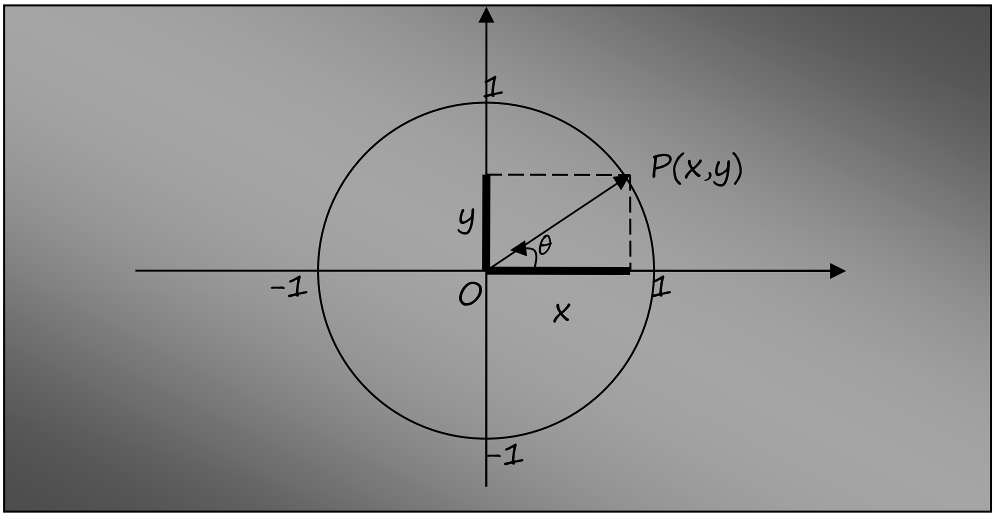
3.4 SINΘ, COSΘ, TANΘ ON THE UNIT CIRCLE

♦ SINΘ, COSΘ

Consider again the **unit circle** (radius r =1) on the Cartesian plane.



Let P(x,y) be moving along the circle,

OP = r = 1

θ = angle between OP and x-axis

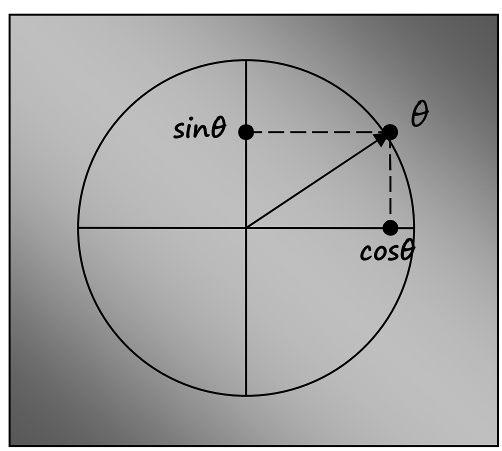
Then

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = y \qquad \text{and} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x$$

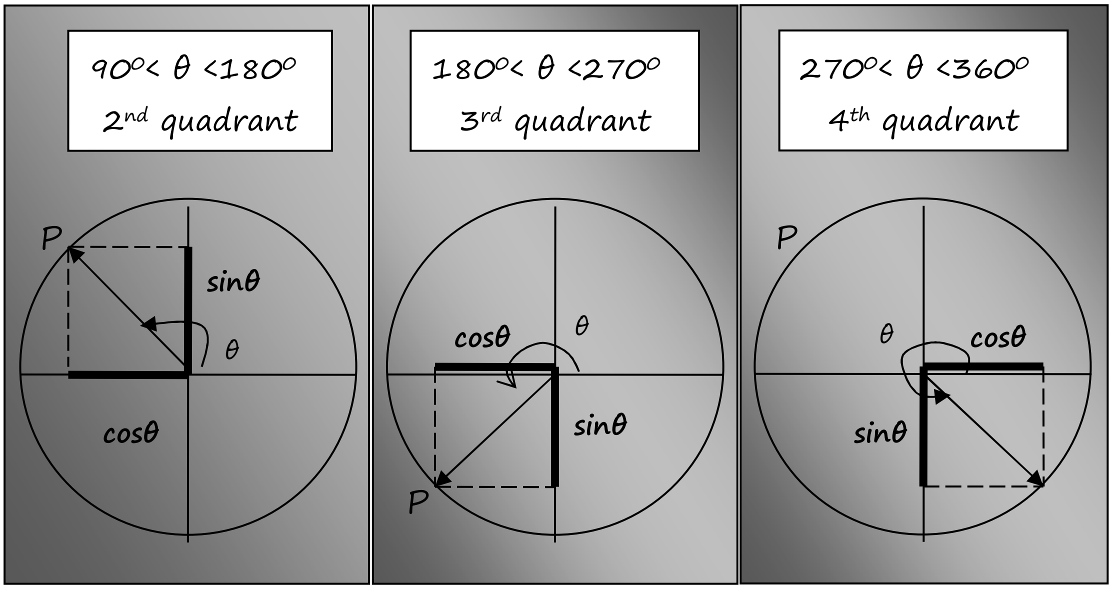
Thus, if we think the angle θ as a point on the circle:

$\sin\theta$ = y coordinate of θ

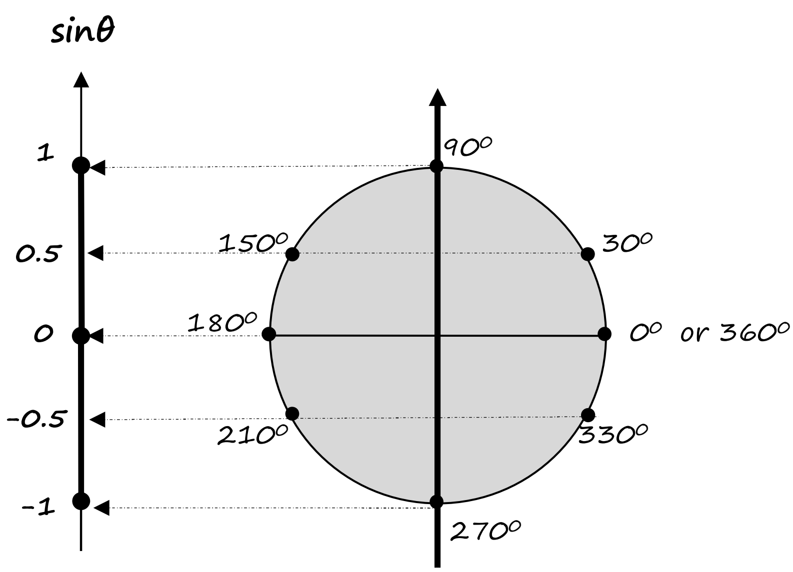
$\cos\theta$ = x coordinate of θ



This description helps us to define sinθ and cosθ not only for angles within $0^o \le \theta \le 90^o$, but for any value of θ on the circumference.



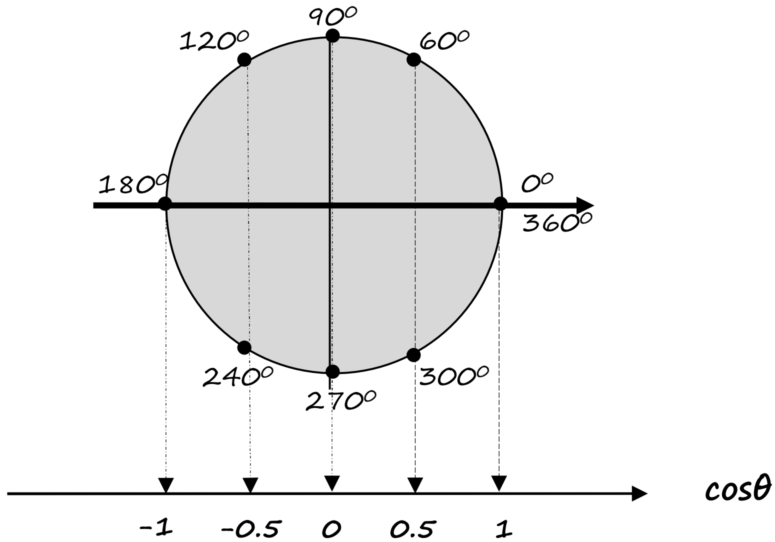
It will be helpful to move the y-axis which represents sinx to the left of the circle:



sinθ is the y-coordinate of θ									
x	0°	30°	90°	150°	180°	210°	270°	330°	360°
sinθ	0	0.5	1	0.5	0	- 0.5	-1	- 0.5	0

This picture explains why supplementary angles have equal sines.

Similarly, it will be helpful to move the x-axis which represents cosx under the circle:



cosθ is the **x**-coordinate of θ									
x	0°	60°	90°	120°	180°	240°	270°	300°	360°
cosθ	1	0.5	0	– 0.5	-1	– 0.5	0	0.5	1

This picture explains why opposite angles have equal cosines.

NOTICE

- The value of θ can be any real number positive or negative.

In that sense, any point on the circumference has infinitely many names! For example, the point where 30° is situated, is also called

$$390^o,\ 750^o,\ \ldots \quad \text{(we add } 360^o\text{)}$$

$$\text{or}\quad -330^o,\ -690^o,\ \ldots \quad \text{(we subtract } 360^o\text{)}$$

In general, the same point on the circles represents the angles

$$30^o+360^o k$$

Speaking in radians the same point represents the angles

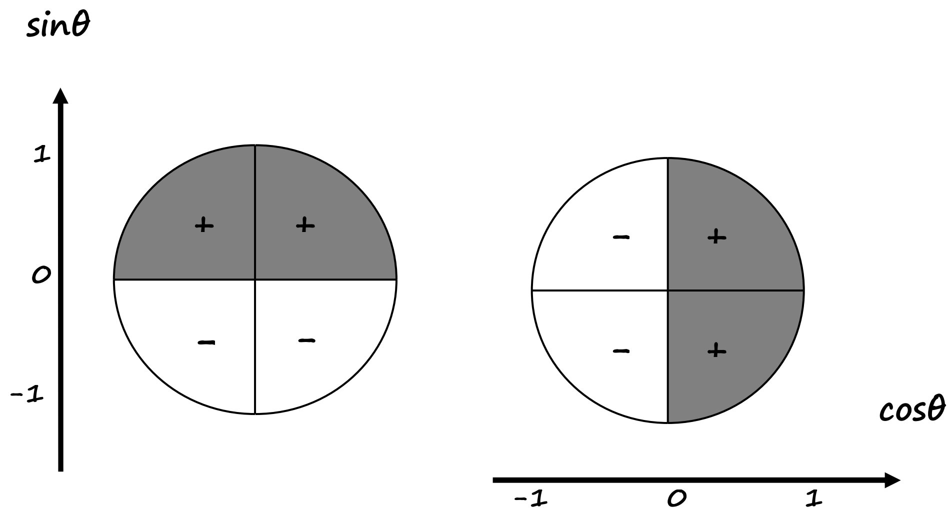
$$\frac{\pi}{6}+2k\pi$$

Since sin30° = 0.5, the sine of all these angles is also 0.5

Similarly, the cosine of all these angles is $\sqrt{3}/2$

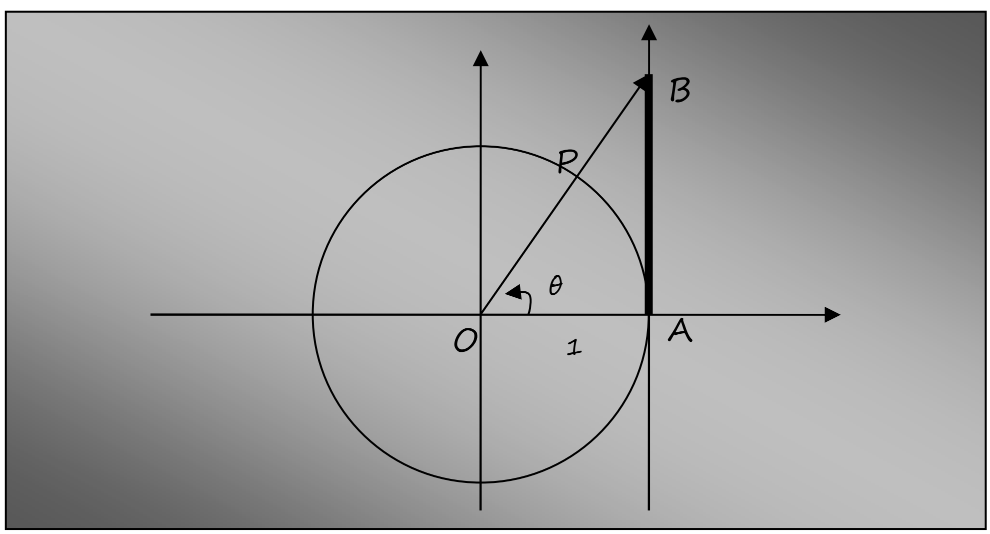
As far as the values of sinθ and cosθ,

$\boxed{-1 \leq \sin\theta \leq 1}$ $\boxed{-1 \leq \cos\theta \leq 1}$



- **TANΘ**

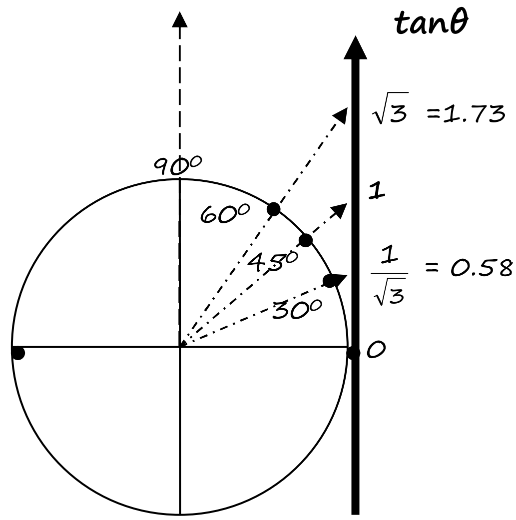
Consider now the **unit circle** below and an additional vertical axis passing through point A.



Then

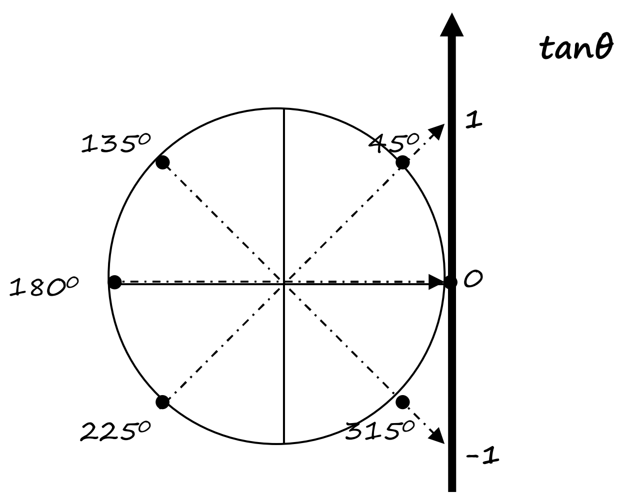
$$\boxed{\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{AB}{1} = AB}$$

Thus



x	0^o	30^o	45^o	60^o	90^o
tanθ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	$+\infty$

Again, this description helps us to define tanθ not only for angles θ within $0^o \leq \theta \leq 90^o$.



x	0^o	45^o	135^o	180^o	225^o	315^o	360^o
tanθ	0	1	−1	0	1	− 1	0

It is clear that diametrically opposite angles have equal tangents.

NOTICE

- Not only θ, but all values

$$\theta + 180k^{\circ} \quad \text{(in degrees)}$$

$$\theta + k\pi \quad \text{(in radians)}$$

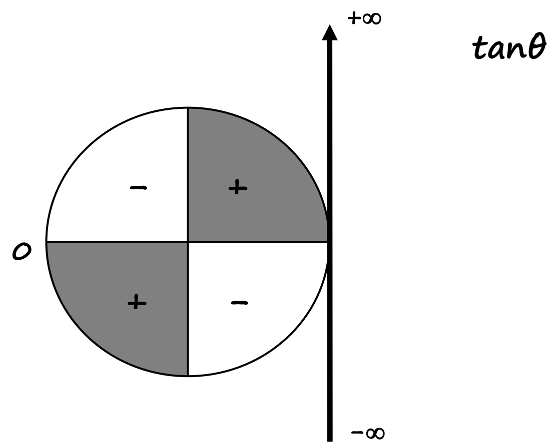
  have equal tangents (we just add or subtract semicircles).

- It is obvious that $\tan\theta$ is not defined for $\theta=90^{\circ}$ or $\theta=-90^{\circ}$. In fact, $\tan\theta$ is not defined for

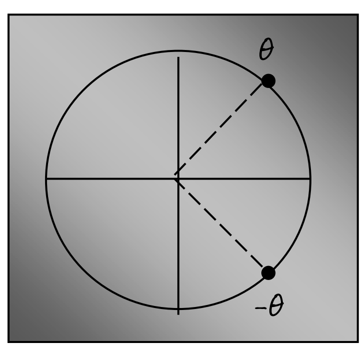
$$90^{\circ} + 180k^{\circ} \quad \text{(in degrees)} \qquad \frac{\pi}{2} + k\pi \text{ (in radians)}$$

  For any other value of θ,

$$\boxed{-\infty < \tan\theta < +\infty}$$



It is worthwhile to notice that for opposite angles, θ and $-\theta$



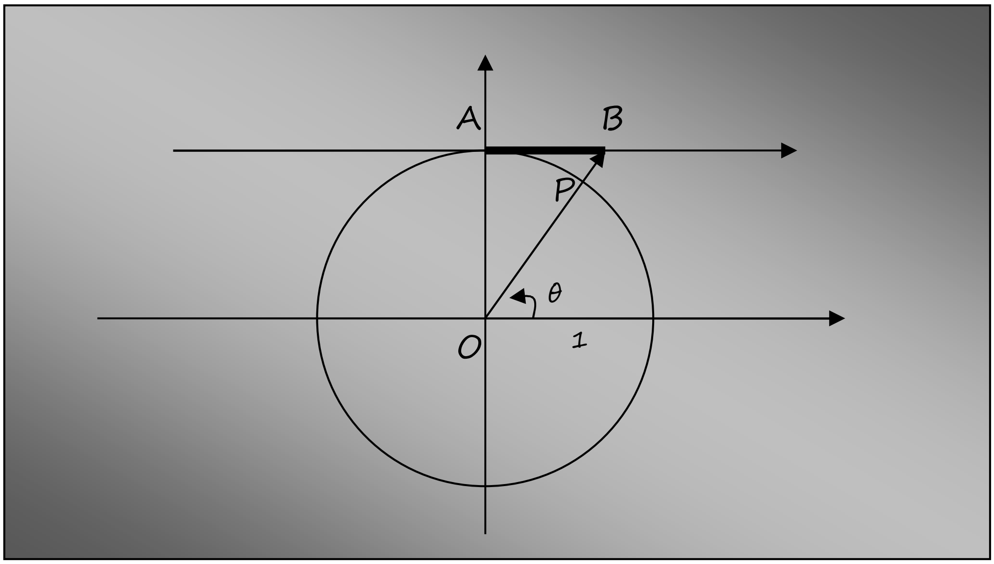
$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\tan(-\theta) = -\tan\theta$$

♦ COTΘ (only for HL)

Working as in $\tan\theta$, consider now the **unit circle** below and an additional horizontal axis passing through point A.



Then

$$\cot\theta = AB$$

Very similar observations to $\tan\theta$ can be made.

♦ SECΘ , CSCΘ (only for HL)

Two new trigonometric numbers are defined as follows

Secant: $\sec\theta = \dfrac{1}{\cos\theta}$

Cosecant: $\csc\theta = \dfrac{1}{\sin\theta}$

Also remember that

Cotangent: $\cot\theta = \dfrac{1}{\tan\theta}$

3.5 TRIGONOMETRIC IDENTITIES AND EQUATIONS

♦ IDENTITIES

We have already seen the fundamental identity

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

If we divide by $\cos^2\theta$ we obtain

$$\boxed{\tan^2\theta + 1 = \frac{1}{\cos^2\theta}}$$

The following identities concerning the **double angle 2θ** are useful:

$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\cos 2\theta = 2\cos^2\theta - 1$ $\cos 2\theta = 1 - 2\sin^2\theta$	$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

EXAMPLE 1

Let $\sin\theta = \frac{3}{5}$. Find

$$\cos\theta,\ \tan\theta,\ \sin 2\theta,\ \cos 2\theta,\ \tan 2\theta$$

if

(a) $\theta < 90^\circ$ (acute)

(b) $90^\circ < \theta < 180^\circ$ (obtuse)

Solution

By the fundamental identity $\sin^2\theta + \cos^2\theta = 1$, we obtain

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25},$$

thus

$$\cos\theta = \pm\frac{4}{5}$$

If θ is acute (1st quadrant) $\boxed{\cos\theta = \frac{4}{5}}$, if θ is obtuse $\boxed{\cos\theta = -\frac{4}{5}}$

(a) Since $\theta < 90^o$

$$\cos\theta = \frac{4}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3}{4},$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\ \frac{3}{5}\frac{4}{5} = \frac{24}{25}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{24}{7}.$$

(b) Since $90^o < \theta < 180^o$

$$\cos\theta = -\frac{4}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{3}{4},$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{3}{5}\right)\frac{4}{5} = -\frac{24}{25}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = -\frac{24}{7}.$$

NOTICE

Consider the double angle identity

$$\sin 2\theta = 2\sin\theta\cos\theta$$

That means

$$\sin 30^o = 2\sin 15^o\cos 15^o$$

$$\sin 100^o = 2\sin 50^o\cos 50^o$$

or

$$\sin 4\theta = 2\sin 2\theta\cos 2\theta$$

$$\sin 10\theta = 2\sin 5\theta\cos 5\theta$$

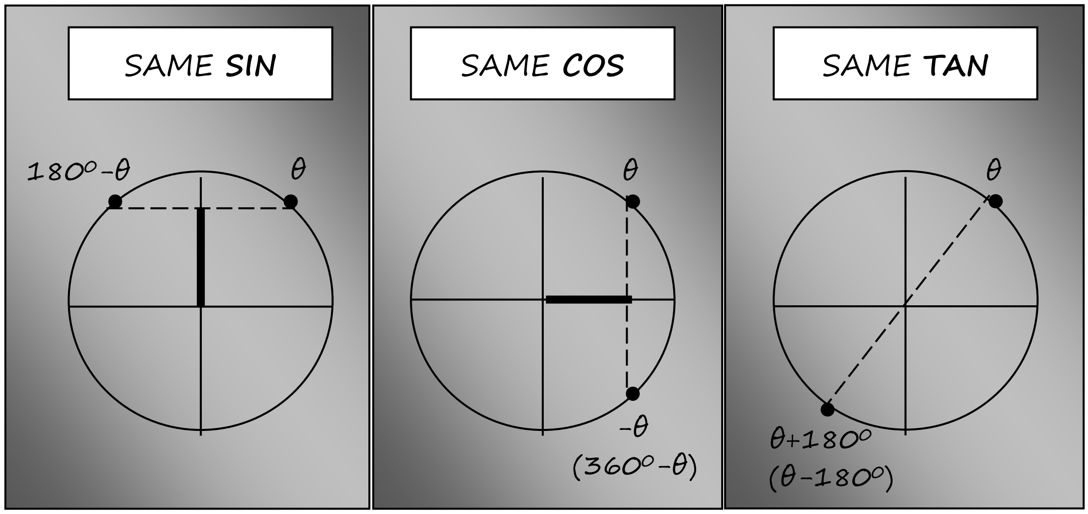
Similar variations can be obtained by the other identities, e.g.

$$\cos 30^o = 1 - 2\sin^2 15^o$$

$$\cos 4\theta = 1 - 2\sin^2 2\theta$$

♦ TRIGONOMETRIC EQUATIONS

Remember that



Working with radians

<u>SAME **SIN**</u>	<u>SAME **COS**</u>	<u>SAME **TAN**</u>
$\boldsymbol{\theta}$ and $\boldsymbol{\pi-\theta}$	$\boldsymbol{\theta}$ and $\boldsymbol{-\theta}$ (or $\boldsymbol{2\pi-\theta}$)	$\boldsymbol{\theta}$ and $\boldsymbol{\theta+\pi}$

These diagrams help us to solve equations of the form

$$\sin x=a$$
$$\cos x=a$$
$$\tan x=a$$

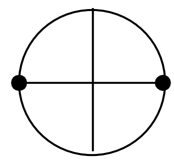
	IN DEGREES	IN RADIANS
sinx=a	the principal solution is $\theta=\sin^{-1}a$	
	$x=\theta+360^{\circ}k$ $x=180^{\circ}-\theta+360^{\circ}k$	$x=\theta+2k\pi$ $x=\pi-\theta+2k\pi$
cosx=a	the principal solution is $\theta=\cos^{-1}a$	
	$x=\theta+360^{\circ}k$ $x=-\theta+360^{\circ}k$	$x=\theta+2k\pi$ $x=-\theta+2k\pi$
tanx=a	the principal solution is $\theta=\tan^{-1}a$	
	$x=\theta+180^{\circ}k$	$x=\theta+k\pi$

EXAMPLE 1

Solve the equation $\sin x=0$, $0^o \leq x < 360^o$

The principal solution is $\boxed{x=0^o}$

It will help to represent the solutions on the unit circle:



There is one more solution in the given interval: $\boxed{x=180^o}$

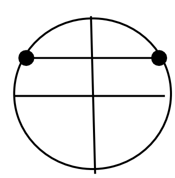
REMARKS:

- If the equation is given in radians under the restriction $0 \leq x < 2\pi$ the solutions are $\boxed{x=0}$, $\boxed{x=\pi}$
- If the restriction is $0^o \leq x \leq 360^o$ or $0 \leq x \leq 2\pi$ there are 3 solutions $\boxed{x=0^o}$, $\boxed{x=180^o}$, $\boxed{x=360^o}$ or $\boxed{x=0}$, $\boxed{x=\pi}$, $\boxed{x=2\pi}$ respectively.
- If the restriction is $-180^o \leq x \leq 180^o$ or $-\pi \leq x \leq \pi$, the solutions are $\boxed{x=0^o}$, $\boxed{x=180^o}$, $\boxed{x=-180^o}$ or $\boxed{x=0}$, $\boxed{x=\pi}$, $\boxed{x=-\pi}$ respectively.

EXAMPLE 2

Solve the equation $\sin x=\frac{1}{2}$, $0^o \leq x \leq 360^o$

The principal solution is $\boxed{x=30^o}$



There is one more solution in the given interval: $\boxed{x=150^o}$

REMARKS:

- If the equation is given in radians under the restriction $0 \leq x \leq 2\pi$, the solutions are $\boxed{x=\pi/6}$, $\boxed{x=\pi-\pi/6=5\pi/6}$
- If the restriction is $-180^o \leq x \leq 180^o$ or $-\pi \leq \theta \leq \pi$ the solutions are still the same.