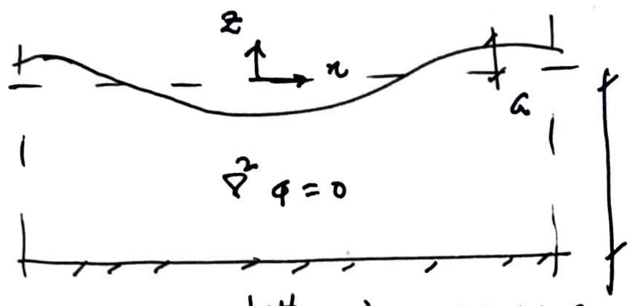
Linear theory of 2D progressive wave & wave dispersion

periodic boundary $x = x$



free surface: $z = 0$

periodic boundary: $x = x + \lambda$.

bottom boundary: $z = -h$

For inviscid, irrotational flow, velocity potential exists, $\phi = \phi(x, z, t)$. For incompressible flow, it satisfies, $\nabla^2 \phi = 0$ —(1)

- The velocity potential must conform to wave elevation of the form $\eta = \eta(x,t) = a\cos(kx - \omega t)$

and boundary conditions,

(1) periodic boundary conditions.

(2) bottom boundary condition, $z = -h$.

(3) free surface boundary condition, $z = 0$.

(1) Periodic boundary conditions.

$\eta = a\cos(kx - \omega t)$

$\eta = \eta(x, t) \equiv \eta(x+\lambda, t)$

or, $a\cos(kx - \omega t) = a\cos(k(x+\lambda) - \omega t)$

or, $a(\cos kx \cos \omega t + \sin kx \sin \omega t)$

~~$= a(\cos kx \cos k\lambda - \sin kx \sin k\lambda$~~

$= a(\cos(kx + k\lambda) . \cos \omega t + \sin(kx + k\lambda) . \sin \omega t)$

or, $\cos kx = \cos(kx + k\lambda)$ —(2)

$\sin kx = \sin(kx + k\lambda)$. —(3)

and,

$H/c \quad H_b \quad C_{9a} \quad . \quad b_a$

for both ② and ③ to be valid,

$$k\lambda = 2\pi \quad \text{or,}$$

$$\boxed{k = \frac{2\pi}{\lambda}}$$

periodic boundary condition for time,

$$\eta(x,t) = \eta(x, t+T)$$

$$a\cos(kx-\omega t) = a\cos(kx-\omega(t+T))$$

$$a\cos kx\cos\omega t + a\sin kx\sin\omega t = a\cos kx\cos(\omega t+\omega T) + a\sin kx\sin(\omega t+\omega T)$$

or,

$$\cos(\omega t) = \cos(\omega t+\omega T) \quad -④$$

$$\text{and}\quad \sin(\omega t) = \sin(\omega t+\omega T) \quad -⑤$$

both ④ and ⑤ to be valid, we must have,

$$\omega T = 2\pi,$$

or,

$$\boxed{T = \frac{2\pi}{\omega}}$$

Thus, the periodic boundary conditions do not give us anything more about the solution, rather than confirming that the 2D wave elevation is periodic in both space and time with wave number (k) and wave period (T) respectively.

Solution of the velocity potential from the Laplace ~~equation~~ equation

using separation of variables, we can get,

$$\phi = \phi(x,z,t) = \mathrm{Re}\left[Z(z)\cdot e^{-ikx+i\omega t}\right]$$

$$\therefore \frac{\partial\phi}{\partial x} = \mathrm{Re}\left[(-ik)\cdot Z(z)\cdot e^{-ikx+i\omega t}\right]$$

$$\frac{\partial^2\phi}{\partial x^2} = \mathrm{Re}\left[-k^2\cdot Z(z)\cdot e^{-ikx+i\omega t}\right]$$

$$\frac{\partial \phi}{\partial z} = Re\left[\frac{d Z}{d z} \cdot e^{-ikx + i\omega t}\right]$$

$$\frac{\partial^2 \phi}{\partial z^2} = Re\left[\frac{d^2 Z}{d z^2} \cdot e^{-ikx + i\omega t}\right]$$

Thus, from Laplace equation ① we get,

$$\frac{d^2 Z}{d z^2} - k^2 Z = 0. \quad \text{———} \quad ⑥$$

~~It~~ It's general solution is of the from,

$$Z = C \cdot e^{kz} + D \cdot e^{-kz} \quad \text{———} \quad ⑦$$

<u>The bottom boundary condition</u>

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=-h} = 0 \quad \text{———} \quad ⑧$$

or, $$\left. Re\left[\frac{d Z}{d z} \cdot e^{-ikx + i\omega t}\right]\right|_{z=-h} = 0$$

or, $$\left.\frac{d Z}{d z}\right|_{z=-h} = 0 \quad \text{———} \quad ⑨$$

<u>The free surface boundary condition</u>

4. <u>Dynamic free surface boundary condition</u>

$$p(\eta) = 0$$

$$p(\eta) = \cancel{p(0)}\ p(z)\Big|_{z=0} + \eta \frac{\partial}{\partial z}(p(z))\Big|_{z=0} + \cdots,$$

~~for~~ for Bernoulli's ~~eq~~ equation for inviscid, irrotation flow, valid throughout the fluid, we get,

H_x $\quad H_b^2 \cdot C_{9a} \cdot b_a$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + \frac{p}{\rho} + gz = 0.$$

or, $$p = -\rho\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2+w^2) + gz\right]$$

$$= \cancel{-\rho\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}\right)\right]}$$

$$= -\rho\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + gz\right]$$

By ~~linearizati~~ linearization,

$$0 = -\rho\left[\frac{\partial \phi}{\partial t} + gz\right]_{z=0} - \eta \cdot g \cdot \rho$$

or, $$\eta \cdot g = -\left.\frac{\partial \phi}{\partial t}\right|_{z=0}$$

or, $$\eta = -\frac{1}{g} \cdot \left.\frac{\partial \phi}{\partial t}\right|_{z=0}. \quad \text{— (10)}$$

3. Kinematic free surface boundary condition

A point on the free surface remains on the free surface.

$$\frac{D}{Dt}\eta = \frac{\partial}{\partial t}\eta + u\frac{\partial \eta}{\partial x} + w\frac{\partial \eta}{\partial z}$$

$\eta = \eta(x, t)$ for 2D wave,

or, $\frac{\partial \eta}{\partial z} = 0$.

$u \cdot \frac{\partial \eta}{\partial x}$ neglected through linearization.

thus, $$w = \frac{\partial \phi}{\partial z} = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t}$$

or, $$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = \frac{\partial \eta}{\partial t} \quad \text{— (11)}$$

Combining (10) and (11), we get,

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \quad \text{— (12)}$$

Equation (12) in brief is known as combined free surface boundary condition or CFSBC.

Finding constant C and D

applying ⑦ into ⑧ ⑨, we get,

$$C \cdot e^{-kh} - D \cdot e^{kh} = 0 \quad -⑬$$

applying ⑦ into ⑩, we get,

$$C + D = \frac{i a g}{\omega} \quad -⑭$$

by solving, we get,

$$C \cdot (e^{kh} + e^{-kh}) = \frac{i a g}{\omega} \cdot e^{kh}$$ and

$$D \cdot (e^{kh} + e^{-kh}) = \frac{i a g}{\omega} \cdot e^{-kh}$$

thus, $$Z(z) = \frac{i a g}{\omega} \cdot \frac{\cosh k(z+h)}{\cosh(kh)}$$

and, $$\phi = \mathrm{Re}\left[Z(z) \cdot e^{-ikx + i\omega t}\right]$$

$$= \frac{a g}{\omega} \cdot \frac{\cosh k(z+h)}{\cosh(kh)} \cdot \sin(kx - \omega t) \quad -⑮$$

Wave dispersion for finite depth

applying ⑮ to ⑫ we get,

$$\omega^2 = g k \tanh(kh) \quad -⑯$$

Wave potential for infinite depth

$$Z(z) = \cancel{\cosh} \cdot \cancel{\cosh} \quad C \cdot e^{kz} \quad -⑯$$

D = 0, since e^{-kz} increases as ~~z~~ ~~deer~~ z decreases. Thus we have at $Z(z)$ $z \to -\infty$

$$= C \cdot e^{kz} = 0.$$

$\frac{H}{k}$ $\frac{H_b^2}{}$ - $\frac{C_{ga}}{}$. $\frac{b_a}{}$

applying (14) into (15), we get,

$$C = \frac{iga}{\omega}.$$

thus, $$\phi(x,z,t) = \frac{ag}{\omega} \cdot e^{kz} \sin(kx-\omega t). \quad — (17)$$

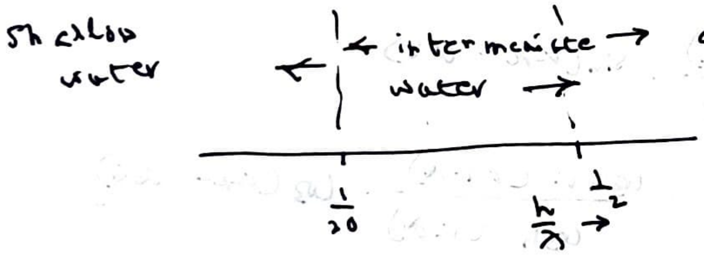
~~applying~~, applying (17) ~~into~~ ~~into~~ into (12), we get,

$$-\cancel{\omega^2}\cdot\frac{ag}{\cancel{\omega}}\cdot \cancel{e^{kz}} \cdot \sin(kx-\cancel{\omega t}) + g\cdot\frac{ag}{\omega}\cdot k\cdot \cancel{e^{kz}} \cdot \sin(kx-\cancel{\omega t}) = 0.$$

$$\omega = \frac{gk}{\omega} \quad \text{or,} \quad \omega^2 = gk \quad — (18).$$

Equation (18) is the wave dispersion relationship in infinite depth case. By using proper approximations for infinite depth for hyperbolic functions it can also be shown that equation (16) reduces to equation (18).

Approximating linear wave dispersion for deep water and shallow water conditions.



$$\omega^2 = gk.\tanh(hk)$$
$$= gk.\frac{e^{kh}-e^{-kh}}{e^{kh}+e^{-kh}}$$

for deep water h is very large, or, $\frac{h}{\lambda} >> 1$, thus $kh >> 1$, since

$$kh = \frac{2\pi h}{\lambda}.$$

$\approx$ Thus, $$\omega^2 = gk.\frac{1-e^{-2kh}}{1+e^{-2kh}}$$

$$= gk.\frac{1-\frac{1}{e^{2kh}}}{1+\frac{1}{e^{2kh}}} \approx gk.(1)$$

or, $$\omega^2 = gk \quad \text{— (1)}$$

Thus we can recove wave dispersion for deep water condition by applying suitable transformation to $\tanh(kh)$.

~~So,~~ On the other hand, for shallow water, $\frac{h}{\lambda} << 1$ or, $kh << 1$.

Thus, $$\omega^2 = gk \cdot \frac{e^{kh} - e^{-kh}}{e^{kh} + e^{-kh}}$$

$$\approx gk \cdot \frac{1 + kh - 1 + kh}{1 + kh + 1 - kh}$$

$$= gk \cdot (kh) = gk^2 h$$

or, $$c^2 = \frac{\lambda^2}{T^2} = gh, \quad \text{or,} \quad c = \sqrt{gh}$$

or, $$\lambda = T \cdot \sqrt{gh}. \quad \text{— (2)}$$

Velocity field under 2D progressive wave

$$\phi = \frac{ag}{\omega} \cdot \frac{\cosh k(z+d)}{\cosh kd} \cdot \sin(kx - \omega t)$$

$$u = \frac{\partial \phi}{\partial x} = \frac{gAk}{\omega} \cdot \frac{\cosh k(z+d)}{\cosh(kd)} \cdot \cos(kx - \omega t)$$

$$w = \frac{\partial \phi}{\partial z} = \frac{gak}{\omega} \cdot \frac{\sinh k(z+d)}{\cosh(kd)} \cdot \sin(kx - \omega t)$$

→ u and w is in phase difference of $\frac{\pi}{2}$.

Thus, when u is maximum, w is minimum and vice-versa.

$\underline{H_a}$ $\underline{H_b^2}$ - $\underline{C_{ga}}$. $\underline{b_a}$

Pressure field under a progressive wave in 2D

from Bernoulli's equation we get,

$$\left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + gz\right)\Bigg|_{z=z} = \left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + gz\right)\Bigg|_{z=\eta} \quad \text{—}(1)$$

Through the process of linearization, we get,

$$\cancel{\left(\frac{\partial\phi}{\partial t}\right)_{z=\eta} \approx \left(\frac{\partial\phi}{\partial t}\right)_{z=0}}$$

$$\left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + gz\right)\Bigg|_{z=\eta} \approx \left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + gz\right)\Bigg|_{z=0} + \eta\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + gz\right)\Bigg|_{z=0}$$

~~from Dynamic free surface boundary condition~~

$\cancel{p(z=\eta)=0}$

From dynamic free surface boundary condition, we get,

$p(z=\eta) = 0$. and, $\eta = -\frac{1}{g}\cdot\frac{\partial\phi}{\partial t}\Big|_{z=0}$

and also, $\frac{\partial\phi}{\partial t}\Big|_{z=\eta} \approx \frac{\partial\phi}{\partial t}\Big|_{z=0}$, since η is very small.

Thus, from equation (1), we get,

$$\left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + gz\right)\Bigg|_{z=z} = \left(\frac{\partial\phi}{\partial t}\right)_{z=0} - g\cdot\frac{1}{g}\cdot\frac{\partial\phi}{\partial t}\Bigg|_{z=0}$$

$$= 0$$

or, $$\frac{p}{\rho} = -gz - \frac{\partial\phi}{\partial t}$$

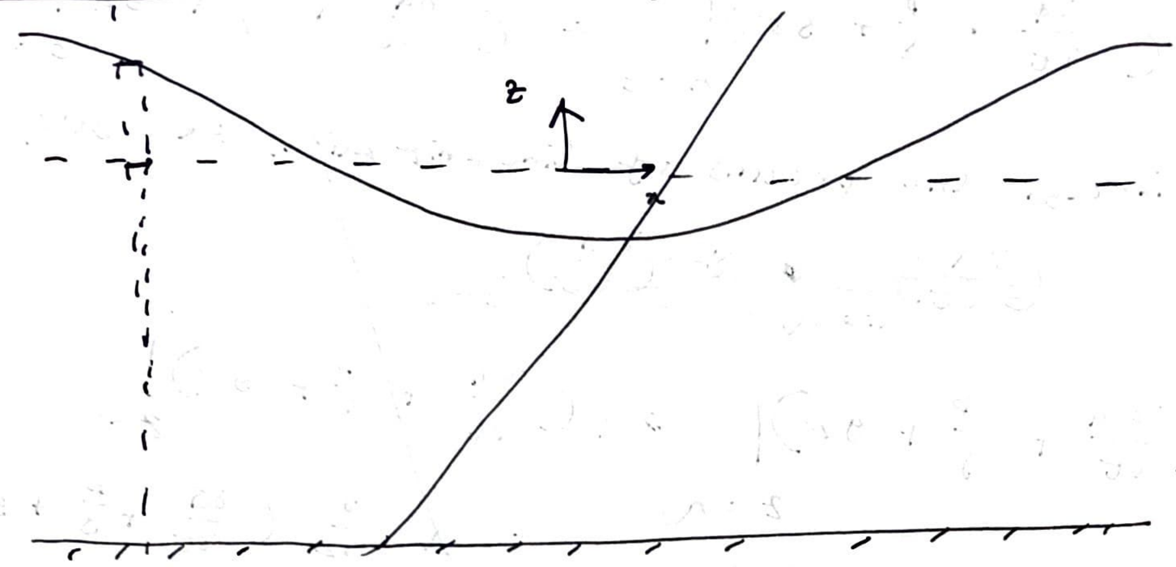
or, $$p = \underbrace{-\rho g z}_{\text{hydrostatic part}} + \underbrace{\rho a g\cdot K_p(z)\cdot\cos(kx-\omega t)}_{\text{Dynamic Part}}$$

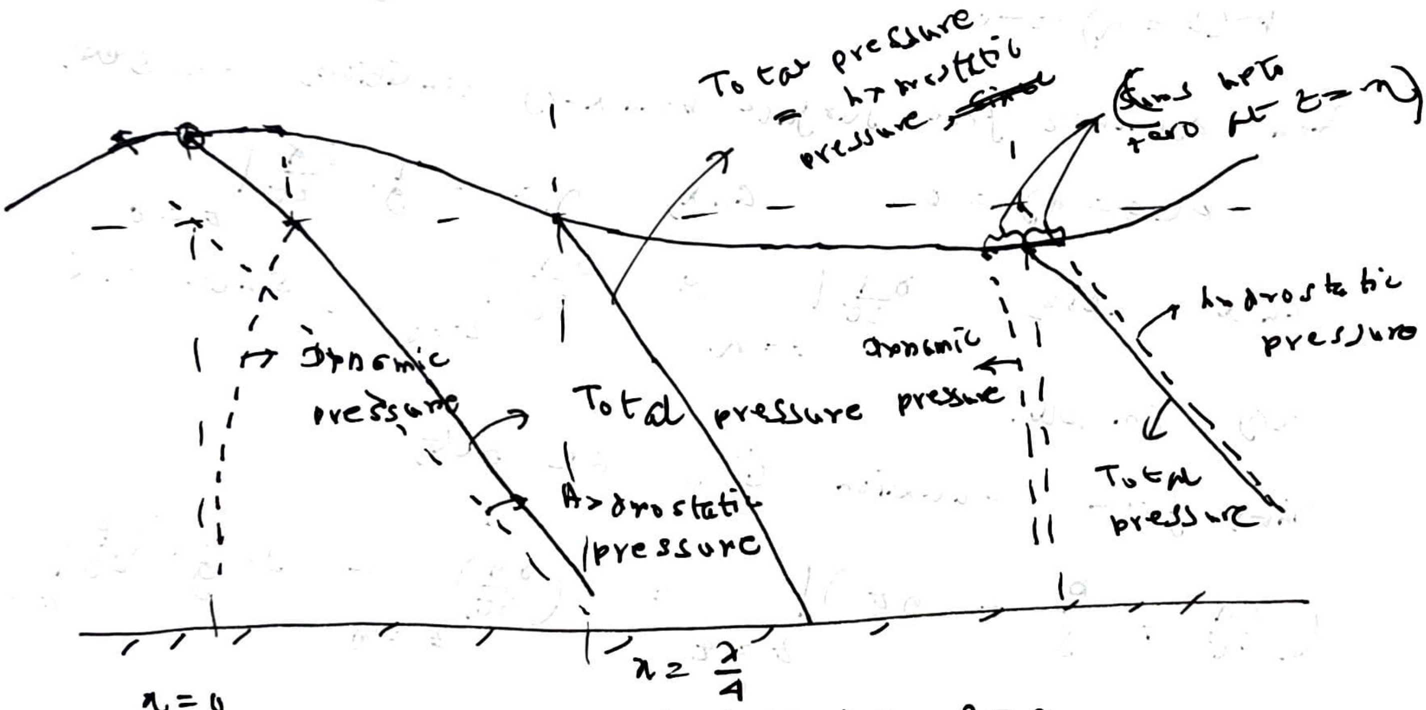
(using the solution of $\phi(x,z,t)$ defined earlier)

where, $K_p(z)$ is pressure response factor.

and define as,

$$k_p(z) = \frac{\cosh k(h+z)}{\cosh(kh)}$$





at free surface, i.e., $z = \eta$,

$$\cancel{p(z=\eta)=0} \neq \cancel{\rho g z} + \frac{\partial \phi}{\partial t}$$

$$\cancel{p(z=\eta) = -\rho g z}$$

$$\cancel{p(z=\eta) = -\rho g \eta + \rho \frac{\partial \phi}{\partial t}\Big|_{z=0}} = 0$$

$$p(z=\eta) = -\rho g \eta - \rho \frac{\partial \phi}{\partial t}\Big|_{z=0} = 0$$

H_a H_b^2 C_{ga} . b_a

- at $z = \eta$, the hydrostatic part is cancelled by the dynamic part and thus we have $p(z=\eta)=0$.

~~under the node, i.e., at $x = \frac{\lambda}{4}$, the total pressure is purely hydrostatic, since,~~

$$\cancel{\cos(kx - \omega t)\Big|_{x=\frac{\lambda}{4}} = \cos\left(\frac{\pi}{2} - \omega t\right) = \cos\frac{\pi}{2} \cdot \sin\omega t + \cos\omega t \cdot \sin\frac{\pi}{2} =}$$

under the node, i.e., at $x = \frac{\lambda}{4}$, the total pressure is purely hydrostatic, since,

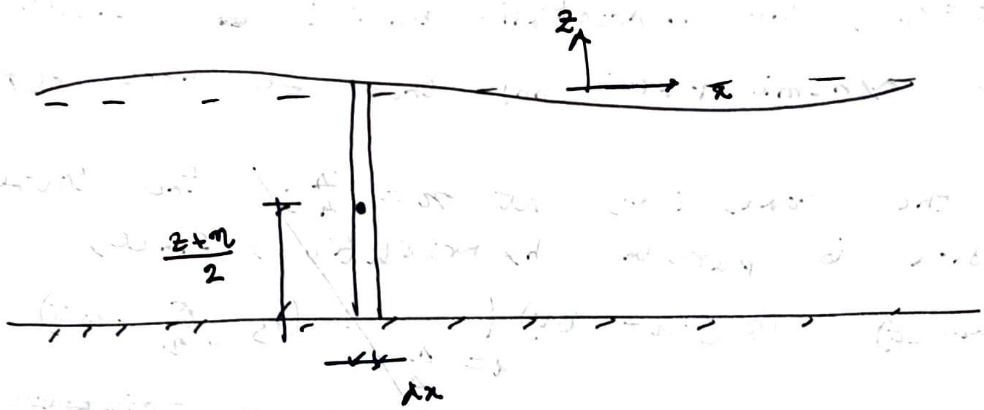
$$\cos(kx - \omega t)\Big|_{x=\frac{\lambda}{4},\, t=0} = 0 ,$$

~~thus~~ for the dynamic pressure is zero across the depth.

Wave Energy

For a progressive periodic wave, there is a movement of the free surface. Thus there is a potential energy. ~~___~~ On the other hand, as the wave propagates, there is a movement ~~___~~ of a fluid element due to the existence of a velocity field. Thus there is a ~~potential~~ kinetic energy.

Estimation of wave energy is required for ~~as assessing~~ assessing power required to drive a wave maker in a flume for doing experiments and for estimating ~~___~~ availability of wave power while planning a device for ~~transforming~~ ~~energy~~ transforming wave energy into other form of energy.



for ~~accont~~ a water column of ~~width~~ width dx and thickness along y axis as 1, the potential energy is

$$d(PE) = dm \cdot g \cdot \frac{h+\eta}{2}$$

$$\& \quad dm = \rho \cdot (h+\eta) \cdot dx$$

$$\therefore \quad d(PE) = \rho g \frac{(h+\eta)^2}{2} \, dx$$

$$= \frac{1}{2} \rho g \left(h^2 + 2h\eta + \eta^2 \right) dx$$

The average potential energy over a wave length

$$(\overline{PE}) = \frac{1}{\lambda} \int_x^{x+\lambda} \left(\frac{1}{2} \rho g \right) \left(h^2 + 2h\eta + \eta^2 \right) dx$$

$$= \left(\frac{1}{2} \right) \cdot \left(\frac{\rho g}{\lambda} \right) \cdot \int_x^{x+\lambda} \left(h^2 + 2h\, a \cos(kx - \omega t) + a^2 \cos^2(kx - \omega t) \right) dx$$

Now,

$$\int_x^{x+\lambda} h^2 \, dx = \cancel{0} \; h^2 . \lambda .$$

$$\int_x^{x+\lambda} 2h \, a \cos(kx - \omega t) \, dx = 0$$

$$a^2 \int_x^{x+\lambda} \cos^2(kx - \omega t) \, dx = \frac{a^2 \pi}{k}$$

$\frac{H_x}{}$ $\frac{H_b^2}{}$ $\frac{C_{ga}}{}$ $\frac{b_a}{}$

$$\therefore \frac{\rho g}{\lambda} \cdot \frac{1}{2} \cdot \frac{a^2 \cancel{\lambda}}{k} = \frac{\rho g}{2\cancel{\lambda}} \cdot \frac{a^2 \cancel{\lambda}}{\frac{2\pi}{\cancel{\lambda}}} = \frac{\rho g a^2}{4}$$

(Thus total energy is $\left(\frac{\rho g h^2}{2} + \frac{\rho g a^2}{4}\right)$)

Similarly, kinetic energy of a small fluid element of size $dx\,dz$ at a depth z and at a distance x is

$$d(KE) = dm \cdot \frac{u^2 + w^2}{2} \cdot \cancel{dx\,dz} = \rho \frac{u^2 + w^2}{2} \cdot dx\,dz$$

per unit thickness across y axis.

Thus total potential energy due to waves is
$$\frac{\rho g h^2}{2} + \frac{\rho g a^2}{4} - \frac{\rho g h^2}{2} = \frac{\rho g a^2}{4}$$

Thus average kinetic energy is

$$(\overline{KE}) = \frac{1}{\lambda} \int_x^{x+\lambda} \int_{-h}^{\eta \approx 0} \rho \frac{u^2 + w^2}{2} \cdot dx\,dz$$

substitution for u and w gives,

$$(\overline{KE}) = \frac{1}{\lambda} \int_x^{x+\lambda} \int_{-h}^{\eta \approx 0} \rho \cdot \left(\frac{1}{2}\right) \left(\frac{g^2 a^2 k^2}{\omega^2} \cdot \frac{\cosh^2 k(z+h)}{\cosh^2 (kh)} \cdot \cos^2 (kx - \omega t) + \frac{g^2 a^2 k^2}{\omega^2} \cdot \frac{\sinh^2 k(z+h)}{\cosh^2 (kh)} \sin^2 (kx - \omega t) \right) dx\,dz$$

$$= \frac{1}{\lambda} \int_x^{x+\lambda} \int_{-h}^{\eta \approx 0} \frac{g^2 a^2 k^2}{\omega^2} \cdot \frac{1}{\cosh^2 (kh)} \cdot \left[\cosh^2 k(h+z) \cos^2 (kx - \omega t) + \sinh^2 k(h+z) \cdot \sin^2 (kx - \omega t) \right] dx\,dz$$

$$= \frac{\rho g a^2}{4}$$

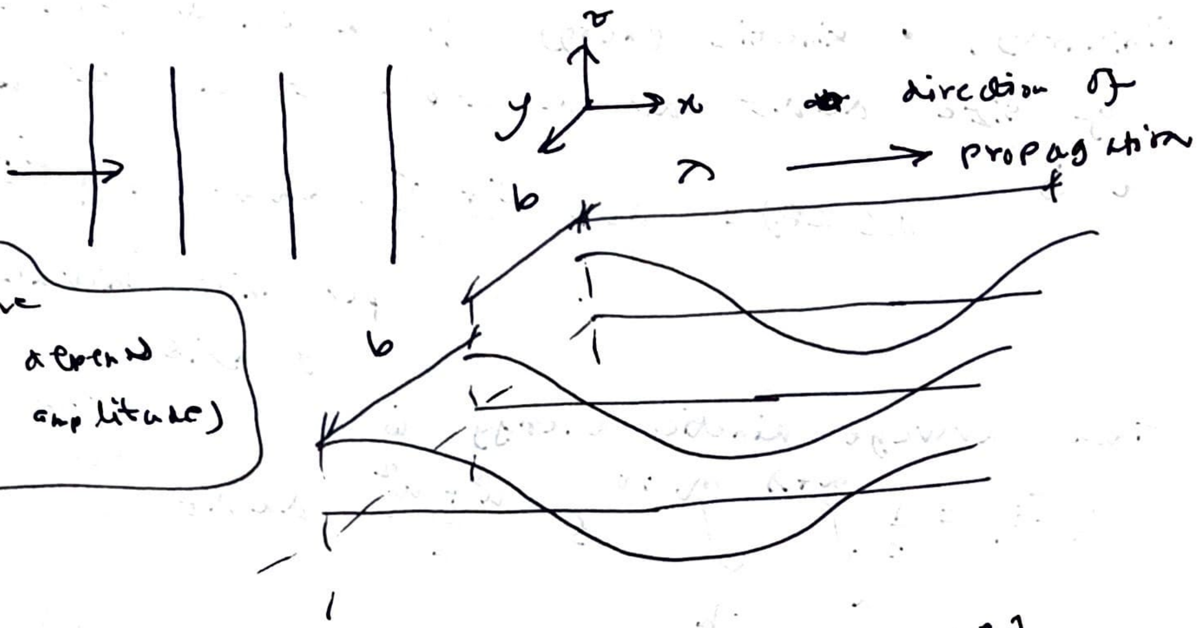
Thus, the total wave energy per wave length and per unit thickness across y axis is

$$E = \frac{\rho g a^2}{4} + \frac{\rho g a^2}{4} = \frac{\rho g a^2}{2} = \frac{\rho g H^2}{8}$$

This ~~is also~~ may also be defined as wave energy per unit surface area (i.e., ~~per~~ $\lambda \cdot b$)

where b may be ~~as~~ a measure of separation in between identical ~~ac~~ wave trains.

+ (Total wave energy only depends on wave amplitude)



$$[E] = [\tfrac{1}{2}\rho g a^2] \equiv [ML^{-3}].[LT^{-2}].[L^2] = [M T^{-2}]$$

$[E]$ as we know is $[MLT^{-2}].[L] \equiv [M L^2 T^{-2}]$

So, $[E][L^{-2}] \equiv [M L^2 T^{-2}][L^{-2}] \equiv [M T^{-2}]$

Some details about the kinetic wave energy derivation

$$(\overline{KE}) = \frac{1}{\lambda}\int_x^{x+\lambda}\int_{-h}^{\eta\approx 0}\left(\frac{\rho}{2}\right).\left[\frac{g^2 a^2 k^2}{\omega^2}.\frac{\cosh^2 k(h+z)}{\cosh^2(kh)}.\cos^2(kx-\omega t) + \frac{g^2 a^2 k^2}{\omega^2}.\frac{\sinh^2 k(h+z)}{\cosh^2(kh)}.\sin^2(kx-\omega t)\right] dx\,dz$$

$$= \left(\frac{\rho}{2\lambda}\right).\frac{g^2 a^2 k^2}{\omega^2}.\frac{1}{\cosh^2(kh)}.\int_x^{x+\lambda}\int_{-h}^{\eta\approx 0}\left[\cosh^2 k(h+z).\cos^2(kx-\omega t) + \sinh^2 k(h+z).\sin^2(kx-\omega t)\right] dx\,dz$$

We observe,

$$\cosh^2\alpha.\cos^2\theta + \sinh^2\alpha.\sin^2\theta$$

$$= \left(\frac{e^{\alpha}+e^{-\alpha}}{2}\right)^2.\cos^2\theta + \left(\frac{e^{\alpha}-e^{-\alpha}}{2}\right)^2.\sin^2\theta$$

$\underline{H_x} \quad \underline{H_b^2} - \underline{C_{9a}} \cdot \underline{b_a}$

$$= \frac{e^{2\alpha} + 2 + e^{-2\alpha}}{4} \cdot \cos^2\theta + \frac{e^{2\alpha} - 2 + e^{-2\alpha}}{4} \cdot \sin^2\theta$$

$$= \frac{e^{2\alpha} + e^{-2\alpha}}{4} \cdot \cos^2\theta + \frac{e^{2\alpha} + e^{-2\alpha}}{4} \cdot \sin^2\theta + \frac{2}{4} \cdot (\cos^2\theta - \sin^2\theta)$$

$$= \frac{1}{2} \cdot \cosh(2\alpha) + \frac{1}{2} (\cos 2\theta)$$

thus, $$(\overline{KE}) = \left(\frac{8}{2\lambda}\right) \cdot \frac{\rho^2 a^2 K^2}{\omega^2} \cdot \frac{1}{\cosh^2(Kh)} \cdot \int_x^{x+\lambda} \int_{-h}^{\eta\approx 0} \frac{1}{2} \left(\cosh 2K(h+z) + \cos 2(Kx - \omega t)\right) dx\, dz$$

Now, $$\int_x^{x+\lambda} \int_{-h}^{0} \cosh 2K(h+z)\, dz\, dx$$

Let, $2K(h+z) = \alpha$.

or, $2K \cdot dz = d\alpha$

$dz = \frac{d\alpha}{2K}$

z	$-h$	0
α	0	$2Kh$

$$= \frac{1}{2K} \int_x^{x+\lambda} \int_0^{2Kh} \cosh(\alpha) \cdot d\alpha \cdot dx$$

$$= \frac{1}{2K} \int_x^{x+\lambda} \left[\sinh(\alpha)\right]_0^{2Kh} dx$$

$$= \frac{1}{2K} \int_x^{x+\lambda} \sinh(2Kh)\, dx$$

$$= \frac{1}{2K} \cdot \sinh(2Kh) \cdot \lambda.$$

and, $$\int_x^{x+\lambda} \int_{-h}^{0} \cos 2(Kx - \omega t)\, dx\, dz$$

$$= \int_{-h}^{0} \int_x^{x+\lambda} \cos 2(Kx - \omega t)\, dx\, dz$$

$2(Kx - \omega t) = \theta.$

$2K\, dx = d\theta$

$dx = \frac{d\theta}{2K}$

x	x	$x+\lambda$
θ	$2(Kx-\omega t)$	$2(2\pi + Kx - \omega t)$

$$= \int_{-h}^{0} \int_{2(Kx - \omega t)}^{2(2\pi + Kx - \omega t)} \cos\theta \quad d\theta \quad dz$$

$$= \int_{-h}^{0} \left[\sin\theta\right]_{2(Kx-\omega t)}^{2(2\pi + Kx - \omega t)} \cdot dz$$

$$= 0$$

$2(Kx + K\lambda - \omega t)$

$= 2(Kx + 2\pi - \omega t)$

$= 2(2\pi + Kx - \omega t)$

$=$

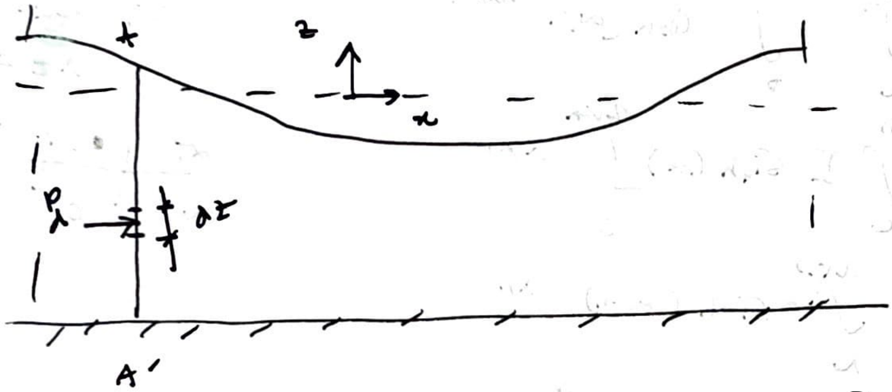
Thus, $(KE) = \left(\frac{\rho}{2}\right) \cdot \frac{g^2 a^2 k^2}{\omega^2} \cdot \frac{1}{\cosh^2(kh)} \cdot \frac{1}{4k} \cdot \sinh(2kh) \cdot \lambda$

$$= \left(\frac{\rho}{\cancel{2}}\right) \cdot \frac{g^{\cancel{2}} a^2 \cancel{k}}{\cancel{gk} \tanh(kh)} \cdot \frac{1}{\cosh^2(kh)} \cdot \frac{1}{\cancel{4}} \cdot \cancel{2} \sinh(kh) \cosh(kh)$$

$$= \frac{\rho g a^2}{4} \cdot \frac{\sinh(kh) \cdot \cosh(kh)}{\sinh(kh) \cdot \cosh(kh)}$$

$$= \frac{\rho g a^2}{4}.$$

Idea about wave energy flux



Rate at which work is done by fluid on one side of section A-A' to the other side is

$d(W) = p_d(z) \cdot u \, dz$ per unit width across y axis.

Thus, the time average rate of work done is defined as energy flux and given as,

$$\hat{E} = \frac{1}{T} \int_t^{t+T} \int_{-h}^{\eta \approx 0} p_d(z) \cdot u \, dz \, dt$$

$$= \frac{1}{T} \int_t^{t+T} \int_{-h}^{\eta \approx 0} \rho g \, K_p(z) \cdot \eta \cdot \frac{g a k}{\omega} \cdot \frac{\cosh k(h+z)}{\cosh k(h)} \cos(kx - \omega t) \, dz \, dt$$

$$= \frac{1}{T} \cdot \frac{\rho a^2 g^2 k}{\omega} \int_t^{t+T} \int_{-h}^{\eta \approx 0} \frac{\cosh^2 k(h+z)}{\cosh^2(kh)} \cdot \cos^2(kx - \omega t) \, dz \, dt$$

$\underline{H_a} \quad \underline{H_b^2} - \underline{C_{ga}} \cdot \underline{b_a}$

$$= \frac{\rho g^2 a^2 k}{2\, \omega T} \frac{1}{\cosh^2(kh)} \cdot \int_t^{t+T} \int_{-h}^{0} \cosh^2 k(h+z) \cdot \cos^2(kx - \omega t)\, dz\, dt$$

We find, $\int_t^{t+T} \int_{-h}^{0} \cosh^2 k(h+z) \cos^2(kx - \omega t)\, dz\, dt$

$$= \int_{kx-\omega t}^{kx-\omega t-2\pi} \int_0^{kh} \cosh^2(\alpha) \cdot \cos^2(\theta) \frac{d\alpha}{k} \left(-\frac{d\theta}{\omega}\right)$$

Let, $k(h+z) = \alpha$.

or, $k\, dz = d\alpha$.

$dz = \frac{d\alpha}{k}$.

z	$-h$	0
α	0	kh

$$= -\frac{1}{\omega k} \cdot \left(\frac{1}{2}\right) \cdot \int_{kx-\omega t}^{kx-\omega t-2\pi} \int_0^{kh} (1 + \cosh(2\alpha)) \cos^2\theta\, d\alpha\, d\theta$$

$kx - \omega t = \theta$

or, $-k\, \omega\, dt = d\theta$

$dt = -\frac{d\theta}{\omega}$

t	t	$t+T$
θ	$kx-\omega t$	$kx-\omega t-2\pi$

$$= -\frac{1}{2\omega k} \cdot \int_{kx-\omega t}^{kx-\omega t-2\pi} \left[\alpha + \frac{\sinh(2\alpha)}{2}\right]_0^{kh} \cdot \cos^2\theta\, d\theta$$

$$= -\frac{1}{2\omega k} \cdot \left(kh + \frac{\sinh(2kh)}{2}\right) \cdot \left(\frac{1}{2}\right) \int_{kx-\omega t}^{kx-\omega t-2\pi} (1 + \cos 2\theta)\, d\theta$$

$$= -\frac{1}{2\omega k} \cdot \frac{1}{2} \cdot \left(kh + \frac{\sinh(2kh)}{2}\right) \cdot \left[\theta + \frac{\sin 2\theta}{2}\right]_{kx-\omega t}^{kx-\omega t-2\pi}$$

$$= -\frac{1}{4\omega k} \cdot \left(kh + \frac{\sinh(2kh)}{2}\right) \left(\cancel{kx} - \cancel{\omega t} - 2\pi + \frac{\sin 2(kx - \omega t - 2\pi)}{2} - \cancel{kx} + \cancel{\omega t} - \frac{\sin 2(kx - \omega t)}{2}\right)$$

$$= \frac{1}{4\omega k} \cdot \left(kh + \frac{\sinh(2kh)}{2}\right) \cdot (2\pi)$$

Thus, $$\hat{E} = \frac{\rho a^2 g^2 k}{\cancel{2\pi}} \cdot \frac{1}{\cosh^2(kh)} \cdot \frac{1}{4\omega \cancel{k}} \cdot \left(kh + \frac{\sinh(2kh)}{2}\right) \cdot \cancel{2\pi}$$

$$= \rho a^2 g^2 \cdot \frac{\omega}{4\omega^2 \cosh^2(kh)} \cdot kh \left(1 + \frac{\sinh(2kh)}{2kh}\right)$$

$$= \frac{\rho a^2 g^2 \omega}{4\omega^2 \cosh^2(kh)} \cdot \cancel{kh} \cdot \frac{\sinh(2kh)}{2\cancel{kh}} \left(1 + \frac{2kh}{\sinh(2kh)}\right)$$

$$= \frac{\rho a^2 s^2 \omega}{\rho \omega^2 \cosh^2(kh)} \cdot \sinh(2kh) \cdot \frac{1}{2}\left(1+\frac{2kh}{\sinh(2kh)}\right)$$

$$= \frac{\rho a^2 s^2 \omega \cdot 2 \sinh(kh)\cosh(kh)}{4 \cdot g \cdot k \cdot \tanh(kh) \cdot \cosh^2(kh)} \cdot \frac{1}{2}\left(1+\frac{2kh}{\sinh(2kh)}\right)$$

$$= \left(\frac{\omega}{k}\right) \cdot \left(\frac{\rho g a^2}{2}\right) \cdot \frac{2 \sinh(kh)\cosh(kh)}{\sinh(kh) \cdot \cosh(kh)} \cdot \frac{1}{2}\left(1+\frac{2kh}{\sinh(2kh)}\right)$$

$$= c \cdot \left(\frac{\rho g a^2}{2}\right) \cdot \frac{1}{2}\left(1+\frac{2kh}{\sinh(2kh)}\right)$$

$$= E \cdot c_g,$$

where, $c_g = \frac{1}{2}\left(1+\frac{2kh}{\sinh(2kh)}\right) \cdot c = n \cdot c$.

c_g is known as the group velocity. This is the velocity at which energy travels through the free surface due to wave propagation.

In shallow water, n asymptotes to 1,

since, $\frac{1}{2}\left(1+\frac{2kh}{\sinh(2kh)}\right) \approx \frac{1}{2}\left(1+\frac{2kh}{\frac{1+2kh-1+2kh}{2}}\right)$

$= 1$ if $kh \ll 1$ or, $h < \lambda$.

On the other hand, in deep water n asymptotes towards $\frac{1}{2}$, since, $\frac{1}{2}\left(1+\frac{2kh}{\sinh(2kh)}\right)$

$$= \frac{1}{2}\left(1+\frac{2kh}{\frac{1-e^{-4kh}}{2} \cdot e^{2kh}}\right)$$

$\approx \frac{1}{2}$ if $h \gg \lambda$ as in deep water.

Thus the wave energy travels much slower in deep water than in shallow water.

$\frac{H}{\cancel{x}}$ H_b^2 - C_{ga} . b_a

why this velocity is called as group velocity?

because, we have, $\omega = \omega(k)$ for water wave and for wave dispersion they according to,

$$\omega^2 = gk\tanh(kh). \quad —①$$

from ① it can be shown that, $\frac{d\omega}{dk} = c_g$.

and now, if we have two wave systems with frequency and ~~amplitude~~ wave number being very close to ω and k as, $k_1 = k - \frac{dk}{2}$ and, $k_2 = k + \frac{dk}{2}$; $\omega_1 = \omega - \frac{d\omega}{2}$ and $\omega_2 = \omega + \frac{d\omega}{2}$, then we can define the two wave systems as,

$$\eta_1 = a\cos(k_1 x - \omega_1 t) \text{ and,}$$
$$\eta_2 = a\cos(k_2 x - \omega_2 t).$$

The superposition of η_1 and η_2 will lead to modulated wave profile over space. The envelope of this modulated wave profile travels at the speed of c_g and carries the energy of the main wave of frequency ω and wave number k or wave length λ.