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Karman–Pohlhausen Momentum integral equations for boundary layer

- An method to find approximate solution for the ~~finding~~ Prandtl boundary layer equations.

~~x-momentum equation~~: – for a steady, two-dimensional incompressible flow,

x-momentum equation:

$$\int_0^\delta u \frac{\partial u}{\partial x}dz + \int_0^\delta w \frac{\partial u}{\partial z}dz = \int_0^\delta -\frac{1}{\rho}\frac{dp}{dx}dz + \int_0^\delta \nu \frac{\partial^2 u}{\partial z^2}dz$$

$$I_1 + I_2 = -I_3 + I_4. \quad — ①$$

$$I_4 \equiv \nu \int_0^\delta \frac{\partial^2 u}{\partial z^2} dz = \nu . \left[\frac{\partial u}{\partial z}\right]_0^\delta$$

$$= \nu . \left[\cancel{\frac{\partial u}{\partial z}}\right]_\delta^{\to 0} - \nu \left[\frac{\partial u}{\partial z}\right]_0 .$$

$$= - \nu . \left(\frac{\partial u}{\partial z}\right)\Big|_{z=0} = -\frac{\tau_{yx}}{\rho}.$$

$$I_2 = \int_0^\delta w \frac{\partial u}{\partial z} dz = [w u]_0^\delta - \int_0^\delta u \frac{\partial w}{\partial z} dz$$

$$= w_\delta . U + \int_0^\delta u \frac{\partial u}{\partial x} dz$$

$$\left[\because \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\right]$$

$$= U. \int_0^\delta \frac{\partial w}{\partial z} dz + \int_0^\delta u \frac{\partial u}{\partial x} dz$$

$$= - U \int_0^\delta \frac{\partial u}{\partial x} dz + \int_0^\delta u \frac{\partial u}{\partial x} dz \quad — ②$$

Substitution of ② into ① gives,

$$\int_0^\delta u \frac{\partial u}{\partial x} dz - U\int_0^\delta \frac{\partial u}{\partial x} dz + \int_0^\delta u \frac{\partial u}{\partial x} dz$$

$$= \int_0^\delta -\frac{1}{\rho}\frac{dp}{dx} dz - \frac{\tau_{yx}}{\rho}.$$

— ③

~~x-momentum equation~~ :

$$\cancel{u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g}$$

x- momentum eqaution at $z=\delta$, (~~at bounda~~ out side boundary layer, $U = U(x)$.)

inviscid zone, $\therefore \mu\frac{\partial^2 U}{\partial x^2} = 0$) since viscosity is zero)

$$U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx}. \quad \text{— (4)}$$

Substitute (4) into (3) and we get,

$$\int_0^\delta u\frac{\partial u}{\partial x}dz - U\int_0^\delta \frac{\partial u}{\partial x}dz + \int_0^\delta u\frac{\partial u}{\partial x}dz$$

$$= \int_0^\delta U\frac{\partial U}{\partial x}dz - \frac{\tau_{yx}}{\rho}.$$

or, $\int_0^\delta 2u\frac{\partial u}{\partial x}dz - \int_0^\delta U\frac{\partial U}{\partial x}dz - U\int_0^\delta \frac{\partial u}{\partial x}dz = -\frac{\tau_{yx}}{\rho}.$

or, $\int_0^\delta \left(2u\frac{\partial u}{\partial x} - u\frac{\partial U}{\partial x} - U\frac{\partial u}{\partial x}\right)dz = -\frac{\tau_{yx}}{\rho}$

Consider Leibnitz rule for integration,

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x}f(x,t)\,dt + f(x,b)\frac{d}{dx}(b) - f(x,a)\frac{d}{dx}(a)$$

$b = b(x)$
$a = a(x)$

we see, $\frac{\partial}{\partial x}[u(U-u)] = \frac{\partial}{\partial x}[uU - u^2]$

$$= u\frac{\partial U}{\partial x} + U.\frac{\partial u}{\partial x} - 2u\frac{\partial u}{\partial x}.$$

or, $(2u - U)\frac{\partial u}{\partial x} = u.\frac{\partial U}{\partial x} - \frac{\partial}{\partial x}[u(U-u)]$ — (5)

Substitute (5) into (4) and get,

$$-\int_0^\delta \frac{\partial}{\partial x}(Uu - u^2)\,dz - \int_0^\delta U\frac{\partial U}{\partial x}dz + \int_0^\delta u\frac{\partial U}{\partial x}dz = -\frac{\tau_{yx}}{\rho}.$$

or, $-\frac{d}{dx}\int_0^\delta (uU - u^2)\,dz - \int_0^\delta (U - u)\frac{dU}{dx}\,dz = -\frac{\tau_{yx}}{\rho}$.

or, $\frac{d}{dx} U^2 \underbrace{\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dz}_{\theta} + \frac{dU}{dx} \underbrace{\int_0^\delta U.\left(1 - \frac{u}{U}\right) dz}_{\delta^* U} = \frac{\tau_{yx}}{\rho}$

$$\boxed{\begin{aligned} uU - u^2 &= U^2 . \frac{uU - u^2}{U^2} = U^2 . \left(\frac{u}{U} - \frac{u^2}{U^2}\right) \\ &= U^2 . \frac{u}{U}\left(1 - \frac{u}{U}\right) \end{aligned}}$$

or, $\frac{d}{dx}\left(U^2 \theta\right) + \left(\frac{dU}{dx}\right) U . \delta^* = \frac{\tau_{yx}}{\rho}$.

(θ is momentum layer thickness, and δ^* is displacement thickness).

$\int_0^\delta \frac{\partial}{\partial x}(Uu - u^2)\,dz$

$$\frac{d}{dx} . \int_0^\delta (Uu - u^2)\,dz = \int_0^\delta \frac{\partial}{\partial x}(Uu - u^2)\,dz + (Uu - u^2)\Big|_{x=\delta} \frac{d}{dx}\delta(x)$$

at, $x = \delta$, $u = U$.

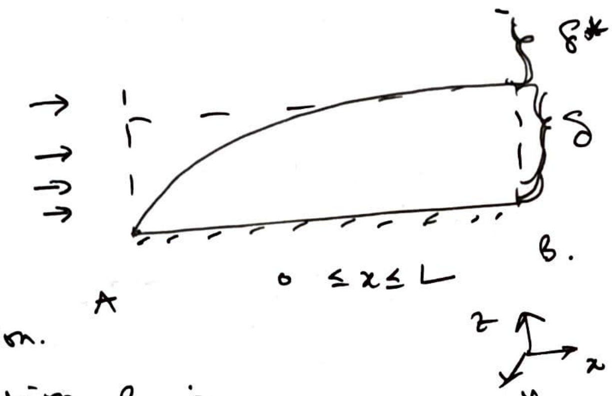
$\therefore\ Uu - u^2\Big|_{x=\delta} = U^2 - U^2 = 0$.

$\therefore\ \int_0^\delta \frac{\partial}{\partial x}(Uu - u^2)\,dz = \frac{d}{dx}\int_0^\delta (Uu - u^2)\,dz$

Physical significance of ~~boundary layer~~

~~Physical significance of boundary~~

physical significance of displacement thickness and momentum thickness



→ mass flow rate at Section A is $\rho U \delta . b$.
where b is the width of the plate along y direction.

→ mass flow rate at Section B is

$$\rho b \int_0^{\delta} u(z)\, dz$$

Since $\int_0^{\delta} u(z)\, dz < U.\delta$, the effective area at Section B has to be higher than boundary layer thickness at section B, in order to ~~...~~ preserve the mass flow rate along the ~~...~~ length of the boundary layer.

Let us denote this extra height above the boundary layer at section B as δ^*.

Thus,
$$\rho U.\delta\, b = \rho b \int_0^{\delta} u(z)\, dz + \rho U \delta^* b.$$

or,
$$U.\delta = \int_0^{\delta} u(z)\, dz + \delta^*$$

or,
$$\delta^* = \cancel{\int_0^{\delta} (u(z) - U)\, dz} \quad \left[\because U.\delta = \int_0^{\delta} U\, dz\right].$$

$$= \int_0^{\delta}$$

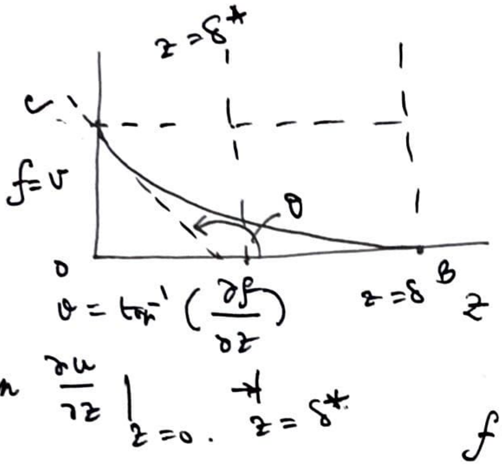
or,
$$U \delta^* = U \int_0^{\delta} dz - \int_0^{\delta} u(z)\, dz \quad \text{——(1)}$$

or,
$$U\delta^* = \int_0^{\delta} U\left(1 - \frac{u(z)}{U}\right) dz$$

$$\text{or,} \quad \delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dz$$

Now, the question is if displacement thickness is larger or smaller than the boundary layer thickness?

To answer that question, let us plot equation ①



$$f = U - u.$$

$$\frac{\partial f}{\partial z} = -\frac{\partial u}{\partial z}$$

$$\therefore \left.\frac{\partial f}{\partial z}\right|_{z=0} = -\left.\frac{\partial u}{\partial z}\right|_{z=0}.$$

$$\theta = \tan^{-1}\left(\frac{\partial f}{\partial z}\right)$$

$$\tau = \mu \left.\frac{\partial u}{\partial z}\right|_{z=0}.$$

$f = 0$, at $z = \delta$,

$f = U$, at $z = 0$.

~~tan~~ ~~τ is~~

τ is in same direction as $\frac{\partial u}{\partial z}$ at $z = 0$.

from distribution of $u(z)$ over z we know as z increases across the boundary layer, u increases and as z decreases, u decreases. Thus $\frac{\partial u}{\partial z}$ is positive for ~~a steady~~ inside a steady boundary layer. Thus, $\frac{\partial f}{\partial z} < 0$ at $z = 0$.

Thus, $\int_0^\delta (U - u)\, dz \equiv \int_0^\delta f(z)\, dz$ is actually area under the curve $\overline{BC}$. This is actually is deficit in the mass transport due to the presence of the boundary layer. By equating this with $U.\delta^*$ we try to balance this deficit in order to preserve conservation of mass.

$$U.\delta^* = \int_0^\delta (U - u)\, dz < U.\delta.$$

$$\text{or,} \quad \delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dz < \delta.$$

It is called displacement thickness ~~because~~ because it denotes the ~~disp~~ extra displacement along z axis we need to account above boundary layer thickness δ to preserve mass flow rate correctly.

Momentum thickness

momentum at section A is $\rho U \delta b U = \rho U^2 \delta b$.

momentum at section B is $\rho \int_0^{\delta} u^2(z)\,dz \; b$

$+ \rho U \delta^* U . b$.

$$= \rho b \left(\int_0^{\delta} u^2(z)\,dz + U^2 \delta^* \right)$$

$\therefore$ deficit in linear momentum is =

$$\rho U^2 \delta b - \rho b \int_0^{\delta} u^2(z)\,dz - \rho b U^2 \int_0^{\delta} \left(1 - \frac{u}{U}\right) dz$$

$$= \rho U^2 \delta b - \rho b \int_0^{\delta} u^2(z)\,dz - \rho b U^2 \int_0^{\delta} \frac{U - u}{U}\,dz$$

$$= \cancel{\rho U^2 \delta b} - \rho b \int_0^{\delta} u^2\,dz - \cancel{\rho b U^2 \delta b} + \rho b \int_0^{\delta} U u\,dz$$

$$= \rho b \int_0^{\delta} (U u - u^2)\,dz = \rho b U^2 \int_0^{\delta} \frac{u}{U^2}(U - u)\,dz$$

$$= \rho b U^2 \int_0^{\delta} \frac{u . U}{U^2} \cancel{\left(\frac{u}{U} - \frac{u}{U}\right)}\,dz$$

$$= \rho b u^2 \int_0^{\delta} \frac{u}{U} \cancel{\left(\frac{u}{}\right.}$$

$$= \rho b U^2 \int_0^{\delta} \frac{u . U}{U^2} \left(1 - \frac{u}{U}\right) dz$$

$$= \rho b U^2 \underbrace{\int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dz}_{\theta.}$$

[this is a way of writing so that the term inside integral is momentum thickness.

$$\theta = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dz$$ is momentum thickness. because in length scale]

(it gives information about deficit in rate of linear momentum at section B.).

Pohlhausen's approximation for solving momentum integral equations

We use a fourth ~~order~~ degree polynomial to approximate the velocity distribution $u(z)$ inside the boundary layer as.

$$\frac{u}{U} = a_0 + a\left(\frac{z}{\delta}\right) + b\left(\frac{z}{\delta}\right)^2 + c\left(\frac{z}{\delta}\right)^3 + d\left(\frac{z}{\delta}\right)^4.$$

Subjected to,

$$u = 0 \quad \text{at } z = 0 \quad -(1)$$

$$u = U \quad \text{at } z = \delta \quad -(2)$$

$$\frac{\partial u}{\partial z} = 0 \quad \text{at } z = \delta \quad -(3)$$

$$\frac{\partial^2 u}{\partial z^2} = 0 \quad \text{at } z = \delta \quad -(4).$$

application of boundary layer equations at $z = \delta$ gives, (x-momentum only)

$$U . \frac{\partial U}{\partial x} + 0 . \frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 U}{\partial z^2}$$

$$U = U(x)$$

$$\therefore \frac{\partial U}{\partial x} = \frac{dU}{dx}.$$

$$\frac{\partial^2 U}{\partial z^2} = 0.$$

or,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = U \frac{\partial U}{\partial x}.$$

if $\frac{\partial p}{\partial x} = \frac{dp}{dx}$ is constant ~~across~~ ~~across~~ across the boundary layer, we have

$$-\frac{1}{\rho}\frac{dp}{dx} = U . \frac{\partial U}{\partial x}.$$

On the other hand application of x-momentum equation at $z = 0$ gives,

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2 u}{\partial z^2}\right)$$

$$u, w = 0 \quad \text{at } z = 0.$$

$$\therefore \nu \frac{\partial^2 u}{\partial z^2} = \frac{1}{\rho}\frac{dp}{dx} = -U . \frac{\partial U}{\partial x}.$$

or, $\nu \frac{\partial^2 u}{\partial z^2} = - U \cdot \frac{dU}{dx}$. —(5)

Satisfaction of boundary conditions gives,

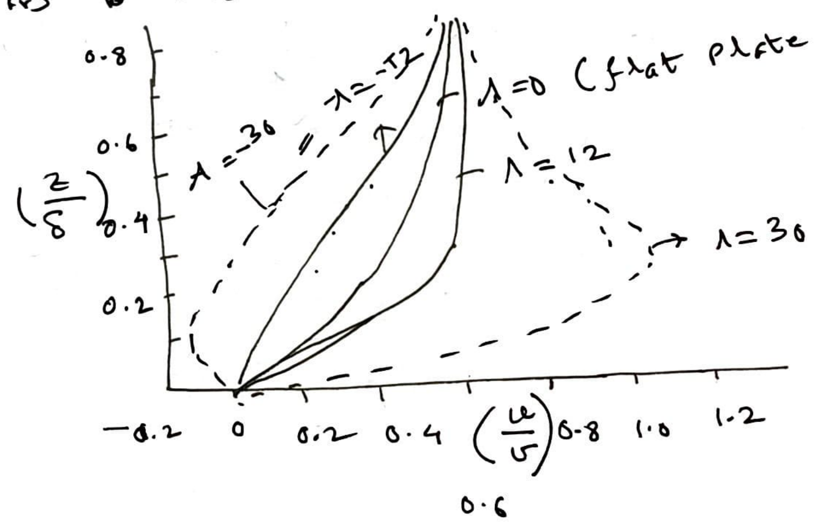
$$a_0 = 0 \; ; \quad b = -\frac{\Lambda}{2}$$
$$a = 2 + \frac{\Lambda}{6} \; ; \quad c = -2 + \frac{\Lambda}{2} \qquad \text{—(6)}$$
$$d = 1 - \frac{\Lambda}{6}$$

where, $\Lambda = \left(\frac{\delta^2}{\nu}\right) \cdot \frac{dU}{dx}$.

(cross check as part of assignment). (L8 A1)

~~With these~~ ~~With these~~

With these solutions of the unknown coefficients, the velocity profile inside the boundary layer is uniquely defined through a parameter known as shape factor. This is a very important parameter in ship hydrodynamics, since with the help of this parameter we can approximate the velocity profile for a given within the boundary layer profile for a ship hull.



- ~~positive~~ positive value of Λ refers to favorable ~~press~~ pressure gradient.

- negative value of Λ refers to adverse pressure gradient.

- $\Lambda < -12$ refers to cases with separation of boundary layer.

(cross check assignment L8A2)

- $\Lambda > 12$ is avoided ~~since~~ since it refers to a case when velocity inside a boundary layer may be higher than U.

with the help of the coefficients in the fourth degree of the polynomial used for approximating the velocity profile inside the boundary layer, we are able to find out the displacement thickness and the momentum thickness as,

$$\delta^* = \int_0^{\delta} \left(1-\frac{u}{U}\right) dz = \delta\left(\frac{3}{10}-\frac{\Lambda}{120}\right),$$

$$\theta = \int_0^{\delta} \frac{u}{U}\left(1-\frac{u}{U}\right) dz = \delta\left(\frac{37}{315}-\frac{\Lambda}{945}-\frac{\Lambda^2}{9072}\right)$$

$$\tau_{yx} = \mu \left.\frac{\partial u}{\partial z}\right|_{z=0} = \frac{\mu U}{\delta}\left(2+\frac{\Lambda}{6}\right)$$

Substitution of δ^*, θ, and τ_{yx} back into momentum integral equations gives an equation we can solve to find out $\delta(x)$. This is much easier if we consider the flat plate case, for which, $\Lambda = 0$, $\frac{dU}{dx} = 0$ and, thus, $\frac{dp}{dx} = 0$.

The momentum integral equation for this case then becomes,

$$\frac{\tau_{yx}}{\rho} = \frac{d}{dx}\left(U^2 \theta\right) \quad \text{—(7)}$$

or, $$\delta \, d\delta = \left(\frac{2\nu U}{\alpha U^2}\right) dx = \left(\frac{2\nu}{\alpha U}\right) dx$$

where, $\alpha = \frac{37}{315}$.

by integrating this equation (7) we get

$$\frac{\delta^2}{2} = \frac{2\nu}{\alpha U} \cdot x + C_1$$

as, $x \to 0$, $\delta \to 0$, thus $C_1 = 0$.

or, $$\delta^2 = \frac{4\nu}{\alpha U} \cdot x.$$

or, $$\delta = 5.8356 \sqrt{\frac{\nu x}{U}}$$

$$= 5.8356 \sqrt{Re} \, . \, x.$$

$$Re = \frac{Ux}{\nu}$$

So, $\frac{\nu x}{U} = \frac{\nu x^2}{U . x} = \frac{x^2}{Re}$.

$\therefore$ $\sqrt{\frac{\nu x}{U}} = \frac{x^2}{Re}$.

So, $\delta^* = 0.3 . . \delta = 1.75 . \sqrt{\frac{\nu x}{U}}$,

- this displacement thickness is very close to value predicted by classing similarity solution.