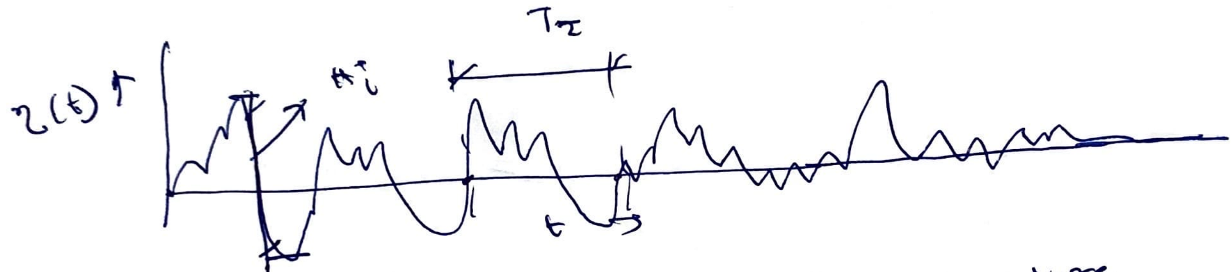
A brief overview on irregular waves

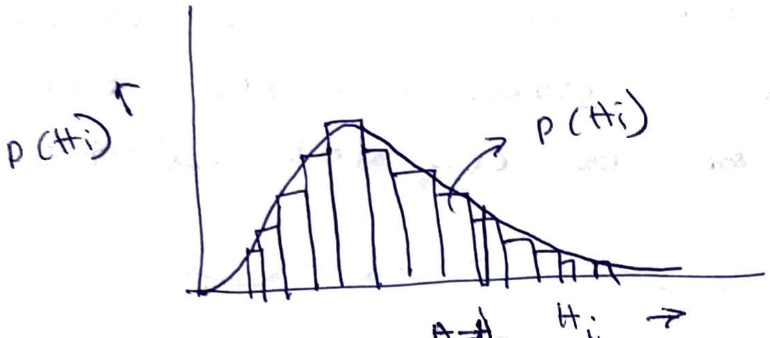
A real ocean field does not look like ~~regular~~, to be composed of regular, monochromatic sinusoidal waves but random or irregular waves. We need to consider means to analyse an irregular wave field ~~from~~ from statistical point of view.

If we measure the wave field by using a wave gauge at a fixed point, we will have a time history of wave amplitude. This ~~can~~ can be as shown in the picture ~~below~~ below:



From this time history, we can measure various wave heights, i.e., H_i; up crossing ~~period~~ period (T_z) etc. We can create a histogram of wave heights based on this time history of wave amplitude as

H_i Wave heights (H_i)		Probability $p(H_i) = \frac{n(H_i)}{N} \times 100\%$.
0 – 1 m	$\rightarrow$	p_1
1 – 2 m	$\rightarrow$	p_2
⋮	⋮	
6 – 12 m	$\rightarrow$	p_N.



It is found from various measurements that the wave heights follow a Rayleigh probability distribution. Similar information we can also obtain if we have a measurement of a irregular wave profile over space, i.e., over x.

$$p(H_i > H_a) = \int_{H_a}^{\infty} p(H_i)\, dH_i \qquad (1)$$

The above expression gives the probability of having waves of height greater than H_a.

The Rayleigh probability distribution is given as

$$p(H_i) = \frac{2H_i}{H_{rms}^2}\, e^{-(H_i/H_{rms})^2}$$

Fourier analysis ~~to fit~~ decompose a random wave signal to several regular monochromatic waves

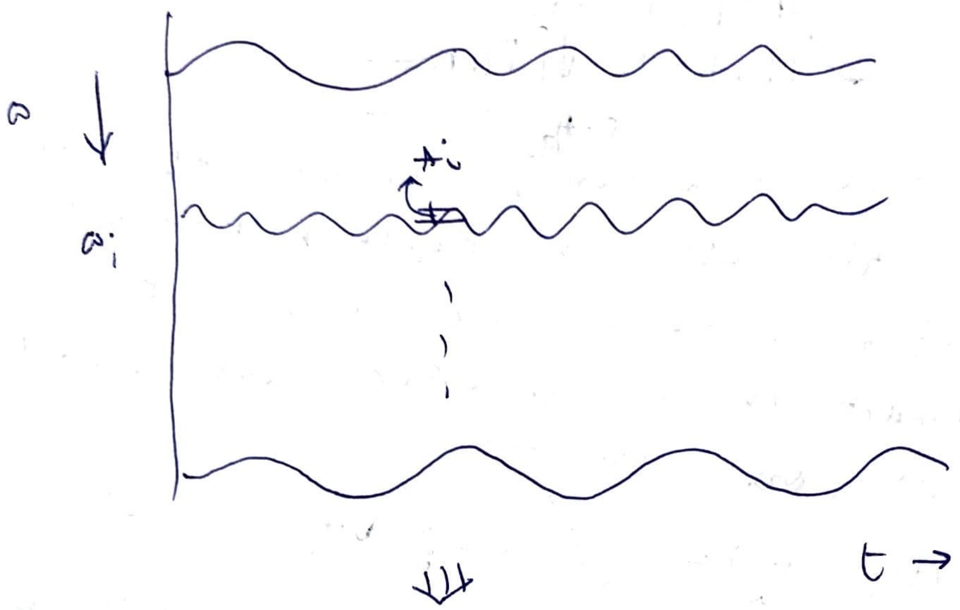
$$\eta(t) = A_0 + \sum_{i=1}^{N} A_i \cos(k_i x - \omega_i t + \epsilon_{Ai}) + B_i \sin(k_i x - \omega_i t + \epsilon_{Bi})$$

$$\equiv A_0 + \sum_{i=1}^{N} \bar{A} \cos(k_i x - \omega_i t + \epsilon_i) \qquad (2)$$

where, ϵ_{Ai}, ϵ_{Bi} are random phases.

The Fourier analysis gives us the various amplitudes A_i, B_i s, that by

Superimposing these various waves we are able to ~~first sort~~ construct a random time history with ~~an~~ ~~an~~ equivalent statistics.



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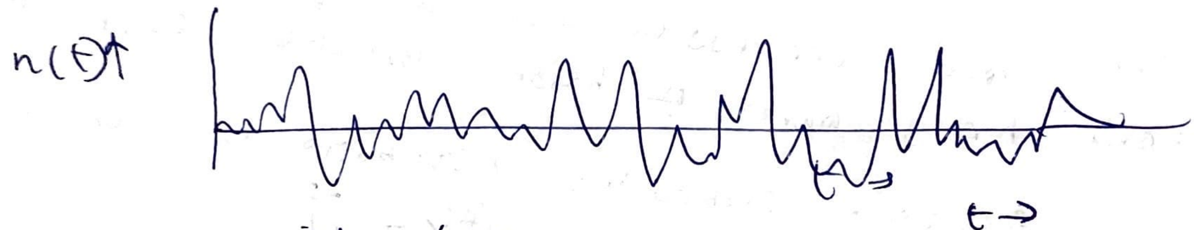
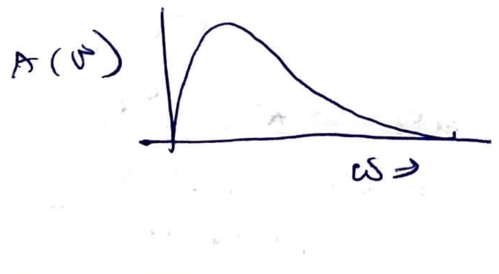


This is possible because the linear wave theory allows solution the linear boundary value problem as superposition of known solutions.

Wave ~~spec~~ Spectrum

$$\bar{E} = \frac{1}{2}\rho g A^2 \quad \text{or,} \quad \left(\frac{E}{\rho g}\right)_i = \frac{1}{2} A_i^2(\omega)$$

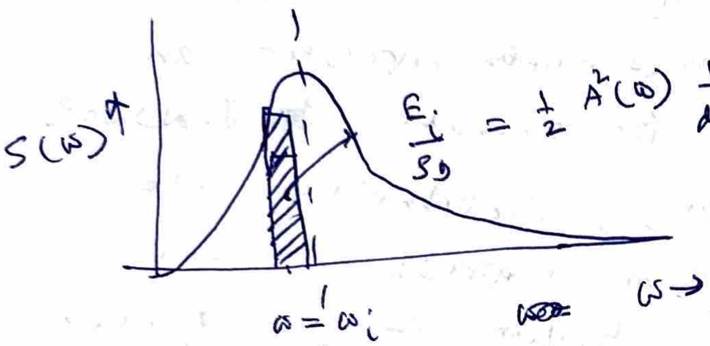
because, from fourier analysis we can get a distribution of $A(\omega)$ over various wave frequency ω. This distribution is known as amplitude spectrum.



a continuous form of discrete fourier transform as shown in equation ① ② could be

$$\cancel{\eta(t) = \int^{\infty} A(\omega)\cos(k_i x - \omega_i t + \epsilon_i)\, d\omega_i}$$

$$\eta(t) = \int_0^{\infty} A(\omega)\cos(kx - \omega t + \epsilon)\, d\omega. \qquad -③$$



$$\frac{E_i}{\frac{1}{SD}} = \frac{1}{2} A^2(\omega) \frac{1}{d\omega} = S(\omega_i)$$

Energy under the wave spectrum gives the total energy of the irregular wave field. That is energy of the irregular wave field. given by ~~m_0~~ $= \int_0^{\infty} S(\omega)\, d\omega = \cancel{\frac{1}{2}\left(\frac{H_{1/3}}{2}\right)^2 = \left(\frac{H_{1/3}}{4}\right)^2}\ m_0.$

if $m_n = \int_0^{\infty} \omega^n S(\omega)\, d\omega$, then, $m_0 = \int_0^{\infty} S(\omega)\, d\omega$.

~~then $H^2_{1/3} = 8 m_0$, or, $H_{1/3} = 2\sqrt{2}\sqrt{m_0}$~~

$H_{1/3} = 4\sqrt{m_0}$; where, $H_{1/3}$ is the average of ~~first~~ heighest 1/3 rd of waves and known as significant wave height.

Zero up crossing period T_Z is given as

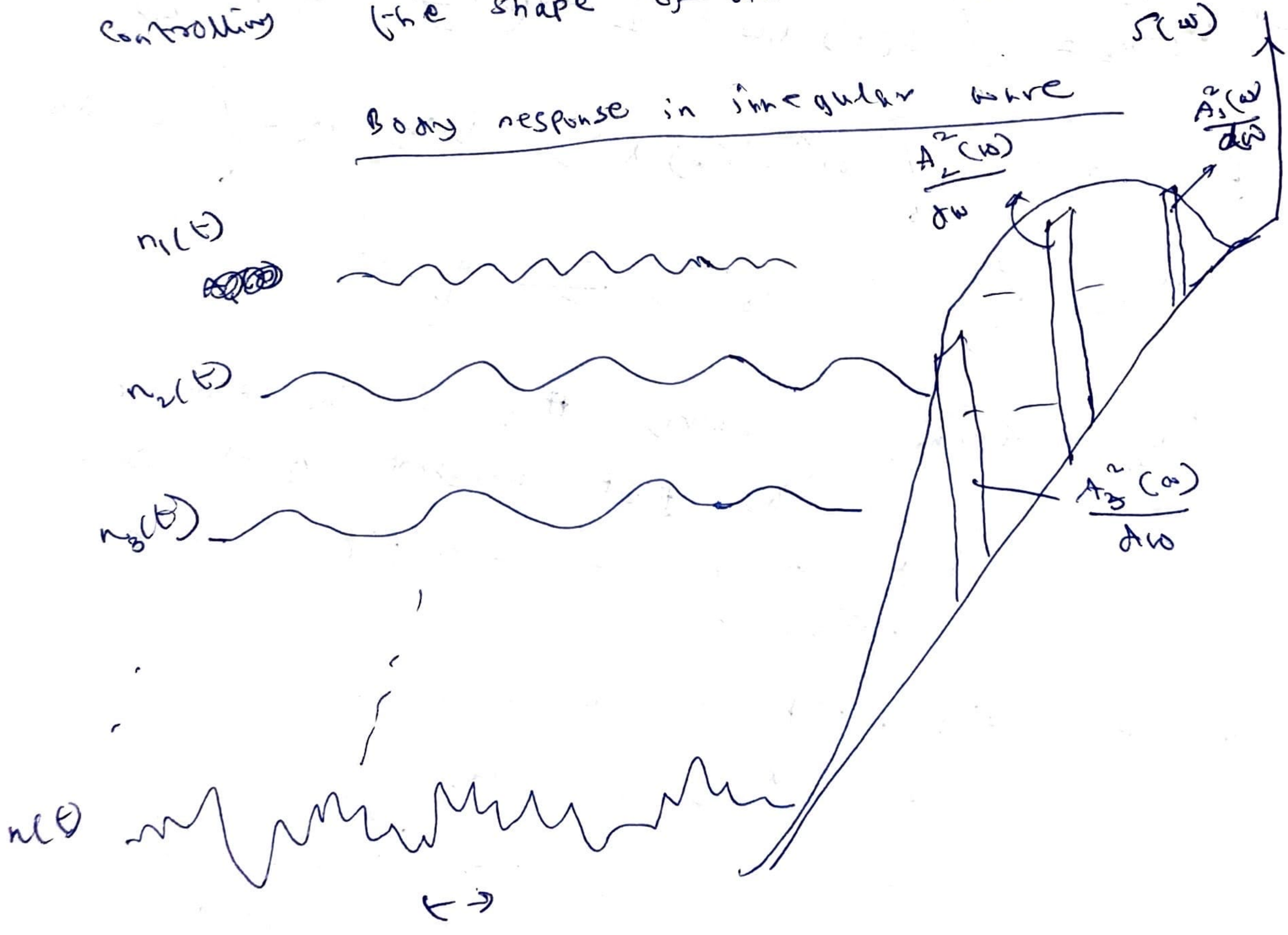
$$T_Z = 2\pi \frac{m_0}{m_1}, \quad \text{since,} \quad \omega_z = \frac{\int_0^{\infty} \omega S(\omega)\, d\omega}{\int_0^{\infty} S(\omega)\, d\omega} = \frac{m_1}{m_0}$$

The ~~period~~ frequency at which $S(\omega)$ is maximum is known as peak wave frequency.

Here, we are considering only waves propagating in x-direction. This is known as long crested or 2D waves. If we consider waves from various other directions, we have short crested or 3D waves.

There are a number of theoretical wave spectrum available to reasonably represent an irregular wave field. for example, JONSWAP, Pierson-Moskowitz (PM), Bretschneider etc. This gives closed form expression of wave spectrum $S(\omega)$ as a function of ω, peak wave frequency (ω_p), wind speed and a few other input parameters controlling the shape of the wave spectrum curve.

Body response in irregular wave



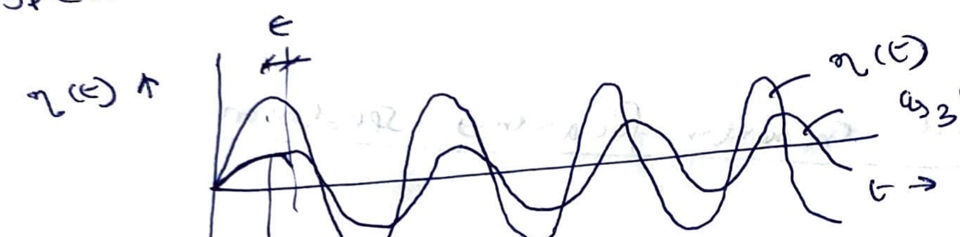
The above picture explains how do we get wave spectrum from a time history measurement at a location in irregular wave field.

For a floating body responding to a regular incident wave field at zero forward speed, we have,

$$\eta(t) = A \cos(kx - \omega t)$$

$$\omega_3(t) = \omega_{a3} \cos(\omega t - \epsilon)$$

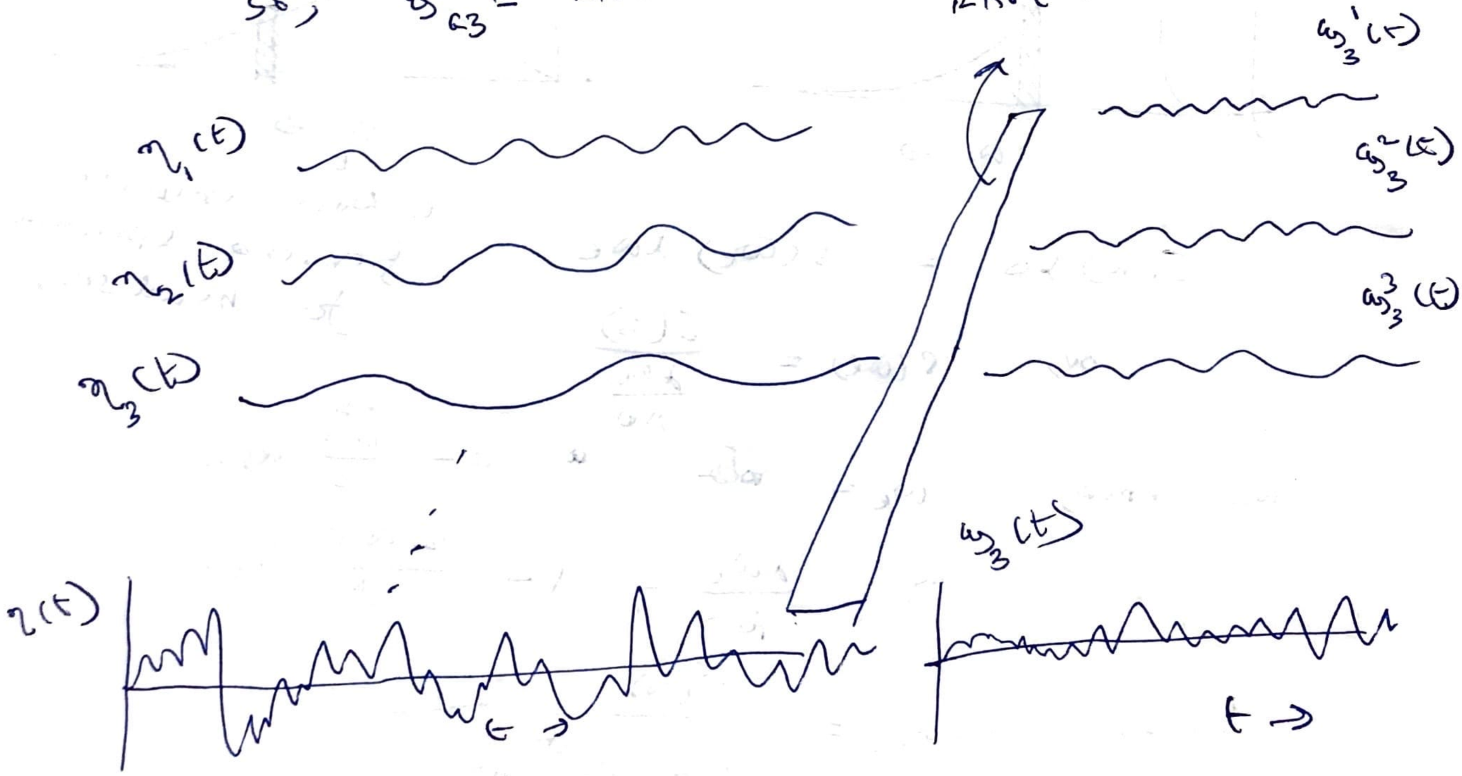
where, ϵ is the phase difference with respect to the incident wave field.

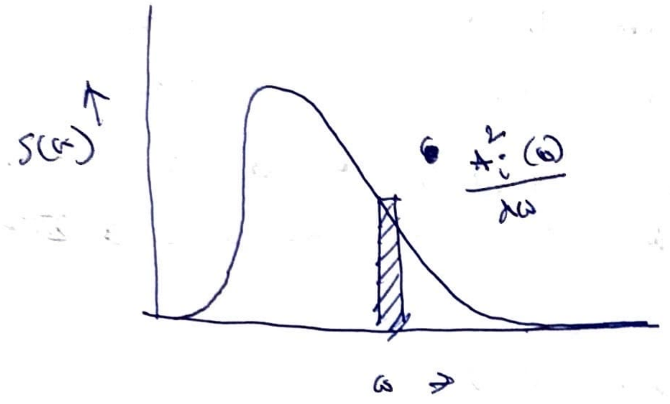


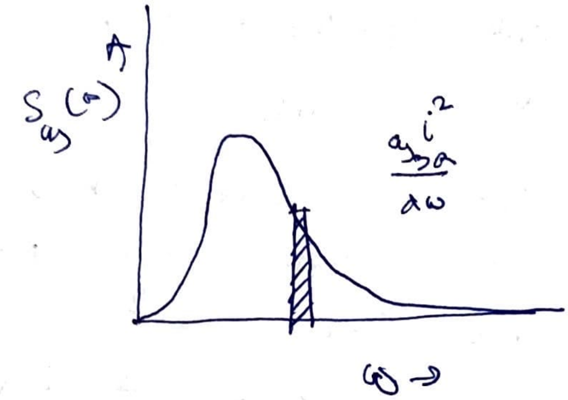
and the RAO is given as,

$$RAO = \frac{\omega_{a3}}{A}$$

So, $\omega_{a3} = RAO \cdot A$.



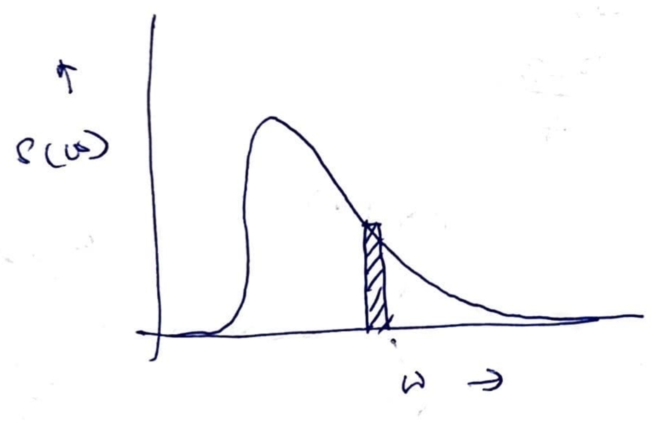


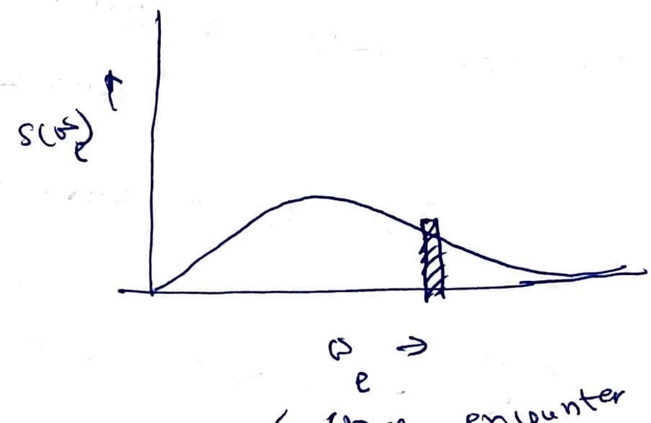


$$\frac{a_{3a}^{i\,2}}{d\omega} = RAO^2 \cdot \frac{A_i^2(\omega)}{d\omega}$$

$$= RAO^2 \cdot S(\omega) = S_{a_3}(\omega)$$

Wave encounter frequency spectrum





(Wave encounter frequency spectrum for head sea)

$$S(\omega)\, d\omega = S(\omega_e)\, d\omega_e$$

or, $$S(\omega_e) = \frac{S(\omega)}{\frac{d\omega_e}{d\omega}}$$

We know, $$\omega_e = \omega - \frac{\omega^2 U}{g} \cos\alpha.$$

or, $$\frac{d\omega_e}{d\omega} = 1 - \frac{2\omega U}{g} \cos\alpha.$$

thus, $$S(\omega_e) = \frac{S(\omega)}{1 - \frac{2\omega U}{g} \cos\alpha}$$