

Advanced Marine Hydrodynamics NA61202 3-1-0

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Lecture 5: Conservation Laws (Part I)

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Spring 2023



1 Stresses in a Fluid Element

2 Transport Theorem

3 Conservation Laws

Idea of Fluid Stresses

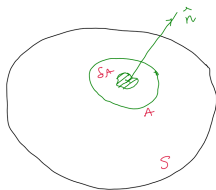


Figure: Traction on an elemental area.

- Let δF be the force exerted by the Fluid exterior to δA in the direction of \hat{n} on the inner side of δA .
- We can define
$$T^n = \lim_{\delta A \rightarrow 0} \frac{\delta F}{\delta A} = \frac{dF}{dA}.$$
- The *traction* T^n is a vector telling us about force per unit area and having components T_j^n .
- There is a relation which relates T^n with stresses.

Stress components on a cube

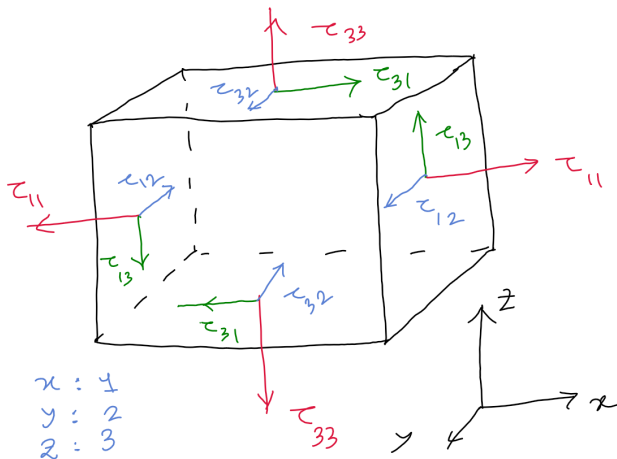


Figure: Stress components.

Stress components on a tetrahedral element

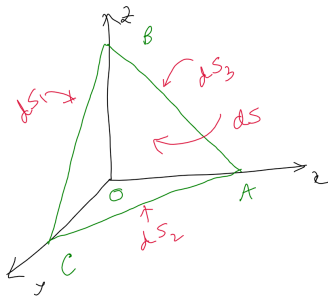


Figure: Stress components on a tetrahedral element.

- $dS_1 = dS n_1$; $dS_2 = dS n_2$ and $dS_3 = dS n_3$.
- volume of the element $dV = h dS / 3$.
- T^n be the traction on the surface dS .
- Let b_1 be the body force per unit mass.

Stress Equilibrium

- Let a_1 is the acceleration of the element along x .
- If the mass of the element is $dm = \rho h dS/3$ and acceleration is a_1 , then as per Newton's 2nd Law,
$$(-\tau_{11}dSn_1 - \tau_{21}dSn_2 - \tau_{31}dSn_3 + T_1^n dS) + b_1 h dS/3 = \rho h dS a_1/3.$$
- As $h \rightarrow 0$, $T_1^n = \tau_{11}n_1 + \tau_{21}n_2 + \tau_{31}n_3 = \sum_{j=1}^3 \tau_{j1}n_j$.
- in matrix-vector form
$$\begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$
- The above is known as Cauchy's Theorem and the stress components are represented by a Tensor.

Transport Theorem

- $I(t) = \int_{V(x,y,z,t)} f(x,y,z,t) dV.$
- $\Delta I = I(t + \Delta t) - I(t) = \int_{V(t+\Delta t)} f(x,y,z,t + \Delta t) dV - \int_{V(t)} f(x,y,z,t) dV.$
- Assuming in Δt time interval, the volume V acquires a new volume $V + dV$. Then, $\Delta I = \int_{V+\Delta V} (f + \Delta t \frac{\partial f}{\partial t}) dV - \int_V f dV = \int_V f dV + \Delta t \int_V \frac{\partial f}{\partial t} dV + \int_{\Delta V} f dV + \Delta t \int_{\Delta V} \frac{\partial f}{\partial t} dV - \int_V f dV.$
- The first and the fifth term together cancels out. If the volume V is surrounded by the surface S then ΔV is the volume contained within $S(t)$ and $S(t + \Delta t)$. This region has a thickness $V_n \Delta t$, where V_n is the component of the velocity in the direction normal to surface S_n . Thus, $\Delta t \int_{\Delta V} \frac{\partial f}{\partial t} dV = \Delta t \int_S \frac{\partial f}{\partial t} (V_n \Delta t) dS = (\Delta t)^2 \int_S \frac{\partial f}{\partial t} (V_n) dS.$ This can be neglected since we are only retaining terms upto order of Δt .

- Using the same principle, the third term becomes

$$\int_{\Delta V} f dV = \int_S f V_n \Delta t dS.$$
- $\Delta I = \Delta t \int_V \frac{\partial f}{\partial t} dV + \int_S (V_n \Delta t) f dS.$
- $\frac{dI}{dt} = \int_V \frac{\partial f}{\partial t} dV + \int_S f V_n dS.$
- Application of Gauss divergence theorem on the second term gives,

$$\frac{dI}{dt} = \int_V \left(\frac{\partial f}{\partial t} + \nabla \cdot (f \mathbf{V}) \right) dV$$

Conservation of Mass

$$\frac{d}{dt} \int_V \rho dV = 0.$$

Conservation of Momentum

- $\frac{d}{dt} \int_V \rho u_i dV = \int_S \tau_{ij} \hat{n}_j dS + \int_V F_i dV.$
- The first term can be simplified using Gauss Divergence Theorem and $\frac{d}{dt} \int_V \rho u_i dV = \int_V \left[\frac{\partial \tau_{ij}}{\partial x_j} + F_i \right] dV.$

Continuity Equation

- By putting $f = \rho$ into transport theorem
$$\frac{d}{dt} \int_V \rho dV = \int_V \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) \right] dV.$$
- $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) = 0.$

Euler's Equation

- By putting $f = \rho u_i$ into transport theorem $\frac{DV}{Dt} = \frac{1}{\rho} \frac{\partial \tau_{ij}}{\partial x_j} + \frac{F_i}{\rho}$.