Advanced Marine Hydrodynamics NA61202 3-1-0

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Lecture 4: Some elements of kinematics of fluid flows (Part II)

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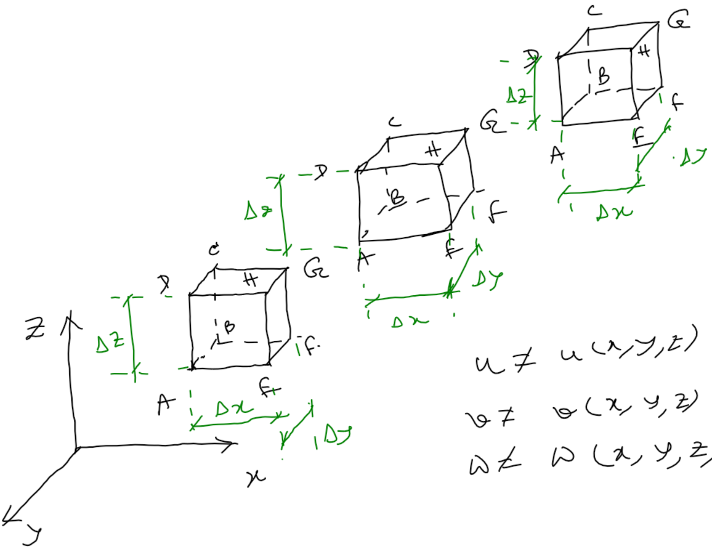
Overview

Translation without deformation

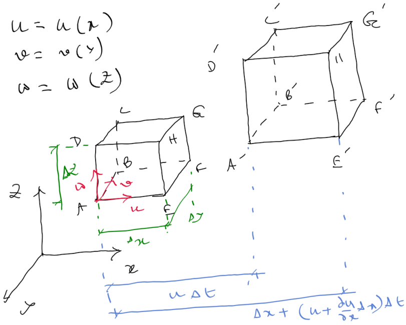


$$u \neq u(x, y, z)$$

$$v \neq v(x, y, z)$$

$$w \neq w(x, y, z)$$

Translation with linear deformation



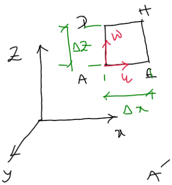
- new length $A'E' = \Delta x + \left(u + \frac{\partial u}{\partial x}\Delta x\right)\Delta t - u\Delta t = \Delta x + \frac{\partial u}{\partial x}\Delta x \Delta t$.
- rate of the change in length along x-axis per unit length, i.e., the linear strain rate $\dot{\epsilon_x} = (A'E' - AE)\,/\,(\Delta x \Delta t) = \frac{\partial u}{\partial x}\Delta x \Delta t / (\Delta x \Delta t) = \frac{\partial u}{\partial x}$.
- Similarly, $\dot{\epsilon_y} = \frac{\partial v}{\partial y}$ and $\dot{\epsilon_z} = \frac{\partial w}{\partial z}$.

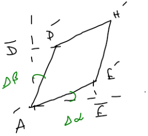
Contd..

- Old volume $\Delta x \Delta y \Delta z$.
- New volume $\left(\Delta x + \frac{\partial u}{\partial x}\Delta x \Delta t\right)\left(\Delta y + \frac{\partial v}{\partial y}\Delta y \Delta t\right)\left(\Delta z + \frac{\partial w}{\partial z}\Delta z \Delta t\right) = \Delta x \Delta y \Delta z + \frac{\partial u}{\partial x}\Delta x \Delta y \Delta z \Delta t + \frac{\partial v}{\partial y}\Delta x \Delta y \Delta z \Delta t + \frac{\partial w}{\partial z}\Delta x \Delta y \Delta z \Delta t$.
- volumetric strain $\dot{\epsilon} = \frac{1}{\bar{v}}\frac{D\bar{v}}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathrm{V}$.
- For an incompressible flow, $\nabla \cdot \mathrm{V} = 0$.

$u = u(x, z)$

$w = w(x, z)$





$A'E' = \Delta x + \frac{\partial u}{\partial x} \Delta x \Delta t$

$EE' = \frac{\partial w}{\partial x} \Delta x \Delta t$

$A'D' = \Delta z + \frac{\partial w}{\partial z} \Delta z \Delta t$

$DD' = \frac{\partial u}{\partial z} \Delta z \Delta t$

- Consider two sides, i.e., AD and AE mutually perpendicular at time t. At time $t + \Delta t$, fluid element deforms so that the same two sides now become $A'D'$ and $A'E'$.
- The old angle is $\pi/2$. The new angle is $\pi/2 - \Delta\alpha - \Delta\beta$. Total change of angle is $\pi/2 - \pi/2 + \Delta\alpha + \Delta\beta = \Delta\alpha + \Delta\beta$.
- $\tan(\Delta\alpha) \approx \Delta\alpha = \frac{\frac{\partial w}{\partial x}\Delta x \Delta t}{\Delta x + \frac{\partial u}{\partial x}\Delta x \Delta t}$.

Contd..

- $\lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t} = \frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\partial w}{\partial x}$.
- $\lim_{\Delta t \to 0} \frac{\Delta \beta}{\Delta t} = \frac{\mathrm{d}\beta}{\mathrm{d}t} = \frac{\partial u}{\partial z}$.
- rate of angular deformation $\dot{\epsilon}_{\theta y} = \left(\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \frac{\mathrm{d}\beta}{\mathrm{d}t}\right) = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$.
- similarly, $\dot{\epsilon}_{\theta x} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$ and $\dot{\epsilon}_{\theta z} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$.
- The components $\epsilon_{\theta x}, \epsilon_{\theta y}and\epsilon_{\theta z}$ comprises the shear strain. The definition is same as what we use for a solid element. The shear strain rate (which is characteristics of a fluid flow) is a gradient of a vector, ∇V. So, the rate of angular deformation is also known as shear strain rate.

Angular velocity

- The angular velocity of a fluid element at point A is defined as $\omega_y = \frac{1}{2}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t} - \frac{\mathrm{d}\beta}{\mathrm{d}t}\right) = \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)$ considering rotation anti-clockwise as positive.
- Similarly, $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$ and $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$, provided we are always at the first octant while viewing these rotations.
- The rate of angular deformation together with the angular velocity comprises the gradient of the velocity vector. This is a *Tensor* given as $\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\dot{\epsilon}_{\theta ij} + \omega_{ij}$.
- $\dot{\epsilon}_{\theta ij}$ is symmetric and ω_{ij} is skew-symmetric.

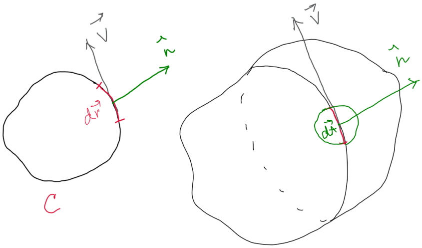
Vorticity

- The rotation in a fluid flow is measured by vorticity $\Omega = 2\omega$ and $\Omega = \nabla \times V$. The final expression depends on the chosen coordinate frame.
- If $\Omega = 0$ we call the flow as irrotational and if $\Omega \neq 0$ we call the flow as rotational.
- This is very powerful: Without resort to a flow visualization, we can say very important feature (i.e., there is some rotation in the flow or not) by simply following some vector operations.

Speical Cases

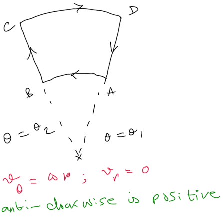
- Case I: $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, $\dot{\epsilon}_{\theta z} = 0$ but $\omega_z = \frac{\partial v}{\partial x}$. This is pure rotation without any angular deformation.
- Case II: $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, $\dot{\epsilon}_{\theta z} = 2\frac{\partial v}{\partial x}$ but $\omega_z = 0$. This is pure angular deformation without any rotation.

Circulation



- The circulation is defined as $\tau = -\oint_C \mathrm{V} \cdot \mathrm{dr}$.
- Following Stokes' theorem, we have $\tau = -\oint_C \mathrm{V} \cdot \mathrm{dr} = -\int_S (\nabla \times \mathrm{V})\, \hat{\mathrm{n}} \mathrm{d}A$.
- For an irrotational flow, $\tau = 0$.

Force vortex



- Assume a scenario, $v_\theta = \omega r$ and $v_r = 0$. This is a forced vortex.
- $\tau = -\oint_{ABCD} \mathrm{V} \cdot \mathrm{ds} = 2\omega r \mathrm{d} r \Delta\theta$, where $\Delta\theta = \theta_2 - \theta_1$.
- Circulation per unit area $= \tau / (r \mathrm{d} r \Delta\theta) = 2\omega$, i.e., the vorticity.

Free vortex

- For the same element, assume, $v_\theta = 0$ and $v_r = c/r$. This is known as free vortex.
- For this case, $\tau = 0$.
- what happens at the center, i.e., $r = 0$?.
- The answer is based on the choice of the contour, i.e., we include the center or not.