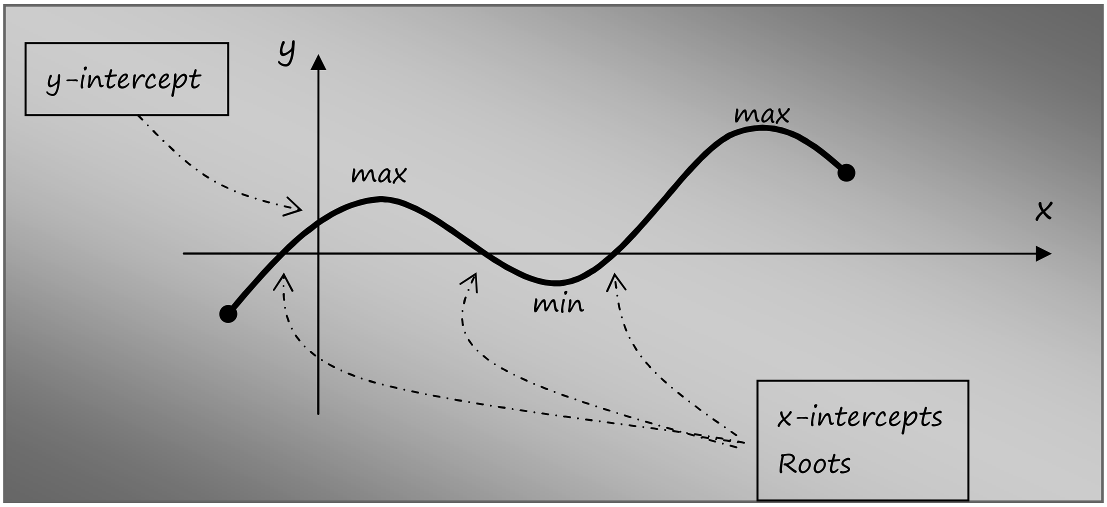
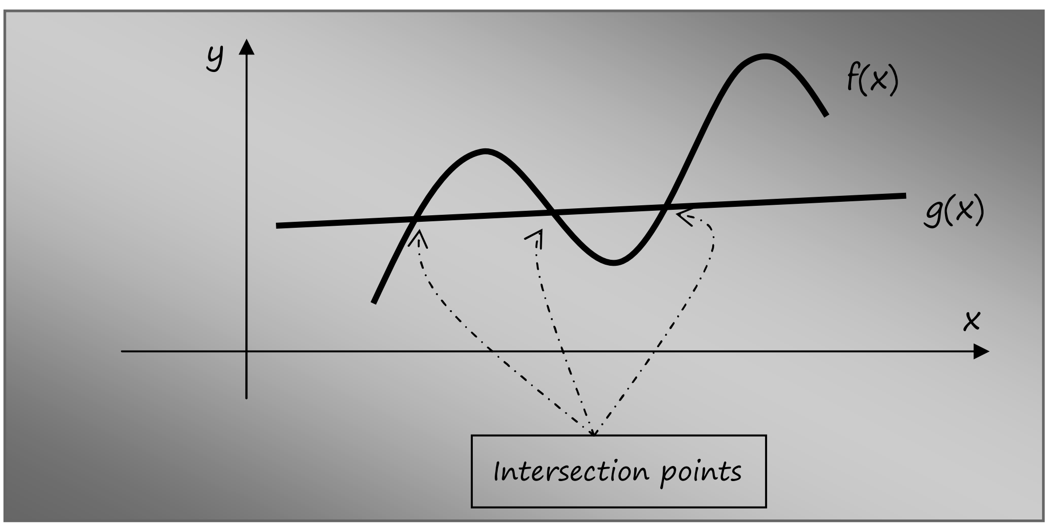
♦ SPECIFIC POINTS ON A GRAPH



For y=f(x)

- ***y-intercept***: Set x=0 and find y
- **x-intercepts (roots)**: Find the roots of the equation f(x)=0
- **max-min:** In Topic 6, we will use <u>derivatives</u> to find them

When we have two graphs y=f(x) and y=g(x), it also useful to know the **intersection points** of the two graphs



These points can be found by solving the equation

f(x)=g(x)

Thus, we obtain the x-coordinates of the intersection points.

If necessary, in order to find the corresponding y-coordinates as well, we may use either y=f(x) or y=g(x).

EXAMPLE 8

a) Consider $f(x)=(x-3)^2-4$

y-intercept: for $x=0$, we obtain $\boxed{y=5}$

x-intercepts or roots: We solve $(x-3)^2-4=0$

$(x-3)^2-4=0 \Leftrightarrow (x-3)^2=4 \Leftrightarrow x-3=\pm 2 \Leftrightarrow x=2+3$ or $x=-2+3$

Hence $\boxed{x=5 \text{ or } x=1}$

max-min: We don't know derivatives yet, but for this particular function (quadratic), we know that there is only a minimum.

We have a min at the vertex, i.e. at $\boxed{\text{point } (3,-4)}$

We say: We have a <u>min at x=3.</u> The <u>min value is y=-4</u>

b) Consider also the linear function $g(x)=x-5$.

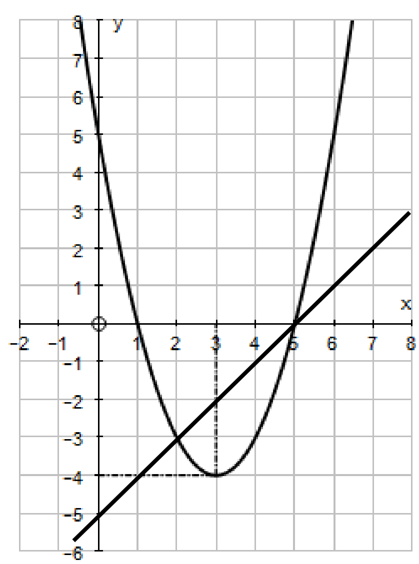
intersection points: we solve the equation $f(x)=g(x)$

$$\begin{aligned}(x-3)^2-4=x-5 \quad &\Leftrightarrow x^2-6x+9-4=x-5\\ &\Leftrightarrow x^2-7x+10=0\\ &\Leftrightarrow x=2 \text{ or } x=5\end{aligned}$$

By using either $f(x)$ or $g(x)$ we find $y=-3$, $y=0$ respectively.

Hence, the curves intersect at the $\boxed{\text{points } (2,-3) \text{ and } (5,0)}$

Indeed, the graphs of $f(x)$ and $g(x)$ are as follows



♦ USING GRAPHS IN THE GDC

- All notions above, namely **y-intercept**, **x-intercepts** (or **roots**), **max**, **min**, **intersection points** can be easily found in the GDC.

- We can easily solve any **equation** $f(x)=g(x)$

  <u>METHOD A:</u> we find the <u>intersection points</u> of the graphs

  $y_1 = f(x)$

  $y_2 = g(x)$

  <u>METHOD B:</u> we find the <u>roots</u> of the graph

  $y_1 = f(x)-g(x)$

  [since the equation may be written as $f(x)-g(x)=0$]

- We can easily solve any **inequality** $f(x)>g(x)$ or $f(x)\geq g(x)$

  <u>METHOD A:</u> we find the intervals where $y_1=f(x)$ is above $y_2=g(x)$

  <u>METHOD B:</u> we find the intervals where $y_1=f(x)-g(x)$ is positive,

  [since the inequality may be written as $f(x)-g(x)>0$ (or ≥ 0)]

<u>EXAMPLE 9</u>

Consider again the functions of Example 8

$$f(x)=(x-3)^2-4 \quad \text{and} \quad g(x)=x-5.$$

a) Solve the equation $f(x)=g(x)$.

<u>METHOD A:</u> Look at the graphs of $y_1=f(x)$ and $y_2=g(x)$ (see Example 8). The intersection points occur at $\boxed{x=2, x=5}$

<u>METHOD B:</u> The equation can be written

$$f(x)-g(x) = (x-3)^2 - 4 - (x-5) = 0$$

Look at the graph of $y_1=f(x)-g(x)$ (see GDC). Roots: $\boxed{x=2, x=5}$

b) Solve the inequality $f(x)>g(x)$.

<u>METHOD A:</u> the graph of $y_1=f(x)$ is above $y_2=g(x)$ (see Example 8) when $\boxed{x<2 \text{ or } x>5}$

<u>METHOD B:</u> the graph of $y_1=f(x)-g(x)$ (see GDC) is positive outside the roots, that is when $\boxed{x<2 \text{ or } x>5}$

2.4 COMPOSITION OF FUNCTIONS: $f \circ g$

- DISCUSSION

Consider the function $f(x)=x^2$

Notice that

$$f(5) = 5^2$$
$$f(a) = a^2$$
$$f(3a+5) = (3a+5)^2$$
$$f(x+1) = (x+1)^2$$
$$f(3x+5) = (3x+5)^2$$

In the last two cases the input value for f is another function of x.

Let us concentrate on

$$f(3x+5) = (3x+5)^2.$$

If we set $g(x)= 3x+5$, our result can be written as follows

$$f(g(x)) = (3x+5)^2$$

In this way, we combine two functions,

$$f(x)=x^2 \qquad \text{and} \qquad g(x)= 3x+5$$

and create a new function $(3x+5)^2$.

This new function is denoted by $\mathbf{f \circ g}$. We write $(f \circ g)(x) = (3x+5)^2$.

- DEFINITION

For two functions f and g, the **composite function $f \circ g$** is a new function defined by

$$(f \circ g)(x)=f(g(x))$$

The action is called **composition**.
We say that $f \circ g$ is the **composite function** of f and g.

Therefore, for the functions $f(x)=x^2$ and $g(x)=3x+5$ given above, the procedure we follow in order to estimate $(f\circ g)(x)$ is

$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(3x+5)\\ &= (3x+5)^2\end{aligned}$$

In the same way we can define the composite function $(g\circ f)(x)$. It is given by

$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g(x^2)\\ &= 3x^2+5\end{aligned}$$

That is

$$(f\circ g)(x)= (3x+5)^2 \qquad \text{while} \qquad (g\circ f)(x)= 3x^2+5$$

<u>**NOTICE**</u>:

- In general

$$f\circ g \neq g\circ f$$

- It is not necessary to write so analytically the answer. You can answer directly. Look at again

$$f(x)=x^2 \qquad \text{and} \qquad g(x)=3x+5$$

  For $f\circ g$ you just plug g into f.

$$(f\circ g)(x)= (3x+5)^2$$

  For $g\circ f$ you just plug f into g.

$$(g\circ f)(x)= 3x^2+5$$

- For three functions $f(x)=x^2$, $g(x)=3x+5$, $h(x)=\sqrt{x}$ we can define

$$(f\circ g\circ h)(x)= (3\sqrt{x}+5)^2$$

  You just plug h into g and the result into f. Notice also that

$$f\circ(g\circ h)=(f\circ g)\circ h$$

EXAMPLE 1

Let $f(x)=2x^2-1$ and $g(x)=x+1$. Find

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$ (c) $(f \circ g)(1)$ (d) $(g \circ f)(1)$

Solution

(a) $(f \circ g)(x) = 2(x+1)^2-1$

(b) $(g \circ f)(x) = (2x^2-1)+1 = 2x^2$

(c) From (a), we have

$$(f \circ g)(1)=7$$

(d) From (b), we have

$$(g \circ f)(1) = 2$$

Notice for questions (c) and (d)

For $(f \circ g)(1)$ and $(g \circ f)(1)$, it is not necessary to find $(f \circ g)(x)$ and $(g \circ f)(x)$ first. Alternatively, we can directly apply the definition as follows

(c) $(f \circ g)(1) = f(g(1)) = f(2) = 7$ [since $g(1)=2$]

(d) $(g \circ f)(1) = g(f(1)) = g(1) = 2$ [since $f(1)=1$]

Of course, if we are given a function f, we may also define the function $f \circ f$ in the obvious way:

$$(f \circ f)(x) = f(f(x))$$

That is, we plug f into itself.

For example, if $f(x)=2x-1$, then

$$(f \circ f)(x) = f(2x-1) = 2(2x-1)-1 = 4x-3$$

EXAMPLE 2

Let $f(x)=\dfrac{x+1}{2}$ and $g(x)=\sqrt{x}$

Find (a) $(f\circ g)(x)$ (b) $(g\circ f)(x)$

(c) $(f\circ f)(x)$ (d) $(g\circ g)(x)$

(e) $(f\circ f\circ f)(x)$ in two ways: as $f\circ(f\circ f)$ and as $(f\circ f)\circ f$

Solution

(a) $(f\circ g)(x)=\dfrac{\sqrt{x}+1}{2}$ (b) $(g\circ f)(x)=\sqrt{\dfrac{x+1}{2}}$

(c) $$(f\circ f)(x)=\frac{\frac{x+1}{2}+1}{2}=\frac{\frac{x+3}{2}}{2}=\frac{x+3}{4}$$

(d) $$(g\circ g)(x)=\sqrt{\sqrt{x}}=\sqrt[4]{x}$$

(e) $$(f\circ f\circ f)(x)=[f\circ(f\circ f)](x)=\frac{\frac{x+3}{4}+1}{2}=\frac{\frac{x+7}{4}}{2}=\frac{x+7}{8}$$

Or $$=[(f\circ f)\circ f](x)=\frac{\frac{x+1}{2}+3}{4}=\frac{\frac{x+7}{2}}{4}=\frac{x+7}{8}$$

- THE IDENTITY FUNCTION $I(x)$

It is the simple function that maps x to itself

$$I(x)=x$$

Notice that

$$(f\circ I)(x)=f(I(x))=f(x)$$

$$(I\circ f)(x)=I(f(x))=f(x)$$

That is

$$f\circ I=f \quad \text{and} \quad I\circ f=f$$

♦ PRESUPPOSITION FOR $f\circ g$ AND $g\circ f$ (**Mainly for HL**)

Let $f: A \to B$ and $g: B \to C$.

Then

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \\ x \arrow[r, mapsto] \arrow[rr, bend right, "g\circ f"'] & f(x) \arrow[r, mapsto] & g(f(x)) \end{tikzcd}$$

That is in $g\circ f$, f is applied first and then g

Notice also that $g\circ f$ can be defined only if the <u>Range of f</u> is inside the <u>Domain of g</u>.

Similar observations may be done for $f\circ g$. Thus,

Function	**Observation**	**Presupposition**
$f\circ g$	g is applied first and then f	$R_g \subseteq D_f$
$g\circ f$	f is applied first and then g	$R_f \subseteq D_g$

2.5 THE INVERSE FUNCTION: f^{-1}

♦ DISCUSSION

Consider the functions $f(x)=x+10$ and $g(x)=x-10$

Notice that

$f(0) = 10$	$g(10) = 0$	
$f(1) = 11$	$g(11) = 1$	
$f(2) = 12$	$g(12) = 2$	
$f(3) = 13$	$g(13) = 3$	and so on.

In other words,

$$f(x)=y \Leftrightarrow g(y)=x.$$

In this case we say that g is the **inverse** of f.

In fact, f and g are **inverse** to each other.

Notice also that

$$(f_o g)(x)=f(g(x))=f(x-10)=(x-10)+10=x$$

$$(g_o f)(x)=g(f(x))=f(x+10)=(x+10)-10=x$$

That is

$$f_o g = I = g_o f \qquad \text{(the identity function)}$$

The inverse function of f, that is g, will be denoted by $\mathbf{f^{-1}}$

♦ DEFINITION

Let $f:R\to R$

The **inverse function** $\mathbf{f^{-1}}$ is a new function such that

$$f(x)=y \Leftrightarrow f^{-1}(y)=x.$$

Moreover,

$$\mathbf{f_o f^{-1} = I = f^{-1}{}_o f}$$

Hence, if $f(x)=x+10$, we showed above that $f^{-1}(x)=x-10$

- HOW DO WE FIND f^{-1} ?

Steps f is given	Example $f(x) = x+10$
1. Set $f(x)=y$	1. $x+10 = y$
2. Solve for x	2. $x = y-10$
3. Keep only the solution but replace y by x	3. $f^{-1}(x)=x-10$

NOTICE:

1. The inverse function of f^{-1} is f itself. That is

$$\boxed{(f^{-1})^{-1} = f}$$

2. The domain of f^{-1} is the range of f and vice-versa:

$$\boxed{D_{f^{-1}} = R_f} \qquad \boxed{R_{f^{-1}} = D_f}$$

EXAMPLE 1

Let $f(x)=3x+5$. Find (a) $f^{-1}(x)$ (b) $f^{-1}(11)$

Solution

(a) We follow the three steps:

- Set $3x+5=y$
- $3x+5=y \Leftrightarrow 3x = y-5 \Leftrightarrow x = \dfrac{y-5}{3}$
- $f^{-1}(x)= \dfrac{x-5}{3}$

(b) Since we know $f^{-1}(x)= \dfrac{x-5}{3}$, it is $f^{-1}(11) = 2$

Alternatively: It is not necessary to find $f^{-1}(x)$ first. We can directly set $3x+5 = 11$, instead of $3x+5=y$:

$3x+5 = 11 \Leftrightarrow 3x = 6 \Leftrightarrow x=2.$

Thus, $f^{-1}(11) =2$

Remark:

Let us verify that the inverse function of $f^{-1}(x)= \frac{x-5}{3}$ is $f(x)= 3x+5$.

- Set $\frac{x-5}{3} = y$
- $\frac{x-5}{3} = y \Leftrightarrow x-5 = 3y \Leftrightarrow x = 3y+5$
- The inverse function is $f(x)= 3x+5$

That is why we also say that f and f^{-1} are inverse to each other.

EXAMPLE 2

Let $f(x)=2x^2-1$ where $x\geq 0$. Find (a) $f^{-1}(x)$ (b) $f^{-1}(49)$

Solution

(a) We follow the three steps:

- Set $2x^2-1=y$
- $2x^2-1=y \Leftrightarrow 2x^2 = y+1 \Leftrightarrow x^2 = \frac{y+1}{2} \Leftrightarrow x = \sqrt{\frac{y+1}{2}}$
- $f^{-1}(x)= \sqrt{\frac{x+1}{2}}$

(b) Since we know $f^{-1}(x)=\sqrt{\frac{x+1}{2}}$, it is

$f^{-1}(49) = \sqrt{25} = 5$

[again, $f^{-1}(49)$ can be estimated directly as follows

$2x^2-1= 49 \Leftrightarrow 2x^2 = 50 \Leftrightarrow x^2 = 25 \Leftrightarrow x = 5$,

thus $f^{-1}(49) = 5$]
