Green's theorem and distribution of singularities for motion of an object in an unbounded fluid

Wherever we have an object undergoing oscillatory motion in a fluid, there is a portion of fluid which always moves along with the object. This should not to be understood as a result of viscosity. But rather from the point of view of inertia. As we move an object inside a fluid, the object displaces some portion of fluid. Thus the movement of an object in a fluid is associated with the movement of some fluid. This leads to the idea of added mass in a ~~very~~ most simplistic form. We do not feel the effect of added mass as such in air because the density of air is much less than that of a fluid like water.

The inviscid flow theory for elementary flows like source, sink, dipole etc becomes very useful to develop theoretical tools to quantify the added mass effect and thus develop better insight of added mass of a given body shape.

Let us consider two velocity potentials ϕ and φ both satisfying Laplace equation in a volume $\forall$ bounded by a closed surface S. Thus from divergence theorem we get,

$$\int_S \left[\phi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \phi}{\partial n}\right] dS = \int_\forall \nabla \cdot \left(\phi \frac{\partial \varphi}{\partial n} - \varphi \cdot \frac{\partial \phi}{\partial n}\right) d\forall$$

$$= \int_\forall \left(\phi \nabla^2 \varphi + \nabla\phi \cdot \nabla\varphi - \varphi \nabla^2 \phi - \nabla\varphi \cdot \nabla\phi\right) d\forall$$

$$= 0 \qquad \text{--- (1)}$$

This result is known as Green's theorem.

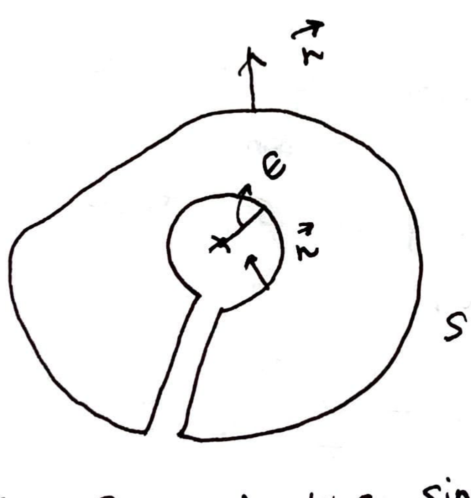
Suppose, we take the velocity potential as,

$$\varphi = \frac{1}{4\pi r} = \left(\frac{1}{4\pi}\right) \cdot \left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{-\frac{1}{2}};$$

— (2)

where, (ξ, η, ζ) is the location of the source in a fixed coordinate system (x, y, z). We can show that this function $\varphi(x, y, z; \xi, \eta, \zeta)$ satisfies Laplace equation no matter we choose (x, y, z); or (ξ, η, ζ) coordinate system. Furthermore, the potential φ has a singularity at r. ~~We say that~~ Point $\vec{\xi}(\xi, \eta, \zeta)$ is a source point and point, $\vec{x}(x, y, z)$ is a field point.

With the ~~velocity~~ velocity potential φ defined in equation (2), we can find the other velocity potential using Green's theorem (1). However this requires a careful ~~singularity~~ choice of the surface S. There is a singularity at r. Thus we ~~construct~~ construct the surface as,



where we surround the singularity at r with a very small surface say S_ε and connect it to the outer surface S as shown in the above picture. The normal vector $\vec{n}$ is ~~always~~ always pointing outwards from the surface enclosing the fluid ~~part~~ portion.

Thus, from equation ①, we have,

$$\frac{1}{4\pi}\int_{S+S_\epsilon}\left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{1}{r}\cdot\frac{\partial\phi}{\partial n}\right]ds=0$$

or,
$$\frac{1}{4\pi}\int_{S}\left(\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial\phi}{\partial n}\right)ds$$

$$=-\frac{1}{4\pi}\int_{S_\epsilon}\left(\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial\phi}{\partial n}\right)ds. \qquad ③$$

Now, we want to examine the integral on the right hand side of equation ②, as the radius ϵ of the inner sphere tends towards zero.

If ϵ is very small, we consider ϕ to change very slowly and take that as almost like a constant. Thus we can take ϕ out of this first integral on the right hand side.

Thus,
$$-\frac{1}{4\pi}\int_{S_\epsilon}\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)ds$$

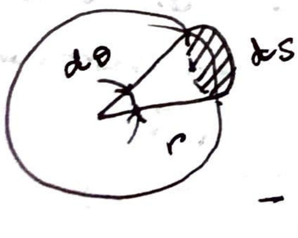
$$=-\frac{1}{4\pi}\cdot\phi\cdot\int_{S_\epsilon}\frac{\partial}{\partial n}\left(\frac{1}{r}\right)ds \qquad ④$$

and the second integral is $-\frac{1}{4\pi}\int_{S_\epsilon}\left(-\frac{1}{r}\right)\frac{\partial\phi}{\partial n}ds$.

Now, since S_ϵ can be viewed as an enclosing surface of a sphere, $\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\equiv\frac{\partial}{\partial r}\left(\frac{1}{r}\right)=-\frac{1}{r^2}$.

and using idea of a solid angle,

$$ds=r^2.\,d\theta.$$

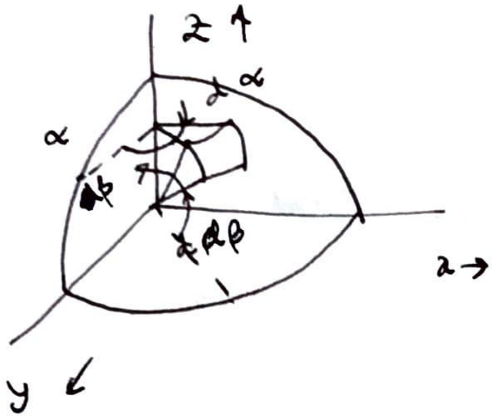


we have,

$$-\frac{1}{4\pi}\phi.\int_{S_\epsilon}\frac{\partial}{\partial n}\left(\frac{1}{r}\right)ds=-\frac{1}{4\pi}.\phi\int_{S_\epsilon}\left(-\frac{1}{r^2}\right)\cdot r^2.\,d\theta$$

$$=-\frac{1}{4\pi}\phi\int_{S_\epsilon}(-d\theta)$$

Note, $d\theta$ is a solid angle.



~~$ds = (r d\beta) \cdot (r\, ds$~~

$$ds = (r\, d\beta)\cdot(r \sin\beta \cdot d\alpha)$$
$$= r^2 \cdot \sin\beta \, d\beta \, d\alpha.$$

$$\therefore \quad d\theta = \frac{ds}{r^2}$$
$$= \sin\beta \, d\beta \, d\alpha.$$

$$\therefore \quad \int_{S_\epsilon} d\theta = 8 \int_{\beta=0}^{\frac{\pi}{2}} \int_{\alpha=0}^{\frac{\pi}{2}} \sin\beta \, d\beta \, d\alpha$$

$$= 8 \int_0^{\frac{\pi}{2}} \left[\cos\beta \right]_0^{\frac{\pi}{2}} d\alpha = -8 \cdot \frac{\pi}{2} = -4\pi.$$

$$\therefore \quad \lim_{\epsilon \to 0} \left(-\frac{1}{4\pi} \phi \int_{S_\epsilon} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) ds \right) = -\phi(x, y, z)$$

Since we take ϕ as almost constant, the second integral $-\frac{1}{4\pi} \int_{S_\epsilon} \left(-\frac{1}{r}\right) \frac{\partial \phi}{\partial n}$ vanishes as $\epsilon \to 0$.

Thus, we have,

$$\phi(x, y, z) = -\frac{1}{4\pi} \int_S \left[\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r} \cdot \frac{\partial \phi}{\partial n} \right] ds \qquad \text{——— (5).}$$

Equation (5) is the representation of velocity potential if the point of evaluation is inside the surface S. On the other hand, if this point is on the surface S, we can surround the singularity as.

The surface S_6 now lies on a hemisphere. And thus, we can show that,

$$\phi(x, y, z) = \cancel{\;} - \frac{1}{2\pi} \int_S \left[\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r} \cdot \frac{\partial \phi}{\partial n} \right] ds. \quad -⑥.$$

Equation ⑥ is frequently used for constructing the velocity potential on the surface of a ship hull or a moving body. In these cases, the derivative $\frac{\partial \phi}{\partial n}$ is known on the surface at any point and thus equation ⑥ is actually an integral equation to solve to find out velocity potential $\phi(x, y, z)$. This integral equation can be solved using a suitable numerical technique.

For a body like a ship hull moving in an unbounded fluid, the surface S must include the body surface, the sea bed and the free surface. In this case, there are additional boundary conditions to be imposed and there is a computational advantage of using equation ⑥ to solve for the velocity potential. In this context, we define the Green function as,

$$G(x, y, z; \xi, \eta, \zeta) = \frac{1}{r} + H(x, y, z; \xi, \eta, \zeta) \quad - ⑦$$

where, H is any function that satisfy Laplace equation. Thus, using ⑦ in ⓒ we have,

$$\int_S \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right) ds = \begin{cases} 0 \\ -2\pi\phi(x, y, z) \\ -4\pi\phi(x, y, z) \end{cases}$$

if the point of evaluation is outside the region S; is on the surface S and inside the surface S respectively.

(In 2D, equation (5) can be shown to be as

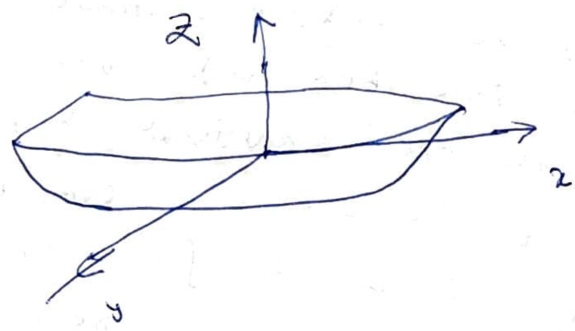
$$\phi(x,y,z) = -\frac{1}{2\pi}\int_S \left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n}\right] ds$$

where the source potential is of the form $\frac{1}{2\pi}\ln(r)$ as we have seen earlier while studying the source flows in 2D. The point of evaluation is inside the curve S.)

$\boxed{\text{Assignment (20 marks)}}$

Linear wave body interaction problem

(wave diffraction + radiation)



Total velocity potential, $\phi(x,y,z,t)$

$$= Re\left[\sum_{j=1}^{6} u_j \phi_j(x,y,z) + A\cdot\phi_A(x,y,z) \cdot e^{i\omega t}\right] \quad -(1)$$

ϕ_j are the radiation potential and ϕ_A is the diffracted wave potential.

Boundary conditions

$$\left.\frac{\partial \phi_j}{\partial n}\right|_{S_B} = \begin{cases} \vec{u}\; i\omega n_j & \text{for } j = 1,2,3 \\ i\omega(r \times n_j) & \text{for } j = 4,5,6 \end{cases} \quad -(2)$$

Since, $U_j(t) = Re(i\omega u_j e^{i\omega t})$

Here, $\vec{n}$ is an unit normal vector pointing into the body from the fluid.

~~for~~ For wave diffraction problem,

$$\left.\frac{\partial \phi_A}{\partial n}\right|_{S_B} = 0 \quad \text{——— (3)}$$

Furthermore, we consider $\phi_A = \phi_0 + \phi_7$, where ϕ_0 is the incident wave potential and ϕ_7 is scattered wave potential.

Thus, we have, $\frac{\partial \phi_0}{\partial n} = -\frac{\partial \phi_7}{\partial n}$ on S_B. —(4)

from combine d free surface boundary conditions we have,

$$-\frac{\omega^2}{g}\phi_j + \frac{\partial \phi_j}{\partial z} = 0 \quad \text{on } S_F \text{, for}$$

$$j = 0, 1, 2, \dots, 7.$$

—(5)

The last boundary condition we consider is the Sommer field radiation boundary condition, which make the radiation potentials, (ϕ_j, $j = 1, 2, 3, \dots 6$) different from and the scattered potential and incident potential as,

$$\phi_j \propto R^{-\frac{1}{2}} e^{-ikR} \quad \text{as } R \to \infty \quad j = 1, 2, 3 \dots 7.$$

—(6)

Total hydrodynamic pressure of the body is given as,

$$p = -\rho\left(\frac{\partial \phi}{\partial t} + gz\right)$$

$$= -\rho \, Re\left[\left(\sum_{j=1}^{6} a_j \phi_j + A(\phi_0 + \phi_7)\right) \cdot i\omega e^{i\omega t}\right]$$

$$- \rho g z$$

—(7)

And the total hydrodynamic force is given as,

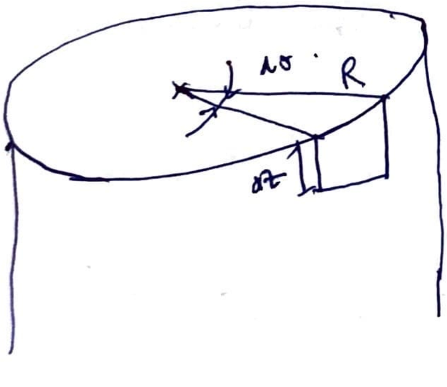
$$\begin{pmatrix} F \\ M \end{pmatrix} = -\rho g \int_{S_B} \begin{pmatrix} \vec{n} \\ \vec{r}\times\vec{n} \end{pmatrix} z\, ds \;-\; \rho\, \mathrm{Re} \sum_{j=1}^{6} i\omega\, \eta_j\, e^{i\omega t} \cdot \int_{S_B} \begin{pmatrix} \vec{n} \\ \vec{r}\times\vec{n} \end{pmatrix} \phi_j \, ds$$

$$- \rho\, \mathrm{Re}\; i\omega A\, e^{i\omega t} \int_{S_B} \begin{pmatrix} \vec{n} \\ \vec{r}\times\vec{n} \end{pmatrix} (\phi_0 + \phi_7)\, ds. \quad (8)$$

Sommerfield radiation boundary condition

* wave energy flux is given by $Q = \frac{\rho g a^2}{2}$. This is the wave energy per unit wave length in 2D. And this is energy per unit area in 3D.

Suppose there is an elemental surface ds at large distance from the source of wave generation. The total wave energy out of this ~~elem~~ elemental area ds is



$$dE = \left(\frac{\rho g a^2}{2}\right) . ds$$

$$= \left(\frac{\rho g a^2}{2}\right) . R . d\theta . dz$$

$$\therefore E = \int_{\theta=0}^{2\pi} \int_{z=-h}^{0} \frac{\rho g a^2}{2} . R\, d\theta\, dz$$

$$= h . \frac{\rho g a^2}{2} . 2\pi R .$$

If this energy is to be preserved with increasing R order constant depth h, then, we must have, $a^2 \propto R^{-1}$, or, $a \propto R^{-\frac{1}{2}}$.

We know from ~~kinematic~~ kinematic free surface boundary condition that, $\eta = -\frac{1}{g} \cdot \frac{\partial \phi}{\partial t}\Big|_{z=0}$;

$$\phi \propto Re\{\phi(x,y,z) \cdot e^{i\omega t}\}. \quad — ①$$

Thus, $\phi(x,y,z) \propto \eta \propto a \propto R^{-\frac{1}{2}}$.

On the other hand, by ~~applying~~ applying Laplace equation in a cylindrical coordinate system we can find a solution of the velocity potential and show that

$\phi(R, \theta, z) = R(R) \cdot \Theta(\theta) \cdot Z(z)$ and $R(r)$ can be represented by Hankel function $(H_m^{(2)}(R))$ i.e., of order m of second type. At large R, ~~when~~ when $R \to +\infty$, $H_m^{(2)}(R)$ can be shown to behave asymptotically as,

$$H_m^{(2)}(R) \propto e^{-ikR}. \quad — ②$$

Thus, combining both ① and ② we have,

$\phi(x,y,z) \equiv G(r, \theta, z) \propto R^{-\frac{1}{2}} \cdot e^{-ikR}$ as $R \to \infty$. This is Sommer field radiation boundary condition as $R \to \infty$.

Added mass and damping coefficients

From expression on wave radiation force, we have,

$$F_i = \mathrm{Re}\left\{ \sum_{j=1}^{6} \xi_j e^{i\omega t} f_{ij} \right\}, \quad \text{for } i = 1, 2, \ldots 6.$$

where,

$$f_{ij} = -\rho \int_{S_B} \phi_j \frac{\partial \phi_i}{\partial n} \, ds.$$

Since, from body boundary condition we have,

$$\left. \frac{\partial \phi_j}{\partial n} \right|_{S_B} = i\omega n_j \quad \text{for } j = 1, 2, 3$$

we write $f_{ij} = \omega^2 a_{ij} - i\omega b_{ij}$ — ①

thus, $f_{ij} \in \mathbb{C}$; $a_{ij} \in \mathbb{R}$; $b_{ij} \in \mathbb{R}$.

Since motion of the object is proportional to $e^{i\omega t}$, we can interprete equation ① as the over all hydrodynamic force f_{ij} is comprised of two parts. One part is in phase with the acceleration and another part is in phase with velocity.

The coefficient a_{ij} is known as added mass, since it represents the part of the force which is proportional to the acceleration. b_{ij} is known as damping coefficient.

The various components of added mass and damping coefficient can be interpreted as force component evaluated in j th direction due to motion in i th direction. These coefficients are specified as a function of frequency or wave number.

Wave-exciting forces and moments

The first term of total wave exciting forces and moments are given as,

$$\begin{Bmatrix} F_{ox} \\ M_{ex} \end{Bmatrix} = -\rho \, Re \, i\omega A \, e^{i\omega t} \int_{S_B} (\phi_0 + \phi_7) \cdot \begin{pmatrix} \vec{n} \\ \vec{r} \times \vec{n} \end{pmatrix} ds$$

It is convenient to ~~co~~ rewrite above expressions as,

$$(F_{ex})_i = Re \{ A \, e^{i\omega t} \, X_i \} \quad i = 1, 2, \dots 6.$$

where,

$$\cancel{X_i = -i\omega\rho \int_S (\phi_0 + \phi_7) \begin{pmatrix} \vec{n}_i \\ \vec{r} \times \vec{n}_i \end{pmatrix} ds}$$

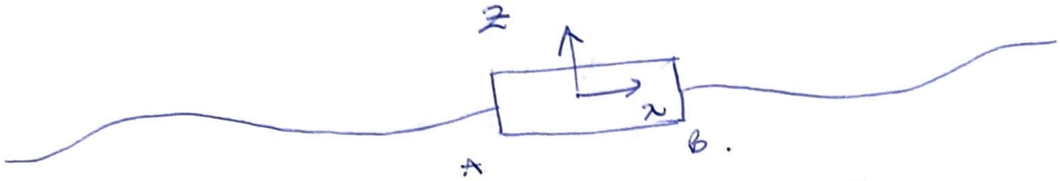
$$X_i = -i\omega\rho \int_{S_B} (\phi_0 + \phi_7) \begin{pmatrix} \vec{n}_i \\ \vec{r} \times \vec{n}_i \end{pmatrix} ds$$

is known as amplitude of exciting force or moment and clearly, ~~$\{X\} \in$~~ $\{X\} \in \mathbb{C}$.

In order to evaluate $\{X\}$, we must solve for the scattered wave potential ϕ_7 for the given boundary conditions.

The problem of finding ~~excitat.~~ exciting force amplitude can be greatly simplified if we assume that the size and shape of the ship hull is such that the presence of the object does not affect the overall pressure distribution under the otherwise uninterrupted incident wave. A progressive incident wave has its own pressure distribution across the water depth and thus we can evaluate the ~~the~~ wave exciting force and moment on the ship hull using this pressure distribution alone provided the ship hull does not alter the incident wave. This approach is known as Froude-Krylov approximation.

Demonstration of Froude Krylov Approach



$$F_z = \int_{x=x_A}^{x=x_B} p \cdot dx$$

$$p = -\rho g z + \rho a g K_p(z) \cdot \cos(kx - \omega t)$$

where, $$K_p(z) = \frac{\cosh k(h+z)}{\cosh(kh)}$$

Similar to added mass and damping coefficients, the wave exciting force and moment amplitudes are specified as a function of wave frequency.

Body response in regular wave

For a floating body at zero speed oscillations in response to an incident wave experience the following force components:

1> The inertia force component. This is given as

$$[m]\{\ddot{\xi}\} = -\omega^2 [m]\,\mathrm{Re}\left[\{\xi_j\} e^{i\omega t}\right]$$

$$= -\omega^2 [m]\,\mathrm{Re}\left[\{\xi_j\} \cdot e^{i\omega t}\right]$$

2> The restoring force component given as

$$[C]\,\mathrm{Re}\left[\{\xi_j\} e^{i\omega t}\right]$$

3> The wave radiation and exciting force component $\cancel{\omega^2 [a]\,\mathrm{Re}\left[\{\xi_j\} e^{i\omega t}\right]}$

$$\omega^2 [a] \, Re\left[\{\eta\} \cdot e^{i\omega t}\right] - i \cdot \omega [b] \cdot Re\left[\{\eta\} \cdot e^{i\omega t}\right] + Re\left[A \cdot \{x\} \cdot e^{i\omega t}\right]$$

Thus, balancing these force components we have,

$$-\omega^2 [M] \, Re\left[\{\eta\} e^{i\omega t}\right] + [c] \, Re\left[\{\eta\} e^{i\omega t}\right] = \omega^2 [a] \, Re\left[\{\eta\} e^{i\omega t}\right] - i\omega [b] \, Re\left[\{\eta\} e^{i\omega t}\right] + Re\left[A \{x\} e^{i\omega t}\right]$$

or, $$\left\{-\omega^2([M]+[a]) + i\omega[b] + [c]\right\}\{\eta\} = A\{x\}$$

or, $$\{\eta\} = A \, [\alpha]^{-1} \{x\},$$ where,

$$[\alpha] \equiv -\omega^2([M]+[a]) + i\omega[b] + [c]$$

or, $$Z_j(\omega, \theta) = \frac{\eta_j}{A}, \quad \text{for } j = 1, 2, 3, \ldots, 6.$$

vector $\{\eta\}$ is known as frequency response and the ratio $Z_j(\omega, \theta)$ is known as Response amplitude operator (RAO). It depend upon ~~incident wave direction~~ frequency (ω) and incident wave direction (θ).

If we take only one member of the equation of motion for a specific degree of freedom say heave and consider that to be uncoupled from any other degree of freedom we may have

$$\left\{-\omega^2(m + a_{33}) + i\omega b_{33} + c_{33}\right\}\{\eta\} = A\{x_3\}$$

This type of relationship is ~~very~~ quite analogous to a single degree of freedom spring mass damper system. However, the difficulty here lies in the fact that a_{33}, b_{33} these are functions of frequency ω, and therefore it is not straightforward to solve for ~~this~~ the motion response η_3 even for the case when the motion is presumed to be uncoupled.

All the other members of the other matrices depend upon geometry of the vessel, location of center of gravity, center of buoyancy and center of floatation.

Wave encountering frequency for a ship with a forward speed

For a ship moving with a forward speed U, the frequency of the incident waves will be different from what the ship experiences when it is at rest.