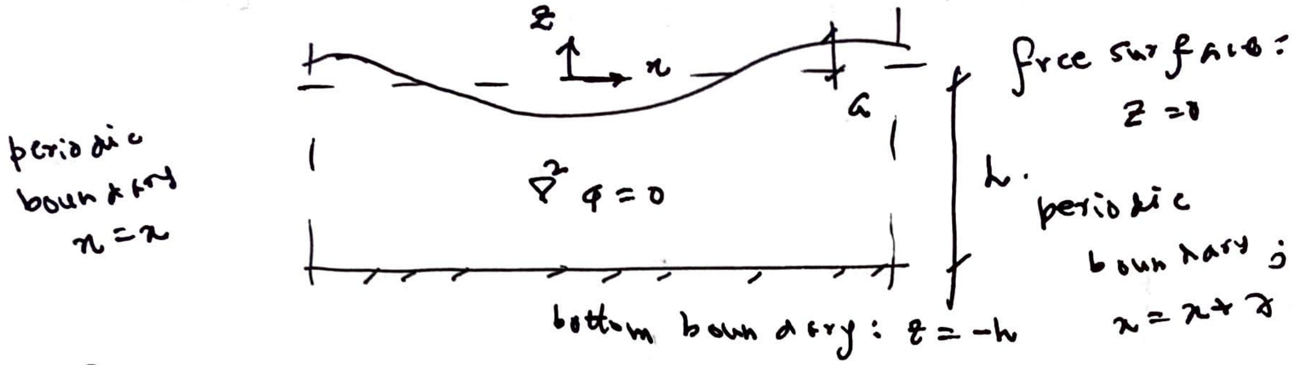
Linear theory of 2D progressive wave and wave dispersion



For inviscid, irrotational flow, velocity potential exists, $\phi = \phi(x, z, t)$. For incompressible flow, it satisfies, $\nabla^2 \phi = 0$ — ①

- The velocity potential must conform to wave elevation of the form $\eta = \eta(x,t) = a\cos(kx - \omega t)$ and boundary conditions,

① periodic boundary conditions.

② bottom boundary condition, $z = -h$.

③ free surface boundary condition, $z = 0$.

① Periodic boundary conditions.

$\eta = a\cos(kx - \omega t)$

$\eta = \eta(x, t) \equiv \eta(x+\lambda, t)$

or, $a\cos(kx - \omega t) = a\cos(k(x+\lambda) - \omega t)$

or, $a(\cos kx \cos \omega t + \sin kx \sin \omega t)$

~~$= a(\cos kx \cos k\lambda - \sin kx \sin k\lambda$~~

$= a(\cos(kx+k\lambda) \cdot \cos \omega t + \sin(kx+k\lambda) \cdot \sin \omega t)$

or, $\cos kx = \cos(kx + k\lambda)$ — ②

and, $\sin kx = \sin(kx + k\lambda)$. — ③

for both (2) and (3) to be valid,

$$k\lambda = 2\pi \cdot 1\pi,$$

$$\boxed{k = \frac{2\pi}{\lambda}}$$

periodic boundary condition for time,

$$\eta(x,t) = \eta(x, t+T)$$

$$a \cos(kx - \omega t) = a \cos(kx - \omega(t+T))$$

$$a \cos kx \cos \omega t + a \sin kx \sin \omega t = a \cos kx \cos(\omega t + \omega T)$$
$$+ a \sin kx \sin(\omega t + \omega T)$$

or,

$$\cos(\omega t) = \cos(\omega t + \omega T) \quad -(4)$$

$$\text{and} \quad \sin(\omega t) = \sin(\omega t + \omega T) \quad -(5)$$

both (4) and (5) to be valid, we must have,

$$\omega T = 2\pi,$$

$$\text{or,} \quad \boxed{T = \frac{2\pi}{\omega}}$$

Thus, the periodic boundary conditions do not give us anything more about the solution, rather than confirming that the 2D wave elevation is periodic in both space and time with wave number (k) and wave period (T) respectively.

Solution of the velocity potential from the Laplace equation

using separation of variables, we can get,

$$\phi = \phi(x,z,t) = Re\left[Z(z) \cdot e^{-ikx + i\omega t}\right]$$

$$\therefore \frac{\partial \phi}{\partial x} = Re\left[(-ik) \cdot Z(z) \cdot e^{-ikx + i\omega t}\right]$$

$$\frac{\partial^2 \phi}{\partial x^2} = Re\left[-k^2 \cdot Z(z) \cdot e^{-ikx + i\omega t}\right]$$

$$\frac{\partial \phi}{\partial z} = Re\left[\frac{dZ}{dz} \cdot e^{-ikx + i\omega t}\right]$$

$$\frac{\partial^2 \phi}{\partial z^2} = Re\left[\frac{d^2 Z}{dz^2} \cdot e^{-ikx + i\omega t}\right]$$

Thus, from Laplace equation ① we get,

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \quad \text{—— ⑥}$$

~~Et~~ Its general solution is of the form,

$$Z = C \cdot e^{kz} + D \cdot e^{-kz} \quad \text{—— ⑦}$$

<u>The bottom boundary condition</u>

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=-h} = 0 \quad \text{—— ⑧}$$

or, $$Re\left[\frac{dZ}{dz} \cdot e^{-ikx + i\omega t}\right]\Big|_{z=-h} = 0$$

or, $$\left.\frac{dZ}{dz}\right|_{z=-h} = 0 \quad \text{—— ⑨}$$

<u>The free surface boundary condition</u>

4. <u>Dynamic free surface boundary condition</u>

$$p(\eta) = 0$$

$$p(\eta) = \left.p(z)\right|_{z=0} + \eta \left.\frac{\partial}{\partial z}(p(z))\right|_{z=0} + \cdots,$$

~~for~~ for Bernoulli's ~~eq~~ equation for inviscid, irrotation flow, valid throughout the fluid, we get,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + \frac{p}{\rho} + gz = 0.$$

or, $$p = -\rho\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + gz\right]$$

$$= -\rho\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + gz\right]$$

By ~~linear~~ linearization,

$$0 = -\rho\left[\frac{\partial \phi}{\partial t} + gz\right]_{z=0} - \eta \cdot g \cdot \rho$$

or, $$\eta \cdot g = -\left.\frac{\partial \phi}{\partial t}\right|_{z=0}$$

or, $$\eta = -\frac{1}{g}\cdot\left.\frac{\partial \phi}{\partial t}\right|_{z=0}. \quad \text{— (10)}$$

3. <u>Kinematic free surface boundary condition</u>

A point on the free surface remains on the free surface.

$$\frac{D}{Dt}\eta = \frac{\partial}{\partial t}\eta + u\frac{\partial \eta}{\partial x} + w\frac{\partial \eta}{\partial z}$$

$\eta = \eta(x, t)$ for 2D wave,

or, $\frac{\partial \eta}{\partial z} = 0.$

$u \cdot \frac{\partial \eta}{\partial x}$ neglected through linearization.

thus, $$w = \frac{\partial \phi}{\partial z} = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t}$$

or, $$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = \frac{\partial \eta}{\partial t} \quad \text{— (11)}$$

Combining (10) and (11), we get,

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \quad \text{— (12)}$$

Equation (12) in brief is known as combined free surface boundary condition or CFSBC.

Fixing constant C and D

applying ⑦ into ⑨, we get,

$$C \cdot e^{-kh} - D \cdot e^{kh} = 0 \quad — (13)$$

applying ⑦ into ⑩, we get,

$$C + D = \frac{i a g}{\omega} \quad — (14)$$

by solving, we get,

$$C \cdot \left(e^{kh} + e^{-kh} \right) = \frac{i a g}{\omega} \cdot e^{kh}$$

and

$$D \cdot \left(e^{kh} + e^{-kh} \right) = \frac{i a g}{\omega} \cdot e^{-kh}$$

thus, $$Z(z) = \frac{i a g}{\omega} \cdot \frac{\cosh k(z+h)}{\cosh(kh)}$$

and, $$\phi = \mathrm{Re}\left[Z(z) \cdot e^{-ikx + i\omega t} \right]$$

$$= \frac{a g}{\omega} \cdot \frac{\cosh k(z+h)}{\cosh(kh)} \cdot \sin(kx - \omega t) \quad — (15)$$

wave dispersion for finite depth

applying (15) to (12) we get,

$$\omega^2 = gk \tanh(kh) \quad — (16)$$

wave potential for infinite depth

$$Z(z) = \cancel{\cosh} \cdot \cancel{\cosh} \; C \cdot e^{kz} \quad — (16)$$

$D = 0$, since e^{-kz} increases as ~~z inc~~ z decreases. Thus we have at $Z(z)$ $z \to -\infty$

$$= C \cdot e^{kz} = 0.$$

H_s H_b^2 C_{ga} . b_a

applying (16) into (15), we get,

$$C = \frac{i g a}{\omega}.$$

thus, $\phi(x,z,t) = \frac{ag}{\omega} . e^{kz} \sin(kx - \omega t)$. — (17)

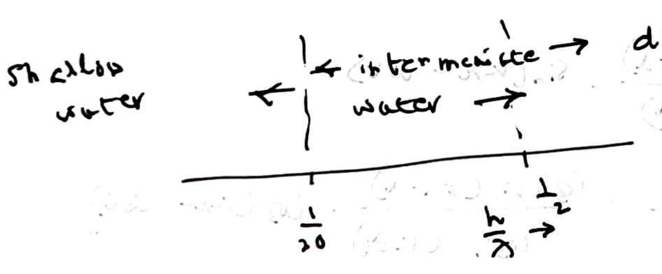
~~applying,~~ applying (17) ~~into~~ ~~into~~ into (12), we get,

$$-\omega^2 . \frac{ag}{\omega} . e^{kz} . \sin(kx - \omega t) + g . \frac{ag}{\omega} . k . e^{kz} . \sin(kx - \omega t) = 0.$$

$$\omega = \frac{gk}{\omega} \quad \text{or,} \quad \omega^2 = gk \quad — (18)$$

Equation (18) is the wave dispersion relationship in infinite depth case. By using proper approximations for infinite depth for hyperbolic functions it can also be shown that equation (16) reduces to equation (18).

Approximating linear wave dispersion for deep water and shallow water conditions.



$$\omega^2 = gk . \tanh(hk)$$

$$= gk . \frac{e^{kh} - e^{-kh}}{e^{kh} + e^{-kh}}$$

for deep water h is very large, or, $\frac{h}{\lambda} \gg 1$, thus $kh \gg 1$, since

$$kh = \frac{2\pi h}{\lambda}.$$

Thus, $$\omega^2 = gk . \frac{1 - e^{-2kh}}{1 + e^{-2kh}}$$

$$= gk . \frac{1 - \frac{1}{e^{2kh}}}{1 + \frac{1}{e^{2kh}}} \approx gk. \ (1)$$

or, $\omega^2 = gK$ — ①

Thus we can recover wave dispersion for deep water condition by applying suitable transformation to $\tanh(Kh)$.

On the other hand, for shallow water, $\frac{h}{\lambda} \ll 1$ or, $Kh \ll 1$.

Thus,
$$\omega^2 = gK \cdot \frac{e^{Kh} - e^{-Kh}}{e^{Kh} + e^{-Kh}}$$
$$\approx gK \cdot \frac{1 + Kh - 1 + Kh}{1 + Kh + 1 - Kh}$$
$$= gK \cdot (Kh) = gK^2 h$$

or, $c^2 = \frac{\lambda^2}{T^2} = gh$, or, $c = \sqrt{gh}$

or, $\lambda = T \cdot \sqrt{gh}$. — ②

Velocity field under 2D progressive wave

$$\phi = \frac{ag}{\omega} \cdot \frac{\cosh K(z+d)}{\cosh Kd} \cdot \sin(Kx - \omega t)$$

$$u = \frac{\partial \phi}{\partial x} = \frac{gAK}{\omega} \cdot \frac{\cosh K(z+d)}{\cosh(Kd)} \cdot \cos(Kx - \omega t)$$

$$w = \frac{\partial \phi}{\partial z} = \frac{gaK}{\omega} \cdot \frac{\sinh K(z+d)}{\cosh(Kd)} \cdot \sin(Kx - \omega t)$$

→ u and w is in phase difference of $\frac{\pi}{2}$.

Thus, when u is maximum, w is minimum and vice-versa.

Pressure field under a progressive wave in 2D

From Bernoulli's equation we get,

$$\left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz\right)\Big|_{z=z} = \left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz\right)\Big|_{z=\eta} \quad \text{— ①}$$

Through the process of linearization, we get,

$$\cancel{\left(\frac{\partial \phi}{\partial t}\right)_{z=\eta} \approx \left(\frac{\partial \phi}{\partial t}\right)_{z=0}}$$

$$\left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz\right)\Big|_{z=\eta} \approx \cancel{\left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz\right)\Big|_{z=0} + \eta \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz\right)\Big|_{z=0}}$$

~~from Dynamic free surface boundary condition~~

$$\cancel{p(z=\eta) = 0}$$

From dynamic free surface boundary condition, we get,

$p(z=\eta) = 0$. and, $\eta = -\frac{1}{g} \cdot \frac{\partial \phi}{\partial t}\Big|_{z=0}$

and also, $\frac{\partial \phi}{\partial t}\Big|_{z=\eta} \approx \frac{\partial \phi}{\partial t}\Big|_{z=0}$, since η is very small.

Thus, from equation ①, we get,

$$\left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz\right)\Big|_{z=z} = \left(\frac{\partial \phi}{\partial t}\right)_{z=0} - g \cdot \frac{1}{g} \cdot \frac{\partial \phi}{\partial t}\Big|_{z=0}$$

$$= 0$$

or, $$\frac{p}{\rho} = -gz - \frac{\partial \phi}{\partial t}$$

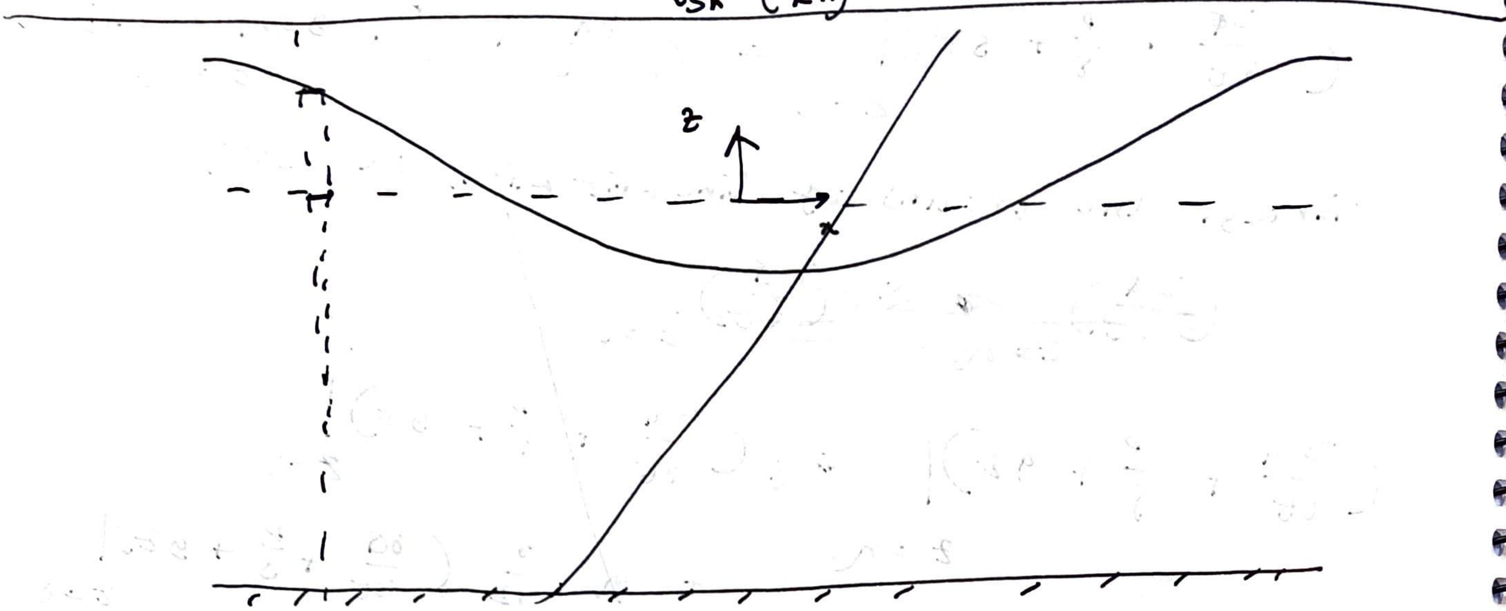
or, $$p = \underbrace{-\rho g z}_{\text{hydrostatic part}} + \underbrace{\rho a g \cdot K_p(z) \cdot \cos(kx - \omega t)}_{\text{Dynamic part}}$$

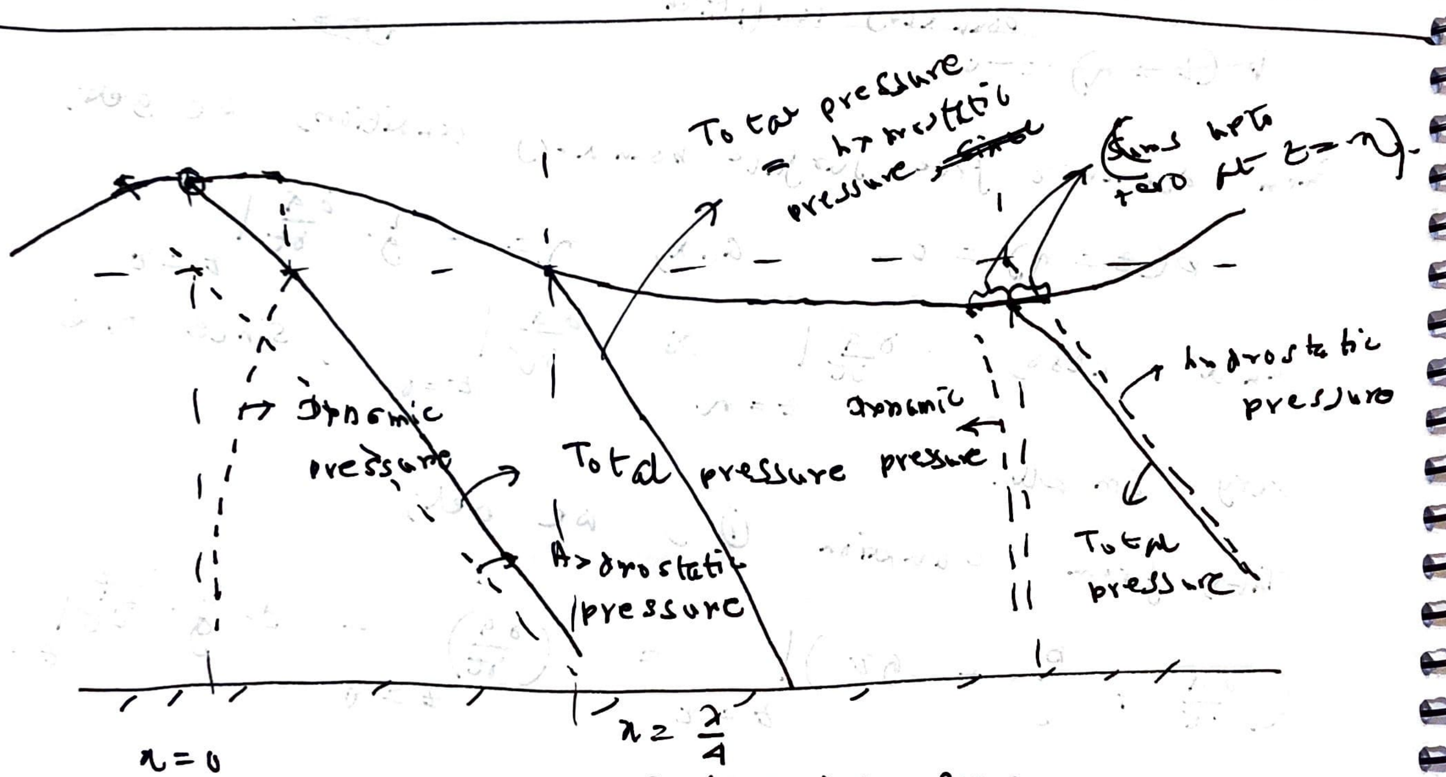
(using the solution of $\phi(x, z, t)$ defined earlier)

where, $K_p(z)$ is pressure response factor.

and define k_p as,

$$k_p(z) = \frac{\cosh k(h+z)}{\cosh(kh)}$$





at free surface, i.e., $z = \eta$,

$$\cancel{p(z=\eta)=0} \neq \cancel{\rho g z + \left(\frac{\partial\phi}{\partial t}\right)}$$

$$\cancel{p(z=\eta) = -\rho g z}$$

$$\cancel{p(z=\eta) = -\rho g \eta + \rho \cdot \frac{\partial\phi}{\partial t}\Big|_{z=0}} = 0.$$

$$p(z=\eta) = -\rho g \eta - \rho \cdot \frac{\partial\phi}{\partial t}\Big|_{z=0} = 0$$

$H_b \quad H_b \quad C_{ga}$

- At $z = \eta$, the hydrostatic part is cancelled by the dynamic part and thus we have $p(z=\eta)=0$.

~~under the node, i.e., at $x = \frac{\lambda}{4}$; the total pressure is purely hydrostatic, since,~~

~~$\cos(kx - \omega t)\Big|_{x=\frac{\lambda}{4}} = \cos\left(\frac{\pi}{2} - \omega t\right)$~~

~~$= \cos\frac{\pi}{2} \cdot \sin\omega t + \cos\omega t \cdot \sin\frac{\pi}{2}$~~

~~$=$~~

under the node, i.e., at $x = \frac{\lambda}{4}$, the total pressure is purely hydrostatic, since,

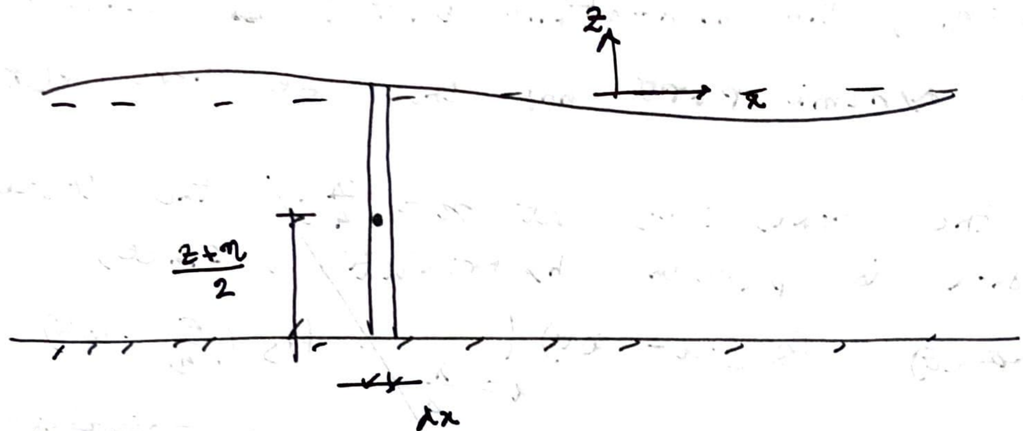
$$\cos(kx - \omega t)\Big|_{x=\frac{\lambda}{4},\, t=0} = 0$$

~~thus~~ for the dynamic pressure is zero across the depth.

Wave Energy

For a progressive periodic wave, there is a movement of the free surface. Thus there is a potential energy. On the other hand, as the wave propagates, there is a movement of a fluid element due to the existence of a velocity field. Thus there is a ~~potential~~ kinetic energy.

Estimation of wave energy is required for ~~assessing~~ assessing power required to drive a wave maker in a flume for doing experiments and for estimating availability of wave power while planning a device for ~~transforming~~ transforming wave energy into other form of energy.



for ~~aconst~~ a water column of ~~width~~ width dx and thickness along y axis as 1, the potential energy is

$$d(PE) = dm \cdot g \cdot \frac{h+\eta}{2}$$

& $$dm = \rho \cdot (h+\eta) \cdot dx$$

$$\therefore \quad d(PE) = \rho g \frac{(h+\eta)^2}{2} \cdot dx$$

$$= \frac{1}{2} \rho g \left(h^2 + 2h\eta + \eta^2 \right) dx$$

The average potential energy over a wave length

$$(\overline{PE}) = \frac{1}{\lambda} \int_x^{x+\lambda} \left(\frac{1}{2} \rho g \right) \left(h^2 + 2h\eta + \eta^2 \right) dx$$

$$= \left(\frac{1}{2} \right) \cdot \left(\frac{\rho g}{\lambda} \right) \cdot \int_x^{x+\lambda} \left(h^2 + 2h a \cos(kx - \omega t) + a^2 \cos^2(kx - \omega t) \right) dx$$

Now,

$$\int_x^{x+\lambda} h^2 \, dx = 0$$

$$\int_x^{x+\lambda} 2h\, a \cos(kx - \omega t)\, dx = 0$$

$$a^2 \int_x^{x+\lambda} \cos^2(kx - \omega t)\, dx = \frac{a^2 \pi}{k}$$

$$\therefore \frac{\rho g}{\lambda} \cdot \frac{1}{2} \cdot \frac{a^2 \cancel{x}}{k} = \frac{\rho g}{2\cancel{\lambda}} \cdot \frac{a^2 \cancel{\lambda}}{\frac{2\cancel{\pi}}{\cancel{\lambda}}} = \frac{\rho g a^2}{4}$$

Similarly, kinetic energy of a small fluid element of size $dx\,dz$ at a depth z and at a distance x is

$$d(KE) = dm \cdot \frac{u^2 + w^2}{2} \cdot \cancel{dx\,dz} = \rho \frac{u^2 + w^2}{2} \cdot dx\,dz$$

per unit thickness across y axis.

Thus average kinetic energy is

$$(\overline{KE}) = \frac{1}{\lambda} \int_{x}^{x+\lambda} \int_{-h}^{\eta \approx 0} \rho \frac{u^2 + w^2}{2} \cdot dx\,dz$$

substitution for u and w gives,

$$(\overline{KE}) = \frac{1}{\lambda} \int_{x}^{x+\lambda} \int_{-h}^{\eta \approx 0} \rho \cdot \left(\frac{1}{2}\right) \left(\frac{g^2 a^2 k^2}{\omega^2} \cdot \frac{\cosh^2 k(z+h)}{\cosh^2(kh)} \cdot \cos^2(kx - \omega t) + \frac{g^2 a^2 k^2}{\omega^2} \cdot \frac{\sinh^2 k(z+h)}{\cosh^2(kh)} \cdot \sin^2(kx - \omega t) \right) dx\,dz$$

$$= \frac{1}{\lambda} \int_{x}^{x+\lambda} \int_{-h}^{\eta \approx 0} \rho \frac{g^2 a^2 k^2}{\omega^2} \cdot \frac{1}{\cosh^2(kh)} \cdot \left[\cosh^2 k(h+z) \cos^2(kx - \omega t) + \sinh^2 k(h+z) \cdot \sin^2(kx - \omega t) \right] dx\,dz$$

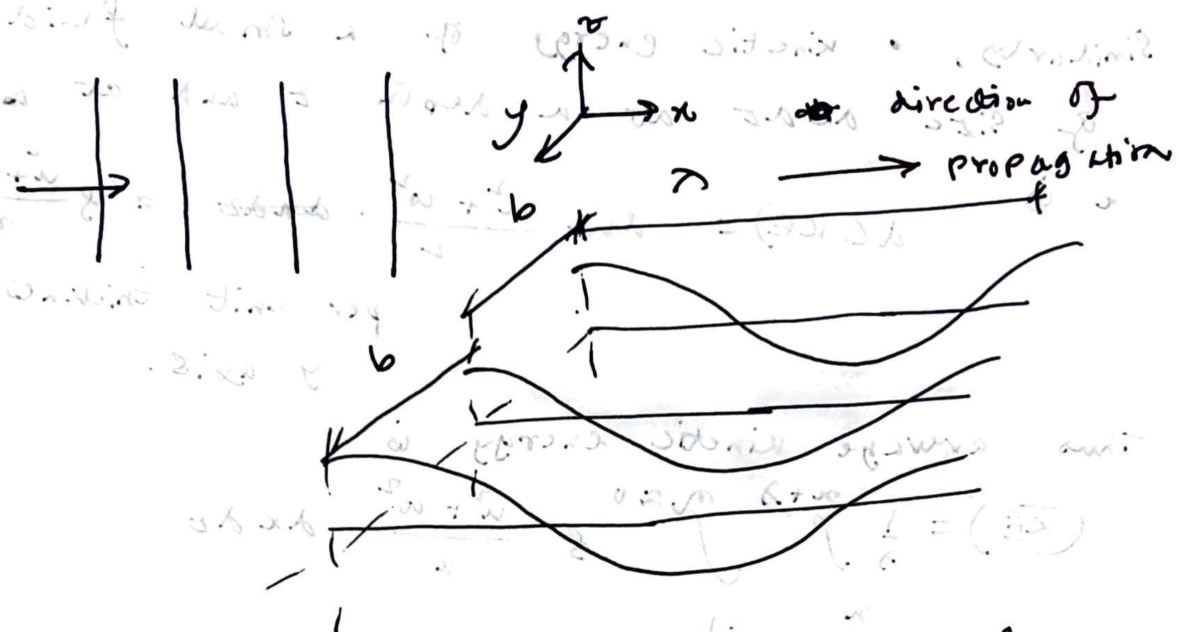
$$= \frac{\rho g a^2}{4}$$

Thus, the total wave energy per wave length and per unit thickness across y axis is

$$E = \frac{\rho g a^2}{4} + \frac{\rho g a^2}{4} = \frac{\rho g a^2}{2} = \frac{\rho g H^2}{8}$$

This ~~is also~~ may also be defined as wave energy per unit surface area (i.e., $\lambda \cdot b$)

where b may be a measure of separation is between identical wave trains.



$[E] = [\frac{1}{2} \rho g a^2] \equiv [ML^{-3}].[LT^{-2}].[L^2] = [MT^{-2}]$

$[E]$ as we know is $[MLT^{-2}].[L] \equiv [ML^2T^{-2}]$

so, $[E][L^{-2}] \equiv [ML^2T^{-2}][L^{-2}] \equiv [MT^{-2}]$