Advanced Marine Hydrodynamics NA61202 3-1-0

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Lecture 5: Conservation Laws (Part I)

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Spring 2023

Overview

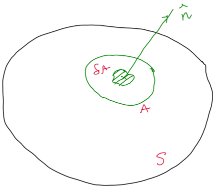

Figure: Traction on an elemental area.

- Let δF be the force exerted by the Fluid exterior to δA in the direction of n̂ on the inner side of δA.
- We can define $\mathrm{T}^n = \lim_{\delta A \to 0} \frac{\delta \mathrm{F}}{\delta \mathrm{A}} = \frac{\mathrm{dF}}{\mathrm{dA}}$.
- The *traction* T^n is a vector telling us about force per unit area and having components T_j^n.
- There is a relation which relates T^n with stresses.

Stress components on a cube

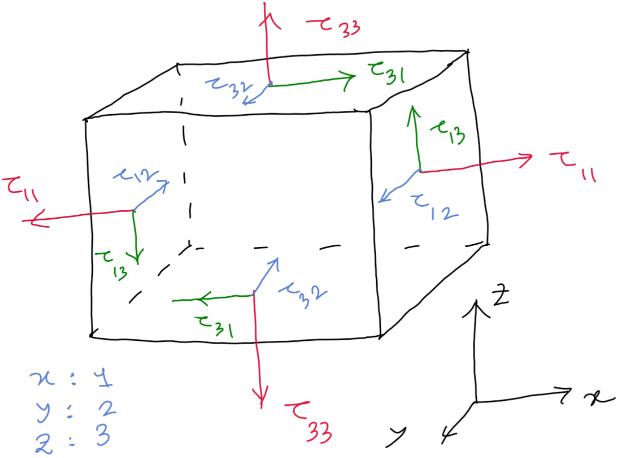

Figure: Stress components.

Stress components on a tetrahedral element

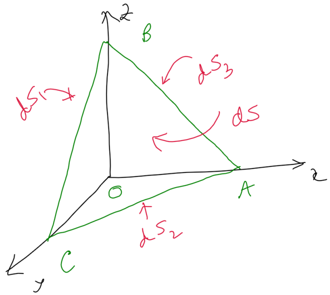

Figure: Stress components on a tetrahedral element.

- $\mathrm{d}S1 = \mathrm{d}Sn_1; \mathrm{d}S2 = \mathrm{d}Sn_2$ and $\mathrm{d}S3 = \mathrm{d}Sn_3$.
- volume of the element $\mathrm{d}V = h\mathrm{d}S/3$.
- T^n be the traction on the surface $\mathrm{d}S$.
- Let b_1 be the body force per unit mass.

Stress Equilibrium

- Let $a1$ is the acceleration of the element along x.
- If the mass of the element is $\mathrm{d}m = \rho h \mathrm{d}S/3$ and acceleration is a_1, then as per Newton's 2nd Law,
  $(-\tau_{11}\mathrm{d}Sn_1 - \tau_{21}\mathrm{d}Sn_2 - \tau_{31}\mathrm{d}Sn_3 + T_1^n\mathrm{d}S) + b_1 h\mathrm{d}S/3 = \rho h\mathrm{d}Sa_1/3.$
- As $h \to 0$, $T_1^n = \tau_{11}n_1 + \tau_{21}n_2 + \tau_{31}n_3 = \sum_{j=1}^{3} \tau_{j1}n_j$.
- in matrix-vector form $\begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$.
- The above is known as Cauchy's Theorem and the stress components are respresnted by a Tensor.

Transport Theorem

- $I\left(t\right) = \int_{V(x,y,z,t)} f\left(x, y, z, t\right) \mathrm{d}V.$
- $\Delta I = I\left(t + \Delta t\right) - I\left(t\right) = \int_{V(t+\Delta t)} f\left(x, y, z, t + \Delta t\right) \mathrm{d}V - \int_{V(t)} f\left(x, y, z, t\right) \mathrm{d}V.$
- Assuming in Δt time interval, the volume V acquires a new volume $V + \mathrm{d}V$. Then, $\Delta I = \int_{V+\Delta V} \left(f + \Delta t \frac{\partial f}{\partial t}\right) \mathrm{d}V - \int_V f \mathrm{d}V = \int_V f \mathrm{d}V + \Delta t \int_V \frac{\partial f}{\partial t} \mathrm{d}V + \int_{\Delta V} f \mathrm{d}V + \Delta t \int_{\Delta V} \frac{\partial f}{\partial t} \mathrm{d}V - \int_V f \mathrm{d}V.$
- The first and the fifth term together cancels out. If the volume V is surrounded by the surface S then ΔV is the volume contained within $S\left(t\right)$ and $S\left(t + \Delta t\right)$. This region has a thickness $\mathrm{V}_n \Delta t$, where V_n is the component of the velocity in the direction normal to surface S_n. Thus, $\Delta t \int_{\Delta V} \frac{\partial f}{\partial t} \mathrm{d}V = \Delta t \int_S \frac{\partial f}{\partial t} \left(\mathrm{V}_n \Delta t\right) \mathrm{d}S = \left(\Delta t\right)^2 \int_S \frac{\partial f}{\partial t} \left(\mathrm{V}_n\right) \mathrm{d}S.$ This can be neglected since we are only retaining terms upto order of Δt.

Contd..

- Using the same principle, the third term becomes $\int_{\Delta V} f \mathrm{d}V = \int_S f \mathrm{V}_n \Delta t \mathrm{d}S$.
- $\Delta I = \Delta t \int_V \frac{\partial f}{\partial t} \mathrm{d}V + \int_S (\mathrm{V}_n \Delta t) f \mathrm{d}S$.
- $\frac{\mathrm{d}I}{\mathrm{d}t} = \int_V \frac{\partial f}{\partial t} \mathrm{d}V + \int_S f \mathrm{V}_n \mathrm{d}S$.
- Application of Gauss divergence theorem on the second term gives, $\frac{\mathrm{d}I}{\mathrm{d}t} = \int_V \left(\frac{\partial f}{\partial t} + \nabla f \cdot \mathrm{V}_n \right) \mathrm{d}V$

Conservation of Mass

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \,\mathrm{d}V = 0.$$

Conservation of Momentum

- $\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho u_i \mathrm{d}V = \int_S \tau_{ij}\hat{\mathrm{n}}_j \mathrm{d}S + \int_V F_i \mathrm{d}V$.
- The first term can be simplified using Gauss Divergence Theorem and $\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho u_i \mathrm{d}V = \int_V \left[\frac{\partial \tau_{ij}}{\partial x_j} + F_i\right] \mathrm{d}V$.

Continuity Equation

- By putting $f = \rho$ into transport theorem
  $\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathrm{d}V = \int_V \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right] \mathrm{d}V.$
- $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0.$