Advanced Marine Hydrodynamics NA61202 3-1-0

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Lecture 9: Elementary Flows Part III

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Overview

A Free Vortex Flow

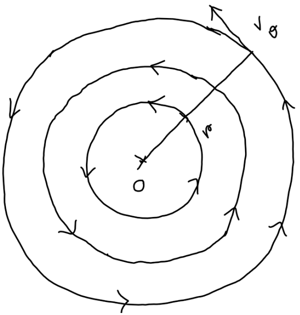

Figure: A Free Vortex Flow

- For this flow type, we recall from Lecture 4 that $V_r = 0$ and $V_\theta = \text{constant}/r$.
- It can be easily verified that a vortex flow is a physically possible incompressible flow, i.e., $\nabla \cdot \mathrm{V} = 0$ and irrotational, i.e., $\nabla \times \mathrm{V} = 0$ in every point except the origin in the flow-field.

Evaluation of the constant

- We recall that the circulation is $\hat{\tau} = -\oint_C \mathrm{V} \cdot \mathrm{d}r = -V_\theta 2\pi r$. This gives $V_\theta = -\frac{\hat{\tau}}{2\pi r}$.
- Comparing we get constant $= -\hat{\tau}/2\pi$.
- Thus we can write $\hat{\tau} = -2\pi C$ for circulation. By convention we call $\hat{\tau}$ as the strength of the vortex flow. For the sign convention vortex of positive strength means negative V_θ i.e., a vortex of positive strength rotates clockwise.
- Vorticity is a tendency in a flow: this clearly results in circulation which we can measure. Example: Feeling hot or cold and measuring temperature.

What happens at the origin ?

- We know $\hat{\tau} = -\oint_C \mathrm{V} \cdot \mathrm{d}r = -\int_S (\nabla \times \mathrm{V}) \cdot \mathrm{d}S$ by Stokes theorem.
- By comparison, $2\pi C = \int_S (\nabla \times \mathrm{V}) \cdot \mathrm{d}S$.
- Since both $\nabla \times \mathrm{V}$ and $\mathrm{d}S$ are in the same direction, we can take $(\nabla \times \mathrm{V}) \cdot \mathrm{d}S = |\nabla \times \mathrm{V}| \mathrm{d}S$.
- Thus, $2\pi C = \int_S |\nabla \times \mathrm{V}| \mathrm{d}S$. The surface integral is taken over the circular area inside the streamline along which the circulation $\hat{\tau} = -2\pi C$.

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- In the limit as $r \to 0$, $\int_S |\nabla \times \mathrm{V}| \mathrm{d}S = |\nabla \times \mathrm{V}| \mathrm{d}S$.
- This gives $|\nabla \times \mathrm{V}| = \frac{2\pi C}{\mathrm{d}S}$ as $r \to 0$.
- However, as $r \to 0, \mathrm{d}S \to 0$ thus $|\nabla \times \mathrm{V}| \to \infty$.
- The vortex flow is *not irrotational* at the origin. This is a singular point in the flow field.

The velocity field due to a vortex flow

- We had $V_r = \frac{\partial \phi}{\partial r} = 0$ and $V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\hat{\tau}}{2\pi r}$. Integrating and then comparing for constant we get $\phi = -\frac{\hat{\tau}}{2\pi}\theta$.
- Similarly, $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$ and $V_\theta = \frac{\partial \psi}{\partial r} = -\frac{\hat{\tau}}{2\pi r}$. Again integrating and comparing we have $\psi = \frac{\hat{\tau}}{2\pi} \ln\left(r\right)$.

Force components on an Aerofoil under a uniform flow

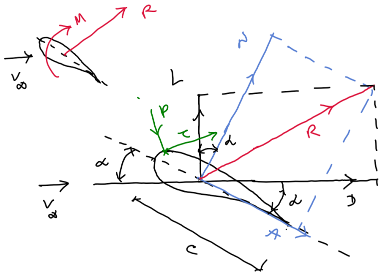

Figure: Force components on an aerofoil

- The total force experienced by the aerofoil section is due to pressure p and a the shear stress τ. Combination of these two leads to a net resultant force R and a moment M.
- The angle at which the flow incidents on the aerofoil is the angle of atack α.
- The chord c is the distance between the leading edge and the trailing edge.
- The force components L and D are called the Lift and the Drag respectively for a Cartesian basis; whereas the forces N and A are the normal and axial forces defined with respect to the chord line.

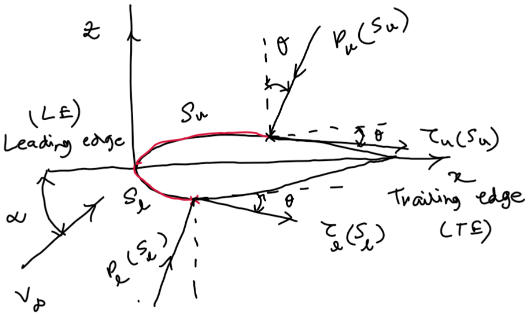

Figure: Pressure and stress definitions on the aerofoil.

Total lift and drag forces are found by simply resolving the forces along an axis and then integrating over the upper and lower surfaces.

Some useful dimensionless parameters

Assuming $q_\infty = \rho V_\infty^2/2$ is the dynamic pressure and S is a reference area; we have the following

- Lift coefficient $C_L = \frac{L}{q_\infty S}$.
- Drag coefficient $C_D = \frac{D}{q_\infty S}$.
- Normal force coefficient $C_N = \frac{N}{q_\infty S}$.
- Axial force coefficient $C_A = \frac{A}{q_\infty S}$.
- pressure coefficient $C_p = \frac{p - p_\infty}{q_\infty}$.

For 2D flows, we take the width perpendicular to the plane of the flow as 1 and re-express these above parameters using lowercase letter (i.e. c_l inplace of C_L) to denote forces per unit span.

Expression of lift and drag coefficients using pressure coefficient

After the force integration and non-dimensionalization by $q_\infty S$ and $S = c\,(1)$, we can get the final result for the lift coefficient $c_l = \frac{1}{c}\int_0^c C_{p,l}\mathrm{d}x - \frac{1}{c}\int_0^c C_{p,u}\mathrm{d}x.$

Superposition of nonlifting flow with a vortex flow

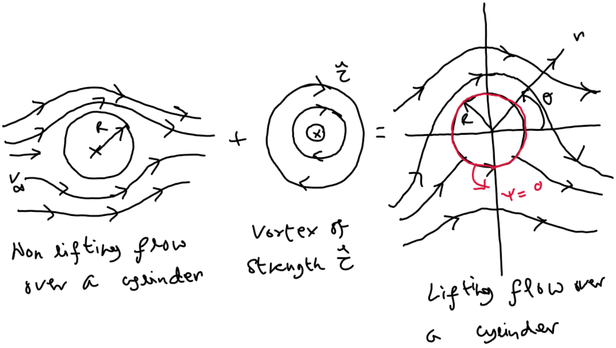

Figure: Superposition of nonlifting flow with a vortex flow.

The stream function is $\psi = V_\infty r \sin(\theta)\left(1 - \frac{R^2}{r^2}\right) + \frac{\hat{\tau}}{2\pi}\ln\left(\frac{r}{R}\right)$, where we take the arbitrary constant as $-\ln(R)$.

Stagnation points for the lifting flow over a cylinder

- The velocity components are $V_r = V_\infty \cos(\theta)\left(1 - \frac{R^2}{r^2}\right)$ and $V_\theta = -V_\infty \sin(\theta)\left(1 + \frac{R^2}{r^2}\right) - \frac{\hat{\tau}}{2\pi r}$.
- For stagnation points are found by setting V_θ and V_r as zero. For this case, we find the solution (R, θ) where $\theta = \sin^{-1}\left(-\frac{\hat{\tau}}{4\pi V_\infty R}\right)$.

Different cases for the stagnation points

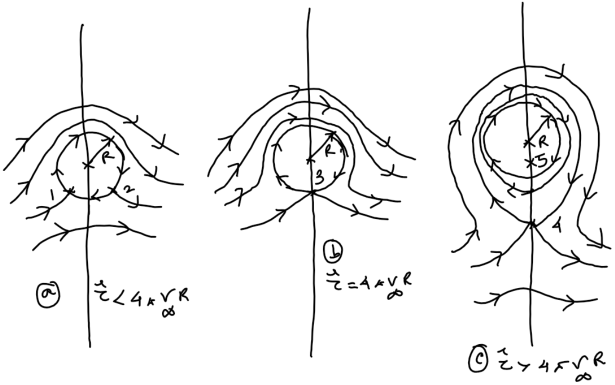

Figure: Possible cases for the stagnation points for the lifting flow over a cylinder.

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- For case a, the θ is either of third or fourth quadrants.
- For case b, there is only position.
- For case c, we take $\theta = -\pi/2$ into V_θ and that leads to a quadratic equation for r as $r^2 V_\infty - \frac{\hat{\tau}}{2\pi} r + R^2 V_\infty = 0$ which after solving gives $r = \frac{\hat{\tau}}{4\pi V_\infty} \pm \sqrt{\left(\frac{\hat{\tau}}{4\pi V_\infty}\right)^2 - R^2}$. The point which is outside the cylinder, i.e., point 4 in the figure is only realistic.
- There is no drag due to symmetry.

The lift force

- Velocity of the surface of the cylinder is obtained as $V = V_\theta\,(r = R) = -2V_\infty \sin(\theta) - \frac{\hat{\tau}}{2\pi R}$.
- We use the results obtained for the aerofoil section and with simplifications for the cylinder we get $c_l = -\frac{1}{2}\int_0^{2\pi} C_p \sin(\theta) \mathrm{d}\theta$.
- Substitution for C_p gives $c_l = \frac{\hat{\tau}}{RV_\infty}$.
- Thus the lift force per unit span is $\hat{L} = q_\infty S c_l$. If we take $S = 2R\,(1)$ as the projected area then it can be shown that $\hat{L} = \rho_\infty V_\infty \hat{\tau}$. This is the Kutta-Joukowski theorem.