Strassen's Matrix Multiplication

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Strassen's Matrix Multiplication

- Belongs to Divide and Conquer Paradigm
- How Matrix Multiplication done Normally
- How divide and Conquer strategy applied for matrix multiplication
- What Strassen had done to improve matrix multiplication using divide and conquer strategy

Matrix Multiplication

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \; B \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

2× 2 2× 2

$$= C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

2 × 2

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \times \; B_{ik}$$

Matrix Multiplication (contd.)

```
for(i=0;i<n;i++)
    {
     for(j=0;j<n;j++)
       {
        C[i,j]=0;
         C[i,j] = A[i,k] ×B[k,j];
         }
     }
```

Time Complexity of Matrix multiplication is $O(n^3)$

Divide and Conquer strategy for matrix multiplication

$$A = \underset{2\times 2}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \times B \underset{2\times 2}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}$$

$$= C = \underset{2\times 2}{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}$$

- Multiplying a 2×2 matrix is a small problem
- $c_{11} = a_{11} \times b_{11} + a_{12} \times b_{21}$
- $c_{12} = a_{11} \times b_{12} + a_{12} \times b_{22}$
- $c_{21} = a_{21} \times b_{11} + a_{22} \times b_{21}$
- $c_{22} = a_{21} \times b_{12} + a_{22} \times b_{22}$

Divide and Conquer strategy for matrix multiplication (contd.)

- If each addition term takes one unit of time, then total time is 4 unit, it's a constant
- If there are 8 multiplications and take 8 unit of time, it is also a constant.
- These 4 formulas take constant time.
- If we want to reduce the problem into 1x1 matrix

$$A = [a_{11}],\ B = [b_{11}]$$

$$A \times B = C$$

$$C = [a_{11} * b_{11}]$$

Divide and Conquer strategy for matrix multiplication (contd.)

- If the row/column size is grater than 2, we need to divide the problem and solve each single problem
- We take the problem or assume the problem as power of 2.
- If the matrix is not power of 2 then we make it as power of 2 by adding 0's

Divide and Conquer strategy for matrix multiplication (contd.)

$$
A = \begin{array}{c}
\begin{array}{cc} A_{11} & A_{12} \end{array} \\
\left[\begin{array}{cc|cc}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\ \hline
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{array}\right] \\
\begin{array}{cc} A_{21} & A_{22} \end{array}
\end{array}_{4/2 \times 4/2}
\times B
\begin{array}{c}
\begin{array}{cc} B_{11} & B_{12} \end{array} \\
\left[\begin{array}{cc|cc}
b_{11} & b_{12} & b_{13} & b_{14} \\
b_{21} & b_{22} & b_{23} & b_{24} \\ \hline
b_{31} & b_{32} & b_{33} & b_{34} \\
b_{41} & b_{42} & b_{43} & b_{44}
\end{array}\right] \\
\begin{array}{cc} B_{21} & B_{22} \end{array}
\end{array}_{4/2 \times 4/2}
$$

$$
= C = \begin{array}{c}
\begin{array}{cc} A_{11} & A_{12} \end{array} \\
\left[\begin{array}{cc|cc}
c_{11} & c_{12} & c_{13} & c_{14} \\
c_{21} & c_{22} & c_{23} & c_{24} \\ \hline
c_{31} & c_{32} & c_{33} & c_{34} \\
c_{41} & c_{42} & c_{43} & c_{44}
\end{array}\right] \\
\begin{array}{cc} A_{11} & A_{12} \end{array}
\end{array}_{4/2 \times 4/2}
$$

Here A and B are divided into four 2x2 matrices

Divide and Conquer strategy for matrix multiplication (contd.)

- Algorithm of divide and conquer matrix multiplication

```
Algorithm MM(A,B,n)
   {
    if(n≤2)
     {
     c11 = a11 × b11 + a12 × b21
     c12 = a11 × b12 + a12 × b22
     c21 = a21 × b11 + a22 × b21
     c22 = a21 × b12 + a22 × b22
     }
    else
     {
      MM(A11,B11,n/2)+ MM(A12,B21,n/2)
      MM(A11,B12,n/2)+ MM(A12,B22,n/2)
      MM(A21,B11,n/2)+ MM(A22,B21,n/2)
      MM(A21,B12,n/2)+ MM(A22,B22,n/2)
      }
    }
```

Divide and Conquer strategy for matrix multiplication (contd.)

- Time Complexity

8 times recursive call is required.

$$T(n)=\begin{cases} 1 & n \leq 2 \\ 8T\left(\frac{n}{2}\right) + n^2 & n > 2 \end{cases}$$

Time complexity is $\theta(n^3)$

- If we reduce the number of multiplication we can make the algorithm faster

Strassen's Matrix Multiplication

- Reduce 8 to 7 multiplication
- Different equations are applied on the four formulas
- No. of multiplication is reduced but addition and subtraction increased.

$P= (A_{11}+A_{22})(B_{11}+B_{22})$

$Q=(A_{21}+A_{22})$

$R= A_{11}(B_{12}-B_{22})$

$S=A_{22}(B_{21}-B_{11})$

$T=(A_{11}+A_{12})B_{22}$

$U=(A_{21}-A_{11})(B_{11}+B_{12})$

$V=(A_{12}-A_{22})(B_{21}+B_{22})$

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Strassen's Matrix Multiplication(contd)

C_{11} = P+S-T+V

C_{12} = R+T

C_{21} = Q+S

C_{22} = P+R-Q+U

Recurrence relation

$$T(n)=\begin{cases} 1 & n\leq 2 \\ 8T\left(\frac{n}{2}\right) + n^2 & n>2 \end{cases}$$

Time complexity is $O(n^{2.81})$