Advanced Marine Hydrodynamics NA61202 3-1-0

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Lecture 2: Some Background: Elements of Vector Analysis

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Overview

Field definitions

A scalar field

If to each point $P(\mathbf{R})$ of a region say E in space there corresponds a definite scalar denoted by $f(\mathbf{R})$, then $f(\mathbf{R})$ is a scalar field.

A vector field

If to each point $P(\mathbf{R})$ of a region say E in space there corresponds a definite vector denoted by $F(\mathbf{R})$, then $F(\mathbf{R})$ is a vector field.

Gradient of a scalar

The Gradient Operator

The gradient of a scalar f is a *vector* defined in a Cartesian Coordinate Frame as
$\nabla f = \frac{\partial f}{\partial x}\hat{\mathrm{i}} + \frac{\partial f}{\partial y}\hat{\mathrm{j}} + \frac{\partial f}{\partial z}\hat{\mathrm{k}}$

- Geometrical Interpretation ?
- Usage in constructing BCs in a basic wave body interaction problem.
- What if we take gradient of a vector ?
- Example, $f\left(x, y, z\right) = \sin\left(xy\right) + x^3 - x^2 z$. Find $\mathbf{F} = \nabla f$ at $(0, 1, 2)$.

The Divergence Operator

The divergence of a vector

The divergence of a vector $\mathbf{F} = F_x\hat{\text{i}} + F_y\hat{\text{j}} + F_z\hat{\text{k}}$ is a scalar *scalar* defined in a Cartesian Coordinate Frame as

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\hat{\text{i}} + \frac{\partial}{\partial y}\hat{\text{j}} + \frac{\partial}{\partial z}\hat{\text{k}}\right) \cdot \left(F_x\hat{\text{i}} + F_y\hat{\text{j}} + F_z\hat{\text{k}}\right) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

- Meaning of this operation will be explained in detail in the context of deformation in fluid flow in a separate lecture.
- Example, Find $\nabla \cdot \mathbf{F}$ at $(0, 1, 2)$.

The Curl Operator

The curl of a vector

The curl of a vector $\mathbf{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ is another *vector* defined in a Cartesian Coordinate Frame as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

- Meaning of this operation will be explained in detail in the context of deformation in fluid flow in a separate lecture.
- Example, L2A1 Find $\nabla \times \mathbf{F}$ at $(0, 1, 2)$.

Some Useful Operations in with ∇

For vectors **F** and **G** and scalars ϕ and ψ,

1. $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$.
2. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla\mathbf{F} + \nabla\mathbf{G}$.
3. $\nabla \cdot (\phi\mathbf{F}) = (\nabla\phi) \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F})$.
4. $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.
5. L2A2 From First two examples on ∇f and $\nabla \cdot \mathbf{F}$, can you check what is the relation between $\nabla \cdot (\nabla f)$ and $\nabla^2 f$? What you find is another relation in this list!

A line integration

For a continuous vector $\mathbf{F}$ well defined on a curve C if we divide C into n parts say $P_0, P_1, P_2, ..., P_n$ with position vectors $\mathbf{R}_0, \mathbf{R}_1, ..., \mathbf{R}_n$ and let the position vector be $\mathbf{G}_i$ for any point on arc through $P_{i-1}P_i$ then let S be $S = \sum_{n=1}^{n} \mathbf{F}(\mathbf{G}_i) \cdot \delta\mathbf{R}_i$, where $\delta\mathbf{R}_i = \mathbf{R}_i - \mathbf{R}_{i-1}$.
In the limit as $n \to \infty$, such that $|\delta\mathbf{R}_i| \to 0$ if exists, then $\int_C \mathbf{F} \cdot \mathbf{R}$, is what we call as line integral of $\mathbf{F}$ along C. If C is a closed curve, we use $\oint_C$ instead $\int_C$.

- Physical significance of line integral: Path function, Point function and a conservative force field.
- Example, $\mathbf{F} = \left(5xy - 6x^2\right)\hat{\mathrm{i}} + (2y - 4x)\hat{\mathrm{j}}$. Find $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{R}$ on a curve $C : y = x^3$ in the z plane from $(1, 1)$ to $(2, 8)$.

A surface integral

Let $\mathbf{F}$ be a vector and S be any continuous surface. Let $\delta\mathbf{s}$ be an elemental surface at a point on the surface. The elemental area $\delta\mathbf{s}_i$ is a vector with an normal say $\hat{\mathrm{n}}_i$ pointing *outwards* from its mid-point $\mathbf{G}_i$(i.e., $\delta\mathbf{s}_i = \delta s_i \hat{\mathrm{n}}_i$). Let $\bar{S} = \sum_i^n \mathbf{F}(\mathbf{G}_i) \cdot \delta\mathbf{s}_i$ be such that $\delta\mathbf{s} \to 0$ as $n \to \infty$, then $\bar{S} = \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{s}$ is a surface integral of $\mathbf{F}$ over the surface S.

- Physical significance of a surface integral: The pressure force while formulating a typical wave-body problem. L2A3 Can you formulate the calculation of flux across a surface ?
- Example, Find $\int_S \mathbf{F} \cdot \hat{\mathrm{n}} \mathrm{d}s$ where $\mathbf{F} = 6z - 4 + y$ and $S : 2x + 3y + 6z = 12$ in the first octant.

A volume integral

Let $\mathbf{F}$ be a vector well defined on continuous surface S which encloses a finite region E. Let δv_i be an elemental volume inside E enclosing a point $\mathbf{G}_i$. Let the sum $\bar{V}$ be defined as $\sum_i^n \mathbf{F}\left(G\right)_i \delta v_i$. If $\delta v_i \to 0$ as $n \to \infty$, $\bar{V} = \int_E \mathbf{F}\mathrm{d}v$ is called a volume integral of $\mathbf{F}$ over E.

- Physical significance: anytime we refer to a control volume.

Stoke's theorem

For an open surface S bounded by a closed curve C and a vector $\mathbf{F}$ continuously differentiable over S
$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{R} = \int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \mathrm{d}s.$
Example, understanding the irrotationality in terms of circulation.

Gauss Divergence theorem

For an open surface S enclosing a region E and a vector $\mathbf{F}$ continuously differentiable over S
$\int_S \mathbf{F} \cdot \hat{\mathbf{n}} \mathrm{d}s = \int_E \nabla \cdot \mathbf{F} \mathrm{d}v.$
Example, analysis of continuity in a control volume: Relating what happens inside a control volume to what happens around its enclosing surface.

Green's theorem

If ϕ and ψ are scalar point functions possessing continuous 1st and 2nd derivatives, then for a region E bounded by a surface S
$\int_E \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \int_S \left(\phi\nabla\psi - \psi\nabla\phi\right).$
Example, this is extremely important in developing concept of reciprocity and far-field in wave body interaction theory.