Advanced Marine Hydrodynamics NA61202 3-1-0

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Lecture 8: Elementary Flows Part I

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Overview

What is superposition ?

A superposition is a process of combining two or more objects to construct a new object. The objects being added fully retain their features in the new object. Example, standing waves on water, light scattering.

- If y_1 and y_2 are the solutions for a differential equation $f(x) = Z$ then $y_1 + y_2$ is also a solution.
- Consistent with the idea of reversibility/ inviscid flow.
- adding ink into water is *not* superposition.

Uniform Flow

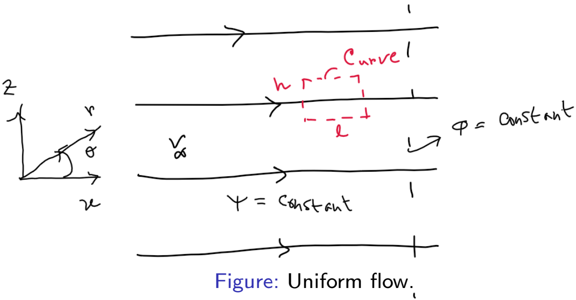

Figure: Uniform flow.

- Satisfies $\nabla \cdot V = 0$ and $\nabla \times V = 0$.
- We can construct both the ϕ and ψ.

The velocity potential and the stream function for Uniform flow

- For velocity potential we have, $\frac{\partial \phi}{\partial x} = u = V_\infty$ and $\frac{\partial \phi}{\partial z} = w = 0$. Integrating with respect to x, we have $\phi = V_\infty x + f(z)$ and with respect to z, we have $\phi = \text{constant} + g(x)$.
- By comparison, we have, $\phi = V_\infty x$.
- For stream function, $\frac{\partial \psi}{\partial z} = u$ and $\frac{\partial \psi}{\partial x} = -w$. Integrating and comparing we have $\psi = V_\infty z$.
- For transformation to polar coordinates $\phi = V_\infty r\cos(\theta)$ and $\psi = V_\infty r\sin(\theta)$. In either coordinate frame, they satisfy Laplace equation.
- ϕ and ψ are mutually perpendicular.

Circulation and Irrotationality

- The circulation is defined as $\hat{\tau} = -\oint_C \mathrm{V} \cdot \mathrm{dr}$. For a closed curve as shown in the Figure, we have $\hat{\tau} = -V_\infty \cdot l + 0 \cdot h + V_\infty \cdot l - 0 \cdot h = 0$.
- Thus, an uniform flow is irrotational and $\hat{\tau} = -\int_S (\nabla \times \mathrm{V})\, \mathrm{dS} = 0$.

A source/ sink flow

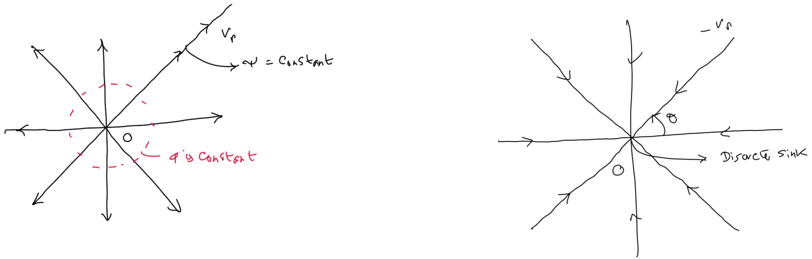

Figure: Source and sink flow.

- Consider everything in a polar coordinate system.
- For the velocity components, $V_\theta = 0$ and V_r is inversely proportional to the radial distance from the origin at O.
- With this information, we can prove that $\nabla \cdot \mathrm{V} = 0$ everywhere except at the origin. The origin is also referred as a singularity. The flow is irrotational excluding the origin.

Relations in polar coordinate system

- For velocity potential $V_r = \frac{\partial \phi}{\partial r}; V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$ and $V_z = \frac{\partial \phi}{\partial z}$.
- For stream function $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}; V_\theta = -\frac{\partial \psi}{\partial r}$.
- The continuity equation is $\frac{\partial}{\partial r}(V_r r) + \frac{\partial}{\partial \theta} V_\theta = 0$.

Utilization

- We can view the situation as a flow being induced by a source/ sink of a given strength.
- A flow over an object of any shape (i.e., an aerofoil) can be represented by a suitable distribution of source and sinks.

Physical interpretation of source strength

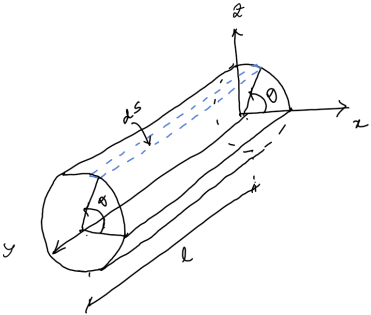

Figure: Source strength derivation.

- We recall $V_r = \Lambda/r$ and $V_\theta = 0$ with Λ being a constant and known as source strength.
- Consider a cylinder where there are many such sources along its axis, i.e., form $y = 0$ to $y = l$.

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- For a circular cross section as in the origin, the mass flux across an elemental curve $rd\theta$ is $V_r \cdot rd\theta$ for an unit width perpendicular to that cross section.
- Total flux $\dot{m}$ across the surface is $\int_0^{2\pi} \rho V_r r l d\theta = 2\pi \rho r l V_r$.
- $\hat{v} = \frac{\dot{m}}{\rho} = 2\pi r l V_r$.
- If we use $\Lambda = \hat{v}/l$ we get $\Lambda = 2\pi r V_r$ or $V_r = \frac{\Lambda}{2\pi r}$.
- Source strength is rate of volume flux from the source per unit depth perpendicular to the cross section.

Velocity potential and stream function for the source flow

- $\frac{\partial \phi}{\partial r} = V_r = \frac{\Lambda}{2\pi r}$ and $\frac{1}{r}\frac{\partial \phi}{\partial \theta} = V_\theta = 0$. Integrating both and comparing we get $\phi = \frac{\Lambda}{2\pi} \ln r$.
- $\frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_r = \frac{\Lambda}{2\pi r}$ and $\frac{\partial \psi}{\partial r} = V_\theta = 0$. Integrating and comparng we have $\psi = \frac{\Lambda}{2\pi}\theta$.
- With these we can prove that both ϕ and ψ satisfies Laplace's equation and $\hat{\tau} = 0$.

Uniform+Source flow

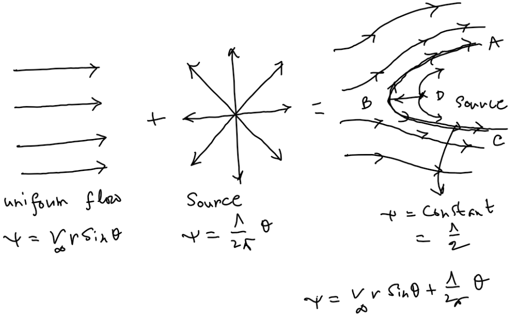

Figure: Superposition of uniform and a source flow.

- Owing to superposition, we have $\psi = V_\infty r\sin(\theta) + \frac{\Lambda}{2\pi}\theta$. for the flow on the right. This also satisfies the Laplace equation and irrotationality.
- The equation of a stream line of the combined flow is $\psi = V_\infty r\sin(\theta) + \frac{\Lambda}{2\pi}\theta = \text{constant}$.

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- For the velocity field, we have $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_\infty \cos(\theta) + \frac{\Lambda}{2\pi r}$ and $V_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin(\theta)$.
- If we set velocity as zero, we get $(r, \theta) = (\Lambda / 2\pi V_\infty, \pi)$. This is the stagnation point.
- Substitution for the stagnation point into the equation for ψ gives $\psi = \frac{\Lambda}{2} =$ constant. This is shown as a stream line along the body surface.

Observations

- The flow outside $\psi = \frac{\Lambda}{2}$ or ABC forms the freestream; whereas the flow inside ABC is due to the source. The stream line given by $\psi = \frac{\Lambda}{2}$ is a dividing stream line.
- If we take an uniform flow and superimpose a source of strength Λ on it at a point D then what we can achieve is a flow over an object which is semi infinite.

Uniform+source+sink flow

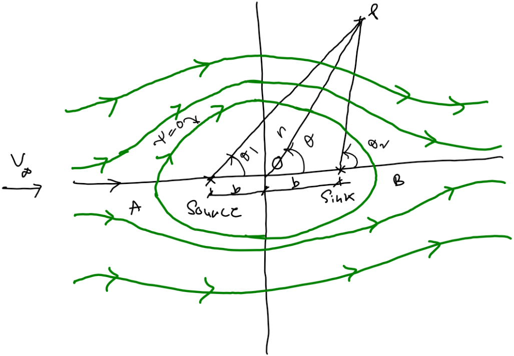

Figure: A Rankine oval.

- Consider a source and a sink of equal strengths kept on similar distance from the origin on either sides.
- For this case, $\psi = V_\infty r\sin(\theta) + \frac{\Lambda}{2\pi}\left(\theta_1 - \theta_2\right)$.
- $\tan(\theta_1) = \frac{r\sin(\theta)}{r\cos(\theta)+b}$ and $\tan(\theta_2) = \frac{r\sin(\theta)}{r\cos(\theta)-b}$.

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- with the above information, we can show that $OA = OB = \sqrt{b^2 + \frac{\Lambda b}{\pi V_\infty}}$ are the two stagnation points.
- The stagnation streamline (i.e., the dividing streamline) is $\psi = V_\infty r \sin(\theta) + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) = 0$. This gives the shape of a Rankine Oval.