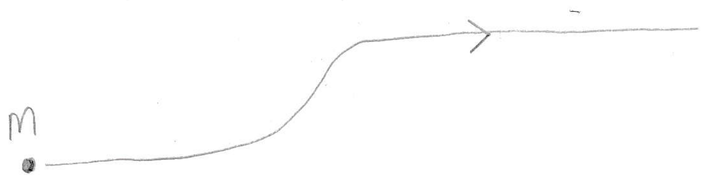
Persamaan Diferensial Biasa
(Ordinary Differential Equation)

Pers. Diff : Pers. matematika yang mengandung suku derivatif

Mekanika :



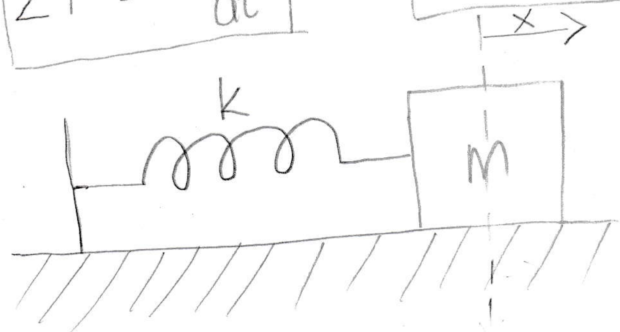
$x(t=0)=x_0$

$$\frac{dx}{dt}=\dot{x}(t=0)=V_0$$

$$\boxed{\sum \bar{F} = m\bar{a}} \quad , \quad \bar{a} = \frac{d\bar{V}}{dt} = \frac{d^2\bar{r}}{dt^2}$$

$\downarrow$

$$\boxed{\sum \bar{F} = m\frac{d\bar{V}}{dt}} \longrightarrow \boxed{m\frac{d^2\bar{r}}{dt^2} - \sum \bar{F} = 0}$$



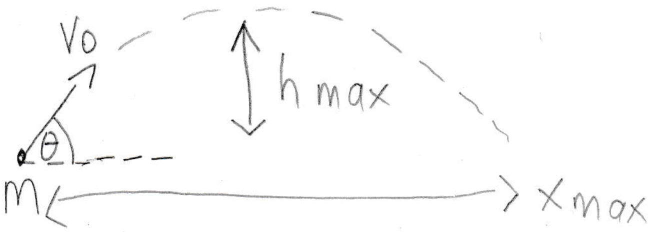
$\Rightarrow$ F_k = Proposional dg simpangan

$= -kx \rightarrow [k] = N/m$

$$\sum F = -kx \Rightarrow \sum F = m\frac{d^2x}{dt^2}$$

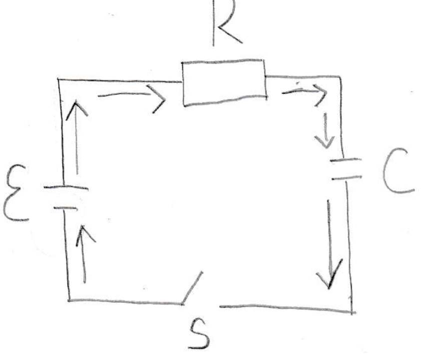
$$-kx = m\frac{d^2x}{dt^2} \Rightarrow \boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

Mathematical Model



$$\Sigma F_x = 0$$
$$\Sigma F_x = m\frac{dV_x}{dt}$$
$$0 = m\frac{dV_x}{dt}$$
$$\boxed{\frac{dV_x}{dt} = 0}$$

$$\Sigma F_y = m a_y$$
$$\Sigma F_y = -mg$$
$$\cancel{m}\frac{dV_y}{dt} = -\cancel{m}g$$
$$\boxed{\frac{dV_y}{dt} = -g}$$



$$\Sigma \mathcal{E} + \Sigma V = 0$$
$$\mathcal{E} - (V_R + V_C) = 0$$

$$C = q/V$$
$$i = \frac{dq}{dt}$$

$$\mathcal{E} = V_R + V_C$$
$$\mathcal{E} = iR + \frac{q}{C}$$
$$\mathcal{E} = \frac{dq}{dt}R + \frac{q}{C}$$
$$\frac{dq}{dt} + \frac{q}{RC} = \frac{\mathcal{E}}{R}$$
$$\boxed{\frac{dq}{dt} + \frac{1}{RC}q = \frac{\mathcal{E}}{R}}$$

PD $\xrightarrow{\text{klasifikasi}}$ Jumlah variabel bebas

- PD Biasa (1)
- PD Parsial (>1)

$$\hookrightarrow \bar{\nabla}\cdot\bar{E} = \frac{\rho}{\varepsilon_0} \quad \Big| \quad \bar{\nabla}\cdot\bar{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$
$$\bar{\nabla}\cdot\bar{B} = 0$$
$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$$
$$\bar{\nabla}\times\bar{B} = \mu_0\bar{J} + \varepsilon_0\frac{\partial \bar{E}}{\partial t}$$

$$PD \begin{cases} \to \text{Orde derivatif} \begin{cases} \to \text{PD orde-1} \\ \to \text{PD orde } n\ (>1) \end{cases} \\ \to \text{linearitas} \begin{cases} \to \text{PD linear} \\ \to \text{PD non-linear} \end{cases} \end{cases}$$

linear : Pangkat dri variabel terikat / derivatifnya adlh 1

$\frac{dy}{dx} + y = x^2 \to$ PDB orde-1 linear

$\frac{dy}{dx} + y^2 = 2x \to$ PDB orde-1 non-linear

$\frac{d^2y}{dx^2} + xy = 0 \to$ PDB orde-2 linear

$\left(\frac{dy}{dx}\right)^2 + xy = 0 \to$ PDB orde-1 non-linear

$\frac{dy}{dx} + \sin y = x^2 \to \sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \dots$

PDB orde-1

$x \Rightarrow$ Variabel bebas

$y \Rightarrow$ — " — terikat, $y = y(x)$

Linear

$\to$ solusi : $y(x)$

$\boxed{a_1 y' + a_0 y = b}$, $y' \equiv \frac{dy}{dx}$ $a_1 = a_1(x)$, $a_0(x) = a_0$

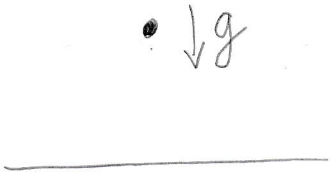
$b = b(x)$

cth: $\dot{q} + \frac{1}{RC} q = \frac{\varepsilon}{R}$, $\dot{q} \equiv \frac{dq}{dt}$

$y \to q, x \to t$

$a_1 = 1, a_0 = \frac{1}{RC}, b = \frac{\varepsilon}{R}$

Teknik Integrasi



$$\Sigma F_y = m\frac{d^2y}{dt^2}$$

$$\cancel{m}g = \cancel{m}\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dt^2} = g$$

$$\Sigma F_y = ma_y = m\frac{dV_y}{dt}$$

$$\cancel{m}g = \cancel{m}\frac{dV_y}{dt}$$

$$g = \frac{dV_y}{dt} \rightarrow V_y = V_y(t)$$

$\times dt$

$$\int_0^{V_y} dV_y = \int_0^t g\,dt$$

$$\hookrightarrow \boxed{V_y = gt},\ V_y \equiv \frac{dy}{dt}$$

$$\downarrow$$

$$\frac{dy}{dt} = gt$$

$\times dt$

$$\int_{y_0}^{y} dy = \int_0^t gt\,dt$$

$$y - y_0 = \frac{1}{2}gt^2 \rightarrow y = y_0 + \frac{1}{2}gt^2$$

Separasi Variabel

$$\boxed{y' + ay = c} \qquad a \equiv a_0/a_1,\quad c \equiv b/a_1$$

Tinjau a = konst. , c = konst.

$$\frac{dy}{dx} = c - ay$$

$\times \frac{dx}{c-ay}$

$$\frac{dy}{c-ay} = dx$$

$$\int \frac{dy}{c-ay} = \int dx \quad \Bigg| \quad \begin{aligned} u &= c-ay \\ du &= -a\,dy \\ \frac{dy}{c-ay} &= -\frac{1}{a}\frac{du}{u} \end{aligned}$$

$$-\frac{1}{a}\int \frac{du}{u} = x+K$$

$$-\frac{1}{a}\ln(c-ay) = x+K$$

$$\ln(c-ay) = -ax + aK$$

$$\ln(c-ay) = -ax + K', \quad K' \equiv aK$$

$$\cancel{e}^{\cancel{\ln}(c-ay)} = e^{-ax} \cdot e^{K'}$$

$$c-ay = e^{-ax}e^{K'} = e^{-ax}B, \quad B \equiv e^{K'}$$

$$\cancel{c}-ay = Be^{-ax}$$

$$\cancel{-}ay = Be^{-ax} - c$$

$$\times -1$$

$$ay = -Be^{-ax} + c$$

$$y = \frac{1}{a}(c - Be^{-ax})$$

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{\varepsilon}{R} \longrightarrow \frac{dq}{dt} = \frac{\varepsilon}{R} - \frac{1}{RC}q$$

$$\times \frac{dt}{\frac{\varepsilon}{R} - \frac{q}{RC}}$$

$$\frac{dq}{\frac{\varepsilon}{R} - \frac{q}{RC}} = dt$$

$$\int \frac{dq}{\frac{\varepsilon}{R} - \frac{q}{RC}} = \int dt \quad \Bigg| \quad \begin{aligned} u &= \frac{\varepsilon}{R} - \frac{q}{RC} \\ u &= \frac{\varepsilon}{R} - \frac{1}{RC}q \\ du &= 0 - \frac{1}{RC}dq \\ du &= -\frac{1}{RC}dq \end{aligned} \longrightarrow -RC\,du = dq$$

$$-RC\int \frac{du}{u} = \int dt$$

$$-RC\ln\left(\frac{\varepsilon}{R} - \frac{q}{RC}\right)\Bigg|_{q=0}^{q} = t \longrightarrow \ln\left(\frac{\varepsilon}{R} - \frac{q}{RC}\right) - \ln\left(\frac{\varepsilon}{R}\right) = -\frac{t}{RC}$$

$$\ln\left[\frac{\varepsilon/R - q/RC}{\varepsilon/R}\right] = -t/RC$$

$$\ln\left[\frac{\varepsilon/R}{\varepsilon/R} - \frac{q/RC}{\varepsilon/R}\right] = -t/RC$$

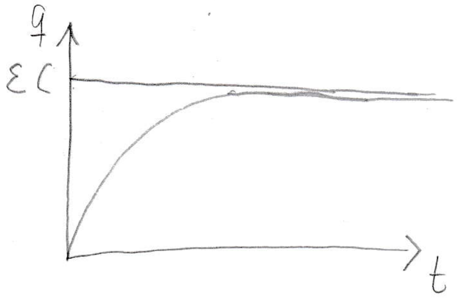
$$e^{\cancel{\ln}}\left[1 - \frac{q/RC}{\varepsilon/R}\right] = e^{-t/RC}$$

$$1 - \frac{q/RC}{\varepsilon/R} = e^{-t/RC} \longrightarrow \frac{q}{RC} = \frac{\varepsilon}{R}\left(1 - e^{-t/RC}\right)$$

$$q = \varepsilon C\left(1 - e^{-t/RC}\right)$$

$$q(t) = \varepsilon C\left(1 - e^{-t/RC}\right)$$

$t \to \infty$



$$x y' = y \Rightarrow x \frac{dy}{dx} = y \quad \times \frac{dx}{xy}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + K$$

$$y = x e^{K} = \underline{Ax}, \quad A \equiv e^{K}$$

$$y' = A$$

$$xA = \underline{\underline{Ax}}$$

$$y' + 2xy^2 = 0 \quad , \quad y(2) = 1$$

$$\frac{dy}{dx} + 2xy^2 = 0$$

$$\frac{dy}{dx} = -2xy^2 \qquad \times \frac{dx}{y^2}$$

$$\frac{dy}{y^2} = -2x\,dx$$

$$\int \frac{dy}{y^2} = -2\int x\,dx$$

$$-\frac{1}{y} = -x^2 - C \qquad \times -1$$

$$\frac{1}{y} = x^2 + C$$

$$y = \frac{1}{x^2 + C}$$

$$1 = \frac{1}{(2)^2 + C}$$

$$1 = \frac{1}{4 + C} \rightarrow 4 + C = 1$$

$$C = -3$$

$$\boxed{y(x) = \frac{1}{x^2 - 3}}$$

Tentukan solusi unik dari PDB berikut:

1) $xy' - xy = y, \quad y(1) = 1$

2) $y' \sin x = y \ln y, \quad y\left(\frac{\pi}{3}\right) = e$

3) $2y' = 3(y-2)^{1/3}, \quad y(1) = 3$

4) $y' = \dfrac{2xy^2 + x}{x^2 y - y}, \quad y(\sqrt{2}) = 0$

Solution

1) $x\dfrac{dy}{dx} - xy = y$

$x\dfrac{dy}{dx} = y + xy$

$x\dfrac{dy}{dx} = y(1+x) \quad \times \dfrac{dx}{xy}$

$\dfrac{dy}{y} = \dfrac{1+x}{x}\,dx$

$\displaystyle\int \frac{dy}{y} = \int \frac{1}{x}\,dx + \int \frac{x}{x}\,dx$

$\ln|y| = \ln|x| + x + C$

$\cancel{e}^{\ln|y|} = \cancel{e}^{\ln|x|} e^x e^C$

$y(x) = x e^x A, \quad A \equiv e^C$

$y(x) = Ae^x x$

Proof:

$y(x) = Ae^x x$

$y'(x) = A\,(e^x x)'$

$= A\,(e^x x + e^x)$

$= Ae^x x + Ae^x$

$= Ae^x (x+1)$

Enter to the eq'n:

$x\left(Ae^x(x+1)\right) - x\left(Ae^x x\right) = Ae^x x$

$Ae^x x(x+1) - Ae^x x^2 = Ae^x x$

$\cancel{Ae^x x^2} + Ae^x x - \cancel{Ae^x x^2} = Ae^x x$

$Ae^x x = Ae^x x \quad \square$

unique solution:

$y(1) = 1$

$1 = Ae^1 \cdot 1$

$1 = Ae$

$A = \dfrac{1}{e}$

$y(x) = \dfrac{1}{e} e^x x$

$y(x) = e^{x-1} x$

$\boxed{y(x) = xe^{x-1}}$

2) $y' \sin x = y \ln y$, $y\left(\frac{\pi}{3}\right) = e$

$$\frac{dy}{dx} \sin x = y \ln y \quad \times \frac{dx}{(\sin x)(y \ln y)}$$

$$\frac{dy}{y \ln y} = \frac{dx}{\sin x}$$

$$\int \frac{dy}{y \ln y} = \int \frac{dx}{\sin x}$$

$$\int \frac{du}{u} = \int \csc x \, dx$$

$u = \ln y$

$du = \frac{1}{y} dy$

$du = \frac{dy}{y}$

$$\ln|\ln|y|| = -\ln|\csc(x) + \cot(x)| + C$$

$$\cancel{e}^{\cancel{\ln}|\ln|y||} = \cancel{e}^{\cancel{\ln}|\csc(x)+\cot(x)|^{-1}} e^{C}$$

$$\ln|y| = (\csc(x) + \cot(x))^{-1} A, \quad A \equiv e^{C}$$

$$\ln|y| = \left(\frac{1}{\csc(x) + \cot(x)}\right) A$$

$$\ln|y| = \left(\frac{1}{\frac{1 + \cos(x)}{\sin(x)}}\right) A$$

$$\ln|y| = \left(\frac{\sin x}{1 + \cos(x)}\right) A$$

$$\ln|y| = A \tan\left(\frac{x}{2}\right)$$

$$\cancel{e}^{\cancel{\ln}|y|} = e^{A \tan\left(\frac{x}{2}\right)}$$

$$y = e^{A \tan\left(\frac{x}{2}\right)}$$

unique solution:

$$y\left(\frac{\pi}{3}\right) = e$$

$$y(x) = e^{A \tan\left(\frac{x}{2}\right)}$$

$$e = e^{A \tan\left(\frac{\pi}{6}\right)}$$

$$1 = A \tan\left(\frac{\pi}{6}\right)$$

$$A = \frac{1}{\tan\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{1}{3}\sqrt{3}} = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{\cancel{3}}$$

$$= \sqrt{3}$$

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1}{3}\sqrt{3}$$

$$\boxed{y(x) = e^{\sqrt{3} \tan\left(\frac{x}{2}\right)}}$$

3) $2y' = 3(y-2)^{1/3}$, $y(1) = 3$

$$\frac{dy}{dx} = \frac{3}{2}(y-2)^{1/3}$$

$\times\, dx$

$$dy = \frac{3}{2}(y-2)^{1/3}\, dx$$

$\times \frac{1}{(y-2)^{1/3}}$

$$\frac{dy}{(y-2)^{1/3}} = \frac{3}{2}\, dx$$

$$\int \frac{dy}{(y-2)^{1/3}} = \int \frac{3}{2}\, dx \quad \Big| \begin{array}{l} u = y-2 \\ du = dy \end{array}$$

$$\int \frac{du}{u^{1/3}} = \frac{3}{2}x + C$$

$$\frac{3}{2}(y-2)^{\frac{2}{3}} = \frac{3}{2}x + C$$

$\times\, 2$

$$3(y-2)^{\frac{2}{3}} = 3x + 2C$$

$\times \frac{1}{3}$

$$(y-2)^{\frac{2}{3}} = x + \frac{2C}{3}$$

$$(y-2)^{\frac{2}{3}} = x + B, \quad B \equiv \frac{2C}{3}$$

$$\left((y-2)^{\frac{2}{\cancel{3}}}\right)^{\cancel{3}} = (x+B)^3$$

$$\left((y-2)^{\cancel{2}}\right)^{\frac{1}{\cancel{2}}} = \left((x+B)^3\right)^{\frac{1}{2}}$$

$$y-2 = (x+B)^{\frac{3}{2}}$$

$$y(x) = (x+B)^{\frac{3}{2}} + 2$$

unique solution:

$$y(x) = (x+B)^{\frac{3}{2}} + 2$$

$$3 = (1+B)^{\frac{3}{2}} + 2$$

$$3 = 1 + B^{\frac{3}{2}} + 2$$

$$3 = 3 + B^{\frac{3}{2}}$$

$$3 - 3 = B^{\frac{3}{2}}$$

$$0 = B^{\frac{3}{2}}, \quad B = 0$$

$$\boxed{y(x) = x^{\frac{3}{2}} + 2}$$

4) $y' = \frac{2xy^2+x}{x^2y-y}$, $y(\sqrt{2})=0$

$$\frac{dy}{dx} = \frac{2xy^2+x}{x^2y-y}$$

$$\frac{dy}{dx} = \frac{x(2y^2+1)}{y(x^2-1)} \quad \times dx$$

$$dy = \frac{x(2y^2+1)}{y(x^2-1)}dx \quad \times y$$

$$y\,dy = \frac{x(2y^2+1)}{(x^2-1)}dx \quad \times \frac{1}{(2y^2+1)}$$

$$\int \frac{y}{(2y^2+1)}dy = \int \frac{x}{(x^2-1)}dx$$

$u = 2y^2+1$
$du = 4y\,dy$
$\frac{1}{4}du = y\,dy$

$w = x^2-1$
$dw = 2x\,dx$
$\frac{1}{2}dw = x\,dx$

$$\frac{1}{4}\int \frac{du}{u} = \frac{1}{2}\int \frac{dw}{w}$$

$$\frac{1}{4}\ln|2y^2+1| = \frac{1}{2}\ln|x^2-1| + C \quad \times 4$$

$$\ln|2y^2+1| = 2\ln|x^2-1| + C$$

$$\cancel{e}^{\cancel{\ln}|2y^2+1|} = \cancel{e}^{\cancel{\ln}|x^2-1|^2} e^C$$

$$2y^2+1 = (x^2-1)^2 A, \quad A \equiv e^C$$

$$2y^2 = (x^2-1)(x^2-1)A - 1$$

$$y^2 = \frac{(x^2-1)^2A-1}{2}$$

$$y(x) = \sqrt{\frac{(x^2-1)^2A-1}{2}}$$

Unique Solution:

$$y(x) = \sqrt{\frac{(x^2-1)^2A-1}{2}}$$

$$0 = \sqrt{\frac{((\sqrt{2})^2-1)^2A-1}{2}}$$

$$0 = \sqrt{\frac{(2-1)^2A-1}{2}}$$

$$0 = \sqrt{\frac{A-1}{2}}$$

$$0 = \sqrt{\frac{A}{2}} - \sqrt{\frac{1}{2}}$$

$$\left(\sqrt{\frac{1}{2}}\right)^2 = \left(\sqrt{\frac{A}{2}}\right)^2$$

$$\frac{1}{2} = \frac{A}{2}$$

$$A = 1$$

$$\boxed{y(x) = \sqrt{\frac{(x^2-1)^2-1}{2}}}$$