$\Rightarrow$ $b = \frac{1}{r}$.

$$c\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial y}\,\hat{j}\cdot\hat{e}_y = \frac{\partial\phi}{\partial y}. \quad (1)$$

or, $c = 1$.

~~thus, $(\nabla\phi)$~~

~~cylindrical~~

Coord

Thus, in cylindrical coordinate system,

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\cdot\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{\partial\phi}{\partial y}\hat{e}_y.$$

Kutta-Jukowski theorem and expression of lift per unit span for a low speed flow over a cylinder

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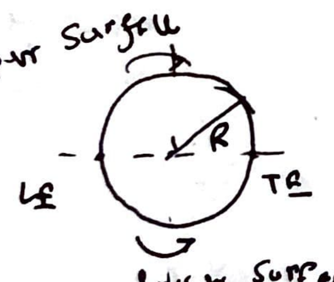
We have,
$$c_l = \frac{1}{c}\int_0^c c_{p,l}\,dx - \frac{1}{c}\int_0^c c_{p,u}\,dx$$

Where, c_l is lift coefficient; $c_{p,l}$ is pressure coefficient on the lower surface and $c_{p,u}$ is the pressure coefficient on the upper surface of the aerofoil.

Let us consider what happens when apply above result in the case of a cylinder of radius R.



$$I_1 = \frac{1}{c}\int_0^c c_{p,l}\,dx$$

$$= \frac{1}{2R}\int_\pi^{2\pi} c_{p,l}\,(-R)\sin\theta\,d\theta$$

$$= -\frac{1}{2}\int_\pi^{2\pi} c_{p,l}\sin\theta\,d\theta$$

$x = R\cos\theta$

$\therefore\ dx = -R\sin\theta\,d\theta$

$$I_2 = -\frac{1}{c}\int_{\pi}^{c} c_{p,u}\, dx = -\frac{1}{2\pi}\int_{\pi}^{0} c_{p,u}\,(-\pi)\sin\theta\, d\theta$$

$$= \frac{1}{2}\int_{\pi}^{0} c_{p,u}\sin\theta\, d\theta = -\frac{1}{2}\int_{0}^{\pi} c_{p,u}\sin\theta\, d\theta.$$

$$\therefore I_1 + I_2 = -\frac{1}{2}\int_{\pi}^{2\pi} c_{p,l}\sin\theta\, d\theta - \frac{1}{2}\int_{0}^{\pi} c_{p,u}\sin\theta\, d\theta.$$

$$= -\frac{1}{2}\left[\int_{\pi}^{2\pi} c_{p,l}\sin\theta\, d\theta + \int_{0}^{\pi} c_{p,u}\sin\theta\, d\theta\right]$$

$$= -\frac{1}{2}\int_{0}^{2\pi} c_{p,l}\sin\theta\, d\theta$$

for cylinder, $c_{p,l} = c_{p,u} = 1 - \left(\frac{v}{v_\infty}\right)^2$

$$= 1 - \left(\frac{v_\theta}{v_\infty}\right)^2$$

Recall, $v_\theta = -\left(1+\frac{R^2}{r^2}\right)v_\infty \sin\theta - \frac{\Gamma}{2\pi R}$

$$\therefore v_\theta^2 = \left(1+\frac{R^2}{r^2}\right)^2 v_\infty^2 \sin^2\theta + 2\left(1+\frac{R^2}{r^2}\right)v_\infty \sin\theta \cdot \frac{\Gamma}{2\pi R} + \frac{\Gamma^2}{4\pi^2 R^2}$$

$$\therefore \frac{v_\theta^2}{v_\infty^2} = \left(1+\frac{R^2}{r^2}\right)^2\sin^2\theta + 2\left(1+\frac{R^2}{r^2}\right)\frac{1}{v_\infty}\cdot\frac{\Gamma}{2\pi R}\cdot\sin\theta + \frac{\Gamma^2}{4\pi^2R^2v_\infty^2}$$

$$\therefore 1 - \frac{v_\theta^2}{v_\infty^2} = 1 - \left(1+\frac{R^2}{r^2}\right)^2\sin^2\theta - 2\left(1+\frac{R^2}{r^2}\right)\frac{1}{v_\infty}\cdot\frac{\Gamma}{2\pi R}\cdot\sin\theta - \frac{\Gamma^2}{4\pi^2R^2v_\infty^2}$$

$$\therefore I_1 + I_2 = -\frac{1}{2}\int_0^{2\pi} c_{p,l}\sin\theta\, d\theta = -\frac{1}{2}\int_0^{2\pi} c_{p,u}\sin\theta\, d\theta$$

Now, $\int_0^{2\pi}\sin\theta\, d\theta = 0$;

$$\int_0^{2\pi}\sin^3\theta\, d\theta = \frac{1}{4}\int_0^{2\pi}(3\sin\theta - \sin 3\theta)\, d\theta = 0 \quad [\because 3\sin\theta - 4\sin^3\theta = \sin 3\theta]$$

$$\int_0^{2\pi}\sin^2\theta\, d\theta = \pi$$

$$\therefore c_l = -\frac{1}{2}\int_0^{2\pi}(-2)\cdot(2)\cdot\frac{1}{v_\infty}\cdot\frac{\Gamma}{2\pi R}\cdot\sin^2\theta\, d\theta \quad [\because r = R]$$

$$= 2\,\frac{1}{v_\infty}\cdot\frac{\Gamma}{2\pi R}\cdot\pi = \frac{\Gamma}{R\,v_\infty}$$

Thus, lift per unit span $L' = c_l \cdot q_\infty \cdot S$

$$= \frac{\vec{\tau}}{\cancel{R} V_\infty} \cdot \frac{1}{\cancel{2}} \rho V_\infty^{\cancel{2}} \cdot \cancel{2R}(1)$$

$$= \rho V_\infty \vec{\tau}.$$

$\therefore$ $\boxed{L' = \rho V_\infty \vec{\tau}}$ → This is known as Kutta-Jukowski theorem.

→ The above result is valid for an aerofoil shape as well.

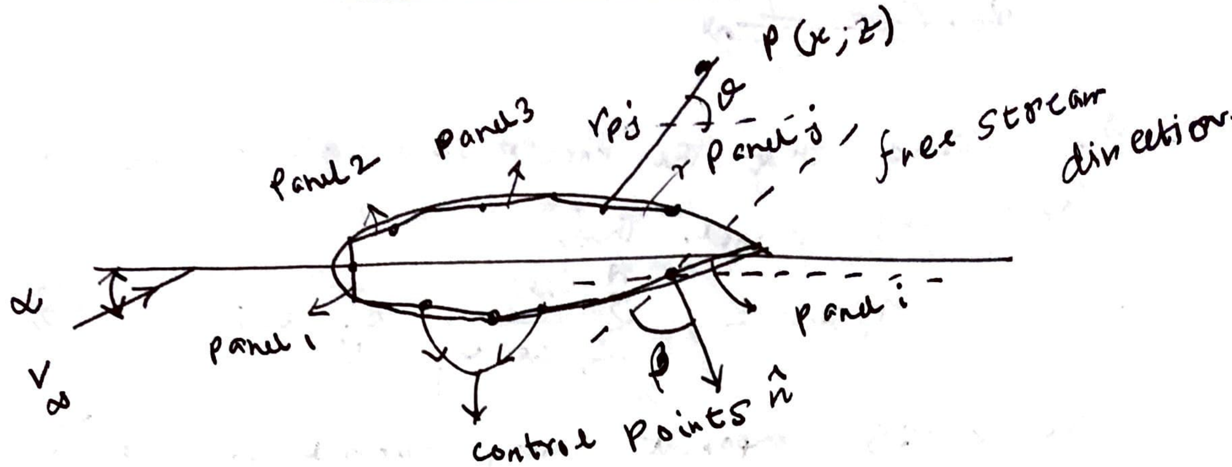
→ From experiments we know lift per unit span depends upon angle of attack. The Kutta-Jukowski theorem as such does not say anything about how $\vec{\tau}$ varies with angle of attack.

→ Still the result from Kutta-Jukowski theorem ~~appears~~ appears to fit very well at low ~~angle~~ angle of attack.

→ The Kutta-Jukowski theorem alone can not give a unique value of lift for a given angle of attack. We need additional consideration.

→ The Kutta-condition gives us the restriction needed to impose on circulation strength $\vec{\tau}$ so that we can obtain an unique lift for a given angle of attack.

Vortex panel method



Velocity potential due to a vortex of strength $\vec{\Gamma}$ is

$$\phi = -\frac{\vec{\Gamma}}{2\pi}\theta \quad \text{—(1)}$$

$$\left[\; v_r = 0 \; ; \; v_\theta = \frac{1}{r}\cdot\frac{\partial\phi}{\partial\theta} = -\frac{\vec{\Gamma}}{2\pi r} \; ; \; \phi = -\frac{\vec{\Gamma}}{2\pi}\theta \; \right]$$

For a differential element of length ds_j on panel j, the contribution to the velocity potential at a point $P(x, z)$ in the flow field is

$$\Delta\phi_j = -\frac{\gamma_j \, ds_j}{2\pi}\cdot\theta \, , \quad \text{—(2)}$$

where, γ_j is vortex strength at a point on j^{th} panel per unit length.

$$\phi_j = -\frac{1}{2\pi}\int_j \gamma_j \, \theta_{pj} \, ds_j \quad \text{—(3)}$$

where, $\theta_{pj} = \tan^{-1}\frac{z - z_j}{x - x_j}$,

where, (x_j, z_j) is ~~the control~~ a point on the j^{th} Panel.

Now, let the point be on another element i^{th}, then, the total velocity potential measured at the control point of i^{th} Panel is

$$\phi_i = \sum_{j=1}^{N} -\frac{1}{2\pi}\int_j \gamma_j \, \theta_{ij} \, ds_j \quad \text{—(4)}$$

We ~~can represent the overall distribution of vortex strength on a panel say j^{th} panel by a single vortex strength γ_j measured at the control point of j^{th} panel.~~

we assume that the vortex strength remains constant over any panel. Thus,

$$\phi_i = - \sum_{j=1}^{N} \frac{\gamma_j}{2\pi} \int_j \theta_{ij} \, ds_j \quad — (5)$$

The velocity component along normal n_i at the control point of i^{th} panel is

$$V_n = - \sum_{j=1}^{N} \frac{\gamma_j}{2\pi} \int_j \frac{\partial}{\partial n_i} \theta_{ij} \, ds_j \quad — (6)$$

If the aerofoil shape has to be a stream line then,

$$V_\alpha \cos\beta_i + V_n = 0$$

or,

$$V_\alpha \cos\beta_i - \sum_{j=1}^{N} \frac{\gamma_j}{2\pi} \int_j \frac{\partial \theta_{ij}}{\partial n_i} \, ds_j = 0.$$

Let,

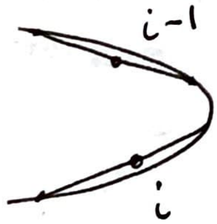
$$\int_j \frac{\partial \theta_{ij}}{\partial n_i} \, ds_j = I_{ij} \quad — (7)$$

thus we can rewrite equation (7) as,

$$V_\alpha \cos\beta_i - \sum_{j=1}^{N} \frac{\gamma_j}{2\pi} I_{ij} = 0. \quad — (8)$$

Now, we need to impose Kutta-condition in order to obtain a physical solution.

Kutta condition gives, $\gamma(TE) = 0$



$$\frac{1}{2} (\gamma_{i-1} + \gamma_i) = 0$$

or, $$\gamma_i = - \gamma_{i-1} \quad — (9)$$

Equation (8) is applied to $(n-1)$ panels and equation (9) is applied to i, j panel, thus using equation (8) together with ~~eq~~ equation (9), we have n unknown γ_j to solve from n equations.

$$\hat{\Gamma} = \sum_{j=1}^{N} \gamma_j \, \Delta s_j \; ;$$

$$L' = \rho V_\infty \hat{\Gamma} \, .$$

→ clearly ~~the~~ solution of γ_j will depend on ~~the~~ angle of attack. Thus, using vortex-panel method we can show that the lift per unit span depend on angle of ~~attack~~ attack.