) = P_{ij}$ with $\sum_j P_{ij} = 1$and $p_{ii} = 0$
- Associated with each state is a sojourn time distribution .
  $d_j(u) :=$
  $P(S_{t+u+1} \neq j, S_{t+u-v} = j, v = 0, \dots, u-2 | S_{t+1} = j, S_t \neq j)$
- Survival function (last visited state)

$$D_j(u) := \sum_{v \geq u} d_j(v)$$

Hidden Markov Models model with J states

- S_t :unobserved states of a homogeneous and irreducible finite-state Markov chain .
- X_t :a state-dependent process which generates the observation depending on the current state of S_t. $S_t \in \{0, 1, \ldots, J-1\}$
- $[X_t | S_t = i] \xrightarrow{d} Gen(p_i)$



Hidden Markov Models model with J states

- Initial probabilities :
  $\pi_j := P(S_0 = j)$with $\sum_j \pi_j = 1$
- Transition probability matrix (TPM) :
  $p_{i,j} = P(S_t = j|S_{t-1} = i)$

$$\Pi = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0J-1} \\ p_{10} & p_{11} & \cdots & p_{1J-1} \\ \vdots & \vdots & \vdots & \ddots \\ p_{J-10} & p_{J-11} & \cdots & p_{J-1J-1} \end{bmatrix}$$

- Complete likelihood function
  $L_c(s_0^{\tau-1}, x_0^{\tau-1}|\theta) =$
  $P(S_0|\theta) \prod_{t=1}^{\tau-1} P(S_t|S_{t-1}, \theta) \prod_{t=0}^{\tau-1} P(X_t = x_t|S_t, \theta)$

Hidden Semi-Markov Models

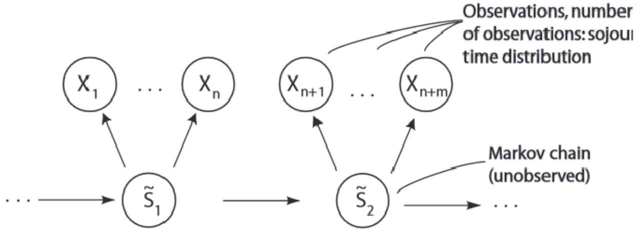
HSMM

- $\{X_t\}, \{S_t\}$: pair of discrete-timestochastic processes.
- X_t :a state-dependent process which generates the observation depending on the current state of S_t.
- S_t :unobserved states of a homogeneous and irreducible finite-state Semi Markov chain . $S_t \in \{0, 1, \ldots, J-1\}$
- X_0^{T-1} the observed sequence of length T.
- $b_j(x_t) = P(X_t = x_t | S_t = j) =$
  $P(X_t = X_t | X_0^{\tau-1} = x_0^{\tau-1}, S_0^{t-1} = s_0^{t-1}, S_t = j, S_{t+1}^{\tau-1} = S_{t+1}^{\tau-1}))$

Hidden Semi-Markov Models

HSMM

- $\tilde{S}_t \in \{0, 1, \ldots, J-1\}$
- $\tilde{S}_i := s_{u_0+\ldots+u_{i-1}}, s_{u_0+\ldots+u_{i-1}+1}, \ldots, , s_{u_0+\ldots+u_{i-1}-1}$



Hidden Semi-Markov Models

The likelihood of a HSMM

- Complete data likelihood

$$L_C\left(s_0^{\tau-1+u}, x_0^{\tau-1} \mid \theta\right) = P\left(S_0^{\tau-1} = s_0^{\tau-1}, \underline{S_{\tau-1+v} = s_{\tau-1}, \ v = 1, \ldots, u-1,}\right.$$
$$\left. S_{\tau-1+u} \neq s_{\tau-1}, X_0^{\tau-1} = x_0^{\tau-1} \mid \theta\right).$$

right-sencoring

$$= \pi_{\tilde{s}_0} d_{\tilde{s}_0}(u_0) \prod_{r=1}^{R} p_{\tilde{s}_{r-1}\tilde{s}_r} d_{\tilde{s}_r}(u_r)\, I\left(\sum_{r=0}^{R-1} u_r < \tau \leqslant \sum_{r=0}^{R} u_r\right)$$

- Likelihood of the observations

$$L(\theta) = \sum_{s_0,\ldots,s_{\tau-1}} \sum_{u_{\tau+}} L_c(s_0^{\tau-1+u}, x_0^{\tau-1} | \theta)$$

EM in Hidden Semi-Markov Models

E-step: Compute the Q-function

$$Q\left(\theta, \theta^{(k)}\right) = \mathbf{E}\left[L_C\left(s_0^{\tau-1+u}, x_0^{\tau-1} | \theta\right) \mid X_0^{\tau-1} = x_0^{\tau-1}, \theta^{(k)}\right],$$

the conditional expectation of the complete-data log-likelihood, where $\theta^{(k)}$ denotes the current estimate of the parameter vector θ.

M-step: Compute $\theta^{(k+1)}$, the parameter values that maximize the function Q w.r.t. θ, i.e.,

$$\theta^{(k+1)} = \underset{\theta}{\operatorname{argmax}}\ Q\left(\theta, \theta^{(k)}\right).$$

EM in Hidden Semi-Markov Models

$$L_C\left(s_0^{\tau-1+u}, x_0^{\tau-1} \mid \theta\right) = \pi_{\tilde{s}_0} d_{\tilde{s}_0}(u_0) \prod_{r=1}^{R} p_{\tilde{s}_{r-1}\tilde{s}_r} d_{\tilde{s}_r}(u_r) \prod_{t=0}^{\tau-1} b_{s_t}(x_t) .$$

$$Q\left(\theta, \theta^{(k)}\right) = \sum_{s_1,\ldots,s_{\tau-1}} \sum_{u_{\tau+}} \left[\log L_C\left(s_0^{\tau-1+u}, x_0^{\tau-1} \mid \theta\right) P\left(S_0^{\tau-1+u} = s_0^{\tau-1+u} \mid X_0^{\tau-1} = x_0^{\tau-1}, \theta^{(k)}\right)\right] .$$

$$= \sum_{s_1,\ldots,s_{\tau-1}} \sum_{u_{\tau+}} \left[\log \pi_{\tilde{s}_0} + \left(\sum_{r=1}^{R} \log p_{\tilde{s}_{r-1}\tilde{s}_r}\right) + \left(\sum_{r=0}^{R} \log d_{\tilde{s}_r}(u_r)\right)\right.$$

$$\left. + \left(\sum_{t=0}^{\tau-1} \log b_{s_t}(x_t)\right)\right] P\left(S_0^{\tau-1+u} = s_0^{\tau-1+u} \mid X_0^{\tau-1} = x_0^{\tau-1}, \theta^{(k)}\right) .$$

EM in Hidden Semi-Markov Models

$$\sum_{j=0}^{J-1} P\left(S_0 = j \mid X_0^{\tau-1} = x_0^{\tau-1}, \theta^{(k)}\right) \log \pi_j,$$

$$\sum_{i=0}^{J-1} \sum_{j \neq i} \sum_{t=0}^{\tau-2} P\left(S_{t+1} = j, S_t = i \mid X_0^{\tau-1} = x_0^{\tau-1}, \theta^{(k)}\right) \log p_{ij},$$

$$\sum_u \left\{ \sum_{t=0}^{\tau-2} \left(S_{t+u+1} \neq j, S_{t+u-v} = j, \; v = 0, \ldots, u-1, S_t \neq j \mid X_0^{\tau-1} = x_0^{\tau-1}, \theta^{(k)}\right) \right.$$

$$\left. + P\left(S_u \neq j, S_{u-v} = j, \; v = 1, \ldots, u \mid .X_0^{\tau-1} = x_0^{\tau-1}, \theta^{(k)}\right) \right\} \log d_j(u),$$

$$\sum_{j=0}^{J-1} \sum_{t=0}^{\tau-1} P\left(S_t = j \mid X_0^{\tau-1} = x_0^{\tau-1}, \theta^{(k)}\right) \log b_j\left(x_t\right).$$

Models for daily return series

Ryden Hidden Markov Models 2 states

- X_0^{T-1} the observed daily return serie.
- S_t :unobserved states of a homogeneous and irreducible finite-state Markov chain with 2 states . $S_t \in \{0,1\}$
- $[X_t|S_t = 0] \xrightarrow{d} N(\mu_0, \sigma_0^2)$
  $[X_t|S_t = 1] \xrightarrow{d} N(\mu_1, \sigma_1^2)$
- P(sojourn in state j of length u)=
  $= p_{jj}^{u-1}(1 - p_{jj})$

est.par.

Bulla-Bulla Hidden Semi-Markov Models (Normal) 2 states

- X_0^{T-1} the observed daily return serie.
- $\tilde{S}_t \in \{0, 1\}$: state sequence
- $[X_t|S_t = 0] \xrightarrow{d} N(\mu_0, \sigma_0^2)$
  $[X_t|S_t = 1] \xrightarrow{d} N(\mu_1, \sigma_1^2)$
- P('sojourn in state j of length u')= $\binom{u-2+r_j}{u-1} p_j^{r_j} (1 - p_j)^{u-1}$

est.par.

Models for daily return serie

Bulla-Bulla Hidden Semi-Markov Models (t-Student) 2 states

- X_0^{T-1} the observed daily return serie.
- $\tilde{S}_t \in \{0,1\}$: state sequence
- $[X_t|S_t = 0] \xrightarrow{d} t(\tau_0, \nu_0)$
  $[X_t|S_t = 1] \xrightarrow{d} t(\tau_1, \nu_1)$
- P('sojourn in state j of length u')= $\binom{u-2+r_j}{u-1} p_j^{r_j} (1-p_j)^{u-1}$

est.par.

Data

- Sector indices from STOXX Ltd
- 1st January 1987 to 5th September 2005
- Currency : Euro
- All data are obtained from Thomson financial datastream.
- All sector indices are leptokurtic and negatively skewed
- The Jarque–Bera statistic confirms the departure from normality Stylized facts

Empirical Results

Basic statistics for the return series and the fitted models

- Empirical mean and the mean of the fitted models lie very close to zero for all 18 sectors.
- The empirical standard deviation is also reproduced very well by the three models . S.D.
- The 3 models exhibit a clear tendency towards the kurtosis.(SM_t provides the best results) Kurtosis Kurtosis of SMt

Compares by AIC

AIC

- Average, - 28388 for the HMM to - 28456 for SMn and -28522 for SMt .

AIC

Sectors	MRy	SMn	SMt
Automobiles	−28 473	−28 537	−28 594
Banks	−31 114	−31 203	−31 258
Basics	−30 061	−30 132	−30 224
Chemicals	−29 645	−29 736	−29 814
Construction	−31 058	−31 104	−31 170
Financials	−31 483	−31 569	−31 625
Food	−31 398	−31 464	−31 639
Healthcare	−29 407	−29 468	−29 564
Industrials	−30 931	−30 979	−31 066
Insurance	−29 360	−29 464	−29 512
Media	−28 989	−29 055	−29 121
Oil and gas	−29 955	−30 006	−30 066
Personal	−31 418	−31 454	−31 483
Retail	−27 222	−27 331	−27 386
Technology	−27 351	−27 423	−27 497
Telecom	−20 559	−20 640	−20 679
Travel	−20 337	−20 361	−20 382
Utilities	−22 230	−22 279	−22 313

RMt - SMn- SMt Model-stylized facts

- TP1: Returns r_t are not autocorrelated (except for, possibly, at lag one). TP1
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- TP3: The Taylor effect $corr(|r_t|, |r_{t-k}|) > corr(|r_t|^{\theta}, |r_{t-k}|^{\theta}), \theta \neq 1$ (Taylor, 1986). taylor effect
- TP4: The autocorrelations of sign $(r_t)$are negligibly small . TP4.

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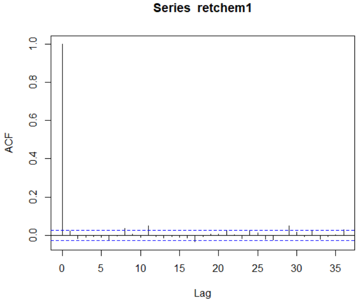
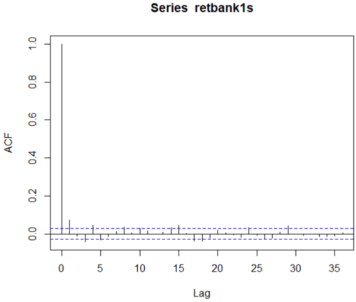
- PED3: $\frac{E[x_t - E[X_t]]^4}{(Var(x_t))^2} = 9$ PED3

Do you want more ?

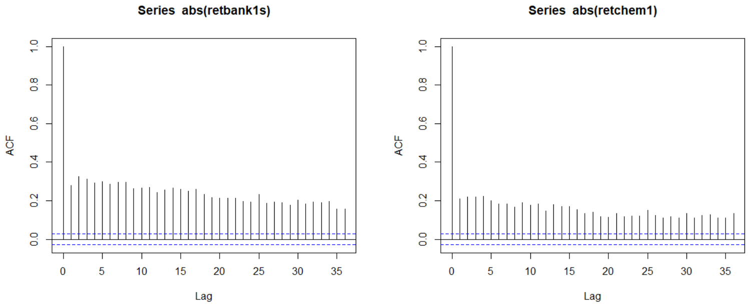
Do you want more ?
Cofee break!!!

Returns r_t are not autocorrelated

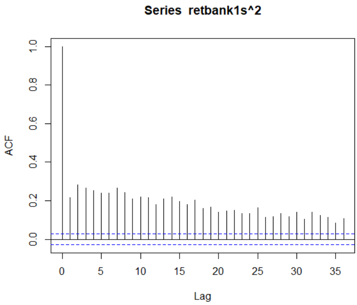


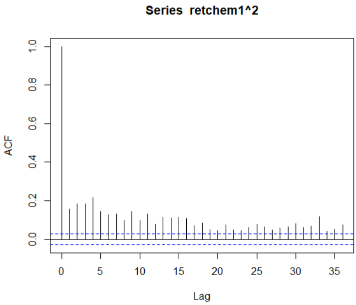


$|r_t|$ autocorrelation functions decay slowly



r_t^2 autocorrelation functions decay slowly





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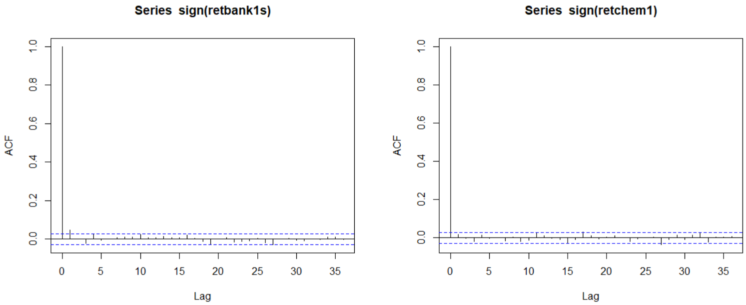
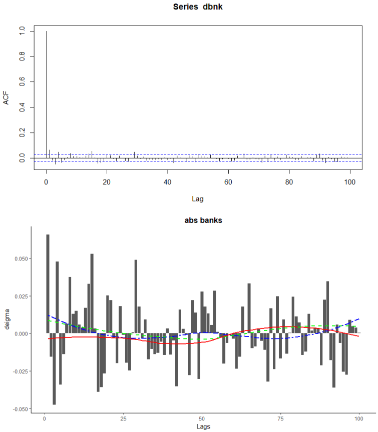
The autocorrelations of sign $(r_t)$are negligibly small



Transition Matrix

$$\begin{bmatrix} 0 & p_{01} & \cdots & p_{0J-1} \\ p_{10} & 0 & \cdots & p_{1J-1} \\ \vdots & \vdots & \vdots & \ddots \\ p_{J-10} & p_{J-11} & \cdots & 0 \end{bmatrix}$$

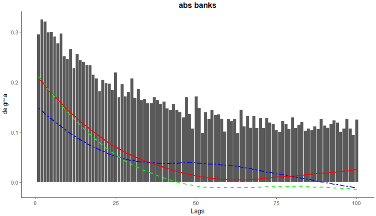
- HMM / HSMM N / HSMM t

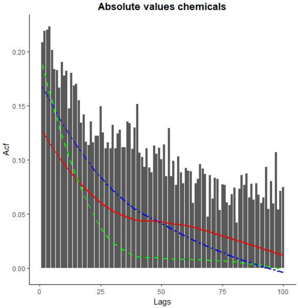


Ryden-bulla R-Bulla t

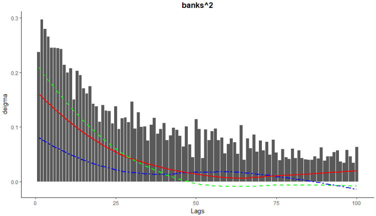
TP2 HMM /HSMM N /HSMM t

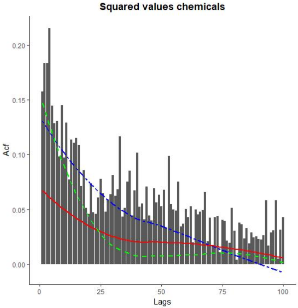
- HMM / HSMM N / HSMM t Ryden-bulla R-Bulla t





- HMM / HSMM N / HSMM t Ryden-bulla R-Bulla t





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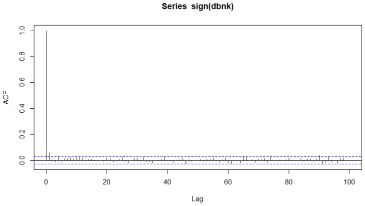
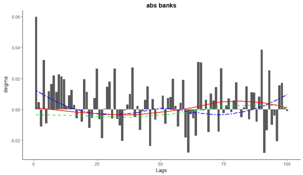


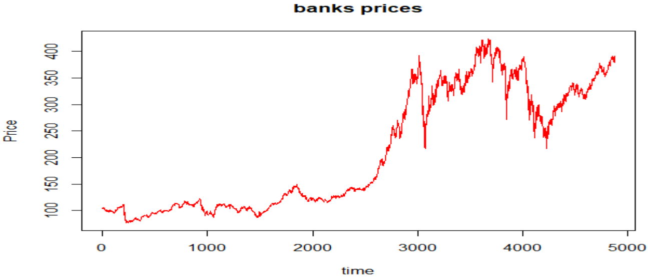


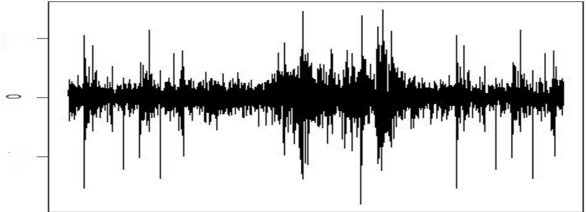
Ryden-bulla R-Bulla t

Prices/ Returns

Stylizing facts of daily returns series and HSMM

start





Data

Sector	N^a	Mean $\times 10^4$	S.D. $\times 10^2$	Skew.	Kurt.	JB[b]
Automobiles	4824	0.534	1.48	−0.363	8.99	7315
Banks	4824	2.28	1.2	−0.29	9.98	9863
Basics	4824	2.09	1.25	−0.396	9.82	9474
Chemicals	4824	2.54	1.29	−0.125	8.6	6314
Construction	4824	2.88	1.09	−0.56	9.39	8478
Financials	4824	1.99	1.16	−0.508	11	13 217
Food	4824	2.84	1.1	−0.247	17.2	40 877
Healthcare	4824	3.54	1.31	−0.388	9.3	8096
Industrials	4824	2.74	1.16	−0.416	9.42	8423
Insurance	4824	1.25	1.45	−0.265	10.4	11 041
Media	4824	2.42	1.49	−0.443	10.3	10 990
Oil and gas	4824	3.89	1.24	−0.251	7.48	4098
Personal	4824	3.32	1.05	−0.162	7.78	4627
Retail	4824	1.68	1.87	−0.258	8.26	5614
Technology	4824	3.33	1.63	−0.134	7.27	3683
Telecom	3521	2.25	1.48	−0.363	8.38	4339
Travel	3521	2.38	1.48	−0.186	5.47	916
Utilities	3521	2.07	1.19	−0.428	10.2	7740

Standard deviation of the data and the fitted models

The empirical standard deviation of the 18 DJ STOOXX sectors in comparison to the standard deviation of the fitted HMM and the standard deviation of the two fitted HSMMs. All results are multiplied by 100.

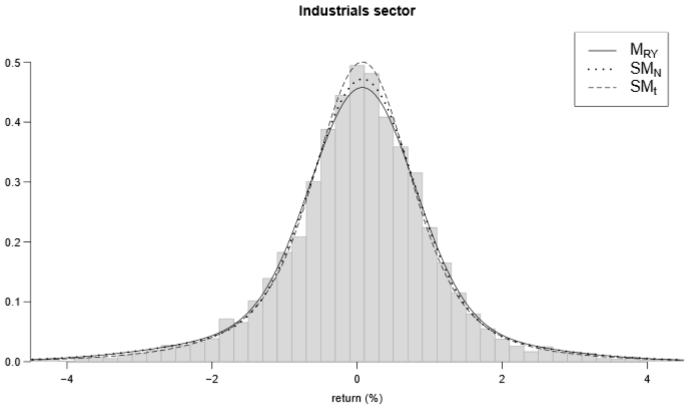
Sector	*Obs.*	M_{RY}	SM_N	SM_t
Automobiles	1.48	1.47	1.47	1.46
Banks	1.20	1.19	1.19	1.18
Basics	1.25	1.24	1.24	1.24
Chemicals	1.29	1.29	1.28	1.27
Construction	1.09	1.09	1.09	1.08
Financials	1.16	1.16	1.15	1.15
Food	1.10	1.10	1.09	1.08
Healthcare	1.31	1.31	1.31	1.30
Industrials	1.16	1.16	1.16	1.15
Insurance	1.45	1.45	1.44	1.44
Media	1.49	1.49	1.48	1.48
Oil & Gas	1.24	1.24	1.24	1.23
Personal	1.05	1.05	1.04	1.04
Retail	1.87	1.87	1.86	1.85
Technology	1.63	1.63	1.63	1.60

Kurtosis of the data and the fitted models

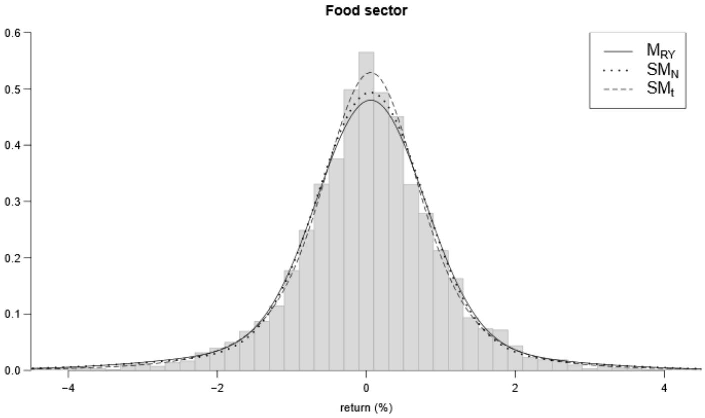
The empirical excess kurtosis of the 18 DJ STOOXX sectors in comparison to the excess kurtosis of the fitted HMM and of the two fitted HSMMs.

Sector	*Obs.*	M_{RY}	SM_N	SM_t
Automobiles	5.99	3.15	3.70	6.53
Banks	6.98	4.10	4.66	8.42
Basics	6.82	3.00	3.39	8.78
Chemicals	5.60	2.50	2.95	5.70
Construction	6.39	2.60	3.19	7.25
Financials	8.04	4.16	5.00	9.87
Food	14.25	3.48	4.26	17.56
Healthcare	6.30	2.26	2.58	5.17
Industrials	6.42	2.70	2.96	7.07
Insurance	7.39	4.31	5.29	9.02
Media	7.34	3.79	4.11	9.30
Oil & Gas	4.48	2.13	2.49	4.41
Personal	4.78	2.86	3.44	4.96
Retail	5.26	3.07	3.40	5.51
Technology	4.27	1.99	2.32	3.67
Telecom	5.38	2.66	3.17	5.26
Travel	2.47	1.80	2.04	2.53

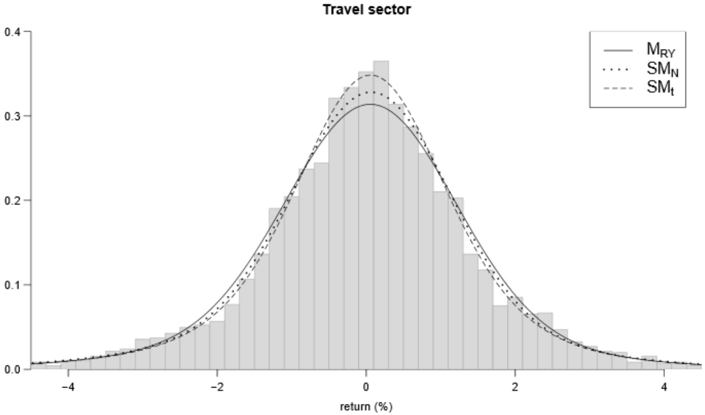
Empirical Results



Empirical Results



Empirical Results



Stylized facts

Sector	Mean/S.D.			
	Data	M_{RY}	SM_N	SM_t
Automobiles	0.965	1.04	1.01	0.965
Banks	0.896	0.962	0.931	0.892
Basics	0.955	1.03	1	0.953
Chemicals	0.97	1.05	1.02	0.973
Construction	1	1.08	1.05	1.01
Financials	0.886	0.956	0.926	0.883
Food	0.918	1.02	0.986	0.938
Healthcare	0.992	1.06	1.04	0.997
Industrials	0.966	1.03	1.01	0.962
Insurance	0.881	0.953	0.918	0.88
Media	0.908	0.973	0.956	0.903
Oil and gas	1.01	1.08	1.06	1.01
Personal	1.02	1.09	1.06	1.02
Retail	0.893	0.954	0.931	0.889
Technology	0.996	1.08	1.04	0.996
Telecom	0.974	1.05	1.02	0.977
Travel	1.08	1.14	1.11	1.08
Utilities	0.994	1.06	1.03	1

Stylized facts

Skewness			
Data	M_{RY}	SM_N	SM_t
2.81	2.19	2.33	2.83
2.97	2.39	2.51	3.14
2.96	2.12	2.22	3.1
2.67	1.97	2.1	2.66
2.9	2.03	2.2	2.93
3.14	2.4	2.6	3.36
4.08	2.27	2.47	3.85
2.8	1.89	1.98	2.56
2.88	2.02	2.08	2.85
3.03	2.44	2.67	3.23
3.06	2.32	2.39	3.24
2.44	1.87	1.97	2.42
2.55	2.11	2.29	2.59
2.52	2.05	2.13	2.56
2.31	1.79	1.89	2.22
2.62	2.03	2.17	2.62
1.93	1.78	1.84	1.98
2.93	2.01	2.19	2.58

Stylized facts

Sector	Original data			
	Data	M_{RY}	SM_N	SM_t
Automobiles	1.27	1.2	1.1	1.0
Banks	1.13	1.0	1.0	0.8
Basics	1.06	1.1	1.1	0.8
Chemicals	1.08	1.1	1.0	0.8
Construction	1.08	1.3	1.2	1.0
Financials	1.05	1.0	1.0	0.8
Food	1.12	1.1	1.1	0.8
Healthcare	0.946	1.0	1.0	0.8
Industrials	0.89	1.0	1.0	0.8
Insurance	1	1.0	1.0	0.8
Media	1.15	1.0	1.0	0.8
Oil and gas	1.58	1.1	1.1	0.9
Personal	1.26	1.4	1.3	1.2
Retail	0.744	0.8	0.8	0.7
Technology	0.95	1.0	0.9	0.8
Telecom	1.22	1.1	1.1	1.0
Travel	1.58	1.4	1.2	1.2
Utilities	1.22	1.1	1.1	0.9

The wMSE reweights the error at lag i by 0.95^{100-i} .

Criterion	Original data		
	M_{RY}	SM_N	SM_t
$\overline{MSE} \times 10^3$	12	8.87	17.5
$\overline{wMSE} \times 10^3$	2.34	1.6	2.19

Stylized facts

Parameter estimation of HMM

Sector	$1-p_1$	$1-p_2$	$\mu_1 \times 10^3$	$\mu_2 \times 10^3$	$\sigma_1^2 \times 10^4$	$\sigma_2^2 \times 10^4$
Automobiles	0.968	0.991	−1.78	0.585	6.29	0.978
Banks	0.957	0.987	−1.2	0.667	4.43	0.502
Basics	0.948	0.982	−1.49	0.803	4.16	0.635
Chemicals	0.953	0.98	−0.925	0.761	3.95	0.657
Construction	0.962	0.989	−1.42	0.766	3.26	0.601
Financials	0.957	0.987	−1.23	0.64	4.17	0.46
Food	0.946	0.985	−0.87	0.608	3.65	0.514
Healthcare	0.963	0.981	−0.622	0.842	3.83	0.663
Industrials	0.975	0.989	−0.845	0.757	3.29	0.506
Insurance	0.964	0.989	−1.14	0.496	6.79	0.739
Media	0.968	0.989	−1.27	0.74	6.52	0.778
Oil and gas	0.983	0.992	−0.195	0.651	3.48	0.666
Personal	0.956	0.991	−1.3	0.681	3.37	0.599
Retail	0.978	0.987	−1.17	0.923	8.17	0.822
Technology	0.981	0.989	−0.546	0.843	5.51	1.01
Telecom	0.94	0.977	−0.493	0.495	5.56	0.922
Travel	0.986	0.996	−0.962	0.598	5.27	1.25
Utilities	0.975	0.991	−0.743	0.547	3.59	0.627

Stylized facts

Parameter estimation of HSMM-Normal

Sector	$1-p_1$	$1-p_2$	$r_1 \times 10$	$r_2 \times 10$	$\mu_1 \times 10^3$	$\mu_2 \times 10^3$	$\sigma_1^2 \times 10^4$	$\sigma_2^2 \times 10^4$
Automobiles	0.983	0.995	0.518	0.746	−1.87	0.576	6.76	0.924
Banks	0.969	0.993	0.847	0.775	−1.1	0.625	4.66	0.459
Basics	0.969	0.991	0.922	0.966	−1.57	0.812	4.37	0.592
Chemicals	0.979	0.99	0.544	0.821	−1.11	0.804	4.21	0.611
Construction	0.965	0.994	0.995	0.824	−1.55	0.748	3.61	0.575
Financials	0.971	0.994	0.89	0.848	−1.41	0.631	4.65	0.448
Food	0.919	0.995	1.75	0.629	−0.95	0.581	4.11	0.5
Healthcare	0.981	0.991	0.703	0.844	−0.801	0.897	4.04	0.631
Industrials	0.988	0.993	0.728	1.08	−0.94	0.793	3.38	0.475
Insurance	0.977	0.995	0.558	0.622	−1.23	0.469	7.63	0.707
Media	0.992	0.993	0.471	1.43	−1.33	0.749	6.71	0.742
Oil and gas	0.986	0.996	0.644	0.476	−0.367	0.705	3.7	0.633
Personal	0.976	0.993	0.615	1.18	−1.62	0.711	3.71	0.575
Retail	0.996	0.99	0.28	1.23	−1.32	0.96	8.53	0.777
Technology	0.99	0.987	0.495	1.18	−0.503	0.825	5.7	0.873
Telecom	0.991	0.975	0.231	2.31	−0.568	0.502	5.99	0.859
Travel	0.996	0.988	0.183	1.55	−0.996	0.71	5.07	1.07
Utilities	0.98	0.996	0.488	0.429	−0.793	0.522	3.97	0.6

Stylized facts

Parameter estimation of HSMM-tl

Sector	$1-p_1$	$1-p_2$	$r_1 \times 10$	$r_2 \times 10$	$\mu_1 \times 10^3$	$\mu_2 \times 10^3$	$\sigma_1^2 \times 10^4$	$\sigma_2^2 \times 10^4$	v_1	v_2
Automobiles	0.985	0.997	0.588	0.511	−1.56	0.622	4.52	0.769	7.55	10.9
Banks	0.977	0.996	0.791	0.464	−0.685	0.628	3.04	0.382	7.05	10.3
Basics	0.983	0.996	0.962	0.55	−1.04	0.816	2.28	0.482	5.53	9.94
Chemicals	0.987	0.995	0.516	0.401	−0.585	0.739	2.5	0.485	6.71	8.86
Construction	0.982	0.996	1.15	0.935	−1.03	0.805	1.97	0.49	5.9	13.5
Financials	0.979	0.996	0.902	0.678	−1.01	0.671	2.81	0.366	6.33	11
Food	0.994	0.993	1.4	3.28	−0.263	0.658	1.35	0.37	4.37	10.4
Healthcare	0.997	0.991	0.238	1.05	−0.32	0.932	2.3	0.516	6.61	11.9
Industrials	0.994	0.997	0.722	0.823	−0.441	0.787	2	0.401	6.21	12.1
Insurance	0.979	0.997	0.659	0.399	−0.892	0.516	4.9	0.575	7.2	11.1
Media	0.993	0.995	0.499	1.2	−1.08	0.767	4.38	0.642	6.32	12.3
Oil and gas	0.994	0.998	0.397	0.289	−0.0489	0.64	2.52	0.526	8.15	10.6
Personal	0.977	0.996	0.868	0.863	−1.46	0.71	2.62	0.495	9.26	12.8
Retail	0.996	0.995	0.328	0.815	−1.08	0.983	6.55	0.631	9.35	7.99
Technology	0.99	0.997	0.616	0.294	−0.538	0.834	4.36	0.709	10.5	6.83
Telecom	0.99	0.992	0.34	0.981	−0.0699	0.465	4.35	0.742	8.29	7.9
Travel	0.989	0.999	0.419	0.25	−1.04	0.619	5.31	0.987	47	10.2
Utilities	0.984	0.997	0.562	0.285	−0.288	0.5	2.43	0.503	7.98	20.7

Stylized facts