## 6.1 THE LIMIT limf(x) – CONTINUITY OF f(x): A ROUGH IDEA[1]

♦ THE LIMIT limf(x)

Consider the function

$$f(x) = 3x+5$$

It is clear that when x=2, then f(2) = 11.

But what is the behavior of f(x) when x approaches 2?

| x | f(x) |
|---|---|
| 1.8 | 10.4 |
| 1.9 | 10.7 |
| 1.99 | 10.97 |
| 1.999 | 10.997 |
| 2.001 | 11.003 |

That is,

if $x \to 2$ [x tends to 2]

then $f(x) \to 11$ [f(x) tends to 11]

In order to express this fact we write

$$\boxed{\lim_{x\to 2} f(x) = 11}$$

and say that <u>the limit of f(x), as x tends to 2, is 11.</u>

In this example

$$\lim_{x\to 2} f(x) = 11 = f(2)$$

$$\lim_{x\to 3} f(x) = 14 = f(3) \qquad \text{and so on!}$$

The fact that $\lim_{x\to a} f(x) = f(a)$ occurs very often, however, this is not always the case (otherwise the limit would be nothing more than a simple substitution!!).

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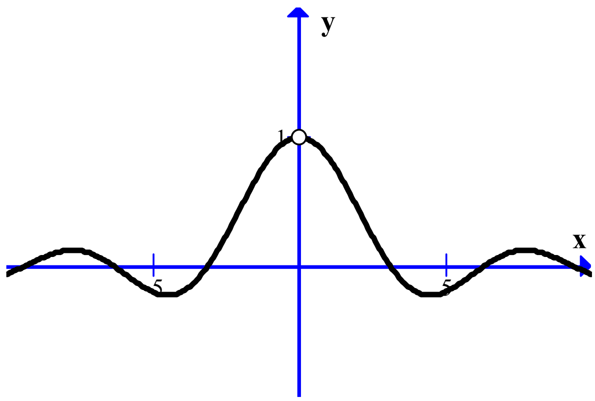
[1] Paragraphs 6.1 and 6.2 may look very "technical". Do not pay much attention on your first reading! You may skip them and proceed to paragraph 6.3; you will realize that the derivatives, in practice, are much easier than they appear here! SL students may have a rough idea of the limit only!

Let' s see a case where the limit is not a simple substitution!

We will find informally (by using our calculator) the limit

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

Notice that the function is not defined at x=0. The graph looks like




Let's approach 0 from positive values of x:

| x | f(x) |
|---|---|
| 0.1 | 0.998334 |
| 0.01 | 0.999983 |
| 0.001 | 0.999999 |

It seems that the limit is 1. We express this fact by

$$\boxed{\lim_{x\to 0^+}\frac{\sin x}{x}=1}$$

Let us approach 0 from negative values of x:

| x | f(x) |
|---|---|
| – 0.1 | 0.998334 |
| – 0.01 | 0.999983 |
| – 0.001 | 0.999999 |

It seems again that that the limit is 1. We express this fact by

$$\boxed{\lim_{x\to 0^-}\frac{\sin x}{x}=1}$$

The two limits $\lim\limits_{x\to 0^+}$ and $\lim\limits_{x\to 0^-}$ are called **side limits**.

Since the two side limits are equal we say that the limit when x tends to 0 is 1 and we write

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}$$

When we studied asymptotes, we saw that the value of f(x) may sometimes tend to +∞ or −∞.

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**EXAMPLE 1**

Find informally (by using your calculator) the limit $\lim_{x\to 0}\frac{1}{x}$.

**Solution**

Again, we have two cases:

- If x is +tive

| x | f(x) | |
|---|---|---|
| 0.1 | 10 | |
| 0.001 | 1000 | |
| 0.000001 | 1000000, | so $f(x)=\frac{1}{x}\to+\infty$ |

- If x is -tive

| x | f(x) | |
|---|---|---|
| -0.1 | -10 | |
| -0.001 | -1000 | |
| -0.000001 | -1000000, | so $f(x)=\frac{1}{x}\to-\infty$ |

Here we only have **side-limits:**

$$\boxed{\lim_{x\to 0^+}\frac{1}{x}=+\infty} \quad \text{and} \quad \boxed{\lim_{x\to 0^-}\frac{1}{x}=-\infty}$$

[$x\to 0^+$ means that x tends to 0 for values above 0,
$x\to 0^-$ means that x tends to 0 for values below 0]

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**Remember:** If $\lim_{x\to a^+}f(x)=\pm\infty$ or $\lim_{x\to a^-}f(x)=\pm\infty$ we say that x=a is a vertical asymptote. So in our example x=0 is a vertical asymptote.

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We also define limits of the form $\lim_{x\to+\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$. Thus, we study the behavior of the function when x approaches $+\infty$ or $-\infty$.

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**EXAMPLE 2**

Find informally (by using your calculator) the limits $\lim_{x\to+\infty}\frac{1}{x}$, $\lim_{x\to-\infty}\frac{1}{x}$.

**Solution**

| x | f(x) |
|---|---|
| 1000 | 0.001 |
| 1000000 | 0.000001 |

Also,

| x | f(x) |
|---|---|
| -1000 | - 0.001 |
| -1000000 | - 0.000001 |

Thus, both limits are 0, namely $\lim_{x\to+\infty}\frac{1}{x}=0$ and $\lim_{x\to-\infty}\frac{1}{x}=0$.

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**Remember:** If $\lim_{x\to+\infty} f(x)=a$ or $\lim_{x\to-\infty} f(x)=a$ then y=a is a horizontal asymptote. So in our example y=0 is a horizontal asymptote.

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**EXAMPLE 3**

Investigate informally (by using your calculator) that $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=e$.

**Solution**

| x | f(x) |
|---|---|
| 1000 | $1.001^{1000}$ = 2.7169239... |
| 1000000 | 2.7182804... |
| $10^{10}$ | 2.7182818... |

The resulting limit is in fact the number **e=2.7182818...** known from logarithms! That is,

$$\boxed{\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e\,.}$$

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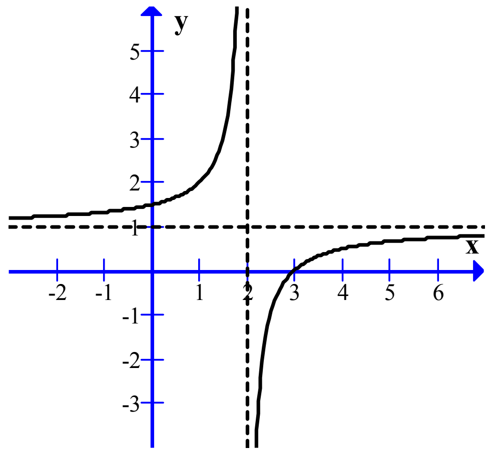
For rational functions we have already seen how we recognize the asymptotes. For example, for

$$f(x)=\frac{x-3}{x-2}$$




we know that

- ***x=2 is a vertical asymptote.***

The formal justification is that $\lim_{x\to 2^-} f(x)=+\infty$ and $\lim_{x\to 2^+} f(x)=-\infty$

- ***y=1 is a horizontal asymptote.***

The formal justification is that $\lim_{x\to +\infty} f(x)=1$ and $\lim_{x\to -\infty} f(x)=1$

We know that a vertical asymptote occurs at some point where the function is not defined.

But let us see a strange situation.

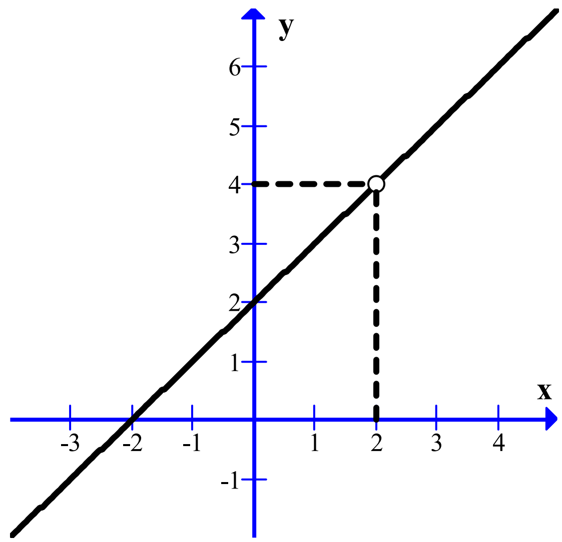
$$f(x)=\frac{x^2-4}{x-2}$$

The function is not defined at x=2. However x=2 is not a vertical asymptote. It is interesting to see what happens to f(x) as x →2.

| x | f(x) |
|---|---|
| 1.9 | 3.9 |
| 1.99 | 3.99 |
| 1.999 | 3.999 |
| 2.001 | 4.001 |
| 2.002 | 4.002 |

It seems that $\lim_{x\to 2} f(x)=4$.

Indeed, look at the graph of f (there is an "empty" point on it)




When x approaches 2, the value f(x) approaches 4. Thus

$$f(2) \text{ is not defined} \quad \text{but} \quad \lim_{x\to 2} f(x) = 4.$$

In fact, we can simplify the function as

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2, \quad \text{where } x \neq 2$$

That is why we obtain the graph of the straight line y=x+2 with some "discontinuity" at x=2. Moreover,

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4$$

(after the simplification we are allowed to set x=2).

This is a nice opportunity to introduce the notion of continuity.

- CONTINUITY (**only for HL**)

We say that a function is **continuous** at x=a, when

- The value f(a) exists;
- The limit $\lim_{x\to a} f(x)$ exists;
- They are equal, i.e. $\lim_{x\to a} f(x) = f(a)$

For example, the function f(x) = 3x+5 is continuous at x=2, since

$$\lim_{x\to 2} f(x) = f(2) = 11$$

**Notice**. In fact, the function f(x) = 3x+5 is continuous everywhere.

Most of the known functions are continuous everywhere (since they look like uninterrupted curves!). For example, lines, quadratics, polynomials in general, exponentials are continuous functions.

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For the continuity at any particular point we must check all three presuppositions above.

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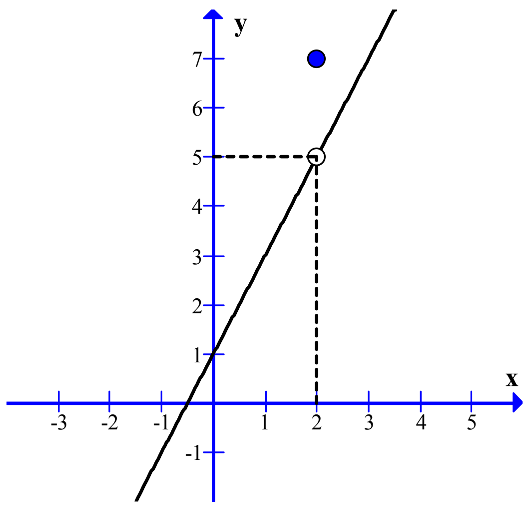
<u>**EXAMPLE 4**</u>

Let $f(x)=\begin{cases}2x+1, & \text{if } x\neq 2\\ 7, & \text{if } x=2\end{cases}$




- $\lim\limits_{x\to 2} f(x)=5$ (when x approaches 2,the value f(x) approaches 5);
- f(2) =7.
- But $\lim\limits_{x\to 2} f(x)\neq f(2)$

Thus the function is not continuous at x=2.

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Remember that at some x=a we may have different side limits

$$\lim_{x\to a^-} f(x) \text{ and } \lim_{x\to a^+} f(x)$$

If they are equal, say $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = b$, then we can say that

$$\lim_{x\to a} f(x) = b$$

It is worthwhile to see the following example of "step" functions to further clarify the notions of limit and continuity.

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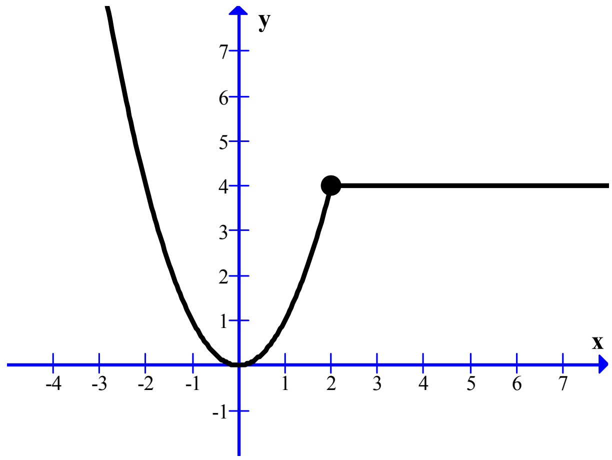
**EXAMPLE 5**

(a) Consider the function

$$f(x)=\begin{cases} x^2, & \text{if } x\le 2 \\ 4, & \text{if } x>2 \end{cases}$$



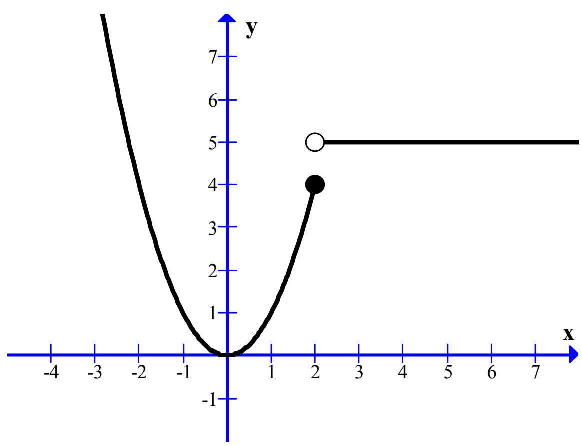

We can see that f(2)=4 and also $\lim\limits_{x\to 2} f(x)=4$.

Since $\lim\limits_{x\to 2} f(x)=f(2)$ the function is **continuous at x=2**.

(in fact the function is continuous everywhere).

(b) Consider the function

$$f(x)=\begin{cases} x^2 & \text{if } \quad x\le 2 \\ 5 & \text{if } \quad x>2 \end{cases}$$




We can see that f(2)=4 but $\lim\limits_{x\to 2} f(x)$ does not exist.

[In fact, only side limits exist: $\lim\limits_{x\to 2^-} f(x)=4$ and $\lim\limits_{x\to 2^+} f(x)=5$ ]

Therefore, the function is **not continuous at x=2**.

(thus the functions is not continuous in general).

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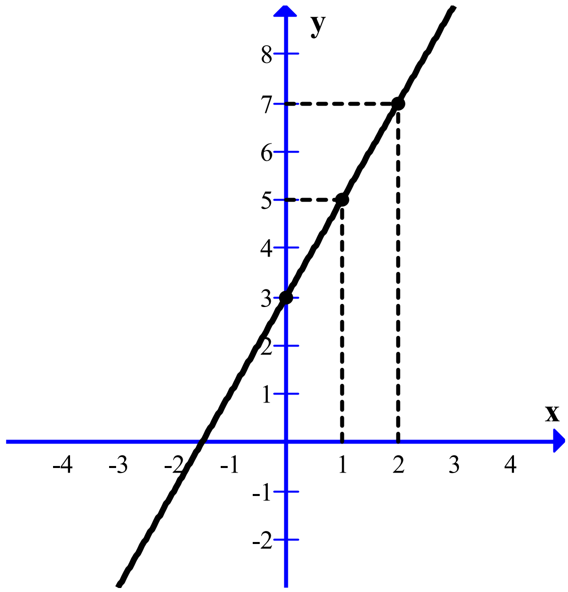
## 6.2 THE FORMAL DEFINITION OF THE DERIVATIVE

♦ RATE OF CHANGE (OR GRADIENT) IN A STRAIGHT LINE

Consider the line $f(x)=2x+3$.




Notice that

when $\quad$ x changes from 0 to 1

then $\quad$ y changes from 3 to 5

Hence, the corresponding rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{f(1)-f(0)}{1-0} = \frac{5-3}{1-0} = 2$$

Notice that

when $\quad$ x changes from 0 to 2

then $\quad$ y changes from 3 to 7

Hence, the corresponding rate of change is still

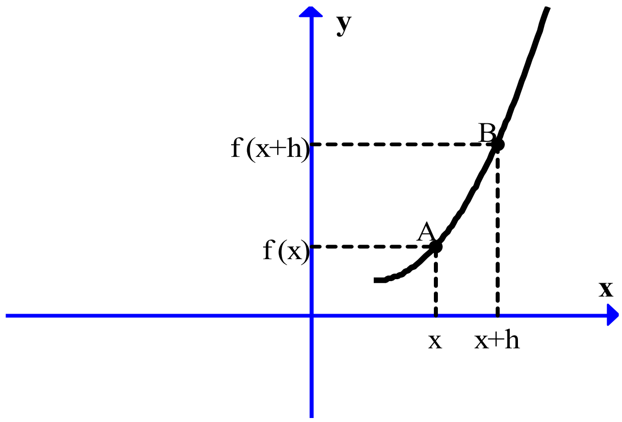
$$\frac{\Delta y}{\Delta x} = \frac{f(2)-f(0)}{2-0} = \frac{7-3}{2-0} = 2$$

We understand that the rate of change between any two points is always the same. This common value is the **gradient** of the line.

- RATE OF CHANGE IN A CURVE

In a curve which is not a straight line, the rate of change between any two points is not always the same.

However, we can measure the "instantaneous" rate of change at any particular point with coordinates A(x,f(x)) as follows:




We select a neighboring point B with

x-coordinate = x+h (where h is very small)

y-coordinate =f(x+h)

As we move from point A to point B, the rate of change is

$$\frac{\Delta y}{\Delta x}=\frac{f(x+h)-f(x)}{h}$$

If we let h become very small, that is h→0, the result will be the rate of change at point A, thus

.

$$\boxed{\text{Rate of change at point } x = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

Let us apply this formula to the function $f(x) = x^2$.

$$\frac{\Delta y}{\Delta x}=\frac{(x+h)^2-x^2}{h}=\frac{x^2+2xh+h^2-x^2}{h}=\frac{2xh+h^2}{h}=2x+h$$

Since $\lim_{h\to 0}(2x+h)=2x$

**Rate of change** of $y=x^2$ at point x is 2x

We also say that

**Gradient** of $y=x^2$ at point x is 2x

If we apply the definition for $f(x)=x^3$ we will find that

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=3x^2 \quad \text{(exercise!)}$$

That is,

The gradient (or rate of change) of $y=x^3$ at point $x$ is $3x^2$

For example, the gradient at point $x=2$ is 12

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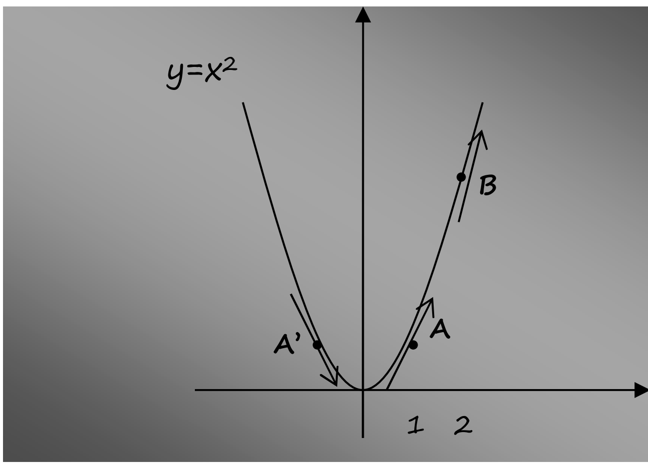
**<u>EXAMPLE 1</u>**




$f(x)=x^2$

gradient at point x: $m=2x$

Thus,

gradient at $x=1$ (A): $m=2$

gradient at $x=2$ (B): $m=4$

gradient at $x=3$ : $m=6$

gradient at $x=-1$ : $m=-2$

Therefore, the notion of the **gradient** (or **slope**) applies not only for straight lines, as we knew, but for any "smooth" curve in general. It shows in an analogous way the inclination of the curve at any particular point. For example, in the graph above

at $x=1$ the function increases with gradient $m=2$

at $x=-1$ the function decreases with gradient $m=-2$

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♦ THE DERIVATIVE $f'(x)$

It is exactly the same thing!!! The new function derived from $f(x)$, which is also in terms of $x$, is called **derivative**; It's denoted by $f'(x)$. Thus,

**DERIVATIVE = RATE OF CHANGE = GRADIENT**

The formal definition of the derivative is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

But don't worry! We will never use it in practice! An easier and more mechanical way to find the derivatives will be used!
However, just for practice, let's see one more example.

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**EXAMPLE 2**

Show from first principles (that is by using the formal definition), that the derivative of the function

$$f(x) = x^3 + 2x$$

is

$$f'(x) = 3x^2 + 2.$$

**Solution**

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{[(x+h)^3 + 2(x+h)] - [x^3 + 2x]}{h} \\ &= \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - x^3 - 2x}{h} \\ &= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3 + 2h}{h} \\ &= \lim_{h\to 0}\frac{h(3x^2 + 3xh + h^2 + 2)}{h} \\ &= \lim_{h\to 0}(3x^2 + 3xh + h^2 + 2) \end{aligned}$$

Now, we are able to set h=0 and obtain,

$$f'(x) = 3x^2 + 2$$

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## 6.3 DERIVATIVES OF KNOWN FUNCTIONS - RULES

The derivative of a function f(x) is a new function denoted by f′(x). As explained in the preceding section, f′(x) indicates the rate of change, or otherwise the gradient of f(x) at any particular point x.

We have seen for example that

for $f(x)=x^2$, we obtain $f'(x)=2x$.

We can also write directly:

$$(x^2)'=2x$$

We present the derivatives of the most common functions:

| f(x) | f ′(x) |
|---|---|
| $x^n$ | $nx^{n-1}$ |
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $e^x$ | $e^x$ |
| $\tan x$ | $\frac{1}{\cos^2 x}$ |
| $\ln x$ | $\frac{1}{x}$ |
| $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ |
| c (constant) | 0 |

Let us especially elaborate on the first formula $(x^n)' = nx^{n-1}$

| $f(x)= x^n$ | $f'(x) =nx^{n-1}$ |
|---|---|
| $x^{10}$ | $10x^9$ |
| $x^4$ | $4x^3$ |
| $x^3$ | $3x^2$ |
| $x^2$ | $2x$ |
| $x$ | 1 |
| 1 | 0 |

This formula also applies for –tive values of n:

| $f(x)=x^n$ | $f'(x)=nx^{n-1}$ |
|---|---|
| $x^{-10}$ | $-10x^{-11}$ |
| $x^{-3}$ | $-3x^{-4}$ |
| $x^{-2}$ | $-2x^{-3}$ |
| $x^{-1}$ | $1x^{-2}$ |

It also applies for rational values of n:

| $f(x)=x^n$ | $f'(x)=nx^{n-1}$ |
|---|---|
| $x^{6.4}$ | $6.4x^{5.4}$ |
| $x^{3/2}$ | $\frac{3}{2}x^{1/2}$ |
| $x^{5/3}$ | $\frac{5}{3}x^{2/3}$ |
| $x^{1/2}$ | $\frac{1}{2}x^{-1/2}$ |

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**EXAMPLE 1**

Show that (a) $\left(\frac{1}{x^2}\right)' = \frac{-2}{x^3}$ (b) $\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$

**Solution**

(a) $\frac{1}{x^2} = x^{-2}$, so the derivative is $-2x^{-3} = \frac{-2}{x^3}$

(b) $\sqrt{x} = x^{1/2}$, so the derivative is $\frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

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**EXAMPLE 2**

Let $f(x)=x^7$. Find

(a) $f(0)$, $f(1)$, $f(2)$

(b) $f'(x)$

(c) $f'(0)$, $f'(1)$, $f'(2)$

(d) the rate of change of $f(x)$ at $x=2$

(e) the gradient of $f(x)$ at $x=2$

**Solution**

(a) $f(0)=0$, $f(1)=1$, $f(2)=128$

(b) $f'(x) = 7x^6$

(c) $f'(0)=0$, $f'(1)=7\cdot 1^6=7$, $f'(2)=7\cdot 2^6=448$

(d) it is $f'(2) = 448$

(e) it is $f'(2) = 448$

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- SYMBOLS

If $y=f(x)$, the derivative is denoted by the following symbols

$$y' \quad \text{or} \quad f'(x) \quad \text{or} \quad \frac{dy}{dx} \quad \text{or} \quad \frac{d}{dx}f(x)$$

The derivative at some specific value of x, say x=2, is denoted by

$$f'(2) \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=2}$$

For example, if $y=f(x)=x^3$, we can write

$$y'=3x^2 \quad \text{or} \quad (x^3)' = 3x^2 \quad \text{or} \quad \frac{dy}{dx}=3x^2 \quad \text{or} \quad \frac{d}{dx}x^3=3x^2$$

Moreover,

$$f'(2)=12 \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=2}=12$$

The procedure of finding the derivative is called **differentiation**.

- RULES OF DIFFERENTIATION

**Rule (1):** $(f+g)' = f' + g'$ $(f-g)' = f' - g'$

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**EXAMPLE 3**

For $f(x) = x^5+x^3$ , $f'(x) = 5x^4+3x^2$

For $g(x) = x^5-x^3$, $g'(x) = 5x^4-3x^2$

For $h(x)= x^7+e^x-\ln x+\sin x-x+5$, $h'(x) = 7x^6+e^x-\frac{1}{x}+\cos x-1$

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**Rule (2):** $\boxed{(af)' = af'}$ (a=constant number)

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**EXAMPLE 4**

For $f(x) = 3\sin x$, $f'(x) = 3\cos x$

For $g(x) = 7e^x$, $g'(x) = 7e^x$

For $h(x) = 5x^3$, $h'(x) = 5(3x^2) = 15x^2$

For $k(x) = 10\ln x$, $k'(x) = 10\dfrac{1}{x} = \dfrac{10}{x}$

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Let us combine Rules (1) and (2):

$$\boxed{(af + bg)' = af' + bg'}$$

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**EXAMPLE 5**

For $f(x) = 2x^3 - 3x^2 + 7x + 5$, $f'(x) = 6x^2 - 6x + 7$

For $g(x) = 5x^7 + 3\ln x - 7\cos x$, $g'(x) = 35x^6 + \dfrac{3}{x} + 7\sin x$

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**NOTICE:**

The differentiation rules above may also be expressed as follows

$$\boxed{\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)}$$

$$\boxed{\frac{d}{dx}[af(x)] = a\frac{d}{dx}f(x)}$$

$$\boxed{\frac{d}{dx}[af(x) + bg(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)}$$

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**EXAMPLE 6**

For $f(x) = x^3$ and $g(x) = \sin x$, $\dfrac{d}{dx}[4f(x) + 5g(x)] = 12x^2 + 5\cos x$

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**Rule (3):** $$\boxed{(f\cdot g)' = f'\cdot g + f\cdot g'}$$ **(product rule)**

Be careful !!!

If $f(x) = x^5\sin x$ then $f'(x)$ **is not** **$(5x^4)(\cos x)$**

We must follow the product rule above.

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**EXAMPLE 7**

For $f(x) = x^5\sin x$, $f'(x)=(x^5)'\sin x + x^5(\sin x)' = 5x^4\sin x + x^5\cos x$

For $g(x) = x\ln x$, $g'(x)= (x)'\ln x + x(\ln x)' = 1\ln x + x\dfrac{1}{x} = \ln x + 1$

For, $h(x) = x^2e^x$, $h'(x) = 2xe^x+x^2e^x$

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**Rule (4):** $$\boxed{\left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2}}$$ **(quotient rule)**

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**EXAMPLE 8**

For $f(x) = \dfrac{x^3}{\sin x}$, $f'(x)=\dfrac{(x^3)'\sin x - x^3(\sin x)'}{(\sin x)^2} = \dfrac{3x^2\sin x - x^3\cos x}{\sin^2 x}$

For $g(x) = \dfrac{x^3-5x}{x^2+1}$, $g'(x) = \dfrac{(3x^2-5)(x^2+1)-(x^3-5x)2x}{(x^2+1)^2}$

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Sometimes, we can avoid the quotient rule. Look at the following

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**EXAMPLE 9**

For $f(x) = \dfrac{x^3-2x+1}{x}$

method A: The quotient rule gives

$$f'(x) = \frac{(3x^2-2)x-(x^3-2x+1)1}{x^2} = \frac{2x^3-1}{x^2} = 2x - \frac{1}{x^2}$$

<u>method B</u>: we may modify f(x) by splitting into three fractions

$$f(x) = \frac{x^3}{x} - \frac{2x}{x} + \frac{1}{x} = x^2 - 2 + x^{-1} \text{ , so that}$$

$$f'(x) = 2x - x^{-2} = 2x - \frac{1}{x^2}$$

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♦ HIGHER DERIVATIVES

The second derivative $f''(x)$ is in fact the derivative of $f'(x)$.
The third derivative $f'''(x)$ is the derivative of $f''(x)$. And so on!

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**<u>EXAMPLE 10</u>**

- For $f(x) = x^5$, $f'(x) = 5x^4$ $f''(x) = 20x^3$ $f'''(x) = 60x^2$
- For $g(x) = \sin x$ $g'(x) = \cos x$ $g''(x) = -\sin x$ $g'''(x) = -\cos x$
- For $h(x) = e^x$, $h'(x) = e^x$ $h''(x) = e^x$ $h'''(x) = e^x$

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Alternative notation:

$f''(x)$ can also be written as $\frac{d^2y}{dx^2}$ or $\frac{d^2}{dx^2}f(x)$

$f'''(x)$ can also be written as $\frac{d^3y}{dx^3}$ or $\frac{d^3}{dx^3}f(x)$

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**<u>EXAMPLE 11</u>**

Let $f(x)=\frac{1}{3}x^3-\frac{3}{2}x^2+2x+7$. Solve the equations

(a) $f'(x) = 0$ (b) $f''(x) = 0$

**<u>Solution</u>**

(a) $f'(x) = x^2-3x+2$, hence $f'(x) = 0 \Leftrightarrow x^2-3x+2=0 \Leftrightarrow \boxed{x=1 \text{ or } x=2}$

(b) $f''(x) = 2x-3$, hence $f''(x) = 0 \Leftrightarrow 2x-3=0 \Leftrightarrow \boxed{x=3/2}$

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