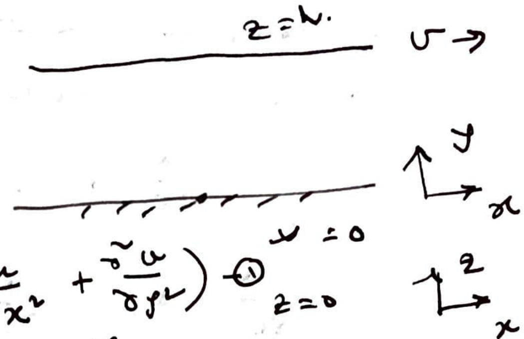
plane Couette flow (steady, 3-D incompressible flow).

- flow between two parallel plates.
- flow only in x-direction.
- no body force is present.

$\vec{v} = (u, 0, 0)$



Navier-Stokes equation for this case,

x-momentum equation -

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (1)$$

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$u(y)$

for steady flow, $\frac{Du}{Dt} = 0$. so,

$u = u(z)$ ∴ $\frac{\partial u}{\partial x} = 0$;

from continuity equation, $\frac{\partial w}{\partial z} = 0$. → {variation of w across z}

$w = 0$ at $z = 0$ and $z = h$.

so, $w = 0$, $0 \le z \le h$.

thus, equation (1) reduces to,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} = 0$$

or, $$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\frac{\partial^2 u}{\partial z^2}$$

or, $$\frac{\partial p}{\partial x} = \mu \cdot \frac{\partial^2 u}{\partial z^2} \quad (2)$$

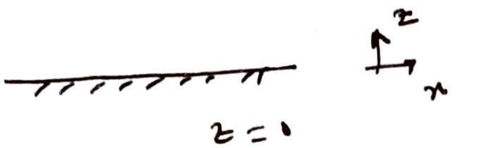
from y and z momentum equation we get,

$$\frac{\partial p}{\partial y} = 0 \; ; \; \frac{\partial p}{\partial z} = 0.$$

∴ equation (2) further reduces to,

$$\frac{dp}{dx} = \mu\frac{d^2 u}{dz^2}$$

Subjected to $u = 0$ at $z = 0$ and $u = U$ at $z = h$.



~~x momentum~~

$v = 0;\ u = 0.$

x momentum ~~equation~~ equation in the absence of any pressure ~~grad~~ gradient, and body force

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad -(1)$$

$$\frac{\partial u}{\partial x} = 0; \quad \frac{\partial u}{\partial z} = 0;$$

then (1) becomes,
$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial z^2}. \quad -(2)$$

Subjected to $u = 0$ at $z = 0$

and $u = U(t)$

$= U_0 \cos\omega t$ at $z = h$.

- Assume a $u(z, t)$

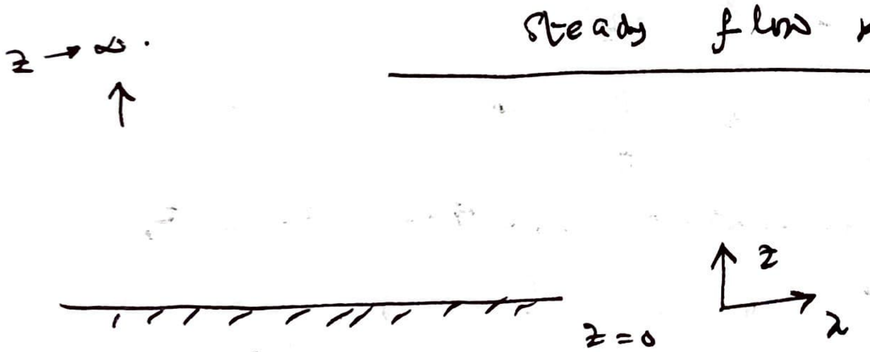
as, $u(z, t) = Re[f(z)\, e^{i\omega t}]$

- real part is in phase with $\cos(\omega t)$ and imaginary part is in phase with $\sin(\omega t)$.

from (2) we get,
$$i\omega f = \nu\frac{d^2 f}{dz^2}$$

Laminar Boundary Layers:

Steady flow past a flat plate



$u = U$ at $z \to \infty$.

$z = 0$

$u = u(x, z)$

$w = w(x, z)$

$0 < x < l$.

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad —— (1)$$

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$-(2)$

(we need laminar boundary layer theory because we can not retain certain flow properties for flow past a flat plate using plane Couette flow solution while letting $z \to \infty$.)

(~~viz~~ plane Couette flow solutions suggest viscous effect will persist at far away from the plate).

(the unsteady motion past a flat plate allowed us to develop necessary solutions to take into account viscous diffusion as we go away from the plate).

(Next, we study steady flow past a flat plate).

$$u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) \quad \text{— (3)}.$$

subjected to $(u, w) = 0$ at $z = 0$

and, $(u, w) = (U, 0)$ at large z.

To make any further progress from this point onward we use order of magnitude analysis.

~~this gi~~ - boundary layer thickness is δ.

- $u \simeq U$ immediately outside the boundary layer, u vary very rapidly from $u = 0$ at $z = 0$ to $u = U$. On the other hand, w vary from 0 to 0.

- $\frac{\partial u}{\partial z}$ is $O\left(\frac{U}{\delta}\right) \gg \frac{\partial u}{\partial x}$ is $O\left(\frac{U}{l}\right)$.

- $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \to \frac{\partial u}{\partial z} \gg \frac{\partial w}{\partial z}$

- this gives, $w \approx 0$.

Thus, we have,

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}$$

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial z}$$

At very large z, $\frac{\partial p}{\partial x}$ is what gives U.

So, there,

$$0 + 0 = -\frac{1}{\rho} \frac{\partial p}{\partial x} + 0$$

or, $\frac{\partial p}{\partial x} = 0$ at very large z.

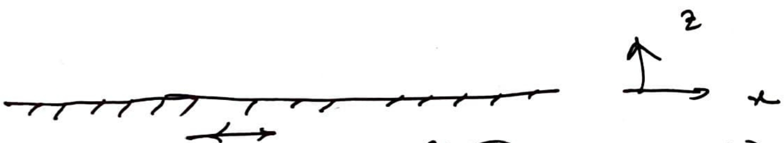
- pressure outside the boundary layer is impressed upon the boundary layer.
- thus ①, ② and ③ becomes,

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} - \nu \frac{\partial^2 u}{\partial z^2} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

<u>unsteady motion over flat plate</u>

($w = 0$ at $z = 0$;
$\frac{\partial w}{\partial z} = 0$, gives, $w = 0$)



$U(t) = U_0 \cos(\omega t)$

$u = U(t)$ at $z = 0$,
$u = 0$ as $z \to \infty$.
} Boundary conditions.

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \quad — ①$$

$u = u(z, t)$

$\therefore \frac{\partial u}{\partial x} = 0.$

$$\therefore \frac{\partial w}{\partial z} = 0.$$

$$u(z,t) = \text{Re}\left[f(z) \cdot e^{i\omega t} \right]$$

$$\frac{\partial u}{\partial t} + \cancel{u \frac{\partial u}{\partial x}}^{0} + \cancel{w \frac{\partial u}{\partial z}}^{0} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{0} + \frac{\partial^2 u}{\partial z^2} \right) \quad —②.$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}. \quad —③$$

$$u(z,t) = Re\left[f(z).\, e^{i\omega t} \right] \quad — \quad (A)$$

$$f(z) = A.e^{K_1.z} + B.\, e^{K_2.z} \qquad \begin{cases} K_1 = (1+i)\sqrt{\frac{\omega}{2\nu}} \\ K_2 = -(1+i)\sqrt{\frac{\omega}{2\nu}} \end{cases}$$

Substitution of $f(z)$ into (A) gives,

$$u(z,t) = Re\left[A.\, e^{K_1.z}.\, e^{i\omega t} + B.\, e^{K_2 z}.\, e^{i\omega t} \right]$$

$$= Re\left[A.\, e^{\sqrt{\frac{\omega}{2\nu}}.z}.\, e^{i\left(\sqrt{\frac{\omega}{2\nu}}z + \omega t\right)} + B.\, e^{-\sqrt{\frac{\omega}{2\nu}}.z}.\, e^{i\left(-\sqrt{\frac{\omega}{2\nu}}z + \omega t\right)} \right]$$

$$= Re\left[A.\, e^{\sqrt{\frac{\omega}{2\nu}}.z} \left\{ \cos\left(\sqrt{\frac{\omega}{2\nu}}z + \omega t\right) + i \sin\left(\sqrt{\frac{\omega}{2\nu}}z + \omega t\right) \right\} + B.\, e^{-\sqrt{\frac{\omega}{2\nu}}.z} \left\{ \cos\left(-\sqrt{\frac{\omega}{2\nu}}.z + \omega t\right) + i \sin\left(-\sqrt{\frac{\omega}{2\nu}}.z + \omega t\right) \right\} \right].$$

$$= A.\, e^{\sqrt{\frac{\omega}{2\nu}}.z} \cos\left(\sqrt{\frac{\omega}{2\nu}}.z + \omega t\right) + B.\, e^{-\sqrt{\frac{\omega}{2\nu}}.z}.\, \cos\left(-\sqrt{\frac{\omega}{2\nu}}z + \omega t\right)$$

for, $z=0$, $u = U_0 = U_0 \cos(\omega t)$.

or, $$U_0 \cos(\omega t) = A.(1).\cos(\omega t) + B.(1).\cos(\omega t).$$

for, $z \to \infty$, $u = 0$.

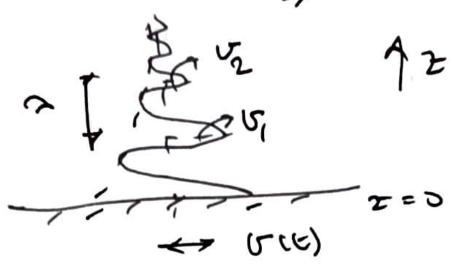
$$0 = A.\, e^{\sqrt{\frac{\omega}{2\nu}}.z}.\cos\left(\sqrt{\frac{\omega}{2\nu}}.z + \omega t\right) + 0.$$

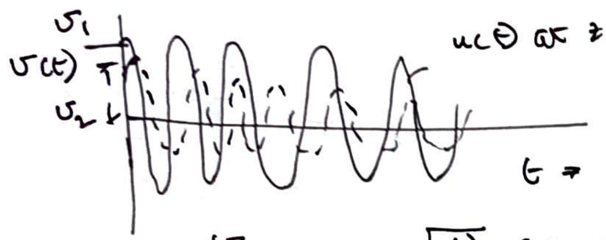
$\therefore$ $A = 0$, $B = U_0$.

$$\therefore u(z,t) = U_0\, e^{-\sqrt{\frac{\omega}{2\nu}}.z} \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}.z\right).$$

$$= U_0 .\, e^{-\sqrt{\frac{\omega}{2\nu}}.z} \cos\left(\sqrt{\frac{\omega}{2\nu}}.z - \omega t\right)$$

- ~~for water, $\nu = 10^{-6}\ m^2/s$;~~
- the velocity profile refers to a traverse damped oscillatory motion with wave length $\frac{2\pi}{\sqrt{\frac{\omega}{2\nu}}}$ or, $2\pi\sqrt{\frac{2\nu}{\omega}}$.





$v_2 < v_1$

$$\frac{v_2}{v_1} = e^{-\sqrt{\frac{\omega}{2\nu}}\cdot(z_2 - z_1)}.$$

$$\frac{v_2}{v_1} = e^{-\sqrt{\frac{\omega}{2\nu}}\cdot\lambda}.$$

$$= e^{-\sqrt{\frac{\omega}{2\nu}}\cdot 2\pi\sqrt{\frac{2\nu}{\omega}}} = e^{-2\pi}.$$

$$= 0.002.$$

for water $\nu = 10^{-6}\ m^2/s$; and if $\omega = 2\pi$ rad/s;

then $\lambda = 2\pi\sqrt{\frac{2\nu}{2\pi}}$

$= 2\pi \times \sqrt{\frac{\nu}{\pi}}$

$= 2\times10^{-3} \times \sqrt{\frac{1}{\pi}} = 2\times10^{-3} \times \sqrt{0.318} = 0.56$

$=$

for water $\nu = 10^{-6}\ m^2/s$ and if $\omega = 2\pi$ rad/s; then

$\lambda = 2\pi\sqrt{\frac{2\nu}{2\pi}} = 2\sqrt{10^{-6}\times\pi} = 2\sqrt{\pi}\times10^{-3} = 2\times1.77\times10^{-3}$

$= 3.54\times10^{-3}\ m = 3.54\times10^{-3}\times10^{2}\ cm$

$= 0.35\ cm$; and $\frac{v_2}{v_1} = e^{-2\pi}$

$= 0.002$;

- thus, the thickness of the layer in which the effect of the velocity of the oscillating plate is present is very small, unless, the frequency of oscillation of the plate is very small.

- For most of our hydrodynamic application the motion is ~~...~~ reasonably ~~...~~ high. Thus, the boundary layer thickness is generally very small.

- Thus the unsteady flow problem past a ~~...~~ oscillating flat plate suggests that the boundary layer where we expect to have most of viscous effect is very thin.

- This ~~...~~ information we can not obtain from studying the solutions for plain Couette flow.

$$\tau = \mu \left.\frac{\partial u}{\partial z}\right|_{z=0} = \mu \frac{\partial}{\partial z}\left[U_0 \cdot e^{-\sqrt{\frac{\omega}{2\nu}}\cdot z} \cdot \cos\left(\sqrt{\frac{\omega}{2\nu}}\cdot z - \omega t\right)\right].$$

$$= \mu \left[U_0 \left(-\sqrt{\frac{\omega}{2\nu}}\right) \cdot e^{-\sqrt{\frac{\omega}{2\nu}}\cdot z} \cdot \cos\left(\sqrt{\frac{\omega}{2\nu}}\cdot z - \omega t\right) - U_0 \cdot e^{-\sqrt{\frac{\omega}{2\nu}}\cdot z} \sin\left(\sqrt{\frac{\omega}{2\nu}}\cdot z - \omega t\right) \cdot \sqrt{\frac{\omega}{2\nu}} \right]$$

$$= -\mu U_0 \cdot e^{-\sqrt{\frac{\omega}{2\nu}}\cdot z} \cdot \sqrt{\frac{\omega}{2\nu}} \left\{ \cos\left(\sqrt{\frac{\omega}{2\nu}}\cdot z - \omega t\right) + \sin\left(\sqrt{\frac{\omega}{2\nu}}\cdot z - \omega t\right)\right\}$$

- Due to oscillatory motion of plate, the direction of stress oscillates as well.

- τ is maximum at $z = 0$.

Prandtl boundary layer equations

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \quad \Big\} - (1).$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

Since the order of magnitude analysis gives $\frac{\partial p}{\partial z} = 0$. across the boundary layer, $p = p(x)$ and thus we can substitute $\frac{\partial p}{\partial x}$ with $\frac{dp}{dx}$ in equation (1).

Using ~~equ~~ ~~eq~~ equation (1), we ~~want~~ want to describe a boundary layer whose thickness $\delta(x)$ ~~to~~ increased ~~mono~~ ~~monoton~~ monotonously with x. This is not ~~only~~ possible if we have an adverse pressure gradient. ~~If~~ On the other hand, if the pressure gradient ~~φ~~ is favourable, then only $\delta(x)$ will increase along x. If there is an adverse pressure gradient present inside the boundary layer, the pressure downstream will try to prevent the flow ~~and~~ and also the viscous drag tries to retard the motion. In this case, boundary layer separation may take place. Boundary layer theory does not work if there is a boundary layer separation.

~~Further~~ ~~furthermore~~

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$O\left(\frac{U^2}{L}\right) + (?) = O\left(\nu \cdot \frac{U}{\delta^2}\right) \quad -(2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$O\left(\frac{U}{L}\right) + O\left(\frac{V}{\delta}\right) = 0$$

$$V \sim O\left(\frac{U}{L}\right)\delta.$$

$$\sim O\left(U\frac{\delta}{L}\right)$$

Thus, if $\delta \ll L$;

$V \ll U$.

Thus, order of magnitude analysis from ⑧ gives,

$$o\left(\frac{U^2}{L}\right) + o\left(U\left(\frac{\delta}{L}\right)\left(\frac{U}{\delta}\right)\right) = o\left(\nu \cdot \frac{U}{\delta^2}\right)$$

or, $$o\left(\frac{U^2}{L}\right) = o\left(\nu \cdot \frac{U}{\delta^2}\right)$$

or, $$o(\delta^2) = o\left(\frac{\nu L}{U}\right) = o\left(\frac{\nu L^2}{UL}\right)$$

$$= o\left(\frac{L^2}{Re}\right) \qquad Re = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$$

or, $$o(\delta) = o\left(Re^{-\frac{1}{2}} \cdot L\right).$$

or, $$\boxed{o\left(\frac{\delta}{L}\right) = o\left(Re^{-\frac{1}{2}}\right)}.$$

Key assumption behind Prandtl boundary layer equations.

1. $\frac{\delta}{L} \ll 1$; (Re is large)
2. Boundary layer separation does not take place, or pressure gradient is favourable.

- Boundary layer equations are simplified version of Navier-Stokes equation applicable only within the boundary layer when thickness of the boundary layer is very small compared to the length along x,

i.e.