Position-Based Social Choice Methods
for Intransitive Incomplete Pairwise Vote Sets

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Abstract

Combining the decisions of multiple agents into a final deci-
sion requires the use of social choice mechanisms. Pairwise
decisions are often incomplete and intransitive, preventing
the use of Borda count and other position-based social choice
mechanisms. We propose and compare multiple methods for
converting incomplete intransitive pairwise vote sets to com-
plete rankings, enabling position-based social choice meth-
ods. The algorithms are evaluated on their output's Kendall's
τ similarity when implementing pairwise social choice mech-
anisms. We show that there is only a small difference between
the outputs of social choice methods on the original pairwise
vote set and the generated ranking set on a real-world pair-
wise voting dataset. Source code for the analysis is available.[1]

Introduction

When combining the preferences of multiple agents, we may
ask them to specify their preferences as an ordering (a first
choice, second choice, ..., nth choice) or as a set of pairwise
preferences (choice 1 is better than choice 2, 2 worse than 3,
and so on). In some cases, collecting the entire ordering for
an agent is infeasible, and so we must make do with pairwise
comparisons (Saaty 2008). When collecting pairwise com-
parisons, "intransitive and incomplete preference relations
can be quite reason based, hence rational" (Nurmi 2014).
Intransitive preferences can stem from stochastic decision-
making processes or multi-objective decision making frame-
works, and incomplete preferences from having limited time
to vote. Running pair-based social choice mechanisms on
these vote sets is simple. Pair-based social choice mecha-
nisms (such as Ranked Pairs) operate on win/loss/tie ratios
in matchups between choices, so intransitivity and incom-
pleteness play no role in the final output. On the other hand,
position-based social choice mechanisms (such as Borda
count) operate on voter's orderings over all the possible
choices, so using a position-based social choice mechanism
on pairwise vote sets requires converting the pairwise votes
into an ordering. This is simple for complete, transitive vote
sets. By representing the votes as graph, where each node

[1]https://github.com/julian-zucker/vote-induction

represents a possible choice and each edge points from the
winner of a pairwise comparison to the loser, a topological
sort will yield the voter's ordering. However, for intransitive
incomplete pairwise vote sets (hereon "messy" vote sets),
you need a more sophisticated method.

Ordering Induction

This paper presents a pattern for algorithms that produce or-
derings from messy vote sets, called "ordering induction"
algorithms. This algorithm will take a graph of one voter's
messy vote set as its input, and produce a ranking over all
choices based on the vote set. The algorithm proceeds in
three steps:

1. Remove votes such that the resulting vote set is transitive.

2. Add votes such that the resulting vote set is complete.

3. Topologically sort the vote graph.

This generic class of algorithm has many different imple-
mentations, using different methods for resolving intransi-
tivity and resolving incompleteness.

Resolving Intransitivity   An intransitive set of votes pro-
duces a matchup graph containing cycles, which cannot be
topologically sorted to produce a unique ordering. So, we
must remove the transitive votes from our messy pairwise
vote sets in order to produce a unique ordering. One trivial
algorithm for resolving intransitivity is to remove every vote
from the set. However, this is obviously not making good
use of the information that the voter has given us. The aim
is to produce the intransitive win-graph that is as close as
possible to the input set. We can measure the distance be-
tween a generated vote set and the original by counting the
number of edges removed. However, finding the minimum
number of edges that must be removed to remove all cycles
from a graph is the feedback arc set problem, a famously
NP-complete problem. (Karp 1972).

Instead of attempting to solve this NP-complete problem
once per voter, an impossible task for votes with many pos-
sible choices, we will explore three ways to remove intran-
sitivity. First is breaking random cycle by removing a ran-
dom edge, until there are no cycles left. The second method
breaks a random cycle by removing the edge that has the

lowest win-ratio across the entire voting population, assuming that it's the matchup that is the most likely to be mistaken while trying to rationalize the intransitive votes. The third method is conceptually derived from the Ranked Pairs social choice method. First, order each edge by win-ratio in the overall population, and then add them in order, only adding those that will not form cycles. While is similar to the second method, it will remove the weakest edge in the set of all cycle-forming edges first, while breaking the weakest edge in each cycle provides no such guarantee.

Resolving Incompleteness An incomplete set of votes does not have edges connecting to each node, and so produces an incomplete ordering when topologically sorted. Prior work has attempted to address this problem by asking their voters to submit more votes (Conitzer and Sandholm 2002). However, often there is no way to get additional votes from the voters, so having methods for filling in votesets without access to the voter is useful.

We have no information on the voter's preferences between the candidates we have to generate votes for, so we must resort to approximate methods. We will look at two. The simplest method would be to fill in missing edges randomly, ensuring that no cycles form. If the graph is only missing a few edges, this can be quite viable, as much of the structure of the ordering already exists, and we would just be deciding how ties are broken. However, if much of the graph is missing, such as if only one vote was cast, this method introduces a lot of noise. Another choice is to add in missing edges in order, starting from the edge that had the highest win ratio in the entire voting set. This assumes that the voter whose vote set we are filling would have been more likely to choose that edge.

Evaluating Ordering Induction

To evaluate ordering induction methods, we compare the output of different social choice mechanisms on the original pairwise vote set and the set of induced orderings produced by running an ordering induction method on each voter's vote set. This will not let us compare the scores for positional social choice mechanisms such as Borda count, as positional social choice mechanisms cannot be run directly on the pairwise vote set. However, if we can obtain similar results on the original dataset and the transformed one, we have at least some evidence that the preferences expressed by the orderings are similar to the preferences expressed by the original pairwise votes. We use Kendall's τ (Kendall 1938) to evaluate the distance between the Ranked Pairs result on the original pairwise voteset and the Ranked Pairs result on the induced ordering set.

Results We evaluate each combination of the proposed methods for resolving intransitivity and incompleteness. However, we must first choose a dataset to evaluate these methods on. PrefLib no datasets with voters who made both incomplete and intransitive pairwiser votes. So, we use the Northeastern Cute Dog Project's dataset (Zucker, d'Leeuwen, and Rassaby 2019). This dataset is from an online survey which offered users the opportunity to decide

Intransitivity	Incompleteness	τ (σ)
Break Random	Add Random	0.896 (0.020)
Break Random	Add By Win Ratio	0.988 (0.000)
Break Weakest	Add Random	0.917 (0.014)
Break Weakest	Add By Win Ratio	0.988 (0.000)
Add In Order	Add Random	0.936 (0.013)
Add In Order	Add By Win Ratio	0.988 (0.000)

Table 1: Results on the Dog Project dataset for each ordering induction method, averaged over 30 runs. τ and σ rounded to three places. 0.000 represents all runs having the same τ.

which of two dogs was cuter. Around one-third of voters in this dataset submitted intransitive preferences, and almost all voters did not vote on every pair.

Adding edges by win ratio in the overall population leads to the highest τ, and always produces a ranking with the same τ regardless of what intransitivity resolver is used. This indicates that, for this dataset at least, resolving incompleteness has a larger impact than resolving intransitivity. Seeing as only a third of voters had intransitive votes, while almost all has incomplete votes, this is easy to believe.

Conclusion

We have provided a way to construct ordering induction methods from intransitivity and incompleteness resolvers. Ordering induction was shown to be robust on a real-world dataset. This is a first step in resolving intransitive and incomplete vote sets without having access to the voters to gather more preferences, and enables the use of position-based social choice mechanisms on complex, real-world problems. This method can be applied to resolving disagreements between agents in multi-agent systems, even when the agents have intransitive and incomplete preferences.

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