Advanced Marine Hydrodynamics NA61202 3-1-0

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Lecture 7: Velocity Potential and Stream Functions

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Overview

Idea of a conservative field

From Lecture 4, we have $\oint_C \mathrm{V} \cdot \mathrm{d}r = \int_S (\nabla \times \mathrm{V})\, \mathrm{d}A$. and for an irrotational flow, $\oint_C \mathrm{V} \cdot \mathrm{d}r = 0$. Here C may be a closed curve as in *abcd*.

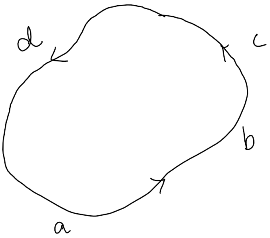

Figure: A closed contour.

- Suppose, $\mathrm{V} = \nabla\phi$, therefore,
$\int_a^c \nabla\phi \mathrm{d}r = \int_a^c \left(\frac{\partial\phi}{\partial x}\hat{\mathrm{i}} + \frac{\partial\phi}{\partial y}\hat{\mathrm{j}} + \frac{\partial\phi}{\partial z}\hat{\mathrm{k}}\right) \cdot \left(\mathrm{d}x\hat{\mathrm{i}} + \mathrm{d}y\hat{\mathrm{j}} + \mathrm{d}z\hat{\mathrm{k}}\right) =$
$\int_a^c \left(\frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z\right) = \int_a^c \mathrm{d}\phi = \phi_c - \phi_a.$
- the scalar function ϕ does not change at all for $\oint_C \nabla\phi \mathrm{d}r$. It is *conserved* as we start from *a*; travel across *abcd* and comes back to *a*.
- Therefore, we say there exists a conservative field for ϕ; corresponding to a velocity field V which satisfies $\oint_C \mathrm{V} \cdot \mathrm{d}r = 0$, i.e., an irrotational field.

The velocity potential

- Conventionally, the scalar ϕ is known as the velocity potential.
- Substitution of relation $\mathrm{V} = \nabla\phi$ into continuity equation gives $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi = 0.$
- This is the Laplace equation for ϕ.

The stream function

- Let us consider velocity components $u = u(x, z, t)$ and $w = w(x, z, t)$ for an incompressible flow in 2D. From continuity equation we get $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$.
- Suppose, there exists a scalar called stream function ψ such that $u = \frac{\partial \psi}{\partial z}$ and $w = -\frac{\partial \psi}{\partial x}$. In this case, the continuity equation is satisfied.
- The condition of irrotationality, i.e., $\nabla \times \mathrm{V} = 0$ for 2D gives $\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0$. Substitution for ψ gives $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \nabla^2 \psi = 0$. This is the Laplace equation for ψ.

Physical picture for stream function

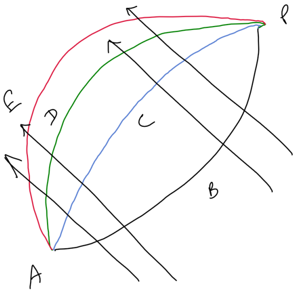

Figure: Various paths between point A and P.

- Consider two points A and P in a flow field. For an incompressible steady flow, the flow rate into the region $ABPCA$ through ABP should be same as through ACP.
- There may any arbitrary paths ADP and AEP, for all these the volume flow rates are all same.

Contd..

- The volume flow rate does not depend on the path joining A and P but only depends on the relative position of P with respect to A. We say that there is a *point* function named stream function for the flow rate.

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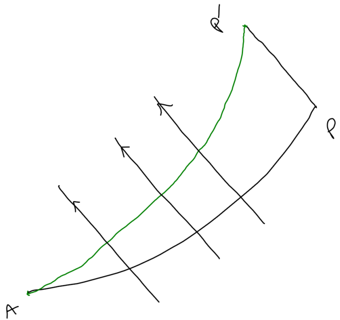


Figure: Stream function and the stream line.

- The flow rate is same through AP and AP', if PP' is steam line. Therefore, ψ is constant across stream line.
- We may use different stream lines at some intervals to imagine the flow.

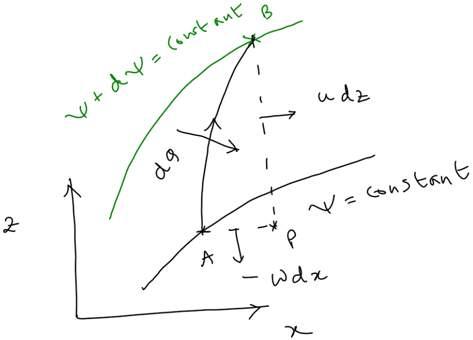


Figure: Stream function and the velocity.

- For the volume flow rate dq; d$q =$ dψ.
- From continuity, d$q = u\mathrm{d}z - w\mathrm{d}x$.

- $\mathrm{d}\psi = \frac{\partial \psi}{\partial x}\mathrm{d}x + \frac{\partial \psi}{\partial z}\mathrm{d}z$.
- By comparison, we get $u = \frac{\partial \psi}{\partial z}$ and $w = -\frac{\partial \psi}{\partial x}$.

The relation between stream function and the velocity potential

- Since both velocity potential and the stream functions are continuously differentiable functions of (x, z) , we can choose any arbitrary point P where both velocity potential and stream function exist.
- Imagine two contours passing through a common point P. In one contour ϕ is constant (i.e., the equipotential line) and in another, ψ is constant (i.e., the stream line).
- Since both ϕ and ψ are scalar; $\nabla\phi$ and $\nabla\psi$ represents two vectors which we can get at point P.
- $\nabla\phi \cdot \nabla\psi = \left(u\hat{\imath} + w\hat{k}\right) \cdot \left(\frac{\partial\psi}{\partial x}\hat{\imath} + \frac{\partial\psi}{\partial z}\hat{k}\right) = u\frac{\partial\psi}{\partial x} + w\frac{\partial\psi}{\partial z} = -uw + uw = 0.$
- The equipotential and the stream lines are mutually orthogonal.

Some facts I

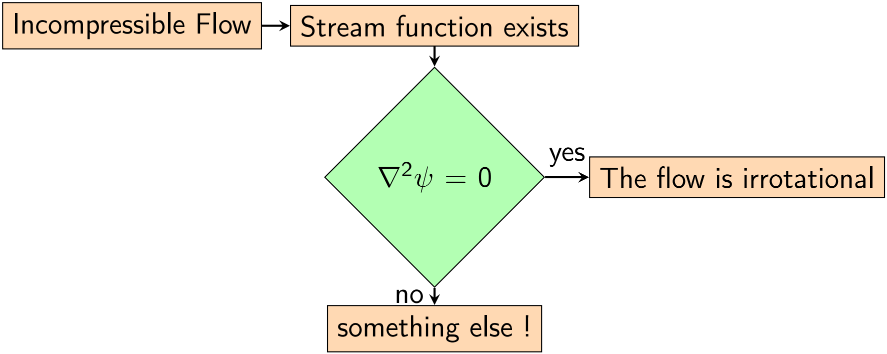
Question

Is there a stream function in 3D?

Answer

- A stream line has to be a plane in 3D. Thus at any flow field, there may be many directions (all of which are in a plane tangential to the hypothetical stream plane) for the flow velocities at a single point. This violates a fundamental property of the flow.
- If the flow is axisymmetric; we can imagine the flow to be in the plane of symmetry as 2D. The selection of the plane of symmetry conform to a selection of coordinate frame.
- The stream function defined in a plane of symmetry is known as the stokes stream function.

Some Facts II



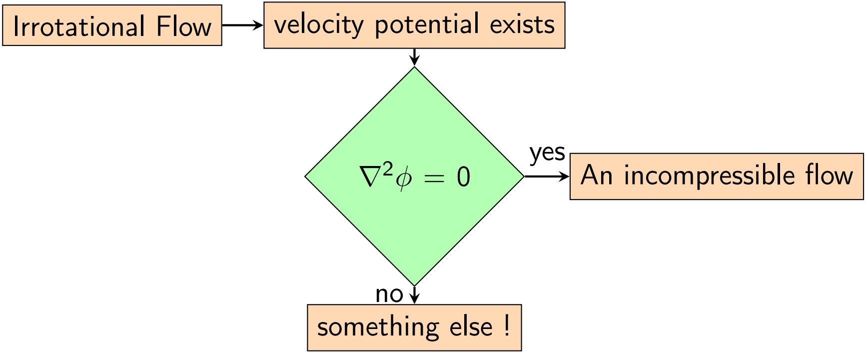

Irrotational Flow
velocity potential exists
$\nabla^2\phi = 0$
yes
An incompressible flow
no
something else !

Figure: Schematic of the problem

For the variation of density over time as measured in a test as shown above, is it possible to determine if the flow is nondivergent or not from this information alone ?

Assignment L6A2

The velocity components in a 2D flow of an inviscid fluid are $u = \frac{Kx}{x^2+z^2}$ and $w = \frac{Kz}{x^2+z^2}$. Assume K to be a constant.

- Is the flow nondivergent ?
- Is the flow irrotational ?
- plot the two streamlines passing through points $A\,(1,1)$ and $B\,(1,2)$.

Bernoullis equation: Form generic to irrotational, inviscid and unsteady flow

- Owing to irrotationality, there exists a velocity potential ϕ. Substitution of ϕ into the Euler's equation gives $\frac{\partial}{\partial t}\frac{\partial \phi}{\partial x_i} + \frac{\partial \phi}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial \phi}{\partial x_i} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p + \rho g x_3\right).$
- again, $\frac{\partial \phi}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial \phi}{\partial x_i} = \frac{1}{2}\frac{\partial}{\partial x_i}\frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_j}$ by rule of differentiation.
- This gives $\frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_j}\right) = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p + \rho g x_3\right).$
- This after integration gives, $\frac{\partial \phi}{\partial t} + \frac{1}{2}\frac{\partial \phi}{\partial x_j}\frac{\partial \phi}{\partial x_j} = -\frac{1}{\rho}\left(p + \rho g x_3\right) + C\left(t\right).$
- By adopting a new potential $\phi' = \phi - \int_0^t C\left(t\right) \mathrm{d}t$, the constant $C\left(t\right)$ is left to be taken as arbitrary. Thus we may omit it.