Advanced Marine Hydrodynamics NA61202 3-1-0

S De Chowdhury [1]

[1]Assistant Professor

Department of Ocean Engineering and Naval Architecture
IIT Kharagpur

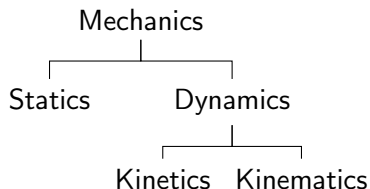Lecture 2: Some Background: Elements of Vector Analysis

sdip@naval.iitkgp.ac.in

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Overview

Where does the kinematics fit in ?

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              Mechanics
          ┌───────┴───────┐
       Statics         Dynamics
                    ┌──────┴──────┐
                 Kinetics    Kinematics
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- Kinematics is the study of motion without accounting for the causes, i.e., the forcing.
- Example, if we know fluid velocity at one point, how we can predict pressure at the same point and vice-versa ?
- key assumptions that we make to develop theories in wave-body interaction are based on principles of flow kinematics.

A control volume

- What is a control volume ? Typical Representation.
- Why do we need a control volume ?

- We fix our focus on a particular point in space and time.
- Then we track whatever is happening in this point.

The Lagrangian view point $\mathcal{L}$

- We fix our focus on a particle with an *identity*.
- Then we simply sit over this point and experience the journey with it.

An example

In a 1-D flow field, the velocity at a point may be given in $\mathcal{E}$ by $u = x + t$. Determine the displacement of a fluid particle whose initial position is x_0 at initial time t_0 in $\mathcal{L}$.
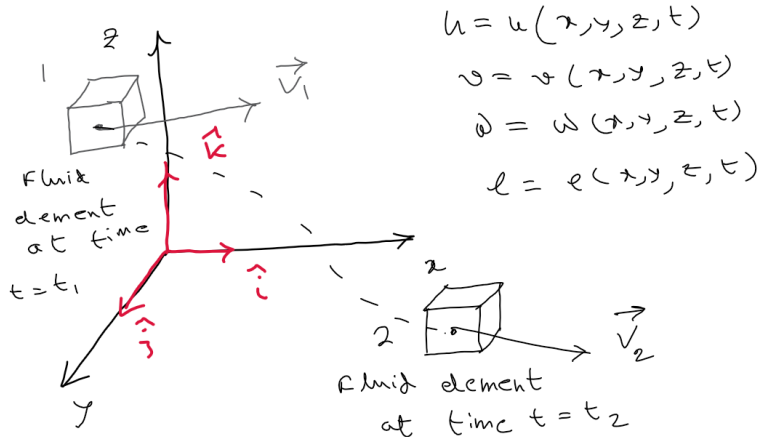
Total/ substantial derivative operator in the Cartesian Coordinate Frame

$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \mathsf{V} \cdot \nabla.$

- Derivation (physical+pure mathematical) in Cartesian coordinate frame.
- The first term is the local derivative showing local rate of change and the second terms is the convective derivative showing rate of change due to change in location.
- The final expression for total/ substantial derivative depends on the base coordinate system.

$$u = u(x, y, z, t)$$
$$v = v(x, y, z, t)$$
$$w = w(x, y, z, t)$$
$$\rho = \rho(x, y, z, t)$$

Fluid element at time $t = t_1$

Fluid element at time $t = t_2$

Contd..

- since, $\rho = \rho(x, y, z, t)$; we can write ρ_2 as an expansion around ρ_1 using a Taylor series as
$\rho_2 = \rho_1 + \left(\frac{\partial \rho}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_1 (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z}\right)_1 (z_2 - z_1) + \left(\frac{\partial \rho}{\partial t}\right)_1 (t_2 - t_1) + \text{higher order terms}$

- substracting ρ_1 from both sides and then dividing by $t_2 - t_1$, we get
$\frac{\rho_2 - \rho_1}{t_2 - t_1} = \left(\frac{\partial \rho}{\partial x}\right)_1 \frac{x_2 - x_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial y}\right)_1 \frac{y_2 - y_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial z}\right)_1 \frac{z_2 - z_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial t}\right)_1$

Contd..

- $\lim_{t_2 \to t_1} \frac{\rho_2 - \rho_1}{t_2 - t_1} = \frac{D\rho}{Dt}$; $\lim_{t_2 \to t_1} \frac{x_2 - x_1}{t_2 - t_1} = u$; $\lim_{t_2 \to t_1} \frac{y_2 - y_1}{t_2 - t_1} = v$ and $\lim_{t_2 \to t_1} \frac{z_2 - z_1}{t_2 - t_1} = w$.

- Thus, we have $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$.

- $\frac{D\rho}{Dt}$ is the instantaneous time rate of change of density of the fluid element as it moves trough point 1.

- This is different from $\frac{\partial \rho}{\partial t}$ which is the time rate of change of density at a fixed point 1.

- since $\rho = \rho(x, y, z, t)$ from chain rule we have
  $d\rho = \frac{\partial \rho}{\partial x}dx + \frac{\partial \rho}{\partial y}dy + \frac{\partial \rho}{\partial z}dz + \frac{\partial \rho}{\partial t}dt$.
- which with some manipulations becomes
  $\frac{d\rho}{dt} = \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt} = \frac{\partial \rho}{\partial x}u + \frac{\partial \rho}{\partial y}v + \frac{\partial \rho}{\partial z}w + \frac{\partial \rho}{\partial t}$.

- For a steady flow, $\frac{\partial}{\partial t} = 0$.
- For a uniform flow, any flow variable say $u \neq u(x, y, z)$.

Flow Type	Temporal derivative	Convective derivative
steady and uniform	0	0
steady and non-uniform	0	$\neq 0$
Unsteady and uniform	$\neq 0$	0
Unsteady and non-uniform	$\neq 0$	$\neq 0$

A simple example

L3A1 A 2D pressure field $p = 4x^3 - 2y^2$ is associated with a velocity field given by $V = (x^2 - y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j}$. What is the rate of change of pressure at a point $(2, 1)$?

- This is an imaginary line in the flow field where the tangent at any point gives us the velocity at this point.
- Equation of a stream line: $V \times \delta R$ for a stream line segment with end-points at R_{i-1} and R_i.
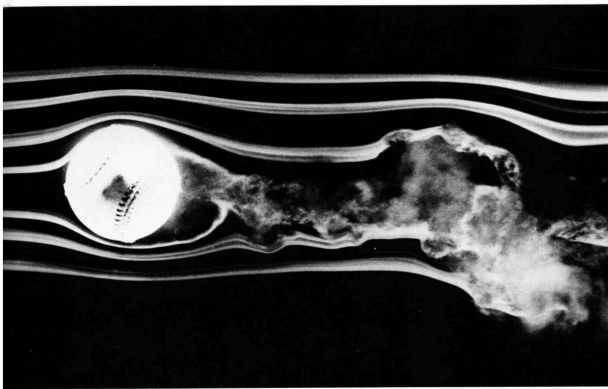- Why don't we use $V \cdot \delta R$ as an equation of a stream line ?

The path lines

A path line is the trajectory of a fluid particle for which we capture its location at various instants of time.

The streak lines

A Streak line (at a given instant of time) is the locus of many fluid particles which cross a common point in the flow field at some earlier time.
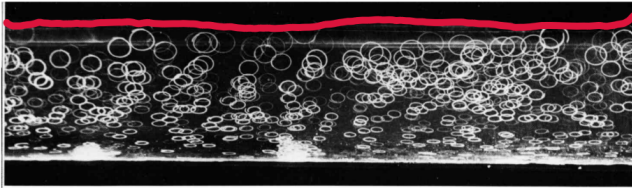
Example: Streak Lines

Figure: Streak lines over a spinning (in the same plane as in the image) basket ball rotating at 630 rpm in a outside flow with 77ft/s. Source: An Album of Fluid Motion by Milton Van Dyke.
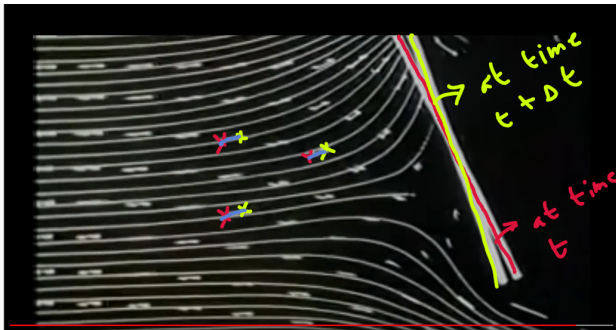
Example: Path Lines
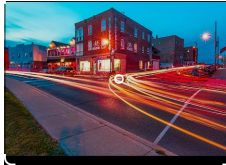
Figure: Photograph taken at a street corner using a DSLR camera and with long exposure. Source YouTube: Visual Art Photography Tutorials-10 Tips For Long Exposure Photography Light Trails.

Figure: Pathlines beneath a progressive wave. Source: An Album of Fluid Motion by Milton Van Dyke.

Example: Stream Lines

Figure: Streamlines for a flow over an oscillating plate. Source YouTube: Flow Visualization by Stephen J Kline.

L3A2 Why for a steady flow, the steam lines, the path lines and the streak lines all has to be the same ?